作业：

1. 推导四足机器人单刚体模型的mpc的QP形式.（轨迹跟踪任务）.

reference：$x_{ref} = (x_{1,ref}, x_{2,ref}, \cdots, x_{N,ref})^T$. $x_{0,ref} = x_0$.

cost：$J = \frac{1}{2}\sum_{i=0}^{N-1}\left[(x_i - x_{i,ref})^T Q_i (x_i - x_{i,ref}) + u_i^T R_i u_i\right] + \frac{1}{2}(x_N - x_{N,ref})^T Q_N (x_N - x_{N,ref})$.

dynamics：$x_{i+1} = Ax_i + Bu_i$. $i = 0, 1, 2, \cdots, N-1$.

$X = [x_1, x_2, \cdots x_N]^T$, $U = [u_0, u_1, \cdots, u_{N-1}]^T$.

$x_1 = Ax_0 + Bu_0$

$x_2 = Ax_1 + Bu_1 = A^2 x_0 + ABu_0 + Bu_1$.

$\vdots$

$x_k = A^k x_0 + A^{k-1}Bu_0 + A^{k-2}Bu_1 + \cdots + ABu_{k-2} + Bu_{k-1}$.

$\vdots$

$x_N = A^N x_0 + A^{N-1}Bu_0 + A^{N-2}Bu_1 + \cdots + ABu_{N-2} + Bu_{N-1}$.

$$\Rightarrow X = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{C} x_0 + \underbrace{\begin{bmatrix} B & & & 0 \\ AB & B & & \\ \vdots & & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}}_{D} U. \qquad X = Cx_0 + DU.$$

又 $J = \frac{1}{2}\sum_{i=0}^{N-1}\left[(x_i - x_{i,ref})^T Q_i (x_i - x_{i,ref}) + u_i^T R_i u_i\right] + \frac{1}{2}(x_N - x_{N,ref})^T Q_N (x_N - x_{N,ref})$.

$= \frac{1}{2}(x_1 - x_{1,ref})^T Q_1 (x_1 - x_{1,ref}) + \frac{1}{2}(x_2 - x_{2,ref})^T Q_2 (x_2 - x_{2,ref}) + \cdots + \frac{1}{2}(x_N - x_{N,ref})^T Q_N (x_N - x_{N,ref})$

$+ \frac{1}{2}u_0^T R_0 u_0 + \frac{1}{2}u_1^T R_1 u_1 + \cdots + \frac{1}{2}u_{N-1}^T R_{N-1} u_{N-1}$.

$= \frac{1}{2}\left[(X - X_{ref})^T Q (X - X_{ref}) + U^T R U\right]$, 其中 $Q = \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_N \end{bmatrix}$, $R = \begin{bmatrix} R_0 & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_{N-1} \end{bmatrix}$.

$= \frac{1}{2}\left[(Cx_0 + DU - X_{ref})^T Q (Cx_0 + DU - X_{ref}) + U^T R U\right]$.

$= \frac{1}{2}U^T \underbrace{(R + D^T Q D)}_{H} U + U^T \underbrace{D^T Q (Cx_0 - X_{ref})}_{g} + \underbrace{\frac{1}{2}(Cx_0 - X_{ref})^T Q (Cx_0 - X_{ref})}_{与U无关，舍去}$.

Dense QP：$\min\limits_{U} \frac{1}{2}U^T H U + U^T g$

2. 参考LQR的推导过程，推导 iLQR 的迭代公式.

cost： $\min\limits_{u,x} \sum_{i=0}^{N-1} f_i(x_i, u_i) + f_N(x_N)$.

dynamics： $x_{i+1} = F(x_i, u_i)$.

对 cost 二阶近似，对 dynamics 一阶近似：

$$\Longrightarrow \min_{\delta u, \delta x} q_N + \delta x_N^T \hat{q}_N + \frac{1}{2}\delta x_N^T Q_N \delta x_N + \sum_{i=0}^{N-1} q_i + \delta x_i^T \hat{q}_i + \delta u_i^T r_i + \frac{1}{2}\delta x_i^T Q_i \delta x_i + \frac{1}{2}\delta u_i^T R_i \delta u_i + \delta u_i^T P_i \delta x_i.$$

s.t. $\delta x_{i+1} = A_i \delta x_i + B_i \delta u_i$.

define：

$$\begin{cases} C_i(\delta x_i, \delta u_i) = q_i + \delta x_i^T \hat{q}_i + \delta u_i^T r_i + \frac{1}{2}\delta x_i^T Q_i \delta x_i + \frac{1}{2}\delta u_i^T R_i \delta u_i + \delta u_i^T P_i \delta x_i. \\ Q_i(\delta x_i, \delta u_i) = C_i(\delta x_i, \delta u_i) + V_{i+1}(\delta x_{i+1}). \\ V_i(\delta x_i) = \min\limits_{\delta u_i} Q_i(\delta x_i, \delta u_i). \\ V_N(\delta x_N) = \frac{1}{2}\delta x_N^T Q_N \delta x_N + \delta x_N^T \hat{q}_N + q_N. \end{cases}$$

dynamic programming：

for $i = N-1$, $V_{N-1} = \min\limits_{\delta u_{N-1}} q_{N-1} + \delta x_{N-1}^T \hat{q}_{N-1} + \delta u_{N-1}^T r_{N-1} + \frac{1}{2}\delta x_{N-1}^T Q_{N-1} \delta x_{N-1} + \frac{1}{2}\delta u_{N-1}^T R_{N-1} \delta u_{N-1} + \delta u_{N-1}^T P_{N-1} \delta x_{N-1}$

$$+ q_N + (A_{N-1}\delta x_{N-1} + B_{N-1}\delta u_{N-1})^T \hat{q}_N + \frac{1}{2}(A_{N-1}\delta x_{N-1} + B_{N-1}\delta u_{N-1})^T Q_N (A_{N-1}\delta x_{N-1} + B_{N-1}\delta u_{N-1}).$$

舍去与u无关的项

$$\min_{\delta u_{N-1}} \delta u_{N-1}^T (r_{N-1} + P_{N-1}\delta x_{N-1} + B_{N-1}^T \hat{q}_N + B_{N-1}^T Q_N A_{N-1} \delta x_{N-1}) + \frac{1}{2}\delta u_{N-1}^T (R_{N-1} + B_{N-1}^T Q_N B_{N-1}) \delta u_{N-1}.$$

$$\Longrightarrow \delta u_{N-1}^* = -(R_{N-1} + B_{N-1} Q_N B_{N-1})^{-1} [r_{N-1} + B_{N-1}^T \hat{q}_{N-1} + (P_{N-1} + B_{N-1}^T Q_N A_{N-1}) \delta x_{N-1}].$$

令 $H_{N-1} = R_{N-1} + B_{N-1} Q_N B_{N-1}$, $h_{N-1} = r_{N-1} B_{N-1}^T \hat{q}_{N-1}$, $G_{N-1} = P_N + B_{N-1}^T Q_N A_{N-1}$.

$$\Longrightarrow \delta u_{N-1}^* = -H_{N-1}^{-1} h_{N-1} - H_{N-1}^{-1} G_{N-1} \delta x_{N-1}.$$

令 $l_{N-1} = -H_{N-1}^{-1} h_{N-1}$, $L_{N-1} = -H_{N-1}^{-1} G_{N-1}$.

$$\Rightarrow \delta u_{N-1}^* = l_{N-1} + L_{N-1} \delta x_{N-1}.$$

代入 $V_{N-1}(\delta x_{N-1})$：

$$V_{N-1}^*(\delta x_{N-1}) = \underbrace{q_{N-1} + q_N + l_{N-1}^T h_{N-1} + \frac{1}{2} l_{N-1}^T H_{N-1} l_{N-1}}_{s_{N-1}}.$$

$$+ \delta x_{N-1}^T (\underbrace{\hat{q}_{N-1} + A_{N-1}^T \hat{q}_N + G_{N-1}^T l_{N-1} + L_{N-1}^T h_{N-1} + L_{N-1}^T H_{N-1} l_{N-1}}_{\hat{s}_{N-1}}).$$

$$+\delta x_{N-1}^T \underbrace{\left(\frac{1}{2}Q_{N-1}+\frac{1}{2}A_{N-1}^T Q_N A_{N-1}-\frac{1}{2}G_{N-1}^T H_{N-1}^{-1} G_{N-1}\right)}_{S_{N-1}} \delta x_{N-1}.$$

$$= s_{N-1} + \delta x_{N-1}^T \hat{s}_{N-1} + \frac{1}{2}\delta x_{N-1}^T S_{N-1}\delta x_{N-1}.$$

注意到：$V_N(\delta x_N) = q_N + \delta x_N^T \hat{q}_N + \frac{1}{2}\delta x_N^T Q_N \delta x_N$，令 $s_N = q_N$, $\hat{s}_N = \hat{q}$, $S_N = Q_N$.

$$\Rightarrow V_N(\delta x_N) = s_N + \delta x_N^T \hat{s}_N + \frac{1}{2}\delta x_N^T S_N \delta x_N.$$

由归纳法可见，$V_0, V_1, \cdots, V_N$ 具有相同形式：

$$V_i(\delta x_i) = s_i + \delta x_i^T \hat{s}_i + \frac{1}{2}\delta x_i^T S_i \delta x_i.$$

$$\delta u_i^* = l_i + L_i \delta x_i.$$

于是得到更新公式：

$$u_i^{[k+1]} = u_i^{[k]} + l_i + L_i(x_i^{[k+1]} - x_i^{[k]}).$$

$$x_{i+1}^{[k+1]} = F\left(x_i^{[k]}, u_i^{[k+1]}\right).$$

其中：$S_i = Q_i + A_i^T S_{i+1} A_i - L_i^T H_i L_i$

$$\hat{s}_i = \hat{q}_i + A_i^T \hat{s}_{i+1} + G_i^T l_i + L_i^T(h_i + H_i l_i).$$

$$h_i = r_i + B_i^T \hat{s}_{i+1}$$

$$G_i = P_i + B_i^T S_{i+1} A_i$$

$$H_i = R_i + B_i^T S_{i+1} B_i$$

$$l_i = -H_i^{-1} h_i$$

$$L_i = -H_i^{-1} G_i.$$