P1

From HW4 :

$$m_1 \ddot{y} = -k_1 y + k_2 (z-y) - c_1 \dot{y}$$
$$m_2 \ddot{z} = k_2 (y - z)$$

choose $\dot{y} = x_1$
$\dot{z} = x_2$
$y = x_3$
$z = x_4$

thus equation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{c_1}{m_1} & 0 & -\frac{(k_1+k_2)}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & \frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

total energy $= \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{z}^2 + \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 (z-y)^2$

$$= \frac{1}{2} m_1 x_1^2 + \frac{1}{2} m_2 x_2^2 + \frac{1}{2} k_1 x_3^2 + \frac{1}{2} k_2 x_4^2 + \frac{1}{2} k_2 x_3^2 - k_2 x_3 x_4$$

$$= [x_1 \ x_2 \ x_3 \ x_4] \underbrace{\begin{bmatrix} \frac{m_1}{2} & 0 & 0 & 0 \\ 0 & \frac{m_2}{2} & 0 & 0 \\ 0 & 0 & \frac{k_1+k_2}{2} & -\frac{k_2}{2} \\ 0 & 0 & -\frac{k_2}{2} & \frac{k_2}{2} \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

P is symmetric, its eigen values are $\frac{m_1}{2}$, $\frac{m_2}{2}$, $\frac{k_1}{4} + \frac{k_2}{2} - \frac{\sqrt{k_1^2 + 4k_2^2}}{4} = \sqrt{(\frac{k}{4}+\frac{k_2}{2})^2} - \sqrt{(\frac{k_1}{4})^2 + (\frac{k_2}{2})^2} > 0$ and $\frac{k_1}{4} + \frac{k_2}{2} + \frac{\sqrt{k_1^2 + 4k_2^2}}{4}$, all of its eigen value are positive. thus P is postive definite

Thus $A^T P + PA = Q$

$$Q = \begin{bmatrix} -c_1 & 0 & \frac{k_1+k_2}{2} - \frac{m_1(k_1+k_2)}{2m_2} & \frac{k_2 m_1}{2m_2} - \frac{k_2}{2} \\ 0 & 0 & 0 & 0 \\ \frac{k_1+k_2}{2} - \frac{m_1(k_1+k_2)}{2m_2} & 0 & 0 & 0 \\ \frac{k_2 m_1}{2m_2} - \frac{k_2}{2} & 0 & 0 & 0 \end{bmatrix}$$

symmetric

The determant of $-Q$

$\det(-Q(1,1)) = c_1$

and the rest are zero

thus Q is negative semidefinite. Since it is 4×4 system

From Matlab $\operatorname{rank}\left(\begin{bmatrix} Q \\ QA \\ QA^2 \\ QA^3 \end{bmatrix}\right) = 4$

(A, Q) is observable, thus the system is asymptotically stable (Invariance principle)

P2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad -1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$[B \quad AB \quad A^2B] = \begin{bmatrix} 2 & -3 & 5 \\ 2 & -3 & 5 \\ 1 & -1 & 1 \end{bmatrix}$. with reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

thus $\mathrm{im}([B \quad AB \quad A^2B]) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix} \right\}$ which is controllable subspace.

$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \\ 9 & -9 & 0 \end{bmatrix}$ $\ker\left(\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\right) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} \right\}$ unobservable subspace

$$W_c = \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} d\tau \qquad W_o = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$$

$$A W_c + W_c A^T = -BB^T \qquad A^T W_o + W_o A = -C^T C$$

$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} W_c + W_c \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix}$

$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}^T W_o + W_o \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

from Matlab.

$W_c = \begin{bmatrix} 1.4167 & 1.4167 & 0.8333 \\ 1.4167 & 1.4167 & 0.8333 \\ 0.8333 & 0.8333 & 0.5 \end{bmatrix}$

$\mathrm{rank}(W_c) = 2$

from Matlas

$W_o = \begin{bmatrix} 0.1667 & -0.1667 & 0 \\ -0.1667 & 0.1667 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\mathrm{rank}(W_o) = 1$

both W_c and W_o are Singular

$\mathrm{im}(W_c) = \mathrm{Span}\left\{ \begin{bmatrix} 1.4167 \\ 1.4167 \\ 0.8333 \end{bmatrix}, \begin{bmatrix} 0.8333 \\ 0.8333 \\ 0.5 \end{bmatrix} \right\}$

which span the same space as the span of $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$ .. which is the controllable subspace from above

$\mathrm{kernel}(W_o) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ a \end{bmatrix} \right\}$

which is same as the unObservable subspace from above.

P3

Step 1: $u_{flip} = u(T-t)$

$$\int_0^T u_{flip}^2(t)\,dt = \int_0^T u^2(T-t)\,dt$$

t from $0 \to T$
τ from $T \to 0$

$\tau = T-t$
$d\tau = -dt$

$$\Rightarrow \int_0^T u^2(T-t)dt = \int_T^0 u^2(\tau)\ -d\tau = \int_0^T u^2(\tau)d\tau$$

thus $\int_0^T u_{flip}^2(t)\,dt = \int_0^T u^2(t)dt$

Step 2: show $x(T) = \int_0^T e^{At} B u_{flip}(t)\,dt$

since $\dot{x} = Ax + Bu$ with $x(0) = 0$

zero initial state solution: $x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

$t = T-\tau$ $\quad dt = -d\tau$

τ from $0 \to T$
t from $T \to 0$

$$x(T) = \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

thus

$$x(T) = \int_T^0 e^{At} B u(T-t)\cdot -dt$$

thus $x(T) = \int_0^T e^{At} B u(T-t) dt$ is proven

Step 3: $J = \int_0^T u^2(t) dt$ $\qquad x(T) = x_f$ $\qquad e^{At}B = V = \begin{bmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_n(t) \end{bmatrix}$

$$x_f = \int_0^T e^{At}B\, u_{flip}(t)dt = \int_0^T \begin{bmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_n(t) \end{bmatrix} u_{flip}(t)\,dt$$

$$= \begin{bmatrix} \int_0^T V_1(t)\, u_{flip}(t)\,dt \\ \int_0^T V_2(t)\, u_{flip}(t)\,dt \\ \vdots \\ \int_0^T V_n(t)\, u_{flip}(t)\,dt \end{bmatrix} = \begin{bmatrix} x_f(1) \\ x_f(2) \\ \vdots \\ x_f(n) \end{bmatrix}$$

thus $\langle V_1, u_{flip} \rangle = x_{f_1}$ ...
$\langle V_n, u_{flip} \rangle = x_{f_n}$

define $\langle u, v \rangle := \int_0^T u(\tau)v(\tau)d\tau$

thus $J = \langle u, u \rangle = \|u\|^2$

$u = u_{\parallel} + u_{\perp}$ where $u_{\parallel} \in \text{span}\{V_1(t), V_2(t) \dots V_n(t)\}$
$u_{\perp}$ is orthogonal to $V_1(t), V_2(t) \dots V_n(t)$

$$\langle u, u \rangle = \langle u_{\parallel}, u_{\parallel} \rangle + 2\langle u_{\parallel}, u_{\perp} \rangle + \langle u_{\perp}, u_{\perp} \rangle$$
$$= \|u_{\parallel}\|^2 + \|u_{\perp}\|^2$$

to minize u, $u_{\perp} = 0$ $\qquad$ from step 1 $\int_0^T u_{flip}^2(t)\,dt = \int_0^T u^2(t)dt$

thus to minize u_{flip}, $u_{flip\perp} = 0$ $\quad u_{flip} \in \text{span}\{V_1(t), V_2(t), \dots V_n(t)\}$

Step 4. $W_c(T) = \int_0^T e^{A\tau} B B^T e^{A^T\tau} d\tau$ $\qquad$ since $x_f = \int_0^T e^{At}B\, u_{flip}(t)dt$ and $u_{flip} \in \{V_1(t), V_2(t) \dots V_n(t)\}$
$= (e^{At}B)^T$

$$x_f = \int_0^T e^{A\tau} B B^T e^{A^T\tau} d\tau\, W_c^{-1}(T) x_f \Rightarrow u_{flip}(t) = B^T e^{A^T\tau} W_c^{-1}(T) x_f$$

$$J = \int_0^T u_{flip}^T u_{flip}\, d\tau = \int_0^T x_f^T (W_c^{-1}(T))^T e^{A\tau} B B^T e^{A^T\tau} W_c^{-1}(T) x_f\, d\tau$$

since
$W_c(T) = W_c^T(T)$

$$= x_f^T W_c^{-1}(T) \int_0^T e^{A\tau} B B^T e^{A^T\tau} d\tau\, W_c^{-1}(T) x_f$$
$$= x_f^T W_c^{-1}(T) x_f$$

Thus proven

P4

	k=0	1	2	3	4	5	6	7	8	9
u(k)	1	0	1	0.5	1	-1	0	1	0	-1
y(k)	0	1	3	2	4	3	5	6	6	5

n=1 first order $y(k) = -a_1 y(k-1) + b_0 u(k) + b_1 u(k-1)$

n=2 second order $y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$

n=1

$$\begin{aligned}
y(1) &= -a_1 y(0) + b_0 u(1) + b_1 u(0)\\
y(2) &= -a_1 y(1) + b_0 u(2) + b_1 u(1)\\
y(3) &= -a_1 y(2) + b_0 u(3) + b_1 u(2)\\
y(4) &= -a_1 y(3) + b_0 u(4) + b_1 u(3)\\
y(5) &= -a_1 y(4) + b_0 u(5) + b_1 u(4)\\
y(6) &= -a_1 y(5) + b_0 u(6) + b_1 u(5)\\
y(7) &= -a_1 y(6) + b_0 u(7) + b_1 u(6)\\
y(8) &= -a_1 y(7) + b_0 u(8) + b_1 u(7)\\
y(9) &= -a_1 y(8) + b_0 u(9) + b_1 u(8)
\end{aligned}$$

$$\underbrace{\begin{bmatrix} 1\\3\\2\\4\\3\\5\\6\\6\\5 \end{bmatrix}}_{Z \; 9\times 1} = \underbrace{\begin{bmatrix} -0 & 0 & 1\\ -1 & 1 & 0\\ -3 & 0.5 & 1\\ -2 & 1 & 0.5\\ -4 & -1 & 1\\ -3 & 0 & -1\\ -5 & 1 & 0\\ -6 & 0 & 1\\ -6 & -1 & 0 \end{bmatrix}}_{H \; 9\times 3} \underbrace{\begin{bmatrix} a_1\\ b_0\\ b_1 \end{bmatrix}}_{X \; 3\times 1}$$

from Matlab $\hat{X} = (H^T H)^{-1} H^T Z$

$$\hat{X} = \begin{bmatrix} -1.0780\\ 1.2455\\ -0.5449 \end{bmatrix}$$

$y(k) = 1.0780 y(k-1) + 1.2455 u(k) - 0.5444 u(k-1)$

n=2

$$\begin{aligned}
y(2) &= -a_1 y(1) - a_2 y(0) + b_0 u(2) + b_1 u(1) + b_2 u(0)\\
y(3) &= -a_1 y(2) - a_2 y(1) + b_0 u(3) + b_1 u(2) + b_2 u(1)\\
y(4) &= -a_1 y(3) - a_2 y(2) + b_0 u(4) + b_1 u(3) + b_2 u(2)\\
y(5) &= -a_1 y(4) - a_2 y(3) + b_0 u(5) + b_1 u(4) + b_2 u(3)\\
y(6) &= -a_1 y(5) - a_2 y(4) + b_0 u(6) + b_1 u(5) + b_2 u(4)\\
y(7) &= -a_1 y(6) - a_2 y(5) + b_0 u(7) + b_1 u(6) + b_2 u(5)\\
y(8) &= -a_1 y(7) - a_2 y(6) + b_0 u(8) + b_1 u(7) + b_2 u(6)\\
y(9) &= -a_1 y(8) - a_2 y(7) + b_0 u(9) + b_1 u(8) + b_2 u(7)
\end{aligned}$$

$$\underbrace{\begin{bmatrix} 3\\2\\4\\3\\5\\6\\6\\5 \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 & 1\\ -3 & -1 & 0.5 & 1 & 0\\ -2 & -3 & 1 & 0.5 & 1\\ -4 & -2 & -1 & 1 & 0.5\\ -3 & -4 & 0 & -1 & 1\\ -5 & -3 & 1 & 0 & -1\\ -6 & -5 & 0 & 1 & 0\\ -6 & -6 & -1 & 0 & 1 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} a_1\\ a_2\\ b_0\\ b_1\\ b_2 \end{bmatrix}}_{X}$$

$\hat{X} = (H^T H)^{-1} H^T Z$

from Matlab

$$\hat{X} = \begin{bmatrix} -0.9485\\ -0.1305\\ 1.3284\\ -0.7414\\ 0.6081 \end{bmatrix}$$

$y(k) = 0.9485 y(k-1) + 0.1305 y(k-2) + 1.3284 u(k)$
$\quad - 0.7414 u(k-1) + 0.6081 u(k-2)$