Introduction to Differential Equations

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1 Background

Differential equations can be useful for modeling how quantities change over time for many different contexts.

- biology: population dynamics, disease spread
- physics: motion or heat flow
- economics: investments

Differential equations relate a function, $y(t)$, to its derivative $\frac{dy}{dt}$ to show how y changes over time.

Ordinary Differential Equations (ODEs): only depend on one variable, t.

Order of ODE: the order depends on the highest derivative involved

- First order ODE: $\frac{dy}{dt}$
- Second order ODE: $\frac{d^2y}{dt^2}$

2 Growth and Decay Equations

The form of a growth or decay equation generally follows the form

$$\frac{dy}{dt} = k \cdot y(t)$$

where

$y(t)$: quantity (species) we are tracking over time
$\frac{dy}{dt}$: rate of change of y over time
k : a constant representing growth or decay

- the sign of k indicates growth (positive) or decay (negative)

3 General Solution for Growth and Decay Equations

Solving the ODE allows you to find the function $y(t)$. Both growth and decay equations have the same general form of the solution

$$y(t) = y_0 \cdot e^{kt}$$

where

$y(0)$: initial value of $y(t)$ at $t = 0$
k : a constant representing growth or decay
t : time

You can rearrange this equation to solve for different quantities.

4 Solving Differential Equations at Steady State

Solving an ODE at **steady state** allows you to see the values of the function when the system is no longer changing over time. To find these quantities, we need to set the derivative of the function equal to zero.
Using the equation:

$$\frac{dy}{dt} = ky - b$$

where

k and b are constants

Defining the steady state condition allows us to set $\frac{dy}{dt} = 0$.

$$0 = ky - b$$

Solve for y to determine the steady state values.

$$b = ky$$

$$y = \frac{b}{k}$$

5 Change of Variables

Change of variables allows us to make differential equations easy to solve.

Lets consider the example:

$$\frac{dw}{dt} = c + w(c_1 - \beta)$$

We want to solve for $w(t)$ where c, c_1, and β are all constants.

The introduction of the various constants makes it a bit more difficult to solve for the differential equation that otherwise would be an easy exponential growth or decay equation. Using a change of variables allows us to eliminate the constants so the equation becomes much easier to solve.

Step 1: Find the Steady-State Solution

We know that steady state occurs when the derivative is zero, $\frac{dw}{dt} = 0$, indicating that the system has reached equilibrium. This allows us to set the equation equal to zero, and call w the steady state.

$$c + w(c_1 - \beta) = 0$$

$$w_{st} = \frac{-c}{c_1 - \beta}$$

Step 2: Define a new variable

Perform a change of variable to simplify the equation by removing the constant term. Let our new variable be $u(t)$ such that it relates the value of w to its steady state without the constant c.

$$w = u + w_{st}$$

We know an expression for w_{st} that can be plugged in.

$$w = u + w_{st} = u - \frac{c}{c_1 - \beta}$$

Step 3: Substitute New Equation Back into the Original Equation
With the expression for w, we can plug it back into the equation for $\frac{dw}{dt}$, making it $\frac{du}{dt}$.

$$\frac{du}{dt} = c + u(c_1 - \beta) - \frac{c}{c_1 - \beta}(c_1 - \beta)$$

We can simplify the equation

$$\frac{du}{dt} = c + u(c_1 - \beta) - c$$

$$\frac{du}{dt} = u(c_1 - \beta)$$

Now we have an equation that is much simpler.

Step 4: Find the General Solution
Now that we have a standard linear differential equation, this is much easier to solve.

$$\frac{du}{dt} = u(c_1 - \beta)$$

Separate the variables and integrate

$$\int \frac{du}{u} = \int (c_1 - beta)dt$$

$$\ln(u) = (c_1 - \beta)t + C$$

Set $t(0) = 0$

$$\ln u(0) = C$$

So it has the solution:

$$u(t) = Ce^{(c_1 - \beta)t}$$

Recall that we said:

$$w = u + w_{st} = u - \frac{c}{c_1 - \beta}$$

Plug the expression back in for $u(t)$

$$w(t) = Ce^{(c_1 - \beta)t} - \frac{c}{c_1 - \beta}$$

If the initial condition is $w(0) = w_0$, we can find the constant C at time $t = 0$

$$w(0) = C - \frac{c}{c_1 - \beta} = w_0$$

Solving for C is

$$C = w_0 + \frac{c}{c_1 - \beta}$$

Thus, the solution is

$$w(t) = \left(w_0 + \frac{c}{c_1 - \beta}\right) e^{(c_1 - \beta)t} - \frac{c}{c_1 - \beta}$$

In your homework, you will have to keep going to solve for t.