ISOMETRIES IN EUCLIDEAN SPACE, PATTERNS & SPACE GROUPS

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ISOMETRIES IN EUCLIDEAN SPACE

[1]If we consider the vector space R^n we can define notions of length and angles using the dot product of vectors, creating a Euclidean Space we can write as E^n. An isometry , or rigid motion, f of E^n is a mapping which preserves distance so that, given two points x and y in R^n:

$$||f(x) - f(y)|| = ||y - x||$$

Two important types of isometries are orthogonal mappings and translations.

ORTHOGONAL TRANSFORMATIONS

Orthogonal mappings form the orthogonal group $O(n)$: these are linear invertible transformations which preserve the inner product, meaning that if $\phi \in O(n)$:

$$x \cdot y = \phi(x) \cdot \phi(y)$$

We can prove any orthogonal transformation is an isometry: given $\phi \in O(n)$ and p, q $\in R^n$:

$$||\phi(p)||^2 = \phi(p) \cdot \phi(p) = p \cdot p = ||p||$$

as ϕ preserves dot product

$$||\phi(p) - \phi(q)|| = ||\phi(p - q)|| = ||p - q||$$

as ϕ is a linear transformation

The last line shows that ϕ preserves distance and is therefore an isometry. One can prove that any isometry f such that $f(0) = 0$ is linear, preserves the inner product and is therefore an orthogonal transformation $(*)$.[2]

TRANSLATIONS

A translation T(v) given a vector v in R^n is a mapping which sends any vector x to $x + v$. Let V be the group of all translations under composition. Any $T(v) \in V$ preserves distance as:

$$||T(v)(x) - T(v)(y)|| = ||(x + v) - (y + v)|| = ||x - y||$$

EUCLIDEAN GROUP

[2]An important result in Euclidean Geometry is that every isometry f can be written as a composition of a translation and an orthogonal mapping $(**)$: T(v)$\circ\phi$. Given an isometry f, consider the translations $T(f(0))$ and $T(-f(0))$. $(T(-f(0)) \circ f)(0) = 0$ so by $(*)$ above, $T(-f(0)) \circ f$ is an orthogonal transformation $\phi \in O(n)$. Then

$$T(f(0)) \circ T(-f(0)) \circ f = f = T(f(0)) \circ \phi$$

Composing two isometries gives another isometry so under composition, isometries form a group Isom(E^n) called the Euclidean group or the group of rigid motions of E^n. Let (v, ϕ) denote the element $T(v) \circ \phi$ in Isom(E^n) where $T(v) \in V$ and $\phi \in O(n)$ so that:

$$(v, \phi)(x) = (T(v) \circ \phi)(x) = v + \phi(x)$$

Composition in the group works in the following way:

$$\begin{aligned}(v, \phi) \circ (v', \phi')(x) &= (v, \phi)(v' + \phi'(x)) \\ &= v + \phi(v' + \phi'(x)) \\ &= v + \phi(v') + \phi(\phi'(x)) \\ &= (v + \phi(v'), \phi\phi')(x)\end{aligned}$$

STRUCTURE OF ISOM(E^n)

[1]Given two groups K and H we define an action α of H on K as $\alpha : H \to Aut(K)$ where Aut(K) is the set of automorphisms of K (isomorphisms from K to K). We can then define the semidirect product group structure $K \rtimes_\alpha H$ on the set $K \times H$ by the multiplication:

$$(k, h) \cdot_\alpha (k', h') = (k + \alpha(h)(k'), hh')$$

We can see that this corresponds to the composition on Isom(E^n) with $K = V$ and $H = O(n)$ as $\phi(v)$ is an automorphism on V . So we can write

$$Isom(E^n) = V \rtimes O(n)$$

ISOMETRIES IN LOWER DIMENSIONS

The result $(**)$ stated above gives us a way of understanding what isometries exist in dimension n once we know what kind of transformations $O(n)$ contains.

1D: $O(1)$ only contains the identity [1] and the reflection about the origin [-1] so the only possible isometries are reflections and translations.

2D: [3]$O(2)$ contains rotations and reflections and consequently the possible isometries are rotations, reflections, translations and glide reflections (reflections about an axis followed by a translation along that axis).

3D: $O(3)$ contains rotations, reflections and a combination of both: improper rotations which are equivalent to a rotation about an axis followed by a reflection in a plane perpendicular to that axis. Composing these transformations with translations gives us the possible isometries in three dimensions.

LATTICES AND PRESERVING TRANSFORMATIONS

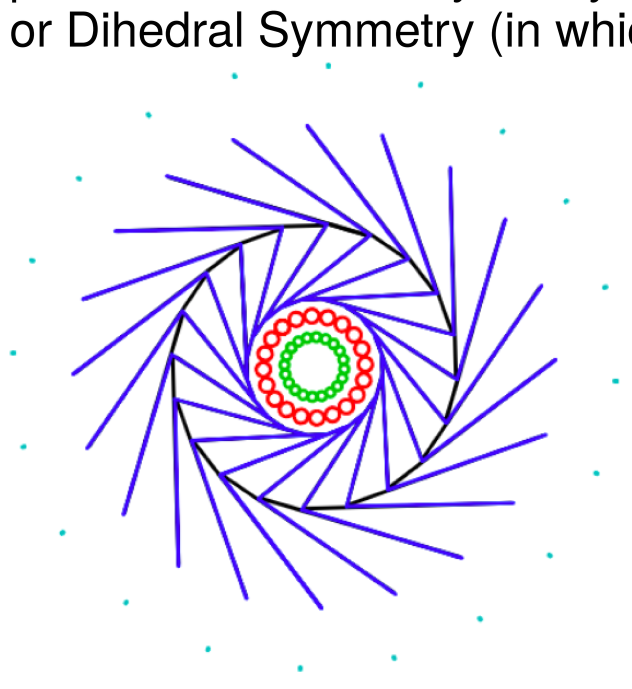
[4]In any dimension n we can construct a lattice: a set T of points generated by linearly independent vectors added with integer coefficients, or in other words the set of points the origin is mapped to by the group of translations M generated by the chosen vectors $(v_1, v_2 \ldots)$ under composition.
The lattice divides E^n into tiles, portions of space with vertices corresponding to lattice points: in each tile of the lattice one can introduce a pattern or structure. Given a certain lattice one can consider the group G of isometries which preserve it with its pattern called the **space group**. Translations in M obviously preserve both as applying a translation to a lattice point moves it to another point in the lattice and the pattern within each tile is translated to a position where it was already present (in another tile) without any other change. The other applicable isometries depend on the symmetries of the lattice and of the pattern or structure within each tile.

TWO DIMENSIONAL LATTICES AND SYMMETRY GROUPS

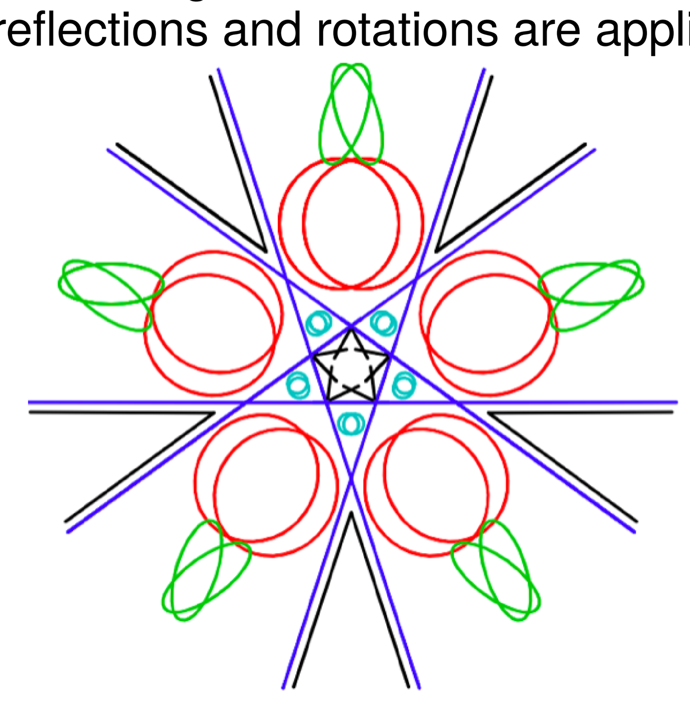
Depending on the how many linearly independent vectors are used to generate M (in two dimensions at most 2) we get different patterns and therefore groups of symmetries.

Rosette Groups

[4]The group of translations for Rosette groups only contains the zero vector and therefore the lattice consists of only the origin. Depending on the pattern on the plane we can have Cyclic Symmetry (preserving transformations are only rotations) or Dihedral Symmetry (in which case reflections and rotations are applicable).[5]



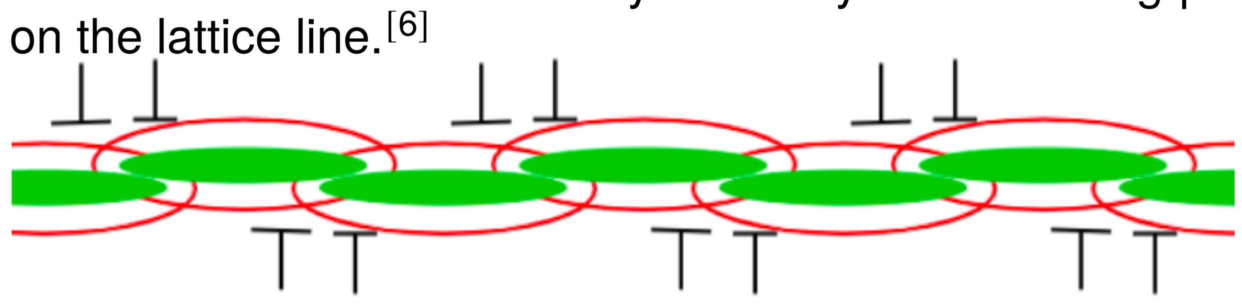
Rosette Group with Cyclic Symmetry of order 20



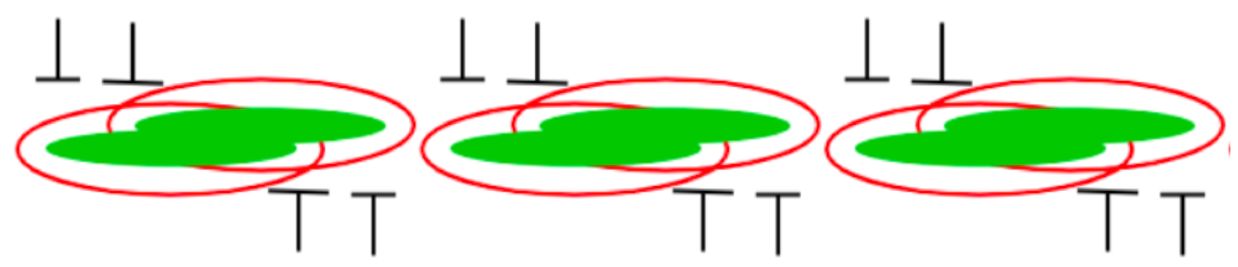
Rosette Group with Dihedral Symmetry of order 5

Frieze Groups

[4]The translation Group M for Frieze groups is generated by a single vector so the lattice is a set of points on a line. We can imagine the 2D plane covered by a series of tiles which extend vertically to infinity: the following pictures only show the part on the lattice line.[6]



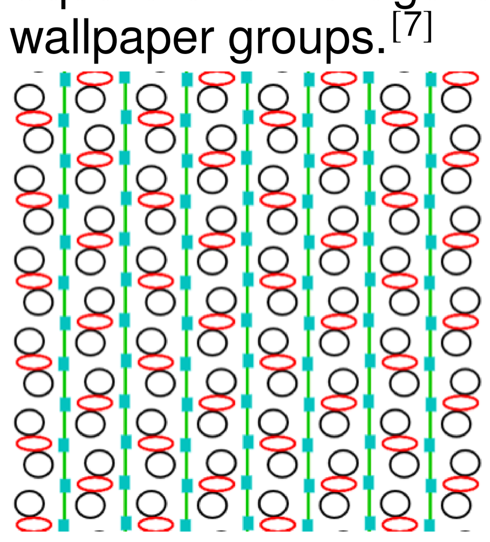
Frieze Group with glide reflection symmetry along axis of translation



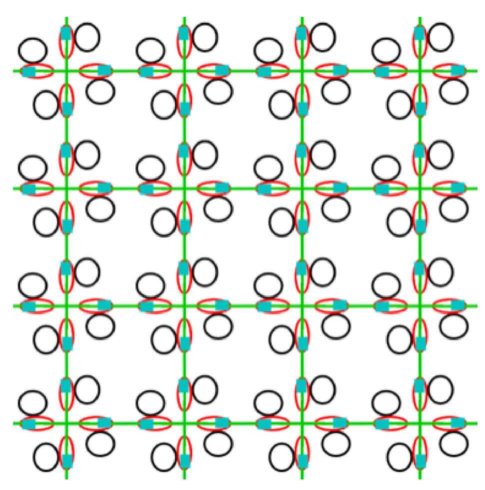
Frieze Group with rotational symmetry

Wallpaper Groups

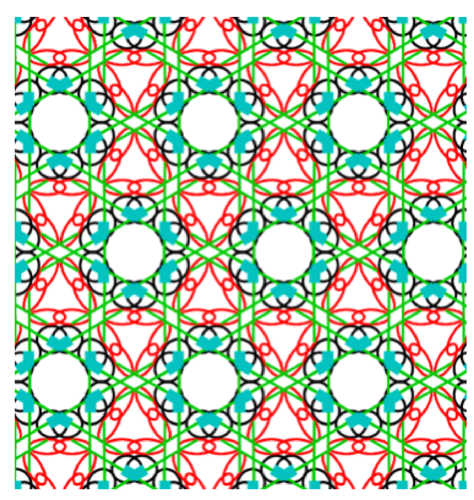
[3]When the group of translations M is generated by two linearly independent vectors there are five possible types of lattices (Oblique, Rectangular, Centred Rectangular, Square and Hexagonal) and, depending on the pattern on each tile, 17 different wallpaper groups.[7]



Wallpaper pattern with glide reflection symmetry along both horizontal and vertical axes and rotational symmetry of order 2



Wallpaper pattern with rotational symmetry of order 4



Wallpaper with rotational symmetry of order 6 and reflectional symmetry along both horizontal and oblique axis

REFERENCES

[1] H. Hiller. Crystallography and Cohomology of Groups. The American Mathematical Monthly, 93:10, 765-779, 1986.

[2] B. O'Neill . Elementary Differential Geometry, Revised 2nd Edition. Academic Press, 2006.

[3] D. Schattschneider. The Plane Symmetry Groups: Their Recognition and Notation. The American Mathematical Monthly, 85:6, 439-450, 1978.

[4] B. Davvaz. Groups and Symmetry : Theory and Applications. Springer, 2021.

[5] Images generated using https://math.hws.edu/eck/js/symmetry/rosette.html.

[6] Images generated using https://math.hws.edu/eck/js/symmetry/frieze.html.

[7] Images generated using https://math.hws.edu/eck/js/symmetry/wallpaper.html.