Problem 1

$$\sin At = At - \frac{(At)^3}{3!} + \frac{(At)^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (At)^{2n+1}}{(2n+1)!}$$

$$\frac{d \sin At}{dt} = A - \frac{A^3t^2}{2!} + \frac{A^5t^4}{5!} + \cdots$$

$$= A\left(I - \frac{A^2t^2}{2!} + \frac{A^4t^4}{4!}\right)$$

$$= A \cos At$$

$$\cos At = I - \frac{A^2t^2}{2!} + \frac{A^4t^4}{4!} + \cdots = I + \sum_{n=1}^{\infty} \frac{(-1)^n (At)^{2n}}{(2n)!}$$

$$\frac{d \cos At}{dt} = -\frac{A^2t}{1!} + \frac{A^4t^3}{3!} + \cdots$$

$$= -A\left(\frac{At}{1!} - \frac{A^3t^3}{3!} + \frac{A^5t^5}{5!}\right)$$

$$= -A \sin At$$

b)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \Delta(\lambda) = \begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{bmatrix} = (\lambda-1)^2 \qquad \lambda_1 = \lambda_2 = 1$$

$$R(A) = a_0 I + a_1 A$$

$$\sin At = a_0 I + a_1 A$$

$$\sin \lambda_1 t = a_0 + a_1 \lambda_1$$

$$\frac{d \sin \lambda_1 t}{d\lambda_1} = \frac{d}{d\lambda_1}(a_0 + a_1\lambda_1) = a_1 \quad \text{or} \quad t \cos \lambda_1 t = a_1$$

$$\begin{cases} \sin \lambda_1 t = a_0 + a_1 \lambda_1 \\ t \cos \lambda_1 t = a_1 \end{cases} \qquad \lambda_1 = 1$$

$$\begin{cases} a_0 = \sin t - t\cos t \\ a_1 = t \cos t \end{cases}$$

$$\sin At = a_0 I + a_1 A$$

$$= (\sin t - t\cos t) I + t\cos t \, A$$

$$= \begin{bmatrix} \sin t - t\cos t & 0 \\ 0 & \sin t - t\cos t \end{bmatrix} + \begin{bmatrix} t\cos t & t\cos t \\ 0 & t\cos t \end{bmatrix}$$

$$= \begin{bmatrix} \sin t & t\cos t \\ 0 & \sin t \end{bmatrix}$$

$$A=\begin{bmatrix}1&1\\0&1\end{bmatrix} \qquad \Delta(\lambda)=\begin{bmatrix}\lambda-1&1\\0&\lambda-1\end{bmatrix}=(\lambda-1)^2 \qquad \lambda_1=1 \quad \lambda_2=1$$

$$\cos At = R(A)=a_0 I + a_1 A \qquad \cos\lambda_1 t = a_0 + a_1\lambda_1$$

$$\frac{d\cos\lambda_1 t}{d\lambda_1} = -t\sin\lambda_1 t = a_1$$

$$\begin{cases} a_0 = \cos t + t\sin t \\ a_1 = -t\sin t \end{cases}$$

$$\cos At = \cos t + t\sin t\begin{bmatrix}1&0\\0&1\end{bmatrix} - t\sin t\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}\cos t + t\sin t & 0\\ 0 & \cos t + t\sin t\end{bmatrix} - \begin{bmatrix}t\sin t & t\sin t\\ 0 & t\sin t\end{bmatrix}$$

$$=\begin{bmatrix}\cos t & -t\sin t\\ 0 & \cos t\end{bmatrix}$$

c)

$$\frac{d^2x}{dt^2} + A^2x(t)=0 \qquad x(0)=\begin{bmatrix}1\\0\end{bmatrix} \qquad \frac{dx(0)}{dt}=\begin{bmatrix}0\\1\end{bmatrix}$$

assume $x(t) = \cos At\cdot a + \sin At\cdot b$ where a, b are vectors

from previous problem:

$$\frac{dx(t)}{dt} = -A\sin At\cdot a + A\cos At\, b$$

$$\frac{d^2x(t)}{dt^2} = -AA\cos At\cdot a + A\cdot -A\sin At\cdot b$$

$$-A^2\cos At\cdot a + -A^2\sin At\, b + A^2\cos At\, a + A^2\sin At\cdot b = 0$$

Satisfy the equation.

$$x(0)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$a=\begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{dx(0)}{dt} = -\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}\sin t & t\cos t\\ 0 & \sin t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}\cos t & -t\sin t\\ 0 & \cos t\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$=-\begin{bmatrix}\sin t\\0\end{bmatrix}+\begin{bmatrix}(b_1+b_2)\cos t - b_2 t\sin t\\ b_2\cos t\end{bmatrix} \qquad t=0$$

$$=\begin{bmatrix}b_1+b_2\\b_2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \begin{matrix}b_2=1\\ b_1=-1\end{matrix}$$

Thus $x(t)=\cos At\begin{bmatrix}1\\0\end{bmatrix}+\sin At\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\cos t + \sin t + t\cos t\\ \sin t\end{bmatrix}$

Problem 2

Given that $M \triangleq \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_n \rangle \\ \vdots & & & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix}$

To prove if M is invertible, then v_1,v_2,...,v_n is linearly independent, we can prove by contradiction that if there is one pair of vectors V_i and V_j that are linearly dependent, then the matrix will not be invertible.

If a matrix is invertible, then det(M) $\neq 0$. Assume there is a scalar 'a' from the field that will satisfy this linear relationship $aV_i = V_j$, so that:

$$det(M) = \begin{vmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_i \rangle & \cdots & \langle v_1, v_j \rangle \cdots \\ \\ \langle v_2, v_1 \rangle & \cdots & \langle v_2, v_i \rangle & \cdots & \langle v_2, v_j \rangle \cdots \\ \\ \vdots & & \vdots & & \vdots \\ \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_i \rangle & \cdots & \langle v_n, v_j \rangle \cdots \end{vmatrix}$$

$$= \begin{vmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_i \rangle & \cdots & \langle v_1, av_i \rangle \cdots \\ \langle v_2, v_1 \rangle & \cdots & \langle v_2, v_i \rangle & \cdots & \langle v_2, av_i \rangle \cdots \\ \vdots & & \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_i \rangle & \cdots & \langle v_n, av_i \rangle \cdots \end{vmatrix} = \begin{vmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_i \rangle & \cdots & \overline{a} \langle v_1, v_i \rangle \cdots \\ \langle v_2, v_1 \rangle & \cdots & \langle v_2, v_i \rangle & \cdots & \overline{a} \langle v_2, v_i \rangle \cdots \\ \vdots & & \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_i \rangle & \cdots & \overline{a} \langle v_n, v_i \rangle \cdots \end{vmatrix}$$

$$= \overline{a} \begin{vmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_i \rangle & \cdots & \langle v_1, v_i \rangle \cdots \\ \langle v_2, v_1 \rangle & \cdots & \langle v_2, v_i \rangle & \cdots & \langle v_2, v_i \rangle \cdots \\ \vdots & & \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_n, v_i \rangle & \cdots & \langle v_n, v_i \rangle \cdots \end{vmatrix} = 0$$

With the property of determinant we can see that if there are vectors $v_i, v_j ... \in V$ that are linearly dependent. Matrix M will be singular and non-invertible. Thus this prove if M is invertible, v_1,v_2,...,v_n must be linearly independent.

Now if v_1,v_2,...,v_n are linearly independent, we need to prove the matrix M is invertible. An inveritble matrix will have zero null space, so we can define a vector a from vector 'a' from vector space A such that $ker(M) = \{a \in A : Ma = 0\}$

Thus $ker(M) = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_n \rangle \\ \vdots & & & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$

$$\begin{bmatrix} a_1 \langle v_1, v_1 \rangle + a_2 \langle v_1, v_2 \rangle + \cdots + a_n \langle v_1, v_n \rangle \\ a_1 \langle v_2, v_1 \rangle + a_2 \langle v_2, v_2 \rangle + \cdots + a_n \langle v_2, v_n \rangle \\ \vdots \\ a_1 \langle v_n, v_1 \rangle + a_2 \langle v_n, v_2 \rangle + \cdots + a_n \langle v_n, v_n \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Notice that from the first row:

$a_1 \langle v_1, v_1 \rangle + a_2 \langle v_1, v_2 \rangle + \cdots + a_n \langle v_1, v_n \rangle = \langle v_1, \overline{a_1} v_1 \rangle + \langle v_1, \overline{a_2} v_2 \rangle + \cdots + \langle v_1, \overline{a_n} v_n \rangle$

$= \langle v_1, \overline{a_1} v_1 \rangle + \langle v_1, \overline{a_2} v_2 + \cdots + \overline{a_n} v_n \rangle = 0$
If $\overline{a_1} \neq 0$, then it is a must that $\overline{a_2} v_2 + \cdots + \overline{a_n} v_n = -\overline{a_1} v_1$
we know that v_1,v_2,...,v_n are linearly independent, $\overline{a_2} v_2 + \cdots + \overline{a_n} v_n \neq -\overline{a_1} v_1$
thus $\overline{a_1} = 0$.
Since $\overline{a_1} = 0$, from the second equation, we can see that
$\langle v_2, \overline{a_2} v_2 + \cdots + \overline{a_n} v_n \rangle = 0$
$\langle v_2, \overline{a_2} v_2 + \cdots + \overline{a_n} v_n \rangle = \langle v_2, \overline{a_2} v_2 \rangle + \langle v_2, \overline{a_3} v_3 + \cdots + \overline{a_n} v_n \rangle = 0$
Again, since $\overline{a_2} v_2$ is not a linear combination of $\overline{a_3} v_3 + \cdots + \overline{a_n} v_n \rangle$: then $\overline{a_3} = 0$
All the way to nth row, we can get that $\overline{a_n} = 0$.

Since $\overline{a_i} = 0$, then its complex conjugate $a_i = 0$. Vector $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$

Thus the $ker(M) = 0$, M is singular and thus invertible matrix if v_1,v_2,...,v_n are linearly independent.

Thus if M is invertible,v_1,v_2,...,$v_n$must be linearly independent, and if v_1,v_2,...,v_n are linearly independent, matrix M is invertible.

Problem 3

$$x(k+1) = Ax(k) + Bu(k) \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$y(k) = Cx(k) + Du(k) \qquad x(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [1 \ 2 \ 0] \qquad D = 1$$

$$[B \,\vdots\, AB \,\vdots\, A^2B] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\text{rank } [B \,\vdots\, AB \,\vdots\, A^2B] = 3$$

$[B \,\vdots\, AB \,\vdots\, A^2B]$ reduced row echelon form

$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{matrix} \to \begin{matrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} \to \begin{matrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 1 & 0 \end{matrix} \to \begin{matrix} 1 & 0 & -1 \\ 0 & -2 & -4 \\ 0 & 1 & 0 \end{matrix} \to \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & 0 \end{matrix}$$

$$\to \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 4 \end{matrix} \to \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \to \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

thus rank 3

Fully controllable

$$x(1) = Ax(0) + Bu(0)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(0)$$

$$= \begin{bmatrix} 0 \\ u(0) \\ u(0) \end{bmatrix}$$

$$x(2) = Ax(1) + Bu(1)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u(0) \\ u(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(1)$$

$$= \begin{bmatrix} u(0) \\ u(1) \\ 2u(0) + u(1) \end{bmatrix}$$

$$x(3) = Ax(2) + Bu(2)$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ 2u(0) + u(1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(2)$$

$$= \begin{bmatrix} u(1) \\ -u(0) + u(2) \\ 3u(0) + 2u(1) + u(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} u(1) = 1 \\ u(2) = 1 + u(0) \\ 3u(0) + 1 + 1 + u(2) = 0 \end{cases} \Rightarrow \begin{matrix} u(0) = -\frac{1}{2} \\ u(1) = 1 \\ u(2) = \frac{1}{2} \end{matrix}$$

Problem 4

P4

k	$u(k)$	$y(k)$
0	0	3
1	1	0
2	0	-1
3	1	3

$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $C = [0 \quad 1 \quad 2]$

$CA = [1 \quad 2 \quad 2]$. $CB = 2$

$C = [1 \; 2 \; 0]$ $D = 1$ $C^2A = [0 \quad 3 \quad 2]$

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ C^2A \end{pmatrix} = 3$$

$y(k) = C\,x(k) + D\,u(k)$

$y(0) = C\,x(0) + D\,u(0)$

$y(1) = C\,x(1) + D\,u(1)$ $x(2) = A\,x(1) + B\,u(1)$

$y(1) = C\left(Ax(0) + Bu(0)\right) + Du(1)$ $x(2) = A\left(Ax(0) + Bu(0)\right) + Bu(1)$

$y(2) = C\,x(2) + D\,u(2)$

$y(2) = CA(Ax(0) + Bu(0)) + CBu(1) + Du(2)$

$$\begin{bmatrix} C \\ CA \\ C^2A \end{bmatrix} x(0) = \begin{bmatrix} y(0) - Du(0) \\ y(1) - Du(1) - CBu(0) \\ y(2) - CABu(0) - CBu(1) - Du(2) \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$$