On the Commerce of Flatland

Landon Buechner

Math 220 - Dr. Sottile - Texas A&M

Abstract

I propose an additional chapter to the book *Flatland: A Romance of Many Dimensions* by E. Abbott that explores the thought processes of A Square, an amateur mathematician, as he considers the optimal way to pack circles in squares. Accused of being a prophet of other dimensions, confined within a prison cell, and hanging on to his sanity; A Square reflects upon his past and previous accomplishments.

23.— On The Commerce of Flatland

1.— Introduction

It has been ten years since I was chosen to be a prophet of higher dimensions, yet I am no prophet. Instead of my insights reaching the ends of Flatland, they only reach as far as my cell walls; all of which seem to reflect back to my ears and add to the cacophony that is my stream of consciousness. Oftentimes I find my self repeating the same idea, endlessly, as if each repetition brings me closer to the reality of my Divine Revelation. "Upward, not northward. Upward, not northward. Upward, not northward..." The irony! Why was I liberated from the constraints of two dimensions just to be confined within these four lines! Not one second goes by where I simulate the most familiar sensation of moving in a direction natural to the inhabitants of Spaceland.

In a sense, my condition is both a blessing and a curse. My most regular counterparts decision to imprison me has liberated my mind from biases inherent to living in a two dimensional society and allowed me to hone my mind for geometric thought. Nonsense such as Colour and the pleas of my wife no longer weigh upon me and I am free to lose myself in theory. Occasionally, the increasing dissonance of my thoughts stagnates and I am graced with a period of strong memory and logic that brings me closer to the Truth. Hence, I have started a journal as a way to manage the risk of permanently losing the ability to conceptualize beyond two dimensions. I currently write for this very reason as last night, when in limbo between a state of wakefulness and slumber, I had the most pleasant memory and geometric insight.

2.— A Survey of Geology and the Post Colour War Economy

Termed Geocircs, the rarest gems found in Flatland are perfect circles with a theorized infinite number of edges. By the Laws of Nature, they posses the quality that they can shrink from their natural length of one inch to an arbitrarily small size. This transformation is not without penalty, it is known that the Regularity of a Geocirc decays as a function of radius and time. In addition to their Natural properties, they exhibit the unique trait that they are popular among the nobility and aristocrats.

As mentioned previously, mutual contact is taboo in the upper echelons of society. Members of this Priestly class are predominantly virtuous and resist their corporeal urges. But the remaining Priests, likely the most Irregular of their peers, are inherently susceptible to Temptation. It is rumored that private parties are hosted in which Circles rotate around Geocircs until the mutual touch of regularity is too much to bear. Already predisposed to amoral behavior, these Priests are Gluttons in their consumption of Geocircs. Over time, their debauchery shrinks the size of the gems proportional to the pressure applied which ultimately renders gems undesirable in terms of Regularity. Due to the limited supply of these gems, the insatiable urges of the nobility, and their unlimited supply of capital; Geocircs are among the most valuable items in all of Flatland.

The savvy merchant known as Trisideq, a vendor of gems and a dear friend, had the largest supply of Geocircs back when I was a proud square who practiced law. He ran a vertically integrated mining operation which dealt with the acquisition, storage, and distribution of Geocircs. Any native Flatlander would know that clusters of Geocircs are seasonally generated in the southernmost regions of Flatland. The cyclic nature of Geocirc acquisition requires his firm to deal with the issue of optimizing storage from season to season. Acutely awakened to the of the importance of proper Geocirc storage after losing a substantial quantity to the Rebellion, Trideseq was granted a permit to legally build a square warehouse in order to bolster security. Unfortunately, Flatland's economy suffered after the Colour War due the purge of Irregularity. As a result, our nations economy experienced inflated market prices catalyzed by the limited supply of goods due to a literally non-existent labor force. Any Regular Shape of high station would know that the quote for line, a keystone commodity and construction material, was about ten dollars per inch at this point in time; a high rate for even the most well off merchants. Faced with the pressures of new architectural mandates and inflated prices of line, Tridiseq's healthy profit margin was at risk.

Being of utmost regularity, Trisideq had a predisposed understanding of Sight Recognition but did not posses sufficient sides or training to wield the skill in practice. As it was commonly known among my countrymen that I practiced mathematics for recreation, it was no surprise when he sought me out for geometric consultation.

3.— A Problem

One gloomy morning characterised by the dense fog that accentuates the details of Sight Recognition, a courier came to my door and delivered me a letter from my good friend Trisideq.

> Flatland, April 10, 2010
>
> Dearest A Square,
>
> Tis' been too long since we have engaged in stimulating discourse! Though I must confess, I write to you out of necessity. I am in desperate need of your geometric insights! As you know, the price of line is nearing record highs and I am in the business of Geocircs. By Priestly decree, I must buy new line material to build a new square warehouse for the storage my Geocircs, but due to the inflated prices of line I must

limit it's size! This constraint raises the question in which way I should manipulate the size of my Geocircs to generate the most profitable arrangement! There must exist an optimal way to store gems within a fixed area! Despite my limited understanding of the Fog and the Natural Form of my compatriots, I am keen enough to intuit the geometric nature of the problem at hand and seek your aid.

Would you mind lending me your insights?
Wishes of Regularity and good fortune,

Trisideq

3.— First Thoughts

Naturally, the problem piqued my interest and I enjoyed a solitary walk to his place of business. Upon arrival I was greeted by Trideseq, who proceeded to guide me to his newly constructed warehouse. Lying in the open were hundreds of Geocrics each 1 inch in length, freshly harvested. (Note I, A Square, have translated the mathematics of Flatland to a syntax understandable to readers in Spaceland)

$Trideseq$. Feel free to experiment with them as you please. Before commencing the investigation, allow me to outline the constraints of this problem. My research team has derived an accurate model of Geocirc decay as a function of radii r where $0 < r \leq 1$ and time t with units in years. More accurately, Regularity and in turn the sale price of Geocircs is exponentially proportional to a gems radius. Letting k be a fixed quantity in units of dollars per square inch, the value per Geocirc after storage time t is as follows.

$$\rho(t, r) = ke^{\frac{-t}{r}}$$

Additionally, our budget only allowed for a warehouse with side lengths of one inch and an area of one square inch. Do you see the predicament my firm is faced with?

I. Painfully, yes. Without shrinking the size of any of your gems, you would only be able to store and sell one gem per season! My intuition is that there is an optimal way to configure your gems within the given area. With sufficient time I believe I shall be of great use to you my dear friend. Before I commence my investigation, do you mind sharing which packing strategies you have already considered?

$Tridiseq$. Of course. Aside from the tragic one gem configuration, we considered shrinking all of the gems to an equal radius that allowed for the storage of our entire supply in our square warehouse. As you might have already guessed, our experimental sample mutated into irregular shapes within just a few days.

I. A noble attempt indeed, but you have failed to account for the exponential nature at which a gems value decays! On a high level, for any number of gems n, your goal should be to maximize the future value of each geocirc by considering the amount of decay accrued over an arbitrary length of time t.

$$profit_{tot} = \pi \sum_{i=1}^{n} r^2 \rho(t, r) = k\pi \sum_{i=1}^{n} r^2 e^{\frac{-t}{r_i}} \tag{1}$$

Consider both your solution with a single Geocirc and the other where each gem has an equal radii. If value were solely a function of area, then your solution would have been appropriate. In fact, both of your solutions would have made you an equal amount of money if your number of gems were a value such as 9. Notice that without the weighting of the decay function $\rho(t)$ you would make the same profit per unit area k.

$$Value1 = k\pi(\frac{1}{2})^2 = \frac{k\pi}{4} \approx 0.785k$$
$$Value2 = 9k\pi(\frac{1}{6})^2 = \frac{k\pi}{4} \approx 0.785k$$

If you were to hold you assets for one month and then sell at the end of that period, your profit would be substantially less due the the Geocircs loss of value. It is apparent that your primary goal,

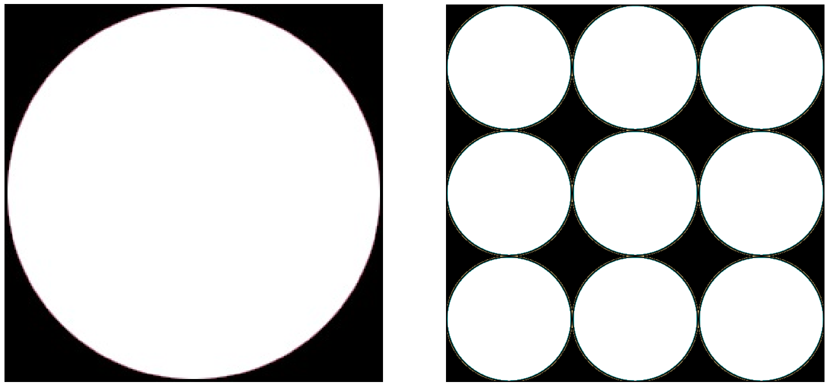


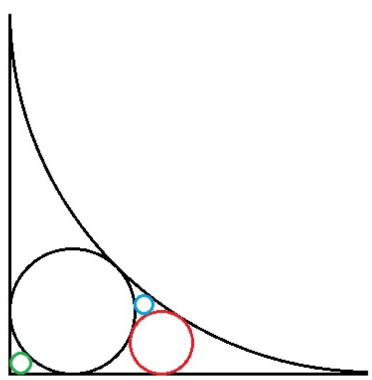
Figure 1: Solution 1 and 2

Tridiseq, should be to try and minimize the degree of which you shrink your Geocircs for storage.

$$ActualValue1 = \pi(\frac{1}{2})^2\rho(\frac{1}{12}, \frac{1}{2}) \approx 0.665k$$
$$ActualValue2 = 9\pi(\frac{1}{6})^2\rho(\frac{1}{12}, \frac{1}{6}) \approx 0.476k$$

3.— Solution

After our first joust at the problem, Trideseq was called away on business leaving me to my own devices. Conceptually, it was apparent that I should place one Geocirc in the center of the square so that it is tangent to each side of the warehouse. Implying a profit of $0.289k$ as discussed. After the initial placement, the goal is to place the next largest possible circle, or shrunk Geocirc, within the box. After experimenting for a while with the resources afforded to me by my dear friend, I discovered 3 different cases which determine the next largest size circle to be placed.

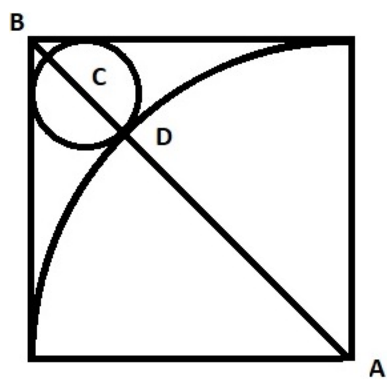


1. One Tangent Circle (Green)
2. Two tangent Circles (Red)
3. Three Tangent Circles (Blue)

Starting with n Geocircs, the optimal solution would be to continue to place each circle in the area that permits the largest radii until out of gems. In order for Tridiseq to implement such an algorithm, he would have to know the maximum size gem that would fit in each available region of area within a warehouse.

One Tangent Circle

Consider the green circle in the above figure. It is apparent that is radius is directly related to its single tangent circle (black). When testing with Trisideqs Geocircs I was able to find out what size Geocirc to place based off of trial and error, but as a Mathematician, that did not suffice.



The following derivation of the maximum radius r that can be placed in a corner, given a radius R is as follows. Let $\overline{AD} = R$. By the nature of 45°triangles, $\overline{AB} = \overline{AD}\sqrt{2}$. Thus for any circle tangent to two perpendicular lines, the quotient of the distance from the center of the circle to the corner, over the radius of that circle, is equal to $\sqrt{2}$ for any given of radii R. Note that $\overline{CB} = \overline{DB} - \overline{DC} = \overline{DB} - r$ where r is the radii of interest. With the following equation we can solve for r.

$$\frac{R(\sqrt{2}-1)-r}{r} = \sqrt{2}$$
$$r = \frac{R(\sqrt{2}-1)}{\sqrt{2}+1}$$

Two and Three Tangent Circles

Luckily for Trideseq, my favorite geometric theorem, derived by René Desquaretes', relates the curvature of any four tangent circles; a true testament to the Regularity and capability of my humble Square brothers. Note that in Flatland, it is common for the Nobility to address each other in terms of their curvature or $k = \frac{1}{r}$.

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$
$$r_{max} = \frac{1}{k4} = k_1 + k_2 + k_3 + \sqrt{k_1 k_2 + k_2 k_3 + k_1 k_3}$$

A known Law of Mathematics is the fact that the curvature of a straight line is 0. This can be seen by noting that the ratio $\frac{1}{r}$ tends to 0 as $r \to \infty$. Thus in the case with two tangent circles and a tangent line (red), Desquaretes' Theorem reduces as follows.

$$(k_1 + k_2 + k_4)^2 = 2(k_1^2 + k_2^2 + k_4^2)$$
$$r_{max} = \frac{1}{k4} = k_1 + k_2 + \sqrt{k_1 k_2}$$

For each of the cases outlined, one could iteratively shrink and place a new gem within the newly defined regions of generated by the previous placement. For example: one could continually place gems into the same corner of a square. Intuitively, this is sub-optimal as there exists other corners of the square in which larger circles can be placed. Thus the stage for an efficient Geocirc packing algorithm is set.

Algorithm 1: Optimal Circle Packing

Result: Profit Maximization

n = number of Geocircs;

while $n \geq 0$ **do**

R = sort(radii of all possible placements);

place_largest(R);

n–;

end

This translates into a general procedure in which you iteratively shrink and place new Geocircs into regions that maximize their area. Theoretically, one could place an infinite number of circles within a given size warehouse as the maximum possible radii that can be placed converges to 0.

4.— Results

I. Tridiseq, rejoice for I have a solution. You will never have to worry about optimal storage again as I have found the optimal packing algorithm. Consider a sample of 9 Geocircs. To demonstrate the methods efficiency, I will calculate present value of these Geocircs at some time t in the future. Here, take note of the rules that guide my placement.

With the given warehouse having sides of length 1 inch, the largest circle that can be placed within has a radius of 0.5. After the initial placement, the next largest possible placements are 4 circles with radii of 0.086. Noting that none of the possible placements in the corners of the square as well as the inner region generated by my last four placements were bigger than the available placements. I placed the remaining 4 tangent to the inner circle, a wall, and one of the previous four. The specific values are as follows.

$$radii = \{0.5,\ 0.086,\ 0.086,\ 0.086,\ 0.086,\ 0.042,\ 0.042,\ 0.042,\ 0.042\}$$

$Tridiseq$. Most impressive my friend. But I lack sufficient sides to compute such a calculation.

What does this mean for my bottom line?

I. Fear not, I will compare this result to the method you initially proposed of 9 circles packed with equal radii. Recall that the value after one month of storage for your configuration was \$0.476 per square inch. The algorithm I devised increases that value by 47.69% to \$0.703 per square inch!

$$value_{tot} = k\pi((0.5^2e^{-t}.5) + (4)(0.086^2e^{-t}0.086) + (4)(0.042^2e^{-t}0.042)) = 0.703\frac{\$}{in^2}$$

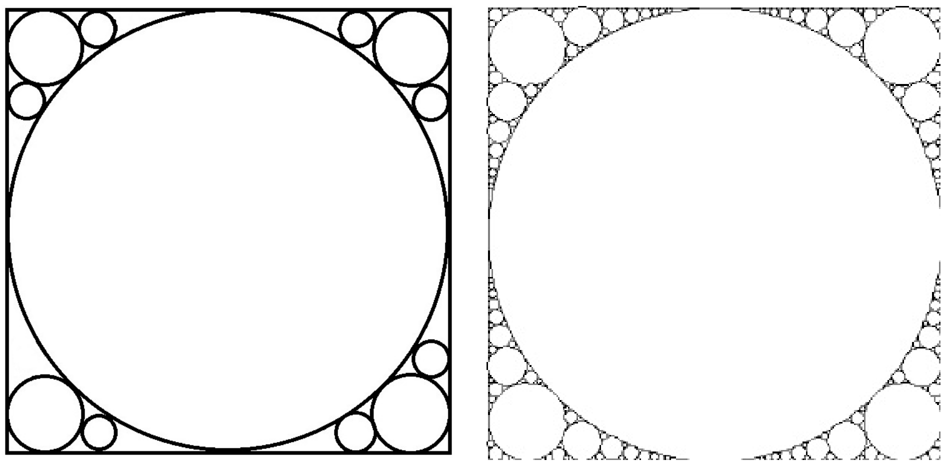


Figure 2: Algorithm approximates this level of efficiency

References

Miyazawa, Flavio K. Approximation Algorithms for Circle Packing. Combinatorics and Optimization. Combinatorics and Optimization, So Paulo.

Mistry, Roshni. Circle Packing, Sphere Packing, and Keplers Conjecture. St. Marys College of California, 2016, math.stmarys-ca.edu/wp-content/uploads/2017/07/Roshni-Mistry.pdf.

Wilker, J. B. ”Four Proofs of a Generalization of the Descartes Circle Theorem.” Amer. Math. Monthly 76, 278-282, 1969.