System Analysis Techniques HW6

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Problem 1

Given the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}}_{B} u$$

Define the controllability matrix $M = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

To show that the system is controllable IF AND ONLY IF b_1, b_2, $\cdots$, b_n are all nonzero. First we have to show if the sytem is controllable, then b_1, b_2, $\cdots$, b_n must be all nonzero:

$$M = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} b_1 & b_1\lambda_1 & b_1\lambda_1^2 & \cdots & b_1\lambda_1^{n-1} \\ b_2 & b_2\lambda_2 & b_2\lambda_1^2 & \cdots & b_2\lambda_2^{n-1} \\ \vdots & & \ddots & & \vdots \\ b_{n-1} & b_{n-1}\lambda_{n-1} & b_{n-1}\lambda_{n-1}^2 & \cdots & b_{n-1}\lambda_{n-1}^{n-1} \\ b_n & b_n\lambda_n & b_n\lambda_n^2 & \cdots & b_n\lambda_n^{n-1} \end{bmatrix}$$

If any of the b_1, b_2, $\cdots$, b_n is zero, there are at least one row contains all zeros in matrix M, thus the determinant of M is zero and it is singular matrix with rank(M) < n. Thus with the proof by contradiction, if the system is controllable, none of the row will be all zeros, and none of the b_1, b_2, $\cdots$, b_n will be zero. Now if b_1, b_2, $\cdots$, b_n are all non zero, we can prove the matrix M is singular with the following:

$$det(M) = \begin{vmatrix} b_1 & b_1\lambda_1 & b_1\lambda_1^2 & \cdots & b_1\lambda_1^{n-1} \\ b_2 & b_2\lambda_2 & b_2\lambda_1^2 & \cdots & b_2\lambda_2^{n-1} \\ \vdots & & \ddots & & \vdots \\ b_{n-1} & b_{n-1}\lambda_{n-1} & b_{n-1}\lambda_{n-1}^2 & \cdots & b_{n-1}\lambda_{n-1}^{n-1} \\ b_n & b_n\lambda_n & b_n\lambda_n^2 & \cdots & b_n\lambda_n^{n-1} \end{vmatrix}$$

$$= b_1b_2b_3\cdots b_n \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & & \ddots & & \vdots \\ 1 & \lambda_{n-1} & \lambda_{n-1}^2 & \cdots & \lambda_{n-1}^{n-1} \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{vmatrix} = b_1b_2b_3\cdots b_n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \vdots & & \ddots & & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-2} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

$= b_1b_2\cdots b_n \prod_{1\le j<i\le n}(\lambda_i-\lambda_j) = b_1b_2\cdots b_n(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_2)\cdots(\lambda_{n-1}-\lambda_{n-2})(\lambda_n-\lambda_{n-2})(\lambda_n-\lambda_{n-1})$

Since all eigen value are distinct, the determinant of M is nonzero. Thus matrix M is singular and the system is controllable if and only if b_1, b_2, $\cdots$, b_n are zero.

b)

Given $y = \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix}}_{C} x$

The testing matrix $N = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_1\lambda_1 & c_2\lambda_2 & c_3\lambda_3 & \cdots & c_n\lambda_n \\ \vdots & & \ddots & & \vdots \\ c_1\lambda_1^{n-2} & c_2\lambda_2^{n-2} & c_3\lambda_3^{n-2} & \cdots & c_n\lambda_n^{n-2} \\ c_1\lambda_1^{n-1} & c_2\lambda_2^{n-1} & c_3\lambda_3^{n-2} & \cdots & c_n\lambda_n^{n-1} \end{bmatrix}$

Similar to the previous question: If any of the c_1, c_2, $\cdots$, c_n are zero, at least one column will be zero, the matrix will be singular and the rank is less than n, thus it is not observable. The determinant of matrix is:

$$det(N) = c_1c_2c_3\cdots c_n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \vdots & & \ddots & & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-2} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

$= c_1c_2\cdots c_n \prod_{1\le j<i\le n}(\lambda_i-\lambda_j) = c_1c_2\cdots c_n(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_2)\cdots(\lambda_{n-1}-\lambda_{n-2})(\lambda_n-\lambda_{n-2})(\lambda_n-\lambda_{n-1})$

Since all eigen value are distinct, and if c_1, c_2, $\cdots$, c_n are zero, the determinant of N is nonzero. Thus matrix N is singular and the system is observable if and only is c_1, c_2, $\cdots$, c_n are zero.

c)

From the previous two problems, we know that the determinant for the two testing matrix for controllablity and observability are:

$det(M) = b_1b_2\cdots b_n \prod_{1\le j<i\le n}(\lambda_i-\lambda_j) = b_1b_2\cdots b_n(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_2)\cdots(\lambda_{n-1}-\lambda_{n-2})(\lambda_n-\lambda_{n-2})(\lambda_n-\lambda_{n-1})$

and $det(N) = c_1c_2\cdots c_n \prod_{1\le j<i\le n}(\lambda_i-\lambda_j) = c_1c_2\cdots c_n(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\cdots(\lambda_n-\lambda_2)\cdots(\lambda_{n-1}-\lambda_{n-2})(\lambda_n-\lambda_{n-2})(\lambda_n-\lambda_{n-1})$

If there are two repeated eigen values in the matrix A, then both the expression above are zero. Thus M and N will be singular, and rank less than n. The system will be unobservable and uncontrollable.

Problem 2

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 5 & 13 \\ 2 & 5 & 13 \end{bmatrix}$ with the reduced row echelon form $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

The is rank 2. Thus the system is uncontrollable.

b)

Choose the first and second column with another random third column to form the matrix $T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix}$, thus

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -6 & -4 \\ 1 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$\tilde{C} = CT = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}$$

We have the controllable state representation of the system

$$\begin{bmatrix} \dot{\tilde{z}}_c \\ \dot{\tilde{z}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} 0 & -6 & -4 \\ 1 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{z}_c \\ \tilde{z}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_c \\ \tilde{z}_{\bar{c}} \end{bmatrix}$$

Thus the controllable subsystem is:

$$\dot{\tilde{z}}_c = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \tilde{z}_{c^+} \begin{bmatrix} -4 \\ 2 \end{bmatrix} \tilde{z}_{\bar{c}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y_c = \begin{bmatrix} 1 & 3 \end{bmatrix} \tilde{z}_c$$

c)

It is unobservable since the testing matrix is $\begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 9 & 27 & 8 \end{bmatrix}$, not full rank (first and second column are linearly dependent).

d)

The testing matrix $N = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 9 & 27 & 8 \end{bmatrix}$ with the reduced row echelon form $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the rank $= 2 < 3$, unobservable.

Choose the first, the second row (linearly independent) and then pick a random third row that's independent from the previous two rows to form the non singular transform matrix

$$T = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\bar{A} = T_O \tilde{A} T_O^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -6 & -4 \\ 1 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\bar{B} = T_O \tilde{B} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\bar{C} = \tilde{C} T_0^{-1} = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 \\ 3 & 9 & 2 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\tilde{z}}_o \\ \dot{\tilde{z}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} \tilde{z}_O \\ \tilde{z}_{\bar{O}} \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_O \\ \tilde{z}_{\overline{O}} \end{bmatrix}$$

The observable space is:

$$\dot{\tilde{z}}_o = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \tilde{z}_o + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y_o = \begin{bmatrix} -2 & 1 \end{bmatrix} \tilde{z}_O$$

e)

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} s-3 & 2 & -2 \\ -1 & s & -2 \\ -1 & 1 & s-3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{s-3}$$

In part d), we can take a look at its transfer function:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s & -1 \\ 3 & s-4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{s-1}{(s-1)(s-3)} = \frac{1}{s-3}$$

whose order is same as the transfer function of the system. With the order of

1 only, there are must be cancellation of pole and zero in the transfer function of the system, thus the original state representation system is not the minimum representation. With the cancellation of pole and zero in transfer function, the system must be uncontrollable or unobservable or both, corresponding to what we have shown.

Problem 3

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3p^2 & 0 & 0 & 2p \\ 0 & 0 & 0 & 1 \\ 0 & -2p & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 2p \\ 0 & 1 \\ -2p & 0 \end{bmatrix}; A^2B = \begin{bmatrix} 0 & 2p \\ -p^2 & 0 \\ -2p & 0 \\ 0 & -4p^2 \end{bmatrix}; A^3B = \begin{bmatrix} -p^2 & 0 \\ 0 & -2p^3 \\ 0 & -4p^2 \\ 2p^3 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2p & -p^2 & 0 \\ 1 & 0 & 0 & 2p & -p^2 & 0 & 0 & -2p^3 \\ 0 & 0 & 0 & 1 & -2p^2 & 0 & 0 & -4p^2 \\ 0 & 1 & -2p & 0 & 0 & -4p^2 & 2p^3 & 0 \end{bmatrix}$$

From the first four columns, we can see that the matrix must be full rank. It is controllable. If the radial thrust fails that $u_1 \equiv 0$, it will not contribute to the controllablity. We can see the matrix formed by column 2, 4, 6 and 8 is full rank matrix, so the system is controllable.

$$\begin{bmatrix} 0 & 0 & 2p & 0 \\ 0 & 2p & 0 & -2p^3 \\ 0 & 1 & 0 & -4p^2 \\ 1 & 0 & -4p^2 & 0 \end{bmatrix}$$

If the tangential thruster fails with $u_2 \equiv 0$, then the matrix is formed by column 1, 3, 5, 7. The third and seventh column are linearly dependent, thus the matrix is not full rank, and it is not controllable.

$$\begin{bmatrix} 0 & 1 & 0 & -p^2 \\ 1 & 0 & -p^2 & 0 \\ 0 & 0 & -2p & 0 \\ 0 & -2p & 0 & 2p^3 \end{bmatrix}$$

b)

The relationship between the output and state can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \cdots \end{bmatrix}$$

Thus from C and CA we can see that it is testing matrix for observability will be full rank, so it will be observable.
If there is only y_1, the third element of C, CA, CA^2, and CA^3 will all be zero, thus it is not full rank and it is not observable.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3p^2 & 0 & 0 & 2p \\ 0 & -p^2 & 0 & 0 \end{bmatrix}$$

Problem 4

Given the $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$ with $x(0) = (0,0)^T$
The solution of the state with zero initial condition: $x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau$
Characteristic polynomial of A: $(\lambda+1)(\lambda+2)$
$\lambda = -1 : e^{-(T-\tau)} = -\alpha_1(\tau) + \alpha_0(\tau)$
$\lambda = -2 : e^{-2(T-\tau)} = -2\alpha_1(\tau) + \alpha_0(\tau)$
$\alpha_1(\tau) = e^{-(T-\tau)} - e^{-2(T-\tau)}$
$\alpha_0(\tau) = e^{-(T-\tau)} + \alpha_1(\tau) = 2e^{-(T-\tau)} - e^{-2(T-\tau)}$
Therefore:
$x(\tau) = \int_0^T \left[\alpha_1(\tau) A + \alpha_0(\tau) I\right] Bu(\tau) d\tau$

$$x(T) = AB \cdot \underbrace{\int_0^T \alpha_1(\tau) u(\tau) d\tau}_{\triangleq V_2} + B \underbrace{\int_0^T \alpha_a(\tau) u(\tau) d\tau}_{\triangleq v_1}$$

Thus for t = 1, the expression of the state can be expressed with input as: $x(1) =$

$$\begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot \underbrace{\int_0^1 (e^{-(1-\tau)} - e^{-2(1-\tau)}) u(\tau) d\tau}_{\triangleq V_2} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{\int_0^1 (2e^{-(1-\tau)} - e^{-2(1-\tau)}) u(\tau) d\tau}_{\triangleq v_1}$$

b)

$$[B : AB] = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$X(T) = [B : AB] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$V_1 = \int_0^T \alpha_0(\tau) u(\tau) d\tau = \langle \alpha_0, u \rangle$

$V_2 = \int_0^T \alpha_1(T)u(\tau)d\tau = \langle \alpha_1, u\rangle$
$u = k_0\alpha_0 + k_1\alpha_1$
$V_1 = 1 = \langle \alpha_0, u\rangle = \langle \alpha_0, k_0\alpha_0 + k_1\alpha_1\rangle = k_0 \langle \alpha_0, \alpha_0\rangle + k_1\langle \alpha_0, \alpha_1\rangle$
$V_2 = 0 = \langle \alpha_1, u\rangle = k_0 \langle \alpha_0,\alpha_1\rangle + k_1 \langle \alpha_1, \alpha_1\rangle$

$$\begin{bmatrix} \langle \alpha_0, \alpha_0\rangle & \langle \alpha_0, \alpha_1\rangle \\ \langle \alpha_0, \alpha_1\rangle & \langle \alpha_1, \alpha_1\rangle \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \int_0^1 (2\cdot e^{-(1-t)} - e^{-2\cdot(1-t)})^2 dt & \int_0^1 (2\cdot e^{-(1-t)} - e^{-2\cdot(1-t)})(e^{-(1-t)} - e^{-2\cdot(1-t)}) dt \\ \int_0^1 (e^{-(1-t)} - e^{-2\cdot(1-t)})(2\cdot e^{-(1-t)} - e^{-2\cdot(1-t)}) dt & \int_0^1 (e^{-(1-t)} - e^{-2\cdot(1-t)})^2 dt \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k_0 \\ k_1 \end{bmatrix}$$

$$\begin{bmatrix} k_0 \\ k_1 \end{bmatrix} = \begin{bmatrix} \frac{6\cdot e^2\cdot(e+3)}{(e^2+4\cdot e+1)\cdot(e-1)} \\ \frac{-18\cdot e^2\cdot\left(e^2+2\cdot e-1\right)}{e^4+2\cdot e^3-6\cdot e^2+2\cdot e+1} \end{bmatrix} = \begin{bmatrix} 7.6596 \\ -27.6561 \end{bmatrix}$$

Thus $u = 7.65952(e^{-(T-\tau)} - e^{-2(T-\tau)}) - 27.6561(e^{-(T-\tau)} - e^{-2(T-\tau)})$ to satisfy the requirement in the space span by $\alpha 1$ and $\alpha 0$. u can be a vector combined with other basis independent from α_0 and α_1 such as $u = k_0\alpha_0+k_1\alpha_1+k_2\alpha_2+...$, but the energy in this case will be higher, since k_0 and k_1 will be the same to move the initial state to the targeted state in the space, while the norm of the vector is larger.