COMP9414: Artificial Intelligence

Lecture 4b: Automated Reasoning

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This Lecture

- Proof systems
  - Soundness, completeness, decidability
- Resolution and Refutation
- Horn clauses and SLD resolution
- Prolog
- Tableau method

Summary So Far

- Propositional Logic
  - Syntax: Formal language built from $\wedge$, $\vee$, $\neg$, $\rightarrow$
  - Semantics: Definition of truth table for every formula
  - $S \models P$ if whenever all formulae in S are True, P is True
- Proof System
  - System of axioms and rules for deduction
  - Enables computation of proofs of P from S
- **Basic Questions**
  - Are the proofs that are computed always correct? (soundness)
  - If $S \models P$, is there always a proof of P from S (completeness) ?

Mechanizing Proof

- A proof of a formula P from a set of premises S is a sequence of steps in which any step of the proof is
  1. An axiom of logic or premise from S, or
  2. A formula deduced from previous steps of the proof using a rule of inference

  and the last line of the proof is the formula P
- Formally captures the notion of mathematical proof
- S proves P ($S \vdash P$) if there is a proof of P from S; alternatively, P follows from S
- Example: Natural Deduction proof

Soundness and Completeness

- A proof system is sound if (intuitively) it preserves truth
  - Whenever $S \vdash P$, if every formula in S is True, P is also True
  - Whenever $S \vdash P$, $S \models P$
  - If you start with true assumptions, any conclusions must be true
- A proof system is complete if it is capable of proving all consequences of any set of premises (including infinite sets)
  - Whenever P is entailed by S, there is a proof of P from S
  - Whenever $S \models P$, $S \vdash P$
- A proof system is decidable if there is a mechanical procedure (computer program) which when asked whether $S \vdash P$, can always answer 'yes' – **or 'no'** – correctly

Resolution

- Another type of proof system based on refutation
- Better suited to computer implementation than systems of axioms and rules (can give correct 'no' answers)
- Decidable in the case of Propositional Logic
- Generalizes to First-Order Logic (see later in term)
- Needs all formulae to be converted to clausal form

Normal Forms

- A literal ℓ is a propositional variable or the negation of a propositional variable (P or $\neg P$)
- A clause is a disjunction of literals $\ell_1 \vee \ell_2 \vee \cdots \vee \ell_n$
- Conjunctive Normal Form (CNF) — a conjunction of clauses, e.g. $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$ – or just one clause, e.g. $P \vee Q$
- Disjunctive Normal Form (DNF) — a disjunction of conjunctions of literals, e.g. $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$ – or just one conjunction, e.g. $P \wedge Q$
- Every Propositional Logic formula can be converted to CNF and DNF
- Every Propositional Logic formula is equivalent to its CNF and DNF

Conversion to Conjunctive Normal Form

- Eliminate $\leftrightarrow$ rewriting $P \leftrightarrow Q$ as $(P \rightarrow Q) \wedge (Q \rightarrow P)$
- Eliminate $\rightarrow$ rewriting $P \rightarrow Q$ as $\neg P \vee Q$
- Use De Morgan's laws to push $\neg$ inwards (repeatedly)
  - Rewrite $\neg(P \wedge Q)$ as $\neg P \vee \neg Q$
  - Rewrite $\neg(P \vee Q)$ as $\neg P \wedge \neg Q$
- Eliminate double negations: rewrite $\neg\neg P$ as P
- Use the distributive laws to get CNF [or DNF] – if necessary
  - Rewrite $(P \wedge Q) \vee R$ as $(P \vee R) \wedge (Q \vee R)$ [for CNF]
  - Rewrite $(P \vee Q) \wedge R$ as $(P \wedge R) \vee (Q \wedge R)$ [for DNF]

Example Disjunctive Normal Form

- $\neg(P \rightarrow (Q \wedge R))$
- $\neg(\neg P \vee (Q \wedge R))$
- $\neg\neg P \wedge \neg(Q \wedge R)$
- $\neg\neg P \wedge (\neg Q \vee \neg R)$
- $P \wedge (\neg Q \vee \neg R)$
- Two clauses: P, $\neg Q \vee \neg R$

Resolution Rule of Inference

$$\frac{A_1 \vee \cdots \vee A_m \vee B \qquad \neg B \vee C_1 \vee \cdots \vee C_n}{A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n}$$

where B is a propositional variable and A_i and C_j are literals

- B and $\neg B$ are complementary literals
- $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is the resolvent of the two clauses
- Special case: If no A_i and C_j, resolvent is empty clause, denoted $\square$

Resolution Rule: Key Idea

- Consider $A_1 \vee \cdots A_m \vee B$ and $\neg B \vee C_1 \vee \cdots C_n$
  - Suppose both are true
  - If B is true, $\neg B$ is false and $C_1 \vee \cdots \vee C_n$ is true
  - If B is false, $A_1 \vee \cdots \vee A_m$ is true
  - Hence $A_1 \vee \cdots \vee A_m \vee C_1 \vee \cdots \vee C_n$ is true

Hence the resolution rule is sound

- Starting with true premises, any conclusion made using resolution must also be true

Applying Resolution: Naive Method

- Convert knowledge base into clausal form
- Repeatedly apply resolution rule to the resulting clauses
- P follows from the knowledge base if and only if each clause in the CNF of P can be derived using resolution from the clauses of the knowledge base (or subsumption)
- Example
  - $\{P \rightarrow Q, Q \rightarrow R\} \vdash P \rightarrow R$
  - Clauses $\neg P \vee Q$, $\neg Q \vee R$, show $\neg P \vee R$
  - Follows from one resolution step

Refutation Systems

- To show that P follows from S (i.e. $S \vdash P$) using refutation, start with S and $\neg P$ in clausal form and derive a contradiction using resolution
- A contradiction is the "empty clause" (a clause with no literals)
- The empty clause $\square$ is unsatisfiable (always False)
- So if the empty clause $\square$ is derived using resolution, the original set of clauses is unsatisfiable (never all True together)
- That is, if we can derive $\square$ from the clausal forms of S and $\neg P$, these clauses can never be all True together
- Hence whenever the clauses of S are all True, at least one clause from $\neg P$ must be False, i.e. $\neg P$ must be False and P must be True
- By definition, $S \models P$ (so P can correctly be concluded from S)

Applying Resolution Refutation

- Negate query to be proven (resolution is a refutation system)
- Convert knowledge base and negated query into CNF
- Repeatedly apply resolution until either the empty clause (contradiction) is derived or no more clauses can be derived
- If the empty clause is derived, answer 'yes' (query follows from knowledge base), otherwise answer 'no' (query does not follow from knowledge base)

Resolution: Example 1

$(G \vee H) \rightarrow (\neg J \wedge \neg K),\ G \vdash \neg J$

Clausal form of $(G \vee H) \rightarrow (\neg J \wedge \neg K)$ is
$\{\neg G \vee \neg J,\ \neg H \vee \neg J,\ \neg G \vee \neg K,\ \neg H \vee \neg K\}$

1. $\neg G \vee \neg J$ [Premise]
2. $\neg H \vee \neg J$ [Premise]
3. $\neg G \vee \neg K$ [Premise]
4. $\neg H \vee \neg K$ [Premise]
5. G [Premise]
6. J [$\neg$ Query]
7. $\neg G$ [1, 6 Resolution]
8. $\square$ [5, 7 Resolution]

Resolution: Example 2

$P \rightarrow \neg Q,\ \neg Q \rightarrow R \vdash P \rightarrow R$

Recall $P \rightarrow R \Leftrightarrow \neg P \vee R$

Clausal form of $\neg(\neg P \vee R)$ is $\{P,\ \neg R\}$

1. $\neg P \vee \neg Q$ [Premise]
2. $Q \vee R$ [Premise]
3. P [$\neg$ Query]
4. $\neg R$ [$\neg$ Query]
5. $\neg Q$ [1, 3 Resolution]
6. R [2, 5 Resolution]
7. $\square$ [4, 6 Resolution]

Resolution: Example 3

$\vdash ((P \vee Q) \wedge \neg P) \rightarrow Q$

Clausal form of $\neg(((P \vee Q) \wedge \neg P) \rightarrow Q)$ is $\{P \vee Q,\ \neg P,\ \neg Q\}$

1. $P \vee Q$ [¬ Query]
2. $\neg P$ [¬ Query]
3. $\neg Q$ [¬ Query]
4. Q [1, 2 Resolution]
5. $\Box$ [3, 4 Resolution]

Soundness and Completeness Again

- Resolution refutation is sound, i.e. it preserves truth (if a set of premises are all true, any conclusion drawn from those premises must also be true)
- Resolution refutation is complete, i.e. it is capable of proving all consequences of any knowledge base (not shown here!)
- Resolution refutation is decidable, i.e. there is an algorithm implementing resolution which when asked whether $S \vdash P$, can always answer 'yes' or 'no' (correctly)

Heuristics in Applying Resolution

- Clause elimination — can disregard certain types of clauses
  - Pure clauses: contain literal L where $\neg L$ doesn't appear elsewhere
  - Tautologies: clauses containing both L and $\neg L$
  - Subsumed clauses: another clause is a subset of the literals
- Ordering strategies
  - Resolve unit clauses (only one literal) first
  - Start with query clauses
  - Aim to shorten clauses

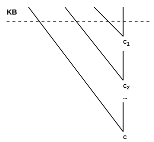
Horn Clauses

Idea: Use less expressive language

- Review
  - literal — atomic formula or negation of atomic formula
  - clause — disjunction of literals
- Definite Clause – exactly one positive literal
  - e.g. $B \vee \neg A_1 \vee \ldots \vee \neg A_n$, i.e. $B \leftarrow A_1 \wedge \ldots \wedge A_n$
- Negative Clause – no positive literals
  - e.g. $\neg Q_1 \vee \neg Q_2$ (negation of a query)
- Horn Clause – clause with at most one positive literal

SLD Resolution – $\vdash_{SLD}$

- Selected literals Linear form Definite clauses resolution
- SLD refutation of a clause C from a set of clauses KB is a sequence
  1. First clause of sequence is C
  2. Each intermediate clause C_i is derived by resolving the previous clause C_{i-1} and a clause from KB
  3. The last clause in the sequence is $\Box$



- **Theorem.** For a definite KB and negative clause query Q: $KB \cup Q \vdash \Box$ if and only if $KB \cup Q \vdash_{SLD} \Box$

Prolog

- Horn clauses in First-Order Logic (see later in term)
- SLD resolution
- Depth-first search strategy with backtracking
- User control
  - Ordering of clauses in Prolog database (facts and rules)
  - Ordering of subgoals in body of a rule
- Prolog is a programming language based on resolution refutation relying on the programmer to exploit search control rules

Prolog Example

```
r.                 # facts
u.
v.

q :- r, u.         # rules
s :- v.
p :- q, r, s.

?- p.              # query
yes
```


Prolog Interpreter

Input: A query Q and a logic program KB

Output: 'yes' if Q follows from KB, 'no' otherwise

Initialise current goal set to $\{Q\}$;

while the current goal set is not empty do

Choose G from the current goal set; (first in goal set)

Choose a copy G' :- $B_1, \ldots, B_n$ of a clause from KB (try all in KB)

(if no such rule, try alternative rules)

Replace G by $B_1, \ldots, B_n$ in current goal set;

if current goal set is empty,

output yes;

else output no;

- Depth-first, left-right with backtracking

Tableau Method

Alpha Rules:

$$\frac{A \land B}{\begin{array}{c}A\\ B\end{array}} \qquad \frac{\neg(A \lor B)}{\begin{array}{c}\neg A\\ \neg B\end{array}} \qquad \frac{\neg(A \rightarrow B)}{\begin{array}{c}A\\ \neg B\end{array}}$$

¬¬-Elimination:

$$\frac{\neg\neg A}{A}$$

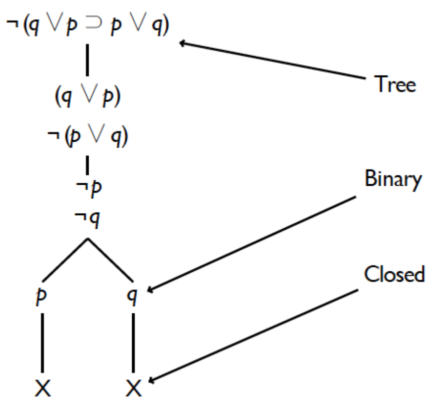
Beta Rules:

$$\frac{A \lor B}{A \mid B} \qquad \frac{A \rightarrow B}{\neg A \mid B} \qquad \frac{\neg(A \land B)}{\neg A \mid \neg B}$$

Branch Closure:

$$\frac{\begin{array}{c}A\\ \neg A\end{array}}{\times}$$

Tableau Method Example



Conclusion: Propositional Logic

- Propositions built from $\land$, $\lor$, $\neg$, $\rightarrow$
- Sound, complete and decidable proof systems (inference procedures)
  - Natural deduction
  - Resolution refutation
  - Prolog for special case of definite clauses
  - Tableau method
- Limited expressive power
  - Cannot express ontologies, e.g. AfPak Ontology
- First-Order Logic can express knowledge about objects, properties and relationships between objects