Práctica 3 – Principio de inducción

Ejercicio 1

Probar que $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- Pruebo $P(1)$

$$P(1) : \sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

$$P(1) : 1 = \frac{1(1+1)}{2}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1) : \sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \underbrace{\sum_{i=1}^{k} i}_{HI} + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Ejercicio 2

Probar que $\forall n \in \mathbb{N}, \sum_{i=1}^{n} (2i-1) = n^2$

- Pruebo $P(1)$

$$P(1) : \sum_{i=1}^{1} (2i-1) = 1^2$$

$$P(1) : 2 * 1 - 1 = 1$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1) : \sum_{i=1}^{k+1} (2i-1) = (k+1)^2$$

$$P(k+1): \underbrace{\sum_{i=1}^{k}(2i-1)}_{HI} + (2(k+1)-1) = (k+1)^2$$

$$P(k+1): k^2 + (2(k+1)-1) = (k+1)^2$$

$$P(k+1): k^2 + (2k+2-1) = k^2+2k+1$$

$$P(k+1): k^2+2k+1 = k^2+2k+1$$

Ejercicio 3

(*Suma de cuadrados y de cubos*) Pobar que $\forall n \in \mathbb{N}$ se tiene

a) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

- Pruebo $P(1)$

$$P(1): \sum_{i=1}^{1} i^2 = \frac{1(1+1)(2*1+1)}{6}$$

$$P(1): 1 = \frac{1*2(2+1)}{6}$$

$$P(1): 1 = \frac{6}{6}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1): \sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$P(k+1): \underbrace{\sum_{i=1}^{k} i^2}_{HI} + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$P(k+1): \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$P(k+1): \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1): \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1): \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1): \frac{(k+1)[2k^2+k+6k+6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1): \frac{(k+1)[2k^2+7k+6]}{6} = \frac{(k+1)(2k^2+3k+4k+6)}{6}$$

$$P(k+1): \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6}$$

b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

- Pruebo $P(1)$

$$P(1): \sum_{i=1}^{1} i^3 = \frac{1^2(1+1)^2}{4}$$

$$P(1): 1 = \frac{1*2^2}{4}$$

$$P(1): 1 = \frac{4}{4}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1) : \sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$P(k+1) : \underbrace{\sum_{i=1}^{k} i^3}_{HI} + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$P(k+1) : \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$P(k+1) : \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$P(k+1) : \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$P(k+1) : \frac{(k+1)^2[k^2 + 4(k+1)]}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$P(k+1) : \frac{(k+1)^2[k^2 + 4k + 4)]}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

$$P(k+1) : \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

Ejercicio 4

Probar que $\forall n \in \mathbb{N}$ se tiene

a) $\sum_{i=1}^{n}(-1)^{i+1}i^2 = \frac{(-1)^{n+1}n(n+1)}{2}$

- Pruebo $P(1)$

$$P(1) : \sum_{i=1}^{1}(-1)^{i+1}i^2 = \frac{(-1)^{1+1}1 * (1+1)}{2}$$

$$P(1) : (-1)^{1+1}1^2 = \frac{(-1)^{1+1}1 * (1+1)}{2}$$

$$P(1) : 1 * 1^2 = \frac{1 * 1 * 2}{2}$$

$$P(1) : 1 = \frac{2}{2}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1) : \sum_{i=1}^{k+1}(-1)^{i+1}i^2 = \frac{(-1)^{(k+1)+1}(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \underbrace{\sum_{i=1}^{k}(-1)^{i+1}i^2}_{HI} + (-1)^{(k+1)+1}(k+1)^2 = \frac{(-1)^{(k+1)+1}(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{(-1)^{k+1}k(k+1)}{2} + (-1)^{(k+1)+1}(k+1)^2 = \frac{(-1)^{(k+1)+1}(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{(-1)^{k+1}k(k+1)}{2} + \frac{2(-1)^{(k+1)+1}(k+1)^2}{2} = \frac{(-1)^{(k+1)+1}(k+1)((k+1)+1)}{2}$$

$$P(k+1) : \frac{(-1)^{k+1}k(k+1) + 2(-1)^{(k+1)+1}(k+1)^2}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

$$P(k+1) : \frac{(-1)^{k+1}(k+1)[k + 2(-1)(k+1)]}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

$$P(k+1): \frac{(-1)^{k+1}(k+1)[k-2k-2]}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

$$P(k+1): \frac{(-1)^{k+1}(k+1)(-k-2)}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

$$P(k+1): \frac{(-1)^{k+1}(k+1)(-1)(k+2)}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

$$P(k+1): \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2} = \frac{(-1)^{(k+1)+1}(k+1)(k+2)}{2}$$

b) $\sum_{i=0}^{n} \frac{-1}{4i^2-1} = \frac{n+1}{2n+1}$

- Prueno $P(1)$

$$P(1): \sum_{i=0}^{1} \frac{-1}{4i^2-1} = \frac{1+1}{2*1+1}$$

$$P(1): \frac{-1}{4*0^2-1} + \frac{-1}{4*1^2-1} = \frac{1+1}{2*1+1}$$

$$P(1): \frac{-1}{-1} + \frac{-1}{3} = \frac{2}{3}$$

$$P(1): 1 + \frac{-1}{3} = \frac{2}{3}$$

$$P(1): \frac{3}{3} + \frac{-1}{3} = \frac{2}{3}$$

$$P(1): \frac{3-1}{3} = \frac{2}{3}$$

$$P(1): \frac{2}{3} = \frac{2}{3}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1): \sum_{i=0}^{k+1} \frac{-1}{4i^2-1} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \underbrace{\sum_{i=0}^{k} \frac{-1}{4i^2-1}}_{HI} + \frac{-1}{4(k+1)^2-1} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{k+1}{2k+1} + \frac{-1}{4(k+1)^2-1} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{k+1}{2k+1} + \frac{-1}{4(k^2+2k+1)-1} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{k+1}{2k+1} + \frac{-1}{4k^2+8k+4-1} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{k+1}{2k+1} + \frac{-1}{4k^2+8k+3} = \frac{(k+1)+1}{2(k+1)+1}$$

Mirando los denominadores, notar que $-\frac{1}{2}$, la raiz de $2k+1$, también es raiz de $4k^2+8k+3$. Entonces:

$$(2k+1)^2 = 4k^2+4k+1$$

Con este resultado vemos cuánto "nos falta"para llegar a $4k^2+8k+3$

$$(2k+1)^2 + 4k + 2 = (4k^2+4k+1) + 4k + 2$$

$$(2k+1)^2 + 2(2k+1) = 4k^2+8k+3$$

$$(2k+1)[(2k+1)+2] = 4k^2+8k+3$$

$$(2k+1)(2k+3) = 4k^2+8k+3$$

Volviendo al problema tenemos:

$$P(k+1): \frac{k+1}{2k+1} + \frac{-1}{(2k+1)(2k+3)} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{(k+1)(2k+3)}{(2k+1)(2k+3)} + \frac{-1}{(2k+1)(2k+3)} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{(k+1)(2k+3)-1}{(2k+1)(2k+3)} = \frac{(k+1)+1}{2(k+1)+1}$$

$$P(k+1): \frac{(k+1)(2k+3)-1}{(2k+1)(2k+3)} = \frac{k+2}{2k+3}$$

Para que se cumpla la igualdad $(k+1)(2k+3)-1$ tiene que ser divisible por $2k+1$:

$$(k+1)(2k+3)-1 = 2k^2+3k+2k+3-1$$

$$(k+1)(2k+3)-1 = 2k^2+5k+2$$

$$\begin{aligned} 2k^2+5k+2 &= (2k^2+k)+4k+2 \\ &= k(2k+1)+2(2k+1) \\ &= (2k+1)(k+2) \end{aligned}$$

Voviendo al problema tenemos:

$$P(k+1): \frac{(2k+1)(k+2)}{(2k+1)(2k+3)} = \frac{k+2}{2k+3}$$

$$P(k+1): \frac{(k+2)}{(2k+3)} = \frac{k+2}{2k+3}$$

c) $\sum_{i=1}^{n}(2i+1)3^{i-1} = n3^n$

- Pruebo $P(1)$

$$P(1): \sum_{i=1}^{1}(2i+1)3^{i-1} = 1*3^1$$

$$P(1): (2*1+1)3^{1-1} = 1*3^1$$

$$P(1): 3*3^0 = 1*3^1$$

$$P(1): 3=3$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1): \sum_{i=1}^{k+1}(2i+1)3^{i-1} = (k+1)3^{k+1}$$

$$P(k+1): \underbrace{\sum_{i=1}^{k}(2i+1)3^{i-1}}_{HI} + (2(k+1)+1)3^{(k+1)-1} = (k+1)3^{k+1}$$

$$P(k+1): k3^k + (2(k+1)+1)3^k = (k+1)3^{k+1}$$

$$P(k+1): 3^k[k+(2(k+1)+1)] = (k+1)3^{k+1}$$

$$P(k+1): 3^k[k+(2k+3)] = (k+1)3^{k+1}$$

$$P(k+1): 3^k[3k+3] = (k+1)3^{k+1}$$

$$P(k+1): 3^k3(k+1) = (k+1)3^{k+1}$$

$$P(k+1): 3^{k+1}(k+1) = (k+1)3^{k+1}$$

d) $\sum_{i=1}^{n}\frac{i2^i}{(i+1)(i+2)} = \frac{2^{n+1}}{n+2} - 1$

- Pruebo $P(1)$

$$P(1): \sum_{i=1}^{1} \frac{i2^i}{(i+1)(i+2)} = \frac{2^{1+1}}{1+2} - 1$$

$$P(1): \frac{1*2^1}{(1+1)(1+2)} = \frac{2^{1+1}}{1+2} - 1$$

$$P(1): \frac{2}{2*3} = \frac{2^2}{3} - 1$$

$$P(1): \frac{1}{3} = \frac{4}{3} - 1$$

$$P(1): \frac{1}{3} = \frac{4}{3} - \frac{3}{3}$$

$$P(1): \frac{1}{3} = \frac{4-3}{3}$$

- Pruebo $P(k) \to P(k+1)$

$$P(k+1): \sum_{i=1}^{k+1} \frac{i2^i}{(i+1)(i+2)} = \frac{2^{(k+1)+1}}{(k+1)+2} - 1$$

$$P(k+1): \underbrace{\sum_{i=1}^{k} \frac{i2^i}{(i+1)(i+2)}}_{HI} + \frac{(k+1)2^{k+1}}{((k+1)+1)((k+1)+2)} = \frac{2^{(k+1)+1}}{(k+1)+2} - 1$$

$$P(k+1): \frac{2^{k+1}}{k+2} - 1 + \frac{(k+1)2^{k+1}}{((k+1)+1)((k+1)+2)} = \frac{2^{(k+1)+1}}{(k+1)+2} - 1$$

$$P(k+1): \frac{2^{k+1}}{k+2} - 1 + \frac{(k+1)2^{k+1}}{(k+2)(k+3)} = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+1}(k+3)}{(k+2)(k+3)} + \frac{(k+1)2^{k+1}}{(k+2)(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+1}(k+3)+(k+1)2^{k+1}}{(k+2)(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+1}[(k+3)+(k+1)]}{(k+2)(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+1}(2k+4)}{(k+2)(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+1}2(k+2)}{(k+2)(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

$$P(k+1): \frac{2^{k+2}}{(k+3)} - 1 = \frac{2^{k+2}}{k+3} - 1$$

Ejercicio 5

Probar que las siguientes desigualdades son verdaderas para todo $n \in \mathbb{N}$

a) $n < 2^n$

- Pruebo $P(1)$

$$P(1): 1 < 2^1$$

$$P(1): 1 < 2$$

- Pruebo $P(k) \rightarrow P(k+1)$

$$\underbrace{k < 2^k}_{P(k)}$$

$$\rightarrow k * 2 < 2^k * 2$$

$$\rightarrow \underbrace{k + 1 \leq k + k}_{usando\ k \geq 1} < 2^{k+1}$$

Entonces probé $P(k+1) : k+1 < 2^{k+1}$

b) $3^n + 5^n \geq 2^{n+2}$

- Pruebo $P(1)$

$$P(1) : 3^1 + 5^1 \geq 2^{1+2}$$

$$P(1) : 8 \geq 2^3 = 8$$

- Pruebo $P(k) \rightarrow P(k+1)$

$$\underbrace{3^k + 5^k \geq 2^{k+2}}_{P(k)}$$

$$\rightarrow 2(3^k + 5^k) \geq 2^{k+2} * 2$$

$$\rightarrow 2 * 3^k + 2 * 5^k \geq 2^{k+2+1}$$

$$\rightarrow 3 * 3^k + 2 * 5^k \geq 2 * 3^k + 5 * 5^k \geq 2 * 3^k + 2 * 5^k \geq 2^{(k+1)+2}$$

$$\rightarrow \underbrace{3^{k+1} + 5^{k+1} \geq 2^{(k+1)+2}}_{P(k+1)}$$

c) $3^n \geq n^3$

- Pruebo $P(1)$

$$P(1) : 3^1 \geq 1^3 \leftrightarrow 3 \geq 1$$

- Pruebo $P(k) \rightarrow P(k+1)$

$$\underbrace{3^k \geq k^3}_{P(k)}$$

$$\rightarrow 3 * 3^k \geq 3k^3$$

$$\rightarrow 3^{k+1} \geq 3k^3 \underbrace{\geq}_{(*)} (k+1)^3$$

Ahora quiero probar que vale $(*)$:

$$3k^3 \geq (k+1)^3 \Leftrightarrow (\sqrt[3]{3}k)^3 \geq (k+1)^3$$

$$\Leftrightarrow \sqrt[3]{3}k \geq (k+1)$$

$$\Leftrightarrow \sqrt[3]{3}k - k \geq 1$$

$$\Leftrightarrow (\sqrt[3]{3} - 1)k \geq 1$$

$$\Leftrightarrow k \geq \frac{1}{\sqrt[3]{3} - 1} \simeq 2,26$$

$(*)$ vale $\forall k \geq 3$. Ahora resta probar que la desigualdad original vale para $k = 2$ también.
Pruebo $P(2)$

$$P(2) : 3^2 \geq 2^3$$

$$P(2) : 9 \geq 8$$

d) $n! \geq \frac{3^{n-1}}{2}$

- Pruebo $P(1)$

$$P(1) : 1! \geq \frac{3^{1-1}}{2}$$

$$P(1) : 1 \geq \frac{3^0}{2}$$

$$P(1) : 1 \geq \frac{1}{2}$$

- Pruebo $P(k) \to P(k+1)$

$$k! \geq \frac{3^{k-1}}{2}$$

$$k!(k+1) \geq \frac{3^{k-1}}{2}(k+1)$$

$$(k+1)! \geq \frac{3^{k-1}}{2}(k+1) \underbrace{\geq}_{(*)} \frac{3^{k-1}}{2} * 3 = \frac{3^{(k+1)-1}}{2}$$

Pruebo $(*)$

$$k+1 \geq 3$$

$$\leftrightarrow k \geq 2$$

Como ya probé que vale $P(1)$ me resta probar que vale $P(2)$

$$P(2) : 2! \geq \frac{3^{2-1}}{2}$$

$$P(2) : 2 \geq \frac{3^1}{2}$$

e) $\sum_{i=1}^{n} \frac{1}{i!} \leq 2 - \frac{1}{2^{n-1}}$

- Pruebo $P(1)$

$$P(1) : \sum_{i=1}^{1} \frac{1}{i!} \leq 2 - \frac{1}{2^{1-1}}$$

$$P(1) : \frac{1}{1!} \leq 2 - \frac{1}{2^0}$$

$$P(1) : 1 \leq 2 - \frac{1}{1}$$

- Pruebo $P(k) \to P(k+1)$

$$\sum_{i=1}^{k} \frac{1}{i!} \leq 2 - \frac{1}{2^{k-1}}$$

$$\to \sum_{i=1}^{k} \frac{1}{i!} + \frac{1}{(k+1)!} \leq 2 - \frac{1}{2^{k-1}} + \frac{1}{(k+1)!}$$

$$\to \sum_{i=1}^{k+1} \frac{1}{i!} \leq \underbrace{2 - \frac{1}{2^{k-1}} + \frac{1}{(k+1)!} \leq 2 - \frac{1}{2^{(k+1)-1}}}_{(*)}$$

Ahora pruebo que vale $(*)$

$$2 - \frac{1}{2^{k-1}} + \frac{1}{(k+1)!} \leq 2 - \frac{1}{2^{(k+1)-1}}$$

$$\leftrightarrow \frac{1}{(k+1)!} \leq 2 - \frac{1}{2^{(k+1)-1}} - 2 + \frac{1}{2^{k-1}}$$

$$\leftrightarrow \frac{1}{(k+1)!} \leq -\frac{1}{2^{k-1}} * \frac{1}{2} + \frac{1}{2^{k-1}}$$

$$\leftrightarrow \frac{1}{(k+1)!} \leq \frac{1}{2^{k-1}} \left[-\frac{1}{2} + 1 \right]$$

$$\leftrightarrow \frac{1}{(k+1)!} \leq \frac{1}{2^{k-1}} * \frac{1}{2}$$
$$\leftrightarrow \frac{1}{(k+1)!} \leq \frac{1}{2^k}$$
$$\leftrightarrow \underbrace{2^k \leq (k+1)!}_{Q(k)}$$

Ahora pruebo $Q(k)$ por inducción:

- Pruebo $Q(1)$

$$Q(1) : 2^1 \leq (1+1)!$$
$$Q(1) : 2 \leq 2$$

- Pruebo $Q(m) \rightarrow Q(m+1)$

$$Q(m) : 2^m \leq (m+1)!$$
$$\leftrightarrow 2^m * (m+2) \leq (m+1)!(m+2)$$
$$\leftrightarrow 2^{m+1} = \underbrace{2^m * 2 \leq 2^m * (m+2)}_{(**)} \leq (m+2)!$$

$(**)$ vale porque $2 \leq (m+2) \leftrightarrow 0 \leq m+2-2 \leftrightarrow 0 \leq m$

Ejercicio 6

Probar que

a) $n! \geq 3^{n-1}, \forall n \geq 5$

- Pruebo $P(5)$

$$P(5) : 5! \geq 3^{5-1}$$
$$P(5) : 120 \geq 3^4 = 81$$

- Pruebo $P(k) \Rightarrow P(k+1)$

$$P(k) : k! \geq 3^{k-1}$$
$$\Rightarrow k!(k+1) \geq 3^{k-1}(k+1)$$
$$\Rightarrow (k+1)! \geq \underbrace{3^{k-1}(k+1) \geq 3^{(k+1)-1}}_{(*)}$$

Pruebo $(*)$

$$3^{k-1}(k+1) \geq 3^{(k+1)-1}$$
$$\Leftrightarrow 3^{k-1}(k+1) \geq 3^{k-1} * 3$$
$$\Leftrightarrow (k+1) \geq 3$$
$$\Leftrightarrow k \geq 3-1$$

b) $3^n - 2^n > n^3, \forall n \geq 4$

- Pruebo $P(4)$

$$P(4) : 3^4 - 2^4 > 4^3, \forall n \geq 4$$
$$\Leftrightarrow 81 - 16 > 64, \forall n \geq 4$$
$$\Leftrightarrow 65 > 64, \forall n \geq 4 \Leftrightarrow True$$

- Pruebo $P(k) \Rightarrow P(k+1)$-

$$3^{n+1} - 2^{n+1} = 3 * 3^n - 2 * 2^n \geq 2 * 3^n - 2 * 2^n = 2(3^n - 2^n) \underbrace{\geq 2n^3}_{HI}$$

$$\Leftrightarrow 3^{n+1} - 2^{n+1} < \underbrace{2n^3 \geq (n+1)^3}_{(*)}$$

Ahora pruebo $(*)$:

$$2n^3 \geq (n+1)^3$$
$$\Leftrightarrow (\sqrt[3]{2}n)^3 \geq (n+1)^3$$
$$\Leftrightarrow \sqrt[3]{2}n \geq n+1$$
$$\Leftrightarrow \sqrt[3]{2}n - n \geq 1$$
$$\Leftrightarrow (\sqrt[3]{2} - 1)n \geq 1$$
$$\Leftrightarrow n \geq \frac{1}{(\sqrt[3]{2} - 1)} \simeq 3,8473221 \Leftrightarrow True$$

c) $\sum_{i=1}^{n} \frac{3^i}{i!} < 6n - 5, \forall n \geq 3$

- Pruebo $P(3)$

$$P(3) : \sum_{i=1}^{3} \frac{3^i}{i!} < 6 * 3 - 5$$
$$\Leftrightarrow 3 + \frac{3^2}{2} + \frac{3^3}{6} < 13$$
$$\Leftrightarrow 12 < 13$$

- Pruebo $P(k) \Rightarrow P(k+1)$

$$\sum_{i=1}^{k+1} \frac{3^i}{i!} = \sum_{i=1}^{k} \frac{3^i}{i!} + \frac{3^{k+1}}{(k+1)!} \underbrace{< (6k-5) + \frac{3^{k+1}}{(k+1)!}}_{HI} \leq 6(k+1) - 5$$

Ahora pruebo la segunda desigualdad

$$(6k-5) + \frac{3^{k+1}}{(k+1)!} \leq 6(k+1) - 5$$
$$\Leftrightarrow (6k-5) + \frac{3^{k+1}}{(k+1)!} \leq 6k + 6 - 5$$
$$\Leftrightarrow \frac{3^{k+1}}{(k+1)!} \leq 6$$

Pruebo por inducción $Q(k) : \frac{3^{k+1}}{(k+1)!} \leq 6$:

- Pruebo $Q(1)$

$$Q(1) : \frac{3^{1+1}}{(1+1)!} = \frac{3^2}{2!} = \frac{9}{2} = 4,5 \leq 6$$

- Pruebo $Q(m) \Rightarrow Q(m+1)$

$$\frac{3^{(m+1)+1}}{((m+1)+1)!} = \frac{3 * 3^{m+1}}{(m+1)!(m+2)} = \underbrace{\frac{3^{m+1}}{(m+1)!} * \frac{3}{m+2} \leq 6 * \frac{3}{m+2}}_{HI} \leq 6$$

Y $\frac{3}{m+2} \leq 1$

Ejercicio 7

a) Quiero probar $P(n) : a_n = 2^n + 3^n$. a_n se define como $a_{n+1} = 3a_n - 2^n$, $a_1 = 5$

- Pruebo $P(1)$

$$a_n = 2^n + 3^n$$
$$\Leftrightarrow a_1 = 2^1 + 3^1$$
$$\Leftrightarrow 5 = 2 + 3$$

- Pruebo $P(k) \Rightarrow P(k+1)$

$$a_{k+1} = 3a_k - 2^k$$
$$\underbrace{\Rightarrow}_{HI} a_{k+1} = 3(2^k + 3^k) - 2^k$$
$$\Rightarrow a_{k+1} = 3 * 2^k + 3^{k+1} - 2^k = 2 * 2^k + 3^{k+1} = 2^{k+1} + 3^{k+1}$$

b) Sea $(a_n)_{n \in n}$ la sucesión de números naturales definida recursivamente por $a_1 = 2$, $a_{n+1} = 2 \cdot n \cdot a_n + 2^{n+1} \cdot n!$, $\forall n \in \mathbb{N}$. Probar que $a_n = 2^n \cdot n!$. HI: $P(n) : a_n = 2^n \cdot n!$

- Pruebo $P(1)$

$$a_1 = 2 = 2^1 \cdot 1!$$

- Pruebo $P(k) \Rightarrow P(k+1)$

$$a_{k+1} = 2 \cdot k \cdot a_k + 2^{k+1} \cdot k! \underbrace{=}_{HI} 2 \cdot k \cdot (2^k \cdot k!) + 2^{k+1} \cdot k! = k \cdot 2^{k+1} \cdot k! + 2^{k+1} \cdot k! = k! \cdot 2^{k+1} \cdot (k+1) = (k+1)! \cdot 2^{k+1}$$

c) Sea $(a_n)_{n \in \mathbb{N}}$ la sucesión de números reales definida recursivamente por $a_1 = 0$, $a_{n+1} = a_n + n(3n+1)$, $\forall n \in \mathbb{N}$. Probar que $a_n = n^2(n-1)$.

HI:

$$P(n) : a_n = n^2(n-1)$$

- Pruebo $P(1)$

$$a_1 = 0 = n^2(1-1)$$

- Pruebo $P(k) \Rightarrow P(k+1)$

$$a_{k+1} = a_k + k(3k+1) \underbrace{=}_{HI} (k^2(k-1)) + k(3k+1) = k[k(k-1) + (3k+1)] = k(k^2 - k + 3k + 1)$$
$$= k(k^2 + 2k + 1) = k(k+1)^2 = ((k+1) - 1)(k+1)^2$$