System Analysis Techniques HW7

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November 2020

Problem 1

Choose $\dot{y}_n = x_n$, $\ddot{y}_n = \dot{x}_n$, $y_n = z_n$, and $x_n = \dot{z}_n$, so that X is the combination of $x_1, x_2, ..., x_n, z_1, z_2, ..., z_n$. Plot with the time from 0 to 1 second:

```
close all;clear all;
N = 100; %random N, now assume it is 100
%create A matrix
First = eye(N)*-2;
First(2:N,1:N-1) = First(2:N,1:N-1)+eye(N-1);
First(1:N-1,2:N) = First(1:N-1,2:N)+eye(N-1);
First = First/(25*0.1/(40^2*10));
A = zeros(N*2);
A(N+1:2*N,1:N) = eye(N);
A(1:N,N+1:2*N) = First;
%find the eigen values of A and sort them
[V, ev] = eig(A,'vector');
[ev, ind] = sort(ev);
V = V(:,ind);
x = zeros(2*N);
for i=1:N
    x(:,2*i-1) = (V(:,2*i-1)+V(:,2*i))/2;
    x(:,2*i) = (V(:,2*i-1)-V(:,2*i))/(2*1i);
end
%get A_hat and C_hat matrix
A_hat = inv(x)*A*x;
C = zeros(N, 2*N);
C(1:N,N+1:2*N) = eye(N);
C_hat = C*x;


%step 3 with q1(0) = 1, q2(0) = 1
q0 = zeros(1,2*N);
q0(1) = 1; q0(2) = 1;
y = [];
```

```
n = 1;
for t=0:0.01:1
    y(:,n) = C_hat*expm(A_hat*t)*q0';
    n = n+1;
end
figure(1)
surf(0:0.01:1,1:N,y);xlabel('Time (t)');ylabel('index k');

%step 4 with q3(0) = 1, q4(0) = 1
q0 = zeros(1,2*N);
q0(3) = 1; q0(4) = 1;
y = [];
n = 1;
for t=0:0.01:1
    y(:,n) = C_hat*expm(A_hat*t)*q0';
    n = n+1;
end
figure(2)
surf(0:0.01:1,1:N,y);xlabel('Time (t)');ylabel('index k');

%step 5 with q5(0) = 1, q6(0) = 1
q0 = zeros(1,2*N);
q0(5) = 1; q0(6) = 1;
y = [];
n = 1;
for t=0:0.01:1
    y(:,n) = C_hat*expm(A_hat*t)*q0';
    n = n+1;
end
figure(3)
surf(0:0.01:1,1:N,y);xlabel('Time (t)');ylabel('index k');
```

N can be randomly chosen, plot for N = 100, and time from zero to one second:

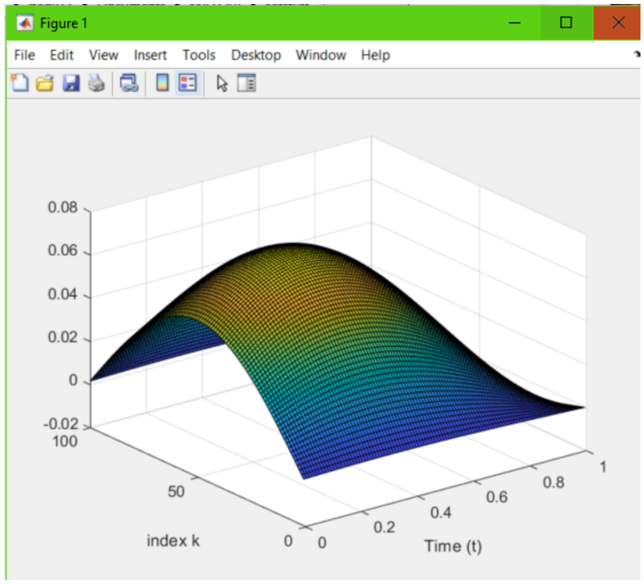

Figure 1: step 3 with initial condition $q_1(0) = 1, q_2(0) = 1$

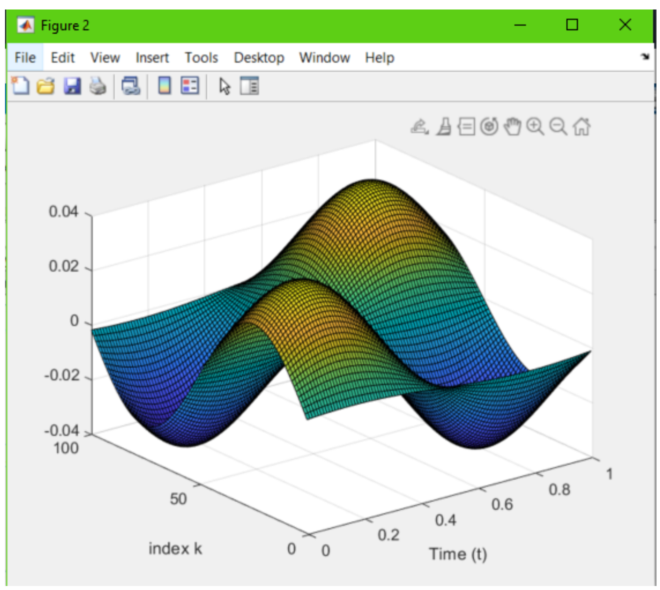

Figure 2: step 4 with initial condition $q_3(0) = 1, q_4(0) = 1$

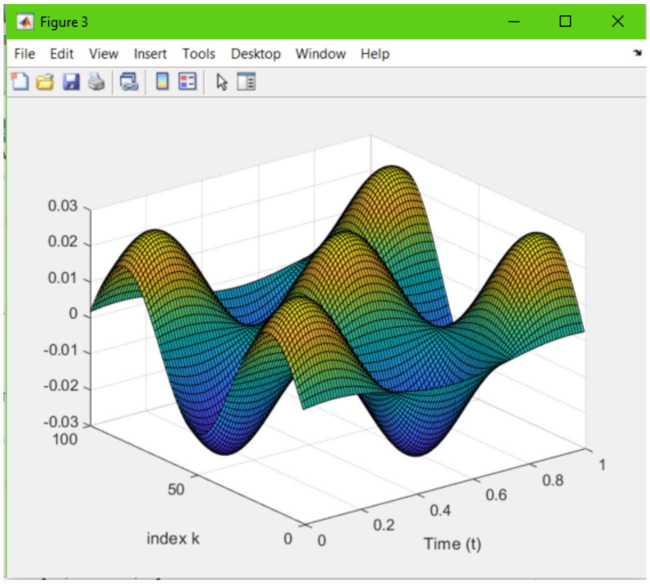

Figure 3: step 5 with initial condition $q_5(0) = 1, q_6(0) = 1$

Problem 2

Given $x(k+2) = x(k) + x(k+1)$, take the z transform:
$z^2(x(z) - x(0) - z^{-1}x(1)) = x(z) + z(x(z) - x(0))$ with $x(0) = 0, x(1) = 1$
thus the z transform of the equation becomes: $z^2x(z) - z = x(z) + zx(z)$
Rearrange: $x(z) = \frac{z}{z^2-z-1} = \frac{z}{(z-\frac{1}{2}-\frac{\sqrt{5}}{2})(z-\frac{1}{2}+\frac{\sqrt{5}}{2})}$
by PDF: $\frac{x(z)}{z} = \frac{A}{z-\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{B}{z-\frac{1}{2}+\frac{\sqrt{5}}{2}}$ such that $A = \frac{\sqrt{5}}{5}$ and $B = -\frac{\sqrt{5}}{5}$
$x(z) = \frac{Az}{z-\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{Bz}{z-\frac{1}{2}+\frac{\sqrt{5}}{2}}$
Apply inverse of z transform: $x(t) = \frac{\sqrt{5}}{5}(\frac{1}{2} + \frac{\sqrt{5}}{2})^n - \frac{\sqrt{5}}{5}(\frac{1}{2} - \frac{\sqrt{5}}{2})^n$
Thus $c_1 = \frac{\sqrt{5}}{5}, c_2 = -\frac{\sqrt{5}}{5}, p = \frac{1}{2} + \frac{\sqrt{5}}{2}, q = \frac{1}{2} - \frac{\sqrt{5}}{2}$

Problem 3

Given that

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 1 \\ -2 & \lambda + 2 \end{bmatrix}$$

$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda + 2 \end{vmatrix} = (\lambda-1)(\lambda+2) + 2 = 0$
$\lambda^2 + \lambda = 0$
we have $\lambda_1 = -1$ $\lambda_2 = 0$
For A-BK:
$|\lambda I - (A - BK)| = \begin{vmatrix} \lambda - 1 + k_1, & 1 + k_2 \\ -2 + k_1 & 2 + k_2 + \lambda\lambda \end{vmatrix}$
$= (\lambda - 1 + k_1)(\lambda + 2 + k_2) - (k_2 + 1)(k_1 - 2)$
$= \lambda^2 + (2 + k_1 + k_2 - 1)\lambda + k_1 + k_2$
$\lambda^2 + (1 + k_1 + k_2)\lambda + k_1 + k_2 = 0$
With $\lambda = 1$, the above equation holds. Thus A-BK will always have an eigenvalue at -1

b)

$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
Rank(M) = 1, not controllable
Choose $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
$\bar{A} = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$
$T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$
$\bar{B} = T^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
The controllable: $\dot{z}_1 = u$ with $A_c = 0 \quad B_c = 1$
The uncontrollable: $\dot{z}_2 = -z_2$ with $A_{\bar{c}} = -1 \quad B_{\bar{c}} = 0$
$det(\lambda I - (A - BK)) = det(\lambda I - (A - BK)_c)\, det(\lambda I - (A - BK)_{\bar{c}})$
$= (\lambda - 0 + k_1)(\lambda + 1 + 0)$
$= (\lambda + k_1)(\lambda + 1)$
There is always an -1 eigenvalue for A-BK

Problem 4

$m\ddot{y} = f - ky - b\dot{y}$
$r = 0 \Rightarrow e(t) = -y$
$\ddot{e} = -\ddot{y}$
$\ddot{e} = \frac{f}{m} - \frac{k}{m}e - \frac{b}{m}\dot{e}$
thus $\begin{bmatrix} \dot{e} \\ \dot{\frac{de}{dt}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-k}{m} & \frac{-b}{m} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \frac{de}{dt} \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} f$
$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-k}{m} & \frac{-b}{m} & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}$; $y = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} e \\ \frac{de}{dt} \\ w \end{bmatrix}$

b)

$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} & \frac{-b}{m^2} \\ \frac{1}{m} & \frac{-b}{m^2} & \frac{b^2}{m^3} - \frac{k}{m^2} \\ 0 & 0 & \frac{1}{m} \end{bmatrix}$

With $rank(M) = rank(\begin{bmatrix} 0 & \frac{1}{m} & \frac{-b}{m^2} \\ \frac{1}{m} & \frac{-b}{m^2} & \frac{b^2}{m^3} - \frac{k}{m^2} \\ 0 & 0 & \frac{1}{m} \end{bmatrix}) = 3$

The rank of M is full rank 3. Thus it is controllable

By pole placement theorem, we can arbitrarily place the poles with appropriate gain k_1, k_2, k_3.

c)

$$\begin{aligned} m &= 1 \\ k &= 2 \\ b &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The desired closed loop at $-1+j, -1-j$, and -2

thus $f_k(\lambda^*) = (\lambda - (-1+j))(\lambda - (-1-j))(\lambda - (-2))$

$= (\lambda + 1 - j)(\lambda + 1 + j)(\lambda + 2)$

$= \lambda^3 + 4\lambda^2 + 6\lambda + 4$

Design $k = \begin{bmatrix} k_1, k_2, k_3 \end{bmatrix}$ such that $|\lambda I - (A - BK)| = 0$

$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 2 + k_1 & \lambda + k_2 + 3 & k_3 \\ -1 & 0 & \lambda \end{bmatrix}$

with $\left| \begin{bmatrix} \lambda & -1 & 0 \\ 2 + k_1 & \lambda + k_2 + 3 & k_3 \\ -1 & 0 & \lambda \end{bmatrix} \right| = 0$

$\lambda^3 + 3\lambda^2 + k_2\lambda^2 + k_1\lambda + k_3 + 2\lambda = 0$

$\lambda^2 + (3 + k_2)\lambda^2 + (k_1 + 2)\lambda + k_3 = 0$

$3 + k_2 = 4$

$k_1 + 2 = 6$

$k_3 = 4$

Solve the above equation: $k_1 = 4$, $k_2 = 1$, and $k_3 = 4$