Numerical Methods
Course Assignment Report

by

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Abstract

In this work we present the analysis of the linear advection equation modelled in one dimension, x, without sources or sinks of the advected variable ϕ. The exact expression of the equation is:

$$\phi_t + u\phi_x = 0 \tag{1}$$

We consider the case of constant and uniform wind, u, and with given initial condition $\phi(x,0) = \phi_0$. It can be shown that the analytic solution of (1) is:

$$\phi(x,t) = \phi_0(x - ut) \tag{2}$$

We have modelled equation (1) using several numerical schemes, currently:

- FTBS

- CTCS

The FTBS scheme has been chosen for its "naivety", being first order accurate in time and space, to show how a "basic" method works. The CTCS method has been chosen because, among the explicit ones, it's the method that guarantees the highest order of accuracy, namely two. A comparison of the two methods has been carried out. This report contains the preliminary results, the full analysis will be presented in the final version of the submission which will contain further developements.

Contents

Chapter 1

FTBS

The first method to be explored is FTBS. Characteristics of the method can be found in [1]. FTBS has been chosen as "naive" method to be compared to a more "sophisticathed" method and show the quantitative and qualitative differences. The initial conditions used are the following:

$$\phi_0(t) = \begin{cases} \frac{1}{2}(1 - cos(4\pi x)) & 0 \le x < \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \tag{1.1}$$

$$\phi_0(t) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ 0 & elsewhere \end{cases} \tag{1.2}$$

The function (1.1) can be used when smooth functions are needed, for example to test order of convergence. On the other hand, function with discontinuities, like the step function (1.2) can be used to check monotonicity of the scheme.

1.1 General characteristics

We will explore the characteristics of the FTBS method. Unless otherwise stated, the results used in this chapter will be taken without proof from [1] and [2].

1.1.1 Conservation of mass

With reference to equation (2), the mass of ϕ is conserved under the linear advection in the exact solution and also in the FTBS numerical implementation, whereas the variance of ϕ is conserved only in the exact solution but not in the FTBS implementation. The variance decreases in FTBS, and this is consistent with what we will find in the subsection 1.1.2, that the method is damping.

1.1.2 Stability

The FTBS method is stable for courant number $c \in [0, 1]$

1.1.3 Accuracy

The FTBS method is first order accurate. We can see in figure 1.1 a qualitative comparison of the output of FTBS scheme vs the exact solution, for initial condition (1.1).
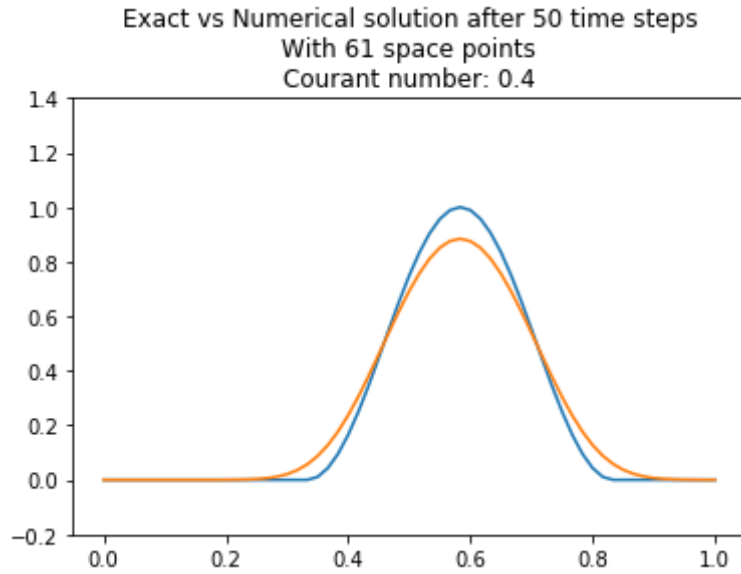

Figure 1.1: *A comparison of FTBS vs exact solution for initial condition (1.1). The blue line is the exact solution, the orange line is the output of the model*

Error for the FTBS have be calculated using norms, the following are the results

for $c = 0.4$, $n_t = 50$, $n_x = 61$:

$$|| \phi_{FTBS} - \phi_{EXACT} ||_2 = 0.1223 \tag{1.3}$$

$$|| \phi_{FTBS} - \phi_{EXACT} ||_\infty = 0.1159 \tag{1.4}$$

1.1.4 Monotonicity

The FTBS is monotonic, it does not add new maxima, it can be seen qualitatively from figure 1.1.

1.1.5 Dispersion errors

1.1.6 Diffusion errors

1.1.7 Computational modes

What are these??? I read they refer to nothing specifically, but in general to modes that can be either physical or coming from numerical implementations. Not sure I understood how to compute these.

1.1.8 Computational Cost

Details of the code used for computing the FTBS scheme can be found in chapter 3.

Do we calculate the cost by memory consumption, time of execution..., and if so, do we do a qualitative analysis such as which order of magnitude these quantities are???

1.1.9 Variable resolution

What does this mean specifically???

Chapter 2

CTCS

The second method to be explored is CTCS. Characteristics of the method can be found in [1].

2.1 General characteristics

We will explore the characteristics of the CTCS method.

2.1.1 Conservation of mass

2.1.2 Stability

2.1.3 Accuracy

2.1.4 Monotonicity

2.1.5 Dispersion errors

2.1.6 Diffusion errors

2.1.7 Computational modes

2.1.8 Computational Cost

2.1.9 Variable resolution

Chapter 3

Python Implementation

3.1 General characteristics

In this section we will explain the python implementation of the numerical schemes. Currently the implementation consists of 4 files, each implementing one of the following:

- Main file used to run the project

- Initial condition file, stores different possible choices of initial conditions that can be used to test different properties

- Advection schemes file

- Diagnostics functions file

3.2 Code Repository

The code repository is:
https://github.com/lr411/Numerics.git
The hash of the commit the report refers to for the code part, is:
43f4315e753289bfaab8d4c84fa88d7124c732dc

3.3   Advection Schemes Implementation

The function used for the method is in the file advectionSchemes.py, and the function in the code is the following, with comments included on the meaning of inputs:

```
def FTCS(phiOld, c, nt):
"Linear advection scheme using FTCS, with Courant number c and"
"                          nt time-steps"
"inputs are:"
"phiOld: initial condition on phi (to save space the array will then"\
"               be used to store values from the previous time step)"
"c: Courant number"
"nt: nr of time steps"
```

Bibliography

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