

System Analysis Techniques HW8

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Problem 1

```
close all;clear all;
load('PCAData.mat');
[U,S,V] = svd(X);
%step 3
figure(1);
for i = 1:200
    plot3(X(1,i),X(2,i),X(3,i),'o')
    hold on
end
xlim([-2 2]);ylim([-2 2]);zlim([-2 2])
hold on;
%principal directions
plot3([0,U(1,1)],[0,U(2,1)],[0,U(3,1)],'r');hold on;
plot3([0,U(1,2)],[0,U(2,2)],[0,U(3,2)],'b');hold on;
plot3([0,U(1,3)],[0,U(2,3)],[0,U(3,3)],'g');hold on;
xlim([-2 2]);ylim([-2 2]);zlim([-2 2]);grid on;
S(3,3) = 0;
Snew = S;
X1 = U*S*transpose(V);

%step 5
figure(2);
for i = 1:200
    plot3(X1(1,i),X1(2,i),X1(3,i),'o')
    hold on
end
hold on;
plot3([0,U(1,1)],[0,U(2,1)],[0,U(3,1)],'r');hold on;
plot3([0,U(1,2)],[0,U(2,2)],[0,U(3,2)],'b');hold on;
plot3([0,U(1,3)],[0,U(2,3)],[0,U(3,3)],'g');hold on;
xlim([-2 2]);ylim([-2 2]);zlim([-2 2]);grid on;
```

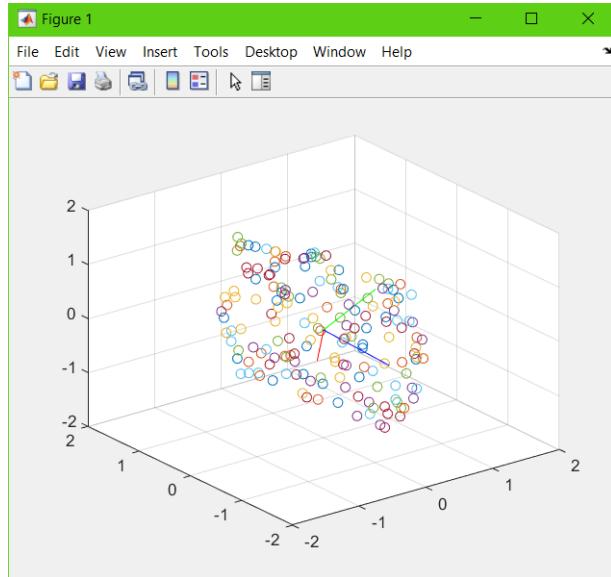


Figure 1: step 3

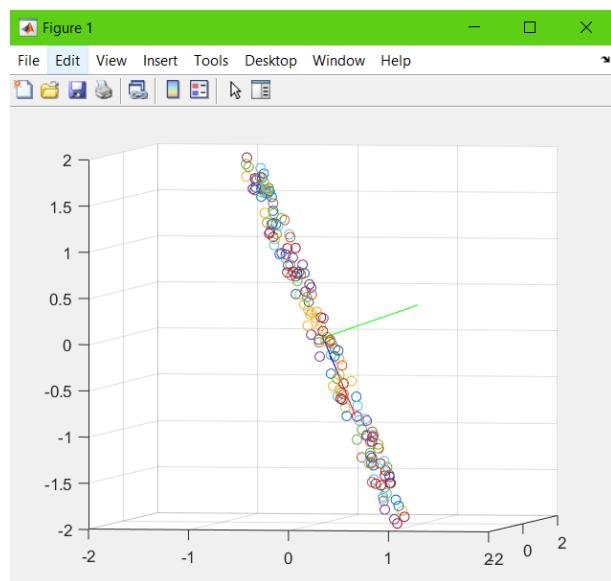


Figure 2: step 3 at a different angle

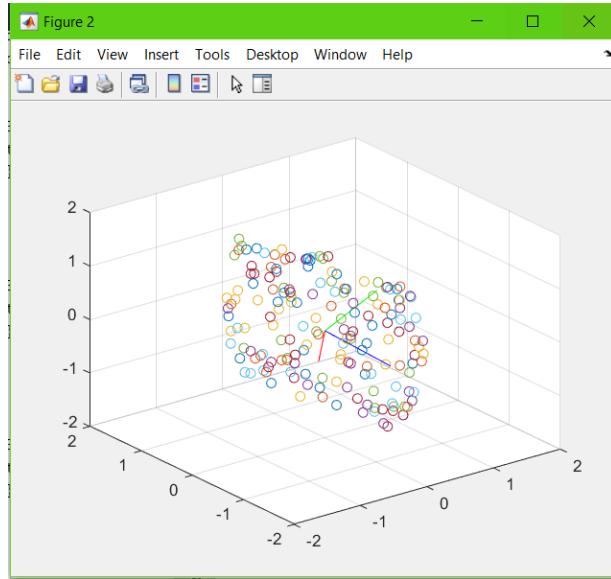


Figure 3: step 5

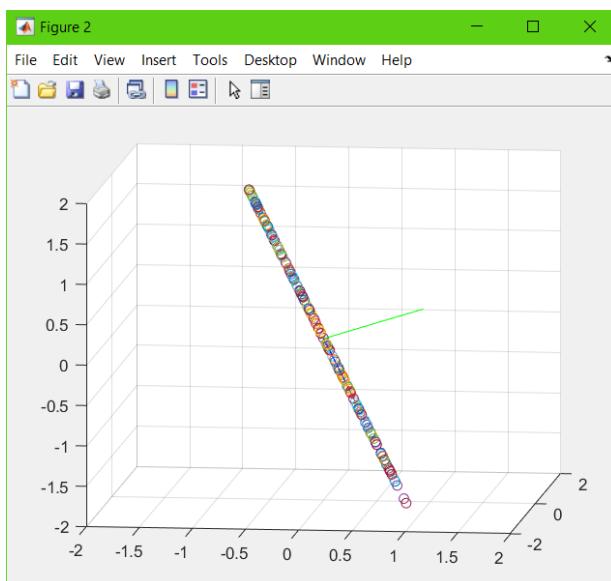


Figure 4: step 5 at a different angle

Problem 2

P2

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Controllable check:

$$[B \quad AB \quad A^T B] = \begin{bmatrix} 2 & 0 & 2 \\ -1 & -1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{rank } 2 < 3$$

not controllable

Observable check:

$$\begin{bmatrix} C \\ CA \\ CA^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 4 & 4 & -3 \end{bmatrix} \quad \text{rank } 2 < 3$$

not observable

Stabilizable check:

$$M = [B \quad AB \quad A^T B] = \begin{bmatrix} 2 & 0 & 2 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{choose } R = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\tilde{A} = R^T A R = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -2 \\ -3 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \tilde{B} = R^T B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u \quad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \quad \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

$$\tilde{A}_{11} = \begin{bmatrix} 0 \\ 1 & 0 \end{bmatrix} \quad \tilde{A}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \tilde{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \tilde{A}_{22} = -2 \quad \tilde{A}_{12} = -2 \quad \lambda_1 = 2 \quad \lambda_2 = -2$$

thus Stabilizable it's Hurwitz matrix

Detectable check:

$$\text{kernel } \begin{bmatrix} C \\ CA \\ CA^T \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{choose } T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\tilde{A} = T^{-1} AT = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -2 \\ -3 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \tilde{C} = CT = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\tilde{B} = T^{-1} B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \tilde{B} u \quad \tilde{A}_{11} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad \tilde{A}_{12} = \begin{bmatrix} 3 & -2 \end{bmatrix}$$

$$Y = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \tilde{A}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \tilde{A}_{22} = \begin{bmatrix} 1 \end{bmatrix} \quad \text{not Hurwitz}$$

$\lambda_1 = 0 \quad \lambda_2 = 1$ not detectable

Figure 5: step 5 at a different angle

Problem 3

P3

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -2x_1 - 3x_2$$

Pd qf: $V(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt}$$

$$\frac{dV(x_1, x_2)}{dt} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2x_1 x_2 + 2x_2 (-2x_1 - 3x_2)$$

$$= 2x_1 x_2 - 4x_1 x_2 - 6x_2^2$$

$$= -2x_1 x_2 - 6x_2^2$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda(\lambda+b)-1 = \lambda^2 + b\lambda - 1 = 0$$

$$\frac{-6 \pm \sqrt{36+4}}{2} = -3 \pm \sqrt{10}$$

not positive definite

not semi positive definite

$$\begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix} < 0$$

not negative definite

not semi negative definite

$\frac{dV(x)}{dt}$ is not positive or negative definite.

We don't know if the system is stable or not based on what we have now.

b) $V(x_1, x_2)$ use Lyapunov Stability theorem

$$\text{define } A^T P + PA = Q$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \begin{aligned} \lambda(a+3)+2 &= 0 \\ \lambda &= -1, \lambda = -2 \end{aligned}$$

choose Q to be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2P_{11} & -2P_{12} \\ P_{11}-3P_{12} & P_{11}-3P_{12} \end{bmatrix} + \begin{bmatrix} -2P_{21} & P_{11}-3P_{12} \\ -2P_{21} & P_{11}-3P_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} -4P_{11} = -1 \\ -2P_{12} + P_{11} - 3P_{12} = 0 \\ P_{11} - 3P_{12} - 2P_{21} = 0 \\ 2P_{21} - 6P_{12} = 1 \end{cases} \Rightarrow \begin{aligned} P_{11} &= \frac{1}{4} \\ P_{12} &= \frac{1}{4} \\ P_{21} &= \frac{5}{4} \end{aligned}$$

Positive definite

Thus the system is stable

$$V(x_1, x_2) = [x_1 \ x_2] \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{5}{4}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{4}x_2^2$$

Figure 6: step 5 at a different angle

Problem 4

$\dot{x} = Ax, x \in R^n$ and Q is n-by-n symmetric negative semidefinite matrix. Given statement 1:

rank($\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix}$) = n, which means full column rank. Thus the kernel of the

observability matrix: $\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x = 0$ if and only if $x = 0$.

$$\text{Notice } \begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x = \begin{bmatrix} Qx \\ QAx \\ \vdots \\ QA^{n-1}x \end{bmatrix} = \begin{bmatrix} Qe^{At}x_0 \\ QAe^{At}x_0 \\ \vdots \\ QA^{n-1}e^{At}x_0 \end{bmatrix}$$

$$\text{At } t = 0, \text{ the above equation becomes } \begin{bmatrix} Qx_0 \\ QAx_0 \\ \vdots \\ QA^{n-1}x_0 \end{bmatrix} = \begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix} x_0 = 0$$

Thus $x_0 = 0, x = e^{At}x_0 = 0 \quad \forall t$, therefore statement 2 is proven

Now given the statement 2: if $Qx(t) = 0 \quad \forall t$, then $x(t) = 0 \quad \forall t$

Q is a n by n square matrix.

If $Qx(t) = 0$ if and only if $x(t) = 0$, then Q is full rank.

$\begin{bmatrix} Q \\ QA \\ \vdots \\ QA^{n-1} \end{bmatrix}$ is the augmented matrix of Q with the maximum rank of n and

with the rank larger or equal to the rank of Q. Since the rank of Q is n, then the rank of the observability matrix is also n. Thus statement 1 is proven.