Systems Analysis Techniques HW2

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Problem 1

$$S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \mid x_1 + x_2 + x_3 = 0 \right\}$$

For a linear vector space, three most important conditions need to be satisfied:

1. Given vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in R^3, \text{ where } x_1 + x_2 + x_3 = 0 + 0 + 0 = 0, \text{ thus } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S_1$$

2. Given vectors $U, V \in S_1$, and define $W = U + V$:

$$u_1 + u_2 + u_3 = 0$$

$$v_1 + v_2 + v_3 = 0$$

$$u_1 + v_1 + u_2 + v_2 + u_3 + v_3 = w_1 + w_2 + w_3 = 0$$

thus $W \in S_1$.

3. Given Vector $U \in S_1$, and define $K = c \cdot U$, where c is a constant from the field, such that:

$$u_1 + u_2 + u_3 = 0;$$

Multiple a constant c to the equation:

$$cu_1 + cu_2 + cu_3 = k_1 + k_2 + k_3 = 0;$$

thus $K \in S_1$.

With above conditions, S_1 is a liner vector space.

b)

$$S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \mid x_1 + x_2 + x_3 = 1 \right\}$$

1. Given vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in R^3, \text{ where } x_1 + x_2 + x_3 = 0 + 0 + 0 \neq 0, \text{ thus } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S_2$$

2. Given vectors $U, V \in S_2$, and define $W = U + V$:

$$u_1 + u_2 + u_3 = 1$$

$$v_1 + v_2 + v_3 = 1$$

$$u_1 + v_1 + u_2 + v_2 + u_3 + v_3 = w_1 + w_2 + w_3 = 2 \neq 1$$

thus $W \notin S_2$.

3. Given Vector $U \in S_2$, and define $K = c \cdot U$, where c is any constant not equal to 1, such that:

$$u_1 + u_2 + u_3 = 1;$$

Multiple a constant c to the equation:

$$cu_1 + cu_2 + cu_3 = k_1 + k_2 + k_3 = c \neq 1;$$

thus $K \notin S_2$.

None of the above conditions met, S_2 is a not liner vector space.

Problem 2

Inner product defined as

$$\langle x, y \rangle := \int_0^{2\pi} x(t)y(t)dt$$

The following functions forms the orthonormal basis

$$\begin{array}{ll} s_1(t) = \frac{1}{\sqrt{\pi}} \sin(t) & c_0(t) = \sqrt{\frac{1}{2\pi}} \\ s_2(t) = \frac{1}{\sqrt{\pi}} \sin(2t) & c_1(t) = \frac{1}{\sqrt{\pi}} \cos(t) \\ \vdots & \vdots \\ s_n(t) = \frac{1}{\sqrt{\pi}} \sin(nt) & c_n(t) = \frac{1}{\sqrt{\pi}} \cos(nt) \end{array}$$

For integer m, n ≥ 1 and m $\neq$ n

$$\langle S_m(t), S_n(t) \rangle = \frac{1}{\pi} \cdot \int_0^{2\cdot\pi} (\sin(m \cdot x) \cdot \sin(n \cdot x))dx$$

$$= \frac{\sin(2 \cdot m \cdot \pi - 2 \cdot n \cdot \pi)}{2 \cdot (m-n) \cdot \pi} - \frac{\sin(2 \cdot m \cdot \pi + 2 \cdot n \cdot \pi)}{2 \cdot (m+n) \cdot \pi} = 0$$

if m = n:

$$\langle S_n(t), S_n(t) \rangle = \frac{1}{\pi} \cdot \int_0^{2 \cdot \pi} (\sin(n \cdot x) \cdot \sin(n \cdot x)) dx$$

$$= \frac{-(\sin(4 \cdot n \cdot \pi) - 4 \cdot n \cdot \pi)}{4 \cdot n \cdot \pi} = 1$$

Similarly

$$\langle C_m(t), C_n(t) \rangle = \frac{1}{\pi} \cdot \int_0^{2 \cdot \pi} (\cos(m \cdot x) \cdot \cos(n \cdot x)) dx$$

$$= \frac{\sin(2 \cdot m \cdot \pi + 2 \cdot n \cdot \pi)}{2 \cdot (m+n) \cdot \pi} + \frac{\sin(2 \cdot m \cdot \pi - 2 \cdot n \cdot \pi)}{2 \cdot (m-n) \cdot \pi} = 0$$

for m = 0

$$\langle C_0(t), C_n(t) \rangle = \frac{1}{\sqrt{2} \cdot \pi} \cdot \int_0^{2 \cdot \pi} \cos(n \cdot x) dx = \frac{\sin(2 \cdot n \cdot \pi) \cdot \sqrt{2}}{2 \cdot n \cdot \pi} = 0$$

Norm of $C_n(t)$

$$\langle C_n(t), C_n(t) \rangle = \frac{1}{\pi} \cdot \int_0^{2 \cdot \pi} (\cos(n \cdot x) \cdot \cos(n \cdot x)) dx$$

$$= \frac{\sin(4 \cdot n \cdot \pi) + 4 \cdot n \cdot \pi}{4 \cdot n \cdot \pi} = 1$$

$$\langle C_0(t), C_0(t) \rangle = \int_0^{2 \cdot \pi} (\frac{1}{2\pi}) dx = 1$$

Inner product between sine group and cosine group

$$\langle S_n(t), C_m(t) \rangle = \frac{1}{\pi} \cdot \int_0^{2 \cdot \pi} (\sin(n \cdot x) \cdot \cos(m \cdot x)) dx$$

$$= \frac{-\cos(2 \cdot m \cdot \pi + 2 \cdot n \cdot \pi)}{2 \cdot (m+n) \cdot \pi} + \frac{\cos(2 \cdot m \cdot \pi - 2 \cdot n \cdot \pi)}{2 \cdot (m-n) \cdot \pi} - \frac{1}{2 \cdot (m-n) \cdot \pi} + \frac{1}{2 \cdot (m+n) \cdot \pi} = 0$$

$$\langle S_n(t), C_0(t) \rangle = \frac{1}{\sqrt{2} \cdot \pi} \cdot \int_0^{2 \cdot \pi} \sin(n \cdot x) dx = \frac{-(\cos(2 \cdot n \cdot \pi) - 1) \cdot \sqrt{2}}{2 \cdot n \cdot \pi} = 0$$

Problem 3

Given

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and the definition of inner product which includes 5 properties.

An inner product on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$.

Thus we can define two nonzero vectors u, v $\in V$, and an element c from field F. Assume $u \neq cv$, and define a new vector w $\in V$, such that:

$$w = u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v \tag{1}$$

The above vector w is a linear combination of the u and v vectors, because the inner product is an element from the Field, as shown in the inner product definition. Now take the inner product of w and v to get:

$$\langle w, v\rangle = \left\langle u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v, v \right\rangle = \langle u, v\rangle - \frac{\langle u, v\rangle}{\langle v, v\rangle}\langle v, v\rangle = 0 \tag{2}$$

Rearrange the equation (1) to get

$$u = w + \frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

denote the $\frac{\langle u, v\rangle}{\langle v, v\rangle}$ as λ, and use the definition of norm of vector, and additivity of the inner product:

$$\|w + \lambda v\| = \sqrt{\langle w + \lambda v, w + \lambda v\rangle} = \sqrt{\langle w, w\rangle + \langle w, \lambda v\rangle + \langle \lambda v, w\rangle + \langle \lambda v, \lambda v\rangle} \tag{3}$$

Multiply a λ to every vector v in equation (2), which is still zero, such that

$$\|w + \lambda v\| = \sqrt{\langle w, w\rangle + \langle \lambda v, \lambda v\rangle} \text{ or } \|w + \lambda v\|^2 = \|w\|^2 + \|\lambda v\|^2 \tag{4}$$

By the norm property:

$$\begin{aligned}\|\lambda v\|^2 &= \langle \lambda v, \lambda v\rangle \\ &= \lambda\langle v, \lambda v\rangle \\ &= \lambda\bar{\lambda}\langle v, v\rangle \\ &= |\lambda|^2\|v\|^2\end{aligned}$$

Equation (4) becomes

$$\|u\| = \|w + \lambda v\| = \|w\|^2 + \|\lambda v\|^2 = \|w\|^2 + |\lambda|^2\|v\|^2$$

thus

$$\|u\|^2 \geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}$$

Change the variable 'u' to 'x', and 'v' to 'y' to match the problem:

$$\|x\|^2\|y\|^2 \geq |\langle x, y\rangle|^2 \tag{5}$$

$$\langle x, x\rangle\langle y, y\rangle \geq \langle x, y\rangle^2$$

b)

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle \\ &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \overline{\langle x,y\rangle} = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\langle x,y\rangle \end{aligned} \tag{6}$$

Since $2\,\mathrm{Re}\langle x,y\rangle \leq 2|\langle x,y\rangle|$, equation (6) $\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle|$
According to the derivation from (5):

$$\|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\| \leq \|x\| + \|y\|$$

Problem 4

Given

$$\pi_x(y) = \langle y, x_1\rangle\, x_1 + \langle y, x_2\rangle\, x_2 + \cdots + \langle y, x_m\rangle\, x_m$$

$$y = \langle y, x_1\rangle\, x_1 + \langle y, x_2\rangle\, x_2 + \cdots + \langle y, x_m\rangle\, x_m + (y - \langle y, x_1\rangle\, x_1 - \langle y, x_2\rangle\, x_2 - \cdots - \langle y, x_m\rangle\, x_m)$$

Define the part inside the parenthesis as vector k, and n = 1 ...m, so that

$$\langle k, x_n\rangle = \langle y, x_n\rangle - \langle y, x_n\rangle = 0$$

Notice: since all the x_n are the orthonormal basis, where $\langle x_i, x_j\rangle = 0$ if $i \neq j$, and

$$k = y - \pi_x(y)$$

For any $z \in \mathrm{span}\,\{x_1, x_2, \cdots, x_m\}$

$$z = \langle z, x_1\rangle\, x_1 + \langle z, x_2\rangle\, x_2 + \cdots + \langle z, x_m\rangle\, x_m$$

and thus

$$\langle k, z\rangle = \langle y, x_n\rangle\, \langle z, x_n\rangle - \langle y, x_n\rangle\, \langle z, x_n\rangle = 0$$

where n = 1, 2, 3 ... m. Since both z and $\pi_x(y) \in \mathrm{span}\,\{x_1, x_2, \cdots, x_m\}$, and define l

$$l = \pi_x(y) - z = (\langle y, x_1\rangle - \langle z, x_1\rangle)x_1 + (\langle y, x_2\rangle - \langle z, x_2\rangle)x_2 + \cdots + (\langle y, x_m\rangle - \langle z, x_m\rangle)x_m$$

now find the inner product between k and l, for n = 1, 2, 3 ... m

$$\langle k, l\rangle = \langle y, x_n\rangle\, (\langle y, x_n\rangle\, \langle z, x_n\rangle - \langle y, x_n\rangle\, \langle z, x_n\rangle) - \langle y, x_n\rangle\, (\langle y, x_n\rangle\, \langle z, x_n\rangle - \langle y, x_n\rangle\, \langle z, x_n\rangle) = 0$$

Now it is true that

$$\|y - \pi_x(y)\|^2 \leq \|y - \pi_x(y)\|^2 + \|\pi_x(y) - z\|^2$$

$$\|k\|^2 \leq \|k\|^2 + \|l\|^2$$

Since $\langle k, l \rangle = 0$, from equation (3) and (4) in problem 3:

$$\|k\|^2 \leq \|k + l\|^2$$

which is

$$\|y - \pi_x(y)\|^2 \leq \|y - \pi_x(y) + \pi_x(y) - z\|^2$$

$$\|y - \pi_x(y)\|^2 \leq \|y - z\|^2$$

thus

$$\|y - \pi_x(y)\| \leq \|y - z\|$$

Problem 5

Verification of the two vectors as orthonomal:

$$\langle x, y \rangle = \int_{-1}^{1} \frac{1}{2} x(t) y(t) dt$$

$$v_1(t) = 1$$
$$v_2(t) = \sqrt{3} \cdot t$$

$$\langle v_1, v_1 \rangle = \int_{-1}^{1} \frac{1}{2} v_1(t) v_1(t) dt = 1$$

$$\langle v_2, v_2 \rangle = \int_{-1}^{1} \frac{3}{2} t^2 = 1$$

$$\langle v_1, v_2 \rangle = \int_{-1}^{1} \frac{\sqrt{3}}{2} t = 0$$

b)

To minimize

$$J(A, B) = \sqrt{\int_{-1}^{1} (\sin t - A - Bt)^2 dt}$$

$$\|y - \pi_x(y)\| \leq \|y - z\|$$

set $z = A + Bt = \pi_x(y)$ and $y = sin(t)$ thus

$$\|y - z\| = \sqrt{\langle y - z, y - z \rangle} = \sqrt{\int_{-1}^{1} \frac{1}{2}(sin(t) - \pi_x(y))^2 dt}$$

In this case: $J(A, B) = \frac{1}{\sqrt{2}} \|y - z\|$, so if $\|y - z\|$ reaches to the minimum, J(A,B) reaches to the minimum.

$\pi_v(y) = \langle y, v_1 \rangle v_1 + \langle y, v_2 \rangle v_2$

$$\langle y, v_1 \rangle = \int_{-1}^{1} \frac{1}{2} \sin(t) dt = 0$$

$$\langle y, v_2 \rangle = \int_{-1}^{1} \frac{1}{2} \sin(t) \sqrt{3} t dt = 0.52163945357205$$

Thus when A = 0, B = 0.52

$$\|y - \pi_x(y)\| = \|y - z\|$$

J(A,B) reach to the minimum