Homework 3 HDP

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Exercise 1. Considering an $m \times n$ matrix A, we want to prove that

$$\max_{x \in S^{n-1}} \|Ax\|_2 = \max_{\substack{x \in S^{n-1} \\ y \in S^{m-1}}} \langle Ax, y \rangle \tag{0.1}$$

We recall that, in general, the dot product between two non zero vectors is maximized when they are parallel; thus, for any $x \in S^{n-1}$ such that $Ax \neq 0$,

$$\max_{y \in S^{m-1}} \langle Ax, y \rangle = \left\langle Ax, \frac{Ax}{\|Ax\|_2} \right\rangle = \|Ax\|_2$$

(and the equality $\max_{y \in S^{m-1}} \langle Ax, y \rangle = \|Ax\|_2$ trivially holds also when $Ax = 0$). Taking the maximum for x over S^{n-1} on both sides, we have the thesis.

Exercise 2. Let A be a real $m \times n$ matrix with $rank(A) = r$. Let us indicate with $\|A\|_F$ and $\|A\|$ the Frobenius and the operator norm of A respectively. We want to prove that $\|A\|_F^2 = tr(A^T A)$, and deduce that

$$\|A\|_F = (\sum_{i=1}^{r} s_i(A)^2)^{1/2}. \tag{0.2}$$

Finally we want to prove that

$$\|A\| \leq \|A\|_F \leq \sqrt{r}\|A\|. \tag{0.3}$$

We recall that if A is an $m \times n$ real matrix, then $A^T A$ is a square matrix of dimension $n \times n$ and its element in position (i, j) is

$$(A^T A)_{i,j} = \sum_{l=1}^{m} a_{l,i} a_{l,j}$$

where $a_{s,t}$ is the element of A in position (s, t) (for any $s = 1, \ldots, m$ and $t = 1, \ldots, n$). Taking the trace of this matrix we recognize the square of the Frobenius norm:

$$tr(A^T A) = \sum_{k=1}^{n} (A^T A)_{k,k} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{l,k}^2 = \|A\|_F^2 \tag{0.4}$$

We can now observe that $A^T A$ is symmetric, and recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues $\lambda_i(A^T A)$ (with multiplicity):

$$tr(A^T A) = \sum_{i=1}^{n} \lambda_i(A^T A) = \sum_{i=1}^{n} s_i^2(A).$$

Hence

$$\|A\|_F^2 = tr(A^T A) = \sum_{i=1}^{n} s_i^2(A) = \sum_{i=1}^{r} s_i^2(A) \tag{0.5}$$

(we stopped at r because the other eigenvalues are equal to zero). Taking the square root on both sides we get 0.2, as we wanted.

Let us now proceed to the second point. The first inequality is clear since the operator norm is equal to the maximum singular value: $\|A\|^2 = s_1(A)^2 \leq \sum_{i=1}^{n} s_i^2(A) = \|A\|_F^2$. Finally, by the relation in 0.5 we have

$$\|A\|_F = \sqrt{\sum_{i=1}^{r} s_i^2(A)} \leq \sqrt{\sum_{i=1}^{r} s_1^2(A)} = \sqrt{r} s_1(A) = \sqrt{r}\|A\| \tag{0.6}$$

and thus $\|A\|_F \leq \sqrt{r}\|A\|$.

Exercise 3. We want to prove that for any s_i singular value of A it holds

$$s_i \leq \frac{1}{\sqrt{i}} \|A\|_F. \tag{0.7}$$

It is an easy calculation: since $s_j \geq s_i$ for any $j \leq i$ and since $s_i = 0$ for any $i > r$,

$$\begin{aligned} i s_i^2 &= s_i^2 + \cdots + s_i^2 \\ &\leq s_1^2 + \cdots + s_i^2 \\ &\leq s_1^2 + \cdots + s_r^2 \\ &= \|A\|_F^2, \end{aligned}$$

where the last equality follows from the previous exercise. This gives the thesis (dividing both sides by i and then taking the square roots).

Exercise 4. We want to prove that the Hamming distance is indeed a metric. We recall the definition: for any $x, y \in \{0,1\}^n$

$$d_H(x,y) = |\{i : x_i \neq y_i\}| = \sum_{i=1}^{n} |x_i - y_i| \tag{0.8}$$

It is clear that $d_H(x,y)$ is always non negative and is equal to zero if and only if $x = y$, since we are counting by how many entries the two vectors differ. Clearly also $d_H(x,y) = d_H(y,x)$ for any $x, y \in \{0,1\}^n$. To prove the triangular inequality we take any $x, y, z \in \{0,1\}^n$ and see that

$$\begin{aligned} d_H(x,y) &= \sum_{i=1}^{n} |x_i - y_i| \\ &= \sum_{i=1}^{n} |x_i - z_i + z_i - y_i| \\ &\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| \\ &= d_H(x,z) + d_H(z,y). \end{aligned}$$

Exercise 5. Let $K = \{0,1\}^n$. We want to prove that for any $m \in [0,n]$

$$\frac{2^n}{\sum_{k=0}^{m} \binom{n}{k}} \leq \mathcal{N}(K, d_H, m) \leq \mathcal{P}(K, d_H, m) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}. \tag{0.9}$$

We saw the second inequality in the Lemma 4.2.8 of the textbook. We proceed with the first inequality. We recall that $\mathcal{N}(K, d_H, m)$ is the number of d_H-balls of radius m necessary to cover K, so we have

$$|K| \leq \mathcal{N}(K, d_H, m) |m B_{d_H}^n|, \tag{0.10}$$

where $B_{d_H}^n$ is the unitary n-dimensional ball with respect to d_H.

Now we notice that

$$|m B_{d_H}^n| = \sum_{k=0}^{m} \binom{n}{k} \tag{0.11}$$

since $\binom{n}{k}$ is the number of elements of K that differ from the origin in exactly k digits (i.e. the points of K at distance k from the origin), for any $k = 0, 1 \ldots, m$. Substituting this and $|K| = 2^n$ into 0.10, we obtain

$$2^n \leq \mathcal{N}(K, d_H, m) \sum_{k=0}^{m} \binom{n}{k}, \tag{0.12}$$

and dividing by the sum we get the first inequality.

At last we study the third inequality. By the definition of $\mathcal{P}(K, d_H, m)$, it follows that there exists, in our space K, $\mathcal{P}(K, d_H, m)$ closed disjoint balls $B(x_i, \lfloor m/2 \rfloor)$ with centers $x_i \in K$. Thus we have

$$\mathcal{P}(K, d_H, m) \left| \left\lfloor \frac{m}{2} \right\rfloor B_{d_H}^n \right| \leq |K| \tag{0.13}$$

and, using formula 0.11,

$$\mathcal{P}(K, d_H, m) \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k} \leq 2^n, \tag{0.14}$$

so that dividing by the sum we have the thesis.

Exercise 6. Let A be an $m \times n$ matrix, $\mu \in \mathbb{R}$ and $\epsilon \in [0, 1/2)$. We have to show that for any ϵ-net $\mathcal{N}$ of the sphere S^{n-1} we have

$$\sup_{x \in S^{n-1}} |\|Ax\|_2 - \mu| \leq \frac{C}{1 - 2\epsilon} \sup_{x \in \mathcal{N}} |\|Ax\|_2 - \mu|. \tag{0.15}$$

We first show that if R is a symmetric $n \times n$ matrix then

$$\sup_{x \in S^{n-1}} |\langle Rx, x\rangle| = \|R\| \leq \frac{1}{1 - 2\epsilon} \sup_{x \in \mathcal{N}} |\langle Rx, x\rangle| \tag{0.16}$$

(Exercise 4.4.3.b of the textbook). The equality follows from the fact that

$$\|R\| = \sup_{x \in S^{n-1}, y \in S^{n-1}} |\langle Rx, y\rangle| \geq \sup_{x \in S^{n-1}} |\langle Rx, x\rangle| \tag{0.17}$$

and that if λ_1 is the eigenvalue with largest absolute value and v_1 is a unitary eigenvector then

$$\|R\| = |\lambda_1| = |\langle Rv_1, v_1\rangle| \leq \sup_{x \in S^{n-1}} |\langle Rx, x\rangle|. \tag{0.18}$$

For the inequality in 0.16 we let $x \in S^{n-1}$ be a point maximizing $|\langle Rx, x\rangle|$ (so that by the previous part $|\langle Rx, x\rangle| = \|R\|$), and let $x_0 \in \mathcal{N}$ be a point such that $\|x - x_0\|_2 \leq \epsilon$. Then,

$$\begin{aligned}
|\langle Rx, x\rangle - \langle Rx_0, x_0\rangle| &= |\langle Rx, x - x_0\rangle + \langle R(x - x_0), x_0\rangle| \\
&\leq |\langle Rx, x - x_0\rangle| + |\langle R(x - x_0), x_0\rangle| \\
&\leq \|Rx\|_2 \|x - x_0\|_2 + \|R(x - x_0)\|_2 \|x_0\|_2 \\
&\leq \|R\|(\|x\|_2 \|x - x_0\|_2 + \|x - x_0\|_2 \|x_0\|_2) \\
&\leq 2\epsilon \|R\|
\end{aligned}$$

and hence

$$\sup_{x \in \mathcal{N}} |\langle Rx, x\rangle| \geq |\langle Rx_0, x_0\rangle| \geq |\langle Rx, x\rangle| - |\langle Rx, x\rangle - \langle Rx_0, x_0\rangle| = \|R\| - |\langle Rx, x\rangle - \langle Rx_0, x_0\rangle| \geq (1 - 2\epsilon)\|R\|, \tag{0.19}$$

which gives the second inequality in 0.16, as we wanted.

We can now go back to our real purpose, i.e. inequality 0.15. Since the matrix A is arbitrary we can change it with μA and divide both sides by $|\mu|$ (if $\mu = 0$ then inequality 0.15 follows directly from Lemma 4.4.1). Thus we can assume $\mu = 1$.

We let $R \coloneqq A^T A - \mathbb{I}_n$, and we observe that R is symmetric and

$$\langle Rx, x\rangle = x^T (A^T A - \mathbb{I}_n)^T x = \|Ax\|_2^2 - \|x\|_2^2 = \|Ax\|_2^2 - 1 \tag{0.20}$$

for any $x \in S^{n-1}$. Hence, by the previous part (inequality 0.16), we get

$$\sup_{x \in S^{n-1}} |\|Ax\|_2^2 - 1| = \|R\| \leq \frac{1}{1 - 2\epsilon} \sup_{x \in \mathcal{N}} |\|Ax\|_2^2 - 1|. \tag{0.21}$$

In particular, since $|\|Ax\|_2^2 - 1|$ is maximized if and only if $\sqrt{|\|Ax\|_2^2 - 1|}$ is maximized, it holds also that

$$\sup_{x \in S^{n-1}} \min\{|\|Ax\|_2^2 - 1|, \sqrt{|\|Ax\|_2^2 - 1|}\} \leq \frac{1}{1 - 2\epsilon} \sup_{x \in \mathcal{N}} \min\{|\|Ax\|_2^2 - 1|, \sqrt{|\|Ax\|_2^2 - 1|}\}. \tag{0.22}$$

Let us now prove that for any $y \in \mathbb{R}_{\geq 0}$ we have

$$|y - 1| \leq \min\{|y^2 - 1|, \sqrt{|y^2 - 1|}\} \leq (\sqrt{2} + 1)|y - 1|. \tag{0.23}$$

The first inequality follows easily from the fact that $|y^2 - 1| = (y + 1)|y - 1|$. For the second we distinguish two cases:

- If $y \le \sqrt{2}$ then $\min\{|y^2-1|, \sqrt{|y^2-1|}\} = |y^2-1| = (y+1)|y-1| \le (\sqrt{2}+1)|y-1|$.
- If $y > \sqrt{2}$ then $\min\{|y^2-1|, \sqrt{|y^2-1|}\} = \sqrt{|y^2-1|} \le (\sqrt{2}+1)|y-1|$ (where the inequality is because $\sqrt{|y^2-1|}/|y-1| = \sqrt{(y+1)/|y-1|}$ is decreasing for $y > 1$ and takes value $\sqrt{2}+1$ when $y = \sqrt{2}$).

Combining 0.22 and 0.23 (with $y = \|Ax\|_2$) we get the thesis:

$$\begin{aligned}\sup_{x\in S^{n-1}} |\|Ax\|_2 - 1| &\le \sup_{x\in S^{n-1}} \min\{|\|Ax\|_2^2 - 1|, \sqrt{|\|Ax\|_2^2-1|}\} \\ &\le \frac{1}{1-2\epsilon} \sup_{x\in\mathcal{N}} \min\{|\|Ax\|_2^2 - 1|, \sqrt{|\|Ax\|_2^2-1|}\} \\ &\le \frac{\sqrt{2}+1}{1-2\epsilon} \sup_{x\in\mathcal{N}} |\|Ax\|_2 - 1|\end{aligned}$$

Exercise 7. We want to deduce from theorem 4.4.5 that

$$\mathbb{E}\|A\| \le CK(\sqrt{m}+\sqrt{n}). \tag{0.24}$$

We recall that the theorem says that given an $m \times n$ random matrix A with independent mean zero sub-Gaussian entries, then

$$\mathbb{P}\{\|A\| \le CK(\sqrt{m}+\sqrt{n}+t)\} \ge 1 - 2\exp(-t^2) \tag{0.25}$$

for any $t > 0$, where $K := \max_{i,j} \|A_{i,j}\|_{\psi_2}$.

By the integral identity and performing the change of variables $s = CK(\sqrt{m}+\sqrt{n}+t)$, we have

$$\mathbb{E}\|A\| = \int_0^{+\infty} \mathbb{P}\{\|A\| > s\} ds = CK \int_{-\sqrt{m}-\sqrt{n}}^{+\infty} \mathbb{P}\{\|A\| > CK(\sqrt{m}+\sqrt{n}+t)\} dt. \tag{0.26}$$

Now clearly

$$\int_{-\sqrt{m}-\sqrt{n}}^{0} \mathbb{P}\{\|A\| > CK(\sqrt{m}+\sqrt{n}+t)\} dt \le \sqrt{m}+\sqrt{n} \tag{0.27}$$

(since the probability is bounded by 1) and by the theorem

$$\int_0^{+\infty} \mathbb{P}\{\|A\| > CK(\sqrt{m}+\sqrt{n}+t)\} dt \le \int_0^{+\infty} 2\exp(-t^2) dt = \sqrt{\pi}. \tag{0.28}$$

Thus

$$\mathbb{E}\|A\| = CK \int_{-\sqrt{m}-\sqrt{n}}^{+\infty} \mathbb{P}\{\|A\| > CK(\sqrt{m}+\sqrt{n}+t)\} dt \le CK(\sqrt{m}+\sqrt{n}+\sqrt{\pi}) \tag{0.29}$$

and with a possibly greater constant C we can get rid of the term $\sqrt{\pi}$ and get the thesis.