# Stochastic Methods for Finance: Report 3

Leonardo Schiavo

March, 2022

**Abstract**

*We want to study the binomial model convergence to the Black Scholes formula and the implementation of Leisen Reimer (1996).*

## 1. Introduction

We chose the call option and studied how in case with no dividends the formula for the call option by the binomial model converges to the formula of the Black Scholes model. This convergence is slow and a huge improvement was done by Leisen Reimer in 1996.

## 2. Binomial versus Black Scholes models

In the binomial model the formula pricing of a call option is

$$C_{Bin} = \frac{1}{(1+r)^n} \sum_{j=0}^{n} \binom{j}{n} q^j (1-q)^{n-j} \left(S_0 u^j d^{n-j} - X\right)^+$$

where $S_0$ is the initial stock price, X is the exercise price, r is the risk free rate, q is the market probability measure, u and d are the movements of the asset in the binomial model and n is the number of steps. In out study of the binomial model we implemented a VBA code and introduced the time of expiration in days, T, the instantaneous variance of the asset returns, $\sigma^2$. We let the number of periods vary and fixed $S = 100$, $X = 100$, $t = 1$, $r = 0.01$, $\sigma = 0.2$.

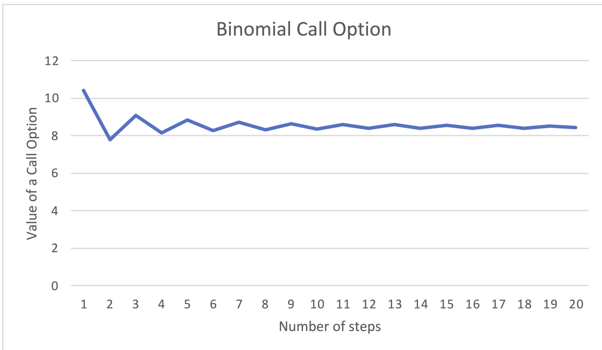
On the other hand by the Black Scholes model the formula pricing a call option is

$$d_1 = \frac{\ln\left[\frac{S_t}{X}\right] + \left(r + \frac{1}{2}\sigma^2\right) t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma_{\sqrt{t}}$$

$$C_{BS} = SN(d_1) - Xe^{-rt}N(d_2)$$

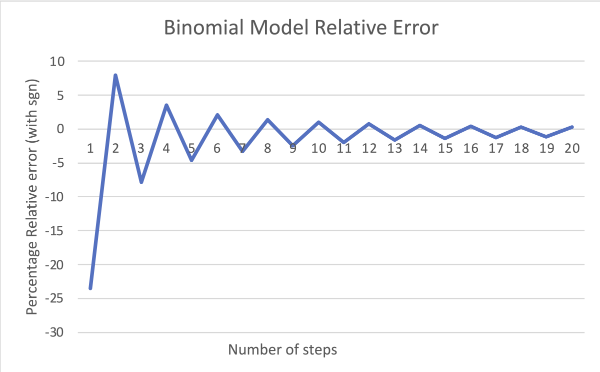
where N stands for the cumulative normal distribution. The Black Scholes formula with the parameters we fixed before gives $C_{BS} = 8.43332$. Now we can see how the binomial models converges to the Black Scholes: the following graph represent the values of a call option with increasing number of steps n.




We can see the convergence is confirmed. Since we are not studying a theoretical way to proof the convergence we are satisfied with an empirical convergence. We computed the relative error (percentage with sign)

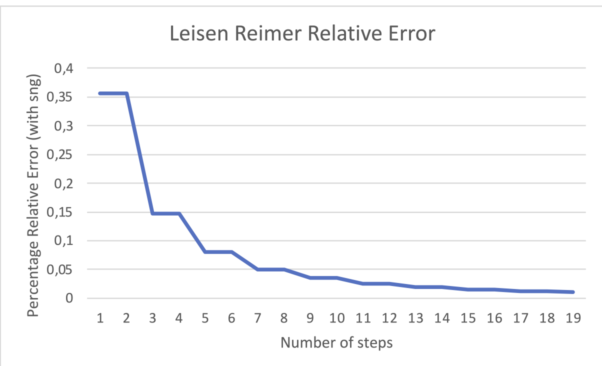
$$E_{rel} = 100 * \frac{C_{Bin} - C_{BS}}{C_{BS}}$$

and saw how it approaches zero:



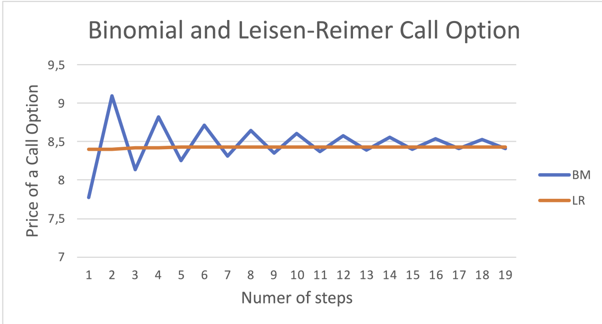

## 3. Leisen Reimer Model

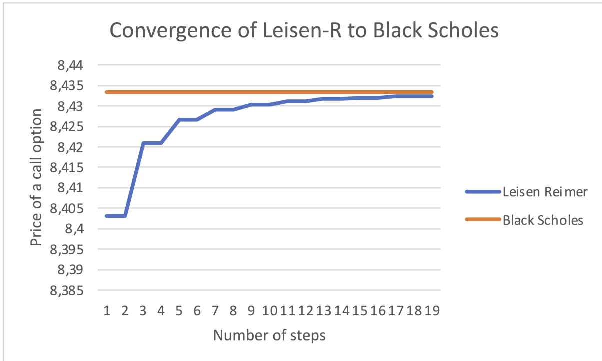
The question now is if it is possible to speed up the convergence. This is possible thanks to the Leisen Reimer method developed in 1996. The goal here is to achieve maximum precision with a minimum number of time steps. Leisen and Reimer developed a method in which the parameters u, d and q of the binomial tree can be altered in order to get better convergence behavior. Instead of choosing the parameters p, u and d to get convergence to the normal distribution Leisen-Reimer suggest to use inversion formulae reverting the standard method – they use normal approximations to determine the binomial distribution B(n, p). In particular, they suggest the following three inversion formulae: Camp-Paulson-Inversion formula, Peizer-Pratt-Inversion formula 1 and 2. We used a VBA implementation optimized to cut the zero region of the Leisen-Reimer tree. This truncation is quite important for large n. The Leisen-Reimer model gives us quite an astonishing convergence considering that since $n = 4$ we have a relative error of 0.14% with respect of Black Scholes.




## 4. Conclusions

If we want to compare the convergence rate of the binomial model with the Leisen-Reimer method we plot them together and the results are enought self speaking.




## References

"Binomial Models for Option Valuation—Examining and Improving Convergence", Leisen D., March 1995, Applied Mathematical Finance 3(4):319-346.

"Arbitrage Theory in Continuous Time", Tomas Björk, 4th Edition, Oxford Univeristy Press 2020.

"Options, Futures, and Other Derivatives", John C. Hull, 11th Edition, Pearson 2022.