Stochastic Methods for Finance: Report 6

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Abstract

We want to build an option pricer based on the Monte Carlo simulation method.

1 Introduction

In the first section we see some simulations of the Geometric Brownian motion and then we try to implement first a pricer for vanilla options and then for exotic (path dependent) options.

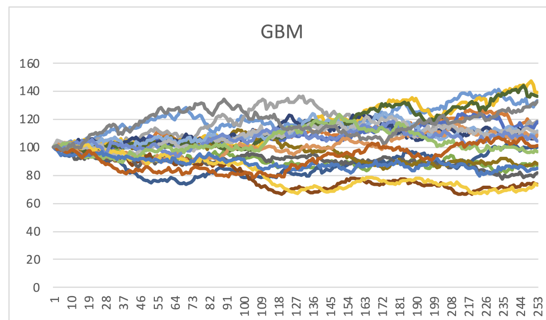

Figure 1: 20 simulations of GBM

2 Simulation of Geometric Brownian motion

A stochastic process S_t is said to follow a Geometric Brownian motion if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Wiener process or Brownian motion, and μ, the percentage drift, and σ, the percentage volatility, are constants. For an arbitrary initial value S_0 the above SDE has the analytic solution, which we used in the simulation:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

For the simulation we fixed some generic parameters for the Black Scholes market model: stock price $S_0 = 100$, volatility $\sigma = 20\%$, risk free interest rate $r = 1\%$, time to maturity $T = 1$ year, strike price $K = 99$, time step for the discretization $dt = 1$ day. In Figure 1 we plot an example of 20 possible evolutions of the stock price. Note that we decided to plot only 20 trajectories to have a better graphic but in the following computation we reach even 10000 trajectories or more.

3 Monte Carlo pricer for Vanilla options

We recall that European (or Vanilla) options are financial instruments that give the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price within a given timeframe. The call and put payoffs are

$$\text{payoff}_T(call) = (S - K)^+$$
$$\text{payoff}_T(put) = (K - S)^+$$

We wanted to price call and put options through Monte Carlo method by 1 step simulation. We took the mean of all the possible discounted payoffs (we took 1000 iterations) and the results are $price(call) = 8.96$ and $price(put) = 7.32$. We noticed that they are quite unstable and they change a lot if we repeat the simulation, this is a sign that many more iterations are needed. We can now compare those results with the Black Scholes previsions: by the BS formula we have that $price_{BS}(call) = 8.92$ and $price_{BS}(put) = 6.93$. Those results as quite good and near the one suggested by the Monte Carlo method.

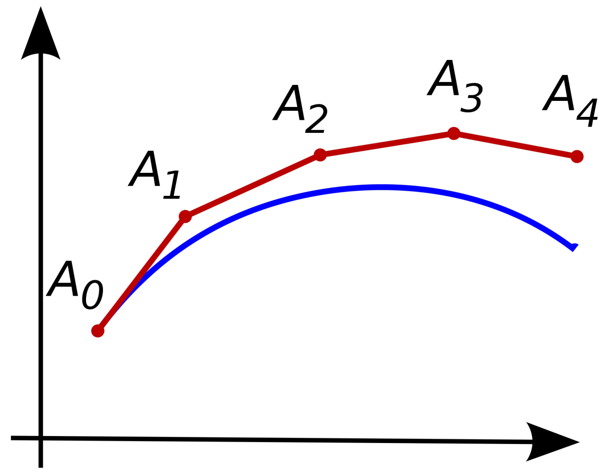

Figure 2: Illustration of the Euler method. The unknown curve is in blue, and its polygonal approximation is in red.

4 Monte Carlo pricer for Vanilla options with Euler method

The Euler method is used to solve ordinary differential equations with a given initial value. It is the most basic explicit method for numerical integration of ordinary differential equations. The Euler method is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size. The principle of the Euler method is the creation of discretized curve of segments, which approximate the original solution curve: given a starting point, by the ODE one calculate the slope of the curve and draw the first segment, then moving to the next point $t+dt$ one can find the slope of the second segment and so on. This procedures creates a broken line graph which approximates the original solution curve as the dt approaches 0. We can see an example in Figure 2.

In this simulation we decided to use a multi step Euler Scheme, so we took 252 steps and 1000 simulations. The discounted results are a payoff of 9.35 for the call option and a payoff of 7.25 for the put option.

5 Price of Exotic options with Monte Carlo method

In this section we give a price to two type of exotic options: the Asian (A) option with fixed strike price K and the Lookback (LB) option with floating strike price. The payoffs of the

Asian options are

$$\text{payoff}_T^A(call) = \left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+$$
$$\text{payoff}_T^A(put) = \left(K - \frac{1}{T}\int_0^T S_t\,dt\right)^+$$

The payoffs of the Lookback options are

$$\text{payoff}_T^{LB}(call) = S_T - \min_{t\leq T} S_t$$
$$\text{payoff}_T^{LB}(put) = \max_{t\leq T} S_t - S_T$$

Applying the Monte Carlo method we got the following results

Option	Price
Asian Call	5.86
Asian Put	4.00
Lookback Call	15.32
Lookback Put	16.88

6 Consistency of the results

In this section we analyze the level of confidence we can give to the results we found until now. First we can compare the price of an European option with the price given by the Black Scholes formula. The Figures 3, 4, 5 and 6 show the convergence of the MC method to the BS model as the number of iterations increases.

We can see a convergence pattern, the only doubt is in the put option where the convergence may need more iterations than what we were able to perform with the computational power we had (or the presence of bugs in the code). Now we analyze the reliability of our results by an empirical point of view: we expect that if the volatility increases or the strike price decreases then the price of the option grows (decreases in the case

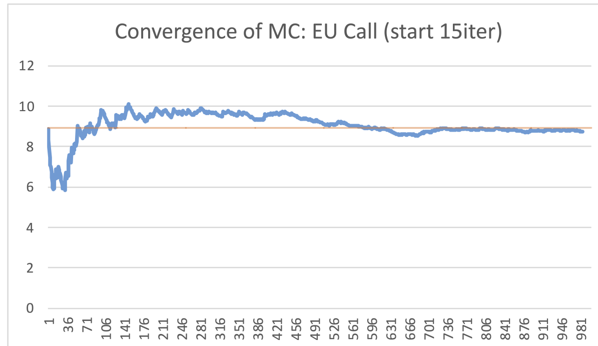

Figure 3: Convergence of the MC to BS for a Call option starting from 15 iterations

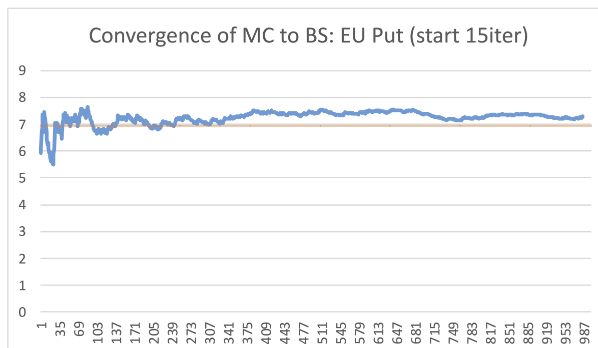

Figure 4: Convergence of the MC to BS for a Put option starting from 15 iterations

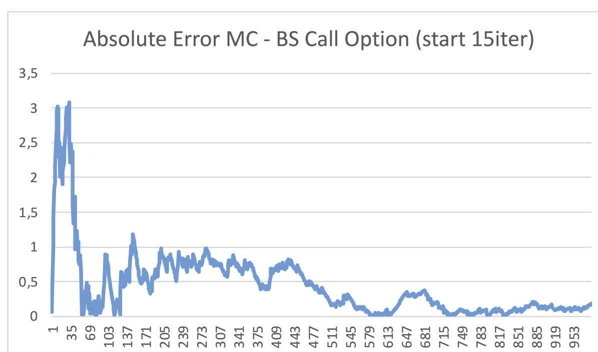

Figure 5: Absolute Error MC - EU Call option

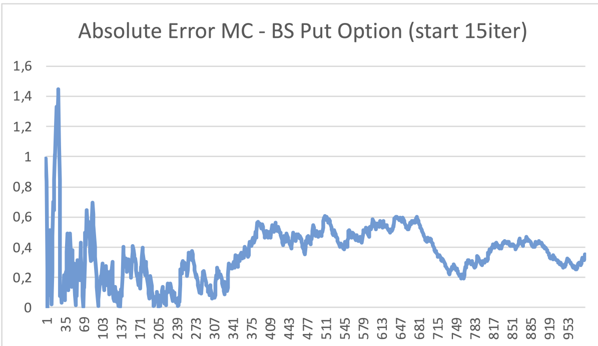

Figure 6: Absolute Error MC - EU Put option

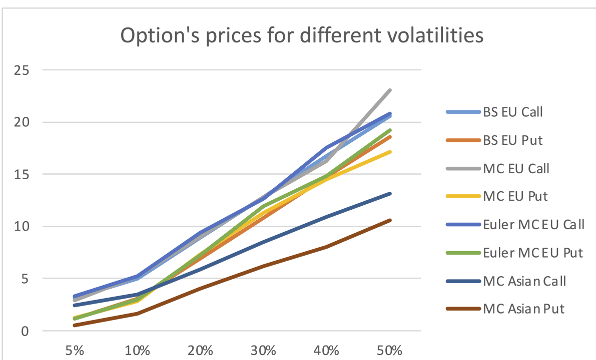

Figure 7: Shock of volatility

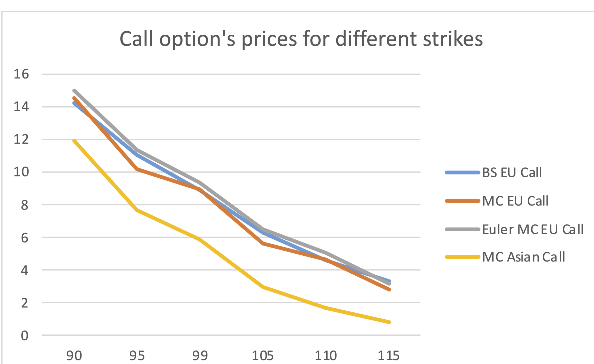

Figure 8: Changing strike price for a Call option

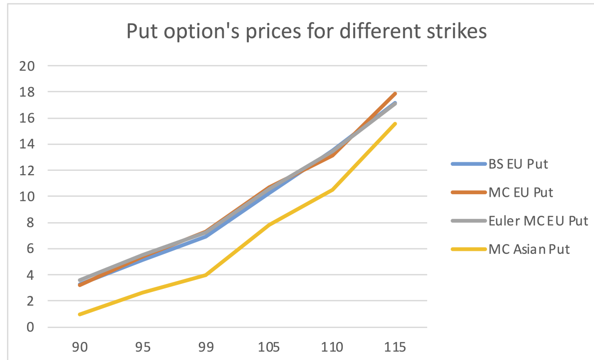

Figure 9: Changing strike price for a Put option

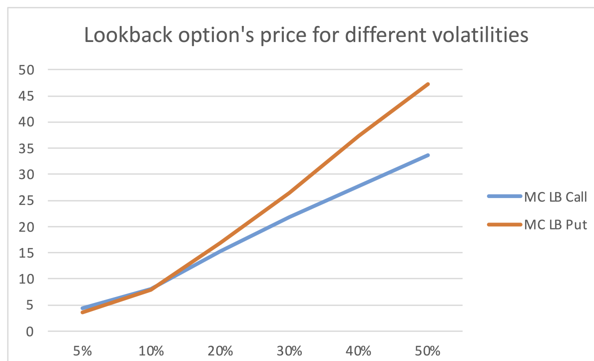

Figure 10: Shock of volatility on Lookback option

of the put option) and vice-versa. This event is verified in by graphs 7, 8 and 9.

For the Loockback option the situation with the shock of volatility is the same but the behavior for the variation of the strike prices is different .

7 Appendix: theoretical foundations

In this section we recall some theoretical results we used in this report. The principles of Monte

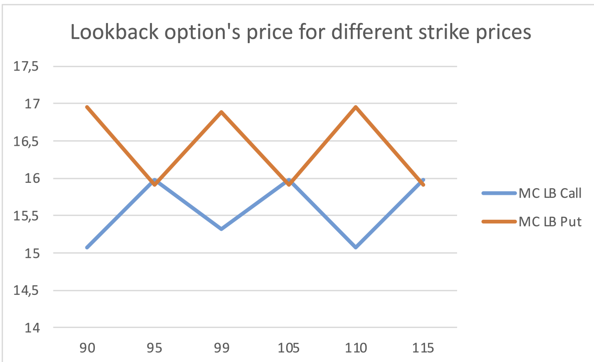

Figure 11: Changing strike price on a Lookback option

Carlo methods based on the Strong Law of Large Numbers.

7.1 Strong Law of Large Numbers

Let $\left(\xi^{(\ell)}, \ell \geq 1\right)$ be a sequence of independent and identically distributed random variables with values in $\mathbb{R}^d$. Assume that $\mathbb{E}\left|\xi^{(1)}\right| < \infty$. For $N \geq 1$, denote the empirical mean of $\left(\xi^{(1)}, \ldots, \xi^{(N)}\right)$ by

$$\hat{S}_N := \frac{1}{N}\sum_{\ell=1}^{N} \xi^{(\ell)}$$

Then, the Strong Law of Large Numbers holds true:

$$\lim_{N\to\infty} \hat{S}_N = \mathbb{E}\left(\xi^{(1)}\right), \ \mathbb{P}\text{ - a.s.}$$

The Strong Law of Large Numbers ensures the convergence of Monte Carlo methods, but does not allow one to describe the accuracy of the method or the number N of simulations in terms of the desired accuracy. We already emphasized that the approximation error is random. The Central Limit Theorem makes precise the limit behavior, when N tends to infinity, of the normalized error probability distribution.

7.2 Central Limit Theorem

Let $\left(\xi^{(\ell)}, \ell \geq 1\right)$ be a sequence of realvalued independent and identically distributed random variables. Assume they are square integrable and set $\sigma^2 := \operatorname{var}\left(\xi^{(1)}\right)$. Consider the random variables

$$Y_N := \frac{\sqrt{N}}{\sigma}\left(\frac{1}{N}\sum_{\ell=1}^{N}\xi^{(\ell)} - \mathbb{E}\left(\xi^{(1)}\right)\right), \qquad N \geq 1$$

The sequence (Y_N) converges in law to the Gaussian law $\mathcal{N}(0,1)$.
Now the question is how many iterations does a Monte Carlo method requires to ensure a certain level of precision in the convergence?

7.3 Absolute Confidence Intervals

Let $\left(\xi^{(\ell)}, \ell \geq 1\right)$ be a sequence of integrable realvalued independent and identically distributed random variables, such that $\mathbb{E}\left(\xi^{(1)}\right)$ is not necessarily null. In order to achieve an accuracy of order ε with a confidence level $1-\delta$ for the Monte Carlo estimation of $\mathbb{E}\left(\xi^{(1)}\right)$, one has to choose a number N of simulations large enough that

$$\mathbb{P}\left(\left|\mathbb{E}\left(\xi^{(1)}\right) - \frac{1}{N}\sum_{\ell=1}^{N}\xi^{(\ell)}\right| \geq \varepsilon\right) \leq \delta$$

Until now every result was already clear but not sufficiently precise. The following Theorem is a little digression from the topic but one can use this result to make sure that the error of the MC method is decreasing at the right rate. First we need the following definition.

7.4 Logarithmic Sobolev inequality

A measure ν on $\mathbb{R}^d$ satisfies the logarithmic Sobolev inequality (LSI) with constant $C > 0$

for the family of function $\mathcal{A}$ if, defined the Entropy of f w.r.t. μ as

$$\mathrm{Ent}_\nu(f) := \mathbb{E}_\nu\left(f\log(f)\right) - \mathbb{E}_\nu\left(f\right)\log\left(\mathbb{E}_\nu\left(f\right)\right),$$

we have that for every function f in $\mathcal{A}$,

$$\mathrm{Ent}_\nu\left(f^2\right) \leq C\mathbb{E}_\nu\left(|\nabla f|^2\right).$$

7.5 Concentration inequalities for the SLLN

Let $\left(X^{(\ell)}, \ell \geq 1\right)$ be a sequence of $\mathbb{R}^d$-valued independent random variables with law μ. Assume that μ satisfies a logarithmic Sobolev inequality with constant C for the family $\mathcal{A}$ of functions with Lipschitz constant smaller than 1. Then, for any f in $\mathcal{A}$ and $r > 0$,

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{\ell=1}^{N} f\left(X^{(\ell)}\right) - \mathbb{E}_\mu(f)\right| \geq r\right) \leq 2\exp\left(-\frac{Nr^2}{C}\right)$$

8 Conclusions

We built a pricer for Vanilla, Asian and Lookback options, the results were quite satisfying but a lot more can be done. Any other pricer can be easily constructed via MC methods as we saw. A huge improvement can be done by switching to Python as the agility of the programming language and the library support would allow to reach probably a Million of iterations. The insertion of the Concentration inequality in the appendix was meant to give an idea of how to study the decreasing error of the put option. That error was quite high and needs a check from a different point of view: the Concentration inequality could be the right mathematical tool to spot the error but more work is needed and not required in this report. As conclusion, for for simplicity of visualization and summary, we give the table of all prices we found (expanding the table shown before)

Option	Price
BS European Call	8.92
BS European Put	6.93
MC European Call	8.96
MC European Put	7.32
Euler-MC European Call	9.35
Euler-MC European Put	7.25
MC Asian Call	5.86
MC Asian Put	4.00
MC Lookback Call	15.32
MC Lookback Put	16.88

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