

Robotics 1 HW2

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1 a)

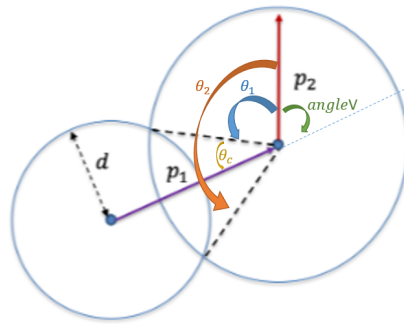


Figure 1: θ_1 in blue and θ_2 in orange

1 b)

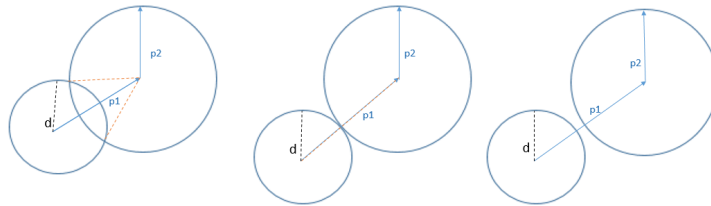


Figure 2: Two solutions, one solution and zero solution

1 c)

P1, P2 and d are given. The norm of the two vectors thus are known; Calculate the angle between P1 and P2, denoting as angleV in part 1a) with arctangent function

$$\text{angleV} = 2 \operatorname{atan} \left(\frac{\|P_1 - P_2\|}{\|P_1 + P_2\|} \right) \quad (1)$$

which is more robust way of finding the angle between two vectors than using the arccosine function. θ_c , the angle between P2 after rotation and P1, can be calculated using the law of cosine

$$\cos \theta_c = \frac{\|P_1\|^2 + \|P_2\|^2 - d^2}{2 \|P_1\| \cdot \|P_2\|}$$

where d and norm of two vectors are known. As indicated in figure 1, there are two possible solutions θ_1 and θ_2

$$\theta_1 = \pi - \text{angleV} - \theta_c \quad (2)$$

$$\theta_2 = \pi - \text{angleV} + \theta_c \quad (3)$$

convert the angles in 0 to 2π range when necessary

1 d)

Implementation of the algorithm and verification:

The algorithm is coded in a function called find_thetas as shown in the box. The calculation of the angle between P1 and P2 using above algorithm requiring an axis of rotation, which will be ez for the planar case, and arctangent function provided by Professor John Wen in subproblem0.m file.

For the verification, the code randomly generates two vectors and a rotation angle between 0 to 2π with variable 'q', and the addition of vectors creates a vector P3, whose norm is denoted as variable 'd'. Passing P1 and P2 vectors and 'd' as the variable to function to output the rotation angle. There are two outputs. If one of them is matching with the 'q', then the algorithm works.

Listing 1: matlab code

```
p1 = randn(2,1); p2 = randn(2,1); q = (rand)*2*pi;
Rotm = [cos(q), -sin(q); sin(q), cos(q)]; d = norm(p1+Rotm*
p2);
disp(q*180/pi); thetas = find_thetas(p1,p2,d); disp(
thetas)
function thetas = find_thetas(p1, p2, d)
    %to excute, make sure to include subproblem0.m in
    path
    %input: two 2-D vectors; output: two possible
    thetas
```

```

p1 = [p1;0]; p2 = [p2;0];
p1norm = norm(p1); p2norm = norm(p2);
ez = [0;0;1];
angleV = subproblem0(p1, p2, ez);%angle between p1
    p2 vectors
thetac = acos((p1norm^2+p2norm^2-d^2)/(2*p1norm*
    p2norm));
theta1 = (pi-angleV-thetac)*180/pi;
theta2 = (pi-angleV+thetac)*180/pi;
if theta2 > 360
    theta2 = theta2 - 360
end
if theta2 < 0
    theta2 = theta2 + 360
end
if theta1 > 360
    theta1 = theta1 - 360
end
if theta1 < 0
    theta1 = theta1 + 360
end
thetas = [theta1, theta2];
end

```

Result (angles expressed in degree): as shown, one of the output angles will always precisely matches to the input angle, with four decimals. The result is still precise as the input is close to 360 degree (0 degree). The algorithm is very robust.

hw2problem1	hw2problem1	hw2problem1
56.3058	154.1711	359.6689
168.7155	56.3058	154.1711
93.7219	267.5808	359.6689

Figure 3: Input and outputs (in degree)

1 e)

This algorithm can be used to find the inverse kinematics for the two linked planar arm. It is known

$$\begin{aligned}
 P_{0T} &= P_{01} + R_{01}P_{12} + R_{02}P_{2T} \\
 P_{1T} &= R_{01}(P_{12} + R_{12}P_{2T}) \\
 \|P_{1T}\| &= \|R_{01}(P_{12} + R_{12}P_{2T})\|
 \end{aligned} \tag{4}$$

$$\|P_{1T}\| = \|P_{12} + R_{12}P_{2T}\|$$

The last equation above is what has already been solved in previous parts. The norm of P_{1T} is known, the length of the arms l_1 and l_2 are given so that P_{12} and P_{2T} are known. The algorithm can be applied to solve θ_1 , θ_2 , and θ_c , and then angleV or q2 can be calculated from equation (2) and (3). Since there are two outputs, there are two q2 as well.

Use the output q2 to find $P_{12} + R_{12}P_{2T}$, which is inside the parenthesis of equation (4). P_{1T} is also known, and thus what the unknown in (4) is just the R_{01} . R_{01} is a rotation matrix rotating the vector from $P_{12} + R_{12}P_{2T}$ to P_{1T} . The angle between these two vectors is q1. This problem thus is to find the angle between two known vectors. The equation (1) in part c) can apply to find the q1. With above algorithm, q1 and q2 can be found out.

Here is a picture to better visualize the meaning of equation (4)

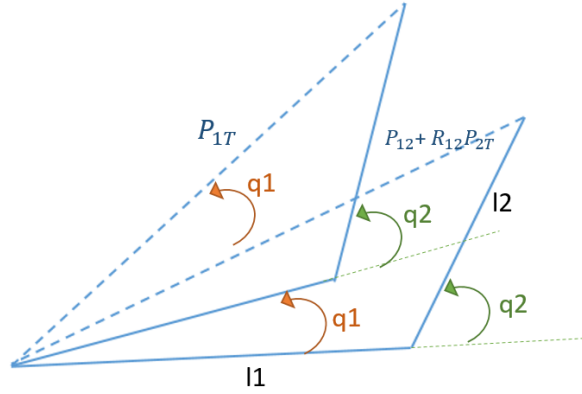


Figure 4: Visualization of equation (4)

The left-hand side of the equation (4) is P_{1T} , shown in the upper left blue dash line. The expression $P_{12} + R_{12}P_{2T}$ inside parenthesis on the right-hand side of the equation is the lower blue dash line. By geometry, it is easy to see that the rotation angle from the lower dash line to the higher one is equal to q1. Thus R_{01} in equation (4) is indeed the a rotation matrix about q1.

Again, two blue dash lines shown in graph is known after finding q2, and equation (1) can be used to find the angle q1 between them.

2

$$w^\times = \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \\ \vec{e}_3 \cdot \end{bmatrix} (\vec{w} \times) \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} \vec{e}_1 \cdot \\ \vec{e}_2 \cdot \\ \vec{e}_3 \cdot \end{bmatrix} \begin{bmatrix} \vec{w} \times \vec{e}_1 & \vec{w} \times \vec{e}_2 & \vec{w} \times \vec{e}_3 \end{bmatrix} \\
&= \begin{bmatrix} \vec{e}_1 \cdot (\vec{w} \times \vec{e}_1) & \vec{e}_1 \cdot (\vec{w} \times \vec{e}_2) & \vec{e}_1 \cdot (\vec{w} \times \vec{e}_3) \\ \vec{e}_2 \cdot (\vec{w} \times \vec{e}_1) & \vec{e}_2 \cdot (\vec{w} \times \vec{e}_2) & \vec{e}_2 \cdot (\vec{w} \times \vec{e}_3) \\ \vec{e}_3 \cdot (\vec{w} \times \vec{e}_1) & \vec{e}_3 \cdot (\vec{w} \times \vec{e}_2) & \vec{e}_3 \cdot (\vec{w} \times \vec{e}_3) \end{bmatrix} \\
&= \begin{bmatrix} \vec{w} \cdot (\vec{e}_1 \times \vec{e}_1) & \vec{w} \cdot (\vec{e}_2 \times \vec{e}_1) & \vec{w} \cdot (\vec{e}_3 \times \vec{e}_1) \\ \vec{w} \cdot (\vec{e}_1 \times \vec{e}_2) & \vec{w} \cdot (\vec{e}_2 \times \vec{e}_2) & \vec{w} \cdot (\vec{e}_3 \times \vec{e}_2) \\ \vec{w} \cdot (\vec{e}_1 \times \vec{e}_3) & \vec{w} \cdot (\vec{e}_2 \times \vec{e}_3) & \vec{w} \cdot (\vec{e}_3 \times \vec{e}_3) \end{bmatrix} \\
&= \begin{bmatrix} 0 & \vec{w} \cdot -\vec{e}_1 & \vec{w} \cdot \vec{e}_2 \\ \vec{w} \cdot \vec{e}_1 & 0 & \vec{w} \cdot -\vec{e}_1 \\ \vec{w} \cdot -\vec{e}_2 & \vec{w} \cdot \vec{e}_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}
\end{aligned}$$

3 a)

$$P_{OT} = \begin{bmatrix} X_T = l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ Y_T = l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

$$\dot{P}_{OT} = \begin{bmatrix} -l_1 \sin(q_1) \dot{q}_1 - l_2 \sin(q_1 + q_2) \dot{q}_1 & -l_2 \sin(q_1 + q_2) \dot{q}_2 \\ l_2 \cos(q_1) \dot{q}_1 + l_2 \cos(q_1 + q_2) \dot{q}_1 & l_2 \cos(q_1 + q_2) \dot{q}_2 \end{bmatrix}$$

$$\dot{P}_{OT} = \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\dot{P}_{OT} = J_T \dot{q}$$

$$J_T = \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

3 b)

$$\dot{R}_{0,1} = \dot{q}_1 S R_{0,1}$$

When i = 1:

$$\dot{q}_1 \begin{bmatrix} -\sin q_1 & -\cos q_1 \\ \cos q_1 & -\sin q_1 \end{bmatrix} = \dot{q}_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{bmatrix}$$

When $i = 2$:

$$\dot{q}_2 \begin{bmatrix} -\sin q_2 & -\cos q_2 \\ \cos q_2 & -\sin q_2 \end{bmatrix} = \dot{q}_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos q_2 & -\sin q_2 \\ \sin q_2 & \cos q_2 \end{bmatrix}$$

$$e_z^\times \begin{bmatrix} p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} Sp \\ 0 \end{bmatrix}$$

3 c)

$$J_T = \begin{bmatrix} S(p_{1T})_0 & S(p_{2T})_0 \end{bmatrix} = \begin{bmatrix} S(R_{01}p_{12} + R_{02}p_{2T}) & SR_{02}p_{2T} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left[\begin{bmatrix} C_1 & -S_1 \\ S_1 & C_1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \end{bmatrix} + \begin{bmatrix} C_{12} & -S_{12} \\ S_{12} & C_{12} \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \right] & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left[\begin{bmatrix} C_{12} & -S_{12} \\ S_{12} & C_{12} \end{bmatrix} \cdot \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \right] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 l_1 + l_2 C_{12} \\ S_1 l_1 + l_2 S_{12} \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} l_2 C_{12} \\ l_2 S_{12} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} -l_1 S_1 - l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} \end{bmatrix} & \begin{bmatrix} -l_2 S_{12} \\ l_2 C_{12} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1) + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix} = J_T$$

Same as in Part 3a)

3 d)

for n-linked arms

$$J_T = e_z^\times [(P_{1T})_0 \quad (P_{2T})_0 \quad (P_{3T})_0 \quad \dots \quad (P_{nT})_0]$$

where

$$(P_{1T})_0 = \begin{bmatrix} l_1 C_1 + l_2 C_{12} + l_3 C_{13} + \dots + l_n C_{12\dots n} \\ l_1 S_1 + l_2 S_{12} + l_3 S_{13} + \dots + l_n S_{12\dots n} \\ 0 \end{bmatrix}$$

$$(P_{2T})_0 = \begin{bmatrix} l_2 C_{12} + l_3 C_{13} + \dots + l_n C_{12\dots n} \\ l_2 S_{12} + l_3 S_{13} + \dots + l_n S_{12\dots n} \\ 0 \end{bmatrix}$$

...

$$(P_{nT})_0 = \begin{bmatrix} l_n C_{12\dots n} \\ l_n S_{12\dots n} \\ 0 \end{bmatrix}$$

$$J_T = [e_z^\times (P_{1T})_0 \quad e_z^\times (P_{2T})_0 \quad e_z^\times (P_{3T})_0 \quad \dots \quad e_z^\times (P_{nT})_0]$$

4 a)

$$R_{31} = \varepsilon_3^* \varepsilon_1 = \varepsilon_3^* \varepsilon_2 \varepsilon_2^* \varepsilon_1 = R_{32} R_{21}$$

$$\varepsilon_i^* \mathcal{R}(\vec{x}_i, \theta) \varepsilon_i = R_x(\theta)$$

$$\varepsilon_0^* \varepsilon_1 = \varepsilon_0^* \mathcal{R}(\vec{x}_0, \theta) \varepsilon_0 = R_{01} = R_x(\theta)$$

$$\varepsilon_1^* \varepsilon_2 = \varepsilon_1^* \mathcal{R}(\vec{z}_1, \phi) \varepsilon_1 = R_{12} = R_z(\phi)$$

$$\varepsilon_2^* \varepsilon_3 = \varepsilon_2^* \mathcal{R}(\vec{y}_2, \psi) \varepsilon_2 = R_{23} = R_y(\psi)$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$$R_y(\psi) = \begin{bmatrix} C\psi & 0 & S\psi \\ 0 & 1 & 0 \\ -S\psi & 0 & C\psi \end{bmatrix}$$

$$R_z(\phi) = \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{31} = R_{32} R_{21} = R_{23}^T R_{12}^T$$

$$\begin{aligned} &= \begin{bmatrix} C\psi & 0 & -S\psi \\ 0 & 1 & 0 \\ S\psi & 0 & C\psi \end{bmatrix} \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C\psi C\phi & S\phi C\psi & -S\psi \\ -S\phi & C\phi & 0 \\ C\phi S\psi & S\psi S\phi & C\psi \end{bmatrix} \end{aligned}$$

4 b)

$$R_{03} = R_{01} R_{12} R_{23} = R_{01} R_{31}^T$$

$$\begin{aligned} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\psi C\phi & -S\phi & C\phi S\psi \\ S\phi C\psi & C\phi & S\psi S\phi \\ -S\psi & 0 & C\psi \end{bmatrix} \\ &= \begin{bmatrix} C\psi C\phi & -S\phi & C\phi S\psi \\ S\phi C\psi C\theta + S\theta S\psi & C\theta C\phi & C\theta S\phi S\psi - S\theta C\psi \\ S\phi S\theta C\psi - S\psi C\theta & C\phi S\theta & S\psi S\phi S\theta - C\theta C\psi \end{bmatrix} \end{aligned}$$

4 c)

$$\varepsilon_1^* \varepsilon_2 = \varepsilon_1^* \mathcal{R}(\vec{z}_1, \phi) \varepsilon_1 = R_{12}$$

$$R_{12}^T \vec{z}_1 = \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5 a)

when i = 1:

$$\begin{aligned} k^\times &= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \\ k^\times k^\times &= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\omega_3^2 - \omega_2^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_3^2 - \omega_1^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_2^2 - \omega_1^2 \end{bmatrix} \\ (k^\times)^3 &= \begin{bmatrix} -\omega_3^2 - \omega_2^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_3^2 - \omega_1^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_2^2 - \omega_1^2 \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\omega_3\omega_2\omega_1 + \omega_3\omega_2\omega_1 & \omega_3(\omega_3^2 + \omega_2^2 + \omega_1^2) & -\omega_2(\omega_3^2 + \omega_2^2 + \omega_1^2) \\ -\omega_3(\omega_3^2 + \omega_2^2 + \omega_1^2) & -\omega_3\omega_2\omega_1 + \omega_3\omega_2\omega_1 & \omega_1(\omega_3^2 + \omega_2^2 + \omega_1^2) \\ \omega_2(\omega_3^2 + \omega_2^2 + \omega_1^2) & -\omega_1(\omega_3^2 + \omega_2^2 + \omega_1^2) & -\omega_3\omega_2\omega_1 + \omega_3\omega_2\omega_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = -k^\times \\ (k^\times)^4 &= \left((k^\times)^2\right)^2 = (kk^T - I)^2 = kk^T kk^T - 2kk^T + I = I - kk^T = -(kk^T - I) = -(k^\times)^2 \end{aligned}$$

$$(k^\times)^{4*2-3} = -(k^\times)^2 k^\times = -(k^\times)^3 = -(-k^\times) = k^\times$$

When k cross raises to the fifth power, it is equivalent of k cross. The cycle begins, and thus

$$(k^\times)^{4i-3} = k^\times$$

$$(k^\times)^{4i-2} = (k^\times)^2$$

$$(k^\times)^{4i-1} = (k^\times)^3 = -k^\times$$

$$(k^\times)^{4i} = -(k^\times)^2$$

as $i > 1$

5 b)

since

$$(k^\times)^{4i} = -(k^\times)^2, \quad (k^\times)^{4i-1} = -(k^\times), \quad (k^\times)^{4i-2} = (k^\times)^2, \quad (k^\times)^{4i-3} = k^\times$$

$$e^{k^\times \theta} = \sum_{i=0}^{\infty} \frac{(k^\times \theta)^i}{i!} = \sum_{i=0}^{\infty} \frac{(k^\times)^i (\theta)^i}{i!}$$

$$= I + k^\times \theta + (k^\times)^2 \frac{\theta^2}{2!} + (k^\times)^3 \frac{\theta^3}{3!} + (k^\times)^4 \frac{\theta^4}{4!} + (k^\times)^5 \frac{\theta^5}{5!} + (k^\times)^6 \frac{\theta^6}{6!} + \dots$$

$$= I + k^\times \theta + (k^\times)^2 \frac{\theta^2}{2!} - k^\times \frac{\theta^3}{3!} - (k^\times)^2 \frac{\theta^4}{4!} + k^\times \frac{\theta^5}{5!} + (k^\times)^2 \frac{\theta^6}{6!} + \dots$$

$$= I + k^\times \theta - k^\times \frac{\theta^3}{3!} + k^\times \frac{\theta^5}{5!} + \dots + (k^\times)^2 \frac{\theta^2}{2!} - (k^\times)^2 \frac{\theta^4}{4!} + (k^\times)^2 \frac{\theta^6}{6!} + \dots$$

$$= I + k^\times \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \right) + (k^\times)^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \dots \right)$$

By Taylor expansion: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$; $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$$e^{k^\times \theta} = I + \sin(\theta) k^\times + (1 - \cos(\theta)) (k^\times)^2$$