System Analysis Techniques hw3

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Problem 1

$$M = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

The matrix $M \in R^{1\times 3}$, is in the reduced echelon form, such that $x_1 = -2X_2 - 3X_3$, and choose $(x_1, x_2) = (1, 0)$ and $(0, 1)$, we can get the solution of the x satisfying $Mx = 0$, where:

$$\vec{x} = C_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

and thus,

$$\text{Null } M = \text{span}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right)$$

b)

$$x = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$$

The span of a vector $x \in R^3$ is a line or zero. To find a full row matrix such that $\text{span}\{x\} = \ker M$ requires the null space of the matrix M to be a line or zero as well. From vector x:

$$x_1 = x_3 \text{ and } x_2 = -x_3$$

thus a reduced echelon form satisfying with above equations can be

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \tag{1}$$

with a rank of 2 equal to the number of rows, thus it is a full row rank matrix, with the free unknown x_3 in null space: $C \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Thus any matrix can be reduced to the echelon form like in (1) such as

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{2}$$

can be the matrix satisfying $= \text{span}\{x\} = \ker M$

Problem 2

The inner product defined as

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$$

for any x and y belongs to the linear vector space S which contains all real-valued sequence x_n and $\sum_{n=1}^{\infty} x_n^2 < \infty$

With the above definition: $\langle y, x \rangle = \sum_{n=1}^{\infty} y_n x_n \ = \sum_{n=1}^{\infty} x_n y_n = \langle x, y \rangle$

The inner product of x and itself: $\langle x, x \rangle = \sum_{n=1}^{\infty} x_n^2$, where $x_n^2 \geq 0$ and only if $x_n = 0$, $\langle x, x \rangle = 0$.

Since it is assumed that vector addition in the space element-wise addition, and vector scaling is element-wise scaling, a new vector $ax_1 + bx_2$ can be defined and the inner product of this new product with y is

$$\langle ax_1 + bx_2, y \rangle = \sum_{n=1}^{\infty} (ax_{1n} + bx_{2n}) y_n$$

$$= a \sum_{n=1}^{\infty} x_{1n} y_n + b \sum_{n=1}^{\infty} x_{2n} y_n = a \langle x_{1n}, y_n \rangle + a \langle x_{1n}, y_n \rangle$$

Thus the proposed inner product operation satisfying the axioms of the inner product.

b)

Since the above axioms of inner product is satisfied, the norm on the space can be defined as $\|x\| = \sqrt{\langle x, x \rangle}$.

Given $x = (x_1, x_2, x_3, \cdots) \in \mathcal{S}$, where $x_n = 2^{-n} \sin \frac{\pi n}{2}$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{n=1}^{\infty} \left(\left(2^{-n} \cdot \sin \left(\frac{\pi \cdot n}{2} \right) \right)^2 \right)}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4^n} \cdot \left(\sin \left(\frac{\pi \cdot n}{2} \right) \right)^2 \right)$$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{4^n} \cdot (1 - \cos(\pi \cdot n))}{2}$$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{4^n} \cdot (1 - \cos(\pi \cdot n))}{2}$$

$$= \sqrt{\frac{1}{6} - \frac{1}{2}\left(-\frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^5 + \cdots\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\left(\frac{1}{4} + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^5 + \cdots - \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^4 - \left(\frac{1}{4}\right)^6 \cdots\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\left(\frac{1}{4} + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^5 + \cdots - \left(\frac{1}{4}\right)\left(\left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^5 \cdots\right)\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\frac{3}{4}\left(\frac{1}{4} + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^5 + \cdots\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\frac{3}{4}\frac{1}{4}\left(1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4 + \cdots\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\frac{3}{4}\frac{1}{4}\left(\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n\right)}$$

$$= \sqrt{\frac{1}{6} + \frac{1}{2}\frac{3}{4}\frac{1}{4}\left(\frac{1}{1 - \frac{1}{16}}\right)} = \sqrt{\frac{4}{15}}$$

Problem 3

Consider the following three vectors: (0,1,0), (1, 1, 0), (w_1,w_2,w_3) to form a matrix such that the determinant is:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ w_1 & w_2 & w_3 \end{vmatrix} = -w_3 = 2^{-3} \neq 0$$

Thus the vectors (0,1,0), (1,1,0), and (w_1, w_2, w_3) are linearly independent.

Since the extension of the linearly independent vectors are also linearly independent,

$$u = (0, 1, 0, 0, 0 \ldots)$$
$$v = (1, 1, 0, 0, 0, \cdots)$$
$$w = (w_1, w_2, w_3, \cdots)$$

are linearly independent.

b)

Given

$$\begin{aligned} u &= (0,1,0,0,0\ldots) \\ v &= (1,1,0,0,0,\cdots) \\ w &= (w_1,w_2,w_3,\cdots) \end{aligned}$$

$a_1 = u = (0,1,0,0,0,...)$; $a_2 = v - \frac{\langle u,v\rangle}{\langle u,u\rangle}u = v - u = (1,0,0,0...)$;

$$a_3 = w - \frac{\langle w,v\rangle}{\langle v,v\rangle}v - \frac{\langle w,u\rangle}{\langle u,u\rangle}u = \langle \frac{w_1 - w_2}{2}, \frac{-w_1 - w_2}{2}, w_3, w_4, ...\rangle$$

$$e_1 = \frac{a_1}{\|a_1\|} = (0,1,0,0,0,...);$$

$$e_2 = \frac{a_2}{\|a_2\|} = (0,1,0,0,0,...);$$

$$e_3 = \frac{a_3}{\|a_3\|} = (1,0,0,0,0,...);$$

$$\|a_3\| = \sqrt{\frac{w_1^2}{2} + \frac{w_2^2}{2} + w_3^2 + w_4^2 + ...} = \sqrt{\frac{1}{8} + \frac{1}{32} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + ...}$$

$$= \sqrt{\frac{1}{8} + \frac{1}{32} + \frac{1}{2^6}(1 + \frac{1}{2^2} + \frac{1}{2^4} + ...)} = \sqrt{\frac{1}{8} + \frac{1}{32} + \frac{1}{2^6}(1 + \frac{1}{4^1} + \frac{1}{4^2} + ...)}$$

$$= \sqrt{\frac{1}{8} + \frac{1}{32} + \frac{1}{64}\frac{4}{3}} = \frac{\sqrt{102}}{24} = 0.4208$$

$$e_3 = \frac{a_3}{\|a_3\|} = w - \frac{\langle w,v\rangle}{\langle v,v\rangle}v - \frac{\langle w,u\rangle}{\langle u,u\rangle}u = \langle \frac{w_1 - w_2}{0.8416}, \frac{-w_1 - w_2}{0.84162}, \frac{w_3}{0.4208}, \frac{w_3}{0.4208}, ...\rangle$$

Problem 4

Given the equation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$$

such that $|\lambda I - A| = 0$:

$$A = \begin{vmatrix} \lambda + 1 & -1 & 2 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = \lambda(\lambda + 1)(\lambda + 2) = 0$$

so that the eigenvalues are $\lambda_1 = 0$; $\lambda_2 = -1$; $\lambda_3 = -2$

For $\lambda_1 = 0$

$$\lambda I - A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}; \text{ with reduced row echelon form: } \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus one of the eigenvector for λ_1 will be $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_2 = -1$

$$\lambda I - A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}; \text{ with reduced row echelon form: } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus one of the eigenvector for λ_2 will be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda_3 = -2$

$$\lambda I - A = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \text{ with reduced row echelon form: } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus one of the eigenvector for λ_3 will be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The matrix formed by linearly independent eigen vectors is: T = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

and its inverse $T^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus $\hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}; \hat{D} = D = 0$$

New state variables representation:

$$\dot{z} = \hat{A}z + \hat{B}u$$
$$y = \hat{C}z$$

b)

From previous question, the new state representation:

$$\dot{z} = \hat{A}z + \hat{B}u$$
$$y = \hat{C}z$$

where $\hat{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\hat{C} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$x(0) = x_0, \quad t \geq 0$

The solution of the state variable as the superposition of zero state and zero input:

$$\boldsymbol{x}(t) = \mathrm{e}^{At}\boldsymbol{x}(0) + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\boldsymbol{\tau}$$

where $u(t) = t$, and $x(0) = 0$

$$x = \int_0^t e^{A(t-\tau)} B\tau d\tau$$

let $t - \tau = l$, then $d\tau = -dl$

$$x = -\int_t^0 e^{Al} B(t-l) dl = \int_0^t e^{Al} B(t-l) dl = \int_0^t e^{Al} Btdl - \int_0^t e^{Al} Bldl$$

$$= A^{-1}\left(e^{At} - I\right)Bt - (\,A^{-1}(le^{Al})\big|_0^t B - A^{-1}\left[\int_0^t e^{Al}dl\right]B)$$

$$= A^{-1}\left(e^{At} - I\right)Bt - A^{-1}te^{At}B + A^{-1}\left[A^{-1}(e^{At} - I)\right]B$$

$$= -A^{-1}IBt + A^{-2}e^{At}B - A^{-2}IB = \left[A^{-2}\left(e^{At} - I\right) - A^{-1}t\right]B$$

It is unfortunate that the A matrix is singular, the above equation is not usable. A new method is needed:

$$x = \int_0^t e^{A(t-\tau)} B\tau d\tau = e^{At}\int_0^t e^{-A\tau} B\tau d\tau$$

We can calculate $\mathbf{e}^{At} = \boldsymbol{T}\mathbf{e}^{\boldsymbol{\Lambda}}\boldsymbol{T}^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2\cdot t} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & 1 - e^{-t} & e^{-2\cdot t} - 1 \\ 0 & 1 & e^{-2\cdot t} - 1 \\ 0 & 1 & e^{-2\cdot t} - 1 \end{bmatrix}$$

substitute t with $-\tau$ to get:

$$\mathbf{e}^{A(-\tau)} = \begin{bmatrix} e^{\tau} & 1 - e^{\tau} & e^{2\cdot\tau} - 1 \\ 0 & 1 & e^{2\cdot\tau} - 1 \\ 0 & 1 & e^{2\cdot\tau} - 1 \end{bmatrix}$$

$$\mathbf{e}^{A(-\tau)}B\tau = \begin{bmatrix} \tau \cdot \left(e^{2\cdot\tau} + e^{\tau} + 1\right) \\ \tau \cdot \left(e^{2\cdot\tau} + 1\right) \\ \tau \cdot \left(e^{2\cdot\tau} + 1\right) \end{bmatrix}$$

$$\int_0^t e^{-A\tau}B\tau d\tau = \begin{bmatrix} \int_0^t \tau \cdot \left(e^{2\cdot\tau} + e^{\tau} + 1\right) d\tau \\ \int_0^t \tau \cdot \left(e^{2\cdot\tau} + 1\right) d\tau \\ \int_0^t \tau \cdot \left(e^{2\cdot\tau} + 1\right) d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(2\cdot t-1)\cdot e^{2\cdot t}}{4} + (t-1)\cdot e^{t} + \frac{t^2}{2} + \frac{5}{4} \\ \frac{(2\cdot t-1)\cdot e^{2\cdot t}}{4} + \frac{t^2}{2} + \frac{1}{4} \\ \frac{(2\cdot t-1)\cdot e^{2\cdot t}}{4} + \frac{t^2}{2} + \frac{1}{4} \end{bmatrix}$$

$$x = e^{At}\int_0^t e^{-A\tau}B\tau d\tau = \begin{bmatrix} \frac{e^{-2\cdot t}\cdot\left((6\cdot t-5)\cdot e^{2\cdot t}+4\cdot e^{t}+2\cdot t^2+1\right)}{4} \\ \frac{e^{-2\cdot t}\cdot((2\cdot t-1)\cdot e^{2\cdot t}+2\cdot t^2+1)}{4} \\ \frac{e^{-2\cdot t}\cdot\left((2\cdot t-1)\cdot e^{2\cdot t}+2\cdot t^2+1\right)}{4} \end{bmatrix}$$

Finally we get the express of x, with $z = T^{-1}x$:

$$z = \begin{bmatrix} 0 \\ ((t-1)\cdot e^{t}+1)\cdot e^{-t} \\ \frac{e^{-2\cdot t}\cdot\left((2\cdot t-1)\cdot e^{2\cdot t}+2\cdot t^2+1\right)}{4} \end{bmatrix}$$

$$y = \hat{C}z = \frac{e^{-2\cdot t}\cdot\left((6\cdot t-5)\cdot \mathrm{e}^{2\cdot t}+4\cdot \mathrm{e}^{t}+2\cdot t^2+1\right)}{4}$$

A plot of y for t from 0 to 1: