Systems Analysis Techniques Homework 1

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Given that $y\left(0^{-}\right)=1$ and $\frac{dy}{dt}\left(0^{-}\right)=-1$, and the following differential equation,

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+2y(t)=\frac{du}{dt}+u(t)$$

applying Laplace transform to the differential equation generates,

$$s^2Y(s)-sy\left(0^{-}\right)-y'\left(0^{-}\right)+2sY(s)-2y(0)+2Y(s)=sU(s)-u\left(0^{-}\right)+\mathrm{U}(s)$$

rearrange and simplify it to

$$s^2Y(s)-s-1+2sY(s)+2Y(s)=1+\frac{1}{s}$$

$$Y(s)\cdot\left(s^2+2s+2\right)=2+\frac{1}{s}+s$$

get the Laplace transform of y(t)

$$Y(s)=\frac{2s+1+s^2}{s^3+2s^2+2s}=\frac{(s+1)^2}{s\left(s^2+2s+2\right)}$$

With partial fraction decomposition

$$Y(S)=\frac{A}{S}+\frac{BS+C}{S^2+2S+2}=\frac{As^2+2As+2A+Bs^2+Cs}{s\left(s^2+2s+2\right)}$$

The following system of equations

$$\begin{cases} A+B=1 \\ 2A+C=2 \\ 2A=1 \end{cases}$$

yields

$$\begin{cases} A=0.5 \\ B=0.5 \\ C=1 \end{cases}$$

and thus Y(s) becomes

$$Y(s) = \frac{1}{2S} + \frac{1}{2}\frac{S+2}{S^2+2S+2} == \frac{1}{2S} + \frac{1}{2}\left[\frac{S+1}{(S+1)^2+1} + \frac{1}{(S+1)^2+1}\right]$$

By applying the inverse of Laplace transform, y(t) becomes

$$y(t) = L^{-1}(Y(s)) = \left(\frac{1}{2} + \frac{1}{2}\left[e^{-t}\cos t + e^{-t}\sin t\right]\right) u(t)$$

2

Prove that the system described by $y(t) = \frac{du}{dt} + 2u(t)$ is time invariant

Given the input u_1

$$y_1(t) = \frac{du_1(t)}{dt} + 2u_1(t)$$

and input u_2

$$y_2(t) = \frac{du_2(t)}{dt} + 2u_2(t)$$

where $u_2 = u_1(t-\tau)$, $y_2(t)$ is equal to

$$L(u_2) = \frac{du_1(t-\tau)}{dt} + 2u_1(t-\tau)$$

With the change of variable $t' = t - \tau$, $dt' = dt$, the above equation becomes

$$y_2(t) = \frac{du_1(t')}{dt'} + 2u_1(t')$$

Now take a look at $y_1(t-\tau)$

$$y_1(t-\tau) = \frac{du_1(t-\tau)}{d(t-\tau)} + 2u_1(t-\tau)$$

With the same substitution of variables $t' = t-\tau$, $dt' = dt$, $y_1(t-\tau)$ becomes

$$y_1(t') = \frac{du_1(t')}{dt'} + 2u(t')$$

Thus $y_1(t-\tau) = L(u_2(t))$, when $u_2 = u_1(t-\tau)$. The system is time invariant.

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Write the system $2\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y(t) = \frac{d^2u}{dt^2} - u(t)$ in state-space form

Rearrange the differential equation $2\dddot{y} + \ddot{y} + 2\dot{y} + 2y = \ddot{u} - u$

$$2\dddot{y} = -\ddot{y} + \ddot{u} - 2\dot{y} - 2y - \mu$$

$$2\ddot{y} = -\dot{y} + \dot{u} - 2y + \int -2y - udt$$
$$2\dot{y} = -y + u + \int (-2y + \int (-2y - udt))dt'$$
$$y = \int (-\frac{y}{2} + \frac{u}{2} + \int (-y + \int (-y - \frac{u}{2})dt)dt')dt''$$

Choose the state variables such that

$$\begin{aligned}
&y = x_3 \\
&\dot{x_1} = -\tfrac{1}{2}u - y \\
&\dot{x_2} = x_1 - y \\
&\dot{x_3} = \tfrac{1}{2}u + x_2 - \tfrac{1}{2}y
\end{aligned}$$

and it generates the system with the state form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0 \cdot u$$

4

To prove the system is linear if and only if with initial conditions all being zero, suggesting such condition implies linearity, while other conditions implies non-linearity.

First take a look at the first order DFQ, with an initial condition:

$$a_1 \frac{dy}{dt} + a_0 y(t) = u(t)$$

which can be modified to

$$a_1 \frac{dy}{dt} + a_0 \left(\int_0^\infty \frac{dy}{dt} dt + y_0 \right) = u(t)$$

where y_0 is a constant (the initial value of output y).

Consider the input u_1,

$$a_1 \frac{dy_1}{dt} + a_0 \left(\int_0^\infty \frac{dy_1}{dt} dt + y_0 \right) = u_1$$

and zero input denoting u_2,

$$a_1 \frac{dy_2}{dt} + a_0 \left(\int_0^\infty \frac{dy_2}{dt} dt + y_0 \right) = u_2$$

adding two equations together $u_1 + u_2$

$$a_1\frac{dy_2}{dt} + a_0\left(\int_0^\infty \frac{dy_2}{dt}dt + y(0)_2\right) + a_1\frac{dy_1}{dt} + a_0\left(\int_0^\infty \frac{dy_1}{dt}dt + y(0)\right) = u_2 + u_1$$

rearrange it to be:

$$a_1\frac{dy_1 + y_2}{dt} + a_0\left(\int_0^\infty \frac{dy_1 + y_2}{dt}dt + 2y(0)\right) = u_2 + u_1 \quad (1)$$

Now consider a new input u_3 for the system

$$a_1\frac{dy_3}{dt} + a_0\left(\int_0^\infty \frac{dy_3}{dt}dt + y(0)\right) = u_3 \quad (2)$$

where $u_3 = u_2 + u_1$. In order to satisfy superposition, (1) needs to be equal to (2), such that

$$a_0(y(0)) = 2a_0y(0)$$

is true, if and only if the initial condition y(0) = 0. Substitute y(0) with a non-zero such as 1, the above equation does not hold. If the initial condition is nonzero, the system fail to satisfy $L(u1 + u2) = y1 + y2$. This proof by contradiction suggests linearity implies zero condition.

Similar idea can be applied to the second order differential equation, with two initial conditions $\frac{dy}{dt}(0)$ and $y(0)$ given:

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y(t) = u(t)$$

that can be modified to

$$a_2\frac{d^2y}{dt^2} + a_1\left(\int_0^\infty \frac{d^2y}{dt}dt + \frac{dy}{dt}(0)\right) + a_0\left(\int_0^\infty \frac{dy}{dt}dt + y(0)\right) = u(t)$$

Given u1 as input to the system, and

$$a_2\frac{d^2y_1}{dt^2} + a_1\left(\int_0^\infty \frac{d^2y_1}{dt}dt + \frac{dy}{dt}(0)\right) + a_0\left(\int_0^\infty \frac{dy_1}{dt}dt + y(0)\right) = u_1 \quad (3)$$

u2 as input the system

$$a_2\frac{d^2y_2}{dt^2} + a_1\left(\int_0^\infty \frac{d^2y_2}{dt}dt + \frac{dy}{dt}(0)\right) + a_0\left(\int_0^\infty \frac{dy_2}{dt}dt + y(0)\right) = u_2 \quad (4)$$

multiply an arbitrary constant denoting 'a' to (3) and 'b' to (4), and then sum them up

$$a_2\frac{d^2\left(ay_1 + by_2\right)}{dt^2} + a_1\left(\int_0^\infty \frac{d^2\left(ay_1 + by_2\right)}{dt}dt + (a+b)\frac{dy}{dt}(0)\right) + a_0\left(\int_0^\infty \frac{dy_1}{dt}dt + (a+b)y(0)\right) = au_1 + bu_2$$

A new input u3 applied to the system

$$a_2 \frac{d^2 y_3}{dt^2} + a_1 \left(\int_0^\infty \frac{d^2 y_3}{dt} dt + \frac{dy}{dt}(0) \right) + a_0 \left(\int_0^\infty \frac{dy_3}{dt} dt + y(0) \right) = u_3$$

where $u_3 = au_1 + bu_2$. If $(a+b)\frac{dy}{dt}(0) = \frac{dy}{dt}(0)$, and $(a+b)y(0) = y(0)$ for any a and b, that y(0) = 0 and $\frac{dy}{dt}(0)$ are zero.

If the two initial conditions are both zero, the system satisfies, as proved above,

$L(au_1 + bu_2)$ or $y_3 = ay_1 + by_2$, implying this is a linear system.

Thus it is proved that linearity implies zero initial conditions, and initial conditions being zero implies linearity. Zero initial condition is the necessary and sufficient condition for the system depicted by the differential equations with such form to be linear.

For nth order differential equations,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) = u(t)$$

The system is linear if and only if

$$\begin{aligned} & y_0 = (a+b)y_0 \\ & y_1 = (a+b)y_1 \\ & \cdots \\ & y_{n-1} = (a+b)y_{n-1} \end{aligned}$$

with a and b being any arbitrary constants

$$y_0 = y_1 = \cdots = y_{n-1} = 0$$