Data And Uncertaininity
Assignment 8 November 2017

Leonardo Ripoli

Ex1.1)
Let $\Omega = \mathbb{N}$, show that $\boldsymbol{A}$, the set of all cofinite subsets in Ω, forms an algebra.

The proof is done by checking that the 3 properties of an algebra are satisfied: empty set, closeness to complement and closeness to finite unions.

1. $\varnothing \in \boldsymbol{A}$ trivially because it is finite.

2. By definition if $A \in \boldsymbol{A}$ it is cofinite, therefore either it is finite and its complement is infinite, or the other way around: in either case $A \in \boldsymbol{A} \implies A^{\complement} \in \boldsymbol{A}$.

3. Let $A_1, A_2, A_3... \in \boldsymbol{A}$, then if all of the A_i are finite, clearly $\cup_{k=1}^{n} A_i$ is finite and therefore $\cup_{k=1}^{n} A_i \in \boldsymbol{A}$. If at least one of the A_i is infinite, then $\cup_{k=1}^{n} A_i$ is infinite, we have to show that $(\cup_{k=1}^{n} A_i)^{\complement}$ is finite. Let's say at least A_k is infinite (and therefore $A_k^{\complement}$ is finite because A_k is cofinite by hypothesis), we have that $(\cup_{k=1}^{n} A_i)^{\complement} = \cap_{k=1}^{n} A_i^{\complement}$, and since $\cap_{k=1}^{n} A_i^{\complement} \subseteq A_k^{\complement}$ and $A_k^{\complement}$ is finite, it must be that $\cap_{k=1}^{n} A_i^{\complement}$ is finite and therefore we have shown that $\cup_{k=1}^{n} A_i \in \boldsymbol{A}$.

Ex1.2)
Let $\boldsymbol{A}$ be the algebra of cofinite sets and define the set function $\mu(A) = 1$ if A is finite, and 0 otherwise: show that μ is normalised and additive.

1. $\mu(\mathbb{N}) = 1$ since $\mathbb{N}$ is infinite, therefore μ is normalised.

2. For additivity we have to show that $\mu(\cup_{k=1}^{n} A_i) = \sum_{k=1}^{n} \mu(A_i), \forall A_k \in \boldsymbol{A}$ s.t. $A_i \cap A_j = \varnothing \ i \neq j$. If all A_k are finite, then $\cup_{k=1}^{n} A_i$ is finite and therefore $\mu(\cup_{k=1}^{n} A_i) = 0 = \sum_{k=1}^{n} \mu(A_i)$, since $\mu(A_k) = 0 \ \forall k$. To address the case where at least one of the A_k is infinite, firstly we show that the condition $A_1, A_2 \in \boldsymbol{A}, A_1 \cap A_2 = \varnothing, A_1 \cup A_2$ is infinite means that only one between A_1 and A_2 can be infinite, and not both. In fact, if we assume that A_1 and A_2 are both infinite and that $A_1 \cap A_2 = \varnothing$, we have that $A_1^{\complement}$ and $A_2^{\complement}$ are both finite (A_1 and A_2 being cofinite sets) and therefore we would have: $A_1 \cap A_2 = \varnothing \implies A_1^{\complement} \cup A_2^{\complement} = \mathbb{N}$ and this is an absurd as the union of two finite sets would give the whole set

of natural numbers which is infinite. By applying the previous result to each pair $A_i \cap A_j = \varnothing, i \neq j$, we know that the condition $A_k \in \boldsymbol{A}$ $\forall k = 1, ..n, A_i \cap A_j = \varnothing$ $i \neq j, \cup_{k=1}^{n} A_i$ is infinite, implies that only one of the A_k can be infinite and therefore we have that $\mu(\cup_{k=1}^{n} A_i) = 1$ since $\cup_{k=1}^{n} A_i$ is infinite, and $\sum_{k=1}^{n} \mu(A_i) = 1$ because only one of the k terms of the sum will be 1 (the contribution from the only infinite set) and all other contributions will be zero. Therefore we have shown that $\mu(\cup_{k=1}^{n} A_i) = \sum_{k=1}^{n} \mu(A_i), \forall A_k \in \boldsymbol{A}$.

Ex1.3)
Show that μ is not continuous at the empty set.

We take $A_k = \mathbb{N} \setminus \{0, 1, ...k\}$, all A_k are cofinite, in fact $A_k^{\complement} = \{0, 1, ...k\}$, a finite set, and in addition, by construction, $\cap_{k \in \mathbb{N}} = \varnothing$ and $A_1 \supset A_2....$ We have $\mu(A_k) = 1$ as A_k is infinite $\forall$ k.

We Let's fix n and consider the set $B_n = \{0, 2, ...2^{n-1}\}$, and put $A_k = B_n \setminus \{0, 2, ..., 2^{k-1}\}$, $k = 0, ...n$: clearly $A_k \in \boldsymbol{A}$ in fact $A_k \subset \mathbb{N}$, $k = 0, ...n$ and is finite. In addition, by construction we have that $A_1 \supset A_2... \supset A_n$ and $\cap_j^n A_j = \varnothing$. As $n \to \infty$ $A_n \to \mathbb{N} \setminus \{0, 2, 4, ...\}$, i.e. to the set of odd numbers which is infinite and would therefore have measure 1. The set of cofinite sets is not a sigma algebra, in fact if we consider the $\cup_{n=1}^{\infty} B_n = C$ where $B_n = \{0, 2, ...2^{n-1}\}$, we have that C is the set of odd numbers, which is not cofinite as it is infinite and its complement (i.e. the set of even numbers) is infinite as well.

Ex2.1)
Let $(\Omega, \boldsymbol{A}, P)$ be a probability space. Let f be a non negative random variable, and suppose that $\int f dP = 1$, on $\boldsymbol{A}$ define the set function F by $F(A) = \int 1_A \cdot f dP$.

Show that F is a probability on $(\Omega, \boldsymbol{A})$.

The proof is done by checking that the 3 properties of a probability: normalisation, additivity and continuity at $\varnothing$ (this latter is proved using the equivalent sigma additivity).

1. $F(\Omega) = \int 1_\Omega \cdot f dP$, remembering the definition of the indicator func-

tion, we have that $F(\Omega) = \int 1_\Omega \cdot f dP = \int_\Omega f dP = 1$.

2. Let $A_1, A_2, ...A_n \in \boldsymbol{A}$, with $A_i \cap A_j = \varnothing$ for $i \neq j$. $\sum_{k=1}^n F(A_k) = \sum_{k=1}^n \int 1_{A_k} \cdot f dP = \sum_{k=1}^n \int_{A_k} \cdot f dP$. On the other hand, $F(\cup_{k=1}^n A_k) = \int 1_{\cup_{k=1}^n A_k} \cdot f dP = \int_{\cup_{k=1}^n A_k} f dP$. Since all the domains A_k are disjoint we have that $\int_{\cup_{k=1}^n A_k} f dP = \sum_{k=1}^n \int_{A_k} f dP$, and therefore $\sum_{k=1}^n F(A_k) = \sum_{k=1}^n \int 1_{A_k} \cdot f dP = F(\cup_{k=1}^n A_k)$, which proves the additivity.

3. Let $A_1, A_2, ...A_n \in \boldsymbol{A}$, with $A_i \cap A_j = \varnothing$ for $i \neq j$. $\sum_k P(A_k) = \sum_k F(A_k) = \sum_k \int 1_{A_k} \cdot f dP$. Using monotone convergence theorem, we know we can build $f_n \uparrow f$ with f_n simple functions and so that $\int f_n dP \uparrow \int f dP$. It's $\displaystyle\sum_k \int 1_{A_k} \cdot f dP = \lim_{n\to\infty} \sum_k \int 1_{A_k} \cdot f_n dP$ (in fact $\int_{A_k} f_n dP \uparrow \int_{A_k} f dP \forall A_k$). Expressing explicitly f_n as simple function we have: $f_n = \sum_{j=1} f_{n_j} \cdot 1_{B_{nj}}$. <u>In the following expressions, for simplicity of notation, we w</u>
<u>$\lim_{n\to\infty}$or $\uparrow$ whenever there is f_n, but it is intended each time there is f_n.</u>
So expressing in terms of f_n, $F(A_k) = \int \sum_j f_{n_j} \cdot 1_{B_{nj}} \cdot 1_{A_k} dP = \int \sum_j f_{n_j} \cdot 1_{B_{nj} \cap A_k} dP$, and therefore $\sum_k F(A_k) = \sum_k \int \sum_j f_{n_j} \cdot 1_{B_{nj} \cap A_k} dP =$
$= \sum_k \sum_j f_{n_j} \mu(B_{nj} \cap A_k)$. On the other hand,
$F(\cup_k A_k) = \int \sum_j f_{n_j} 1_{B_{nj}} \cdot 1_{\cup_k A_k} dP = \int \sum_j f_{n_j} 1_{B_{nj} \cap (\cup_k A_k)} dP =$
$= \sum_j f_{n_j} \mu(B_{nj} \cap (\cup_k A_k)) = \sum_j f_{n_j} \mu(\cup_k (B_{nj} \cap A_k))$. From the sigma additivity property of μ, and since $A_i \cap A_j = \varnothing$ for $i \neq j \implies (B_{nj} \cap A_k) \cap (B_{nj} \cap A_z) = \varnothing$ for $k \neq z$, we have that $\mu(\cup_k (B_{nj} \cap A_k)) = \sum_k \mu(B_{nj} \cap A_k)$.
Putting all together we have that $\sum_k F(A_k) = \sum_k \sum_j f_{n_j} \mu(B_{nj} \cap A_k) = F(\cup_k A_k)$.

Ex2.2)
Show that $P(A) = 0 \implies F(A) = 0$.

We start by proving for simple f. Let $f = \sum_{j=1}^p f_j \cdot 1_{B_j}$, then we have $F(A) = \int f \cdot 1_A dP = \int \sum_{j=1}^p f_j \cdot 1_A \cdot 1_{B_j} dP = \int \sum_{j=1}^p f_j \cdot 1_{A \cap B_j} dP = \sum_{j=1}^p f_j P(A \cap B_j) = 0$, the last equality coming from the fact that we have $P(A) = 0$ by hypothesis and therefore it must be $0 <= P(A \cap B_j) <= P(A) = 0$. We have therefore proved that that for

f simple $P(A) = 0 \implies F = 0$.
We now consider the case of a positive function that is not simple. We consider $f_n = \sum_{j=1}^{p_n} f_{jn} \cdot 1_{B_jn}$, with $\{f_n\}$ an increasing sequence of simple functions s.t. $f_n \uparrow f$, and therefore $\int_A f_n dP \uparrow \int_A f dP$. We have demonstrated in the first part of this exercise 2.2 that for a simple function, if $P(A) = 0$, it will be $\int f_n \cdot 1_A dP = \int \sum_{j=1}^{p_n} f_{jn} \cdot 1_A \cdot 1_{B_{jn}} dP = \sum_{j=1}^{p_n} f_j P(A \cap B_{jn}) = 0$, therefore $\int f_n \cdot 1_A dP = 0 \; \forall n$, and $0 = \lim_{n\to\infty} \int f_n \cdot 1_A dP = \int f \cdot 1_A dP = F(A)$, so we have demonstrated that $P(A) = 0 \implies F(A) = 0$.

Ex2.3)
Show that $\int g dP = F = \int f dP \implies f = g$ a.e. ($\Longleftarrow$ is trivial).

We can use the result that for $z \geq 0$, $\int z dP = 0 \implies z = 0$ a.e. We have by hypothesis that $\int (f - g) dP = 0$, let $h = f - g$, using h_+ =max$\{f, 0\}$ and h_- =min$\{f, 0\}$, we have $h = h_+ - h_-$. We consider $\Omega_1 = \{x \in \Omega \mid f \geq g\}$ and $\Omega_2 = \{x \in \Omega \mid f < g\}$, it's $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cup \Omega_2 = \Omega$. Since the property $\int_A h dP = 0$ holds on any $A \subseteq \Omega$, we have $\int_{\Omega_1} h dP = 0$ and $\int_{\Omega_2} h dP = 0$. So, using definitions of h_+ and h_-, we have $\int_{\Omega_1} h dP = \int_{\Omega_1} h_+ dP = 0$ and $\int_{\Omega_2} h dP = -\int_{\Omega_2} h_- dP = 0 \implies \int_{\Omega_2} h_- dP = 0$: in both cases we can use the result $z \geq 0$, $\int z dP = 0 \implies z = 0$ a.e. and therefore we have $f = g$ a.e. on $\Omega_1 \cup \Omega_2 = \Omega$.