Robotics HW3

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a)

$$(P_{B1})_B = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}, (P_{B2})_B = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix}, (P_{B3})_B = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

$$(P_{B4})_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, (P_{B5})_B = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

b)

$$(p_{45})_0 = R_{0B}(\vec{p}_{45})_B;$$
$$(p_{24})_0 = R_{0B}(\vec{p}_{24})_B;$$
$$(p_{12})_0 = R_{0B}(\vec{p}_{12})_B;$$
$$[(\vec{p}_{45})_0, (\vec{p}_{24})_0, (\vec{p}_{12})_0] = R_{0B}\,[(\vec{p}_{45})_B, (\vec{p}_{24})_B, (\vec{p}_{12})_B]$$

$[(\vec{p}_{45})_0; (\vec{p}_{24})_0; (\vec{p}_{12})_0]$ is known from the graph, and if the measurements $[(\vec{p}_{45})_0; (\vec{p}_{24})_0; (\vec{p}_{12})_0]$ are also given, R_{0B} can be found with the inverse of body frame matrix:

$$R_{0B} = [(\vec{p}_{45})_0, (\vec{p}_{24})_0, (\vec{p}_{12})_0]\,[(\vec{p}_{45})_B, (\vec{p}_{24})_B, (\vec{p}_{12})_B]^{-1}$$

c)

Given

$$(p_{45})_0 = \begin{bmatrix} -0.5 \\ 0.31 \\ -0.8 \end{bmatrix}, \quad (p_{24})_0 = \begin{bmatrix} -0.31 \\ 0.8 \\ 0.5 \end{bmatrix}, \quad (p_{12})_0 = \begin{bmatrix} 2.41 \\ 1.51 \\ -0.93 \end{bmatrix}$$

$$(p_{45})_B = (p_{B5})_B - (p_{B4})_B = \begin{bmatrix} 0 \\ 0 \\ -c \end{bmatrix}, (p_{24})_B = (p_{B4})_B - (p_{B2})_B = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

$$(p_{12})_B = (p_{B2})_B - (p_{B1})_B = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

Since a = 3, b = 1, c = 1,

$$[(\vec{p}_{45})_B\,, (\vec{p}_{24})_B\,, (\vec{p}_{12})_B] = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_{0B} = \begin{bmatrix} -0.5 & -0.31 & 2.41 \\ 0.31 & 0.8 & 1.51 \\ -0.8 & 0.5 & -0.93 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0.803333 & -0.31 & 0.5 \\ 0.503333 & 0.8 & -0.31 \\ -0.31 & 0.5 & 0.8 \end{bmatrix}$$

$$R_{0B}R_{0B}^T = \begin{bmatrix} 0.991444 & 0.001344 & -0.004033 \\ 0.001344 & 0.989444 & -0.004033 \\ -0.004033 & -0.004033 & 0.9861 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

An article "Finding the Nearest Orthonormal Matrix" from MIT [1] estimates the nearest orthonormal matrix given a matrix with such equation:

$$R = M(I+\Lambda)^{-1} = M\left(M^TM\right)^{-1/2}$$

$$R_{0B}^{(appr)} = R_{0B}(R_{0B}^TR_{0B})^{-1/2} = \begin{bmatrix} 0.805822 & -0.310853 & 0.504005 \\ 0.504836 & 0.805497 & -0.310347 \\ -0.309502 & 0.504524 & 0.806017 \end{bmatrix}$$

To find the error between R_{0B} and $R_{0B}^{(appr)}$, the second norm of the error matrix $R_{0B} - R_{0B}^{(appr)}$:

$$Merror = R_{0B} - R_{0B}^{(appr)} = \begin{bmatrix} -0.0025 & 0.0009 & -0.0040 \\ -0.0015 & -0.0055 & 0.0003 \\ -0.0005 & -0.0045 & -0.0060 \end{bmatrix}$$

is computed with Matlab function norm(Merror, 2) calculating the 2-norm of the error:

$$\|Merror\|_2 = 0.0088$$

d)

Since there are four vectors:

$$[(\vec{p}_{45})_0, (\vec{p}_{24})_0, (\vec{p}_{12})_0, (\vec{p}_{14})_B] = R_{0B}\,[(\vec{p}_{45})_B, (\vec{p}_{24})_B, (\vec{p}_{12})_B, (\vec{p}_{14})_B] \qquad (1)$$

the matrix on the right-handed side is 3 by 4. To invert a non-square matrix, the pseudo inverse is applied to find the the rotation matrix, denoting the matrix on the right hand side of equation (1) as B and the left hand side matrix as A, thus:

$$AB^T = R_{0B}BB^T$$
$$R_{0B} = AB^T(BB^T)^{-1}$$

e)

$$P_{0i} = P_{0B} + R_{0B}P_{Bi}$$
$$P_{0i} = \begin{bmatrix} R_{0B} & P_{0B} \end{bmatrix} \cdot \begin{bmatrix} P_{Bi} \\ 1 \end{bmatrix}$$

To uniquely determine the 3 by 4 matrix of R_{0B} and P_{0B}, with one P_{0b} unknown and three DOF for R_{0B}, four vectors are needed. Thus $P_{01}P_{02}P_{03}P_{04}$ can be chosen as the set to determine the rotation matrix and translation vector.

f)

Given $\theta = \frac{\pi}{4}$ and rotation vector:

$$\vec{k} = (\vec{x}_0 + \vec{y}_0 + \vec{z}_0)\,/\sqrt{3}$$

translation:

$$\vec{x}_0 + 5\vec{y}_0$$

$$e^{k^x\theta} = I + \sin(\theta)k^\times + (1-\cos(\theta))\left(k^\times\right)^2$$

$$R_{0B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} + (1-\frac{\sqrt{2}}{2})\begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}^2$$

$$R_{0B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} + (1-\frac{\sqrt{2}}{2})\begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}^2$$

$$R_{0B} = \begin{bmatrix} 0.804738 & -0.310617 & 0.505879 \\ 0.505879 & 0.804738 & -0.310617 \\ -0.310617 & 0.505879 & 0.804738 \end{bmatrix}$$

g)

$P_{0i} = P_{0B} + R_{0B}P_{Bi}$;

$$\frac{dP_{0i}}{dt} = \dot{P_{0B}} + \omega^\times R_{0B}P_{Bi}$$
$$\frac{dP_{0i}}{dt} = \dot{P_{0B}} + (\dot{\theta}k)^\times R_{0B}P_{Bi}$$

$$\dot{P_{0B}} = \begin{bmatrix} \frac{5}{\sqrt{26}}v \\ \frac{v}{\sqrt{26}} \\ 0 \end{bmatrix};\ (\dot{\theta}k^\times) = \begin{bmatrix} 0 & -\frac{\dot{\theta}}{\sqrt{3}} & \frac{\dot{\theta}}{\sqrt{3}} \\ \frac{\dot{\theta}}{\sqrt{3}} & 0 & -\frac{\dot{\theta}}{\sqrt{3}} \\ -\frac{\dot{\theta}}{\sqrt{3}} & \frac{\dot{\theta}}{\sqrt{3}} & 0 \end{bmatrix}$$

Thus

$$\frac{dP_{01}}{dt} = \dot{P_{0B}} + (\dot{\theta}k)^\times R_{0B}P_{B1}$$

$$= \begin{bmatrix} \frac{5}{\sqrt{26}}v \\ \frac{v}{\sqrt{26}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{\dot{\theta}}{\sqrt{3}} & \frac{\dot{\theta}}{\sqrt{3}} \\ \frac{\dot{\theta}}{\sqrt{3}} & 0 & -\frac{\dot{\theta}}{\sqrt{3}} \\ -\frac{\dot{\theta}}{\sqrt{3}} & \frac{\dot{\theta}}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-b\dot{\theta}}{\sqrt{3}} + \frac{c\dot{\theta}}{\sqrt{3}} + \frac{5v}{\sqrt{26}} \\ \frac{a\dot{\theta}}{\sqrt{3}} + \frac{-c\dot{\theta}}{\sqrt{3}} + \frac{v}{\sqrt{26}} \\ \frac{-a\dot{\theta}}{\sqrt{3}} + \frac{b\dot{\theta}}{\sqrt{3}} \end{bmatrix}$$

References

[1] B. K. Horn, "Finding the nearest orthonormal matrix," http://people.csail.mit.edu/bkph/articles/Nearest$_{O}rthonormal_{M}atrix.pdf$.