Numerical Computing HW9

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Problem 1

Trapezoid n = 4: $\int_0^1 e^x dx$
$\int_0^1 e^x dx \approx \frac{1}{8}\left[e^0+e^1+2\left(e^{\frac{1}{4}}+e^{\frac{2}{4}}+e^{\frac{3}{4}}\right)\right]=1.72722$
$\frac{(b-a)h^2}{12} \mid f''(c) \mid = \frac{1\left(\frac{1}{4}\right)^2}{12}\left|e^c\right| \leq \frac{\left(\frac{1}{4}\right)^2}{12} e^1 = 0.014158$
b) $\int_0^1 e^x dx = 1.71828$
$exact \quad error: 0.00894 < 0.014158$
n = 6; Trapezoid

x_0	x_1	x_2	x_3	x_4	x_5	x_6
0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

$[0,1] \quad h=\frac{1}{6}$
$\int_0^1 e^x dx = \frac{1}{12}\left[e^0+e^1+2(e^{\frac{1}{6}}+e^{\frac{2}{6}}+e^{\frac{3}{6}}+e^{\frac{4}{6}}+e^{\frac{5}{6}})\right]$
$=1.72226$
$\frac{\left(\frac{1}{6}\right)^2}{12}\left|e^c\right| \leqslant \frac{\left(\frac{1}{6}\right)^2}{12}\left|e^1\right| = 0.006292$
exact error: $0.003976 < 0.006292$
composite Simpson's rule: n = 4; $h=(b-a)/2n=1/8$
$\int_0^1 e^x dx = \frac{1}{24}\left[e^0+e^1+4\left(e^{\frac{1}{8}}+e^{\frac{3}{3}}+e^{\frac{5}{8}}+e^{\frac{7}{8}}\right)+2\left(e^{\frac{3}{8}}+e^{\frac{4}{8}}+e^{\frac{5}{8}}\right)\right]=1.71828$
$\frac{b-a}{180}h^2 f^4(c) = \frac{1}{180}(\frac{1}{8})^2 e^c \leq \frac{1}{180}\left(\frac{1}{8}\right)^2 e^1 = 0.000236$
exact error $0.000002 \quad < 0.000236$
n = 6; $h=\frac{1}{12}$ $\int_0^1 e^x dx = \frac{1}{36}\left[e^0+e^1+4\left(e^{\frac{1}{12}}+e^{\frac{3}{12}}+e^{\frac{5}{12}}+e^{\frac{7}{12}}+e^{\frac{9}{12}}+e^{\frac{11}{12}}\right)+2\left(e^{\frac{2}{12}}+e^{\frac{9}{12}}+e^{\frac{6}{12}}+e^{\frac{8}{12}}+e^{\frac{10}{12}}\right)\right.$
1.71828
$\frac{b-a}{180}\left(\frac{1}{12}\right)^2 f^4(c) \leqslant \frac{1}{180}\left(\frac{1}{12}\right)^2 e^1 = 0.000105$
$exact \quad error: 4.5491 \times 10^{-7} \quad \leqslant 0.00105$

Problem 2

$\int_0^1 f(x)dx = C_1 f(0) + C_2 f(0.5) + C_3 f(1)$
$f(x) = 1, x, x^2$
$1 = \int_0^1 1 dx = C_1 + C_2 + C_3$
$0.5 = \int_0^1 x dx = 0.5C_2 + C3$

$\frac{1}{3}=\int_0^1 x^2dx=c_2\left(\frac{1}{2}\right)^2+C_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0.5 \\ \frac{1}{3} \end{bmatrix}$$

$C_1=\frac{1}{6}$
$C_2=\frac{2}{3}$
$C_3=\frac{1}{6}$

Simpson's rule: $\frac{1}{2}\cdot\frac{1}{3}\big(f(a)+f(b)+4f(a+h)\big)$

Problem 3

$\int_0^\pi \sin^2 xdx$
$f'(x)=2\sin x\cos x$
$f''(x)=2\cos^2 x-2\sin^2 x$
$f'''(x)=-4\sin x\cos x-4\sin x\cos x=-8\sin x\cos x$
$f^4(x)=-8\cos^2 x+8\sin^2 x$
$\frac{\pi h^4}{180}8\left(\sin^2 c-\cos^2 c\right)\leqslant 0.5\times 10^{-6}$
$\frac{\pi\left(\frac{\pi}{2n}\right)^4\cdot 8}{180}\leqslant 0.5\times 10^{-6}$
$n\geqslant 36.1093$
$n=37$ is enongh

Problem 4

$\int_a^b f(x)dx=\frac{h}{2}(f(a)+f(b))-\frac{h^3}{12}f''(c),\quad h=b-a$
$\sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}} f(x)dx=\sum_{i=0}^{m-1}\frac{h}{2}\left(f\left(x_i\right)+f\left(x_{i+1}\right)\right)-\frac{h^3}{12}f''(c_j)$ where $C_j\in[x_i,\quad x_{i+1}]$
$=\frac{h}{2}\left(f\left(x_0\right)+f\left(x_n\right)+2\sum_{j=1}^{m-1}f\left(x_1\right)\right)-\sum_{i=0}^{m-1}\frac{h^3}{12}f''\left(c_j\right)$
$\sum_{i=0}^{m-1}\frac{h^3}{12}f''\left(c_j\right)=\frac{h^3}{12}mf''(c)$
$=\frac{h}{2}\left(f\left(x_0\right)+f\left(x_0\right)+2\sum_{j=1}^{m-1}f\left(x_i\right)\right)-\frac{h^2}{12}(6-a)f''(c)$
error term $=-\frac{h^2}{12}(6-a)f''(c)$

Problem 5

```
%Computes composite Simpson
%f = @(x) x/sqrt(x^2+9)
function result=Simpson(f, m, x0,xn)
h = (xn-x0)/(2*m);
mpoints = x0+h:h:xn-h;
```

```
oddpoints = mpoints(1:2:length(mpoints));
evenpoints = mpoints(2:2:length(mpoints));
%mm = zeros(1:length(mpoints));
for i = 1:length(oddpoints)
    odd(i) = f(oddpoints(i));
end
for i = 1:length(evenpoints)
    even(i) = f(evenpoints(i));
end
osm = sum(odd);
esm = sum(even);
result = h/3*(f(x0)+f(xn)+4*(osm)+2*(esm));
end
```

```
%Computes composite Trapzoid
%f = @(x) x/sqrt(x^2+9)
function result=Trapezoid(f, m, x0,xn)
h = (xn-x0)/m;
mpoints = x0+h:h:xn-h;
%mm = zeros(1:length(mpoints));
for i = 1:length(mpoints)
    mm(i) = f(mpoints(i));
end
sm = sum(mm);
result = h/2*(f(x0)+f(xn)+2*(sm));
end
```

```
>> m = 8;
>> xn = 4;
>> x0 = 0;
>> f = @(x) x/sqrt(x^2+9);
>> Trapezoid(f,m,x0,xn)

ans =

    1.9945
```

error = 0.0054558358681

```
>> f = @(x) x/sqrt(x^2+9);
>> m = 8;
>> xn = 4;
>> x0 = 0;
>> Simpson(f,m,x0,xn)

ans =

   2.000002853916421
```

error $= 2.8539164 \times 10^{-6}$

```
>> f = @(x) x^3/(x^2+1);
>> m = 8;
>> xn = 1;
>> x0 = 0;
>> Trapezoid(f,m,x0,xn)

ans =

   0.154731051562777
```

error $= 0.00130464184275$

```
>> f = @(x) x^3/(x^2+1);
>> m = 8;
>> xn = 1;
>> x0 = 0;
>> Simpson(f,m,x0,xn)

ans =

   0.153425769127771
```

error $= 6.4059225 \times 10^{-7}$