Data And Uncertainity
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Ex1.1)

Let $\Omega = \mathbb{N}$, show that $\boldsymbol{A}$, the set of all cofinite subsets in Ω, forms an algebra.

The proof is done by checking that the 3 properties of an algebra are satisfied: empty set, closeness to complement and closeness to finite unions.

1. $\varnothing \in \boldsymbol{A}$ trivially because it is finite.

2. By definition if $A \in \boldsymbol{A}$ it is cofinite, therefore either it is finite and its complement is infinite, or the other way around: in either case $A \in \boldsymbol{A} \implies A^{\complement} \in \boldsymbol{A}$.

3. Let $A_1, A_2, A_3... \in \boldsymbol{A}$, then if all of the A_i are finite, clearly $\cup_{k=1}^n A_i$ is finite and therefore $\cup_{k=1}^n A_i \in \boldsymbol{A}$. If at least one of the A_i is infinite, then $\cup_{k=1}^n A_i$ is infinite, we have to show that $(\cup_{k=1}^n A_i)^{\complement}$ is finite. Let's say at least A_k is infinite (and therefore $A_k^{\complement}$ is finite because A_k is cofinite by hypothesis), we have that $(\cup_{k=1}^n A_i)^{\complement} = \cap_{k=1}^n A_i^{\complement}$, and since $\cap_{k=1}^n A_i^{\complement} \subseteq A_k^{\complement}$ and $A_k^{\complement}$ is finite, it must be that $\cap_{k=1}^n A_i^{\complement}$ is finite and therefore we have shown that $\cup_{k=1}^n A_i \in \boldsymbol{A}$.

Ex1.2)

Let $\boldsymbol{A}$ be the algebra of cofinite sets and define the set function $\mu(A) = 1$ if A is finite, and 0 otherwise: show that μ is normalised and additive.

1. $\mu(\mathbb{N}) = 1$ since $\mathbb{N}$ is infinite, therefore μ is normalised.

2. For additivity we have to show that $\mu(\cup_{k=1}^n A_i) = \sum_{k=1}^n \mu(A_i), \forall A_k \in \boldsymbol{A}$ s.t. $A_i \cap A_j = \varnothing$ $i \neq j$. If all A_k are finite, then $\cup_{k=1}^n A_i$ is finite and therefore $\mu(\cup_{k=1}^n A_i) = 0 = \sum_{k=1}^n \mu(A_i)$, since $\mu(A_k) = 0$ $\forall k$. To address the case where at least one of the A_k is infinite, firstly we show that the condition $A_1, A_2 \in \boldsymbol{A}, A_1 \cap A_2 = \varnothing, A_1 \cup A_2$ is infinite means that only one between A_1 and A_2 can be infinite, and not both. In fact, if we assume that A_1 and A_2 are both infinite and that $A_1 \cap A_2 = \varnothing$, we have that $A_1^{\complement}$ and $A_2^{\complement}$ are both finite (A_1 and A_2 being cofinite sets) and therefore we would have: $A_1 \cap A_2 = \varnothing \implies A_1^{\complement} \cup A_2^{\complement} = \mathbb{N}$ and this is an absurd as the union of two finite sets would give the whole set of natural numbers which is infinite. By applying the previous result

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to each pair $A_i \cap A_j = \varnothing, i \neq j$, we know that the condition $A_k \in \boldsymbol{A}$ $\forall k = 1, ..n, A_i \cap A_j = \varnothing \; i \neq j, \cup_{k=1}^n A_i$ is infinite, implies that only one of the A_k can be infinite and therefore we have that $\mu(\cup_{k=1}^n A_i) = 1$ since $\cup_{k=1}^n A_i$ is infinite, and $\sum_{k=1}^n \mu(A_i) = 1$ because only one of the k terms of the sum will be 1 (the contribution from the only infinite set) and all other contributions will be zero. Therefore we have shown that $\mu(\cup_{k=1}^n A_i) = \sum_{k=1}^n \mu(A_i), \forall A_k \in \boldsymbol{A}$.

Ex1.3)
Show that μ is not continuous at the empty set.

We take $A_k = \mathbb{N} \setminus \{0, 1, ...k\}$, all A_k are cofinite, in fact $A_k^{\complement} = \{0, 1, ...k\}$, a finite set, and in addition, by construction, $\cap_{k \in \mathbb{N}} = \varnothing$ and $A_1 \supset A_2....$ We have $\mu(A_k) = 1$ as A_k is infinite $\forall$ k.

   We Let's fix n and consider the set $B_n = \{0, 2, ...2^{n-1}\}$, and put $A_k = B_n \setminus \{0, 2, ..., 2^{k-1}\}$, $k = 0, ...n$: clearly $A_k \in \boldsymbol{A}$ in fact $A_k \subset \mathbb{N}$, $k = 0, ...n$ and is finite. In addition, by construction we have that $A_1 \supset A_2... \supset A_n$ and $\cap_j^n A_j = \varnothing$. As $n \to \infty$ $A_n \to \mathbb{N} \setminus \{0, 2, 4, ...\}$, i.e. to the set of odd numbers which is infinite and would therefore have measure 1. The set of cofinite sets is not a sigma algebra, in fact if we consider the $\cup_{n=1}^\infty B_n = C$ where $B_n = \{0, 2, ...2^{n-1}\}$, we have that C is the set of odd numbers, which is not cofinite as it is infinite and its complement (i.e. the set of even numbers) is infinite as well.

Ex2.1)
Let $(\Omega, \boldsymbol{A}, P)$ be a probability space. Let f be a non negative random variable, and suppose that $\int f dP = 1$, on $\boldsymbol{A}$ define the set function F by $F(A) = \int 1_A \cdot f dP$.
Show that F is a probability on $(\Omega, \boldsymbol{A})$.

The proof is done by checking the properties of a probability on a sigma algebra: normalisation and sigma additivity. Sigma (infinite) additivity proves both simple (non infinite) additivity and continuity at $\varnothing$ (result used without proof from the lecture notes: exercise 2.2, point 2).

1. $F(\Omega) = \int 1_\Omega \cdot f dP$, remembering the definition of the indicator function, we have that $F(\Omega) = \int 1_\Omega \cdot f dP = \int_\Omega f dP = 1$ by hypothesis.

2. Let $A_1, A_2, ... \in \boldsymbol{A}$, $A_i \cap A_j = \varnothing \; i \neq j$. We define $C = \cup_{k=1}^\infty A_k$, $B_n = \cup_{k=1}^n A_k$ we know both $C, B \subseteq \boldsymbol{A}$ since $\boldsymbol{A}$ is a sigma algebra and

therefore closed under infinite unions. Define $g_n = f \cdot 1_{B_n}$, g_n is measurable since f is measurable by hypothesis and the indicator function is measurable too and the product of measurable functions is measurable. Define $g = f \cdot 1_C$. Since $f \geq 0$, we have that $0 \leq g_1 \leq g_2... \leq g$, in fact $f \cdot 1_{B_n} \leq f \cdot 1_{B_{n+1}} \leq f \cdot 1_C$ as $B_n \subseteq B_{n+1}... \subseteq C$. In addition, by construction, $g_n \uparrow g$ as $n \to \infty$. We are therefore in the monotone convergence theorem hypotheses, and we can say that: $F(C) = \int_\Omega f \cdot 1_C dP = \int_\Omega g dP = \lim n \to \infty \int_\Omega g_n dP$ (the last equality comes from the Monotone Convergence Theorem). Remembering that $g_n = f \cdot 1_{B_n}$ and that $B_n = \cup_{k=1}^n A_k$ where all of the A_k are mutually disjoint, we have that $g_n = f \cdot 1_{B_n} = \sum_{k=1}^n f \cdot 1_{A_k}$, and therefore $F(\cup_{k=1}^n A_k) = \int_\Omega g_n dP = \int \sum_{k=1}^n f \cdot 1_{A_k} dP = \sum_{k=1}^n \int f \cdot 1_{A_k} dP = \sum_{k=1}^n F(A_k)$. And it follows from previous results that $F(C) = F(\cup_{k=1}^\infty A_k) = \int_\Omega f \cdot 1_C dP = \int_\Omega g dP = \lim n \to \infty \int_\Omega g_n dP = \lim n \to \infty \sum_{k=1}^n F(A_k) = \sum_{k=1}^\infty F(A_k)$. And this proves the sigma additivity.

Ex2.2)
Show that $P(A) = 0 \implies F(A) = 0$.

- We start by proving for simple f. Let $f = \sum_{j=1}^p f_j \cdot 1_{B_j}$, then we have $F(A) = \int f \cdot 1_A dP = \int \sum_{j=1}^p f_j \cdot 1_A \cdot 1_{B_j} dP = \sum_{j=1}^p \int f_j \cdot 1_{A \cap B_j} dP = \sum_{j=1}^p f_j P(A \cap B_j) = 0$, the zero equality coming from the fact that we have $P(A) = 0$ by hypothesis and $A \cap B_j \subseteq A$ therefore it must be $0 <= P(A \cap B_j) <= P(A) = 0$. We have therefore proved that that for f simple $P(A) = 0 \implies F = 0$.

- We now consider the case of a positive function that is not simple. We consider $f_n = \sum_{j=1}^{p_n} f_{jn} \cdot 1_{B_j n}$, with $\{f_n\}$ an increasing sequence of simple functions s.t. $f_n \uparrow f$, and therefore (thanks to the Monotone Convergence Theorem) $\int f_n dP \uparrow \int f dP$ (an example of the construction of an increasing sequence of simple functions whose limit is f is in the lecture notes, in the paragraph "The integral"). Note that the results $f_n \uparrow f$ and $\int f_n dP \uparrow \int f dP$ hold also for $f_n \cdot 1_A \uparrow f \cdot 1_A$ and $\int f_n \cdot 1_A dP \uparrow \int f \cdot 1_A dP$, in fact we could define $g_n = f_n \cdot 1_A$ and $g = f \cdot 1_A$ and we would have $g_n \uparrow g$ and $\int_\Omega g_n dP \uparrow \int_\Omega g dP$ since we are in the Monotone Convergence Theorem hypotheses. We have demonstrated in the first part of this exercise 2.2 that for a simple function, if $P(A) = 0$ the integral will always be zero, therefore it will be

$\int f_n \cdot 1_A dP = \int \sum_{j=1}^{p_n} f_{jn} \cdot 1_A \cdot 1_{B_{jn}} dP = \sum_{j=1}^{p_n} f_j P(A \cap B_{jn}) = 0$, therefore $\int f_n \cdot 1_A dP = 0 \; \forall n$, and $0 = \lim_{n \to \infty} \int f_n \cdot 1_A dP = \int f \cdot 1_A dP$, the last equality coming from the Monotone Convergence Theorem, and being $\int f \cdot 1_A dP = F(A)$, we have demonstrated that $P(A) = 0 \implies F(A) = 0$.

Ex2.3)
Show that $\int g dP = F = \int f dP \implies f = g$ a.e. ($\impliedby$ is trivial).

We can use the result that for $z \geq 0$, $\int z dP = 0 \implies z = 0$ a.e. We have by hypothesis that $\int (f-g) dP = 0$, let $h = f - g$, using $h_+ = \max\{f, 0\}$ and $h_- = \min\{f, 0\}$, we have $h = h_+ - h_-$. We consider $\Omega_1 = \{x \in \Omega \mid f \geq g\}$ and $\Omega_2 = \{x \in \Omega \mid f < g\}$, it's $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cup \Omega_2 = \Omega$. Since the property $\int_A h dP = 0$ holds on any $A \subseteq \Omega$, we have $\int_{\Omega_1} h dP = 0$ and $\int_{\Omega_2} h dP = 0$. So, using definitions of h_+ and h_-, we have $\int_{\Omega_1} h dP = \int_{\Omega_1} h_+ dP = 0$ and $\int_{\Omega_2} h dP = -\int_{\Omega_2} h_- dP = 0 \implies \int_{\Omega_2} h_- dP = 0$: in both cases we can use the result $z \geq 0$, $\int z dP = 0 \implies z = 0$ a.e. and therefore we have $f = g$ a.e. on $\Omega_1 \cup \Omega_2 = \Omega$.

Ex3.1)
Show that $A_1, A_2, ... \in \boldsymbol{A}$, $A_1 \supset A_2...$ and $P(A_k) \to 0$ as $k \to \infty$, $f \geq 0$, then $\int_{A_k} f dp \to 0$ for $k \to \infty$.

Let $f_n(x) = \begin{cases} n & for \; x \; | \; f(x) \geq n \\ f(x) & for \; x \; | \; 0 \leq f(x) < n \end{cases}$

Proof
We have that $\{f_n\}$ is an increasing sequence of positive measurable functions (since f is measurable), and, by construction, $f_n \uparrow f$ pointwise. For any $A_k \in \boldsymbol{A}$, we have that

$$\int_{A_k} f dP = \int_{A_k} (f - f_n) dP + \int_{A_k} f_n dP \tag{1}$$

$\int_{A_k} f dP = \int_{A_k} (f - f_n) dP + \int_{A_k} f_n dP$. Since $f_n \uparrow f$ and $0 \leq f_n \leq f$, $(f - f_n)$ is a positive quantity, and we have that $\int_{A_k} (f - f_n) dP \leq \int_\Omega (f - f_n) dP$ because we consider a bigger domain of integration for a positive quantity. In addition, since by construction $f_n \leq n$, it's $\int_{A_k} f_n dP \leq n P(A_k)$. By monotone convergence theorem, for n sufficiently high, $\forall \epsilon$ we can have $\int_\Omega (f - f_n) dP \leq \frac{\epsilon}{2}$,

and also, being by hypothesis $P(A_k) \to 0$, for k sufficiently high we can have $P(A_k) \leq \frac{\epsilon}{2n}$, therefore in (1) we have that $\int_{A_k} f dP \leq \frac{\epsilon}{2} + \frac{\epsilon}{2n}$, therefore, ultimately, $P(A_k) \to 0 \implies \int_{A_k} f dP \to 0$.

Ex3.2)
Show that $A_1, A_2, ... \in \boldsymbol{A}$, $\sum_{k=1}^{\infty} P(A_k) < \infty \implies P(\cup_{l=k}^{\infty} A_l) \to 0$ if $k \to \infty$. Let $\sum_{k=1}^{\infty} P(A_k) = z \geq 0$, from sigma additivity, $P(\cup_{l=k}^{\infty} A_l) = \sum_{l=k}^{\infty} P(A_k)$. From the convergence of the series, we have that $\sum_{l=k}^{\infty} P(A_k) = (z - \sum_{l=k}^{\infty} P(A_k)) \to 0$ as $k \to \infty$, and this proves that $P(\cup_{l=k}^{\infty} A_l) \to 0$ if $k \to \infty$.