

# COMP9414: Artificial Intelligence

## Lecture 3a: Constraint Satisfaction

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## Constraint Satisfaction Problems

- **Constraint Satisfaction Problems** are defined by a set of **variables**  $X_i$ , each with a **domain**  $D_i$  of possible values, and a set of **constraints**  $C$
- Aim is to find an assignment to each the variables  $X_i$  (a value from the domain  $D_i$ ) such that all of the constraints  $C$  are satisfied

## This Lecture

- **Constraint Satisfaction Problems (CSPs)**
- **Standard search methods**
  - ▶ Backtracking search and heuristics
  - ▶ Forward checking and arc consistency
  - ▶ Domain splitting and arc consistency
  - ▶ Variable elimination
- **Local search**
  - ▶ Hill climbing
  - ▶ Simulated annealing

## Example: Map-Colouring



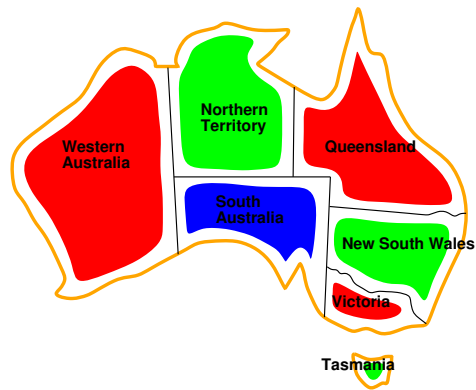
**Variables:** WA, NT, Q, NSW, V, SA, T

**Domains:**  $D_i = \{\text{red, green, blue}\}$

**Constraints:** Adjacent regions have different colours (WA  $\neq$  NT, etc.)

## Example: Map-Colouring

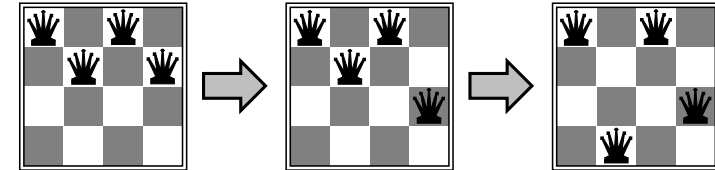
- **Solution** is an assignment that satisfies all the constraints



$\{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green\}$

## n-Queens Puzzle as a CSP

Assume one queen in each column. Domains are possible positions of queen in a column. Assignment is when each domain has one element.



**Variables:**  $Q_1, Q_2, Q_3, Q_4$

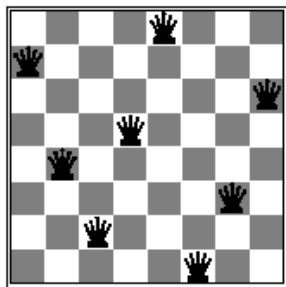
**Domains:**  $D_i = \{1, 2, 3, 4\}$

**Constraints:**

$Q_i \neq Q_j$  (cannot be in same row)

$|Q_i - Q_j| \neq |i - j|$  (or same diagonal)

## Example: n-Queens Puzzle



- Put  $n$  queens on  $n \times n$  board so that no two queens attack one another

## Example: Cryptarithmic

$$\begin{array}{r} \text{S E N D} \\ + \text{M O R E} \\ \hline \text{M O N E Y} \end{array}$$

**Variables:**

DEMNORSY

**Domains:**

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

**Constraints:**

$M \neq 0, S \neq 0$  (**unary** constraints)

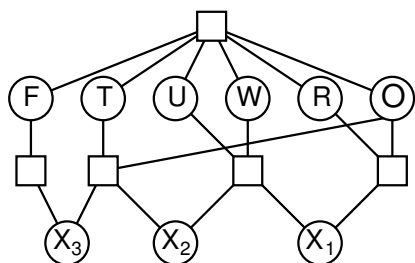
$Y = D + E$  or  $Y = D + E - 10$ , etc.

$D \neq E, D \neq M, D \neq N$ , etc.

## Cryptarithmic with Hidden Variables

We can add “hidden” variables to simplify the constraints.

$$\begin{array}{r} \text{TWO} \\ + \text{TWO} \\ \hline \text{FOUR} \end{array}$$



Variables: F T U W R O  $X_1 X_2 X_3$

Domains:  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints:

$\text{AllDifferent}(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$ , etc.

## Real World CSPs

- Assignment problems (e.g. who teaches what class)
- Timetabling problems (e.g. which class is offered when and where?)
- Hardware configuration (e.g. minimize space for circuit layout)
- Transport scheduling (e.g. courier delivery, vehicle routing)
- Factory scheduling (optimize assignment of jobs to machines)
- Gate assignment (assign gates to aircraft to minimize transit)

Many real world CSPs are also **optimization** problems

## Example: Sudoku

9				6				3
1		5		9	3	2		6
	4			5				9
8						4	7	1
		4	8	7				
7		2	6		1			8
2								
5				3	2		9	4
	8	7		1	6	3	5	

## Varieties of CSPs

Discrete variables

- Finite domains; size  $d \Rightarrow O(d^n)$  complete assignments
  - e.g. Boolean CSPs, incl. Boolean satisfiability (NP-complete)
- Infinite domains (integers, strings, etc.)
  - Job shop scheduling, variables are start/end days for each job
  - Need a **constraint language**, e.g.  $\text{StartJob}_1 + 5 \leq \text{StartJob}_3$
  - **Linear** constraints solvable, **nonlinear** undecidable

Continuous variables

- e.g. start/end times for Hubble Telescope observations
- Linear constraints solvable in polynomial time by LP methods

## Types of Constraint

- **Unary** constraints involve a single variable
  - ▶  $M \neq 0$
- **Binary** constraints involve pairs of variables
  - ▶  $SA \neq WA$
- **Higher-order** constraints involve 3 or more variables
  - ▶  $Y = D + E$  or  $Y = D + E - 10$
- **Inequality** constraints on continuous variables
  - ▶  $EndJob_1 + 5 \leq StartJob_3$
- **Soft** constraints (Preferences)
  - ▶ 11am lecture is better than 8am lecture!

## Path Search vs Constraint Satisfaction

Important difference between path search problems and CSPs

- **Constraint Satisfaction Problems** (e.g.  $n$ -Queens)
  - ▶ Difficult part is knowing the final state
  - ▶ How to get there is easy
- **Path Search Problems** (e.g. Rubik's Cube)
  - ▶ Knowing the final state is easy
  - ▶ Difficult part is how to get there

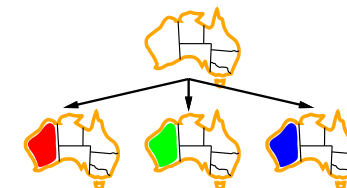
## Backtracking Search

CSPs can be solved by assigning values to variables one by one, in different combinations. Whenever a constraint is violated, go back to the most recently assigned variable and assign it a new value.

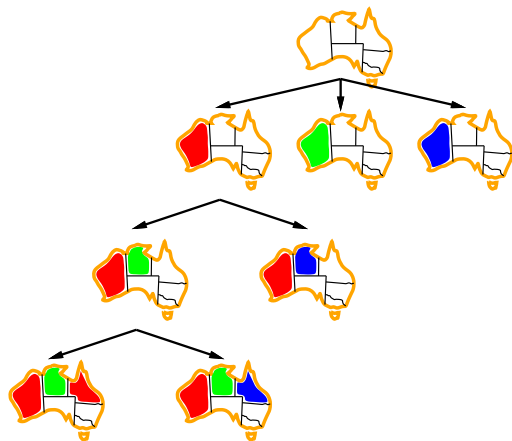
Can be implemented using Depth First Search on a special kind of state space, where states are defined by the values assigned so far

- **Initial state:** Empty assignment
- **Successor function:** Assign a value to an unassigned variable that does not conflict with previously assigned values of other variables
- **Goal state:** All variables are assigned a value and all constraints are satisfied

## Backtracking Search Example



## Backtracking Search Example



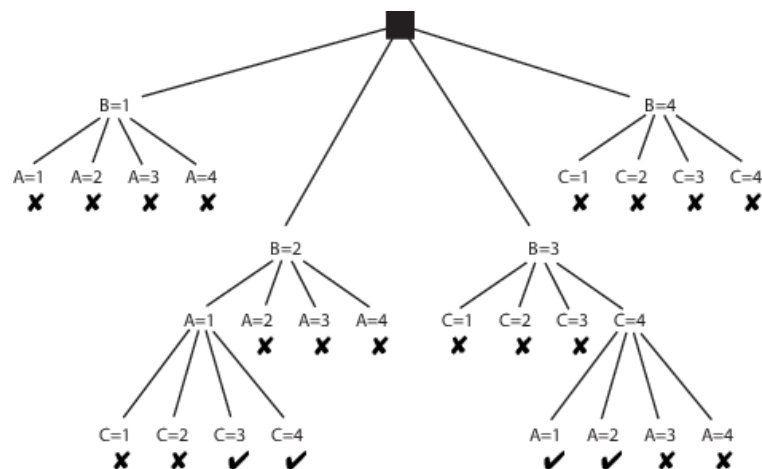
## Backtracking Search Space Properties

The search space has very specific properties:

- If there are  $n$  variables, every solution is at depth exactly  $n$
- Variable assignments are **commutative**  
[WA = red then NT = green] same as [NT = green then WA = red]

Backtracking search can solve  $n$ -Queens for  $n \approx 25$

## Problem with Backtracking Search



## Improvements to Backtracking Search

**General-purpose** heuristics can give huge gains in speed

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can inevitable failure be detected early?

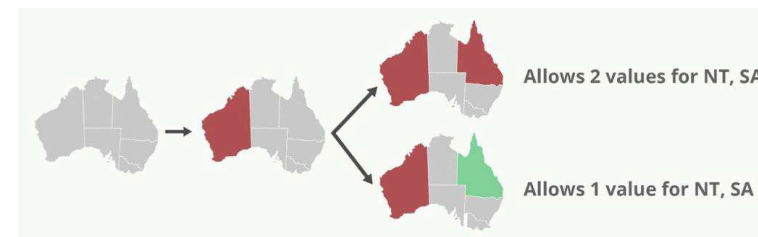
## Minimum Remaining Values

- Minimum Remaining Values (MRV)
  - Choose the variable with the fewest legal values



## Least Constraining Value

- Given a variable, choose the least constraining value:
  - One that rules out the fewest values in the remaining variables

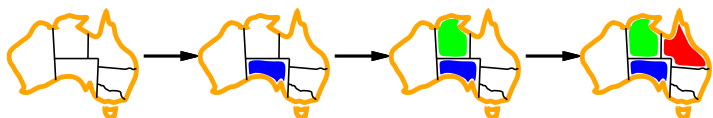


More generally, 3 allowed values would be better than 2, etc.

Combining these heuristics makes 1000-Queens feasible

## Degree Heuristic

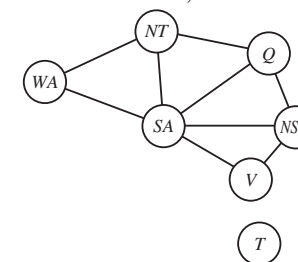
- Tie-breaker among MRV variables
  - Choose variable with most constraints on remaining variables



## Constraint Graph

Binary CSP: Each constraint relates at most two variables

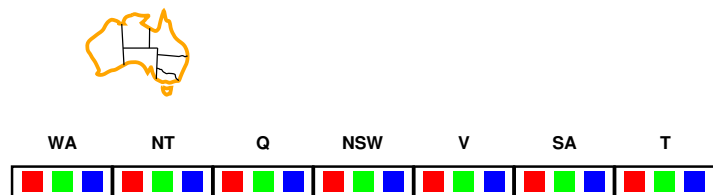
Constraint Graph: Nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure to speed up search, e.g. Tasmania is an independent subproblem!

## Forward Checking

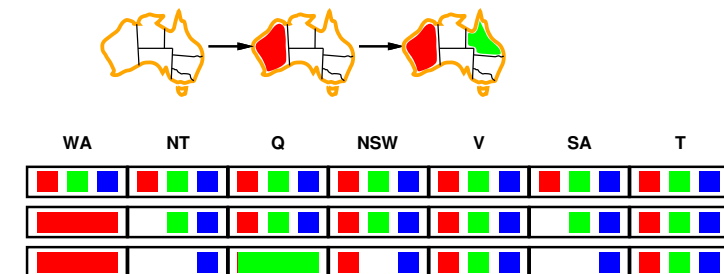
**Idea:** Keep track of remaining legal values for unassigned variables



## Forward Checking

**Idea:** Keep track of remaining legal values for unassigned variables

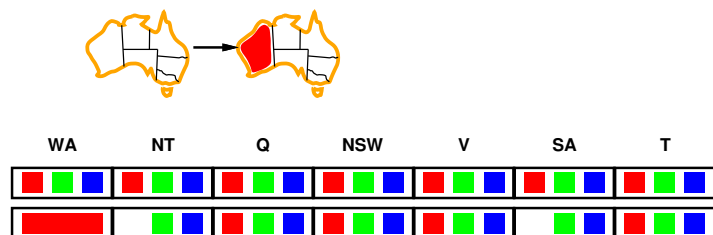
Terminate search when any variable has no legal values



## Forward Checking

**Idea:** Keep track of remaining legal values for unassigned variables

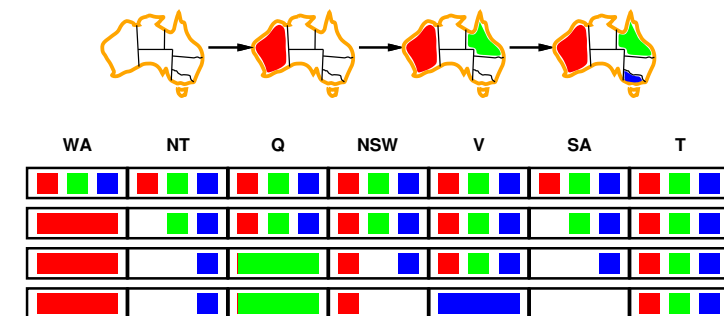
Terminate search when any variable has no legal values



## Forward Checking

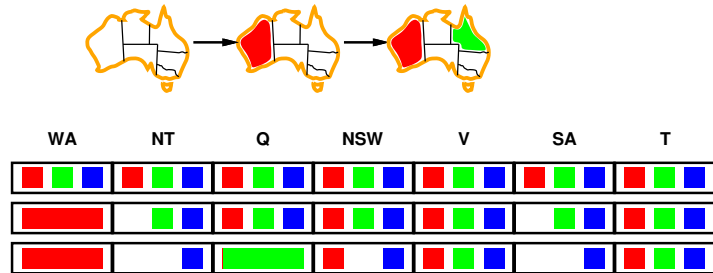
**Idea:** Keep track of remaining legal values for unassigned variables

Terminate search when any variable has no legal values



## Constraint Propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures



NT and SA cannot both be blue!

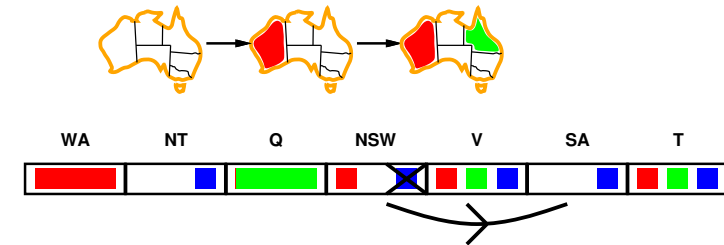
Constraint propagation repeatedly enforces constraints locally

## Arc Consistency

Simplest form of propagation makes every arc **consistent**

$X \rightarrow Y$  is **consistent** if

for **every** value  $x$  of  $X$  there is **some** allowed  $y$  in  $Y$

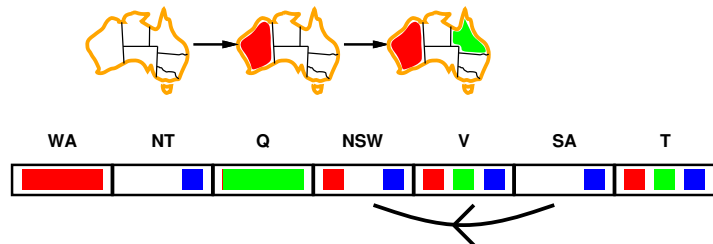


## Arc Consistency

Simplest form of constraint propagation makes each arc **consistent**

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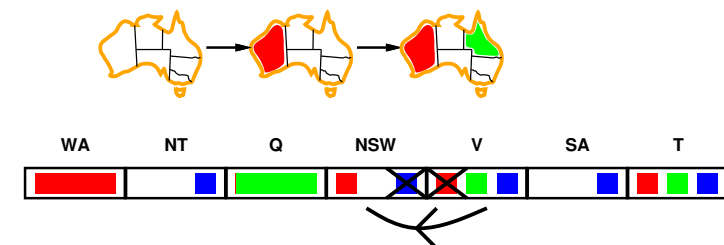


Make  $X \rightarrow Y$  arc consistent by removing any such  $x$  from domain

## Arc Consistency

$X \rightarrow Y$  is **consistent** if

for **every** value  $x$  of  $X$  there is **some** allowed  $y$  in  $Y$



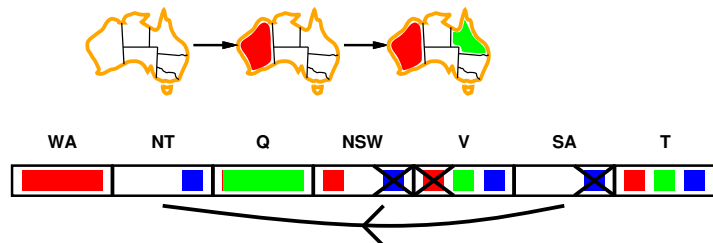
If  $X$  loses a value, neighbours of  $X$  need to be rechecked



## Arc Consistency

$X \rightarrow Y$  is **consistent** if

for **every** value  $x$  of  $X$  there is **some** allowed  $y$  in  $Y$



Arc consistency detects failure earlier than forward checking

For some problems, it can speed up the search enormously

For others, it may slow the search due to computational overheads

## Constraint Optimization Problems

States are whole CSPs (**not** partial assignments) **with costs**

- Make CSP **domain consistent** and **arc consistent**
- Add CSP to priority queue
- To solve CSP using **Greedy Search**
  - ▶ Remove CSP with minimal  $h$  from priority queue
  - ▶ Choose a variable  $v$  with more than one value in domain
  - ▶ Split the domain of  $v$  into two subsets
  - ▶ This gives two smaller CSPs
  - ▶ Make each CSP **arc consistent** – add to priority queue
- $cost(CSP) = \text{sum of costs to violate soft constraints}$

## Domain Splitting and Arc Consistency

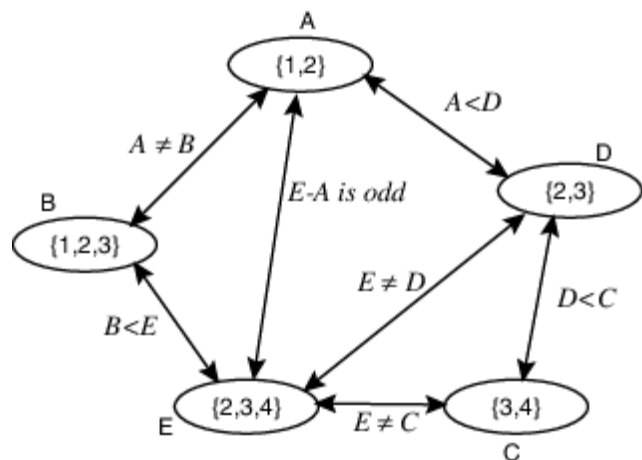
States are whole CSPs (**not** partial assignments)

- Make CSP **domain consistent** and **arc consistent**
  - ▶ Domain consistent = all **unary** constraints are satisfied
- To solve CSP using **Depth First Search**
  - ▶ Choose a variable  $v$  with more than one value in domain
  - ▶ Split the domain of  $v$  into two subsets
  - ▶ This gives two smaller CSPs
  - ▶ Make each CSP **arc consistent**
  - ▶ Solve each resulting CSP (or backtrack if unsolvable)

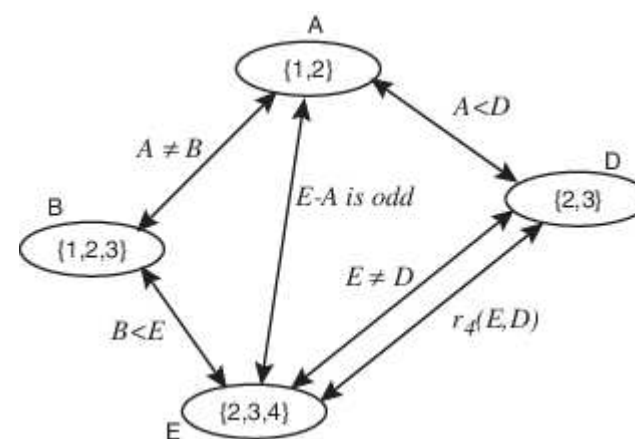
## Variable Elimination

- If there is only one variable, return the intersection of its (unary) constraints
- Otherwise
  - ▶ Select a variable  $X$
  - ▶ Join the constraints in which  $X$  appears, forming constraint  $R1$
  - ▶ Project  $R1$  onto its variables other than  $X$ , forming  $R2$
  - ▶ Replace all of the constraints in which  $X$  appears by  $R2$
  - ▶ Recursively solve the simplified problem, forming  $R3$
  - ▶ Return  $R1$  joined with  $R3$

## Variable Elimination Example



## Variable Elimination Example



## Variable Elimination Example

$r_1 : C \neq E$	C	E
	3	2
	3	4
	4	2
	4	3

$r_2 : C > D$	C	D
	3	2
	4	2
	4	3

$r_3 : r_1 \bowtie r_2$	C	D	E
	3	2	2
	3	2	4
	4	2	2
	4	2	3
	4	3	2
	4	3	3

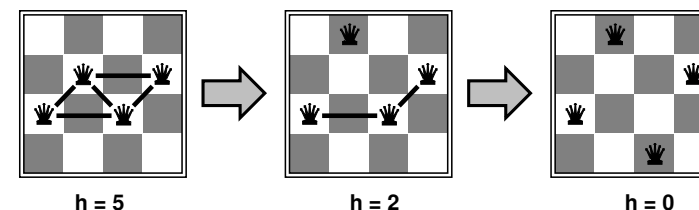
$r_4 : \pi_{\{D,E\}} r_3$	D	E
	2	2
	2	3
	2	4
	3	2
	3	3

↪ new constraint

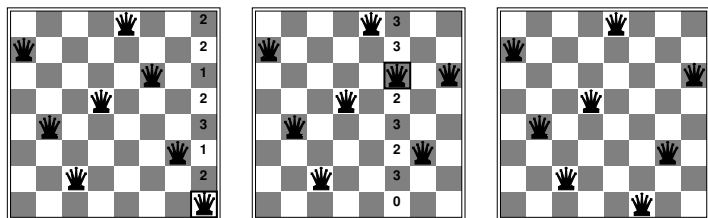
## Local Search

### Iterative Improvement

- Assign all variables randomly (possibly violating constraints)
- Change one variable at a time, trying to reduce the number of violations at each step
- Greedy Search with  $h$  = number of constraints violated

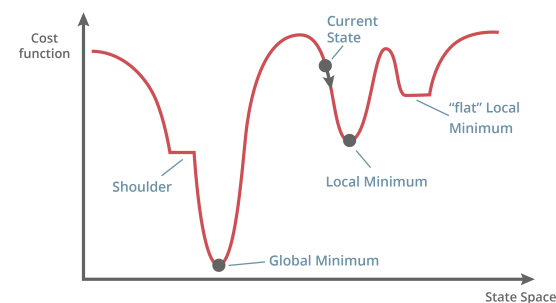


## Hill Climbing by Min-Conflicts



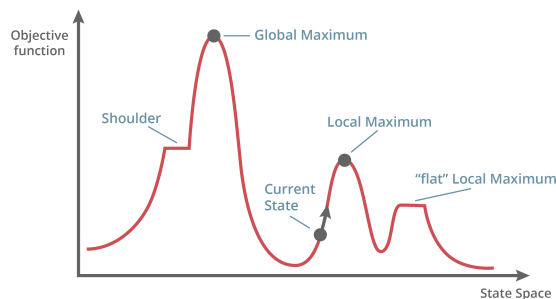
- Variable selection: randomly select any conflicted variable
- Value selection by **min-conflicts** heuristic
  - ▶ Choose value that violates fewest constraints

## Inverted View



When minimizing violated constraints, it makes sense to think of starting at the top of a ridge and climbing **down** into the valleys

## Plateaux and Local Optima



Sometimes, have to go sideways or even backwards in order to make progress towards the actual solution

## Simulated Annealing

- **Stochastic** hill climbing based on difference between evaluation of previous state ( $h_0$ ) and new state ( $h_1$ )
  - ▶ If  $h_1 < h_0$ , definitely make the change
  - ▶ Otherwise, make the change with probability

$$e^{-(h_1 - h_0)/T}$$

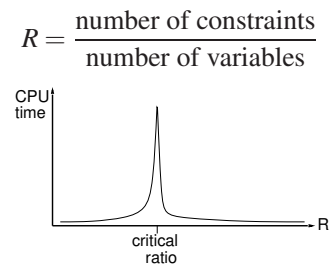
where  $T$  is a "temperature" parameter

- Reduces to ordinary hill climbing when  $T = 0$
- Becomes totally random search as  $T \rightarrow \infty$
- Sometimes, gradually decrease value of  $T$  during search

## Phase Transitions in CSPs

Given random initial state, hill climbing by min-conflicts with random restarts can solve  $n$ -Queens in almost constant time for arbitrary  $n$  with high probability (e.g.  $n = 10,000,000$ )

Randomly-generated CSPs tend to be easy if there are very few or very many constraints, but become extra hard in a narrow range of the ratio



## Summary

- Much interest in CSPs for real-world applications
- Backtracking = depth-first search with one variable assigned per node
- Variable and value ordering heuristics help significantly
- Forward checking helps by detecting inevitable failure early
- Hill climbing by min-conflicts often effective in practice
- Simulated annealing can help to escape from local optima
- Which method(s) are best varies from one task to another!