Data and Uncertainty
Assignment I

Please submit by Wed, 8th Nov 2017 EOB.

1 Cofinite sets in $\mathbb{N}$

In this exercise, we find an algebra and a set function μ which satisfies normalisation and additivity from Definition 2.2 but not the continuity at $\emptyset$. This shows that this requirement is not redundant. Let $\Omega = \mathbb{N}$, the natural numbers. A set $A \subset \mathbb{N}$ is called *cofinite* if either A or A^c is finite.

1. Show that the cofinite subsets of $\mathbb{N}$ form an algebra. (**3 points**)

Now we let $\mathcal{A}$ =the algebra of cofinite sets, and define the set function

$$\mu(A) = 1 \qquad \text{if } A \text{ is infinite and} \qquad \mu(A) = 0 \qquad \text{else.}$$

2. Show that μ is normalised and additive. (**3 points**)
3. Show that μ is *not* continuous at the empty set by finding a sequence of cofinite sets $\{A_k, k \in \mathbb{N}\}$ such that $A_1 \supset A_2 \supset \dots$ with $\cap_{k\in\mathbb{N}} A_k = \emptyset$ but $\mu(A_k)$ does not converge to zero. (**4 points**)

2 The concept of densities

Solve exercise 3.5 from the lecture notes. Item 1: **3 points**, Item 2: **3 points**, Item 3: **4 points**.

3 A continuity property of the integral

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $f : \Omega \to \mathbb{R}_{\geq 0}$ be a nonnegative random variable with $\int_\Omega f \mathrm{d}\mathbb{P} < \infty$. For any set $A \in \mathcal{A}$, we define $\int_A f \mathrm{d}\mathbb{P} := \int_\Omega f \mathbb{1}_A \mathrm{d}\mathbb{P}$. The aim of this exercise is to prove that if $\{A_k\}$ is a sequence of measurable sets with $\mathbb{P}(A_k) \to 0$ for $k \to \infty$, then also $\int_{A_k} f \mathrm{d}\mathbb{P} \to 0$. We do this in a couple of steps.

1. Show that if $A_1, A_2, \dots$ are sets in $\mathcal{A}$ with $A_1 \supset A_2 \supset \dots$ and $\mathbb{P}(A_k) \to 0$ for $k \to \infty$, then $\int_{A_k} f \mathrm{d}\mathbb{P} \to 0$ for $k \to \infty$. (Hint: Show that $f \cdot \mathbb{1}_{A_k} \to 0$ almost surely, then use the Bounded Convergence Theorem.) (**3 points**)
2. Show that if $A_1, A_2, \dots$ are sets in $\mathcal{A}$ (not necessarily nested!) so that $\sum_{k=1}^\infty \mathbb{P}(A_k) < \infty$, then $\mathbb{P}(\cup_{k=l}^\infty A_l) \to 0$ for $k \to \infty$. (**3 points**)

To finally prove our assertion, we argue by contradiction. If the statement is false, then there is an $\epsilon > 0$ and a subsequence $A_{n_1}, A_{n_2}, \dots$ so that $\int_{A_{n_k}} f \mathrm{d}\mathbb{P} \geq \epsilon$ for all k.

3. Show that this is in contradiction with the previous two items. (Hint: argue that the subsequence can be chosen so that $\sum_{k=1}^\infty \mathbb{P}(A_{n_k}) < \infty$.) (**4 points**)