# System Analysis Techniques HW4

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## Problem 1

Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{bmatrix}$ with $|\lambda I - A| = 0$, we get:

$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -a_3 & -a_2 & \lambda - a_1 \end{vmatrix} = \lambda\lambda(\lambda a - a_1) + (-a_3) + 0 - 0 - 0 - \lambda a_2 = \lambda^3 - \lambda^2 a_1 - \lambda 2 - a_3$

The characteristic polynomial is $\lambda^3 - \lambda^2 a_1 - \lambda a_2 - a_3 = 0$

### b)

To prove the matrix $\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$ is singular is to prove the determinant of the matrix is not equal to zero.

$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$

Since $\lambda 1, \lambda 2, \lambda 3$ is non-repeated eigenvalue of the matrix, $(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \neq 0$. Thus the matrix is invertible and it is non-singular.

## Problem 2

Since $\left|A^\top\right| = |A|$ is given:
The eigenvalue of $A^\top$ can be found use: $|\lambda I - A^\top| = 0$
$|\lambda I - A^\top| = \left|\lambda I^\top - A^\top\right| = \left|(\lambda I - A)^\top\right| = |\lambda I - A| = 0$
Thus the $\lambda$ is eigenvalue of A and $A^\top$

### b)

For symmetric matrix: $A^\top = A$, and if A has two distinct eigenvalues with corresponding eigenvectors, denoting them as u and v, such that:
$Au = \lambda_1 u$ and $Av = \lambda_2 v$

From the problem description we know that $u^T A = \lambda_1 u^T$
Multiply $u^\top$ to $Av = \lambda_2 v$: $u^\top A v = \lambda_2 u^\top v$
Substitute: $\lambda_1 u^T v = \lambda_2 u^T v$
Then $(\lambda_1 - \lambda_2)u^T v = 0$. Since $\lambda_1 \neq \lambda 2$, $u^\top v = 0$, $u \perp v$

# Problem 3

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$SI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+2 \end{bmatrix}$ to find the inverse of $(SI - A)^{-1}$: $(SI - A)$

$(SI - A)^{-1} = (SI - A, E) = \left( \begin{array}{ccc|ccc} s & -1 & 0 & 1 & 0 & 0 \\ 0 & s & -1 & 0 & 1 & 0 \\ 0 & 2 & s+2 & 0 & 0 & 1 \end{array} \right)$ make some transform: $= \left( \begin{array}{ccc|ccc} 1 & -\frac{1}{s} & 0 & \frac{1}{s} & 0 & 0 \\ 0 & s & -1 & 0 & 1 & 0 \\ 0 & 2 & s+2 & 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{s^2} & \frac{1}{s} & \frac{1}{s^2} & 0 \\ 0 & 1 & -\frac{1}{s} & 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{2}{s} + s + 2 & 0 & \frac{-2}{s} & 1 \end{array} \right)$

$= \left( \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{s^2} & \frac{1}{s} & \frac{1}{s^2} & 0 \\ 0 & 1 & -\frac{1}{s} & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 1 & 0 & \frac{-2}{2+2s+s^2} & \frac{1}{\frac{2}{s}+s+2} \end{array} \right)$

$= \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{s} & \frac{s+2}{s^3+2s^2+2s} & \frac{1}{s^3+2s^2+2s} \\ 0 & 1 & 0 & 0 & \frac{s+2}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ 0 & 0 & 1 & 0 & \frac{-2}{2+2s+s^2} & \frac{1}{\frac{2}{s}+s+2} \end{array} \right)$

Thus $(SI - A)^{-1} = \left( \begin{array}{ccc} \frac{1}{s} & \frac{s+2}{s^3+2s^2+2s} & \frac{1}{s^3+2s^2+2s} \\ 0 & \frac{s+2}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ 0 & \frac{-2}{2+2s+s^2} & \frac{1}{\frac{2}{s}+s+2} \end{array} \right)$

For each term, take the inverse laplace transform: $\frac{s+2}{s(s+2s+2)} = \frac{-0.5}{s-(-1+j)} + \frac{-0.5}{s-(-1-j)} + \frac{1}{s} => -0.5e^{(-1+j)t} + -0.5e^{(-1-j)t} + e^{0t} = -e^{-t}(\frac{e^{jt}+e^{-jt}}{2}) + 1$
$= 1 - e^t cos(t)$
for $\frac{1}{s^3+2s^2+2s} = \frac{A}{S} + \frac{BS+C}{S^2+2S+2}$ where $A = \frac{1}{2}$, $C = -1$, $B = -\frac{1}{2}$. Thus $\frac{1}{s^3+2s^2+2s} = \frac{1}{2s} - \frac{1}{2}\left(\frac{s+2}{s+1)^2+1}\right) => \frac{1}{2} - \frac{1}{2}(e^t(cos(t) + sin(t)))$
From the laplace transform table, $\frac{s+2}{s^2+2s+2} => e^t(\cos(t) + \sin(t))$
$\frac{1}{s^2+2s+2} => e^{-t} sin(t)$
$\frac{-2}{s^2+2s+2} => -2e^{-t} sin(t)$ $\frac{s}{s^2+2s+2} => e^t(\cos(t) - \sin(t))$

Thus $e^{At} = \begin{bmatrix} 1 & 1 - e^{-t}\cos(t) & \frac{1}{2} - \frac{e^{-t}(\cos(t)+\sin(t))}{2} \\ 0 & e^{-t}(\cos(t) + \sin(t)) & e^{-t}\sin(t) \\ 0 & -2\sin(t)e^{-t} & e^{-t}(\cos(t) - \sin(t)) \end{bmatrix}$

$x(t) = e^{At}x_0$, where $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

thus $x_0 = \begin{bmatrix} \frac{5}{2} - \frac{e^{-t}(\cos(t)+\sin(t))}{2} - e^{-t}\cos(t) \\ e^{-t}\sin(t) + e^{-t}(\cos(t)+\sin(t)) \\ e^{-t}(\cos(t)-\sin(t)) - 2e^{-t}\sin(t) \end{bmatrix}$

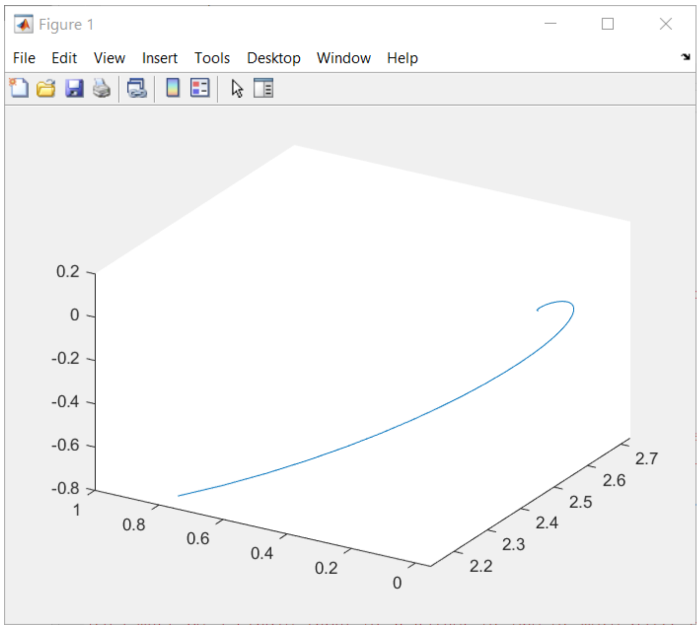


Figure 1: 3D plot, it indeed lines in a 2D frame

# Problem 4

Given $m_1\ddot{y} = -k_1y + k_2(z-y) - c_1\dot{y} - c_2(\dot{y}-\dot{z})$ and $m_2\ddot{z} = k_2(y-z) + c_2(\dot{y}-\dot{z})$
Choose $\dot{y} = x_1$, $\dot{z} = x_2$, $y = x_3$, $x_4 = z$
The equation can be written as:
$m_2\dot{x}_2 = k_2(x_3 - x_4) + c_2(x_1 - x_2)$
$m_1\dot{x}_1 = -k_1x_3 + k_2(x_4 - x_3) - c_1x_1 - c_2(x_1 - x_2)$
Rearrange:
$\dot{x}_1 = -\frac{k_1}{m_1}x_3 + \frac{k_2}{m_1}x_4 - \frac{k_2}{m_1}x_3 - \frac{c_1x_1}{m_1} - \frac{c_2x_1}{m_1} + \frac{c_2x_2}{m_1}$
$\dot{x}_2 = \frac{k_2}{m_2}x_3 - \frac{k_2}{m_2}x_4 + \frac{C_2x_1}{m_2} - \frac{C_2x_2}{m_2}$
and
$\dot{x}_3 = x_1$
$\dot{x}_4 = x_2$

Thus the state representation matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{-(c_1+c_2)}{m_1} & \frac{-c_2}{m_1} & \frac{-(k_1+k_2)}{m_2} & \frac{k_2}{m_2} \\ \frac{c_2}{m_2} & \frac{-c_2}{m_2} & \frac{k_2}{m_2} & \frac{-k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

## b)

with $m_1 = m_2 = 1 \quad k_1 = 1 \quad k_2 = 10 \quad c_1 = c_2 = 0.1$

Matrix A:

$$A = \begin{bmatrix} -0.2 & 0.1 & -11 & 10 \\ 0.1 & -0.1 & 10 & -10 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Find the eigen values using matlab

$\lambda 1 = -0.1262 + 4.5271i$

$\lambda 2 = -0.1262 - 4.5271i$

$\lambda 3 = -0.0238 + 0.6978i$

$\lambda 4 = -0.0238 - 0.6978i$

The corresponding eigen vectors for each eigen value above is:

$$V_1 = \begin{bmatrix} -0.7076 + 0.0000i \\ 0.6727 - 0.0137i \\ 0.0044 + 0.1562i \\ -0.0072 - 0.1484i \end{bmatrix}, V_2 = \begin{bmatrix} -0.7076 + 0.0000i \\ 0.6727 + 0.0137i \\ 0.0044 - 0.1562i \\ -0.0072 + 0.1484i \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 0.0122 - 0.3944i \\ 0.0141 - 0.4145i \\ -0.5651 + 0.0018i \\ -0.5940 + 0.0000i \end{bmatrix}, V_4 = \begin{bmatrix} 0.0122 + 0.3944i \\ 0.0141 + 0.4145i \\ -0.5651 - 0.0018i \\ -0.5940 + 0.0000i \end{bmatrix}$$

Now set the $\bar{X}$: $\bar{X} = \left[ \frac{V_1+V_2}{2} \mid \frac{V_1-V_2}{2j} \mid \frac{V_3+V_4}{2} \mid \frac{V_3-V_4}{2j} \right] = \begin{bmatrix} -0.7076 & 0 & 0.0122 & -0.3944 \\ 0.6727 & -0.0137 & 0.0141 & -0.4145 \\ 0.0044 & 0.1562 & -0.5651 & 0.0018 \\ -0.0072 & -0.1484 & -0.5940 & 0 \end{bmatrix}$

Thus $\bar{X}^{-1} A \bar{X} = \begin{bmatrix} -0.1262 & 4.5271 & -0.0000 & -0.0000 \\ -4.5271 & -0.1262 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0238 & 0.6978 \\ 0.0000 & 0.0000 & -0.6978 & -0.0238 \end{bmatrix}$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -0.1262 & 4.5271 & -0.0000 & -0.0000 \\ -4.5271 & -0.1262 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0238 & 0.6978 \\ 0.0000 & 0.0000 & -0.6978 & -0.0238 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

**c)**

Given that $\begin{bmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ q_4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \\ q_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Matlab code: the script can be run without any external file

Listing 1: matlab code

```
A = [-0.2 0.1 -11 10;0.1 -0.1 10 -10;1 0 0 0;0 1 0 0];
[V, D] = eig(A);
x1 = (V(:,1)+V(:,2))/2;
x2 = (V(:,1)-V(:,2))/(2*1i);
x3 = (V(:,3)+V(:,4))/2;
x4 = (V(:,3)-V(:,4))/(2*1i);
X = [x1, x2, x3, x4];
R = inv(X)*A*X
y1 = [];
z1 = [];
y2 = [];
z2 = [];
n = 0;
for t=0:0.1:10
    n = n + 1;
    At = R*t;
    exR = expm(At);
    q1 = exR*[1;1;0;0];
    q2 = exR*[0;0;1;1];
    xx1 = X*q1;
    xx2 = X*q2;
    y11 = xx1(3);
    z11 = xx1(4);
    y22 = xx2(3);
    z22 = xx2(4);
    y1(n) = y11;
    z1(n) = z11;
    y2(n) = y22;
    z2(n) = z22;
end
figure(1)
plot(z1, y1)
figure(2)
plot(z2,y2)
```

As you can see the difference between the plots, it is due to the difference of the spring constant. First mode has higher imaginary part, which results in a higher damping than the second mode. Output two y-z graphs:

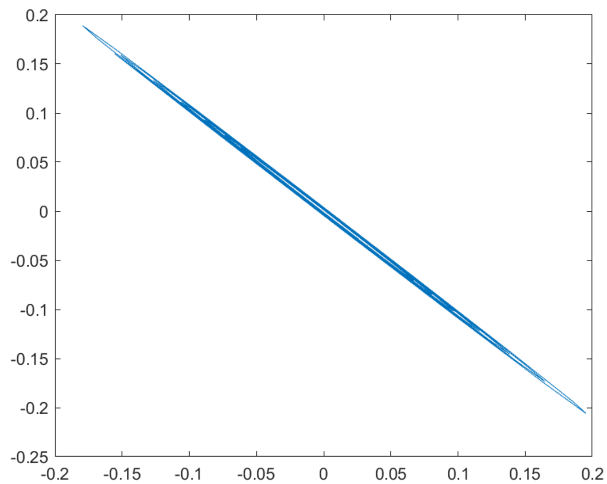


Figure 2: y-z plot of the first initial state

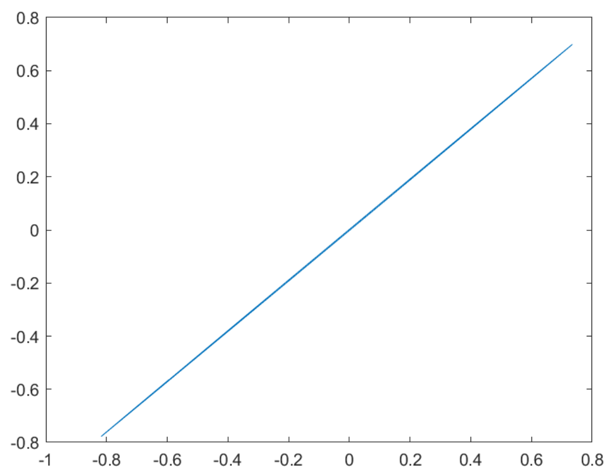


Figure 3: y-z plot of the second initial state