Stochastic Methods for Finance: Report 5

Leonardo Schiavo

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Abstract

We want to study the parametric, historical or simulated Value at Risk through real data.

1. Introduction

We chose two assets, Amazon and Alphabet, we downloaded the historical series for 6 months and worked with it to calculate the V.a.R. of an equibalanced portfolio with different methods: the parametric method, the historical method and the Monte Carlo simulation method. We compared the results and formulated some thoughts on these methods.

2. Companies: Amazon.com, Inc. and Alphabet Inc.

Amazon.com, Inc. engages in the retail sale of consumer products and subscriptions in North America and internationally. The company operates through three segments: North America, International, and Amazon Web Services (AWS). It sells merchandise and content purchased for resale from third-party sellers through physical and online stores. The company also manufactures and sells electronic devices, including Kindle, Fire tablets, Fire TVs, Rings, and Echo and other devices; provides Kindle Direct Publishing, an online service that allows independent authors and publishers to make their books available in the Kindle Store; and develops and produces media content. In addition, it offers programs that enable sellers to sell their products on its websites, as well as its stores; and programs that allow authors, musicians, filmmakers, Twitch streamers, skill and app developers, and others to publish and sell content. Further, the company provides compute, storage, database, analytics, machine learning, and other services, as well as fulfillment, advertising, publishing, and digital content subscriptions. Additionally, it offers Amazon Prime, a membership program, which provides free shipping of various items; access to streaming of movies and series; and other services. The company serves consumers, sellers, developers, enterprises, and content creators. Amazon.com, Inc. was incorporated in 1994 and is headquartered in Seattle, Washington.

Previous Close	2,891.93
Open	2,596.98
Bid	2,474.98 x 1300
Ask	2,475.90 x 800
Volume	13,479,948
Avg. Volume	3,857,733
Market Cap	1.471T
Ex-Dividend Date	N/A

Alphabet Inc. provides various products and platforms in the United States, Europe, the Middle East, Africa, the Asia-Pacific, Canada, and Latin America. It operates through Google Services, Google Cloud, and Other Bets segments. The Google Services segment offers products and services, including ads, Android, Chrome, hardware, Gmail, Google Drive, Google Maps, Google Photos, Google Play, Search, and YouTube. It is also involved in the sale of apps and in-app purchases and digital content in the Google Play store; and Fitbit wearable devices, Google Nest home products, Pixel phones, and other devices, as well as in the provision of YouTube non-advertising services. The Google Cloud segment offers infrastructure, platform, and other services; Google Workspace that include cloud-based collaboration tools for enterprises, such as Gmail, Docs, Drive, Calendar, and Meet; and other services for enterprise customers. The Other Bets segment sells health technology and internet services. The company was founded in 1998 and is headquartered in Mountain View, California.

Previous Close	2,388.23
Open	2,351.56
Bid	2,316.01 x 1000
Ask	2,309.00 x 900
Volume	1,574,343
Avg. Volume	1,562,623
Market Cap	1.508T
Ex-Dividend Date	N/A

3. Construction of an Equibalanced Portfolio

We started working on the data we gathered from Yahoo Finance. Our data are the two historical series of Amazon and Alphabet, 6 months interval from October 25, 2021 to April 22, 2022. By the Adjusted Close we calculated the returns of each asset.

$$R_i = \frac{V_i - V_{i-1}}{V_{i-1}}$$

We decided to invest $100,000\$$, half in the first asset and half in the second. Starting from October 25, 2021 we invested $50,000\$$ in Amazon and the same amount in Alphabet. Let R_i^A and R_i^G be respectively the returns of Amazon and Alphabet (Google) at the i-th day, clearly they are different, then after every day each share of an asset has a countervalue different from the other. To create an equibalanced portfolio we need to move some wealth from the asset with more value to the other one. We can think equivalently to divest all and reinvest half and half to balance the portfolio. In this view the formula we used is

$$V_i = V_{i-1}\frac{2 + R_i^A + R_i^G}{2}$$

Now we can compute the returns of the portfolio as a whole and give some statistics on it.

Initial investment	$100,000.00$
Last countervalue	$87,100.54$
Mean value	$99,190.42$
St dev portfolio	$5,765.52$
Daily Volatility	2.022%
Annual Volatility	32.090%

4. Mathematical definition of Value at Risk and Expected Shortfall

Let X be a profit and loss distribution, the V.a.R. at level $\alpha \in (0,1)$ is the smallest number y such that the probability that $Y := -X$ does not exceed y is at least $1-\alpha$:

$$\begin{aligned} VaR_\alpha(X) &= -\inf\{x \in \mathbb{R} : F_X(x) > \alpha\} \\ &= F_Y^{-1}(1-\alpha) \end{aligned}$$

where F_X is the cumulative distribution function of the random variable X.

The Expected Shortfall at q% level is the expected return on the portfolio in the worst q% of cases. E.S. is an alternative to Value at Risk that is more sensitive to the shape of the tail of the loss distribution. Now if $X \in L^p(\mathcal{F})$ (an Lp space) is the payoff of a portfolio at some future time and $\alpha \in (0,1)$ then we define the Expected Shortfall as

$$ES_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma \qquad (1)$$

The real problem one encounter when he tries to compute the V.a.R. or E.S. is to estimate the distribution of the returns or the losses: in the following sections we show some methods to overcome this issue.

5. Parametric Value at Risk and Expected Shortfall

The parametric method looks at the price movements of investments over a past period and uses probability theory to compute a portfolio's maximum loss. The variance-covariance method for the value at risk calculates the standard deviation of price movements of an investment. Assuming stock price returns follow a normal distribution, the maximum loss within the specified confidence level is calculated. The V.a.R. in percentage is

$$VaR = q_N(p)\sigma_d\sqrt{T}$$

where q_N is the quantile function of the standard normal distribution, $p \in [0,1]$ is the probability took into account and T is the time period. To compute the value in money we need to multiply the percentage with the value of the portfolio today. We can either compute the V.a.R. by taking the joint portfolio or considering the two assets separately. In the first case it is easier since we can just calculate it directly by taking the standard deviation of the returns of the whole portfolio. In the second case we need the covariance matrix, which is

$$\text{Cov} = 10^{-3} \begin{pmatrix} 0.641 & 0.304 \\ 0.304 & 0.385 \end{pmatrix}$$

and the portfolio daily volatility is the square root of the bilinear form C calculated on the vector of the distribution of the wealth. Since we have chosen an equibalanced portfolio we have $\alpha = (0.5, 0.5)^T$ get

$$\sigma_d = \sqrt{\alpha^T C \alpha}$$

This was not just an overview on both methods but they are perfectly equivalent and actually we got the same result. For one day the 99% V.a.R. is 4.70% or 4096.09. By the Definition 1 the Expected Shortfall of level p is calculated by the formula

$$ES_p(X) = \frac{\sigma_d}{\sqrt{2\pi}(1-p)} e^{\frac{-q_N(p)^2}{2}}$$

and the result for 99% and 1 day is 12.07%, which corresponds to 1, 051, 436.98. The plots over a time period of $T = 1, \dots, 80$ and V.a.R. or E.S. of 95%, 99% and 99.5% are following. The vertical axis is a percentage.

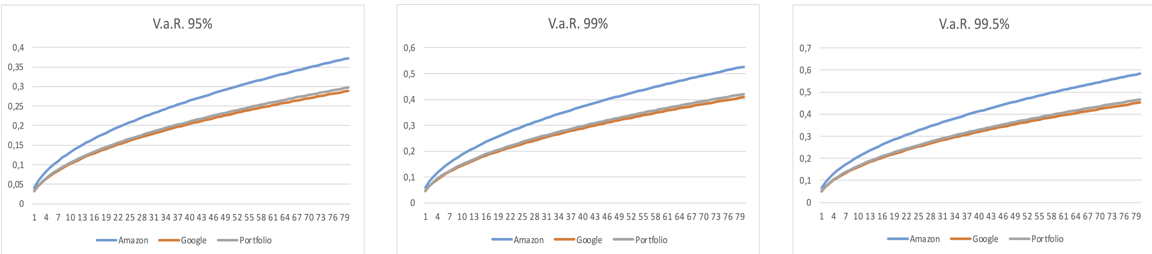


Figure 1: Value at Risk

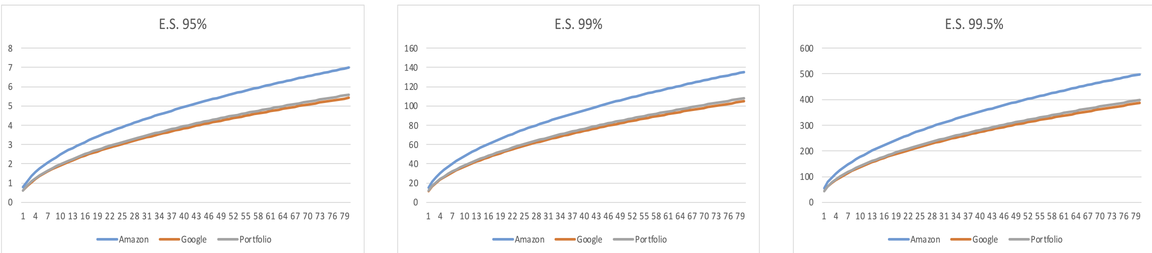


Figure 2: Estimated Shortfall

6. Exponentially Weighted Moving Average (E.W.M.A.)

The weakness of the previous approach is that all returns earn the same weight. The more recent return has no more influence on the variance than last month's return. This problem is fixed by using the Exponentially Weighted Moving Average (E.W.M.A.), in which more recent returns have greater weight on the variance. The E.W.M.A. introduces the smoothing parameter $0 < \lambda < 1$. Under that condition, instead of equal weights, each squared return is weighted by a multiplier as the sequence $1-\lambda$, $(1-\lambda)\lambda$, $(1-\lambda)\lambda^2$,... which sum up to 1 by a geometric series. This approach gives us the formula to compute the variance of the portfolio recursively:

$$\sigma_n^2(EWMA) = \lambda\sigma_{n-1}^2 + (1-\lambda)R_{n-1}^2$$

where the first variance is computed directly on the returns. With the choice of $\lambda = 0.94$ we have

p	95%	99%	99.5%
1 Day VaR	4176.15	5906.40	6539.82
1 Day % VaR	4.79%	6.78%	7.51%
1 Day ES	7112.94	16534.25	23499.29
1 Day % ES	8.17%	18.98%	26.98%

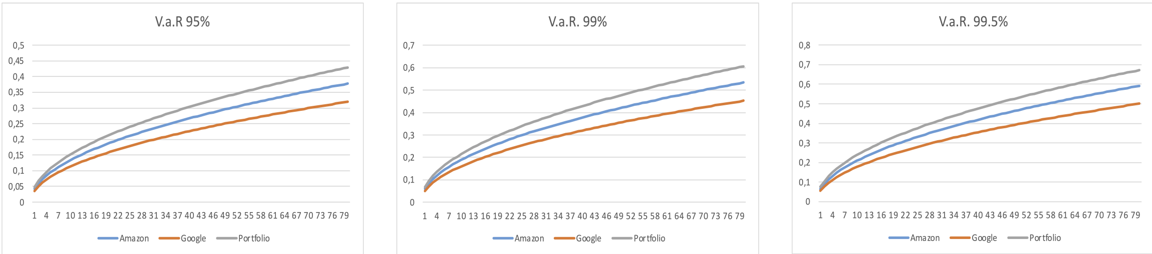


Figure 3: E.W.M.A. - Value at Risk

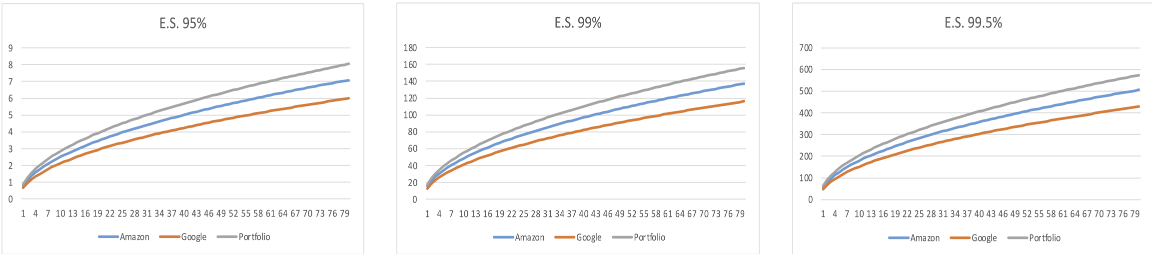


Figure 4: E.W.M.A. - Estimated Shortfall

7. Monte Carlo Method

Now we concentrate to the V.a.R. and we want to use Monte Carlo methods to compute the V.a.R. with N simulations T as free inputs at the levels 95%, 99% and 99.5%. The essence of a Monte Carlo method is the random sampling, in our case to resemble the randomness of the real world. Note that in this paragraph we used the logarithmic return instead of the usual simple one. They are not equal but equivalent in meaning and one can easily check that the multiperiod simple return is the product of simple daily returns of the period and for the logarithmic counterpart the multiperiod return is the sum of the daily returns. In practice to implement the Monte Carlo simulation we created many seed (max was $i = 10000$) values as follows

$$S_i = e^{(r-\frac{1}{2}\sigma_y^2)\frac{T}{252}+\sigma_y^2\sqrt{\frac{T}{252}}q_N(*)}$$

where in the quantile the $*$ is a random number sampled by an uniform distribution over $[0, 1]$ and $r = 0$. After 10000 iteration we got

p	95%	99%	99.5%
1D VaR Am	1772.50	2473.66	2704.47
1D %VaR Am	4.07%	5.68%	6.21%
1D VaR Ggl	1389.25	1955.41	2190.58
1D %VaR Ggl	3.19%	4.49%	5.03%
1D VaR Pf	4172.12	5731.22	5731.22
1D %VaR Pf	4.79%	6.58%	6.58%

and the plots are following.

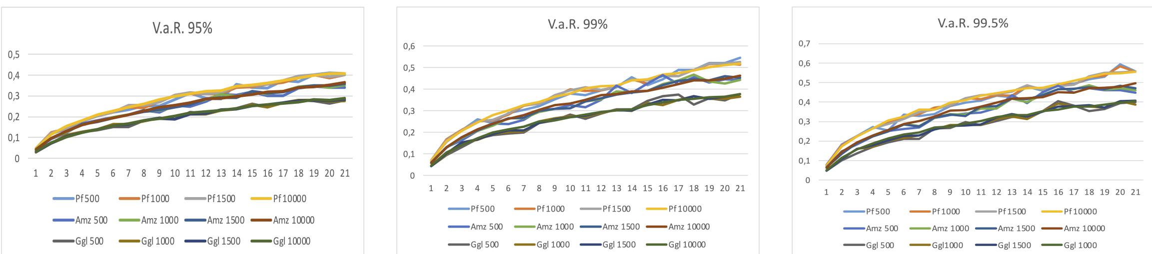


Figure 5: Monte Carlo Value at Risk

8. Historical Var

The Historical V.a.R. is the Value at Risk calculated using the return over a longer period of time, i.e. we define the multiperiod simple return over a ΔT period of time as follows

$$R_i^{\Delta T} = \frac{V_i - V_{i-\Delta T}}{V_{i-\Delta T}}$$

We can now calculate the standard deviation of the vector of the returns obtaining the daily volatility. We chose intervals of 1 month and 3 months. Note that if $\Delta T = 1$ day this method coincides with the Historical Simulation, which is explained in the next section. Again recalling that we defined the (percentage) one day p-V.a.R. as the product of daily volatility and the quantile on level p of the standard normal distribution we obtained:

p	95%	99%	99.5%
Amazon 1M	13.09%	18.52%	20.51%
Google 1M	8.88%	12.50%	13.84%
Portfolio 1M	10.54%	14.91%	16.51%
Amazon 3M	10.19%	14.42%	15.97%
Google 3M	6.01%	8.50%	9.41%
Portfolio 3M	7.65%	10.83%	11.99%

The following graphs show how the Historical V.a.R. evolves in time by the square root of it:

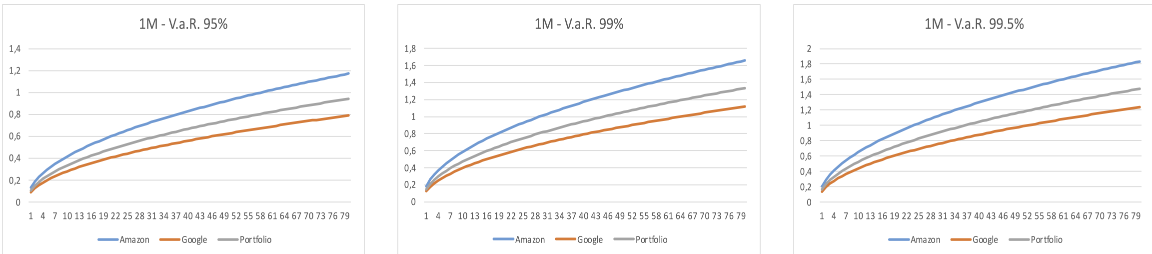


Figure 6: Historical V.a.R. - 1 Month

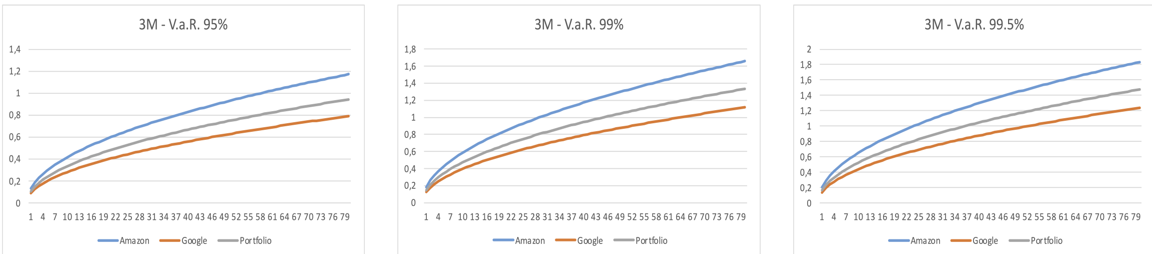


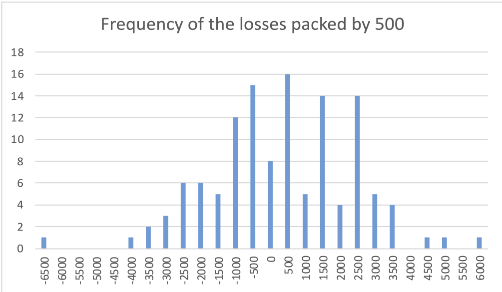
Figure 7: Historical V.a.R. - 3 Months

9. Historical Simulation

The historical simulation is a method of simulating the future based on the past behaviour of the market: we take a period of time in the past and create as many scenarios as many days we have taken. Every scenario represent a possibility of the behaviour of the market for tomorrow. We created the possible scenarios by

$$\text{Value under i-th scenario } = V_n \frac{V_i}{V_{i-1}}$$

Where the n-th value is the portfolio value of today. After this rearrangement we created the equibalanced portfolio and reordered the losses. Looking at the worst losses we see that the 70-th scenario was the worst of all with a loss of 5725.53, this corresponds to a V.a.R. of 99.5%; the second worst was the 91-st scenario with a loss of 4952.04 that corresponds to a V.a.R. of 99% and the sixth worst loss was of 3408.98 corresponding to a V.a.R. of 95%. We can see that the losses follows a sort of normal distribution with mean 91.05, standard deviation of 2021.50, skewness of –0.130 and kurtosis of 0.565:



10. Subadditivity of the Value at Risk

The primitive idea that investing in many different assets reduces the total risk is a fascinating thing. We would like that the Value at Risk in some ways tells us if this is indeed true or if it is not. Unfortunately it is not always true: if the V.a.R. is subadditive, i.e. VaR(X+Y) ≤ V(X)+V(Y), then investing in different assets actually reduces the risk; otherwise if $VaR(X+Y) > V(X)+V(Y)$ investing in different assets increases the risk measured by the V.a.R. We can measure how much is subadditive the V.a.R. by the quantity

$$S = 1 - \frac{V(X+Y)}{V(X)+V(Y)}$$

since if it is positive the V.a.R. is subadditive, if it is negative the V.a.R. is not subadditive. The more positive S is, the better, because we can diversify the portfolio gaining a less risky position. The following table shows the value of S for the different methods used to compute the V.a.R. In the case of the Historical method we displayed here an average over all the outcomes of S and in the case of the Monte Carlo we took the one relative to $N = 10000$; in all other cases S was unique.

Methods	Subadditivity (S)
Parametric	0.55
E.W.M.A.	0.39
Monte Carlo	0.34
Historical	0.52

We can notice that in all cases the quantity S is quite positive and we can deduce that in the case of the two assets we chose it is preferable to diversify the portfolio.

11. Conclusions

We saw the definitions of Value at Risk and some other near things. We calculated the V.a.R. by different methods but some questions now arise. The first is how our result is asset dependent, in fact we were lucky to get the subadditivity but it is not always the case. The second question is which is the best method to estimate the V.a.R.? Probably we should take more methods and different assets and for every combination try to compute the V.a.R. pretending to be in one day of the past and in a back looking way see how wrong is the prediction of every model. Moreover we can try to take a interval of time in the past and shift it day by day instead of using all the 6 months but for this a larger period of time as some year is needed. The third question is if 10000 iteration for the Monte Carlo method are enough, indeed probably 1 Million iteration would be more safe. The forth questions is the real distributions of the returns. Prices usually have a drift, they are not stationary but returns are a stationary process and there is an autocorrelation effect visible on the squared returns. Furthermore the distribution of the returns tends to be leptokurtic, i.e. not gaussian with fatter tails. Moreover negative returns tend to increase volatility by a larger amount than positive returns, this leads to an asymmetry of the distribution. An improvement is done by the GARCH(1,1) model to represent better the distribution of the returns. In this idea Monte Carlo method and GARCH(1,1) are probably the best model to estimate the V.a.R. The parameters of the GARCH(1,1) can be estimated by likelyhood methods but on many different multi asset portfolio they can be fitted to empirical data through Machine Learning algorithms such as Neural Networks.

References

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"Options, Futures, and Other Derivatives", John C. Hull, 11th Edition, Pearson 2022.