Homework 4 HDP

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Exercise 1. Let A be a subset of the sphere $\sqrt{n}S^{n-1}$ such that

$$\sigma(A) > 2\exp(-cs^2) \quad \text{for some } s > 0. \tag{0.1}$$

We want to prove that

$$\sigma(A_s) > \frac{1}{2} \tag{0.2}$$

and deduce that for any $t \geq s$

$$\sigma(A_{2t}) \geq 1 - 2\exp(-ct^2). \tag{0.3}$$

We suppose, by contradiction, that $\sigma(A_s) \leq \frac{1}{2}$. We let $B \coloneqq A_s^c$, so that $\sigma(B) = 1 - \sigma(A_s) \geq \frac{1}{2}$. Hence, we can apply the blow-up lemma to this set B and obtain that

$$\sigma(B_s) \geq 1 - 2\exp(-cs^2). \tag{0.4}$$

Now, for any $x \in B_s$, there exists, by definition, a point $y \in B$ such that $\|x - y\|_2 \leq s$, and since $y \in B \coloneqq A_s^c$, it must be that $\|z - y\|_2 > s$ for any $z \in A$. In particular x cannot be a point in A, i.e. $B_s \subseteq A^c$. It follows that $\sigma(A^c) \geq \sigma(B_s) \geq 1 - 2\exp(-cs^2)$, and hence

$$\sigma(A) = 1 - \sigma(A^c) \leq 2\exp(-cs^2). \tag{0.5}$$

This contradicts our hypothesis, so we conclude that it had to be $\sigma(A_s) > \frac{1}{2}$, as we wanted.

Given $t \geq s$, we have by the previous part that $\sigma(A_t) \geq \sigma(A_s) \geq \frac{1}{2}$. Thus, we can apply the blow-up lemma and conclude that

$$\sigma((A_t)_t) \geq 1 - 2\exp(-ct^2). \tag{0.6}$$

Now, for any $x \in (A_t)_t$, there exists, by definition, a point $y \in A_t$ such that $\|x - y\|_2 \leq t$, and a point $z \in A$ such that $\|z - y\|_2 \leq t$. But then $\|x - z\|_2 \leq \|x - y\|_2 + \|z - y\|_2 \leq 2t$, meaning that $x \in A_{2t}$, and since x was arbitrary, $(A_t)_t \subseteq A_{2t}$. It follows that

$$\sigma(A_{2t}) \geq \sigma((A_t)_t) \geq 1 - 2\exp(-ct^2), \tag{0.7}$$

as we wanted.

Exercise 2. We consider a random vector X taking values in $(\mathbb{R}^n, \|\cdot\|_2)$ and we assume that there exists a $K > 0$ such that

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq K\|f\|_{Lip} \tag{0.8}$$

for every $f\colon \mathbb{R}^n \to \mathbb{R}$, Lipschitz function. Letting σ be a probability measure on $\mathbb{R}^n$, we want to show that if $\sigma(A) \geq \frac{1}{2}$ then for every $t \geq 0$

$$\sigma(A_t) \geq 1 - 2\exp(-ct^2/K^2) \tag{0.9}$$

where $c > 0$ is a constant.

By exercise 5.1.13 (and our assumption 0.8) we have that

$$\|f(X) - M\|_{\psi_2} \leq C\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq CK\|f\|_{Lip} \tag{0.10}$$

where M is the median of $f(X)$, i.e. $\mathbb{P}(f(X) \leq M) \geq \frac{1}{2}$ and $\mathbb{P}(f(X) \geq M) \geq \frac{1}{2}$.

Now, we let $f(x) \coloneqq dist(x, A)$ and notice that f, such defined, is a Lipschitz function with $\|f\|_{Lip} = 1$ and $M = 0$. Indeed, $\mathbb{P}(f(X) \leq 0) \geq \mathbb{P}(X \in A) \geq \frac{1}{2}$ and, in the same way, $\mathbb{P}(f(X) \geq 0) \geq \mathbb{P}(X \in A) \geq \frac{1}{2}$.

We know that the bound on 0.10 (with the function f previously defined and with $M = 0$, $\|f\|_{Lip} = 1$ and $c = \frac{1}{C^2}$) is equivalent to the fact that

$$\mathbb{P}(dist(X, A) \geq t) \leq 2\exp(-ct^2/K^2). \tag{0.11}$$

Since, clearly, $dist(X,A) < t \implies X \in A_t$, we get that

$$\sigma(A_t) = \mathbb{P}(X \in A_t) \geq \mathbb{P}(dist(X,A) < t) = 1 - \mathbb{P}(dist(X,A) \geq t) \geq 1 - 2\exp(-ct^2/K^2), \tag{0.12}$$

and this concludes the proof.

Exercise 3. Let D be an $k \times m$ matrix and B an $m \times n$ matrix. We want to show that

$$||DB||_F \leq ||D||||B||_F. \tag{0.13}$$

We first notice that the square of the Frobenius norm of a matrix is the sum of the squares of the 2-norms of its row-vectors or columns-vectors. This is clear from the definition of Frobenius norm, and it implies that, if we denote by B_i the i-th column of the matrix B, then

$$\begin{aligned} ||DB||_F^2 &= \sum_{i=1}^{n} ||DB_i||_2^2 \\ &\leq \sum_{i=1}^{n} ||D||^2 ||B_i||_2^2 \\ &= ||D||^2 \sum_{i=1}^{n} ||B_i||^2 \\ &= ||D||^2 ||B||_F^2. \end{aligned}$$

Taking the square root on both sides we obtain the thesis.

Exercise 4. We consider i.i.d. random variables $\delta_{ij} \sim Ber(p)$ where $i, j = 1, \ldots, n$. Assuming that $pn \geq \log n$, we show that

$$\mathbb{E} \max_i \sum_j (\delta_{ij} - p)^2 \leq Cpn. \tag{0.14}$$

Now, using Jensen's inequality, the fact that the δ_{ij}'s are i.i.d. and other basic stuff,

$$\begin{aligned} \exp(\mathbb{E} \max_i \sum_j (\delta_{ij} - p)^2) &\leq \mathbb{E} \exp(\max_i \sum_j (\delta_{ij} - p)^2) \\ &= \mathbb{E} \max_i \exp(\sum_j (\delta_{ij} - p)^2) \\ &\leq \mathbb{E} \sum_i \exp(\sum_j (\delta_{ij} - p)^2) \\ &= n\mathbb{E} \exp(\sum_j (\delta_{1j} - p)^2) \\ &= n(\mathbb{E} \exp((\delta_{11} - p)^2))^n \\ &= n[p\exp((1-p)^2) + (1-p)\exp(p^2)]^n. \end{aligned} \tag{0.15}$$

Let us now prove that

$$pe^{(1-p)^2} + (1-p)e^{p^2} \leq e^{2p} \tag{0.16}$$

for any $p \in [0,1]$. First, we notice that if $p = 0$, then both sides are equal to one. Then, we notice that the derivative (over p) of the right-hand side is $2e^{2p}$ and it is grater or equal than 2 for any $p \in [0,1]$. On the other hand, the derivative (over p) of the left-hand side is

$$e^{(1-p)^2} - 2p(1-p)e^{(1-p)^2} - e^{p^2} + 2p(1-p)e^{p^2}$$

and it is always less than the derivative of the right-hand side:

$$e^{(1-p)^2} - 2p(1-p)e^{(1-p)^2} - e^{p^2} + 2p(1-p)e^{p^2} = [1 - 2p(1-p)][e^{(1-p)^2} - e^{p^2}] \leq e - 1 < 2,$$

where we used that $e^{(1-p)^2} \leq e$ and $e^{p^2} \geq 1$ (since $p \in [0,1]$). This is enough to conclude that the inequality in 0.16 holds.

Combining inequality 0.15 with the one just proved (that is inequality 0.16), we get that

$$\exp(\mathbb{E}\max_i \sum_j (\delta_{ij} - p)^2) \le n[p\exp((1-p)^2) + (1-p)\exp(p^2)]^n \le n\exp(2pn), \tag{0.17}$$

which is clearly equivalent to

$$\mathbb{E}\max_i \sum_j (\delta_{ij} - p)^2) \le \log(n\exp(2pn)) = 2pn + \log n. \tag{0.18}$$

By our assumption that $pn \ge \log n$, we are done (with $C = 3$).