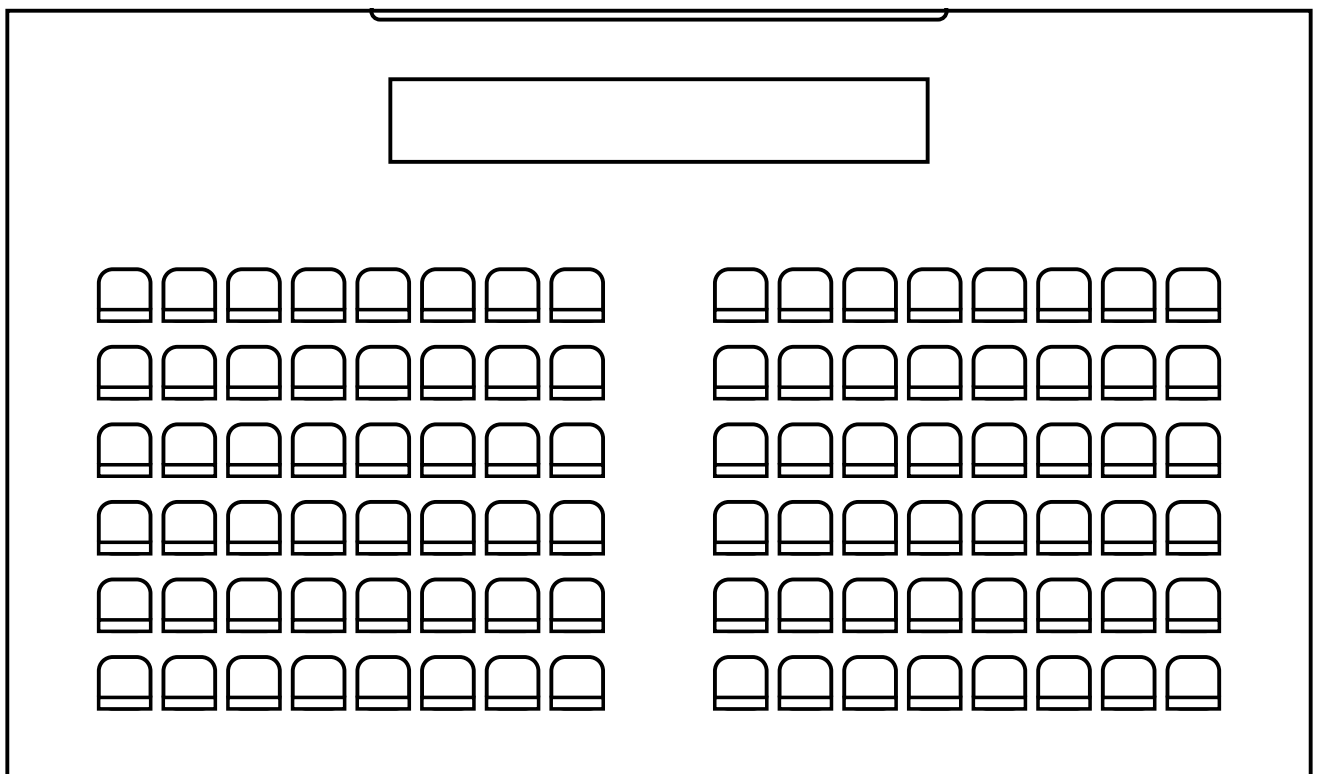
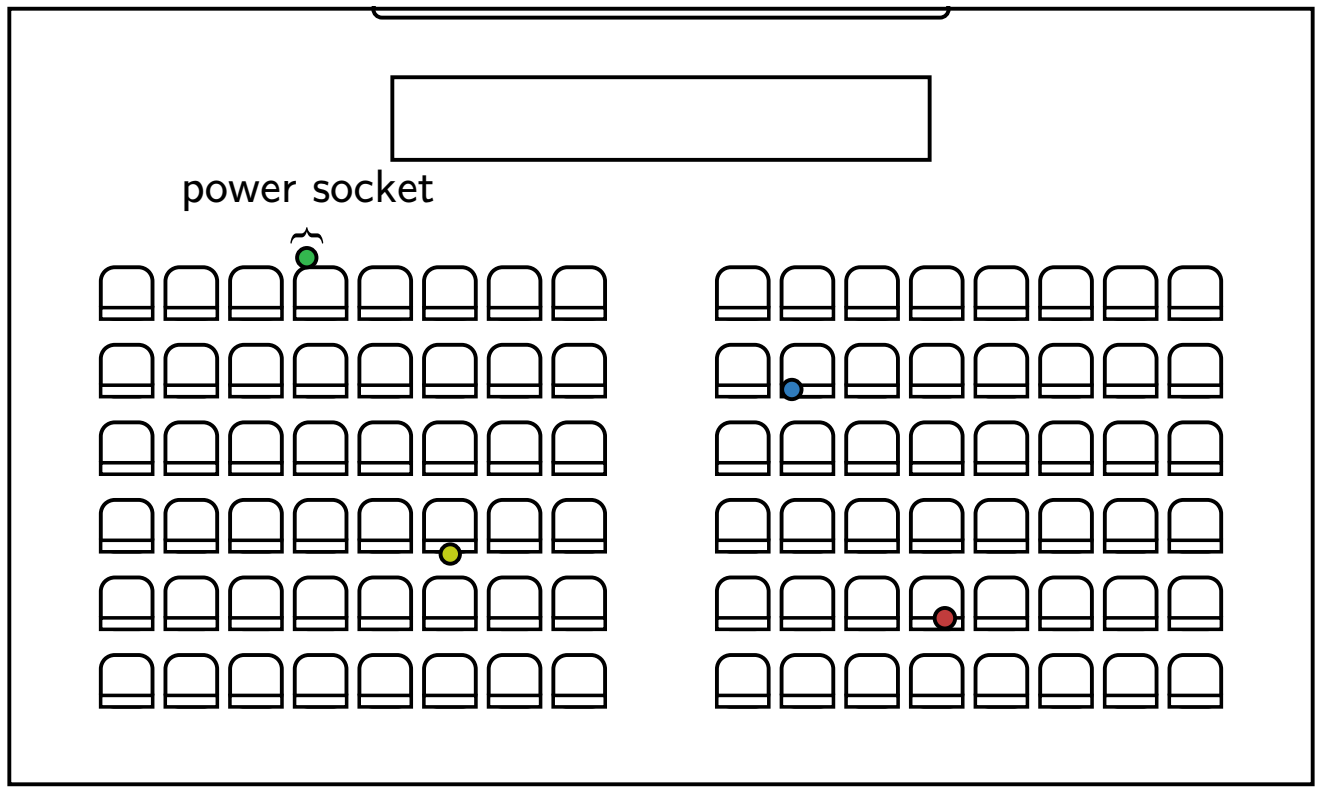
Voronoi Diagrams

An introduction to Voronoi Diagrams and how to compute them

Elias Rodriguez

May 17, 2024

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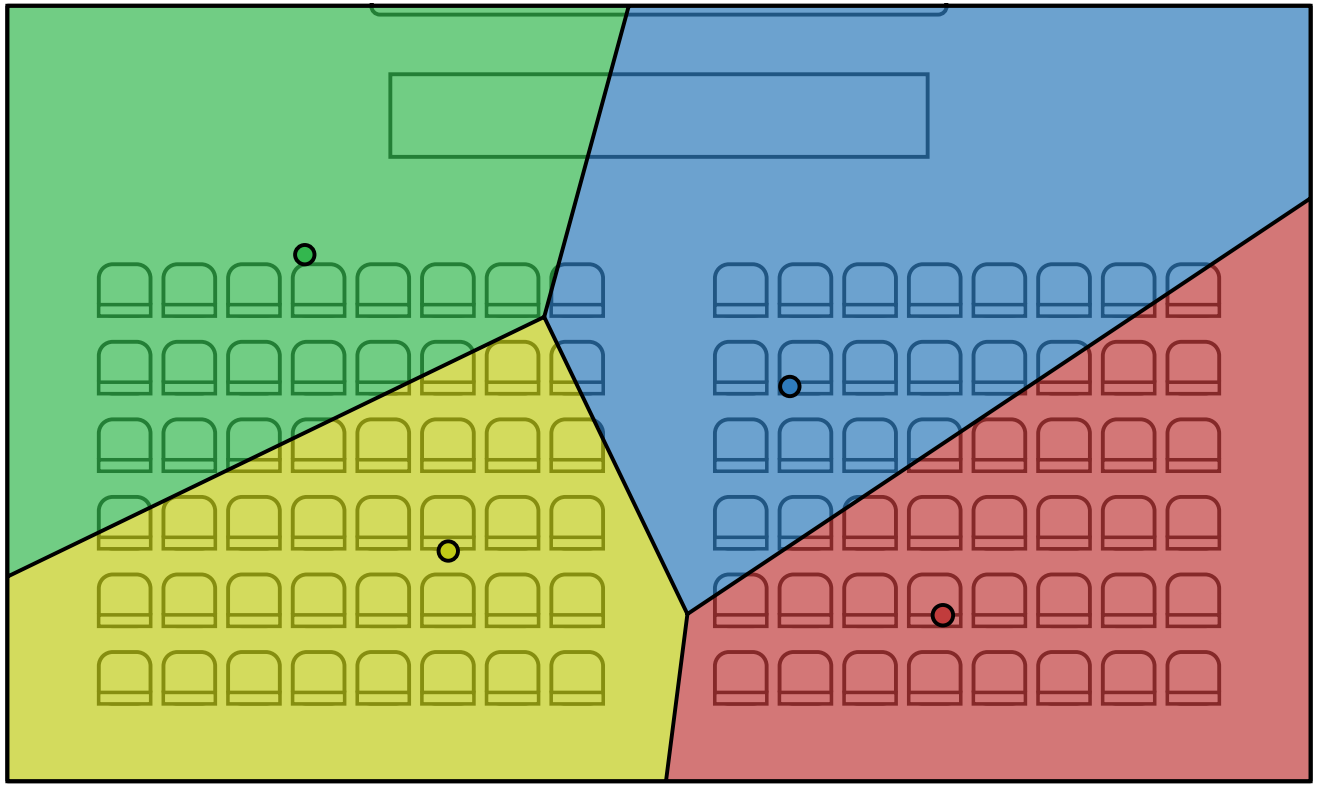
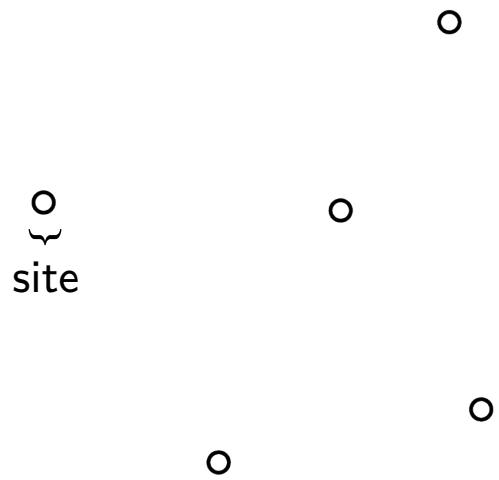


Formalizing our Observations



- Set S of *sites*
- *Voronoi region* of a site p in S is
  $\mathrm{Vor}(p) := \{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\| \text{ for all sites } q \in S\}$
- *Voronoi diagram* of a set S is
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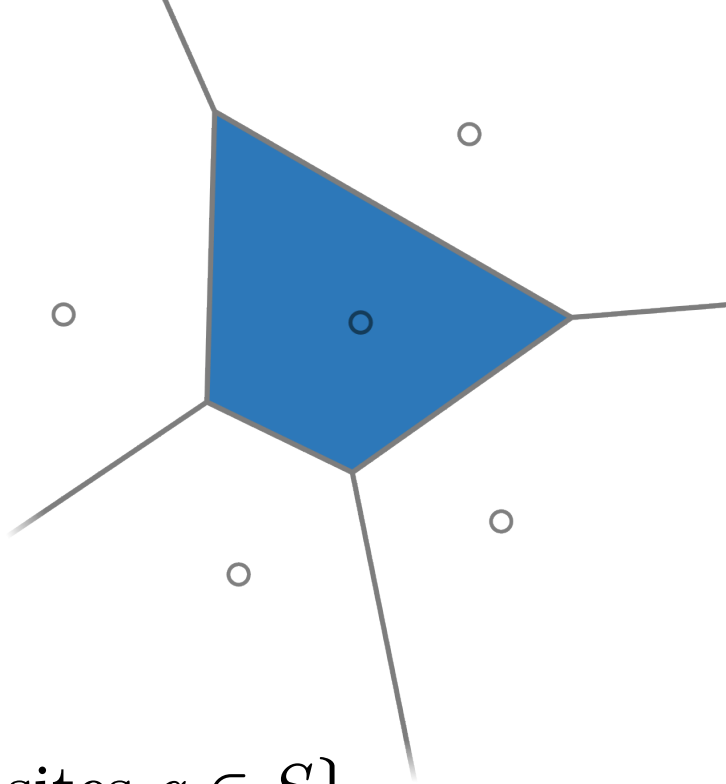

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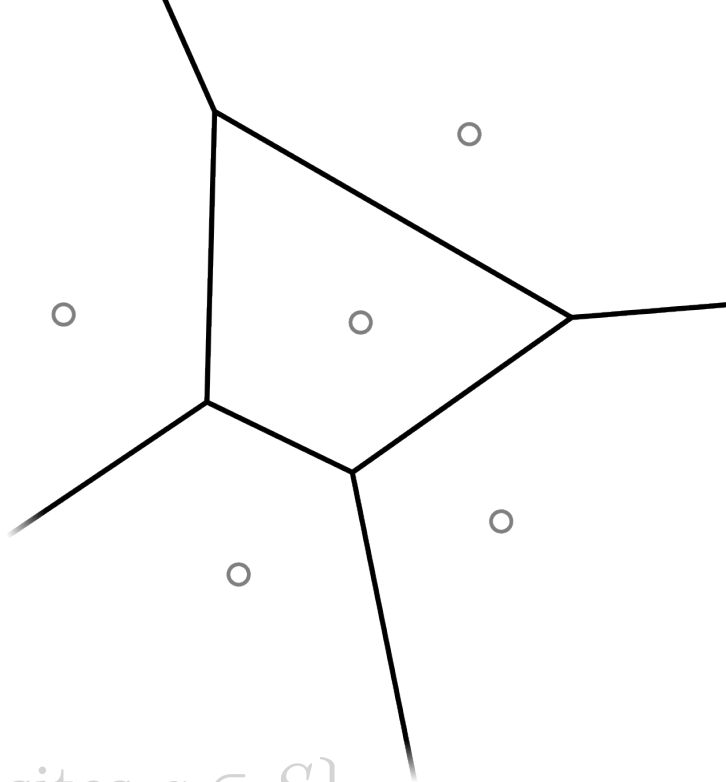

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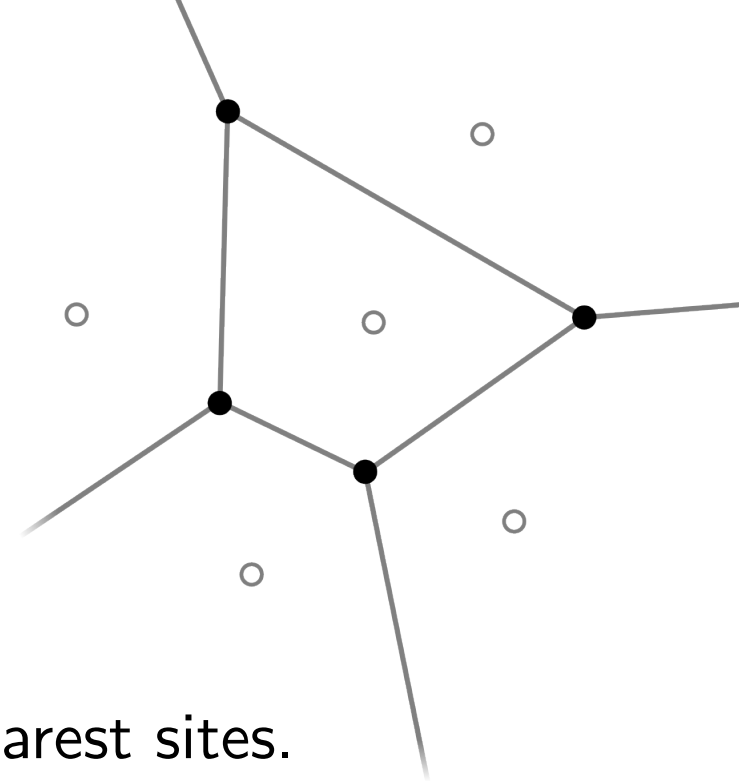
Refining our Definitions



Definition 1.1

A *Voronoi vertex* is a *point* with at least *three* nearest sites.

Definition 1.2

A *Voronoi edge* is a *connected set* with exactly *two* nearest sites.

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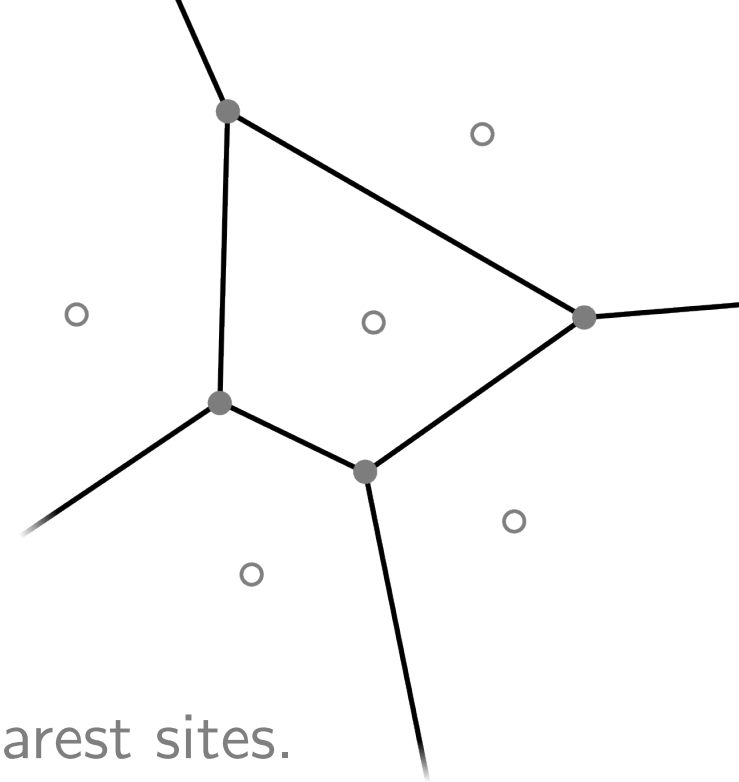

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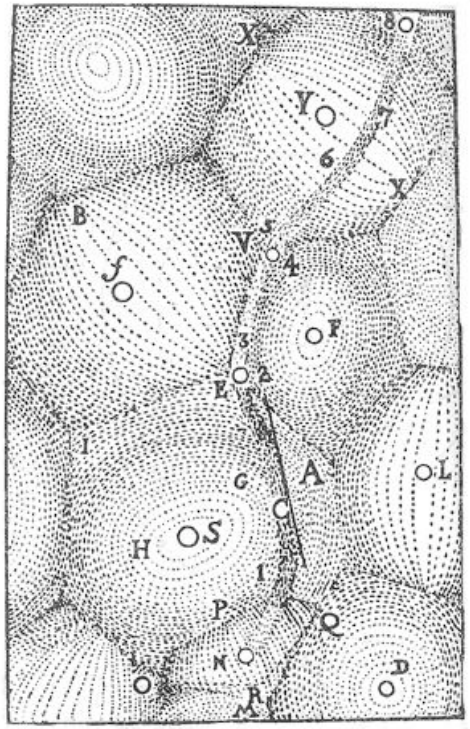

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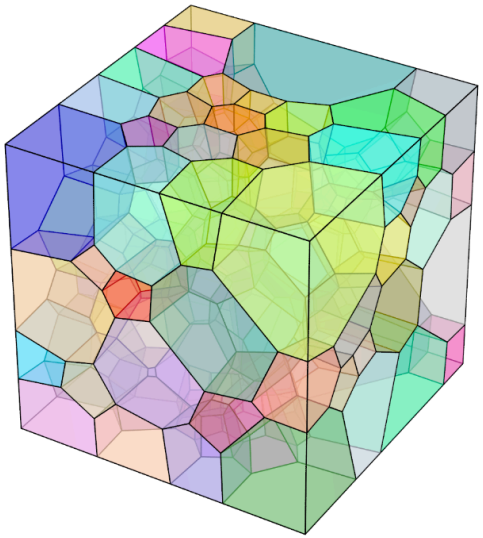

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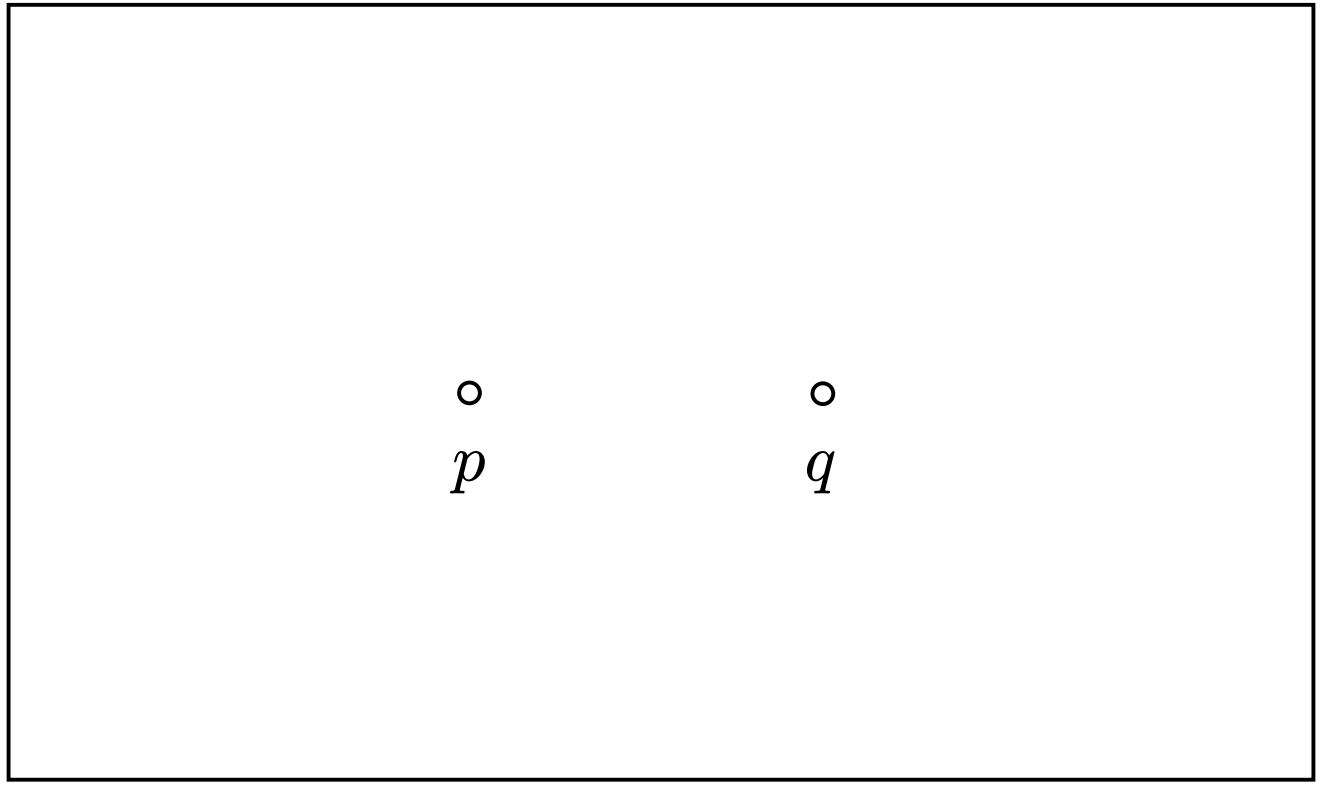
Key Properties

Simplest case



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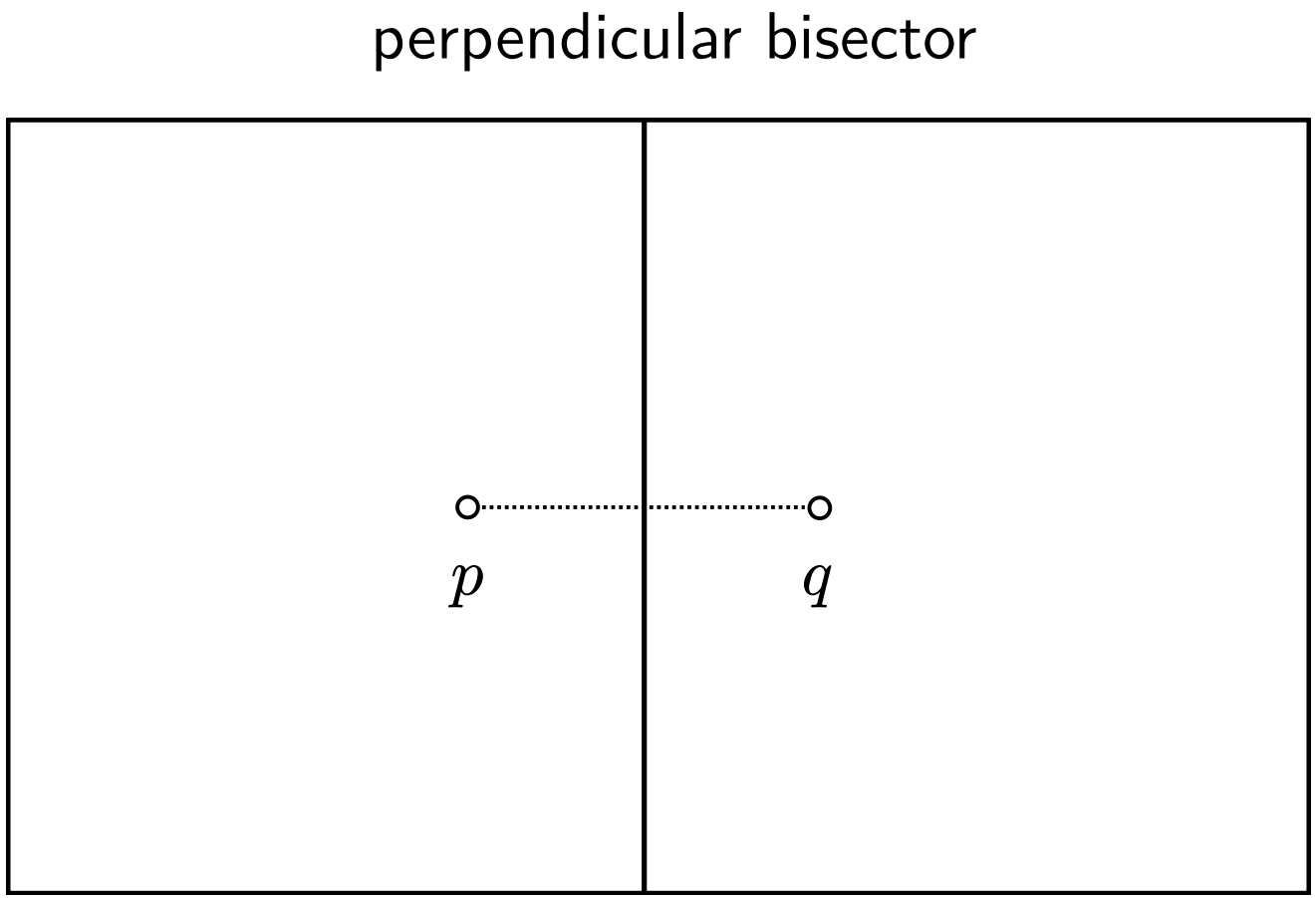
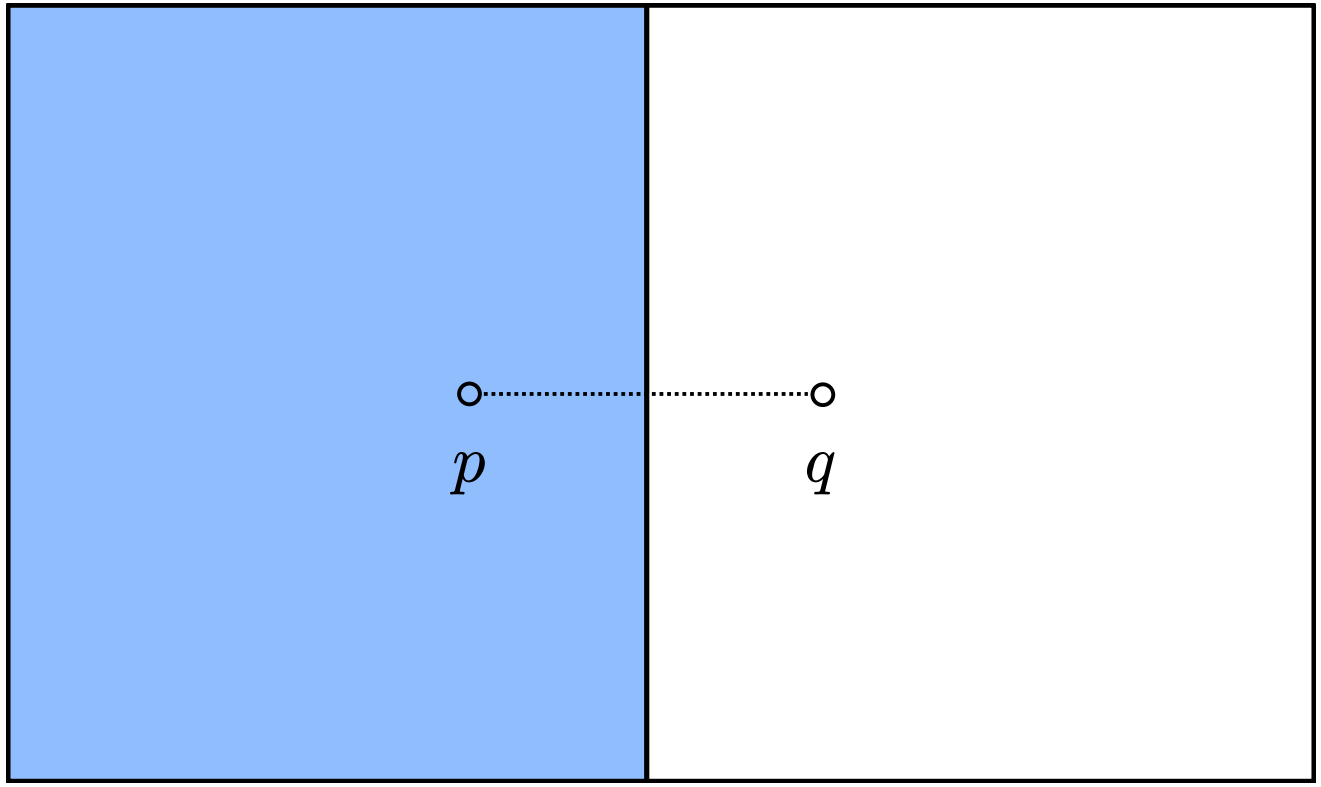


Simplest case

$$H(p,q) := \{x \in \mathbb{R}^2 \mid \|x-p\| \leq \|x-q\|\}$$



Voronoi Regions and their Convexity

Theorem 4.1

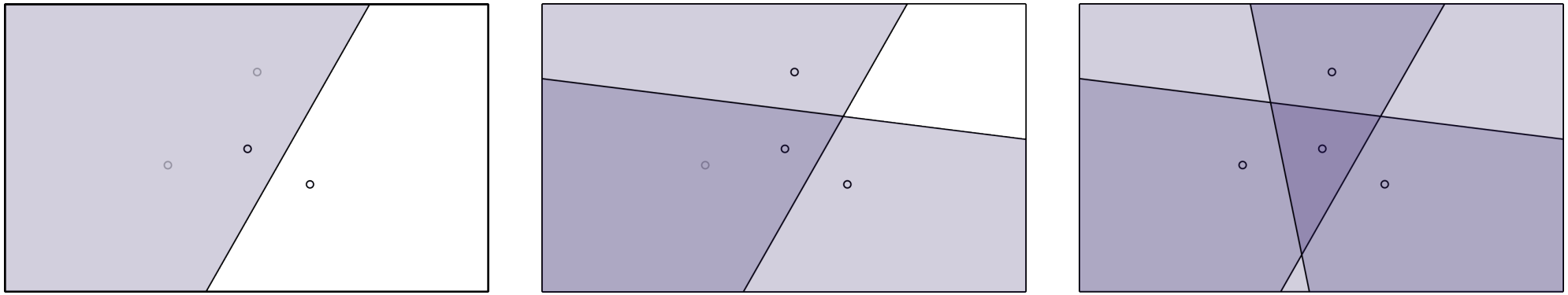
The Voronoi region $\mathrm{Vor}(p)$ is the intersection of all halfplanes $H(p, q)$, where q is any other site in S.

Corollary 4.3

All Voronoi regions are convex.

Proof Theorem 4.1

$$\mathrm{Vor}(p) := \{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\| \text{ for all sites } q \in S\}$$



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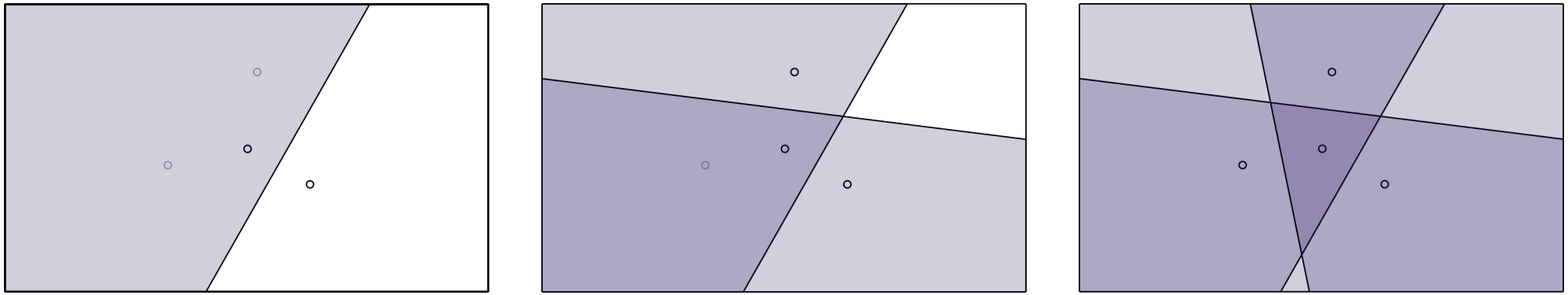
$$\begin{aligned}\mathrm{Vor}(p) &:= \{x \in \mathbb{R}^2 \mid \|x-p\| \leq \|x-q\| \text{ for all sites } q \in S\} \\ &= \bigcap_{q \in S} \{x \in \mathbb{R}^2 \mid \|x-p\| \leq \|x-q\|\}\end{aligned}$$



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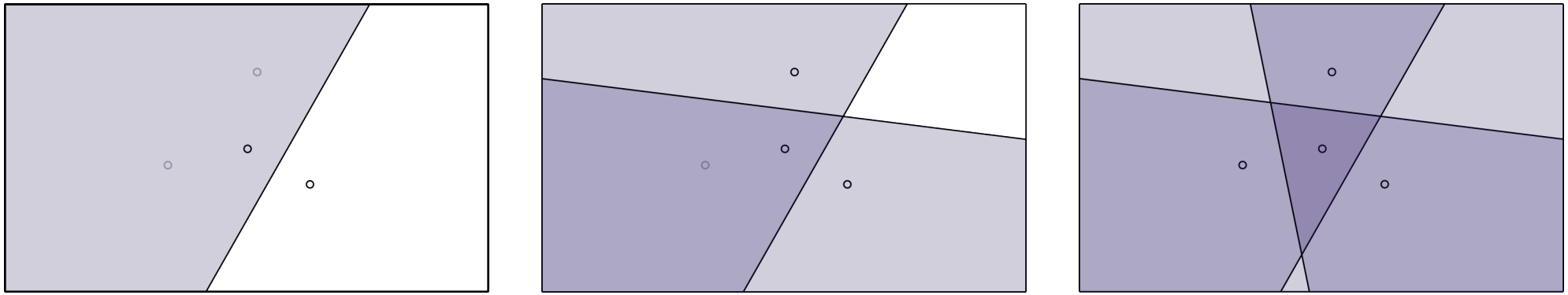

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Proof Corollary 4.3

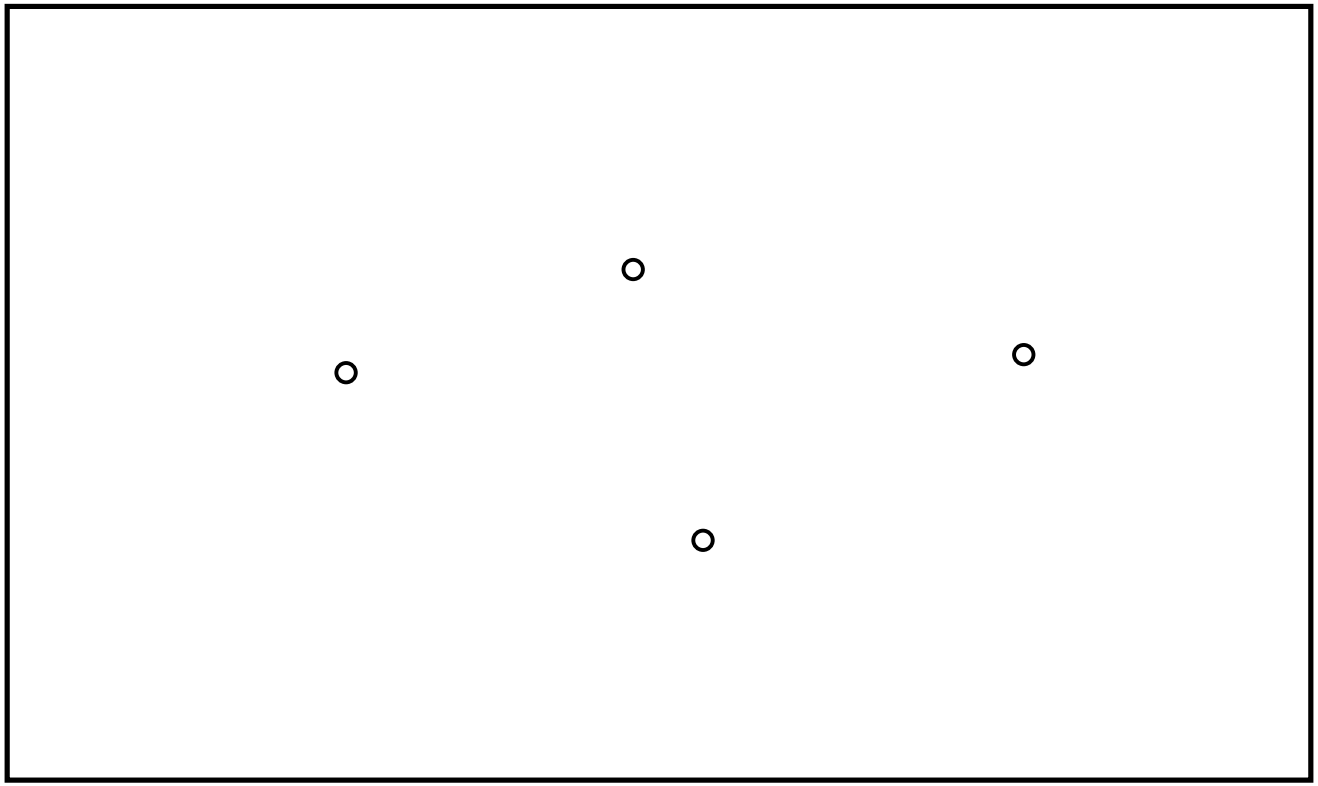
We show the intersection of any set of convex objects is convex.

Assume $X_1, ...X_n$ convex and let $X = X_1 \cap ... \cap X_n$

- By def. of intersection $a, b \in X \Rightarrow a, b \in X_i$ for all i in $[n]$.
- By def. of convexity the entire segment ab lies in every set X_i.

  $\Rightarrow$ Hence X is convex. $\square$

A closer look on perpendicular bisectors



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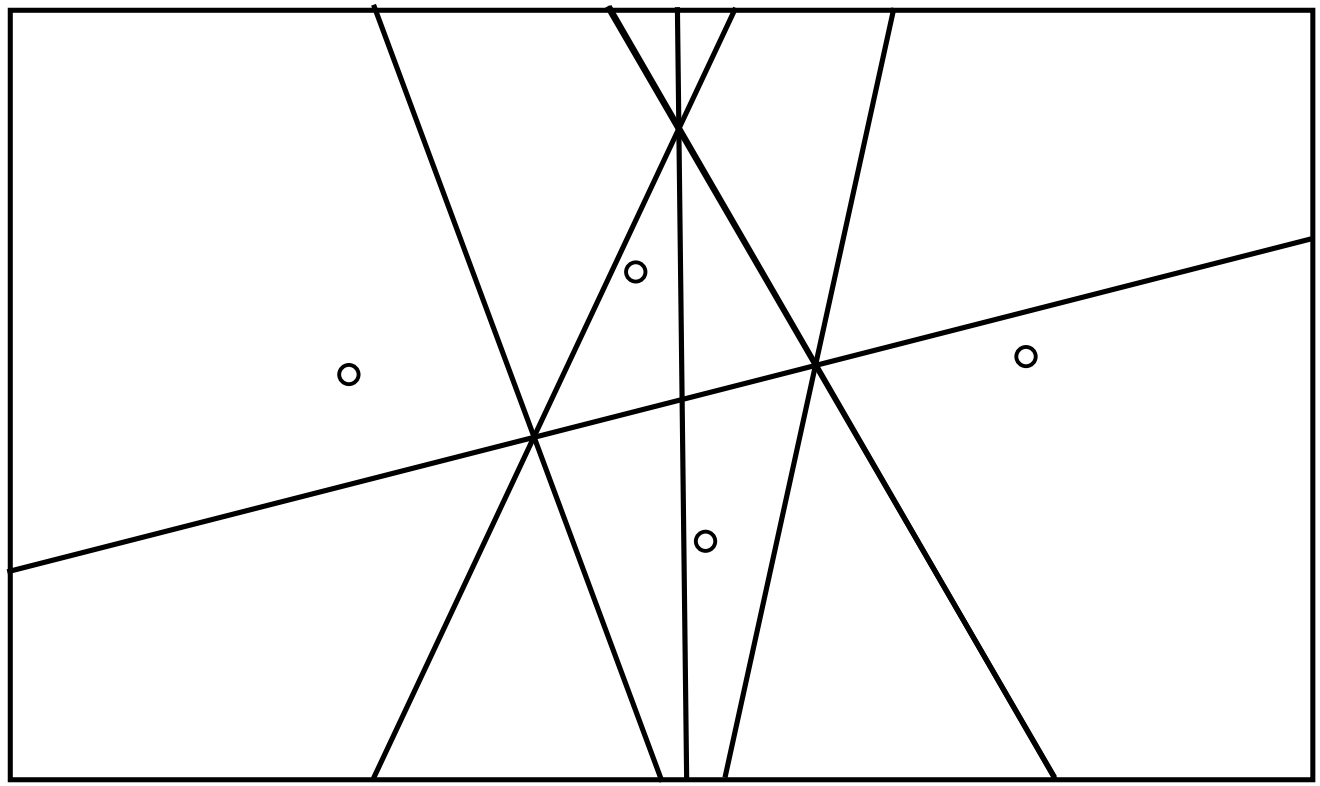

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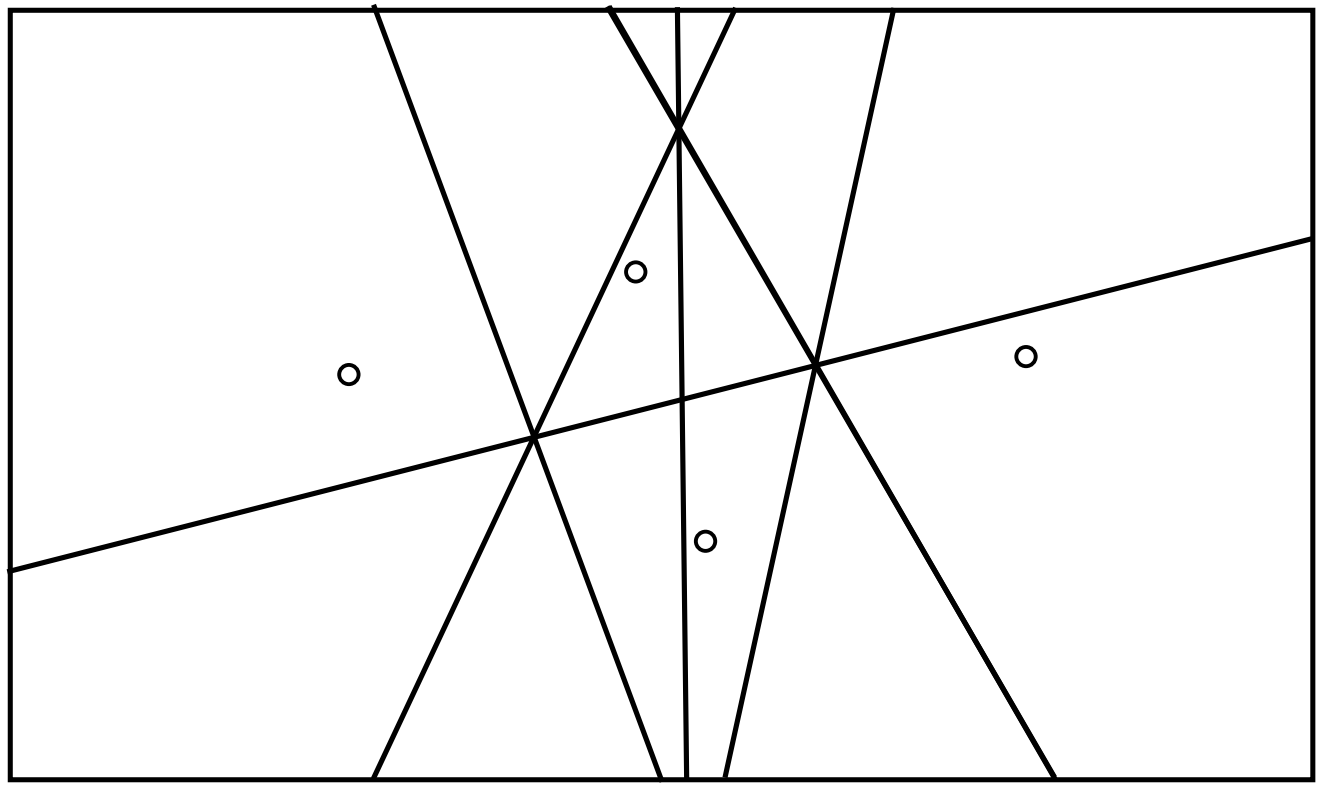

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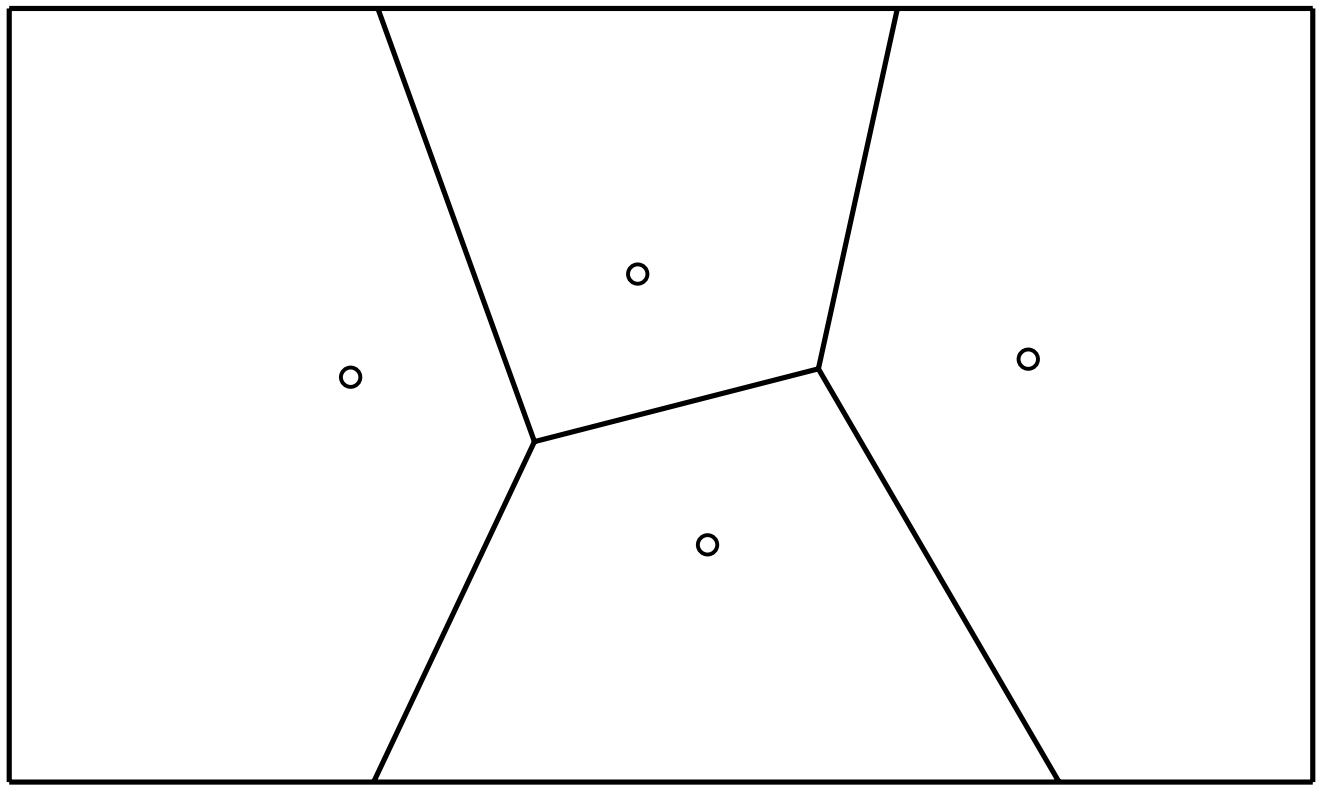


A closer look on perpendicular bisectors

Theorem 4.2

All Voronoi vertices and edges must lie on perpendicular bisectors between sites.

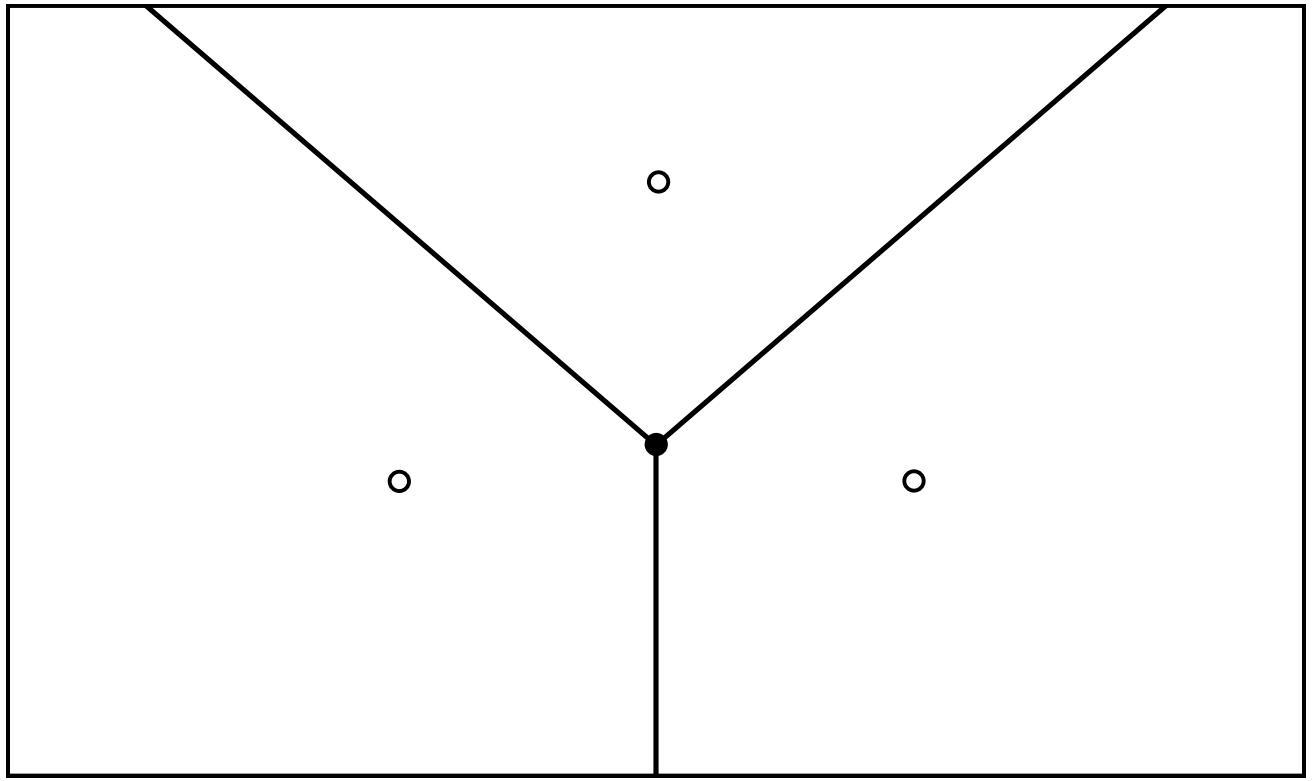
Geometric features of Voronoi Vertices

Theorem 4.6

A point v is a Voronoi vertex of $\mathrm{Vor}(S)$ if and only if there exists a circle centered at v with *three or more* sites on its boundary and *none* in its interior.

Proof Theorem 4.6

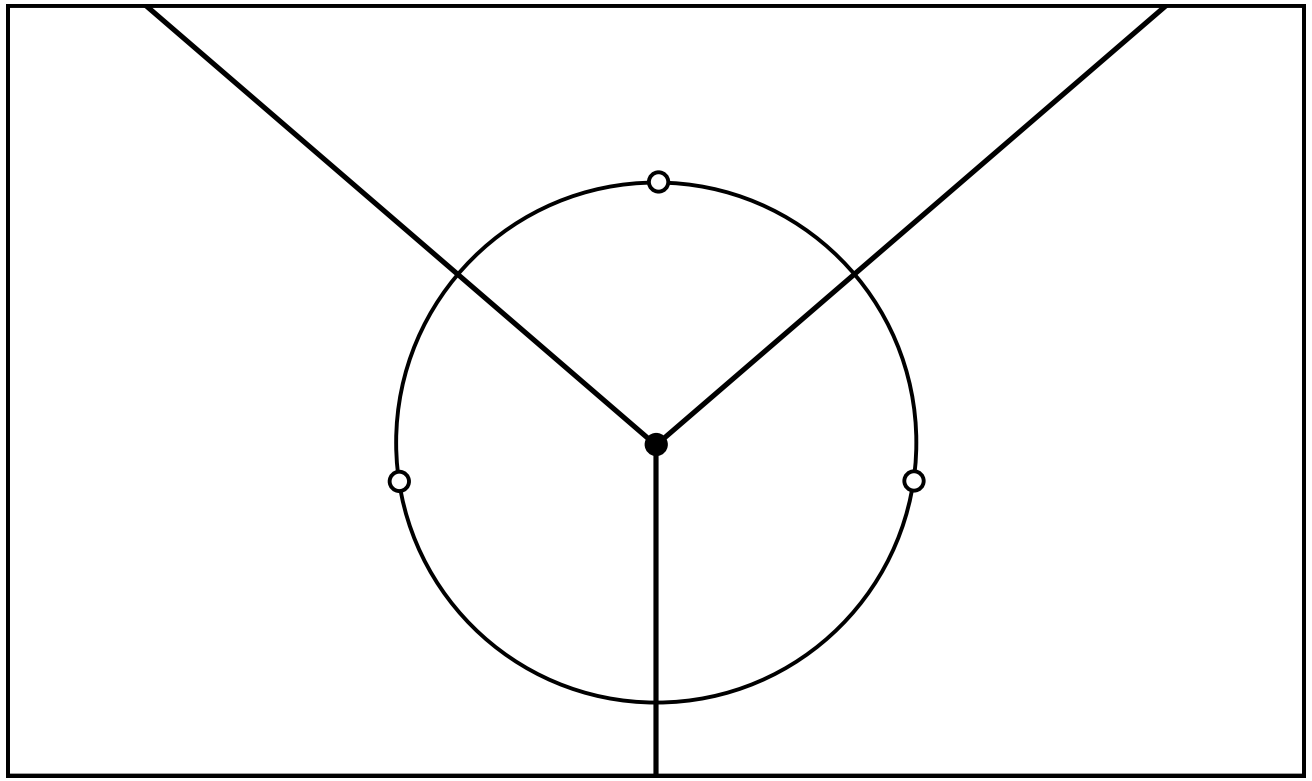
$\Longrightarrow$



Proof Theorem 4.6

$\Longrightarrow$

$$(x - d_x)^2 + (y - d_y)^2 = r^2$$



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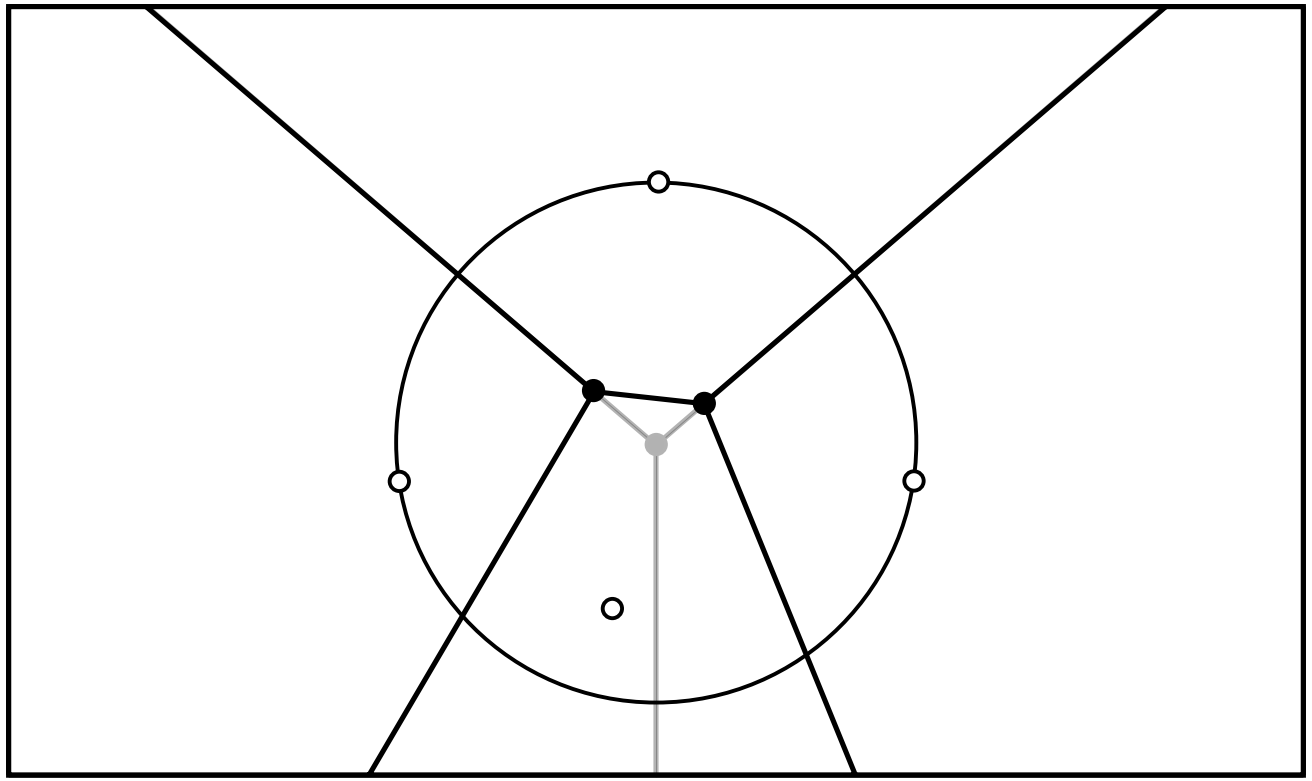


Proof Theorem 4.6

$\Longleftarrow$

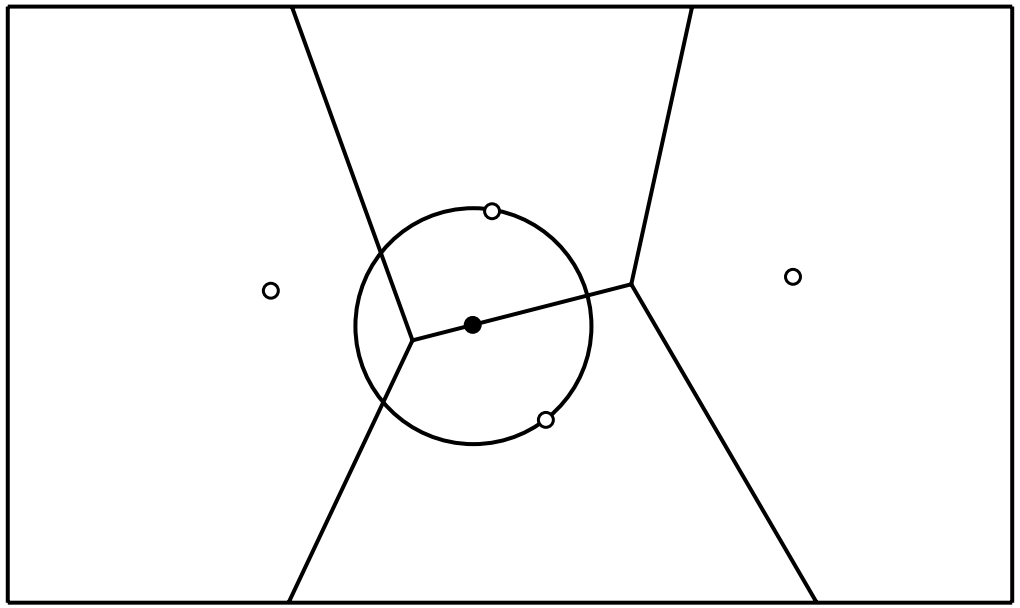
Given a circle with three or more sites on its boundary and none in its interior thus the circle's origin v must have three or more nearest sites thus v is a Voronoi vertex.

Geometric features of Voronoi Edges

Theorem 4.9

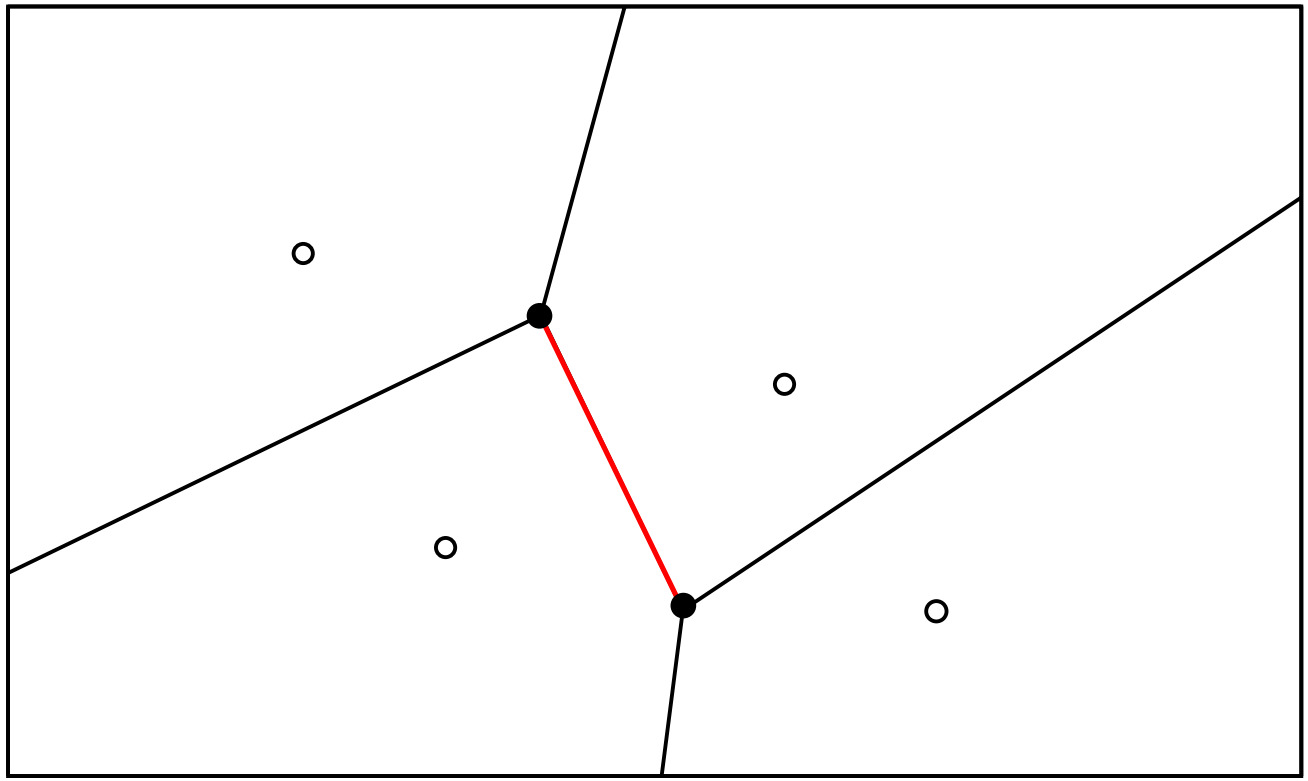
e is a Voronoi edge of $\mathrm{Vor}(S)$ if and only if for each point x in e, the circle centered at x through p and q contains no other sites of S in its interior or boundary.

Proof Theorem 4.9



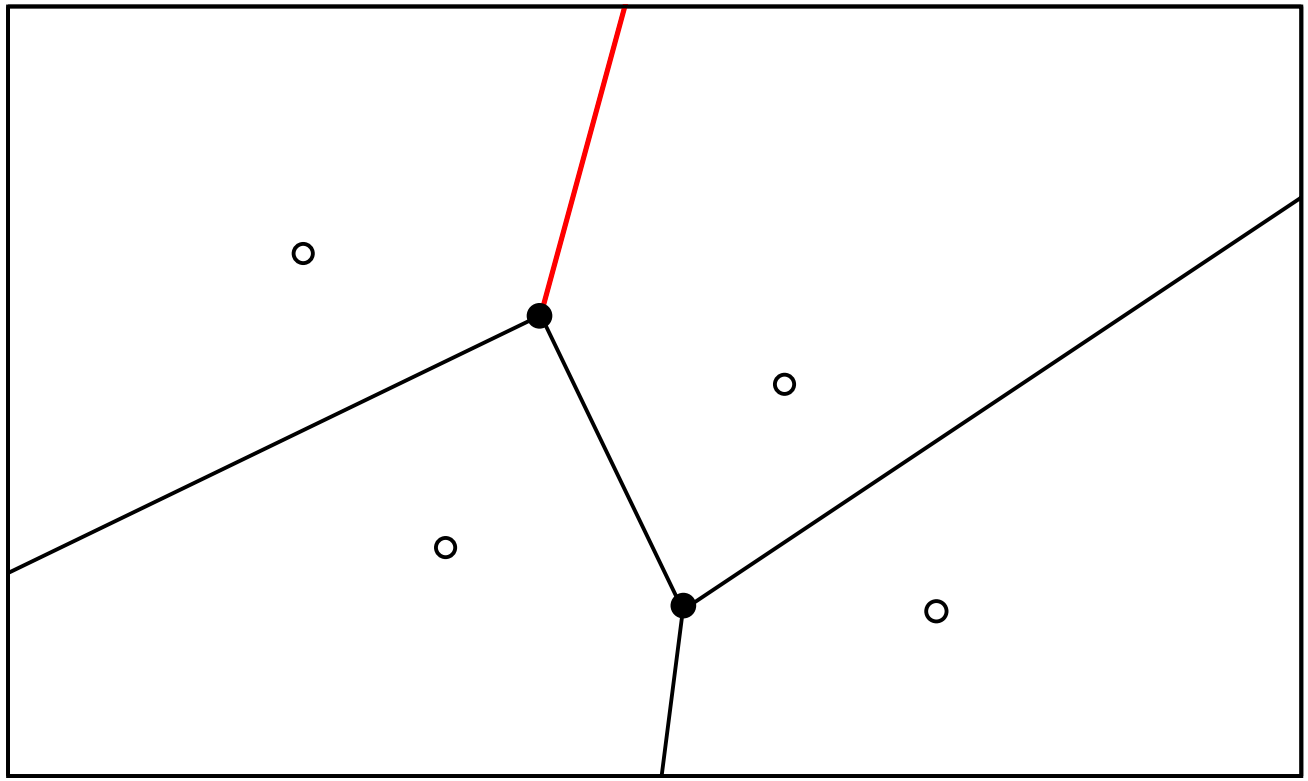
Voronoi Edge Types

finite line segment



Voronoi Edge Types

half line *or* ray



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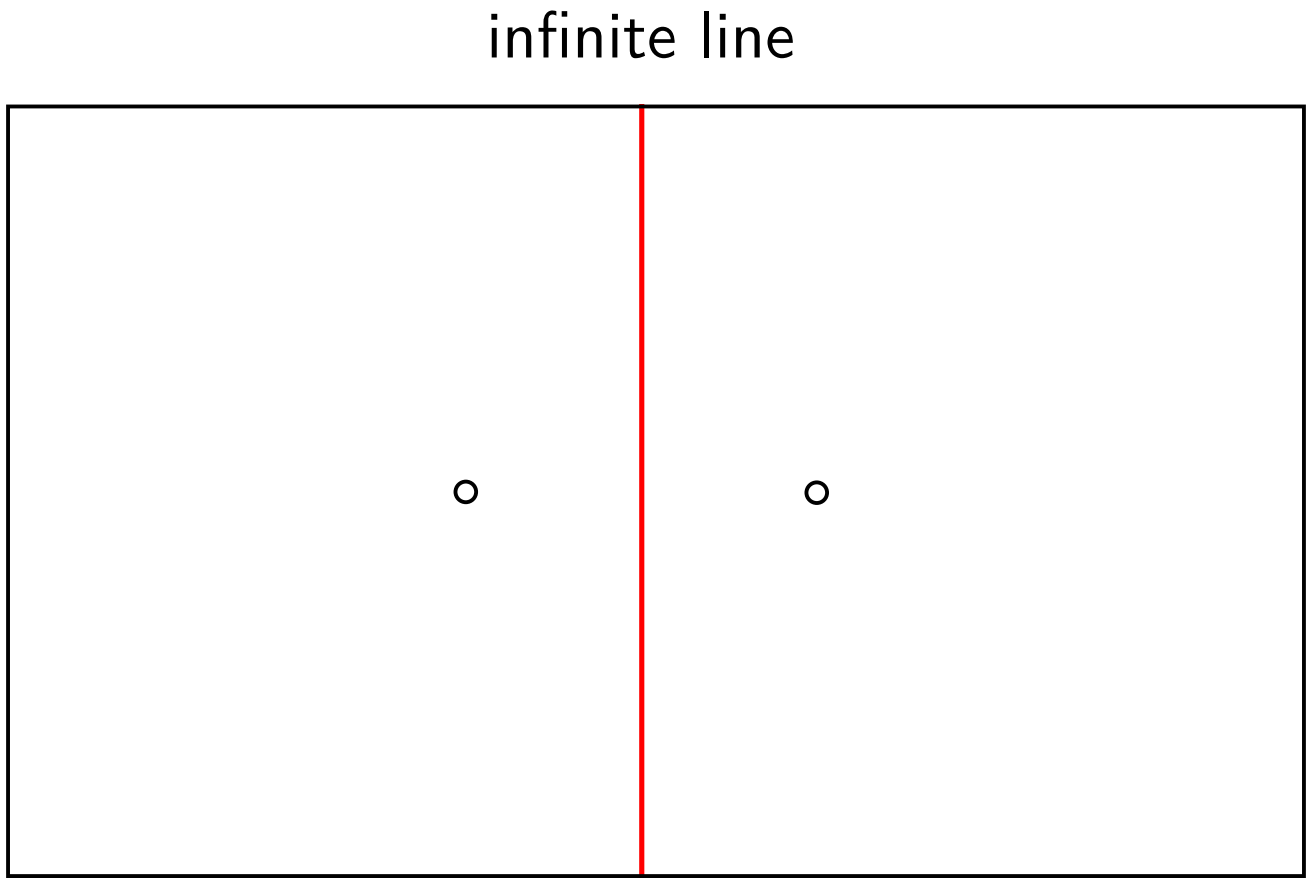

Edge Cases

Theorem 4.10

A Voronoi diagram $\mathrm{Vor}(S)$ for a point set S has infinite line edges if and only if all sites of S are collinear.

Corollary 4.11

For a point set S, the Voronoi diagram $\mathrm{Vor}(S)$ is disconnected if and only if all sites of S are collinear.

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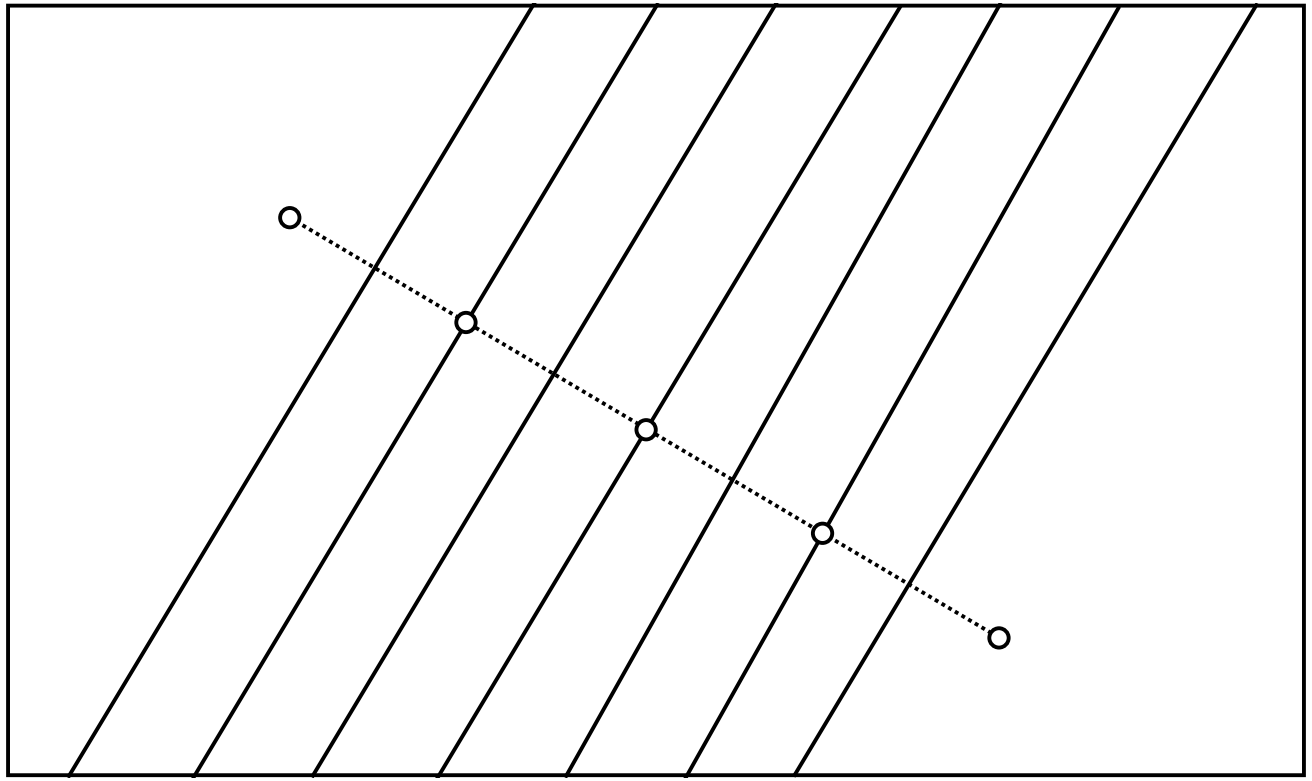

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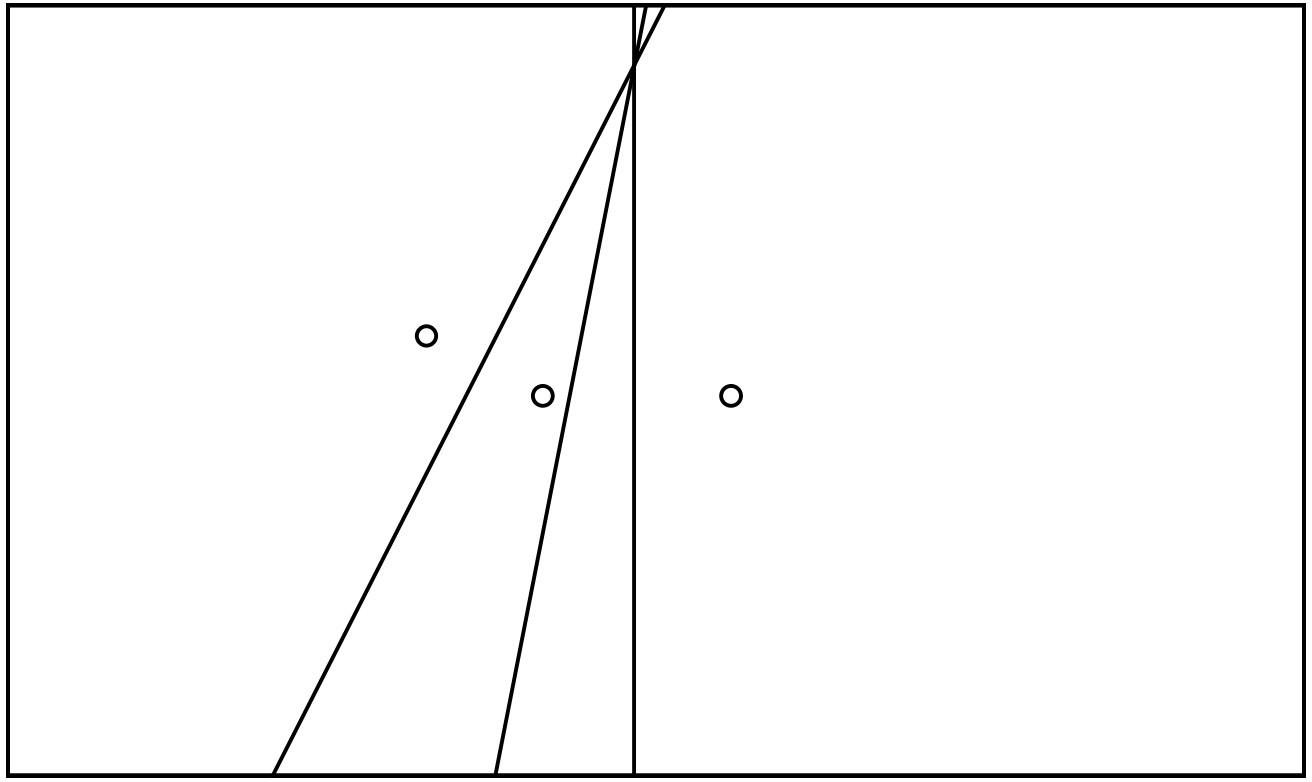

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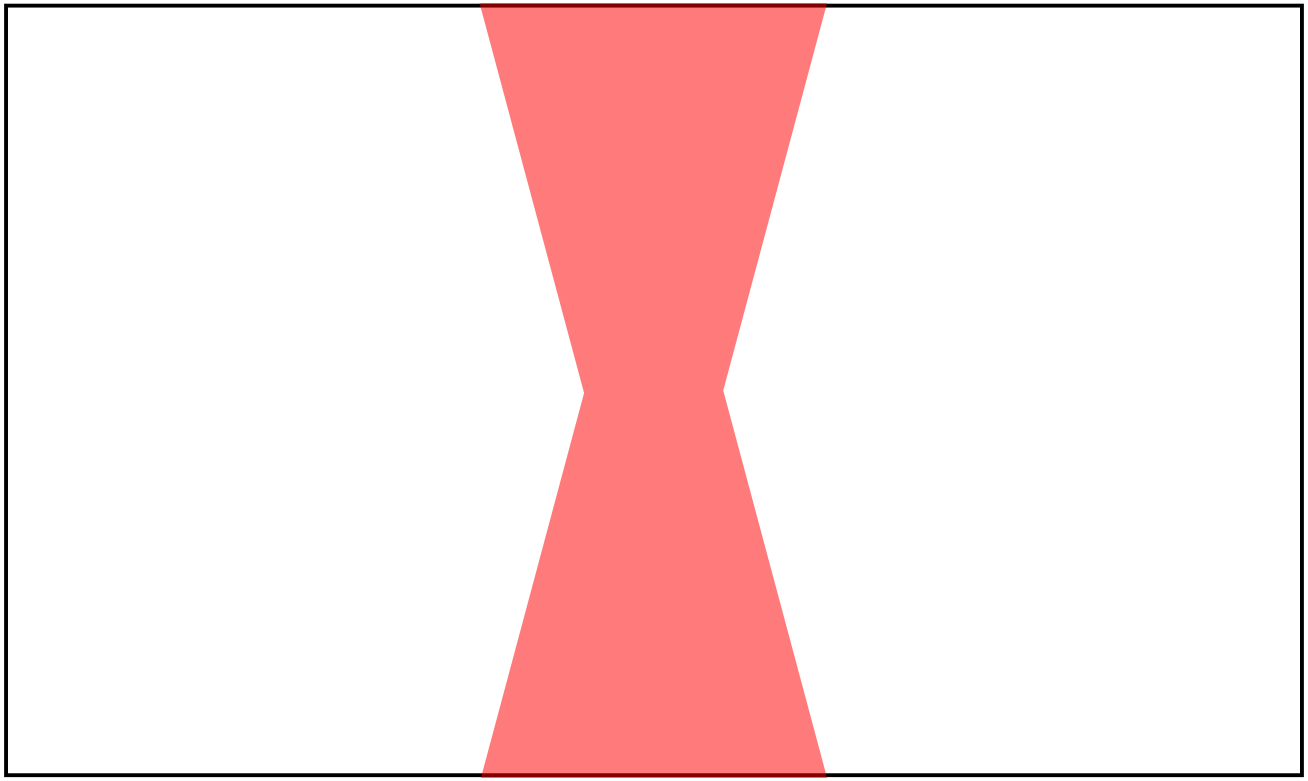

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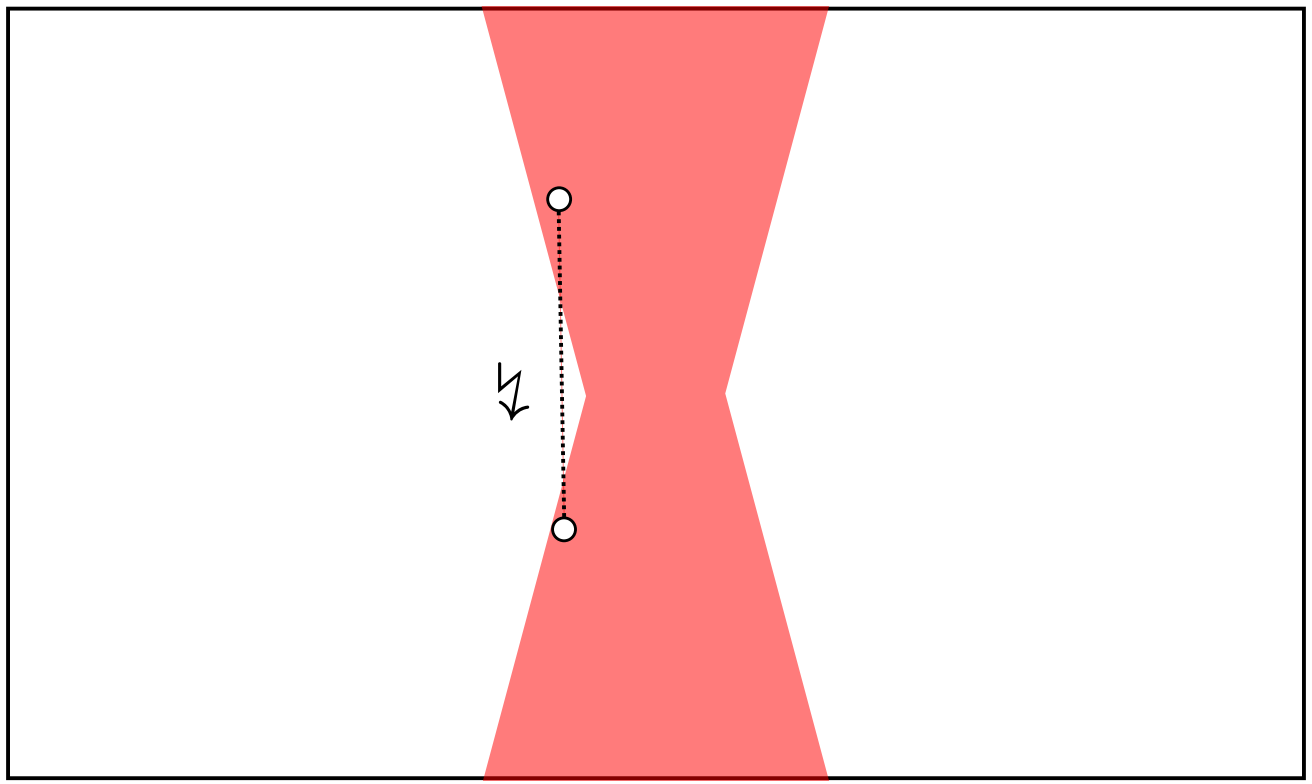
not quite



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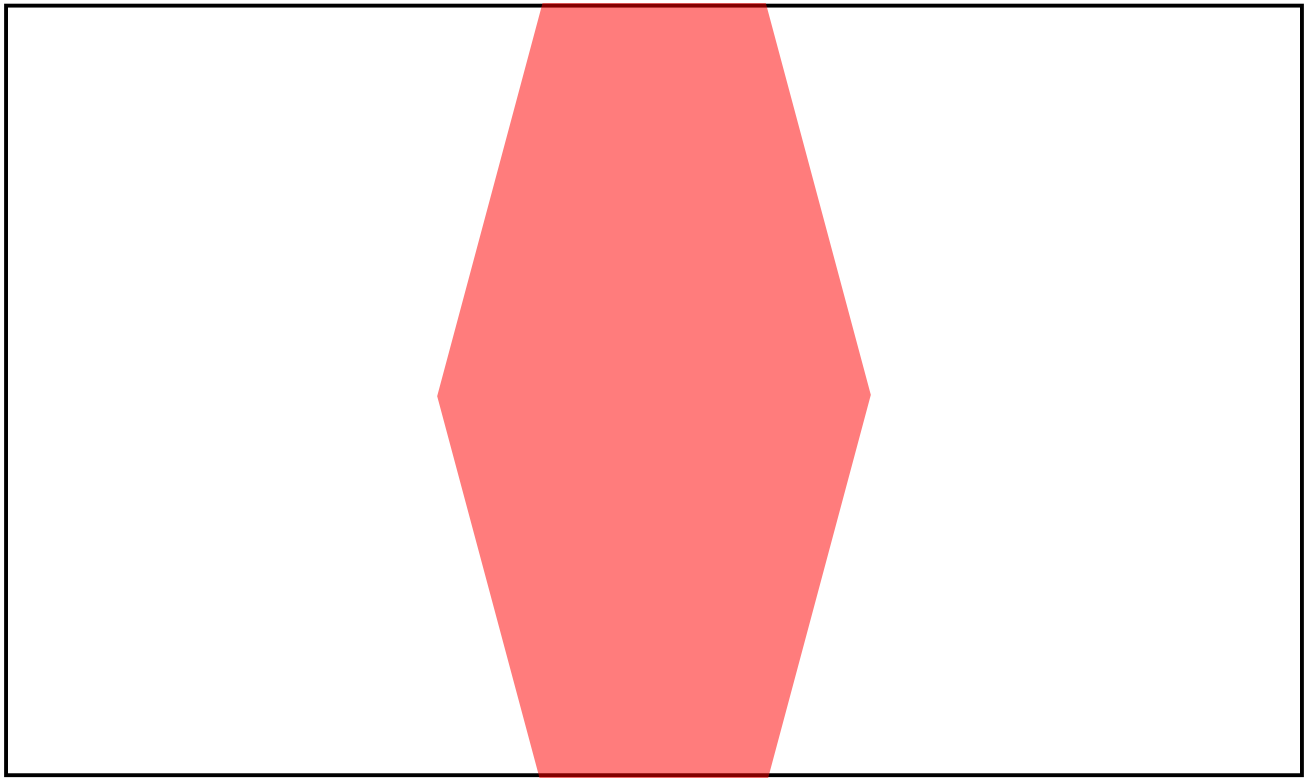

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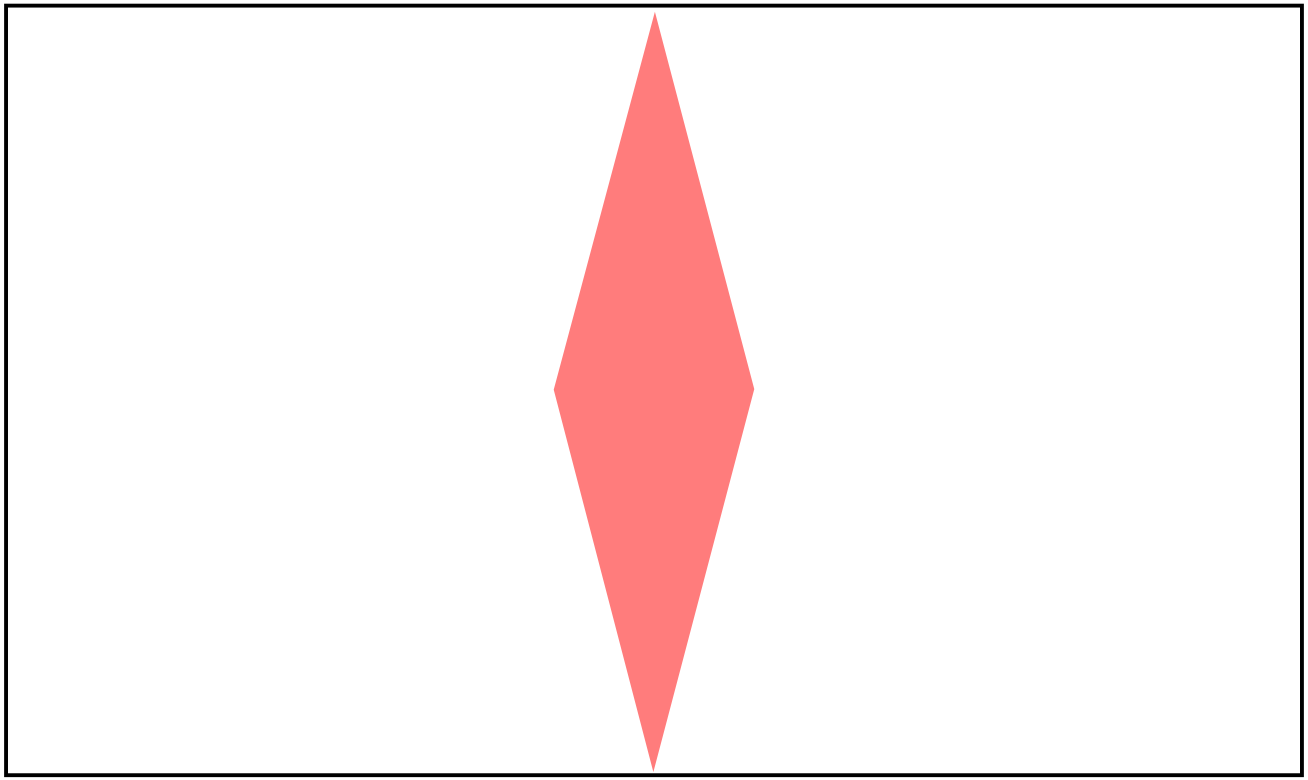

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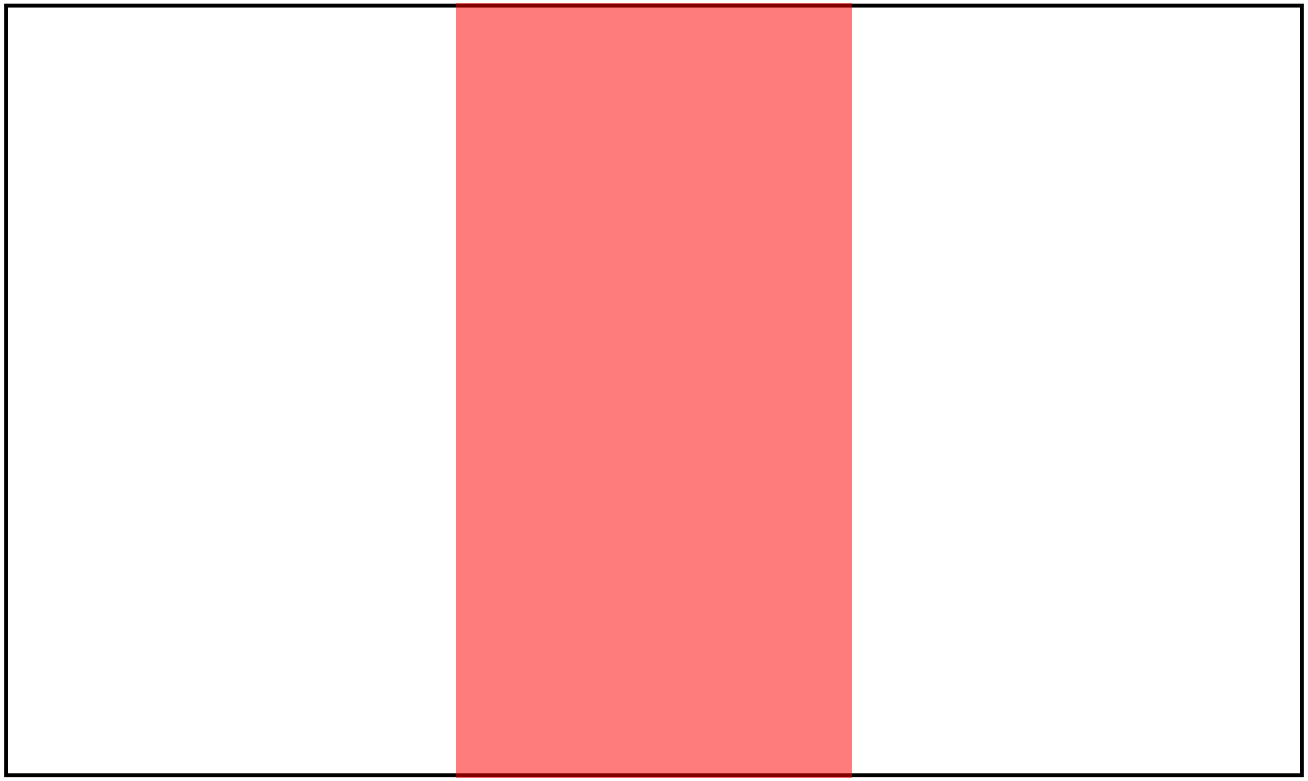

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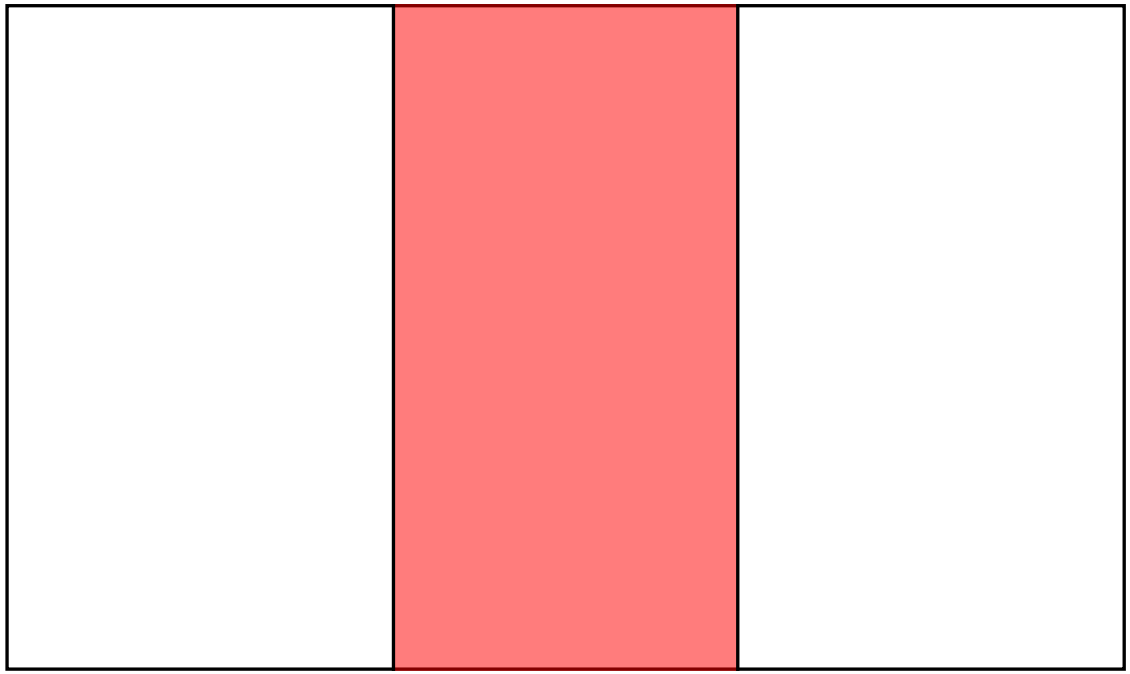


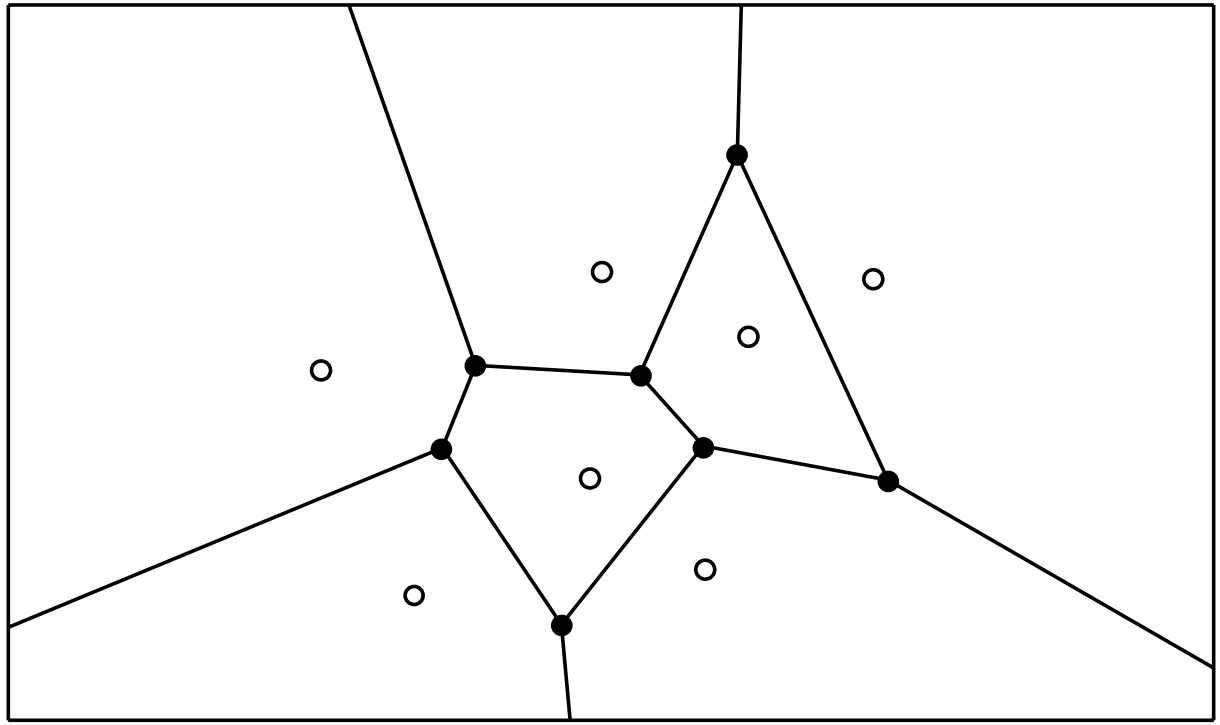
$\xRightarrow{\text{Theorem 4.10}}$ All sites of S are collinear. $\square$

Bounds of Voronoi Edges and Vertices

Theorem 4.12

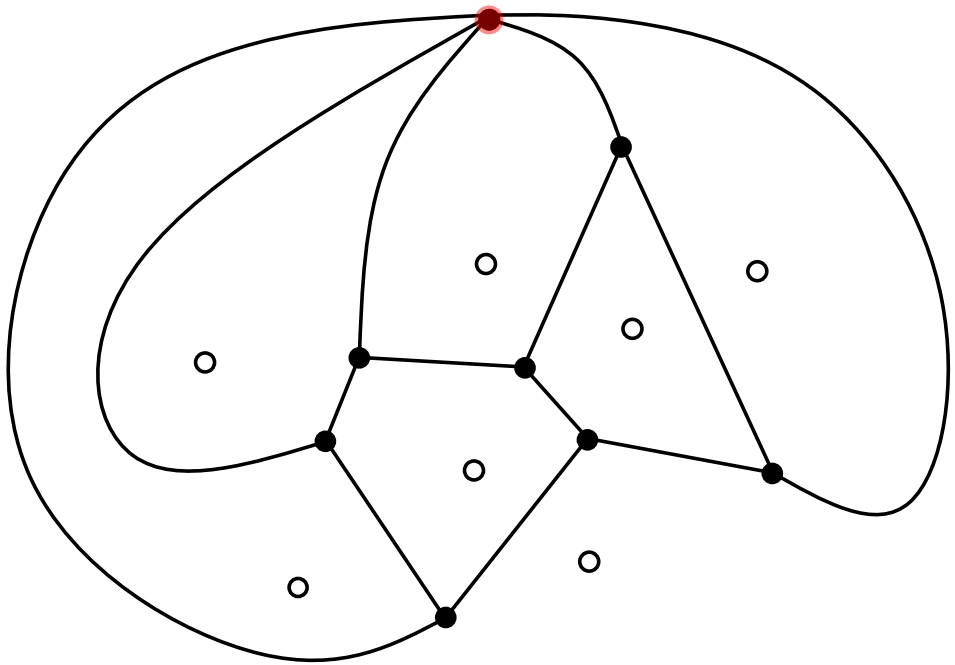
Let S be a point set with $n \geq 3$ sites. Then $\mathrm{Vor}(S)$ has at most $2n - 5$ Voronoi vertices and $3n - 6$ Voronoi edges.

Proof Theorem 4.12



Proof Theorem 4.12

planar graph G



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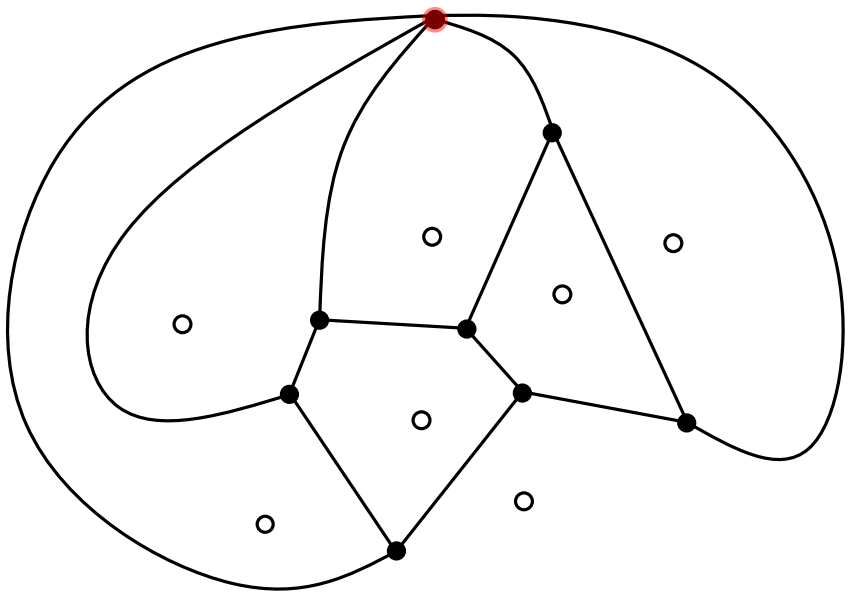

$$V - E + F = 2 \quad \text{(Euler's formula)}$$

Proof Theorem 4.12

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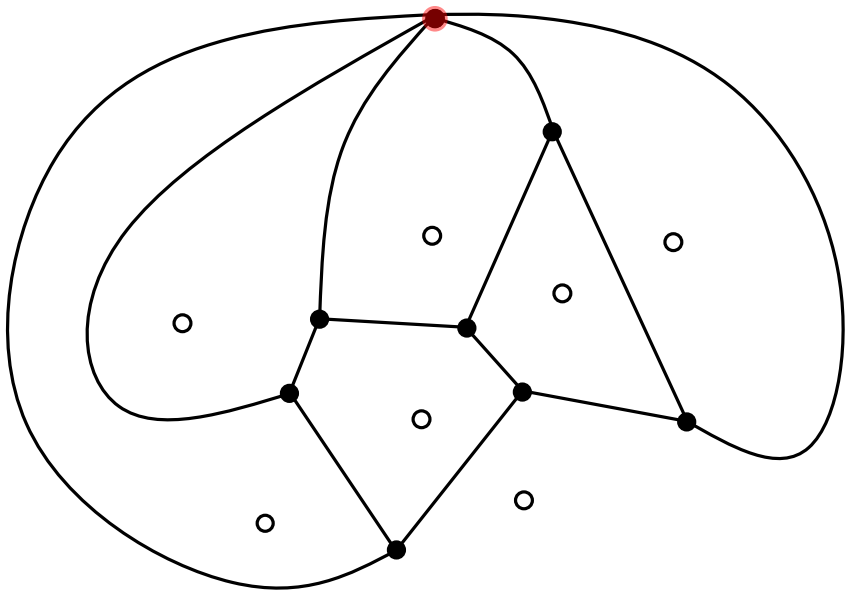

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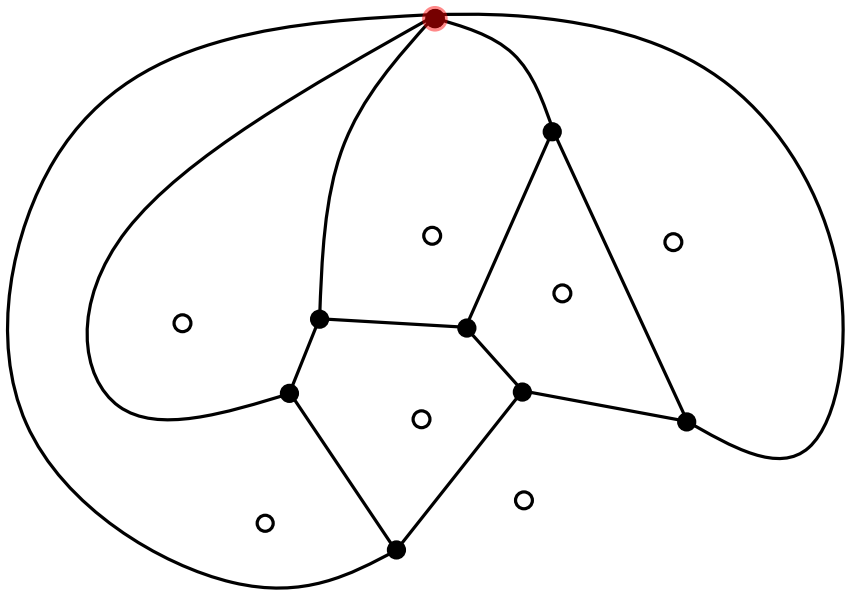

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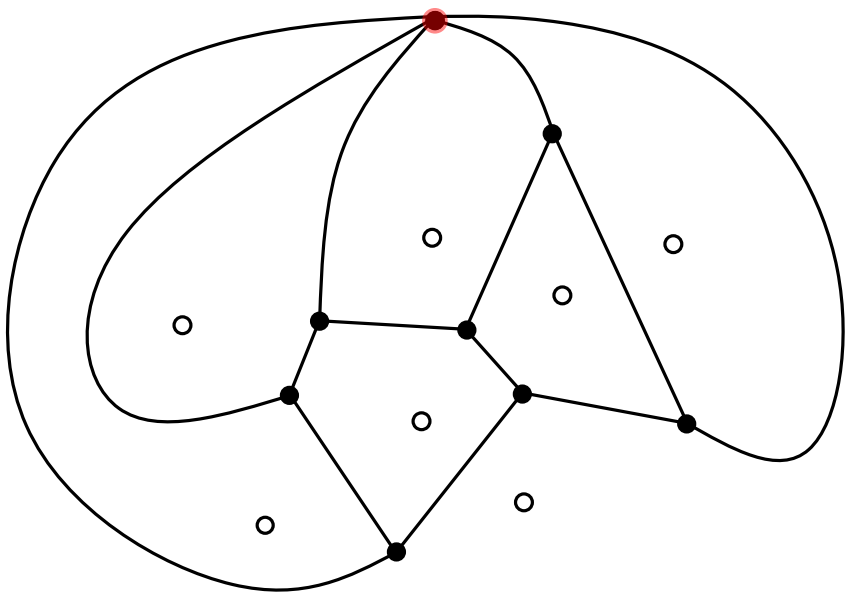

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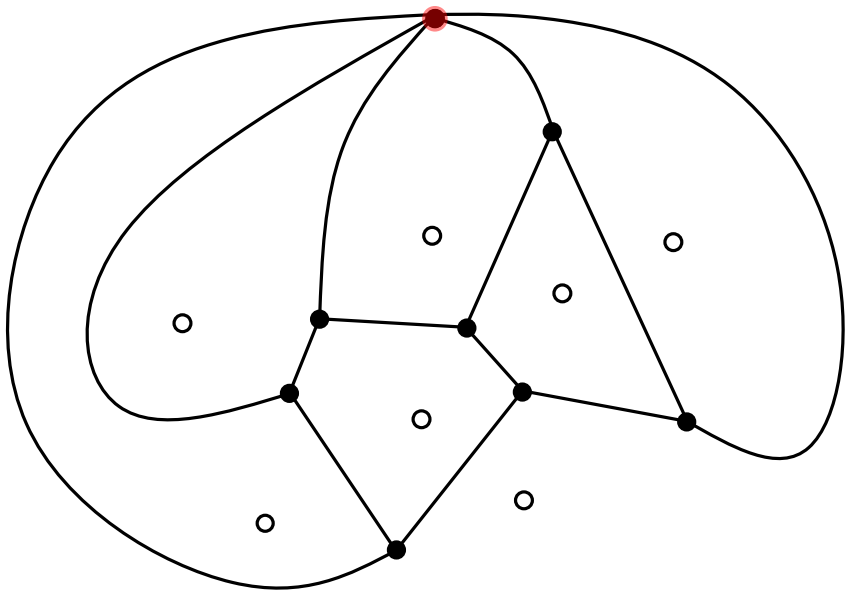
Proof Theorem 4.12

planar graph G



$$V - E + F = 2 \quad \text{(Euler's formula)}$$

$$\Rightarrow (v+1) - e + n = 2$$

$$3(v+1) \leq 2e$$

$$\Rightarrow$$

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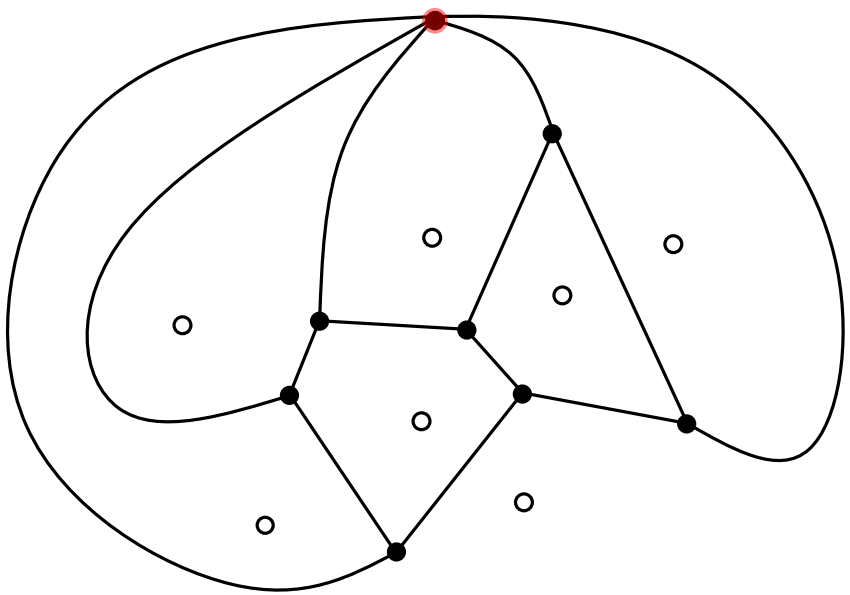
Proof Theorem 4.12

planar graph G



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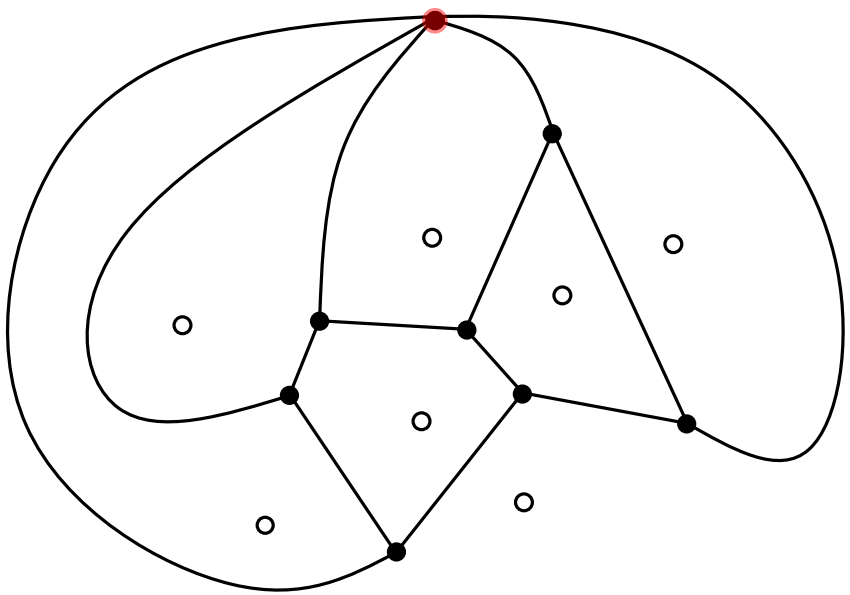
planar graph G



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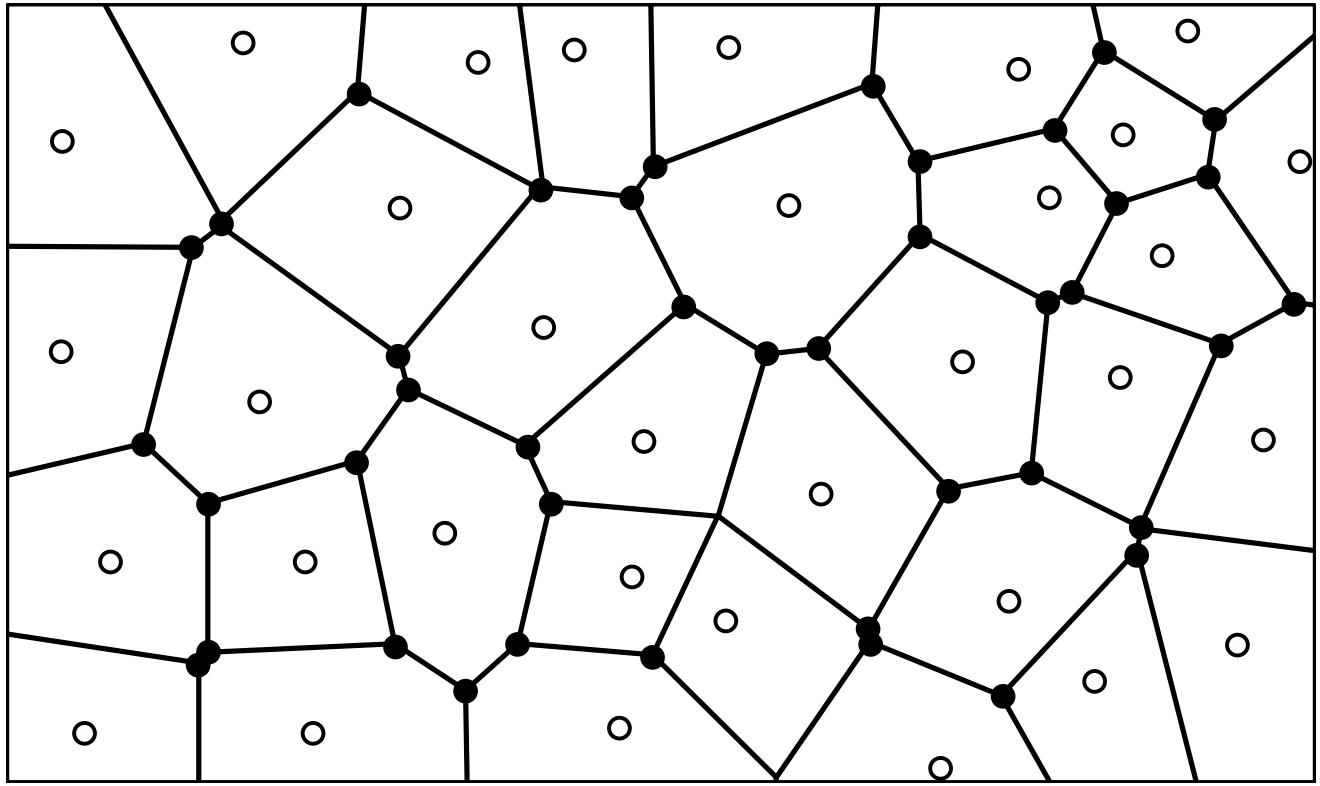
Algorithmic Construction

Walkthrough



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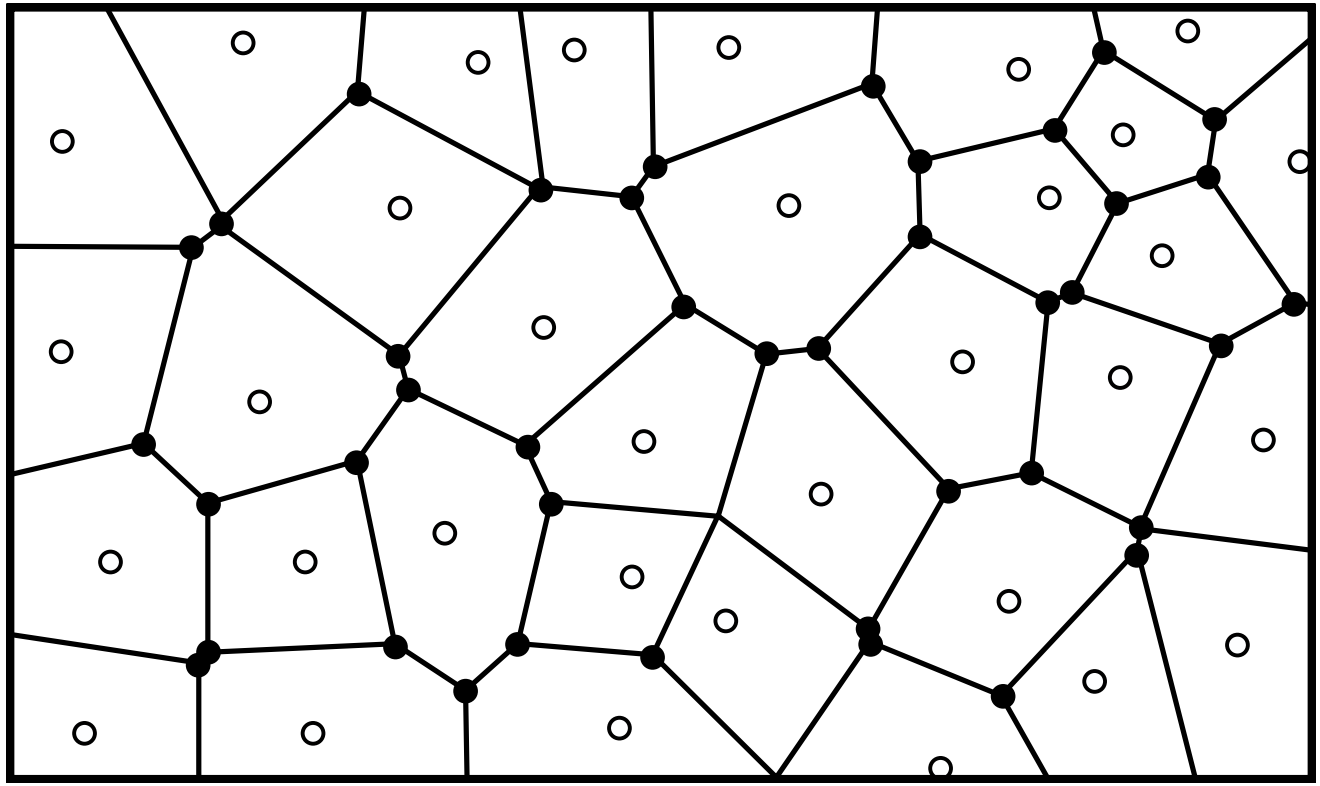

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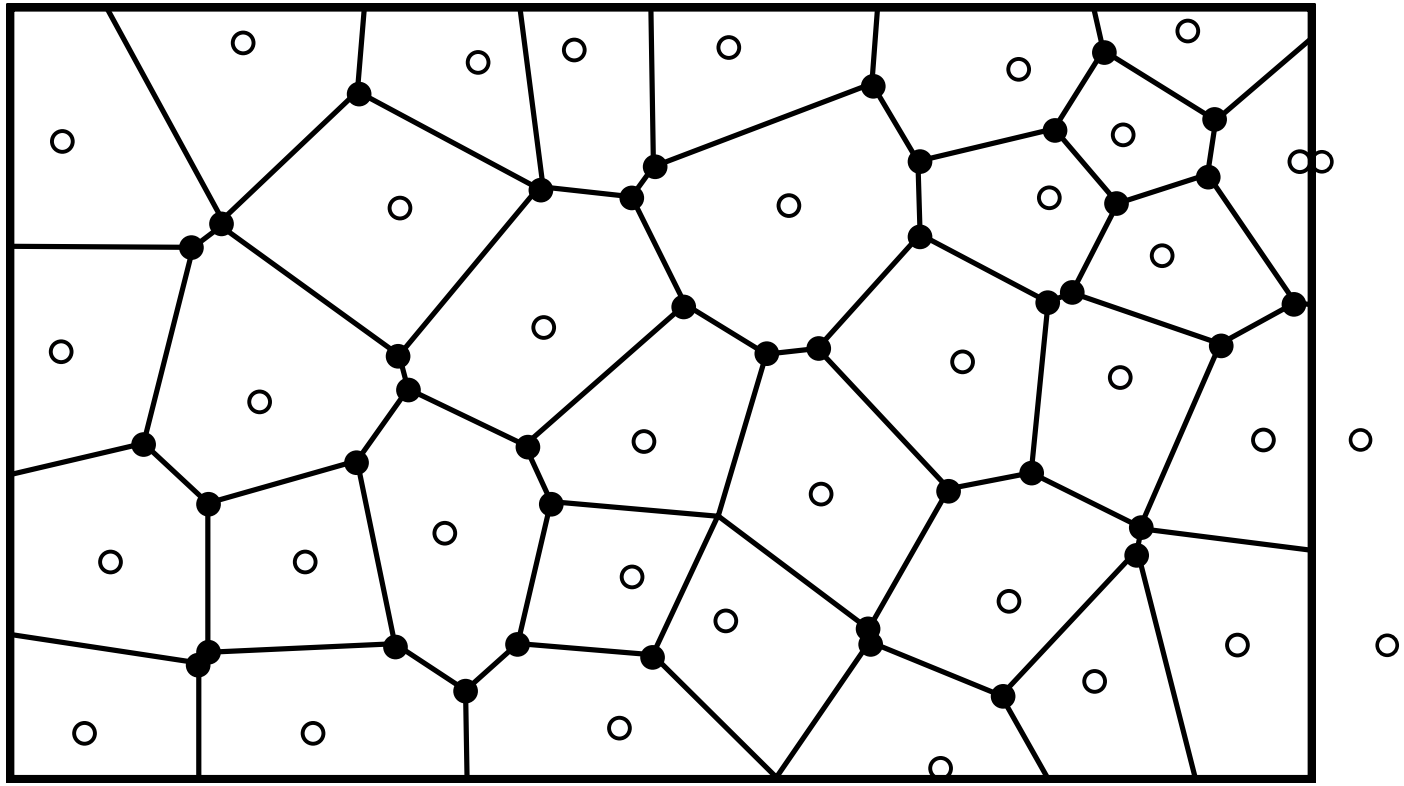

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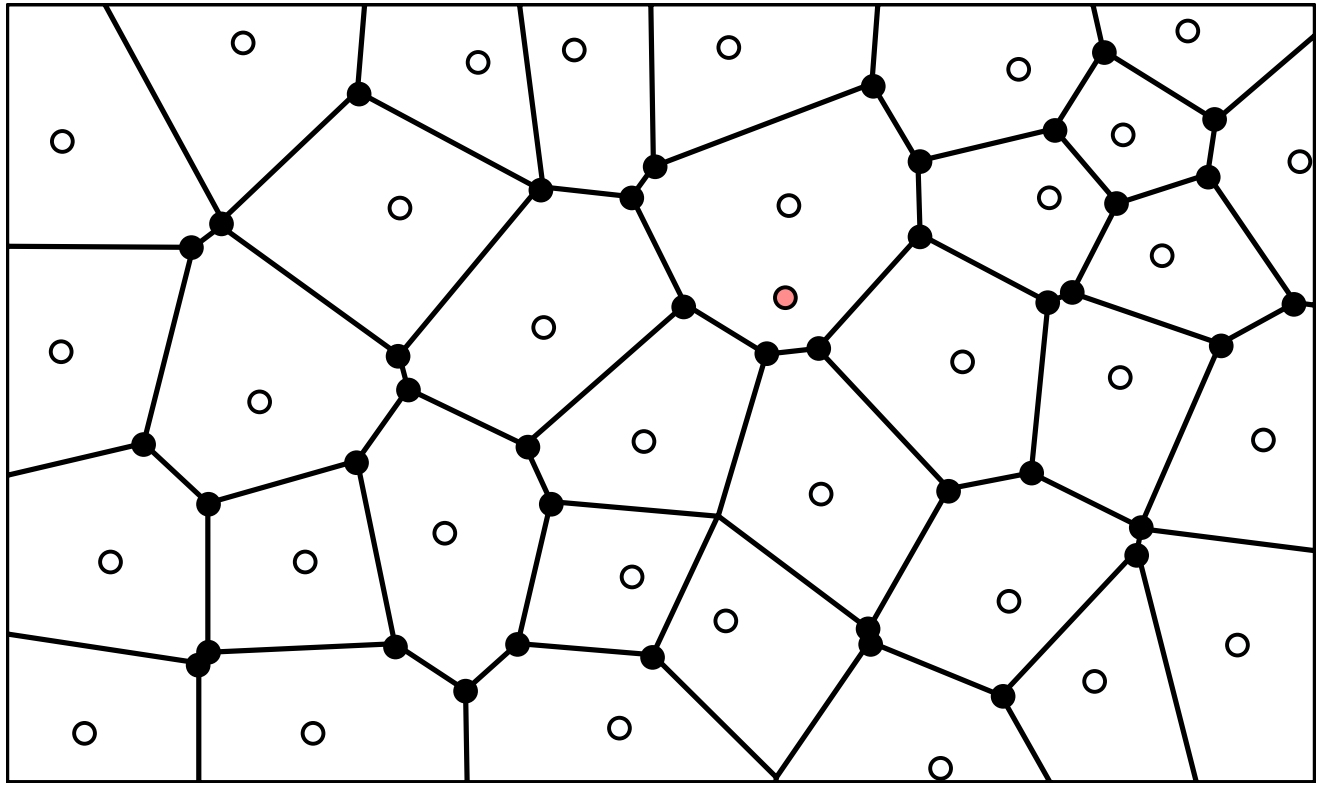

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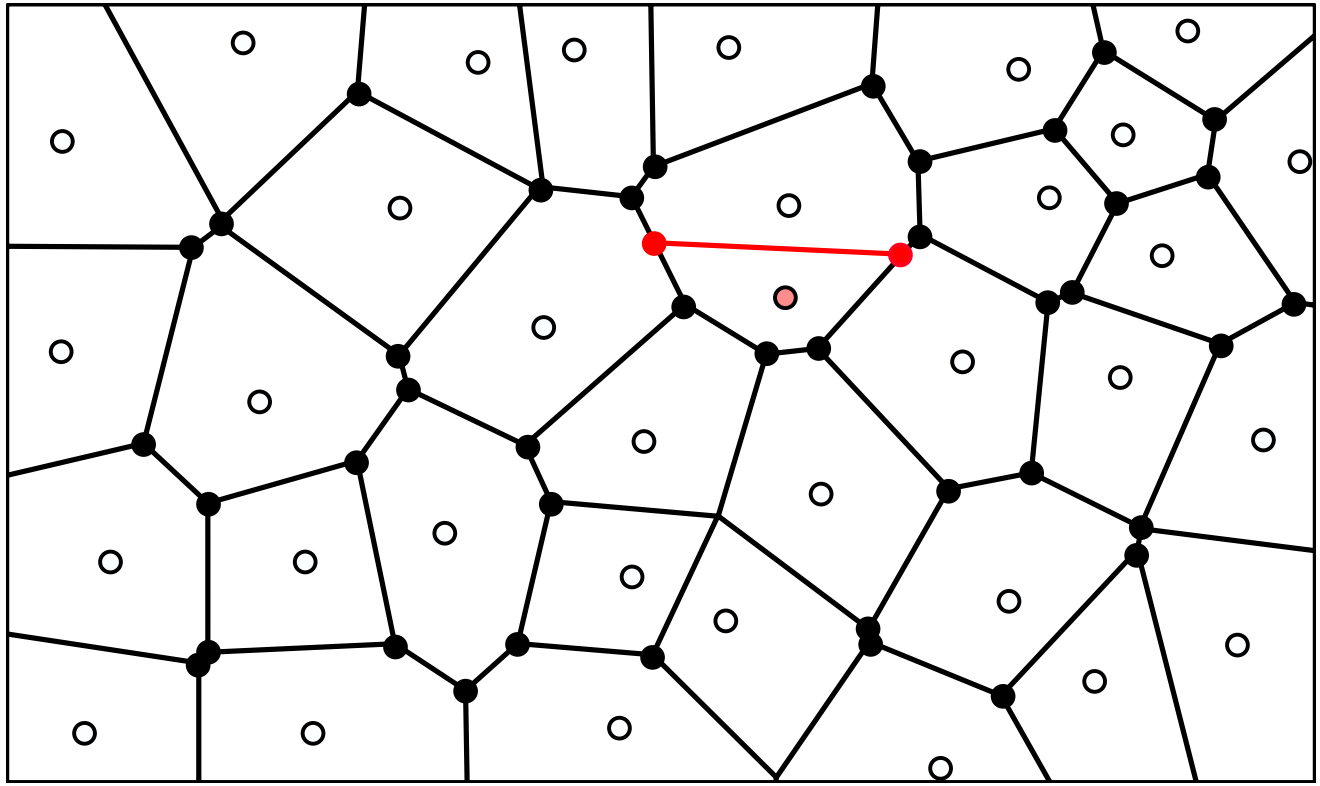

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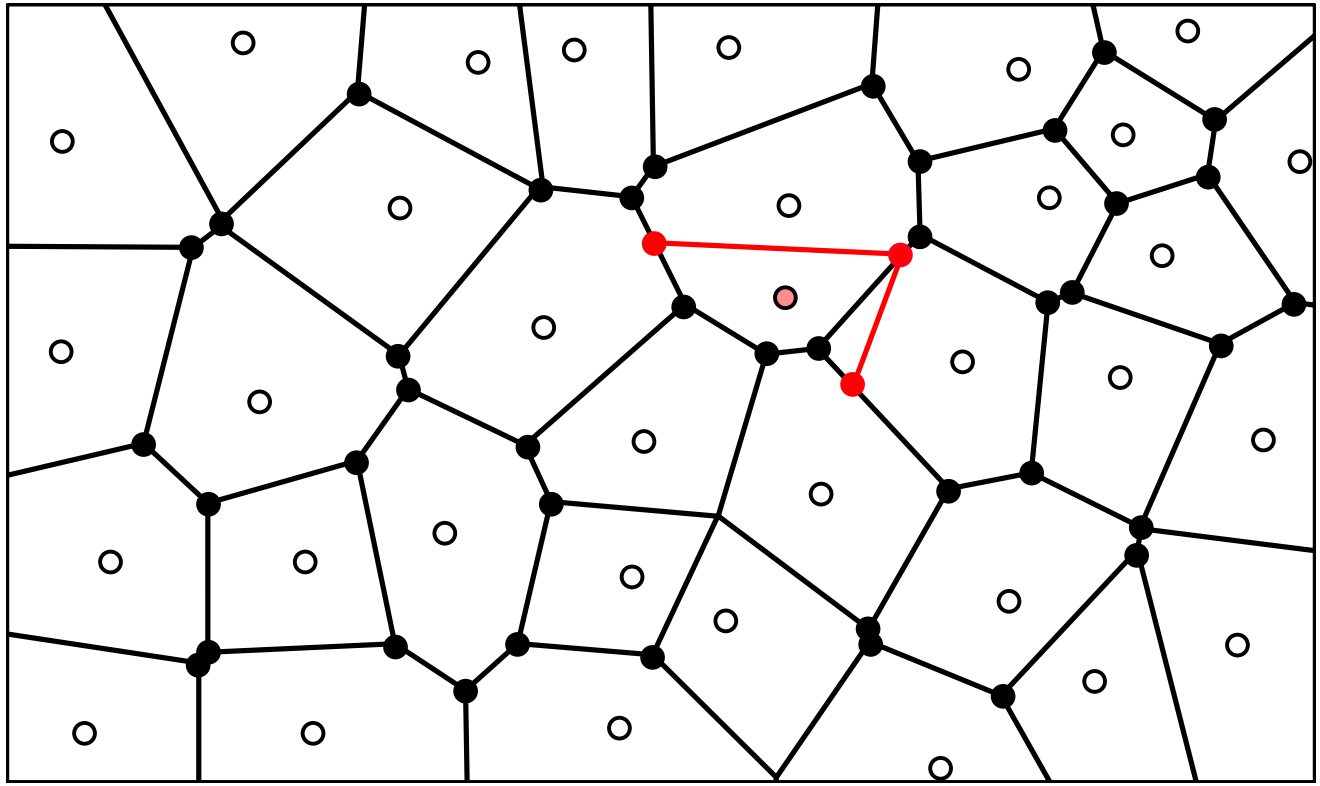

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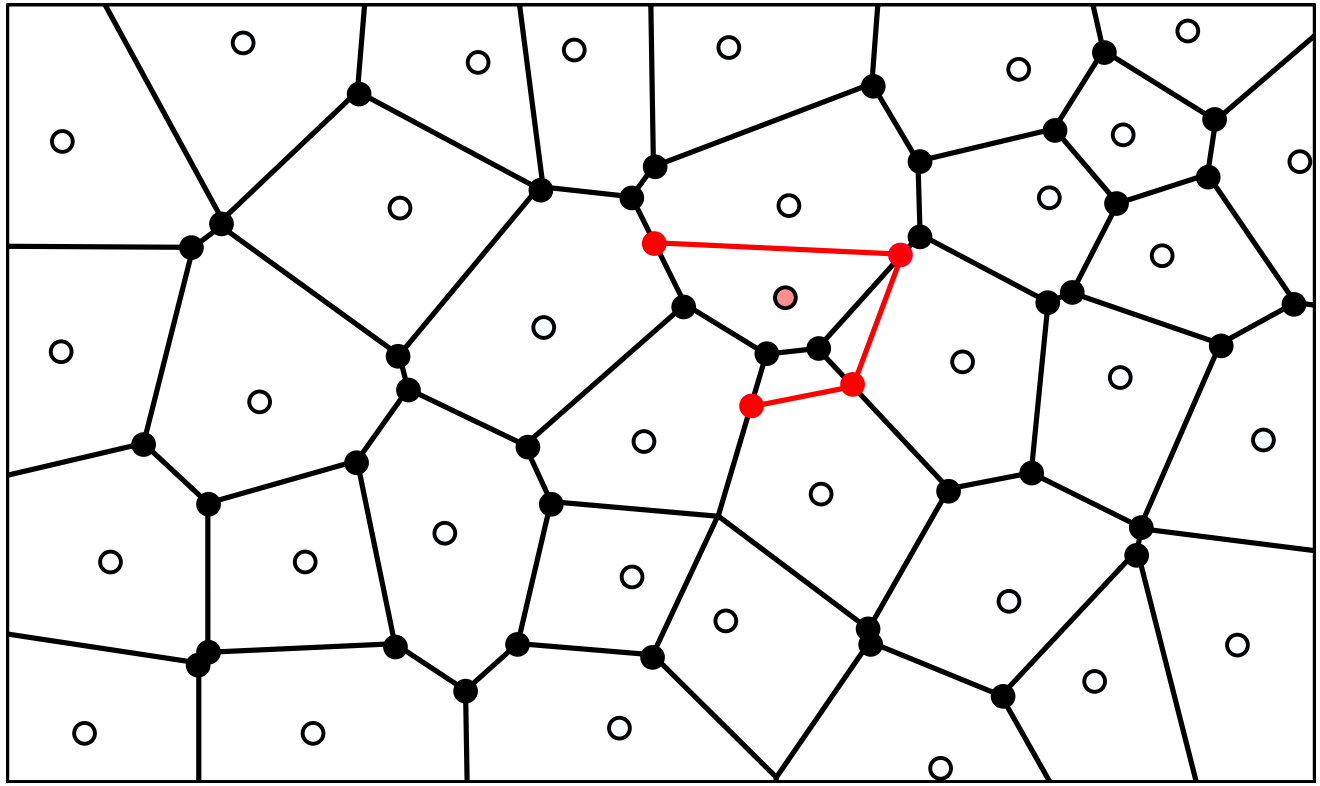

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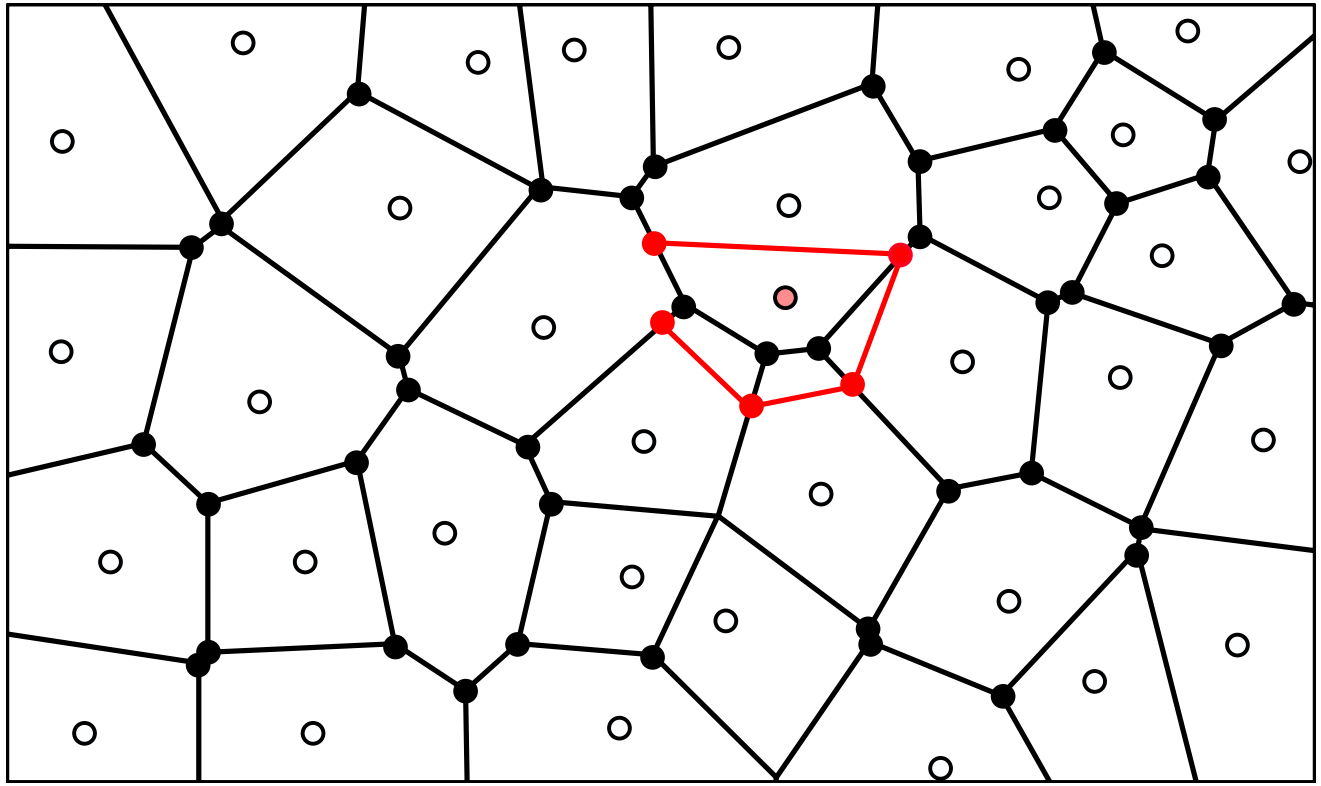

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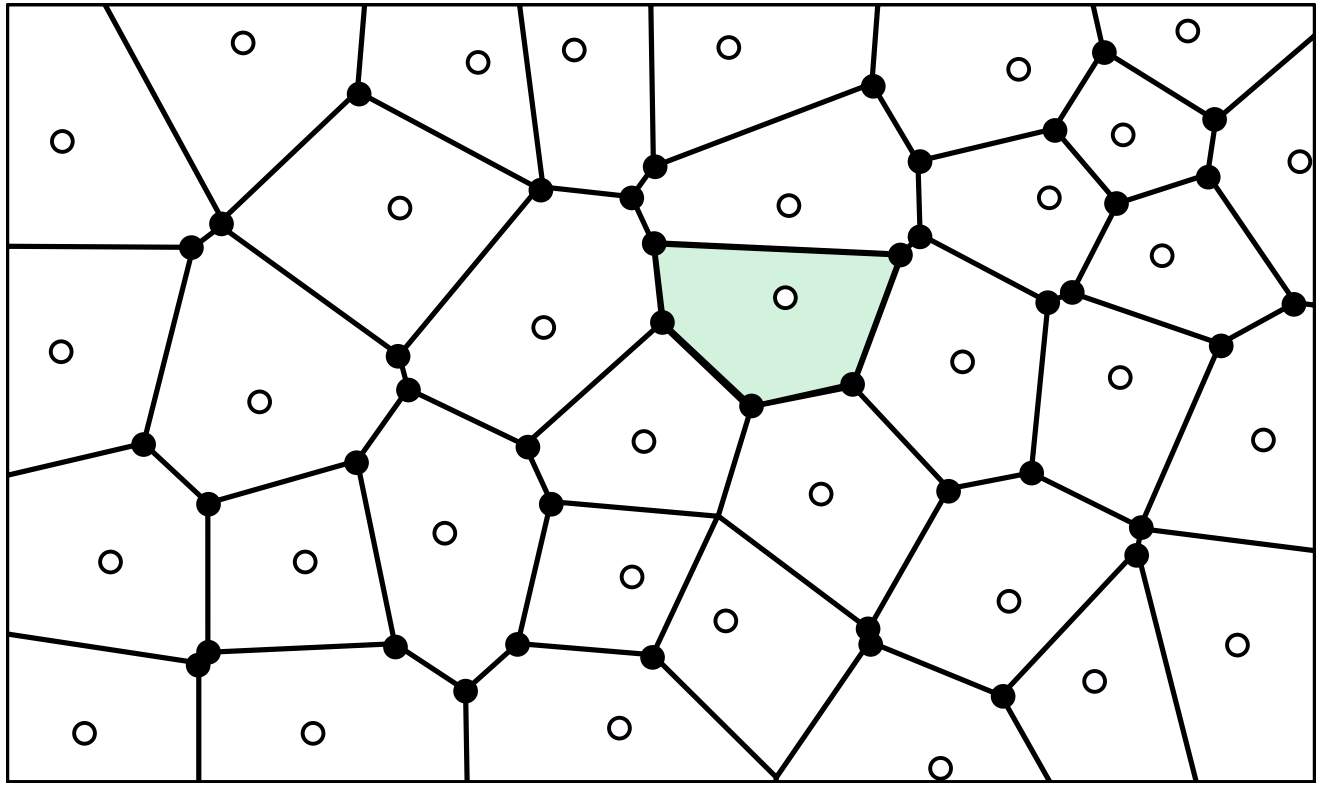

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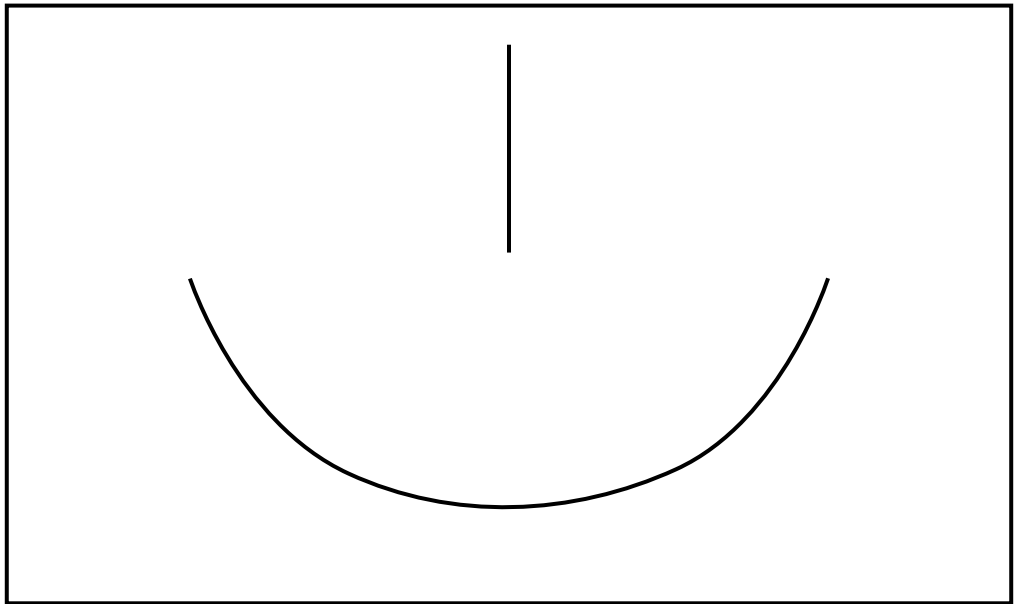

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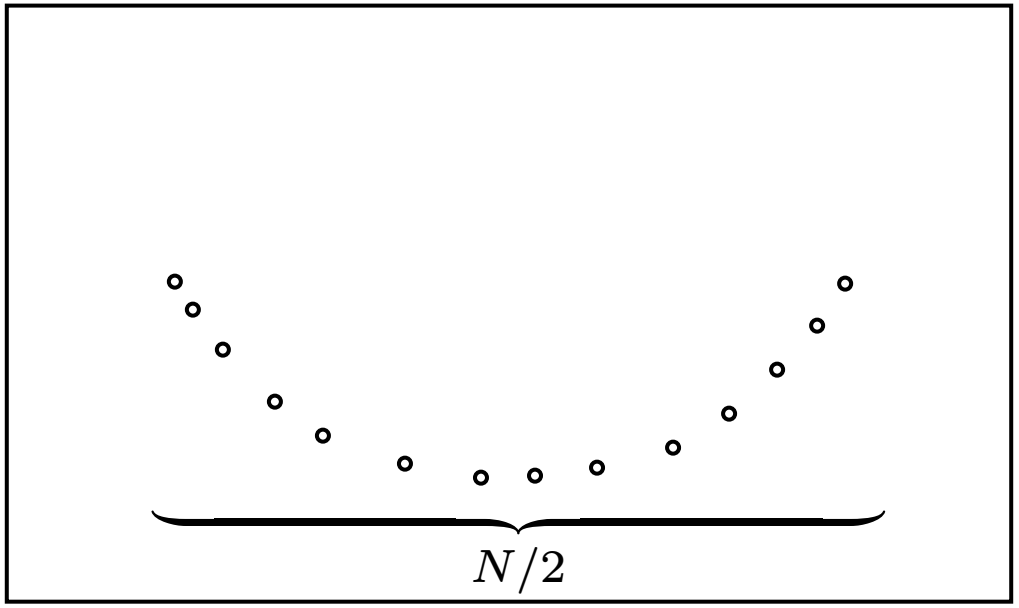

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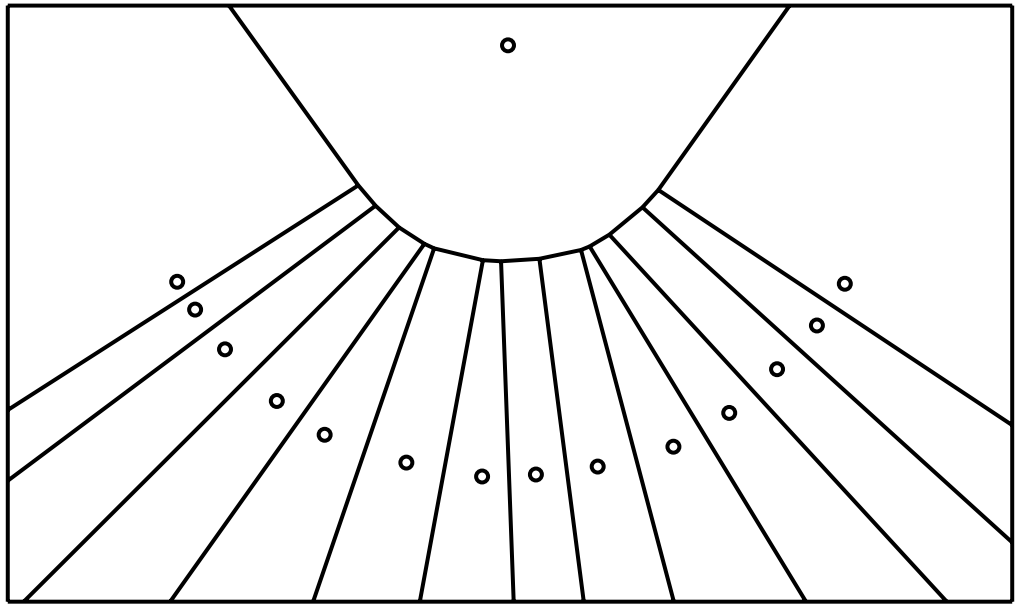

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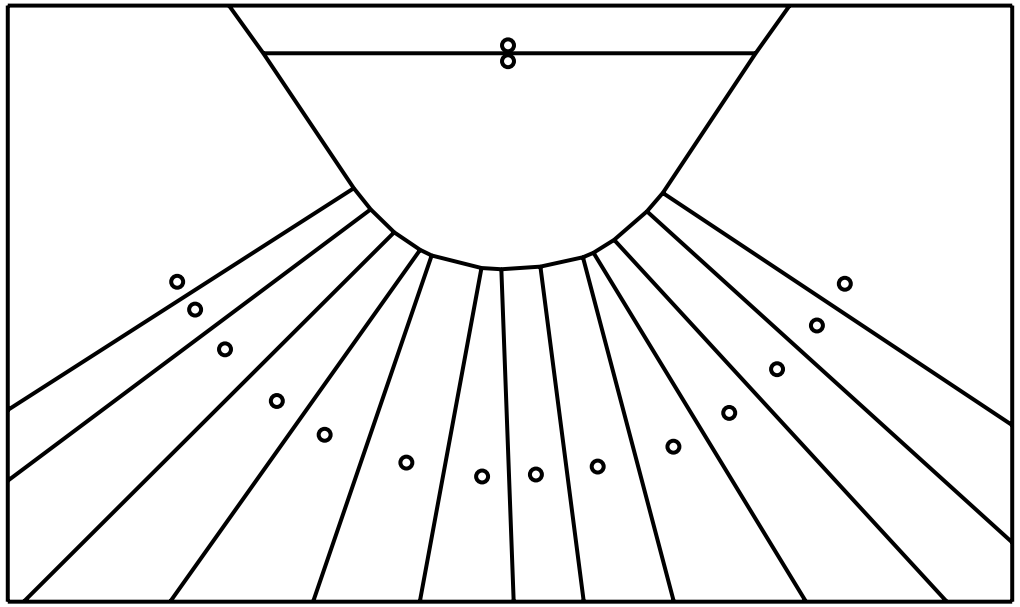

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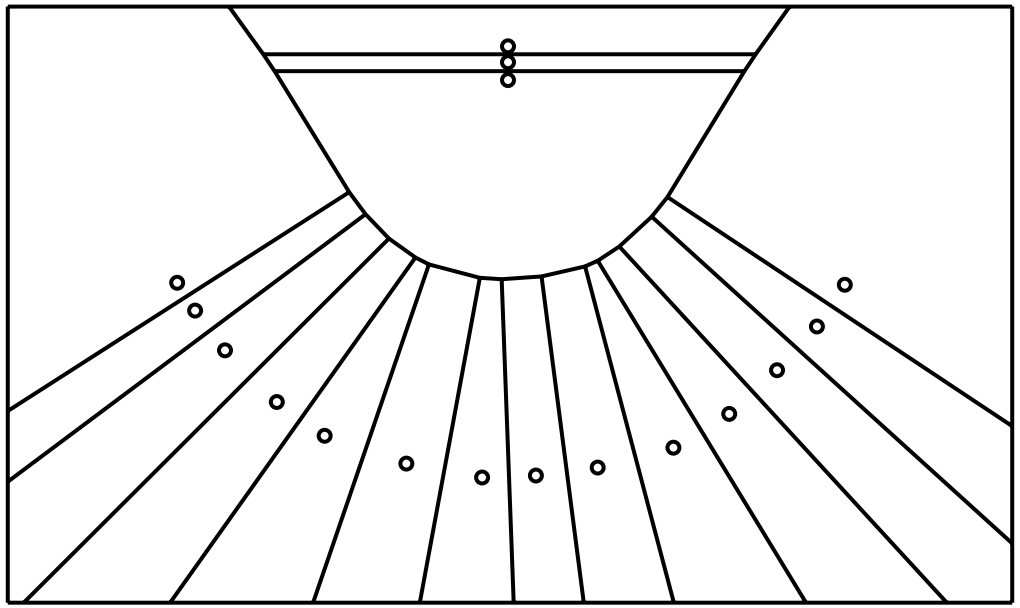

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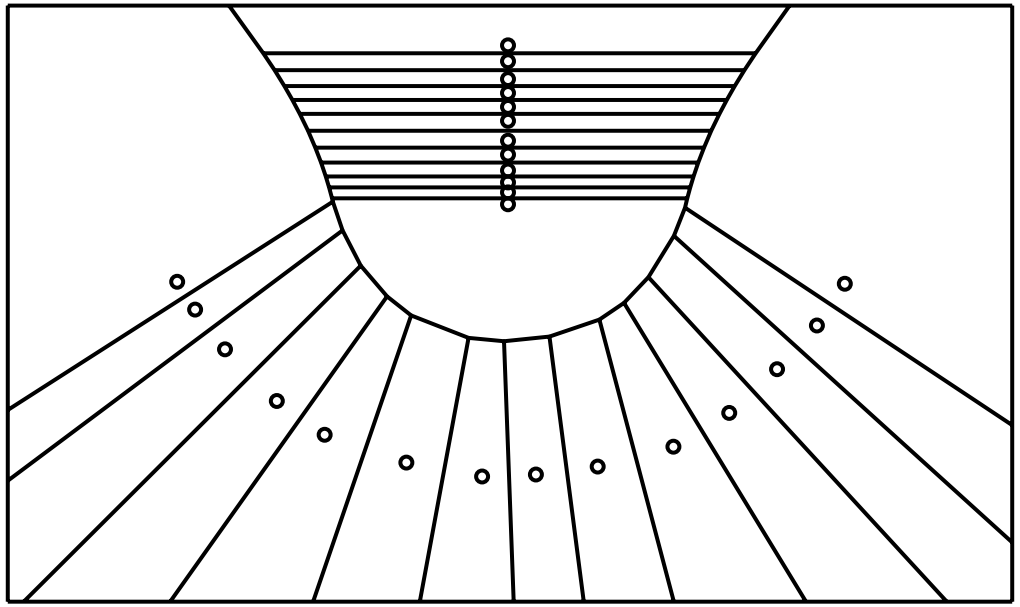

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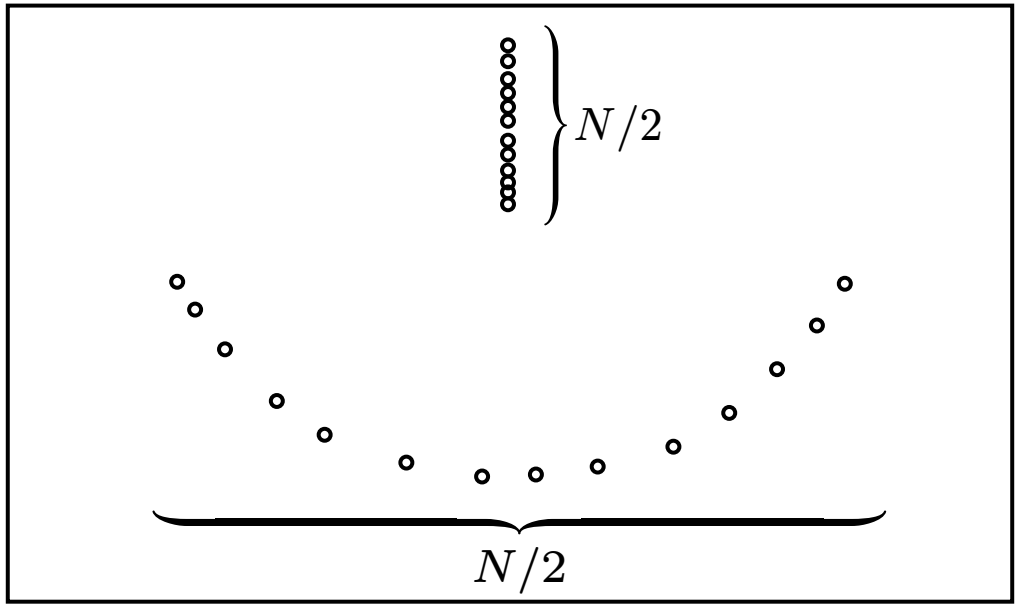


Complexity

Time Complexity: $O(n^2)$

- We insert n sites for each we may have $O(i)$ edits.

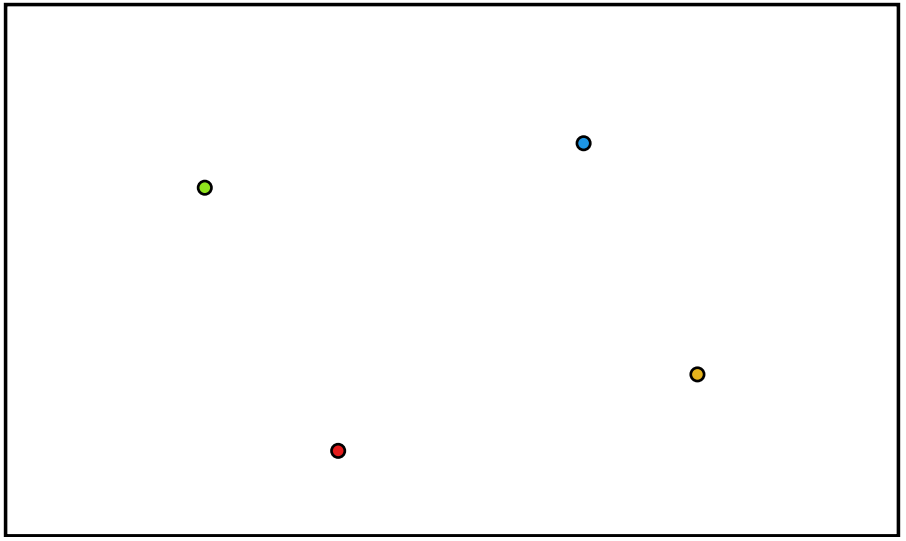
Space Complexity: $O(n)$

- Fixed overhead: Constraints, auxiliary functions etc.
- Data dependent: Voronoi Vertices and Edges, linear in number of sites.

Voronoi Diagrams in the Wild

Voronoi Patterns in Nature

- Voronoi patterns frequently emerge in nature due to the *dynamics of growth* and *competition* among organisms.



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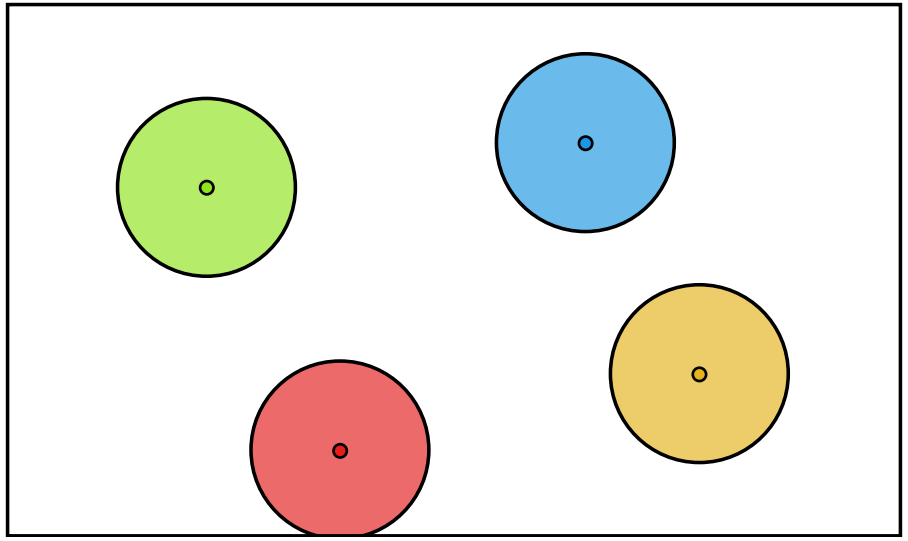

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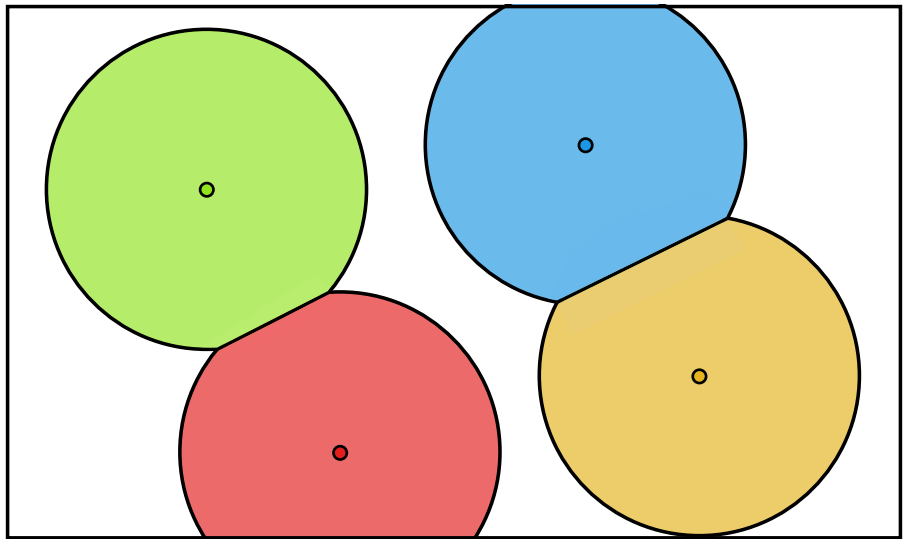

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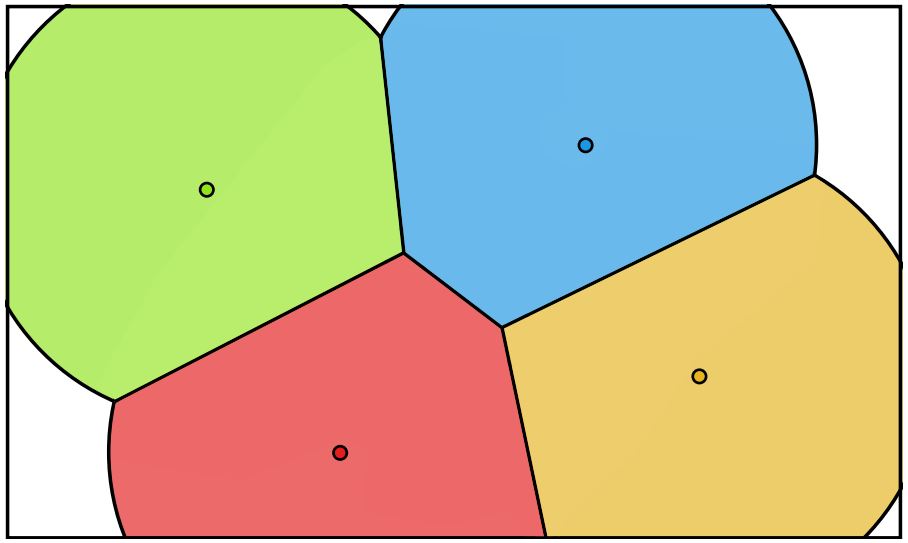

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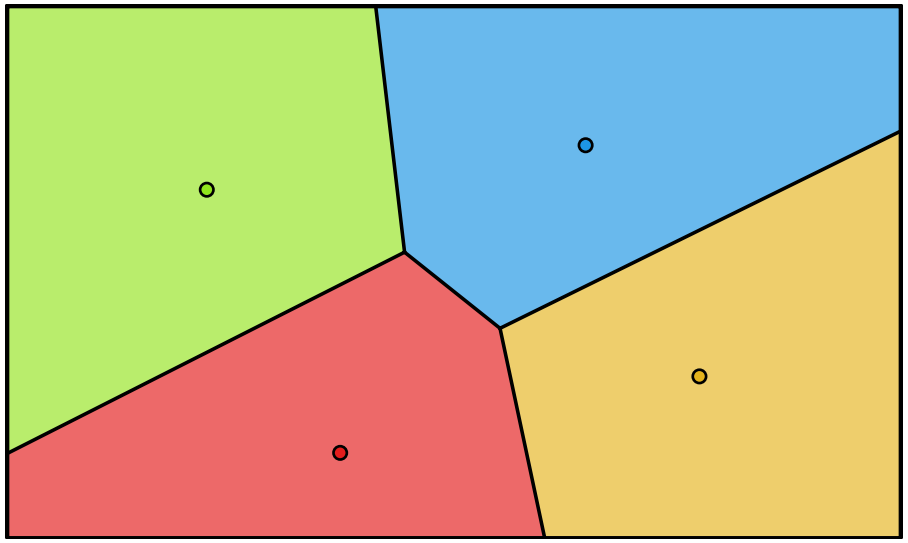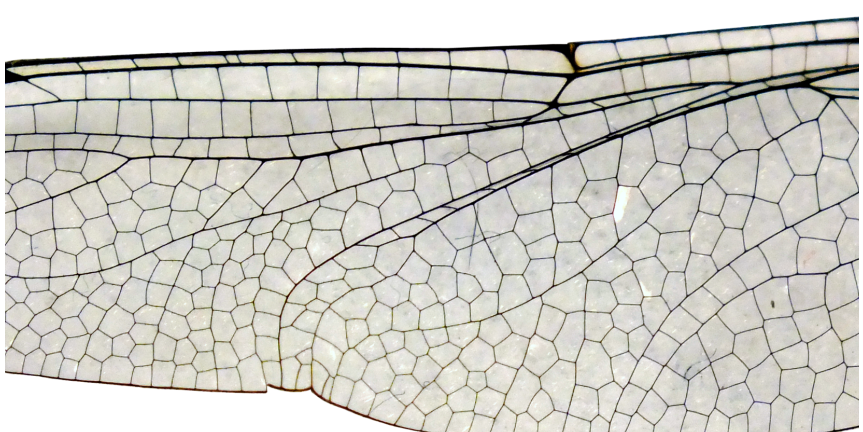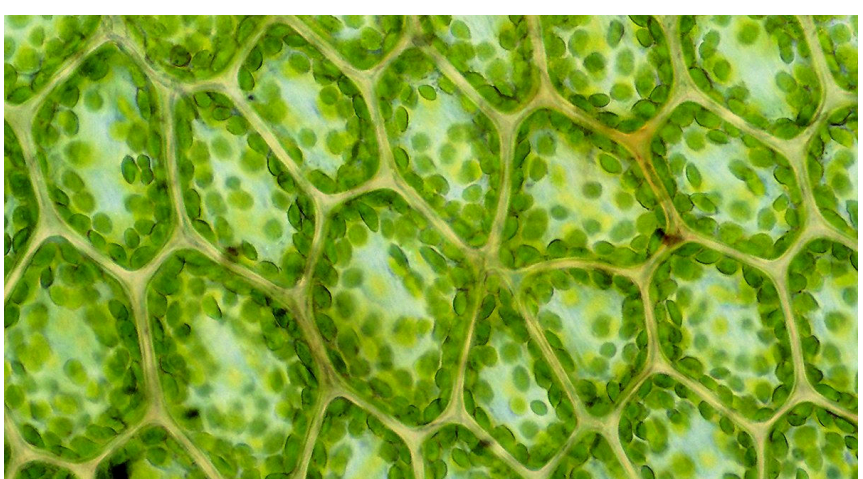
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