System Analysis Techniques HW8

Yunfan Gao

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Problem 1

```
close all;clear all;
load('PCAData.mat');
[U,S,V] = svd(X);
%step 3
figure(1);
for i = 1:200
    plot3(X(1,i),X(2,i),X(3,i),'o')
    hold on
end
xlim([-2 2]);ylim([-2 2]);zlim([-2 2])
hold on;
%principal directions
plot3([0,U(1,1)],[0,U(2,1)],[0,U(3,1)],'r');hold on;
plot3([0,U(1,2)],[0,U(2,2)],[0,U(3,2)],'b');hold on;
plot3([0,U(1,3)],[0,U(2,3)],[0,U(3,3)],'g');hold on;
xlim([-2 2]);ylim([-2 2]);zlim([-2 2]);grid on;
S(3,3) = 0;
Snew = S;
X1 = U*S*transpose(V);

%step 5
figure(2);
for i = 1:200
    plot3(X1(1,i),X1(2,i),X1(3,i),'o')
    hold on
end
hold on;
plot3([0,U(1,1)],[0,U(2,1)],[0,U(3,1)],'r');hold on;
plot3([0,U(1,2)],[0,U(2,2)],[0,U(3,2)],'b');hold on;
plot3([0,U(1,3)],[0,U(2,3)],[0,U(3,3)],'g');hold on;
xlim([-2 2]);ylim([-2 2]);zlim([-2 2]);grid on;
```

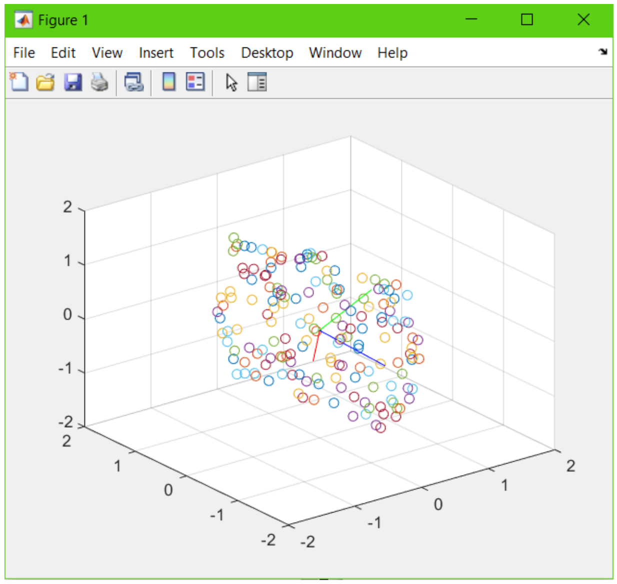


Figure 1: step 3

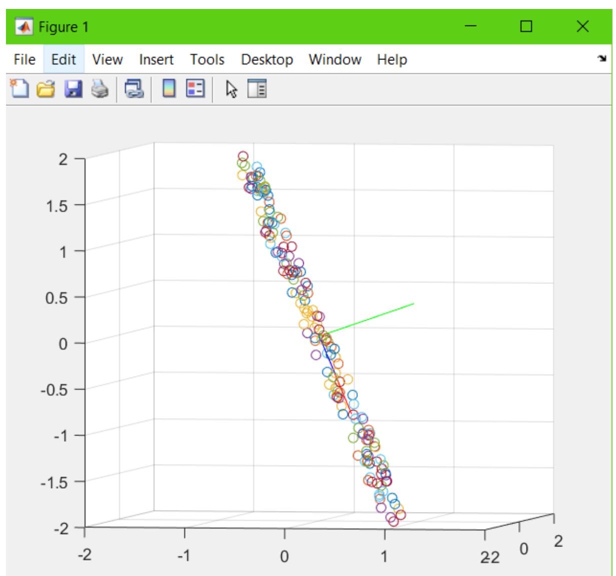


Figure 2: step 3 at a different angle

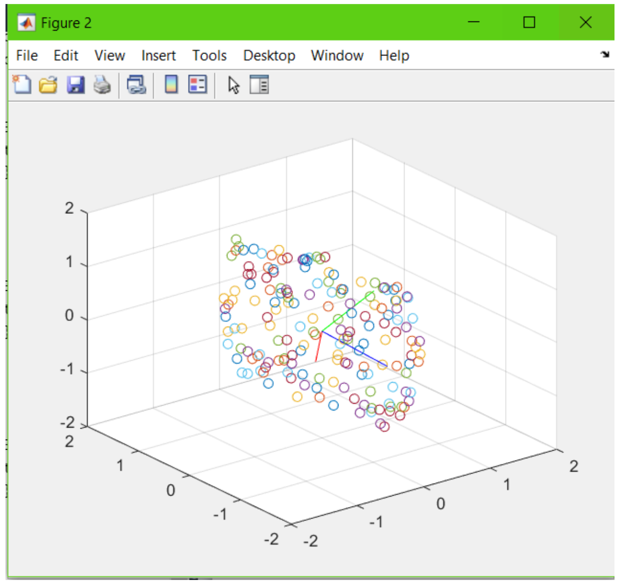


Figure 3: step 5

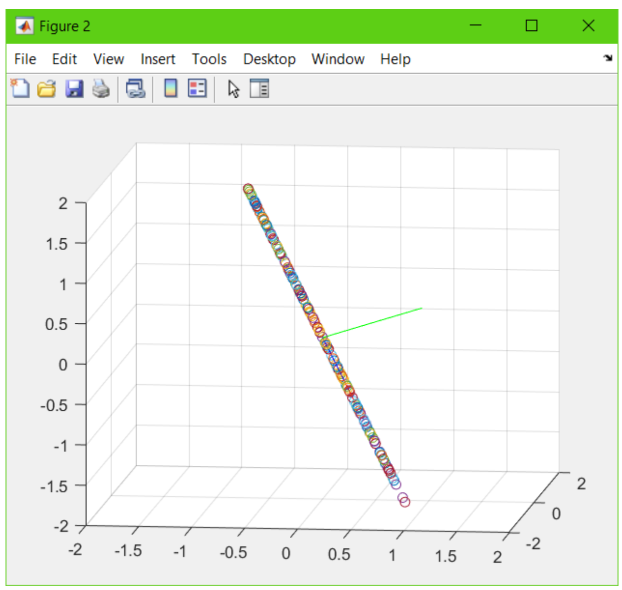


Figure 4: step 5 at a different angle

Problem 2

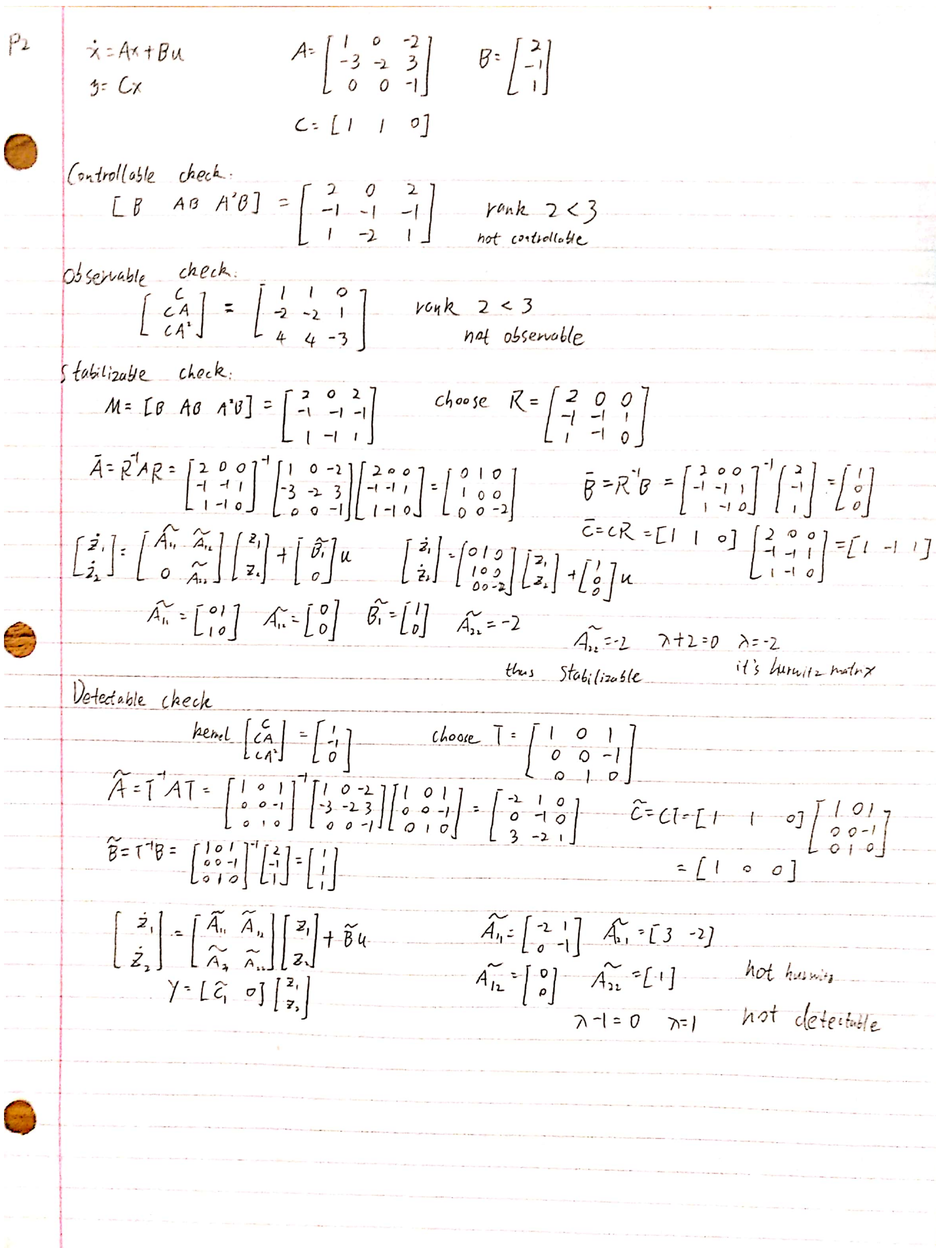
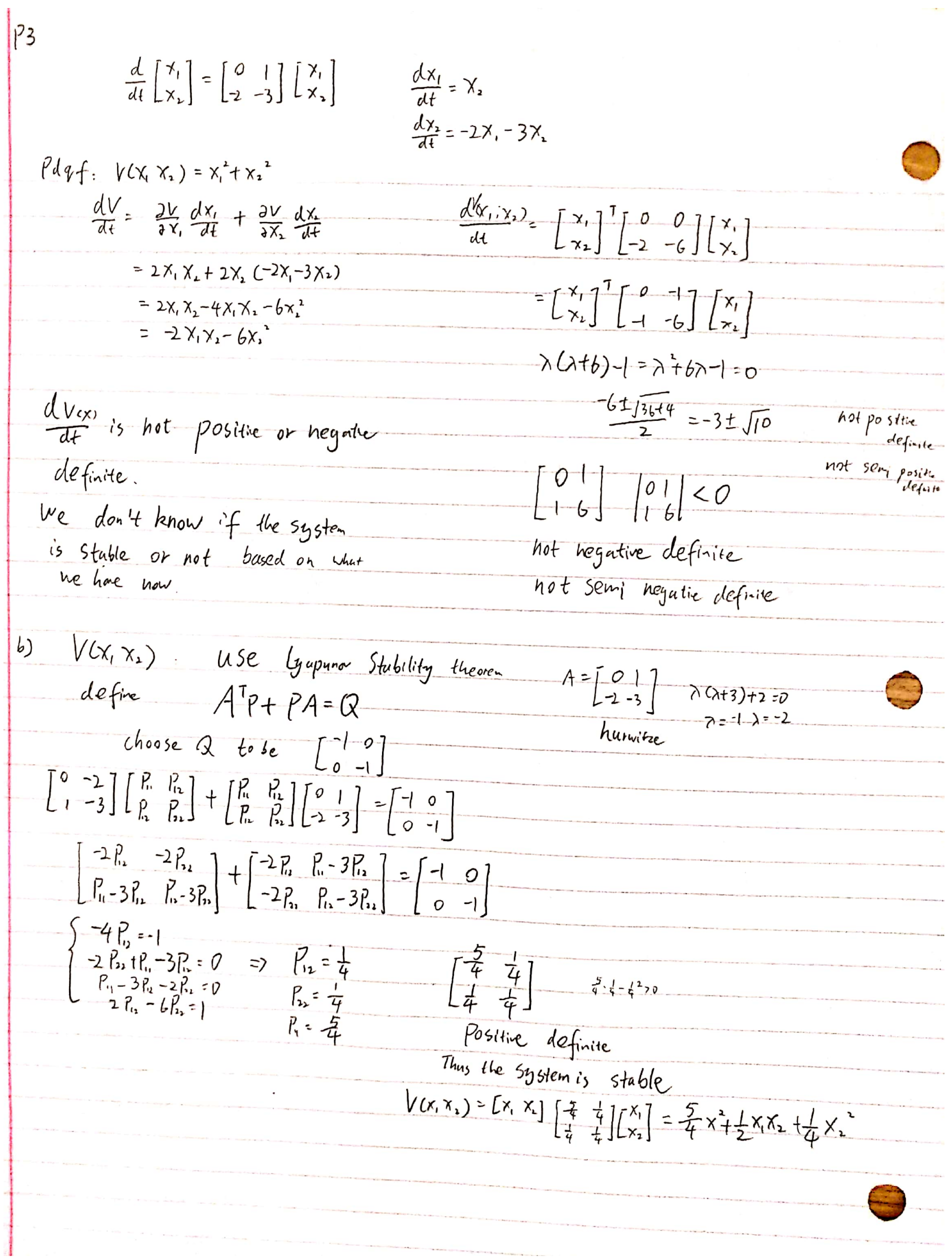


Figure 5: step 5 at a different angle

Problem 3



P3

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -2x_1 - 3x_2$$

Pdqf: $V(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}$$

$$= 2x_1x_2 + 2x_2(-2x_1 - 3x_2)$$

$$= 2x_1x_2 - 4x_1x_2 - 6x_2^2$$

$$= -2x_1x_2 - 6x_2^2$$

$$\frac{dV(x_1, x_2)}{dt} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ -1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda(\lambda+6) - 1 = \lambda^2 + 6\lambda - 1 = 0$$

$$\frac{-6 \pm \sqrt{36+4}}{2} = -3 \pm \sqrt{10}$$

not positive definite

not semi positive definite

$\frac{dV(x)}{dt}$ is not positive or negative definite.

We don't know if the system is stable or not based on what we have now.

$$\begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix} \quad \begin{vmatrix} 0 & 1 \\ 1 & 6 \end{vmatrix} < 0$$

not negative definite

not semi negative definite

b) $V(x_1, x_2)$. use Lyapunov Stability theorem

define $A^T P + PA = Q$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \lambda(\lambda+3)+2=0$$

$$\lambda = -1 \quad \lambda = -2$$

hurwitz

choose Q to be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2P_{12} & -2P_{22} \\ P_{11} - 3P_{12} & P_{12} - 3P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{12} & P_{11} - 3P_{12} \\ -2P_{22} & P_{12} - 3P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{cases} -4P_{12} = -1 \\ -2P_{22} + P_{11} - 3P_{12} = 0 \\ P_{11} - 3P_{12} - 2P_{22} = 0 \\ 2P_{12} - 6P_{22} = 1 \end{cases} \Rightarrow \begin{aligned} P_{12} &= \frac{1}{4} \\ P_{22} &= \frac{1}{4} \\ P_{11} &= \frac{5}{4} \end{aligned}$$

$$\begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad \frac{5}{4}\cdot\frac{1}{4} - \frac{1}{4}^2 > 0$$

Positive definite

Thus the system is stable

$$V(x_1, x_2) = [x_1 \ x_2]\begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{5}{4}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{4}x_2^2$$

Figure 6: step 5 at a different angle

Problem 4

$\dot{x} = Ax, x \in R^n$ and Q is n-by-n symmetric negative semidefinite matrix. Given statement 1:

rank($\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix}$) = n, which means full column rank. Thus the kernel of the observability matrix: $\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x = 0$ if and only if x = 0.

Notice $\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x = \begin{bmatrix} Qx \\ QAx \\ \vdots \\ QA^{n-1}x \end{bmatrix} = \begin{bmatrix} Qe^{At}x_0 \\ QAe^{At}x_0 \\ \vdots \\ QA^{n-1}e^{At}x_0 \end{bmatrix}$

At t = 0, the above equation becomes $\begin{bmatrix} Qx_0 \\ QAx_0 \\ \vdots \\ QA^{n-1}x_0 \end{bmatrix} = \begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x_0 = 0$

Thus $x_0 = 0$, $x = e^{At}x_0 = 0 \quad \forall t$, therefore statement 2 is proven
Now given the statement 2: if $Qx(t) = 0 \quad \forall t$, then $x(t) = 0 \quad \forall t$
Q is a n by n square matrix.
If $Qx(t) = 0$ if and only if $x(t) = 0$, then Q is full rank.

$\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix}$ is the augmented matrix of Q with the maximum rank of n and with the rank larger or equal to the rank of Q. Since the rank of Q is n, then the rank of the observability matrix is also n. Thus statement 1 is proven.