Lecture Notes for

Algebraic Topology I

Lecturer

Stefan Schwede

Notes typed by

Michele Lorenzi

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University of Bonn

This document will (hopefully) contain lecture notes for the course *Algebraic Topology I* given by Prof. Stefan Schwede at Bonn University during the winter semester 2021/22.

The illustrations are made by Álvaro Gutiérrez. Thanks Álvaro!

Everything in these notes should be taken with a grain of salt (at least for now), I'm new to the material, to real time TEXing and I tend to be late to class more often than not.

Eventually I plan to make these notes into a nice reference, maybe adding some (well written) solutions to some important exercises or useful comments, so any feedback on how to improve the notes is appreciated! I have a GitHub repository for the notes, you are welcome to use the Issues tab to report any errors or typos, or make any correction/remark/suggestion (or you can just tell me).

When the "Álvaro pls" Signal /\ appears in the margin, it indicates that I would like some picture to be added in that place eventually.

Also a lot of thanks to Paul, Yikai, Zhu, for lending me their notes/photos of the blackboard when I was missing stuff.

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Introduction

Lecture 1
11^{th} Oct, 2021

In the Topology I class given by Prof. Schwede last year, two important homotopy invariant functors were defined:

- The singular homology groups $H_n(X;\mathbb{Z})$. The definition of these groups is quite involved, but they are relatively easy to compute (e.g. by cellular homology). In the case of the spheres we have:
$$\tilde{H}_n(S^k;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$
- The homotopy groups $\pi_n(S^k, *)$. These groups are instead easy to define, but really difficult to compute. In the case of the spheres their calculation becomes complicated already for $n > k$:
$$\pi_n(S^k) = \begin{cases} 0 & \text{if } n < k \\ \mathbb{Z} & \text{if } n = k \\ ??? & \text{if } n > k \end{cases}$$
As of today (and most likely as of tomorrow too) still a lot is unknown about the higher homotopy groups of the spheres, and those we do know display an apparently erratic behaviour.

Homotopy groups are so hard to compute in general that, as a matter of fact, *there is no non-contractible, simply connected finite CW-complex for which all homotopy groups are known.*

A First Look to Hurewicz Theorem

An important result about homotopy groups is a theorem due to Hurewicz relating the first non-trivial homotopy and homology groups under certain hypotheses:

I.1. Theorem (Hurewicz)**.** — *Let $n \geqslant 2$ and let X be an $(n-1)$-connected based space. Then $H_i(X;A) = 0$ for all $0 < i < n$ and any abelian group A and the Hurewicz map*
$$h : \pi_n(X, x_0) \to H_n(X;\mathbb{Z})$$
is an isomorphism.

Where the **Hurewicz map** is defined in the following way. Let $n \geqslant 1$ and let $c \in H_n(S^n;\mathbb{Z})$ be a generator. For a based space (X, x_0) define

$$h : \pi_n(X, x_0) \to H_n(X;\mathbb{Z}),\ [f : S^n \to X] \mapsto f_*(c)$$

where $f_* : H_n(S^n;\mathbb{Z}) \to H_n(X,\mathbb{Z})$ is the map induced by f on homology groups. i.e. the Hurewicz map h is the evaluation at the fundamental class of S^n.

Proving this theorem will keep us busy for the next few lectures.

Remark. — Choosing the other generator of $H^n(S^n;\mathbb{Z})$, the Hurewicz map changes into its negative which is still an isomorphism, i.e. the map itself slightly depends on the choice of the generator, but the fact that it is an isomorphism does not.

Remark. — Recall: for path connected X, $h : \pi_1(X, x_0) \to H_1(X,\mathbb{Z})$ is surjective with kernel the commutator subgroup, so it factors to an isomorphism $\pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X;\mathbb{Z})$. The Hurewicz theorem is a generalization of this fact, whose first proof is due to Poincaré.

We know prove two properties of the Hurewicz map, namely its naturality and the fact that it is actually a group homomorphism.

Naturality of the Hurewicz map. — Let $f : X \to Y$ be a based map between based spaces. Then the following square commutes

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "h^X"] \arrow[d, "f_*"] & H_n(X;\mathbb{Z}) \arrow[d, "f_*"] \\
\pi_n(Y, f(x_0)) \arrow[r, "h^Y"] & H_n(Y;\mathbb{Z})
\end{tikzcd}$$

Proof. Let $\alpha : S^n \to X$ represent a class in $\pi_n(X, x_0)$. Then

$$f_*(h^X[\alpha]) = f_*(\alpha_*(c)) = (f\alpha)_*(c) = h^Y[f \circ \alpha] = h^Y(f_*[\alpha])$$

□

The Hurewicz map is a group homomorphism. — Let $p : S^n \to S^n \vee S^n$ be a pinch map, i.e. a continuous based map such that both compositions with the projections $S^n \vee S^n \rightrightarrows S^n$ are based-homotopic to the identity. The group structure on $\pi_n(X, x_0)$ (for $n \geqslant 2$) is as follows (thinking of spheres):

The pinch map is the subject of one of the exercises in the first exercise sheet ("The" pinch map, because as we will see it is unique up to homotopy).

$$[f] + [f'] := [(f \vee f') \circ p].$$

It will be an exercise this week to show that if $i_1, i_2 : S^n \to S^n \vee S^n$ are the two summand inclusions the following relations holds:

$$p_*(c) = (i_1)_*(c) + (i_2)_*(c) \text{ in } H_n(S^n \vee S^n;\mathbb{Z})$$

with $c \in H_n(S^n;\mathbb{Z})$ generator. Now we can show that the Hurewicz map is in fact a group homomorphism.

Proof. If $[f], [f'] \in \pi_n(X, x_0)$ we have:

$h([f]+[f']) = h[(f \vee f') \circ p] = ((f \vee f') \circ p)_*(c) = (f \vee f')_*(p_*(c)) = (f \vee f')_*((i_1)_*(c)+(i_2)_*(c)_*)$

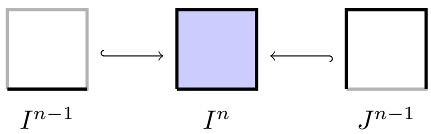
but since $(f \vee f') \circ i_1 = f$ and $(f \vee f') \circ i_2 = f'$,

$$(f \vee f')_*((i_1)_*(c)) + (f \vee f')_*((i_2)_*(c)_*) = f_*(c) + f'_*(c) = h[f] + h[f']$$

□

We will actually prove a stronger version of the Hurewicz theorem, the relative Hurewicz theorem.

Recall the definition of the relative homotopy groups. We identify I^{n-1} with the subspace of I^n with $x_1 = 0$. Define $J^{n-1} = \partial(I^n) \setminus \mathring{I}^{n-1}$. Then $I^{n-1} \cap J^{n-1} = \partial(I^{n-1})$.



The **relative homotopy groups** of the triple (X, A, x_0) are defined as triple homotopy classes of triple maps:

$$\pi_n(X, A, x_0) = [(I^n, I^{n-1}, J^{n-1}), (X, A, x_0)].$$

Addition on $\pi_n(X, A, x_0)$ for $n \geqslant 2$ is defined by "juxtaposition and reparametrization in the first coordinate" as follows:

$$[f] + [g] = [f + g], \ (f + g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, \dots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, \dots, t_n) & t_1 \in [1/2, 1] \end{cases}$$

this is easily seen to be well defined on homotopy classes.

The **relative Hurewicz map** is defined similarly to the absolute one: with $c \in H_n(I^n, \partial I^n; \mathbb{Z})$ a generator, define

$$h : \pi_n(X, A, x_0) \to H_n(X, A; \mathbb{Z}), \ [f] \mapsto f_*(c)$$

Recall that $\pi_1(A, x_0) = [(I', \partial I'), (A, x_0)]$ acts on $\pi_n(X, A, x_0)$ in a non-trivial fashion. This poses a problem, because for all $[f] \in \pi_n(X, A, x_0)$ and $\omega \in \pi_1(S, x_0)$ the maps representing $[\omega] * [f]$ and $[f]$ are pair-homotopic as maps $(I^n, \partial I^n) \to (X, A)$, hence the relative Hurewicz map takes them to the same class in $H_n(X, A; \mathbb{Z})$.

This leads to the definition of a **modified relative Hurewicz map**. For $n \geqslant 2$ define $\pi_n(X, A, x_0)^\dagger$ to be the quotient of $\pi_n(X, A, x_0)$ by the normal subgroup generated by elements of the form $([\omega] * [f])[f]^{-1}$ for all $[\omega] \in \pi_1(A, x_0)$, $[f] \in \pi_n(X, A, x_0)$. By design the relative Hurewicz map factors through this quotient:

$$\begin{tikzcd} \pi_n(X, A, x_0) \arrow[r, "h"] \arrow[d] & H_n(X, A; \mathbb{Z}) \\ \pi_n(X, A, x_0)^\dagger \arrow[ur, dashed, "h^\dagger"'] & \end{tikzcd}$$

Now we can state the relative Hurewicz theorem.

I.2. Theorem (Hurewicz). — *Let (X, A) be a pair of path connected spaces such that for all $x_0 \in A$, the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism. Let $n \geqslant 2$ and suppose that $\pi_i(X, A, x_0) = 0$ for $1 \leqslant i \leqslant n-1$. Then the modified relative Hurewicz map $h^\dagger$ is an isomorphism.*

Remark. — For $A = \{x_0\}$, the relative version recovers the absolute version.

Remark. — The hypothesis of the relative Hurewicz theorem refers to $\pi_i(X, A, x_0)$ but the conclusion refers to $\pi_n(X, A, x_0)^\dagger$. This makes the relative version not as manageable as the absolute one.

Some Consequences of Hurewicz Theorem

Lecture 2
13^{th} Oct, 2021

Before resuming with the proof of the relative Hurewicz theorem we prove an application of it, a version of Whitehead's theorem which uses homology groups in place of homotopy groups.

I.3. Theorem. — *Let $f : X \to Y$ be a map between simply connected CW-complexes such that $f_* : H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is an isomorphism for all $i \geqslant 0$. Then f is an homotopy equivalence.*

Proof. By cellular approximation we can assume f cellular. Let $Z(f) = X \times [0,1] \cup_{X \times 1, f} Y$ be the mapping cylinder of f. This inherits a CW-structure such that $X \cong X \times 0$ and Y are subcomplexes. The projection $Z(f) \to Y$ is a homotopy equivalence (fact check this), hence by replacing Y by $Z(f)$ we can assume wlog that $f : X \to Y$ is the inclusion of a subcomplex. Since X and Y are simply-connected the relative Hurewicz theorem applies for all $n \geqslant 2$, but all relative homology groups vanish because f_* is an isomorphism (by the long exact sequence), hence all relative homotopy groups vanish and by Whitehead's theorem we can conclude that f is an homotopy equivalence. □

An elaboration of the previous results leads to the following proposition.

I.4. Proposition. — *Let $f : X \to Y$ be a map of path-connected CW-complexes. The following are equivalent:*

(i) *f is a homotopy equivalence,*

(ii) *f induces an isomorphism on fundamental groups and the induced map $\tilde{f} : \tilde{X} \to \tilde{Y}$ on universal covers induces an isomorphism on all integral homology groups.*

Proof. $(i) \implies (ii)$ Since f is an homotopy equivalence, f_* is an isomorphism on all homotopy groups. Then $\tilde{f}$ induces an isomorphism on all homotopy groups, hence it is an homotopy equivalence, thus it induces an isomorphism on all homology groups.

$(ii) \implies (i)$ Since $\tilde{f}$ induces an isomorphism on integral homology groups it is a homotopy equivalence by the version of Whitehead's theorem we just proved, hence it induces an isomorphism on all homotopy groups. This in turn means that f induces an isomorphism on all homotopy groups, i.e. it is a homotopy equivalence. □

Getting Rid of the Basepoint

We now return to the proof of the relative Hurewicz theorem

Recall: the **degree** of a map $f : (D^n, \partial D^n) \to (D^n, \partial D^n)$ is the integer $\deg(f)$ such that $f_*(x) = \deg(f)x$ for all $x \in H_n(D^n, \partial D^n; \mathbb{Z})$.

I.5. Lemma. — *Let $n \geqslant 1$. For $n > 1$ assume known that $\pi_{n-1}(S^{n-1}, z)$ is free abelian of rank 1. Let f be a continuous self map of $(D^n, \partial D^n)$ of degree ± 1. Then f is pair-homotopic to the identity if $\deg(f) = 1$ and to any reflection if $\deg(f) = -1$.*

Proof. We first see the case when $\deg(f) = 1$.

$(n = 1)$ Since $\partial D^1 = \{\pm 1\}$ and $\deg(f) = 1$ we have $f|_{\partial D^1} = \mathrm{id}_{\partial D^1}$. Then the linear homotopy $H(x,t) = tf(x) + (1-t)x$ is a relative homotopy between f and the identity id_{D^1}.

$(n \geqslant 2)$ Consider the commutative square:

$$\begin{tikzcd}
H_n(D^n, \partial D^n; \mathbb{Z}) \arrow[r, "\partial", "\cong"'] \arrow[d, "f_* = \mathrm{id}"] & H_{n-1}(S^{n-1}; \mathbb{Z}) \arrow[d, "(f|_{S^{n-1}})_* = \mathrm{id}"] \\
H_n(D^n, \partial D^n; \mathbb{Z}) \arrow[r, "\partial", "\cong"'] & H_{n-1}(S^{n-1}; \mathbb{Z})
\end{tikzcd}$$

Since $\pi_{n-1}(S^{n-1}, z)$ is free of rank 1, the Hurewicz map $h : \pi_{n-1}(S^{n-1}, z) \to H_{n-1}(S^{n-1}; \mathbb{Z})$ is an isomorphism. Then $(f|_{\partial D^n})_* : \pi_{n-1}(S^{n-1}, z) \to \pi_{n-1}(S^{n-1}, z)$ is the identity, therefore $f|_{\partial D^n}$ is homotopic to the identity of S^{n-1}. Now let $H : S^{n-1} \times [0,1] \to S^{n-1}$ be a homotopy. This gives a map $D^n \times 0 \cup S^{n-1} \times [0,1] \cup D^n \times 1 \xrightarrow{f \cup H \cup \mathrm{id}} D^n$. Since $D^n \times [0,1]$ can be obtained from $D^n \times 0 \cup S^{n-1} \times [0,1] \cup D^n \times 1$ by attaching an $(n+1)$-cell and D^n is contractible, there is a continuous extension $\bar{H} : D^n \times [0,1] \to D^n$. This is the desired pair homotopy from f to id_{D^n}.

Why do we need D^n contractible? Think of $S^1 \hookrightarrow D^2$

If $\deg(f) = -1$, we let $r : D^n \to D^n$ be the reflection in the first coordinate. Then $\deg(r \circ f) = 1$, hence $r \circ f$ is pair homotopic to id_{D^n} and so $f = r \circ r \circ f$ is pair homotopic to r. $\square$

Now let (X, A) be a based space. Define the group $\pi_n(X, A)^{\#}$ as the quotient of the free abelian group generated by pair homotopic maps $(I^n, \partial I^n) \to (X, A)$ by the relation $[f] + [f'] = [f + f']$ when the right hand side is defined.

The "forgetful" map $\pi_n(X, A, x_0) \to \pi_n(X, A)^{\#}$ is a group homomorphism and it factors through a homomorphism $\pi_n(X, A, x_0)^{\dagger} \to \pi_n(X, A)^{\#}$ (because $\omega * f$ and f are always pair homotopic).

I.6. Proposition. — *Let (X, A) be a pair of path-connected spaces. Let $n \geqslant 2$ or $n = 1$ and A a point. Then the "forgetful" homomorphism $\pi_n(X, A, x_0)^{\dagger} \to \pi_n(X, A)^{\#}$ is an isomorphism.*

Proof. We will define a homomorphism in the opposite direction. Let $f : (I^n, \partial I^n) \to (X, A)$ be a pair map. f need not send J^{n-1} to x_0, but J^{n-1} is contractible and A path-connected. So $f|_{J^{n-1}}$ is homotopic in A to the constant map at the basepoint. Let $H : J^{n-1} \times [0,1] \to A$ be such a homotopy from $f|_{J^{n-1}}$ to the constant map x_0. The HEP for $(\partial I^n, J^{n-1})$ lets us extend H to a homotopy $H' : \partial I^n \times [0,1] \to A$ from $f|_{\partial I^n}$ to some map $H'(1, -)$

that sends J^{n-1} to x_0. The HEP for $(I^n, \partial I^n)$ with target space X lets us extend H' to $H'' : I^n \times [0,1] \to X$ from f to a map that sends J^{n-1} to x_0. Moreover H'' is a pair homotopy of maps $(I^n, \partial I^n) \to (X, A)$.

We now define a map

$$\Psi : [(I^n, \partial I^n) \to (X, A)] \to \pi_n(X, A, x_0)^\dagger \qquad [f] \longrightarrow [H''(-, 1)],$$

and claim that this is well defined.

Claim. Let $f, f' : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ be triple maps that are pair homotopic as maps $(I^n, \partial I^n) \to (X, A)$. Then they represent the same element in $\pi_n(X, A, x_0)^\dagger$.

Proof of the claim. Let

$$H : I^n \times [0,1] \to X$$

be a pair homotopy from f to f'. We choose a point $z \in J^{n-1}$ and a triple homotopy

$$K : (I^n, \partial I^n, J^{n-1}) \times [0,1] \to (I^n, \partial I^n, J^{n-1})$$

from the identity to a map with $K(J^{n-1} \times 1) = \{z\}$. Formally we are applying two times the HEP (for $(\partial I^n, J^{n-1})$ and for $(I^n, \partial I^n)$, as before). Then f is triple homotopic to $f \circ K(-, 1)$, f' is triple homotopic to $f' \circ K(-, 1)$, so that $f \circ K(-, 1)$ is pair homotopic to $f' \circ K(-, 1)$. In particular

$$\tilde{H} = H \circ (K(-, 1), \mathrm{id}) : I^n \times [0,1] \to X$$

satisfies $\tilde{H}(J^{n-1}, t) = H(z, t)$ for all $t \in [0,1]$. Now for all $x \in J^{n-1}$, the loop at x_0 in A, $\tilde{H}(x, -)$, is independent of the point $x \in J^{n-1}$ and it always agrees with $\omega = \tilde{H}(z, -)$. By identifying I^{n+1} as $I^n \times [0,1]$ we can view $\tilde{H}$ as a triple homotopy between $\omega * (f \circ K(-, 1))$ and $f' \circ K(-, 1)$. In the end we have (the second equality holds in $\pi_n(X, A, x_0)^\dagger$ by construction, the third one is because of the homotopy we found)

I'm not *too* sure I really get this...

$$[f] = [\omega * (f \circ K(-, 1))] = [f' \circ K(-, 1)] = [f'] \quad \text{in } \pi_n(X, A, x_0)^\dagger.$$

.

Corollary of the claim. The map $\Psi : [(I^n, \partial I^n), (X, A)] \to \pi_n(X, A, x_0)^\dagger$ we defined before the claim is well defined, so it has a unique extension on the free abelian group which factors to a homomorphism $\pi_n(X, A)^\# \to \pi_n(X, A, x_0)^\dagger$ which is then an isomorphism by design. $\square$

Punchline: In the situation of the relative Hurewicz theorem it suffices to show that the map $\pi_n(X, A)^\# \to H_n(X, A; \mathbb{Z})$ is an isomorphism (i.e. we don't have to deal with basepoints!).

The Homotopy Addition Theorem

LECTURE 3
18th Oct, 2021

This lecture was given by Tobias Lenz, a PhD student of Schwede. I was late to the class, so most of the notes for this lecture are copied from Qi Zhu's notes, thank you Qi Zhu!

Remark. — There's a standing assumption for all of today's lecture: $\pi_k(S^k) \cong \mathbb{Z}$ for all $1 \leqslant k < n$.

The main goal of today's lesson is to prove the following theorem.

I.7. Theorem (Homotopy Addition Theorem)**.** — *Assume we have $f_1, \dots, f_k : I^n \to I^n$ such that $f_i|_{\mathring{I}^n}$ is an open embedding and the sets $f_i(\mathring{I}^n)$ are pairwise disjoint. Furthermore, let $g : (I^n, \partial I^n) \to (X, A)$ such that $g(I^n \smallsetminus \bigcup_{i=1}^k f_i(\mathring{I}^n)) \subset A$. Then*

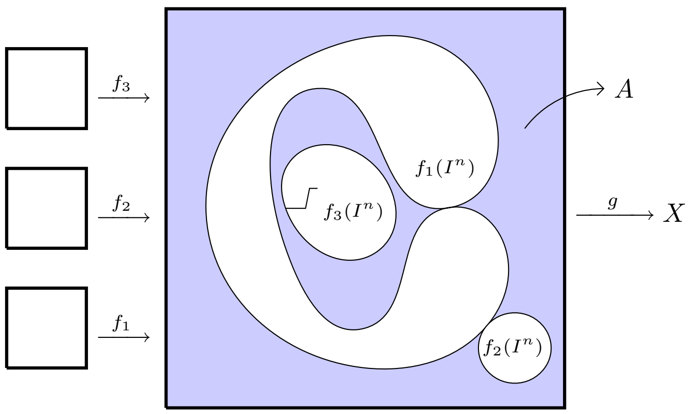
$$[g] = \sum_{i=1}^{k} (\deg f_i)[g \circ f_i]$$

in $\pi_n(I^n, \partial I^n)^\#$.

Remark. — Note that we have $f_i(\partial I^n) \cap f_j(\mathring{I}^n) = \varnothing$ for all i, j. This will play a (small) role later in the lecture.

Remark. — Two remarks about the theorem:

- When homotopy theory was new, people thought this was obvious and didn't feel the need for a proof until Eilenberg suggested so.
- Tobias: *"I'm not sure if I can finish the proof today but I was promised an award if I do!"*



The strategy to prove this theorem is inductive: we want to reduce the problem to the case $k = 1$ which is easy.

Definition. — Let $f : \mathring{I}^n \to \mathring{I}^n$ be an open embedding, $p \in \mathring{I}^n$. We have that f induces a commutative diagram:

$$\begin{tikzcd}
H_n(\mathring{I}^n, \mathring{I}^n \smallsetminus \{p\}) \arrow[r, "i_*"] \arrow[d, "f_*"'] \arrow[dr, "f_*"] & H_n(I^n, I^n \smallsetminus \{p\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"'] \arrow[d, dashed, "d"] \\
H_n(f(\mathring{I}^n), f(\mathring{I}^n) \smallsetminus \{f(p)\}) \arrow[r, "i_*"'] & H_n(f(I^n), f(I^n) \smallsetminus \{f(p)\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

where the maps are all isomorphisms by homotopies and excision, hence they induce the dashed arrow. This is an automorphism of $H_n(I^n, \partial I^n) \cong \mathbb{Z}$ and thus $d = \pm 1$. One can show this is independent of p, hence we call it the **local degree** of f, and write $\deg(f) = d$.

I.8. Lemma. — *For $1 \leqslant i \leqslant k$, let $\mathcal{U}_i \subset f_i(\mathring{I}^n)$ be any non-empty open set. Then g is homotopic relative to $I^n \setminus f(\mathring{I}^n)$ to a map that sends $f(I^n) \setminus \mathcal{U}_i$ to A.*

Remark. — If $[g] = [g']$ in $\pi_n(X, A)^{\#}$ then $[g \circ f_j] = [g' \circ f_j]$ for all $1 \leqslant j \leqslant k$.

Proof. There is a homotopy relative ∂I^n from the identity to a map that sends everything outside of $\mathring{Q}$ to ∂I^n, where Q is a cube inside $f_i^{-1}(\mathcal{U}_i)$ (basically we take a cube Q inside $f_i^{-1}(\mathcal{U}_i)$ and we blow it to the big cube ∂I^n containing $f_i^{-1}(\mathcal{U}_i)$). Composing with gf_i yields a homotopy H of maps of pairs $(I^n, \partial I^n) \to (X, A)$ from gf_i to a map that sends everything outside $\mathring{Q}$ to A. We have a map of sets:

$$H' : I^n \times I \to X, \quad H'(x,t) = \begin{cases} H(f^{-1}(x), t) & x \in f_i(I^n) \\ g(x) & x \notin f_i(\mathring{I}^n) \end{cases}$$

which we can show is well defined. Let $x \in f_i(\partial I^n)$, with preimage y, then

It suffices to check this because $f_i(\partial I^n) \cap f_j(\mathring{I}^n)$ is empty for all i, j.

$$g(x) = H(y,t) = H(y,0) = gf_i(y)$$

so that it is well defined as a map of sets. Then $H'(x,0) = g(x)$ and $H'(x,1) \in A$ for $x \notin f(\mathring{Q})$, in particular $H'(x,1) \in A$ for $x \notin \mathcal{U}_i$. It remains to prove that H' is continuous.

Claim. We have a pushout

$$\begin{tikzcd} \partial I^n \times I \arrow[r, "f_i|_{\partial I^n} \times \mathrm{id}"] \arrow[d, "i"] & (I^n \setminus f_i(\mathring{I}^n)) \times I \arrow[d, "i"] \\ I^n \times I \arrow[r, "f_i \times \mathrm{id}"] & I^n \times I \end{tikzcd}$$

Then,

$$(I^n \times I) \amalg_{\partial I^n \times I} ((I^n \setminus f_i(\mathring{I}^n)) \times I) \xrightarrow{(f_i \times \mathrm{id}, i)} I^n \times I$$

is a homeomorphism.

Proof of the claim. Well-definedness and continuity of the function follow from the universal property of the pushout. One can check that it is bijective by a direct computation. Hence we have a continuous bijection from a quasi-compact space to an Hausdorff space, which is then a homeomorphism. ■

Thus to show that H' is continuous, it suffices to show that the maps $H' \circ (f_i \times I)$ and $H'|_{(I^n \setminus f(\mathring{I}^n)) \times I}$ are continuous, which follows from the construction. □

Not sure how this works, to be honest

Proof of the homotopy addition theorem (I.7). Induction on k.

($k = 1$) Take a cube $Z_1 \subset f_1(\mathring{I}^n)$. By the lemma we may assume that $g(I^n \setminus \mathring{Z}_1) \subset A$. Our goal is to construct some $f_1' : (I^n, \partial I^n) \to (I^n, \partial I^n)$ with $[gf_1] = [gf_1']$. There exists a homotopy Q from id_{I^n} to a map p such that:

The homotopy Q is the same kind of "pushing out" homotopy that we already considered in the proof of the lemma.

- $p|_{Z_1} = \mathrm{id}_{Z_1}$
- $p(I^n \setminus f_1(\mathring{I}^n)) \subset \partial I^n$
- $Q((I^n \setminus \mathring{Z}_1) \times I) \subset I^n \setminus \mathring{Z}_1$

Now, Qf_1 is a homotopy of maps of pairs $f_1, pf_1 : (I^n, \partial I^n) \to (I^n, I^n \smallsetminus \mathring{Z}_1)$. Then gQf_1 will be a homotopy of maps of pairs $(I^n, \partial I^n) \to (X, A)$, hence $[gpf_1] = [gf_1]$ in $\pi_n(X, A)^{\#}$. Then $f_1' := pf_1$ is a map of pairs $(I^n, \partial I^n) \to (I^n, \partial I^n)$ and $f_1 \simeq f_1'$ as maps of pairs $(I^n, \partial I^n) \to (I^n, I^n \smallsetminus \mathring{Z}_1)$.

We are dealing with two different notions of degree, the usual one for f_1' and the local degree for f_1.

Claim. $\deg f_1$ equals the degree of f_1' as a map $(I^n, \partial I^n) \to (I^n, \partial I^n)$.

Proof of the claim. Consider the diagram:

$$\begin{tikzcd}
H_n(\mathring{I}^n, \mathring{I}^n \smallsetminus \{x\}) \arrow[r, "i_*"] \arrow[d, "(f_1)_*"'] & H_n(I^n, I^n \smallsetminus \{x\}) \arrow[d, "(f_1)_*"] & H_n(I^n, \partial I^n) \arrow[l, "i_*"'] \arrow[d, shift right, "(f_1)_*"'] \arrow[d, shift left, "(f_1')_*"] \arrow[dr, "(f_1')_*"] & \\
H_n(\mathring{I}^n, \mathring{I}^n \smallsetminus \{f_1(x)\}) \arrow[r, "i_*"] & H_n(I^n, I^n \smallsetminus \{f_1(x)\}) & H_n(I^n, I^n \smallsetminus \mathring{Z}_1) \arrow[l, "i_*"'] & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

we have that the parallel arrows agree, which proves the claim. ■

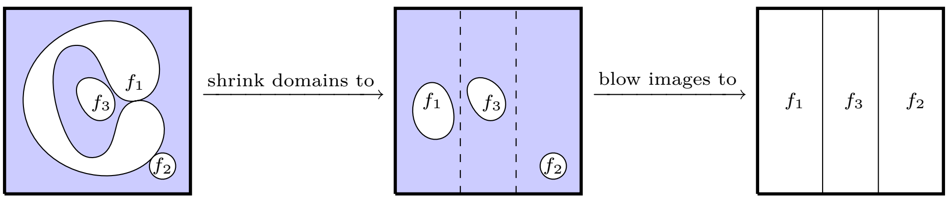
Now, if $\deg f_1 = 1$, then $f_1' \simeq \mathrm{id}$ as maps of pairs, hence $(\deg f_1)[gf_1] = [gf_1'] = [g]$.

If instead $\deg f_1 = -1$, then $f_1' \sim r$ where r is reflection in the first coordinate by lemma I.5, hence $[g] = -[gr] = -[gf_1'] = -[gf_1] = (\deg f_1)[gf_1]$.

To be continued...

LECTURE 4
25th Oct, 2021

$(k \geqslant 2)$ Set $u_i = f_i(\text{center of } I^n) \in I^n$. Assume without loss of generality that the first coordinates of $u_1, \ldots, u_k$ are not all equal (if some of them are, we can "wiggle" f_1).



$\rightsquigarrow [g] = \sum [g \circ f_i]$.

Choose $t \in (0, 1)$ such that

- t is different from the first coordinates of $u_1, \ldots, u_k$,
- for some $1 \leqslant i \leqslant k$, the first coordinate of u_i is smaller than t,
- for some $1 \leqslant i \leqslant k$, the first coordinate of u_i is larger than t.

Choose neighborhoods $\mathcal{U}_i$ of u_i inside $f_i(\mathring{I}^n)$ that do not intersect $\{t\} \times I^{n-1}$, that is such that $\mathcal{U}_i$ lies "on the same side respect to t" as u_i.

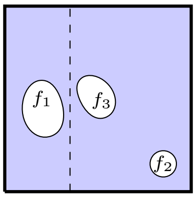
Choose subcubes Q_i inside $\mathring{I}^n$ that contain the center and lie in in $f^{-1}(\mathcal{U}_i)$.

Last time we proved that g is pair-homotopic to some $g' : (I^n, \partial I^n) \to (X, A)$ such that

$$g'(I^n \smallsetminus \bigcup_{i=1,\ldots,k} f_i(Q_i)) \subset A \quad \text{and} \quad [g \circ f_i] = [g' \circ f_i] \text{ in } \pi_n(X, A)^{\#}.$$

We precompose each $f_i : I^n \to I^n$ with the linear shrinking homotopy relative Q_i. Set $f_i' = f_i \circ$ end of shrinking. Then $g' \circ f_i$ is pair homotopic to $g' \circ f_i'$ and $f_i'(I^n) \subset \mathcal{U}_i$.

By replacing g by g' and f_i by f_i' we can therefore assume without loss of generality that $f_i(\mathring{I}^n)$ lies on one side of $\{t\} \times I^{n-1}$.



Write $g = g_1 +_t g_2$ by "cutting along $\{x_1 = t\}$".

Formally, $g_1(x_1, \ldots, x_n) = g(tx_1, \ldots, x_n)$ and $g_2 = g((1-t)x_1 + t, \ldots, x_n)$.

Set $I_1 = \{i \in I \mid u_i \text{ lies left of } \{x_1 = t\}\}$ and $I_2 = \{i \in I \mid u_i \text{ lies right of } \{x_1 = t\}\}$, where $I = \{1, \ldots, k\}$.

Then by the inductive hypothesis:

$$[g] = [g_1] + [g_2] = \sum_{i \in I_1} \deg(f_i)[g_1 \circ f_i] + \sum_{i \in I_2} \deg(f_i)[g_2 \circ f_i] = \sum_{i \in I} \deg(f_i)[g \circ f_i].$$

□

My First Non-Trivial Homotopy Group

I.9. Theorem. — *Let $n \geqslant 2$ and assume the HAT in dimension n as well as $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Then $\pi_n(S^n, *)$ is infinite cyclic.*

Proof. Choose some point $z \in S^n$. Set $U = S^n \smallsetminus \{-z\}$. Then

$$\pi_n(S^n, z) = \pi_n(S^n, \{z\}, z) \underset{U \cong *}{\cong} \pi_n(S^n, U, z) \underset{\pi_1(U,z) = \{1\}}{\cong} \pi_n(S^n, U, z)^{\dagger} \cong \pi_n(S^n, U)^{\#}$$

So we may show that $\pi_n(S^n, U)^{\#} \cong \mathbb{Z}$.

We show that $\pi_n(S^n, U)^{\#}$ is generated by the class of any pair map $\psi : (I^n, \partial I^n) \to (S^n, U)$ such that $\psi(\partial I^n) = \{z\}$ and ψ factors out a homeomorphism $I^n / \partial I^n \cong S^n$.

Let $f : (I^n, \partial I^n) \to (S^n, U)$ be any pair map. Set $V = S^n \smallsetminus \{z\}$ so that $S^n = U \cup V$ is an open cover.

The Lebesgue Number lemma provides an $m \geqslant 1$ so that each subcube of I^n of side length $1/m$ is mapped by f into U or into V. Decompose I^n into m^n subcubes of side length $1/m$.

We define subspaces of I^n in this way:

- $A_{-1} = \partial I^n$
- $A_0 = A_{-1} \cup$ vertices
- $A_1 = A_0 \cup$ edges
- $A_2 = \cdots$
- $A_n = I^n$

We want to "improve" f successively by pair homotopies to maps $f = f_{-1}, f_0, f_1, \dots, f_{n-1}$ such that:

- each f_j is homotopic to f relative ∂I^n,
- each f_j is admissible, i.e. it sends every subcube of side length $1/m$ to U or to V,
- $f_j(A_j) \subset U$ for $j = -1, 0, 1, \dots, n-1$.

We proceed by induction on j. There is nothing to show for $j = -1$. Let now $j \geqslant 0$. We first modify f_{j-1} and the faces of the j-cube. If such a face Q is "good", i.e. sent by f_{j-1} into U, we do not do anything to Q. Otherwise the $(j-1)$-subcube is mapped to V and the restriction of f_{j-1} to it is a pair map $(Q, \partial Q) \to (V, V \cap U)$.

Claim. For $j < n$, any pair map $(I^j, \partial I^j) \to (V, U \cap V)$ is homotopic relative ∂I^j to a map with image in $U \cap V$.

Proof of the claim. By stereographic projection $(V, U \cap V)$ is pair homotopic to $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, i.e. we can costruct a pair map $g : (I^j, \partial I^j) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Because ∂I^j is compact and $0 \notin f(\partial I^j)$, there is an $\varepsilon > 0$ such that $g(\partial I^j) \cap (\varepsilon\text{-ball around } 0) = \varnothing$.

So $g : (I^j, \partial I^j) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \mathring{B}(\varepsilon, 0))$. Now, $\mathbb{R}^n$ can be obtained from $\mathbb{R}^n \setminus \mathring{B}(\varepsilon, 0)$ by attaching an n-cell. The cellular approximation theorem and the fact that I^j is a j-dimensional CW-complex gives us a relative homotopy from g to a cellular map. Since $j < n$, the cellular map has image in $\mathbb{R}^n \setminus \mathring{B}(\varepsilon, 0) \subset \mathbb{R}^n \setminus \{0\}$. ■

We can now change $f_{j-1}|A_j$ into $f_j|A_j$ by a homotopy relative A_{j-1} into a map that sends all j-cells to U.

We use the HEP for (I^n, A_j) to extend f_j to all of I^n; we use the HEP with target U or with target V to ensure that the map f_j is again admissible.

After this inductive construction we can replace f by f_{n-1} and we have arranged without loss of generality that $f(A_{n-1}) \subset U$.

We can now assume that $g : (I^n, \partial I^n) \to (S^n, U)$ satisfies $g(A_{n-1}) \subset U$ and each top-dimensional subcube is mapped to U or to V.

We apply the HAT to this map g with $f_1, \dots, f_k$ the reparametrization of those subcubes that are *not* mapped into U (and hence into V).

Then by the HAT we have:

$$[g] = \sum_i \pm [g \circ f_i] \text{ in } \pi_n(S^n, U)^{\#} \cong \pi_n(S^n, \{z\}, z)^{\dagger} \cong \pi_n(S, z)$$

We have gained that each summand on the right hand side is in the image of the homomorphism $\pi_n(V, V \cap U)^{\#} \to \pi_n(S^n, U)^{\#}$, which is then surjective. By the long exact homotopy group sequence of the pair $(V, V \cap U)$, we obtain:

$$\pi_n(V, V \cap U, z) = \pi_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}, z) \cong \pi_{n-1}(\mathbb{R}^{n-1} \setminus \{0\}, z) \cong \pi_{n-1}(S^{n-1}, z) \cong \mathbb{Z}$$

so $\pi_n(S^n, U)^{\#}$ is infinite cyclic. □

Reminder on Simplicial Sets

We denote by Δ the category with objects the sets $[n] = \{0, 1, \ldots, n\}$, $n \geqslant 0$ and morphisms the weakly monotone maps.

A **simplicial set** is a contravariant functor from Δ to sets, $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. The set of the n-simplices is denoted $X_n = X([n])$, for a morphism $\alpha : [n] \to [m]$ write $\alpha^* = X(\alpha) : X_m \to X_n$.

The **singular complex** (singular simplicial set) of a space Y is the simplicial set

$$\mathcal{S}(Y) : \Delta^{\mathrm{op}} \to \mathrm{Set} \qquad [n] \longmapsto \mathcal{S}(Y)_n := \{\text{continuous maps } f : \nabla^n \to Y\}$$

where

$$\nabla^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \middle| x_i \geqslant 0,\ \sum x_i = 1 \right\}.$$

For $\alpha : [n] \to [m]$, the map $\alpha^* : \mathcal{S}(Y)_m \to \mathcal{S}(Y)_n$ is precomposition with the affine linear map $\alpha_* : \nabla^n \to \nabla^m$ defined by $e_i \mapsto e_{\alpha(i)}$ (i.e. the map $(t_0, \ldots, t_m) \mapsto \alpha_*(t)_j = \sum_{j=\alpha(i)} t_j$).

For a continuous map $\psi : Y \to Z$, a morphism of simplicial sets $\psi_* = \mathcal{S}(\psi) : \mathcal{S}(Y) \to \mathcal{S}(Z)$ is given by $\mathcal{S}(\psi)_n(f) = \psi \circ f$. This yields a functor $\mathcal{S} : \mathrm{Top} \to \mathrm{sSet}$.

The **geometric realization** is a functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ defined as follows. For a simplicial set X we set

$$|X| = \left(\coprod X_n \times \nabla^n \right) / \sim$$

where X_n is endowed with the discrete topology and the equivalence relation is the one generated by:

$$X_m \times \nabla^m \ni (x, \alpha_*(t)) \sim (\alpha^*(x), t) \in X_n \times \nabla^n \quad \text{for all } \alpha : [n] \to [m],\ x \in X_m,\ t \in \nabla^n.$$

The equivalence relation is only *generated* by the condition stated, which is not symmetric. The actual equivalence relation is not easy to understand. We will return on this problem with the theory of minimal representatives in chapter V.

Lecture 5
27th Oct, 2021

Given two simplicial sets X and Y, their product $X \times Y$ is the functor

$$\Delta^{\mathrm{op}} \xrightarrow{(X,Y)} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$$

i.e. $(X \times Y)_n = X_n \times Y_n$ and $\alpha^*_{X \times Y} = \alpha^*_X \times \alpha^*_Y$.

The **simplicial n-simplex** is the represented simplicial set $\Delta[n] := \Delta(-, [n])$. By the Yoneda lemma, for every simplicial set X, the map

$$\mathrm{sSet}(\Delta[n], X) \to X_n \qquad (f : \Delta[n] \to X) \mapsto f_n(\mathrm{id}_{[n]})$$

is bijective.

A **homotopy between morphisms** $f, g : X \to Y$ in sSet is a morphism $H : X \times \Delta[1] \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$, where $i_0, i_1 : X \to X \times \Delta[1]$ (note: the morphism i_j has components $(i_j)_n : X_n \to X_n \times \Delta([n], [1])$, $x \mapsto (x, \mathrm{cost}_j)$).

The topological simplex $\nabla^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0,\ \sum x_i = 1\}$ has a preferred CW-structure with $\mathrm{sk}_k(\nabla^n) = \{(x_0, \ldots, x_n) \in \nabla^n \mid \text{at most } k+1 \text{ cordinates are non-zero}\}$, i.e. $\mathrm{sk}_k(\nabla^n)$ is the union of all k-dimensional faces of ∇^n:

- $\mathrm{sk}_0(\nabla^n) = \{e_0, \ldots, e_n\}$,
- $\mathrm{sk}_j = \ldots$

- $\mathrm{sk}_{n-1}(\nabla^n) = \partial\nabla^n$,
- $\mathrm{sk}_n(\nabla^n) = \nabla^n$.

Let (X, A) be a space pair and $k \geqslant -1$. We define a **simplicial subset** $\mathcal{S}(X, A, k)$ of $\mathcal{S}(X)$ by setting

$$S(X, A, k)_n = \{f : \nabla^n \to X \mid f(\mathrm{sk}_k(\nabla^n)) \subset A\}.$$

This is indeed a simplicial subset because the affine linear maps $\alpha_* : \nabla^n \to \nabla^m$ are cellular.

Note that we have:

$$\mathcal{S}(X) = \mathcal{S}(X, A, -1) \supset \mathcal{S}(X, A, 0) \supset \mathcal{S}(X, A, 1) \supset \cdots \supset \bigcap_{k \geqslant 1} \mathcal{S}(X, A, k) = \mathcal{S}(A).$$

A space pair (X, A) is k**-connected**, $h \geqslant 0$, if the following equivalent conditions hold:

(a) The inclusion $A \hookrightarrow X$ is a bijection on π_i for all $i < k$ and all basepoints in A and a surjection on π_k,

(b) Every pair map $(D^n, \partial D^n) \to (X, A)$ for $0 \leqslant n \leqslant k$ is homotopic relative ∂D^n to a map with image in A,

(c) The map $\pi_0(A) \to \pi_0(X)$ is surjective and $\pi_i(X, A, x) = \{*\}$ for all $1 \leqslant i \leqslant k$ and all $x \in A$.

We want to prove the following theorem.

Theorem. — *Let (X, A) be a k-connected pair of spaces. The inclusion $\mathcal{S}(X, A, k) \hookrightarrow \mathcal{S}(X)$ is then a deformation retraction of simplicial sets.*

This means there is a simplicial homotopy

$$H : \mathcal{S}(X) \times \Delta[1] \to \mathcal{S}(X)$$

from the identity to a morphism with image in $\mathcal{S}(X, A, k)$ that is relative to $\mathcal{S}(X, A, k)$, i.e. the restriction of H to $S(X, A, k)$ is the composite

$$\mathcal{S}(X, A, k) \times \Delta[1] \xrightarrow{\mathrm{proj}} \mathcal{S}(X, A, k) \xhookrightarrow{\mathrm{incl}} \mathcal{S}(X).$$

We first prove a proposition.

Let X be a simplicial set and $x \in X_n$, for $n \geqslant 0$. Then the n-simplex x is **degenerate** if there is a surjective morphism $\sigma : [n] \to [k]$ with $k < n$, and $y \in X_k$ such that $x = \sigma^*(y)$ (i.e. $x \in \mathrm{im}(\sigma^* : X_k \to X_n)$).

I.10. Proposition. — *Let X be a simplicial set and $x \in X_n$ any simplex. Then there is a unique pair (σ, y) consisting of:*

- *a surjective morphism $\sigma : [n] \to [k]$ and*
- *a non-degenerate simplex $y \in X_k$*

such that $x = \sigma^*(y)$.

Proof.

Existence. By induction on n. If $n = 0$ then x is non-degenerate and $(\mathrm{id}_{[0]}, x)$ does the job.

For $n \geqslant 1$: if x is non-degenerate, then $(\mathrm{id}_{[n]}, x)$ does the job. Otherwise $x = \sigma^*(x')$ for some $\sigma : [n] \to [k]$, $k < n$, $x' \in X_k$. Then by induction we have $x' = (\sigma')^*(y)$ for some surjective morphism $\sigma' : [k] \to [l]$ and $y \in X_l$ non-degenerate. Hence

$$x = \sigma^*(x') = \sigma^*((\sigma')^*(y)) = (\sigma' \circ \sigma)^*(y)$$

which is the desired expression.

Uniqueness. Let $x = \sigma^*(y) = \bar{\sigma}^*(\bar{y})$ for surjective morphisms $\sigma : [n] \twoheadrightarrow [k]$, $\bar{\sigma} : [n] \twoheadrightarrow [l]$ and $y \in X_k$, $\bar{y} \in X_l$ non-degenerate.

Let $\delta : [k] \to [n]$ be a morphism in Δ such that $\sigma \circ \delta = \mathrm{id}_{[k]}$. Then

$$y = (\sigma\delta)^*(y) = \delta^*(\sigma^*(y)) = \delta^*(\bar{\sigma}^*(\bar{y})) = (\bar{\sigma}\delta)^*(\bar{y})$$

We write

$$\bar{\sigma}\delta = \delta'\sigma' : [k] \to [l]$$

where $\sigma' : [k] \twoheadrightarrow [a]$ is surjective and $\delta' : [a] \hookrightarrow [l]$ is injective. Then

$$y = (\bar{\sigma}\delta)(\bar{y}) = (\delta'\sigma')^*(\bar{y}) = (\sigma')^*((\delta')^*(\bar{y})) \tag{$*$}$$

Since y is non-degenerate, we must have $a = k$ and $\sigma' = \mathrm{id}_{[k]}$. Hence $k = a \leqslant l$. By interchanging the roles of (σ, y) and $(\bar{\sigma}, \bar{y})$ we obtain $l \leqslant k$, hence $l = k$.

Then by $(*)$ we have $y = (\delta')^*(\bar{y})$ so $\delta' = \mathrm{id}$ and hence $y = \bar{y}$ and $\sigma = \bar{\sigma}$. □

I.11. Theorem. — *Let (X, A) be k-connected. Then $\mathcal{S}(X, A, k) \hookrightarrow \mathcal{S}(X)$ is a simplicial deformation retraction.*

Proof. We will construct the following data: continuous maps $\psi_f : \nabla^n \times \nabla^1 \to X$ for all $f : \nabla^n \to X$, $n \geqslant 0$, such that:

(a) $\psi_f(-, e_0) = f$, $\psi_f(\mathrm{sk}_k(\nabla^n), e_1) \subset A$.

(b) If $f(\mathrm{sk}_k(\nabla^n)) \subset A$, then $\psi_f = f \circ \mathrm{pr}_1$.

(c) The maps ψ_f are compatible in the simplicial direction, i.e. for all $\alpha : [n] \to [m]$, $g : \nabla^m \to X$, the following commutes:

$$\begin{tikzcd} \nabla^n \times \nabla^1 \arrow[r, "\psi_{\alpha^*(g)}"] \arrow[d, "\alpha_* \times \mathrm{id}"'] & X \\ \nabla^m \times \nabla^1 \arrow[ur, "\psi_g"'] & \end{tikzcd}$$

Construction of the ψ_f's. By induction on $n \geqslant 0$.

$(n = 0)$ The map $f : \nabla^0 \to X$ is determined by its image $f(e_0) \in X$. Since $\pi_0(A) \to \pi_0(X)$ is surjective, we can choose a path from $f(e_0)$ to some point in A. We view the path as a continuous map $\psi_f : \nabla^0 \times \nabla^1 \to X$ with $\psi_f(e_0, e_0) = f(e_0)$, $\psi_f(e_0, e_1) \in A$. If $f(e_0) \in A$, we take the constant path at $f(e_0)$.

$(n \geqslant 1)$ We distinguish three cases:

Case 1. $f : \nabla^n \to X$ is degenerate as a simplex of the simplicial set $\mathcal{S}(X)$. Then $f = \sigma^*(g)$ for a unique pair (σ, g) with $\sigma : [n] \twoheadrightarrow [m]$, $m < n$ and $g : \nabla^m \to X$ continuous. By (c) we have to define ψ_f as the composite

$$\nabla^n \times \nabla^1 \xrightarrow{\sigma_* \times \mathrm{id}} \nabla^m \times \nabla^1 \xrightarrow{\psi_g} X$$

Case 2. $f : \nabla^n \to X$ is non-degenerate and $k < n$. We note that by property (c), the map $\psi_f : \nabla^n \times \nabla^1 \to X$ is already fixed on $(\partial\nabla^n) \times \nabla^1$; by (a) it is also determined on $\nabla^n \times e_0$.

We extend the data to $\nabla^n \times \nabla^1$ by a choice of continuous retraction:

$$r : \nabla^n \times \nabla^1 \to (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1)$$

hence set ψ_f to be the composite of r and $\tilde{f} = f \cup \bigcup_{i=0,\ldots,n} \psi_{d_i^*(f)}$:

$d_i : [n-1] \to [n]$ is the unique monotone injection with $i \notin \mathrm{im}(d_i)$).

$$\nabla^n \times \nabla^1 \xrightarrow{r} (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \xrightarrow{\tilde{f}} X$$

The definition satisfies $\psi_f(-, e_0) = f$ by design and $\psi_f(\mathrm{sk}_k(\nabla^n), e_1) \subset \psi_f(\partial\nabla^n, e_1) \subset A$ by induction because $\psi_{d_i^*(f)}$ have property (a).

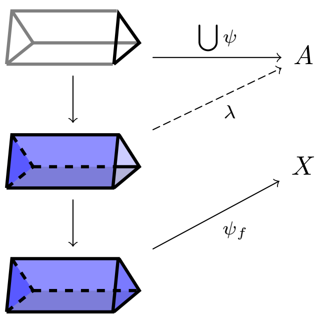
Case 3. $f : \nabla^n \to X$ is non-degenerate and $n \leqslant k$. First, note that we can show the pair $((\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1), \partial\nabla^n \times e_1)$ to be pair homeomorphic to $(D^n, \partial D^n)$.

We assumed that (X, A) is k-connected, so there are a continuous map λ and a homotopy from λ to the map $\tilde{f} = f \cup \bigcup_{i=0,\ldots,n} \psi_{d_i^*(f)}$

$$\lambda : (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \to A$$
$$H : (\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \times [0,1] \to X$$

which combine in the diagram (where the lower triangle is commutative up to homotopy):

$$\begin{tikzcd}
(\partial\nabla^n \times e_1) \arrow[r, "\bigcup_{i=0,\ldots,n} \psi_{d_i^*(f)}"] \arrow[d, hook] & A \arrow[d, hook] \\
(\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \arrow[r, "\tilde{f}"'] \arrow[ur, dashed, "\lambda"'] & X
\end{tikzcd}$$

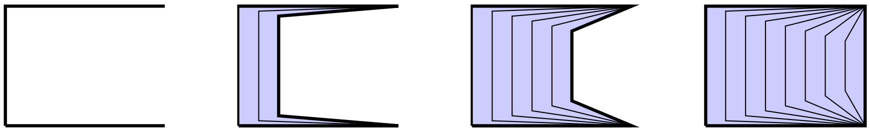


We reparametrize the relative homotopy into the desired map ψ_f as follows:

$$
\begin{tikzcd}
(\nabla^n \times e_0) \cup (\partial\nabla^n \times \nabla^1) \times [0,1] \arrow[r, "H"] \arrow[d] & X \\
\nabla^n \times \nabla^1 \arrow[ur, dashed, "\psi_f"'] &
\end{tikzcd}
$$

considering the continuous quotient map.

The following picture illustrates the reparametrization for the $n = 1$ case.



Now we "adjoin" the continuous maps ψ_f into the simplicial deformation retraction

$$H : \mathcal{S}(X) \times \Delta[1] \to \mathcal{S}(X),$$

that is, in the simplicial dimension n, we need to specify a map

$$\mathcal{S}(X)_n \times \Delta([n],[1]) \to \mathcal{S}(X)_n.$$

We do this via an adjunction bijection:

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{sSet}}(Z \times \Delta[1], \mathcal{S}(X)) &\cong \mathrm{Mor}(|Z \times \Delta[1]|, X) \\
&\cong \{\psi_z : \nabla^n \times \nabla^1 \to X \text{ for all } n \geqslant 0, z \in Z_n, \text{ such that } \psi_{\alpha^*(z)} = \psi_z(\alpha_* \circ \nabla^1)\}
\end{aligned}
$$

Sketch (full argument as an exercise).

Step 1. Given $H : Z \times \Delta[1] \to Y$ and $z \in Z_n$, we define $\psi_z : \Delta[n] \times \Delta[1] \to Y$ by $(\psi_z)_m(\alpha, k) = H_m(\alpha^*(z), k)$ (this is bijective by the Yoneda lemma).

Step 2. $\mathcal{S}$: Top $\to$ sSet is right adjoint to $| - |$: sSet $\to$ Top and there is a preferred homeomorphism $|\Delta[n]| \cong \nabla^n$, $(\alpha : [m] \to [n], t) \mapsto \alpha_*(t)$, with inverse $s \mapsto [\mathrm{id}_{[n]}, s]$.

Step 3. The map $|\Delta[n] \times \Delta[1]| \xrightarrow{(|\,\mathrm{pr}_1\,|,|\,\mathrm{pr}_2\,|)} |\Delta[n]| \times |\Delta[1]| \cong \nabla^n \times \nabla^1$ is a homeomorphism.

Step 4. Combine steps 1-3.

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{sSet}}(Z \times \Delta[1], \mathcal{S}(X)) &\underset{\text{Step 1}}{\cong} \{\psi_z : \Delta[n] \times \Delta[1] \to \mathcal{S}(X) \mid (\psi_z)_m(\alpha, k) = H_m(\alpha^*(z), k)\} \\
&\underset{\text{Step 2}}{\cong} \{\psi_z^{\#} : |\Delta[n] \times \Delta[1]| \to X\} \\
&\underset{\text{Step 3}}{\cong} \{\hat{\psi}_z : \nabla^n \times \nabla^1 \to X\}
\end{aligned}
$$

I might have written some dumb things

$\square$

Proof of Hurewicz Theorem

LECTURE 6
3rd Nov, 2021

Proof of the Hurewicz theorem (I.2). We modify the definition of $\pi_n(X,A)^\#$ by replacing $(I^n, \partial I^n)$ by the homeomorphic pair $(\nabla^n, \partial\nabla^n)$.

Then the fundamental class $i \in H_n(\nabla^n, \partial\nabla^n; \mathbb{Z})$ is represented by the map $\mathrm{id}^n_\nabla \in \mathcal{S}(\nabla^n)_n$. The inclusion of simplicial sets (the $\cong$ comes from theorem I.11 of last lecture)

$$\mathcal{S}(A) \hookrightarrow \mathcal{S}(X, A, n-1) \overset{\cong}{\hookrightarrow} \mathcal{S}(X)$$

induces morphisms of chain complexes

$$C(\mathcal{S}(A)) \to C(\mathcal{S}(X, A, n-1)) \xrightarrow{\sim} C(\mathcal{S}(X))$$

where " $\sim$ " is a chain homotopy equivalence.

We compare the long exact homology sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, equal] & H_n(C(\mathcal{S}(X,A,n-1))) \arrow[r] \arrow[d, "\cong"] & H_n(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}) \arrow[r, "\partial"] \arrow[d, "\cong \text{ (by 5-lemma!)}"] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(\frac{C(\mathcal{S}(X))}{C(\mathcal{S}(A))}) = H_n(X,A) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Conclusion: the inclusion $\mathcal{S}(X, A, n-1) \hookrightarrow \mathcal{S}(X)$ induces an isomorphism

$$H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right) \xrightarrow{\cong} H_n(X,A)$$

so that we reduced the problem to finding an isomorphism

$$\pi_n(X,A)^\# \xrightarrow{"\cong"} H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)$$

We note that $\mathcal{S}(X,A,n-1)_{n-1} = \mathcal{S}(A)_{n-1}$ so

$$\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right)_{n-1} = 0$$

which implies

$$\begin{aligned}
H_n(X,A) &\cong H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right) \\
&= \operatorname{coker}\left(\frac{\mathbb{Z}[\mathcal{S}(X,A,n-1)_{n+1}]}{\mathbb{Z}[\mathcal{S}(A)_{n+1}]} \xrightarrow{d} \frac{\mathbb{Z}[\mathcal{S}(X,A,n-1)_n]}{\mathbb{Z}[\mathcal{S}(A)_n]}\right) \\
&= \mathbb{Z}[f : (\nabla^n, \partial\nabla^n) \to (X,A)]/E''
\end{aligned}$$

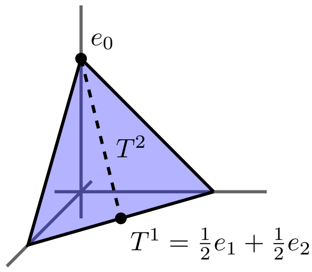
where E'' is the subgroup generated by:

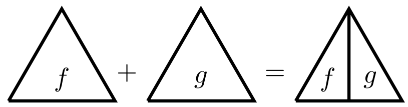
- the classes of all $f : (\nabla^n, \partial\nabla^n) \to (X,A)$ with $f(\nabla^n) \subset A$,
- elements of the form $\sum_0^{n+1}(-1)^i d_i^*(g)$ for all $g : \nabla^{n+1} \to X$ with $g(\mathrm{sk}_{n-1}(\nabla^{n+1})) \subset A$.

On the other hand, $\pi_n(X,A)^\# = \mathbb{Z}[f : (\nabla^n, \partial\nabla^n) \to (X,A)]/E'$ where E' is generated by:

- $f - f'$ for all pair homotopic $f \sim f'$,
- $f_1 + f_2 - (f_1 \oplus f_2)$ whenever f_1 and f_2 are "addible".

To add maps on simplices of the same dimension, we divide ∇^n into two sub-simplices by a procedure defined inductively, using the hyperplane T^n in ∇^n through e_0 and the hyperplane T^{n-1} dividing $d_0(\nabla^n)$. Pictured below, the $n = 2$ case, starting with $T^1 = \frac{1}{2}e_1 + \frac{1}{2}e_2$.





Claim. The canonical homomorphism

$$\mathbb{Z}[f(\nabla^n, \partial\nabla^n) \to (X,A)] \to \pi_n(X,A)^\#, \quad [f] \mapsto [f]$$

factors through a homomorphism

$$\Phi : H_n\left(\frac{C(\mathcal{S}(X,A,n-1))}{C(\mathcal{S}(A))}\right) \to \pi_n(X,A)^\#$$

(which is equivalent to saying that $E'' \subset E'$).

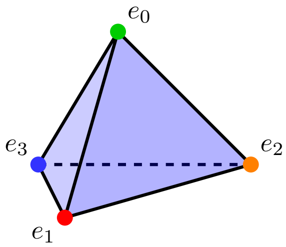
Proof of the claim. We need to show that the two kinds of relations that generate E'' are sent to 0.

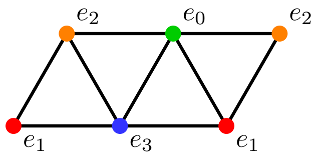
- If $f : (\nabla^n, \partial\nabla^n) \to (X,A)$ has image in A, we contract ∇^n onto e_0 and postcompose this contraction homotopy with f. The result is a pair homotopy from f to a constant map with value $f(e_0)$. So $[f] = [\mathrm{const}_{f(e_0)}]$, which is the zero element in $\pi_n(X,A)^\#$.
- Now we consider all maps $g : \nabla^{n+1} \to X$ with $g(\mathrm{sk}_{n-1}(\nabla^{n+1})) \subset A$. We want to show that $\sum_0^{n+1}(-1)^i[g \circ (d_i)_*] = 0$ in $\pi_n(X,A)^\#$.

We consider the space $B = \nabla^n \cup_{\nabla^{n-1}} \cdots \cup_{\nabla^{n-1}} \nabla^n$. It is a quotient space of a disjoint union of $n+2$ copies of ∇^n. If we number these copies from 0 to $n+1$, we glue the i-th copy to the $(i+1)$-st copy by the maps:

$$\begin{tikzcd}
\nabla^{n-1} \arrow[r, "(d_i)_*"] \arrow[d, "(d_i)_*"] & \nabla^n_{((i+1)\text{-st})} \\
\nabla^n_{(i\text{-th})} &
\end{tikzcd}$$

Informally, B is $\partial\nabla^{n+1}$ "cut open".





We define $p : B \to \partial\nabla^{n+1}$ by defining the restriction to the i-th copy of ∇^n as $(d_i)_*$. The map p is compatible with the equivalence relation (and hence well defined on B) thanks to the simplicial relations:

$$d_i \circ d_i = d_{i+1} \circ d_i$$

The upshot is that p is a quotient map onto $\partial\nabla^{n+1}$.

Since g is defined on all of ∇^{n+1}, its restriction to $\partial\nabla^{n+1}$ represents the 0 element in $\pi_n(X, A)^{\#}$.

$$\nabla^n \cong B \xrightarrow{p} \partial\nabla^{n+1} \hookrightarrow \nabla^{n+1} \xrightarrow{g} X$$

We apply the homotopy addition theorem (in simplex version) for the maps $f_0, \ldots, f_{n+1}$: $\nabla^n \to B = \nabla^n \cup_{\nabla^{n-1}} \cdots \cup_{\nabla^{n-1}} \nabla^n$, where f_i is the inclusion of the i-th copy. By the HAT:

$$0 = [g|_{\partial\nabla^{m+1}} \circ p] = \sum_{i=0}^{m+1} (-1)^i [g \circ (d_i)_*]$$

■

Let's finish once and for all the proof:

$$H_n\left(\frac{C(\mathcal{S}(X, A, n-1))}{C(\mathcal{S}(A))}\right) \xrightarrow{\Phi} \pi_n(X, A)^{\#} \xrightarrow{h^{\#}} H_n(X, A; \mathbb{Z})$$

the composite $h^{\#} \circ \Phi$ is the homomorphism induced by the inclusion $\mathcal{S}(X, A, n-1) \hookrightarrow \mathcal{S}(X)$, which is an isomorphism. So Φ is injective. But Φ is also surjective since it hits all generators. So Φ is an isomorphism, hence $h^{\#}$ is an isomorphism. □

Chapter II.

Fibre Bundles and Fibrations

II

Generalities on Fibre Bundles

A **fibre bundle** over a space B is a continuous map $\pi : E \to B$ that is locally trivial in the following sense: for every point $b \in B$ there is a space F, a neighbourhood $U \subset B$ of b and a homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

B is called the **base**, E the **total space**, F is the **fibre**, π is the **projection**, the maps $\pi^{-1}(U) \xrightarrow{\cong} U \times F$ the **local trivialisations**.

If we fix F, the set of points $b \in B$ such that $F_b = \pi^{-1}(b)$ is homeomorphic to F is open. So in particular, if B is connected, then all fibres are homeomorphic.

Examples. — Trivial fiber bundles: $\pi = \mathrm{pr}_1 : E = B \times F \to B$.

Covering spaces: locally trivial fibre bundle with discrete fiber.

Vector bundles: particular fibre bundles with fibre $\mathbb{R}^n$.

Hopf fibration: $\eta : S^3 \to S^2$.

Remark. — Suppose $\pi : E \to B$ is a locally trivial fibre bundle with fibre $\mathbb{R}^n$. For it to be a **vector bundle** there must be:

- additional structure, as each fibre $F_b = \pi^{-1}(b)$ is given the structure of an $\mathbb{R}$ vector space,
- additional conditions, i.e. the local trivialisation $\pi^{-1}(U)$ are fiberwise linear isomorphisms.

An equivalent perspective is the following. Suppose we chose a cover of B by open subsets $\{U_i\}_{i \in I}$ and local trivializations for each U_i, $u_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$. For each pair of indices i, j the "change of charts"

$$(U_i \cap U_j) \times \mathbb{R}^n \xrightarrow{u_i^{-1}} \pi^{-1}(U_i \cap U_j) \xrightarrow{u_j} (U_i \cap U_j) \times \mathbb{R}^n$$

is a homeomorphism on the projection to the first factors. So $u_j \circ u_i^{-1}$ is of the form

$$(u_j \circ u_i^{-1})(x, v) = (x, \Psi(x))$$

for some map $\Psi : (U_i \cap U_j) \times \mathbb{R}^n \to \mathbb{R}^n$. The map Φ is adjoint to a function

$$U_i \cap U_j \to \mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n), x \mapsto \Psi(x, -)$$

In a vector bundle, the map factors through $\mathrm{GL}_n(\mathbb{R}) =$ linear isos $(\mathbb{R}^n, \mathbb{R}^n)$.

Several related concepts/refinements of fibre bundles can also be conveniently formulated this way, by specifying a **structure group**, for example there is a hierarchy:

- locally trivial fibre bundles with structure group $\mathrm{Homeo}(\mathbb{R}^n, \mathbb{R}^n)$,
- smooth bundles with structure group $\mathrm{Diffeo}(\mathbb{R}^n, \mathbb{R}^n)$,
- vector bundles with structure group $\mathrm{GL}_n(\mathbb{R}^n)$,
  - vector bundles can be equipped with an inner product, in which case the structure group is required to be $\mathrm{O}(n)$,
- oriented bundles with structure group $\mathrm{GL}_n^+(\mathbb{R}^n)$,
  - oriented vector bundles can be equipped with an inner product, in which case the structure group is required to be $\mathrm{SO}(n)$.

Hopf Fibration

Lecture 7
8th Nov, 2021

The goals of the following two lectures (which are given by Markus Hausmann, a PhD student of Schwede) are:

1. to construct the **Hopf fibration**, a fibre bundle $\eta : S^3 \to S^2$ with fibre S^1,
2. to associate to every fibre bundle $p : E \to B$ a long exact sequence of the form

$$\cdots \to \pi_n(p^{-1}(b), e) \xrightarrow{i_*} \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \to \pi_{n-1}(p^{-1}(b), e) \to \cdots$$

When we are done, we will have as a corollary the computation of our first *really* non-trivial homotopy group.

II.1. Corollary. — *For every $n \geqslant 3$ there is an isomorphism $\pi_n(S^3, *) \cong \pi_n(S^2, *)$. In particular, $\pi_3(S^2, *) \cong \mathbb{Z}$, generated by the class of the Hopf fibration.*

Proof. For $n \geqslant 3$ we have

$$\cdots \to \pi_n(S^1, *) = 0 \to \pi_n(S^3, *) \xrightarrow{\eta_*} \pi_n(S^3, *) \to \pi_{n-1}(S^1, *) = 0 \to \cdots$$

which yields the claim.

In particular, for $n = 3$ we get that $\eta_* : \pi_3(S^3, *) \cong \mathbb{Z} \to \pi_3(S^2, *)$ is an isomorphism which sends $[\mathrm{id}_{S^3}]$, generator of $\pi_3(S^3, *)$, to $[\eta \circ \mathrm{id}_{S^3}] = [\eta]$. $\square$

The Hopf fibration is part of a family of fibre bundles.

Let $K = \mathbb{R}$ or $\mathbb{C}$ and recall the projective spaces:

$$K\mathbf{P}^n = (K^{n+1} \smallsetminus \{0\})/K^\times$$

where $x \sim \lambda x$ for all $x, \lambda \in K^\times$. In particular, we have that:

- $\mathbb{R}\mathbf{P}^n$ is an n-dimensional manifold,
- $\mathbb{C}\mathbf{P}^n$ is a $2n$-dimensional manifold.

Moreover, recall that $\mathbb{R}\mathbf{P}^1 \cong S^1$ and $\mathbb{C}\mathbf{P}^1 \cong S^2$.

We consider now the projections $p : K^{n+1} \setminus \{0\} \to K\mathbf{P}^n$.

Let G be a topological group. A principal G-bundle is a G-space E such that:

(1) for every $e \in E$ the map $G \to Ge = \{ge \mid g \in G\}$, $g \to ge$, is a homeomorphism,

(2) the quotient map $p : E \to E/G = E/\sim$, where $e \sim ge$ for all $e \in E$, $g \in G$, is a fibre bundle.

Example. — The group action of the additive group of the real numbers with the discrete topology on itself (with the standard topology):

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is an example of free action which does not satisfy property (2).

II.2. Proposition. — *For $K = \mathbb{R}$ or $\mathbb{C}$, the $K^\times$ action on $K^{n+1} \setminus \{0\}$ is a $K^\times$-principal bundle.*

Proof. Let $e \in K^{n+1} \setminus \{0\}$. Then the map

$$K^\times \to K^\times e, \quad \lambda \mapsto \lambda e$$

is continuous and satisfies $\|\lambda_1 e - \lambda_2 e\| = |\lambda_1 - \lambda_2|\|e\|$. It follows that the inverse is also continuous.

It remains to show that $K^{n+1} \setminus \{0\} \to K\mathbf{P}^n$ is a locally trivial fibre bundle. For $1 \leqslant i \leqslant n+1$ let $X_i \subset K^{n+1} \setminus \{0\}$ be the subspace of tuples $(x_1, \ldots, x_{n+1})$ such that $x_i \neq 0$, i.e. $x_i \in K^\times$. Then $K^{n+1} \setminus \{0\} = \cup_{i=1}^{n+1} X_i$. Let $Y_i = p(X_i) \subset K\mathbf{P}^n$. This is open since $p^{-1}(Y_i) = X_i$. We define $u : p^{-1}(Y_i) = X_i \to Y_i \times K^\times$ by $u(x) = (p(x), x_i)$, with inverse

$$u^{-1}([x], \lambda) = (x_1/x_i, \ldots, x_{i-1}/x_i, \lambda, x_{i+1}/x_i, \ldots, x_{n+1}/x_i).$$

□

For $K = \mathbb{C}$ and $n = 1$ this gives a fibre bundle

$$p : \mathbb{C}^2 \setminus \{0\} \cong S^3 \to \mathbb{C}\mathbf{P}^1 \cong S^2$$

with fibre $\mathbb{C}^\times \cong S^1$. This is already the Hopf fibration "up to homotopy".

Let $S(K^n) \subset K^n$ be the unit sphere ($(n-1)$-dimensional if $K = \mathbb{R}$, $(2n-1)$-dimensional if $K = \mathbb{C}$). Further, let $G(K) \subset K^\times$ be the subgroup of elements of norm 1 ($G(\mathbb{R}) = \{\pm 1\}$, $G(\mathbb{C}) = S^1$). Then the $K^\times$-action on $K^{n+1} \setminus \{0\}$ restricts to a $G(K)$-action on $S(K^{n+1})$ and the induced map

$$\begin{tikzcd}
S(K^{n+1}) \arrow[r] \arrow[dr] & K^{n+1} \setminus \{0\} \arrow[r] & K\mathbf{P}^n \\
& S(K^{n+1})/G(K) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

is a homeomorphism.

II.3. Proposition. — *The $G(K)$-action on $S(K^{n+1})$ defines a $G(K)$-principal bundle with base space $K\mathbf{P}^n$.*

Proof. Let $X_i \subset K^{n+1} \setminus \{0\}$, $Y_i \subset K\mathbf{P}^n$ as before. We obtain a homeomorphism

$$v : q^{-1}(Y_i) = S(K^{n+1}) \cap X_i \to Y_i \times G(K), \quad x \mapsto (q(x), x_i/|x_i|).$$

$\square$

For $K = \mathbb{R}$ we obtain the covering space $S^n \to \mathbb{R}\mathbf{P}^n$ we already knew.

For $K = \mathbb{C}$ we get a fibre bundle $S^{2n+1} \to \mathbb{C}\mathbf{P}^n$ with fibre S^1. For $n = 1$ we get the Hopf fibration $\eta : S^3 \to S^2$.

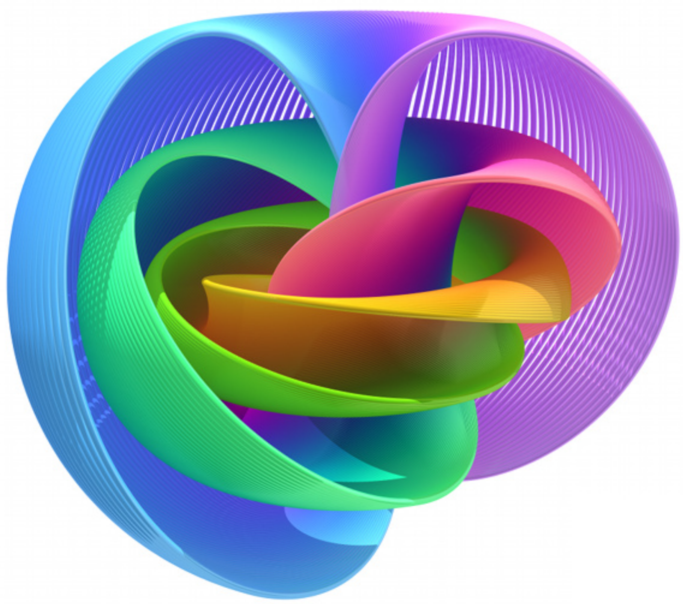
Remark. — The Hopf fibration decomposes S^3 as a disjoint union of circles, continuously indexed over S^2. It can be shown that any two of them are linked!

Considering $\eta : S^3 \to S^2$ and $x_1 \neq x_2 \in S^2$

$$s : \eta^{-1}(x_1) \times \eta^{-1}(x_2) \to S^2, \ (x, y) \to \frac{x - y}{||x - y||}$$

is continuous. Choosing orientations on S^2 and $\eta^{-1}(x_1) \times \eta^{-1}(x_2) \cong S^1 \times S^1$, we can consider the mapping degree of s. This is an example of an invariant for links called the **linking number** and it will be ± 1 in this case.

I would recommand Dimension: a walk through mathematics for the animation of Hopf fibration.



Someday I'll really wrap my head around this picture... (but maybe it's just not *that* illuminating?)

The Long Exact Sequence Associated to a Serre Fibration

We turn now to our second goal. If $p : E \to B$ is a fibre sequence, $b \in B$, $e \in p^{-1}(b)$, then we want to show that there is a long exact sequence of the form:

$$\cdots \to \pi_n(p^{-1}(b), e) \to \pi_n(E, e) \to \pi_n(B, b) \to \cdots$$

Example. — Let $p : E \to B$ be a covering space. We get a long exact sequence (assuming E is connected):

$$\cdots \to \pi_n(p^{-1}(b), e) = 0 \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(p^{-1}(b), e) = 0 \to \cdots$$
$$\cdots \to 0 \to \pi_1(E, e) \to \pi_1(B, b) \to \pi_0(p^{-1}(b), e) \to 0$$

This amounts to the known statements that $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is injective (and that the fiber can be identified, via path lifting, with the set of cosets of $p_*\pi_n(E, e)$ in $\pi_1(B, b)$) and that $p_* : \pi_n(E, e) \to \pi_n(B, b)$ is an isomorphism for $n \geqslant 2$. Both facts have been already proven using the lifting properties of covering spaces.

He adds a reminder on map lifting here.

Let $p : E \to B$ be a continuous map. A **test situation** for the homotopy lifting property (HLP) consists of a space X and a commutative square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

A solution to the test situation is a map $\tilde{H} : X \times [0,1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H} \circ i_0 = f$.

there is also a relative version: a pair of spaces (X, A) and a commutative diagram:

$$\begin{tikzcd} X \times \{0\} \cup (A \times [0,1]) \arrow[r, "f"] \arrow[d, hook, "i_0"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

A solution is again a map $\tilde{H} : X \times [0,1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H} \circ i_0 = f$.

A map $p : E \to B$ is called a **Hurewicz fibration** if it has the HLP with respect to all X and all absolute test-situations for X.

A map $p : E \to B$ is called a **Serre fibration** if it has the HLP with respect to every CW-complex X and all absolute test-situations for X.

For the proof of the existence of the long exact sequence associated to a fibre bundle we need two intermediate results.

Proposition. — *Let $p : E \to B$ be a Serre fibration, $Y \subset B$ a subspace and $x \in p^{-1}(Y)$. Then the projection induces an isomorphism (for $n \geqslant 1$):*

$$p_* : \pi_n(E, p^{-1}(Y), x) \xrightarrow{\cong} \pi_n(B, Y, p(x))$$

Corollary. — *Let $Y = \{b\}$. We get a long exact sequence:*

$$\cdots \to \pi_n(p^{-1}(b), x) \to \pi_n(E, x) \to \pi_n(E, p^{-1}(b), x) \cong \pi_n(B, b) \to \cdots$$

Theorem. — *Every fiber bundle is a Serre fibration.*

LECTURE 8
10th Nov, 2021

Before we get to the promised results, we prove an auxiliary one.

⚠ Didn't sleep much the night before this one, I hope I didn't type anything too stupid!

II.4. Lemma. — *Let $p : E \to B$ be a map. The following are equivalent:*

(1) *p is a Serre fibration,*

(2) *p has the absolute HLP for D^n for all n,*

(3) *p has the relative HLP for $(D^n, \partial D^n)$ for all n,*

(4) *p has the relative HLP for all relative CW-complexes.*

Proof. (1) $\implies$ (2) This is true because D^n is a CW-complex.

(2) $\implies$ (3) The space pairs $(D^n \times [0,1], D^n \times \{0\})$ and $(D^n \times [0,1], D^n \times \{0\} \cup \partial D^n \times [0,1])$ are homeomorphic, hence any test situation for one HLP can be translated into a test situation for the other, and similarly for the solutions.

(3) $\implies$ (4) Let (X, X') be a relative CW-complex. Consider the test situation:

$$\begin{tikzcd}
X \times \{0\} \cup X' \times [0,1] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r] & B
\end{tikzcd}$$

We first assume that $X = X' \cup_{\partial D^n} D^n$ is obtained from X' by attaching a single cell, with characteristic map $\alpha : D^n \to X$.

We obtain:

$$\begin{tikzcd}
D^n \times \{0\} \cup \partial D^n \times [0,1] \arrow[r] \arrow[d] & X \times \{0\} \cup X' \times [0,1] \arrow[r, "f"] \arrow[d] & E \arrow[d, "P"] \\
D^n \times [0,1] \arrow[r, "\alpha \times \mathrm{id}_{[0,1]}"'] \arrow[urr, dashed, "H'"] & X \times [0,1] \arrow[r, "H"'] & B
\end{tikzcd}$$

By assumption, there exists a lift H' as in the diagram.

Then the desired solution is

$$X \times [0,1] = (X' \times [0,1]) \cup_{\partial D^n \times [0,1]} D^n \times [0,1] \xrightarrow{f \cup H'} E.$$

The case where (X, X') has finitely many relative cells follows by induction, the infinite case by passing to the colimit.

(4) $\implies$ (1) This is the special case $(X, \varnothing)$. $\square$

Remark. — Note: CW-complexes are colimits of their skeleta.

$$\begin{tikzcd}
\mathrm{sk}_n X \times [0,1] \arrow[r] \arrow[d, hook] & E \\
\mathrm{sk}_{n+1} X \times [0,1] \arrow[ur]
\end{tikzcd}$$

We have $X \cong \mathrm{colim}_n \, \mathrm{sk}_n X$ and $X \times [0,1] \cong \mathrm{colim}_n(\mathrm{sk}_n X \times [0,1])$.

Now we can prove the results promised at the end of last lecture.

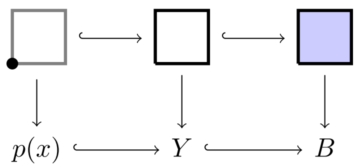
Lemma. — *Let $p : E \to B$ be a Serre fibration, $Y \subset B$ and $x \in p^{-1}(Y)$. Then p induces an isomorphism*

$$p_* \pi_n(E, p^{-1}(Y), x) \xrightarrow{\cong} \pi_n(B, Y, p(x))$$

for all $n \geqslant 1$.

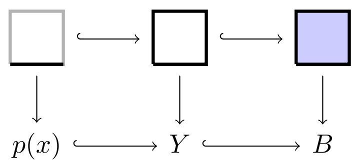
Proof.

Surjectivity. Let $[\beta] \in \pi_n(B, Y, p(x))$ be represented by $\beta(I^n, \partial I^n, s_0) \to (B, Y, p(x))$ with $s_0 = (0, \dots, 0)$.



This already looks like a homotopy but we need to lift $\beta|_{I^{n-1} \times \{0\}}$. There's different ways to go about this, for example repeated applications of the HEP would work, but since we're working with a contractible space, there's an easier and faster way.

Applying the HEP first to the relative CW-complex $(\partial I^n, I^{n-1} \times \{0\})$ (for maps into Y), and second for the relative CW-complex $(I^n, \partial I^n)$ mapping into B, we can replace β by an homotopic map β' which sends all of $I^{n-1} \times \{0\}$ to $p(x)$.

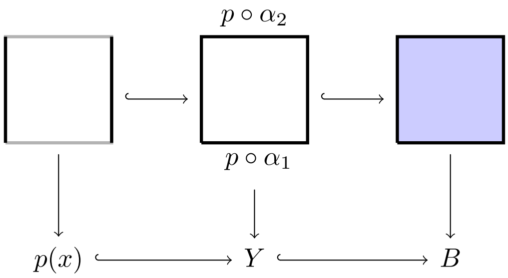


The constant map $c_{p(x)} : I^{n-1} \times \{0\} \to Y$ has a canonical lift to E via the constant map $c_x : I^{n-1} \times \{0\} \to p^{-1}(Y)$. Hence we obtain:

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "c_x"] \arrow[d] & E \arrow[d, "p"] \\
I^n \arrow[r, "\beta'"'] \arrow[ur, dashed, "\exists \tilde{\beta}"] & B
\end{tikzcd}$$

Since p is a Serre fibration, there exists $\tilde{\beta} : I^n \to E$ such that $\tilde{\beta}|_{I^{n-1}\times\{0\}} = c_x$ (in particular $\tilde{\beta}(s_0) = x$) and $p \circ \tilde{\beta} = \beta'$ (in particular $\tilde{\beta}(\partial I^n) \subset p^{-1}(Y)$). Hence $\tilde{\beta}$ represents an element $[\tilde{\beta}]$ of $\pi_n(E, p^{-1}(Y), x)$, which by construction maps to $[\beta'] = [\beta]$ under p_*.

Injectivity. Let $\alpha_1, \alpha_2 : (I^n, \partial I^n, s_0) \to (E, p^{-1}(Y), x)$ represent elements of $\pi_n(E, p^{-1}(Y), x)$ which are sent to the same element under p_*. Then there exists a homotopy of triple maps $H : I^n \times I \to B$ from $p \circ \alpha_1$ to $p \circ \alpha_2$.



Again we can assume that $\alpha_1(I^{n-1} \times \{0\}) = \alpha_2(I^{n-1} \times \{0\}) = \{x\}$. In addition we can assume that H sends $(I^{n-1} \times \{0\}) \times I$ constantly to $p(x)$. We again lift $c_{p(x)}$ to c_x and view α_1 and α_2 as lifts of H on the subspace $I^{n-1} \times \{0\} \times \{0\} \cup I^{n-1} \times \{0\} \times \{1\}$. Since $(I^{n-1} \times \{0\} \times I, I^{n-1} \times \{0\} \times \{0\} \cup I^{n-1} \times \{0\} \times \{1\})$ is a relative CW-complex, we can apply the relative HLP to lift H to a map $\tilde{H} : I^n \times I \to E$, giving a relative homotopy from α_1 to α_2. $\square$

We remember that this has as a consequence the long exact sequence associated to a Serre fibration.

II.5. Theorem. — *Let $Y = \{b\}$, $F = p^{-1}(b)$. We get a long exact sequence:*

$$\cdots \to \pi_n(F, x) \to \pi_n(E, x) \to \pi_n(E, F, x) \cong \pi_n(B, b) \to \cdots$$

Many examples of Serre fibrations come from fibre bundles, thanks to the next theorem.

II.6. Theorem. — *Every fibre bundle is a Serre fibration.*

Proof. Let $p : E \to B$ be a fibre bundle and a lifting problem

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] & B \end{tikzcd}$$

Easy case. Let p be globally trivial, i.e. of the form $\mathrm{pr}_B : B \times F \to B$ for some space F. Then we have

$$\begin{tikzcd} X \times \{0\} \arrow[r, "(f_1, f_2)"] \arrow[d] & B \times F \arrow[d, "\mathrm{pr}_B"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

We can define a lift $\tilde{H}$ explicitly via $\tilde{H}(x,t) = (H(x,t), f_2(x))$ (this works for any space X).

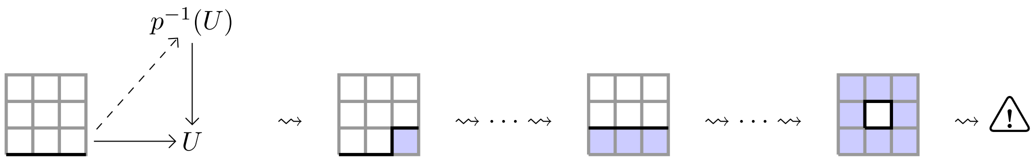
General case. We have to glue local lifts together systematically.

By lemma II.4, it suffices to check the HLP for disks D^n, or equivalently for cubes I^n. Hence we are given:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"] & B \end{tikzcd}$$

Let $\{U_i\}_{i\in I}$ be an open covering of B, such that $p^{-1}(U_i) \xrightarrow{p} U_i$ is a trivial fibre bundle for all i. Pulling back along H, we get an open cover of $I^n \times I$. By Lebesgue's lemma, we can divide $I^n \times I$ into smaller cubes of side length $1/k$, such that each cube is contained in some $p^{-1}(U_i)$.

The drawing is an explanation of how "row by row" is fine and otherwise not: add explanation!



We can then extend H iteratively over the smaller cubes "row by row". In every situation this amounts to choosing a solution to the relative lifting problem for a globally trivial fibre bundle. $\square$

Remark. — Not every fibre bundle is a Hurewicz fibration (but actual counter-examples are complicated). A sufficient condition is that the base space be paracompact.

Remark. — An interesting question: are lifting of homotopies unique? It turns out that they are unique up to homotopy!

I wasn't paying attention, it might be interesting to reconstruct the argument!

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed, "\tilde{H}_1", "\tilde{H}_2"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \times I \times \{0\} \cup X \times I \times \{1\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r] & B \end{tikzcd}$$

More on Fibre Bundles and Fibrations

Lecture 9
15^{th} Nov, 2021

Prof. Schwede is back!

II.7. Theorem. — *Let $p : E \to B$ be a continuous map, with path-connected base.*

(1) *If p is a Hurewicz fibration, then any two fibers of p are homotopy equivalent.*

(2) *If p is a Serre fibration, then any two fibres which are CW-complexes are homotopy equivalent.*

Proof. Let $b_0, b_1 \in B$ be points, set $F_0 = p^{-1}(b_0), F_1 = p^{-1}(b_1)$.

Let $\omega : [0,1] \to B$ be a path from b_0 to b_1. Choose homotopies H and K as liftings in the followings diagrams:

$$\begin{tikzcd}
F_0 \times 0 \arrow[r, "\mathrm{incl}"] \arrow[d, hook] & E \arrow[d, "p"] \\
F_0 \times [0,1] \arrow[r, "\omega\circ\mathrm{pr}_2"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
F_1 \times 0 \arrow[r, "\mathrm{incl}"] \arrow[d, hook] & E \arrow[d, "p"] \\
F_1 \times [0,1] \arrow[r, "\bar{\omega}\circ\mathrm{pr}_2"] \arrow[ur, dashed, "K"] & B
\end{tikzcd}$$

where $\bar{\omega}(t) = \omega(1-t)$ is the inverse path.

We set $f = H(-,1) : F_0 \to F_1$ and $g = K(-,1) : F_1 \to F_0$.

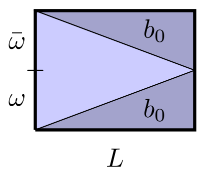
Let $L : [0,1] \times [0,1] \to B$ be a homotopy, relative $\{0,1\}$, from $\omega \times \bar{\omega}$ (the concatenated path) to const_{b_0}.

$$L(-,0) = \omega \times \bar{\omega},$$
$$L(-,1) = \mathrm{const}_{b_0}$$
$$L(0,t) = L(1,t) = b_0$$

Choose another lifting in the following diagram:

$$\begin{tikzcd}
F_0 \times ((0 \times I) \cup (I \times 0) \cup (1 \times I)) \times 0 \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
F_0 \times [0,1] \times [0,1] \arrow[r, "L\circ\mathrm{pr}_{2,3}"'] \arrow[ur, dashed, "\bar{L}"'] & B
\end{tikzcd}$$

where the upper arrow is $(\mathrm{const}_{\mathrm{incl}} \cup H \times (K \circ (f \times \mathrm{id})) \cup \mathrm{const}_{g\circ f})$.



$$\begin{tikzcd}
F_0 \times \sqsubset \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
F_0 \times \square \arrow[r, "L\circ\mathrm{pr}_{2,3}"'] \arrow[ur, dashed, "\bar{L}"'] & B
\end{tikzcd}$$

Then we can set $G = \bar{L}(-,-,1) : F_0 \times [0,1] \to E$ to obtain a homotopy from $\mathrm{incl} : F_0 \hookrightarrow E$ to $\mathrm{incl} \circ g \circ f : F_0 \to E$.

Since $p \circ G = \mathrm{const}_{b_0}$, this homotopy G takes place inside F_0.

So G is a homotopy from id_{F_0} to $g \circ f$. Reversing the roles of b_0 with b_1, F_0 with F_1 and f with g yields a homotopy from id_{F_1} to $f \circ g$. $\square$

Induced fibres bundles/fibrations. — We construct the pullback bundle. Let $p : E \to B$ and $\beta : B' \to B$ be continuous maps. The pullback is

This construction is also called base change sometimes.

$$E' = B' \times_B E = \{(b', e) \in B' \times E \mid \beta(b') = p(e)\}$$

with subspace topology of the product topology.

$$\begin{tikzcd}
 & (b',e) \arrow[r, mapsto] & e \\
(b',e) \arrow[d, mapsto] & B' \times_B E \arrow[r, dashed, "\beta'"] \arrow[d, dashed, "p'"] & E \arrow[d, "p"] \\
b' & B' \arrow[r, "\beta"] & B
\end{tikzcd}$$

This is a pullback in the sense of category theory, i.e. the following universal property holds: for all spaces A and continuous maps $\alpha : A \to B'$ and $\varepsilon : A \to E$ such that $\beta\alpha = p\varepsilon$, there is a unique continuous map $\delta : A \to B' \times_B E$ such that $\beta'\delta = \varepsilon$ and $\alpha\delta = p'$ (namely the map: $\delta(a) = (\alpha(a), \varepsilon(a))$).

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\delta"] \arrow[drr, bend left, "\varepsilon"] \arrow[ddr, bend right, "\alpha"'] & & \\
 & B' \times_B E \arrow[r, "\beta'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
 & B' \arrow[r, "\beta"] & B
\end{tikzcd}$$

Example (Restriction bundle)**.** — Suppose B' is a subspace of B and $\beta : B' \to B$ the inclusion. Then

$$B' \times_B E \cong p^{-1}(B')$$

with homeomorphism given by

$$(b', e) \mapsto e$$

$$(p(e), e) \leftarrow\!\shortmid e.$$

Example (A pretty stupid one)**.** — I lost the explanation, but take it as a little rebus:

$$\begin{tikzcd}
B' \amalg B' \arrow[r] \arrow[d] & B' \arrow[d] \\
* \amalg * \arrow[r] & *
\end{tikzcd}$$

In general taking the constant map $\beta : B' \to B$ to a point $b \in B$ will yield as the pullback bundle the trivial bundle $B' \times p^{-1}(b)$.

II.8. Theorem. — *Let $p : E \to B$ and $\beta : B' \to B$ be continuous maps.*

(i) If p is a fibre bundle, then so is $p' : B' \times_B E \to B'$.

(ii) If p is a Hurewicz fibration, then so is p'.

(iii) If p is a Serre fibration, then so is p'.

Proof. There's two reasonable proofs of the first fact, one direct, one categorical. The Professor seems to be suggesting a blue pill/red pill situation (with the categorical version being *"much more transparent"* to him).

"You take the blue pill the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill you stay in Wonderland, and I show you how deep the rabbit hole goes. Remember: all I'm offering is the truth. Nothing more."

(i) Consider any point $a \in B'$. There is a open neighbourhood U of $\beta(a)$ in B and a local trivialization, i.e. a homeomorphism $\psi : p^{-1}(U) \to U \times F$ for one space F (over U).

We argue that $p' : B' \times_B E \to B'$ is trivializable over $V = \beta^{-1}(U)$ with the same fibre F. The following are mutually inverse homeomorphisms:

$$\begin{array}{rcl} (p')^{-1}(V) & \xleftrightarrow{\cong} & V \times F \\ (b', e) & \mapsto & (b', \psi(\beta(b'))) \\ (b', \psi^{-1}(\beta(b'), f)) & \leftarrow\!\shortmid & (b', f) \end{array}$$

Categorical proof: pullbacks are transitive.

In any category $\mathcal{C}$, we consider a commutative diagram:

$$\begin{tikzcd} B'' \times_B E \arrow[r] \arrow[d] & B' \times_B E \arrow[r, "\beta'"] \arrow[d, "p'"] & E \arrow[d] \\ B'' \arrow[r] & B' \arrow[r, "\beta"] & B \end{tikzcd}$$

If both squares are pullbacks, then the composite square is also a pullback. Symbolic notation:

$$B'' \times_B E \cong B'' \times_{B'} (B' \times_B E).$$

Hence we have:

$$(p')^{-1}(V) = V \times_{B'} (B' \times_B E) \cong V \times_B E = V \times_U (U \times_B E) \cong V \times_U (U \times F) \cong V \times F.$$

(ii)+(iii) We show: whenever $p : E \to B$ has the HPL for some space X, then $p' : B' \times_B E \to B'$ also has the HLP for X.

Consider a lifting square on the left:

$$\begin{tikzcd} X \times 0 \arrow[r, "f"] \arrow[d, hook] & B' \times_B E \arrow[r, "\beta'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "H"'] \arrow[ur, dashed, bend left, "\bar{K}"] \arrow[urr, dashed, "K"] & B' \arrow[r, "\beta"'] & B \end{tikzcd}$$

There is a homotopy $K : X \times [0,1] \to E$ so that $p \circ K = \beta \circ H$ and $K(-,0) = \beta' \circ f$.

The universal property of the pullback provides a unique continuous map

$$\bar{K} : X \times [0,1] \to B' \times_B E$$

such that $p' \circ \bar{K} = H$ and $\beta' \circ \bar{K} = K$.

Two continuous maps $f, \bar{K}(-,0) : X \times 0 \to B' \times_B E$ compose in the same way with p' and with β'. The uniqueness part of the universal property then forces $f = \bar{K}(-,0)$. $\square$

Chapter III.

Mapping Spaces

III

Compact-open Topology

Our aim: to define and study a specific topology on $Z^X = \{f : X \to Z \text{ continuous}\}$.

We would like the "exponential law" to hold: the map

$$Z^{X\times Y} \to (Z^X)^Y$$
$$(f : X \times Y \to Z) \mapsto \{y \mapsto f(-, y)\}$$

should be a homeomorphism.

Unfortunately, it is not. At least not in general, but the good news is that it is whenever Y is Hausdorff and X locally compact.

Remark. — There is a way to arrange the exponential law in complete generality: work in the full subcategory CG of compactly generated spaces. Then CG is a cartesian closed category, i.e. it has finite products and for all $X \in \mathrm{ob}(G)$

$$- \times X : CG \to CG$$

has a right adjoint, written $Z \mapsto Z^X$.

Watch out: the product in CG is not always the usual product topology!

Some examples of classes of spaces in CG are:

- every locally compact Hausdorff spaces,
- every CW-complex,
- every manifold is in CG,
- the realization of every simplicial set,
- for two CW-complexes X and Y, the product in CG, $X \times_{CG} Y$, is again a CW-complex.
- for all simplicial sets A, B, the canonical map:

$$|A \times B| \to |A| \times_{CG} |B|$$

  is a homeomorphism.

Enter: the compact-open topology. For spaces X and Z, write Z^X for the set of continuous maps $f : X \to Z$. Let K be a compact subset of X and let O be an open subset of Z. Set:

$$W(K, O) = \{f \in Z^X : f(K) \subset O\} \subset Z^X.$$

The **compact-open topology** on Z^X is the topology generated by the sets $W(K,O)$ with $K \subset X$ compact and $O \subset Z$ open, i.e. these sets form a subbasis of the compact-open topology.

III.1. Theorem. — *Let X be a compact space and (Z,d) a metric space.*

(1) *There is a metric on Z^X, defined as:*

$$d(g_1, g_2) = \sup_{x \in X} d(g_1(x), g_2(x)), \ \textit{for } g_1, g_2 \in Z^X.$$

called the supremum metric.

(2) *The compact-open topology on Z^X coincides with the metric topology of the supremum metric.*

Proof. (1) Omitted (see [**Hatcher**].

(2) "$\subset$": compact-open is metrically open.

It suffices to show that the generating sets $W(K,O)$ are metrically open. Fix $K \subset X$ compact, $O \subset Z$ open and $f \in W(K,O)$, i.e. $f(K) \subset O$. Then $C = Z \smallsetminus O$ is a closed subset of Z and $f(x) \notin C$ for all $x \in K$ (hence $d(f(x), C) > 0$ for all $x \in K$). Since K is compact, $\varepsilon = \inf_{x \in K} d(f(x), C) > 0$ is positive. We claim that $B_{\sup}(f, \varepsilon) \subset W(K,O)$.

Let $g \in Z^X$ be such that $d(f,g) < \varepsilon$. Then $d(f(x), g(x)) < \varepsilon$ for all $x \in K$, hence:

$$d(f(x), g(x)) \leqslant d(f(x), C),$$

so $g(x) \in O = Z \smallsetminus C$. Therefore $g(K) \subset O$, so $g \in W(K,O)$.

"$\supset$": metrically open is compact-open.

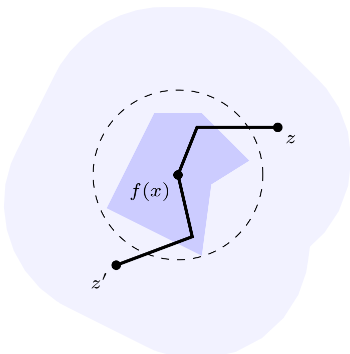
Let $A \subset Z^X$ be open in the sup-topology, $f \in A$. We will construct finitely many compact sets K_i in X and open sets O_i in Z so that:

$$f \in W(K_1, O_1) \cap \cdots \cap W(K_m, O_m) \subset A.$$

Since A is open in the sup-topology, there is an $\varepsilon > 0$ so that A contains the ε-ball around f.

For $x \in X$ the set $f^{-1}(B(f(x), \varepsilon/5))$ is an open neighbourhood of x in X. Since X is compact, this contains a compact neighbourhood K_x of x. Since X is compact, there are finitely many $x_1, \ldots, x_m$ such that $X = K_{x_1} \cup \cdots \cup K_{x_m}$. Set $K_i := K_{x_i}$ and $O_i :=$ open $\varepsilon/5$-neighbourhood of $f(K_i)$ in Z.

Note: compact spaces, meaning quasi-compact *and* Hausdorff, are locally compact, but quasi-compact spaces may not be!



As shown in the picture, for all $z, z' \in O_i$, we have $d(z, z') \leqslant 4\varepsilon/5 < \varepsilon$ (to expand on this, observe we have $f(K_x) \subset B(f(x), \frac{\varepsilon}{5}) \subseteq O_x := \frac{\varepsilon}{5}$-neighborhood of $f(K_x)$, hence both z and z' are at distance less or equal than $\frac{\varepsilon}{5}$ from two points of $f(K_x)$, and both these points are at distance less or equal than $\frac{\varepsilon}{5}$ from $f(x)$).

Suppose that $g \in W(K_1, O_1) \cap \cdots \cap W(K_m, O_m)$. Then for all $x \in X$ there is $1 \leqslant i \leqslant m$ with $x \in K_i$. Since $g \in W(K_i, O_i)$, we have $g(x) \in O_i$. We also have $x \in K_i = K_{x_i}$, so $d(f(x), f(x_i)) \leqslant \varepsilon/5$ so $f(x) \in O_i$. Hence $d(g(x), f(x)) < \varepsilon$. Since this holds for all $x \in X$, $d(f, g) < \varepsilon$. So $W(K_1, O_1) \cap \cdots \cap W(K_m, O_m) \subset B_{\sup}(f, \varepsilon) \subset A$. □

Lecture 10
17th Nov, 2021

I missed this lecture, thanks to Paul for providing photos of the blackboard!

Up to now we have proved the following facts:

- for a Hurewicz fibration (resp. Serre fibration) over a path-connected space, the fibres are homotopy equivalent (resp. the same, but whenever they admit CW-structures),
- fibre bundles, Hurewicz fibrations and Serre fibrations are stable under base change,
- if X is compact and Z is a metric space, the compact-open topology on Z^X agrees with the topology of the supremum metric.

Example. — Let I be any set, endowed with the discrete topology. Let Z be a space and consider $Z^I = \prod_I Z$. Then the compact-open topology on Z^I agrees with the product topology.

- Let J be any subset of I; then J is compact if and only if it is finite. So the sets $W(J, O) = \prod_{j \in J} O \times \prod_{j \notin J} Z$ for J finite and O open form a subbasis of the compact-open topology. These sets are open in the product topology.
- A subbasis in the product topology is given by the sets $\prod_{j \in J} O_j \times \prod_{j \notin J} Z$ for $J \subset I$ finite and O_j open in Z. But then we have that

$$\prod_{j \in J} O_j \times \prod_{j \notin J} Z = \bigcap_{j \in J} (O_j \times \prod_{i \neq j} Z) = \bigcap_{j \in J} W(\{j\}, O_j)$$

  is open in the compact-open topology.

III.2. Theorem. — *Let X and Z be spaces and let $\mathcal{S}$ be a subbasis of the topology on Z. Then the sets $W(K, O)$ for $K \subset X$ compact and $O \in \mathcal{S}$ form a subbasis of the compact-open topology.*

Proof. The "new topology" (generated by $W(K, O)$ for $O \in \mathcal{S}$) is clearly contained in the compact-open topology. We need to show that for all $O \subset Z$ open, the set $W(K, O)$ is open in the new topology.

If $O \in \mathcal{S}$, this is true by definition.

If $O = O_1 \cap \ldots \cap O_n$ with $O_i \in \mathcal{S}$, then

$$W(K, O) = W(K, O_1) \cap \cdots \cap W(K, O_n)$$

which is open in the new topology.

Now, suppose $O = \bigcup_{i \in I} O_i$ with $O_i \in \mathcal{B}$, where $\mathcal{B}$ is the basis generated by $\mathcal{S}$, i.e. the set of all finite intersections of sets in $\mathcal{S}$. Let $f \in W(K, O)$. Then for each $x \in K$ there is a set

$O_x \in \mathcal{B}$ with $f(x) \in O_x \subset O$. Since K is compact, hence locally compact, there is a compact neighbourhood K_x of x with $f(K_x) \subset O_x$. The covering $\{K_x\}_{x\in K}$ of the compact set K has a finite subcover $K \subset K_{x_1} \cup \cdots \cup K_{x_n}$. Then

$$f \in \bigcap_{i=1,\dots,n} W(K_{x_i}, O_{x_i}) \subset W(K, O).$$

□

III.3. Theorem. — *Let X and Z be spaces and let X be locally compact. Then the evaluation map* $\mathrm{ev} : Z^X \times X \to Z,\ \mathrm{ev}(f, x) = f(x)$, *is continuous.*

Proof. Let O be any open subset of Z. We want to show that

$$\mathrm{ev}^{-1}(O) = \{(f, x) \in Z^X \times X : f(x) \in O\}$$

is open. Let $(f, x) \in Z^X \times X$ be in $\mathrm{ev}^{-1}(O)$, so $f(x) \in O$. Since f is continuous, $f^{-1}(O)$ is an open neighbourhood of x. Since X is locally compact, there is a compact neighbourhood K of x inside $f^{-1}(O)$. Then $f \in W(K, O)$ and $W(K, O) \times K$ is a neighbourhood of (f, x) in $Z^X \times X$. Then $\mathrm{ev}(W(K, O) \times K) \subset O$, hence $(f, x) \in W(K, O) \times K \subset \mathrm{ev}^{-1}(O)$, so ev^{-1} is open. □

III.4. Theorem. — *Let X, Y and Z be spaces.*

(0) *For every continuous map $f : X \times Y \to Z$, the adjoint map*

$$\Phi(f) : Y \longrightarrow Z^X \qquad y \longmapsto [\Phi(f)(y) : x \mapsto f(x, y)]$$

is continuous. So Φ defines a map $Z^{X\times Y} \to (Z^X)^Y$.

(1) *The map $\Phi : Z^{X\times Y} \to (Z^X)^Y$ is continuous.*

(2) *If X is locally compact (and Hausdorff), then Φ is bijective.* Locally compact includes Hausdorff for Schwede.

(3) *If X is locally compact and Y is Hausdorff, then Φ is a homeomorphism.*

Proof.

(0) Let $f : X \times Y \to Z$ be continuous and $\Phi(f) : Y \to Z^X$ the adjoint map. To show that $\Phi(f)$ is continuous, it suffices to show that the sets $(\Phi(f))^{-1}(W(K, O))$ are open in Y for all $K \in X$ compact, $O \in Z$ open. Let $y \in (\Phi(f))^{-1}(W(K, O))$, i.e. $f(K \times \{y\}) \subset O$, or $K \times \{y\} \subset f^{-1}(O)$ which is open. By the tube lemma, there is a open neighbourhood U of y in Y such that $K \times U \subset f^{-1}(O)$ or equivalently $\Phi(f)(U) \subset W(K, O)$. Hence $y \in U \subset (\Phi(f))^{-1}(W(K, O))$, so the latter set is open in Y.

(1) The compact-open topology on Z^X has a subbasis consisting of the sets $W(K, O)$ for $K \subset X$ compact and $O \subset Z$ open. By theorem III.2 the compact-open topology on $(Z^X)^Y$ has a subbasis of the form $W(K', W(K, O))$ for $K \subset X$ compact, $K' \subset Y$ compact, $O \subset Z$ open. But $\Phi^{-1}(W(K', W(K, O))) = W(K \times K', O)$ which is open in the compact-open topology on $Z^{X\times Y}$ because $K \times K'$ is again compact.

(2) On the set-theoretic level, every map $Y \to Z^X$ is of the form $\Phi(f)$ for some unique map $f : X \times Y \to Z$. So Φ is continuous by (1) and injective. We have to show that when

X is locally compact and $g : Y \to Z^X$ is continuous, $\Phi^{-1}(g)$ is continuous. But since the evaluation map is continuous, by (0) the composite

$$\Psi(g) : X \times Y \cong Y \times X \xrightarrow{g\times\mathrm{id}} Z^X \times X \xrightarrow{\mathrm{ev}} Z$$

is continuous, hence we have:

$$\Phi(\Psi(g))(y)(x) = \Psi(g)(x,y) = \mathrm{ev}(g(y),x) = g(y)(x) \text{ for all } x \in X, y \in Y$$

so that $\Phi(\Psi(g)) = g$.

(3) Now we suppose that X is locally compact Hausdorff and Y is Hausdorff. As we saw in (1), the sets $W(K', W(K,O))$ with $K \subset X$ and $K' \subset Y$ compact, $O \subset Z$ open, form a subbasis on $(Z^X)^Y$. Then

$$\Phi^{-1}(W(K', W(K,O))) = W(K \times K', O)$$

so that we just need to show that the sets $W(K\times K', O)$ generate the compact-open topology on $Z^{X\times Y}$. This is the content of the following lemma. $\square$

III.5. Lemma. — *Let X and Y be Hausdorff spaces and Z any space. Then the compact-open topology on $Z^{X\times Y}$ is generated by the sets $W(K \times K', O)$ for all $K \subset X$ compact, $K' \subset Y$ compact and $O \subset Z$ open.*

Proof. Let L be any compact subset of $X \times Y$ and O an open subset of Z. We need to show that $W(L,O)$ is open in the potentially small topology generated by the sets $W(K \times K', O)$. Let $f \in W(L,O)$ be arbitrary, we have $L \subset f^{-1}(O)$, which is open in $X \times Y$. For each $(x,y) \in L$ we can choose $U_{x,y}$ open in X and $V_{x,y}$ open in V with $(x,y) \in U_{x,y} \times V_{x,y} \subset f^{-1}(O)$. We set $L_X = \mathrm{pr}_X(L)$ and $L_Y = \mathrm{pr}_Y(L)$, which are quasi-compact subsets of X and Y respectively. Since X and Y are Hausdorff, L_X and L_Y are indeed compact, hence locally compact. For each $(x,y) \in L$ we can find compact neighbourhoods $K_{x,y}$ of x in L_X and $K'_{x,y}$ of y in L_Y. Then the sets $\{K_{x,y} \times K'_{x,y}\}_{(x,y)\in L}$ cover L. Since L is compact, there is a finite subcover $\{K_{x,y} \times K'_{x,y}\}_{(x,y)\in I}$ with I finite. Then

$$f \in \underbrace{\bigcap_{(x,y)\in I} W(K_{x,y} \times K'_{x,y}, O)}_{\text{open in the potentially smaller topology}} = W(\bigcup_{(x,y)\in I} K_{x,y} \times K'_{x,y}, O) \subset W(L,O)$$

where the last inclusion follows from $L \subset \bigcup_{(x,y)\in I} K_{x,y} \times K'_{x,y}$. $\square$

Remark. — The assignment $(Z,X) \mapsto Z^X$ is a contravariant functor for continuous maps in Hausdorff spaces and a covariant functor for continuous map in all spaces, i.e.

$$(-)^{(-)} : \mathrm{Top} \times \mathrm{Top}_H^{\mathrm{op}} \to \mathrm{Top}$$

Contravariant functoriality: let $f : X \to X'$ be a continuous map between Hausdorff spaces. Define $f^* : Z^{X'} \to Z^X$ by precomposition with f, i.e.

$$f^*(\psi : X' \to Z) = \psi \circ f : X \to Z.$$

This map is continuous: let $K \subset X$ be compact, $O \subset Z$ open, then

$$(f^*)^{-1}(W(K,O)) = \{\psi : X' \to Z \mid (\psi \circ f)(K) \subset O\} = W(f(K),O).$$

Since K is compact, $f(K)$ is quasi-compact; since X' is Hausdorff, $f(K)$ is compact.

Covariant functoriality: let $g : Z \to Z'$ be continuous. We define $g_* : Z^X \to (Z')^X$ by postcomposition with g, i.e.

$$g_*(\psi : X \to Z) = g \circ \psi : X \to Z'.$$

This map is continuous: let $K \subset X$ be compact and $O \subset Z'$ open, then

$$g_*^{-1}(W(K,O)) = \{\psi : X \to Z \mid (g \circ \psi)(K) \subset O\} = W(K, g^{-1}(O))$$

which is open in Z^X.

Note also that for $f : X \to X'$ continous, $g : Z \to Z'$ continuous and X, X' Hausdorff, the following square commutes:

$$\begin{array}{ccc} (Z')^{X'} & \xrightarrow{f^*} & (Z')^X \\ {\scriptstyle g_*}\uparrow & & {\scriptstyle g_*}\uparrow \\ Z^{X'} & \xrightarrow{f^*} & Z^X \end{array}$$

Example. — Let $f, g : X \to Z$ be homotopic maps, with $H : X \times [0,1] \to Z$ a homotopy between them. The adjoint $\Phi(H) : [0,1] \to Z^X$ is then continuous. Hence homotopies correspond to paths in Z^X, so that the map:

$$\begin{array}{ccc} [X,Z] & \to & \pi_0(Z^X) \\ [f] & \mapsto & [f] \end{array}$$

where $[X,Z]$ indicates the set of homotopy classes of maps $X \to Z$, is well-defined and a bijection whenever X is locally compact.

Path Spaces and Loop Spaces

For a space X, the space $X^{[0,1]}$ is the **path space** of X. For a pointed space (X, x_0), the **loop space** ΩX is the subspace of $X^{[0,1]}$ consisting of all loops at x_0, i.e. those $\omega \in X^{[0,1]}$ with $\omega(0) = \omega(1) = x_0$.

Let $q : [0,1] \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the quotient map $q(x) = e^{2\pi i x}$, this induces a continuous map

$$q^* : X^{S^1} \to X^{[0,1]},$$

which restricts to a continuous bijection onto ΩX

$$(X, x_0)^{(S^1,1)} = \{f : S^1 \to X \mid f(1) = x_0\}.$$

This is in fact a homeomorphism, as a special case of the following lemma.

Lemma. — *Let $q : X \to Y$ be a quotient map between compact spaces. Then for every space Z, the continuous map $q^* : Z^Y \to Z^X$ is a homeomorphism onto the subspace of all functions $f : X \to Z$ that factor through q.*

Lecture 11
22nd Nov, 2021

We prove the lemma from last lecture.

⚠ I definitely still need to digest lecture 11 and 12, don't count on everything I write.

Proof. First, note that q^* is injective, continuous and has the desired image by the universal property of the quotient topology. We show that q^* is also open as a map onto its image. We let $K \subset Y$ be compact and $O \subset Z$ be open. Then

$$q^*(W(K,O)) = \{g \circ q : X \to Z \mid g \in Z^Y,\ g(K) \subset O\} = W(q^{-1}(K), O) \cap \operatorname{im}(q^*).$$

Because K is compact and Y Hausdorff, K is closed in Y; so $q^{-1}(K)$ is closed in X. Since X is compact, $q^{-1}(K)$ is again compact. So $W(q^{-1}(K), O) \cap \operatorname{im}(q^*)$ is open. □

Example. — The map $q : [0,1] \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $q(x) = e^{2\pi i x}$ is a quotient map, so

$$q^* : X^{S^1} \to \{f \in X^{[0,1]} \mid f(0) = f(1)\}$$

is a homeomorphism. For any base point $x_0 \in X$, this homeomorphism restricts to a homomorphism

$$(X, x_0)^{(S^1,1)} \to \Omega X.$$

If (Y, y_0) is a pointed space, the pointed version of $[Y, Z] = \pi_0(Z^Y)$ yields a well-defined surjective map $[Y, Z]_* \to \pi_0((Z, z_0)^{(Y,y_0)})$. If Y is locally compact, this is a bijection.

For $Y = S^1$, this gives a bijection $\pi_1(Z, z_0) \cong \pi_0((Z, z_0)^{(S^1,1)}) = \pi_0(\Omega Z)$.

Loop Shifts the Homotopy Groups

Let (Y, y_0) be a based space. The reduced suspension is the space:

$$\Sigma Y = \frac{Y \times [0,1]}{Y \times 0 \cup \{y_0\} \times [0,1] \cup Y \times 1}.$$

The quotient map $q : Y \times [0,1] \to \Sigma Y$ induces a continuous injection:

$$q^* : (Z, z_0)^{(\Sigma Y, *)} \to Z^{Y \times [0,1]}$$

whose image consists of all continuous maps $f : Y \times [0,1] \to Z$ that factor through the quotient map, i.e. such that $f(Y \times 0 \cup \{y_0\} \times [0,1] \cup Y \times 1) = \{z_0\}$. If Y is compact, then so are $Y \times [0,1]$ and ΣY, and q^* is even an homeomorphism onto its image:

$$(Z, z_0)^{(\Sigma Y, *)} \cong (\Omega Z, \mathrm{const}_{z_0})^{(Y, y_0)}.$$

In particular:

$$\operatorname{Hom}_{\mathrm{Top}_*}((\Sigma Y, *), (Z, z_0)) \cong \operatorname{Hom}_{\mathrm{Top}_*}((Y, y_0), (\Omega Z, \mathrm{const}_{z_0})).$$

So the functors Σ and Ω are adjoint endofunctors in the category of based spaces.

On path components, we obtain a bijection

$$[\Sigma Y, Z]_* \cong [Y, \Omega Z]_*.$$

For $Y = S^n$ the last bijection specializes to a bijection

$$\pi_{n+1}(Z, z_0) = [S^{n+1}, Z]_* \cong [\Sigma S^n, Z]_* \cong [S^n, \Omega Z]_* = \pi_n(\Omega Z, \mathrm{const}_{z_0})$$

which is even a group homomorphism (as we will see).

Mapping Spaces and Serre Fibrations

III.6. Theorem. — *Let Z be any space, (X, A) a relative CW-complex with X and A finite CW-complexes. Then the restriction map* $\mathrm{incl}^* : Z^X \to Z^A$, $f \mapsto f|_A$ *is a Serre fibration.*

A slightly more general statement was mentioned but I missed it. I guess X and A need not be finite but something weaker?

Proof. We show that incl^* has the HLP with respect to all CW-complexes Q. So we consider a lifting diagram

$$\begin{tikzcd} Q \times 0 \arrow[r, "f"] \arrow[d, hook] & Z^X \arrow[d, "\mathrm{incl}^*"] \\ Q \times [0,1] \arrow[r, "\Phi"] & Z^A \end{tikzcd}$$

Since X and A are locally compact we can use the exponential law to adjoint f and Φ to continuous maps

$$\tilde{f} : X \times Q \times 0 \to Z$$
$$\tilde{\Phi} : A \times Q \times [0,1] \to Z$$

The condition $\mathrm{incl}^* \circ f = \Phi|_{Q\times 0}$ becomes the relation that $\tilde{f}$ and $\tilde{\Phi}$ coincide on $A \times Q \times 0$.

We consider the glued maps

$$\tilde{f} \cup \tilde{\Phi} : X \times Q \times 0 \cup A \times Q \times [0,1] \to Z.$$

Since $(X \times Q \times [0,1], X \times Q \times 0 \cup A \times Q \times [0,1])$ is a relative CW-complex, it has the HEP so there is a continuous extension $H : X \times Q \times [0,1] \to Z$ that extends $\tilde{f}$ and $\tilde{\Phi}$. The adjoint $H^\# : Q \times [0,1] \to Z^X$ of H then solves the original lifting problem. □

For the relative CW-complex $([0,1], \{0,1\})$ the theorem says that the restriction map

$$Z^{[0,1]} \to Z^{\{0,1\}} \cong Z \times Z,\ w \mapsto (w(0), w(1))$$

is a Serre fibration for every space Z.

We let $z_0 \in Z$ be any base point. Define EZ as the subspace of $Z^{[0,1]}$ of all paths that start at z_0.

Equivalently, EZ is the pullback:

$$\begin{tikzcd} w \arrow[d, mapsto] & EZ \arrow[r, hook] \arrow[d] & Z^{[0,1]} \arrow[d] & w \arrow[d, mapsto] \\ w(1) & Z \arrow[r, "{(z_0,-)}"] & Z \times Z & (w(0), w(1)) \end{tikzcd}$$

Since Serre fibrations are stable under base-change, we conclude that the map $EZ \to Z$, $w \mapsto w(1)$ is a Serre fibration.

III.7. Theorem. — *The space EZ is contractible onto the constant path at z_0.*

Proof. Let $H : [0,1] \times [0,1] \to [0,1]$ be a homotopy, relative $\{0\}$, that contracts the interval onto 0, e.g. $H(s,t) = s(1-t)$. Let

$$H^* : Z^{[0,1]} \to Z^{[0,1]\times[0,1]}$$

be the continuous induced map and

$$\tilde{H}^* : Z^{[0,1]} \times [0,1] \to Z^{[0,1]}$$

its adjoint, i.e. $\tilde{H}^*(w,t)(s) = w(H(s,t))$.

We observe that

$$\tilde{H}^*(w,t)(0) = w(H(0,t)) = w(0).$$

So whenever $w(0) = z_0$ (i.e. $w \in EZ$), then also $\tilde{H}^*(w,t)(0) = z_0$. In other words, for all $t \in [0,1]$, the map $\tilde{H}^*(-,t)$ takes EZ to EZ. So we can restrict $\tilde{H}^*$ to a continuous map $\tilde{H}^* : EZ \times [0,1] \to EZ$; this is the desired contracting homotopy:

$$\tilde{H}^*(w,1)(s) = w(H(s,1)) = w(0)$$

so the homotopy $\tilde{H}^*$ ends in the constant map at z_0.

For all $w \in EZ$, $\tilde{H}^*(w,0)(s) = w(t(s,0)) = w(s)$, so $\tilde{H}^*(w,0) = w$. □

Now we can see that $\pi_n(\Omega Z, *) \cong \pi_{n+1}(Z, z_0)$ is a group morphism.

Second proof/construction of the isomorphism $\pi_n(\Omega Z, *) \cong \pi_{n+1}(Z, z_0)$. We have seen that the space $EZ = \{w \in Z^{[0,1]} \mid w(0 = z_0)\}$ is contractible.

The map $e : EZ \to Z$, $e(w) = w(1)$ is a Serre fibration, so for every point in Z, we get a long exact sequence of homotopy groups. For $z_0 \in Z$, the fibre of e at z_0 is ΩZ; hence the long exact sequence of homotopy groups is:

$$\cdots \to \pi_{n+1}(EZ, *) = 0 \xrightarrow{e_*} \pi_{n+1}(Z, z_0) \xrightarrow{\partial} \pi_n(\Omega Z, *) \xrightarrow{\mathrm{incl}_*} \pi_n(EZ, *) = 0 \to \cdots$$

So for $n \geqslant 1$, the connecting morphism $\partial : \pi_{n+1}(Z, z_0) \to \pi_n(\Omega Z, *)$ is an isomorphism. □

Turning Maps into Fibrations, up to Homotopy

This is kind of dual to the mapping cylinder.

III.8. Theorem. — *Every continuous map $f : X \to Y$ can be factored functorially and naturally as a composite*

What does he mean by functorially and naturally?

$$X \xrightarrow{\simeq} Ef \xrightarrow{p} Y$$

of a homotopy equivalence followed by a Serre fibration.

Proof. Let
$$Ef = X \times_Y Y^{[0,1]} = \{(x,w) \in X \times Y^{[0,1]} \mid f(x) = w(0)\}.$$

More precisely, Ef is the pullback:

$$\begin{tikzcd} Ef \arrow[r] \arrow[d] & Y^{[0,1]} \arrow[d] & w \arrow[d, mapsto] \\ X \arrow[r, "f"] & Y & w(0) \end{tikzcd}$$

We define natural continuous maps

$$\begin{aligned} h : X &\longrightarrow Ef & x &\longmapsto (x, \mathrm{const}_{f(x)}) \\ p : Ef &\longrightarrow Y & (x,w) &\longmapsto w(1). \end{aligned}$$

Clearly we have $f = p \circ h$.

We observe that Ef can be described as a slightly different pullback:

$$\begin{tikzcd} (x,w) \arrow[d, mapsto] & Ef \arrow[r] \arrow[d] & Y^{[0,1]} \arrow[d] & w \arrow[d, mapsto] \\ (x, w(1)) & X \times Y \arrow[r, "f \times \mathrm{id}"] & Y \times Y & (w(0), w(1)) \end{tikzcd}$$

Since Serre fibrations are stable under base change, the map $Ef \to X \times Y$, $(x,w) \mapsto (x, w(1))$ is a Serre fibration. Since $\mathrm{pr}_2 : X \times Y \to Y$ is a Serre fibration and Serre fibrations are closed under composition, p is a Serre fibration. The map $h : X \to Ef$ is a homotopy equivalence: the homotopy inverse, which is also a left inverse, is the projection to the first factor.

Claim. The composite $c : Ef \to Ef$ is homotopic to the identity.

Proof of the claim. We define the desired homotopy

$$\bar{H} : Ef \times [0,1] \to Ef = X \times_Y Y^{[0,1]}$$

by specifying its projections to X and to $Y^{[0,1]}$. The first coordinate of $\bar{H}$ is $Ef \times [0,1] \to X$, $(x,w,t) \mapsto x$, i.e. the constant homotopy of the projection to X. The second coordinate is the composite

$$Ef \times [0,1] \xrightarrow{\mathrm{pr}_2 \times \mathrm{id}} Y^{[0,1]} \times [0,1] \xrightarrow{\tilde{H}^*} Y^{[0,1]}$$

where $\tilde{H}^*$ is the homotopy we constructed in the proof of theorem III.7.

This has the following properties:

- $\bar{H}$ starts with the identity,

- for all $t \in [0,1]$, all $w \in Y^{[0,1]}$, the path $\tilde{H}^*(w,t)$ has the same startpoint as w, so $\bar{H}$ really lands in Ef.
- $\tilde{H}^*(w,1)$ is constant at $w(0)$, which means that $\bar{H}(-,1) = h \circ \mathrm{pr}_1$.

$\blacksquare$

$\square$

LECTURE 12
24th Nov, 2021

The **homotopy fibre** of a continuous map $f : X \to Y$ over a point $y_0 \in Y$ is the space:

$$\begin{aligned} \mathrm{Ho}_{y_0}(f) &= X \times_Y Y^{[0,1]} \times_Y \{y_0\} \\ &= \{(x,w) \in X \times Y^{[0,1]} \mid f(x) = w(0), w(1) = y_0\} = p^{-1}(y_0). \end{aligned}$$

Informally the homotopy fibre is what we get when we turn the map into a Serre fibration, then take the actual fibre.

Because $p : Ef \to Y$ is a Serre fibration, the homotopy groups of $\mathrm{Ho}_{y_0}(f)$ participate in a long exact sequence:

$$\cdots \to \pi_{n+1}(Y,y_0) \xrightarrow{\partial} \pi_n(\mathrm{Ho}_{y_0}(f),x) \to \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y_0) \to \cdots$$

The slogan is: the homotopy fibres measure how far f is from a real homotopy equivalence.

If X and Y happen to be path-connected CW-complexes, then f is a homotopy equivalence if and only if $\pi_n(\mathrm{Ho}_{y_0}(f),x) = 0$ for all $n \geqslant 0$.

For $f : X \to Y$ the analogy goes:

$Z(f) = X \times [0,1] \cup_{X \times 1} Y$:

$$\begin{tikzcd} X \arrow[r, "\text{cl. emb.}"] \arrow[dr, "f"'] & Z(f) \arrow[d, "(f\circ \mathrm{pr}_1)\cup \mathrm{id}_Y", "\simeq"'] \\ & Y \end{tikzcd}$$

$Cf = Z(f)/X = * \cup_{X \times 0} X \times [0,1] \cup_{X \times 1} Y$.

Long exact homology sequence.

$Ef = X \times_Y Y^{[0,1]}$:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr, "f"'] & Ef \arrow[d, "p"] \\ & Y \end{tikzcd}$$

$\mathrm{Ho}_{y_0}(f) = X \times_Y Y^{[0,1]} \times_Y \{y_0\}$.

Long exact homotopy sequence.

Where the long exact homology sequence we are referring to is the one associated with the mapping cone:

$$\cdots \to H_n(X;A) \xrightarrow{f_*} H_n(Y;A) \to \tilde{H}_n(Cf;A) \xrightarrow{\partial} H_{n-1}(X;A) \to \cdots$$

and the long exact homotopy sequence the one associated with the homotopy fibre.

Relative Homotopy Groups of a Map

Let X, Y be spaces, $f : X \to Y$ a continuous map and $x_0 \in X$, $y_0 = f(x_0)$. Choose a basepoint $s_0 \in S^{n-1} \subset D^n$.

Elements of the homotopy group $\pi_n(f)$ are represented by commutative squares of continuous maps:

$$\begin{tikzcd}
(S^{n-1}, s_0) \arrow[r, "\alpha"] \arrow[d, hook] & (X, x_0) \arrow[d, "f"] \\
(D^n, s_0) \arrow[r, "\beta"] & (Y, y_0)
\end{tikzcd}$$

Two representatives (α_0, β_0) and (α_1, β_1) define the same class in $\pi_n(f)$ when there is a commutative square of continuous maps

$$\begin{tikzcd}
(S^{n-1}, s_0) \times [0,1] \arrow[r, "H"] \arrow[d, hook] & (X, x_0) \arrow[d, "f"] \\
(D^n, s_0) \times [0,1] \arrow[r, "\bar{H}"] & (Y, y_0)
\end{tikzcd}$$

such that $H(-,0) = \alpha_0$, $H(-,1) = \alpha_1$, $\bar{H}(-,0) = \beta_0$, $\bar{H}(-,1) = \beta_1$.

If $X \subset Y$ and f is the inclusion, then the map α is already defined by β, and it exists precisely when $\beta(S^{n-1}) \subset X$. So $\pi_n(\mathrm{incl} : X \hookrightarrow Y) = \pi_n(Y, X, x_0)$.

There is also an alternative (and isomorphic) description via cubes. We consider equivalence classes of commutative diagrams of continuous maps of pairs:

$$\begin{tikzcd}
(I^{n-1}, \partial I^{n-1}) \arrow[r, "\alpha"] \arrow[d, hook] & (X, x_0) \arrow[d, "f"] \\
(I^n, J^{n-1}) \arrow[r, "\beta"] & (Y, y_0)
\end{tikzcd}$$

The advantage of this description is that for $n \geqslant 2$ we can define a group structure on $\pi_n(f)$ by stacking representatives next to each other in the first coordinate.

The long exact sequence of homotopy groups also generalizes:

$$\cdots \to \pi_{n+1}(f) \xrightarrow{\partial} \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \to \pi_n(f) \to \cdots \tag{$*$}$$

where ∂ is:

$$[\alpha : S^n \to X, \beta : D^{n+1} \to Y] \mapsto \alpha,$$

and the map $\pi_n(Y, y_0) \to \pi_n(f)$ is:

$$[\beta : (I^n, \partial I^n) \to (Y, y_0)] \mapsto \begin{tikzcd}
(I^{n-1}, \partial I^{n-1}) \arrow[r, "\mathrm{const}_{x_0}"] \arrow[d, hook] & (X, x_0) \arrow[d, "f"] \\
(I^n, J^{n-1}) \arrow[r, "\beta"] & (Y, y_0)
\end{tikzcd}.$$

The long exact sequence is natural in $f : X \to Y$, i.e. for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

we get a commutative diagram of long exact sequences.

III.9. Theorem. — *The long exact sequence of homotopy groups* $(*)$ *is exact.*

There are two possible proof, one "short", one instructive.

Proof.

1. Mimic the arguments for $\pi_n(Y, X, x_0)$, i.e. the special case $f = \mathrm{incl} : X \hookrightarrow Y$.

2. For $n \geqslant 1$ there is a natural isomorphism $\pi_{n-1}(\mathrm{Ho}_{y_0}(f), *) \cong \pi_n(f)$ such that the following diagram commutes:

$$\begin{tikzcd} & \pi_n(Y, y_0) \arrow[dl, "\partial"'] \arrow[dr] & \\ \pi_{n-1}(\mathrm{Ho}_{y_0}(f), *) \arrow[rr, "\cong"] \arrow[dr] & & \pi_n(f) \arrow[dl, "\partial"] \\ & \pi_{n-1}(X, x_0) & \end{tikzcd}$$

where the connecting homomorphism on the top left of the diagram comes from the sequence of spaces $\mathrm{Ho}_{y_0}(f) \hookrightarrow Ef \xrightarrow{p} Y$ and the map on the bottom left is induced by $\mathrm{pr}_1 : \mathrm{Ho}_{y_0}(f) \to X$.

Since the long exact sequence of the Serre fibration $p : Ef \to Y$ is exact, we have that the new sequence $(*)$ is also exact.

The isomorphism is constructed as follows.

$\mathrm{Ho}_{y_0}(f) = X \times_Y Y^{[0,1]} \times_Y \{y_0\}$, so elements of $\pi_{n-1}(\mathrm{Ho}_{y_0}(f), *)$ are represented by continuous maps of pairs $\gamma : (I^{n-1}, \partial I^{n-1}) \to (X \times_Y Y^{[0,1]} \times_Y \{y_0\}, *)$. The representative γ consists of two continuous maps $\alpha : (I^{n-1}, \partial I^{n-1}) \to (X, x_0)$ and $\bar{\beta} : (I^{n-1}, \partial I^{n-1}) \to (Y^{[0,1]}, \mathrm{const}_{y_0})$ that satisfy $f \circ \alpha = \mathrm{ev}_0 \circ \bar{\beta}$ and $\mathrm{ev}_1 \circ \bar{\beta} = \mathrm{const}_{y_0}$.

The exponential law lifts $\bar{\beta}$ to a continuous map $\beta : I^n = I^{n-1} \times [0,1] \to Y$ and the previous conditions become the conditions $\beta|_{I^{n-1} \times 0} = f \circ \alpha$ and $\beta(\partial I^{n-1} \times [0,1] \cup I^{n-1} \times 1) = y_0$. Then (α, β) represent a class in $\pi_n(f)$. This process is fully reversible and works in 1-parameter families. $\square$

Chapter IV.

Eilenberg-MacLane Spaces and Representability of Cohomology

IV

Executive summary. Let A be an abelian group, $n \geqslant 1$. Eilenberg-MacLane spaces are spaces such that:

This is a non very rigorous summary, says the Professor.

$$\pi_i(K(A,n)) = \begin{cases} A & i = n \\ 0 & i \neq n \end{cases}.$$

We want to show that they exist and are unique up to homotopy and that they represent cohomology:

$$H^n(X, A) \cong [X, K(A,n)].$$

Let $n \geqslant 1$, let A be a group, abelian if $n \geqslant 2$. An **Eilenberg-MacLane space** of type (A, n), called "a $K(A,n)$", is a path-connected based space together with an isomorphism $\pi_n(X, x_0) \to A$ such that $\pi_i(X, x_0) = 0$ for all $i \geqslant 1$, with $i \neq n$.

Examples. — The circle S^1 is a $K(\mathbb{Z}, 1)$: its universal cover $\mathbb{R}$ is contractible, so we have that $\pi_i(S^1, *) \cong \pi_i(\mathbb{R}, 0) = 0$ for $i \geqslant 2$, and $\pi_1(S^1, *) \cong \mathrm{Deck}(\exp : \mathbb{R} \to S^1) \cong \mathbb{Z}$.

Let X be a path-connected space with a contractible universal cover $\tilde{X}$. Then X is a $K(G, 1)$ for $G = \mathrm{Deck}(p)$.

We have a cover $S^\infty \to \mathbb{R}\mathbf{P}^\infty$ with Deck $\cong \mathbb{Z}/2$. So $\mathbb{R}\mathbf{P}^\infty = K(\mathbb{Z}/2, 1)$.

The torus $S^1 \times S^1$ has R^2 as a universal cover, with deck transformation group $\mathbb{Z}^2$. Then the torus is a $K(\mathbb{Z}^2, 1)$.

Same for the Klein bottle, it is a $K(\mathbb{Z} \rtimes \mathbb{Z}, 1)$.

Example (A more interesting one). — $\mathbb{C}\mathbf{P}^\infty$ is a $K(\mathbb{Z}, 2)$. We have that:

- $\mathbb{C}\mathbf{P}^\infty$ admits a CW-structure with exactly one cell in every even dimension, so $\mathbb{C}\mathbf{P}^\infty$ is simply-connected by cellular approximation.
- $\pi_2(\mathbb{C}\mathbf{P}^\infty, *) = H_2(\mathbb{C}\mathbf{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ by Hurewicz theorem.

We also have that a generator of $\pi_2(\mathbb{C}\mathbf{P}^\infty, *)$ is represented by any choice of homeomorphism $S^2 \cong \mathbb{C}\mathbf{P}^1 \hookrightarrow \mathbb{C}\mathbf{P}^\infty$.

We claim that $\pi_n(\mathbb{C}\mathbf{P}^\infty, *) \cong 0$ for $n \geqslant 3$.

The map $S^{2m+1} \cong S(\mathbb{C}^{m+1}) \to \mathbb{C}\mathbf{P}^m$, $x \mapsto \mathbb{C} \cdot x$, is a fibre bundle with fibre S^1. So we get a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(S^1, 1) \to \pi_n(S(\mathbb{C}^{m+1}, *)) \to \pi_n(\mathbb{C}\mathbf{P}^m, *) \xrightarrow{\partial} \pi_{n-1}(S^1, *) \to \cdots$$

since $\pi_n(S(\mathbb{C}^{m+1}, *)) = 0$ for $n \leqslant 2m$ and $\pi_{n-1}(S^1, *) = 0$ for $n \geqslant 3$, $\pi_n(\mathbb{C}\mathbf{P}^m, *) = 0$ for $3 \leqslant n \leqslant 2m$. Then $\pi_n(\mathbb{C}\mathbf{P}^m, *) \to \pi_n(\mathbb{C}\mathbf{P}^\infty, *)$ is an isomorphism for $n \leqslant 2m$ by cellular approximation.

Example. — If X and Y are EM-spaces for A, B in the same dimension n, then $X \times Y$ is an EM-space for $A \times B$ in dimension n.

LECTURE 13
29th Nov, 2021

There's one last important easy example of EM-space.

Example. — For $n \geqslant 2$, we have $\Omega K(A, n) = K(A, n-1)$.

Sometimes the notation with EM-spaces gets a bit sloppy.

We want to show existence and uniqueness (up to homotopy) of EM-spaces. For the existence part, it is better to treat the dimension 1 case separately (the reason is that in this case we do not have Hurewicz's theorem).

Construction of Classifying Spaces.

A space of type $(G, 1)$ is also denoted BG, and called the **classifying space** for G.

We use a construction that is similar to the one we saw in AT1Sheet4-1.

Contruction (Bar construction)**.** — Let X be a set. We define a simplicial set EX by setting $(EX)_n = \mathrm{Hom}(\{0, 1, \ldots, n\}, X)(\cong X^{n+1}$ via $f \mapsto (f(0), f(1), \ldots, f(n)))$ where the simplicial structure map $\alpha^* : (EX)_n \to (EX)_m$ induced by $\alpha : [m] \to [n]$ is $\alpha^*(f) = f \circ \alpha$. Equivalently we have: $\alpha^*(x_0, \ldots, x_n) = (x_{\alpha(0)}, \ldots, x_{\alpha(m)})$.

If $g : X \to Y$ is any map, a morphism of simplicial sets $g_* : EX \to EY$ is given by

$$g_*(f) = g \circ f \text{ or } g_*(x_0, \ldots, x_n) = (g(x_0), \ldots, g(x_n)).$$

Altogether we get a functor $E : \mathrm{Set} \to \mathrm{sSet}$.

IV.1. Proposition. — *If X is not empty, then EX is simplicially contractible.*

Proof. Pick any element $y \in X$. We will write down an explicit morphism of simplicial sets

$$H : EX \times \Delta[1] \to EX$$

that contracts EX onto y.

For $0 \leq i \leq n+1$, let $k_i : [n] \to [1]$ be the weakly monotone map with

$$k_i(i-1) = 0 \text{ and } k_i(i) = 1.$$

We define $H_n : (EX)_n \times \Delta[1]_n = X^{n+1} \times \Delta([n], [1]) \to X^{n+1}$ as

$$H_n((x_0, \ldots, x_n), k_i) = (x_0, \ldots, x_{i-1}, y, \ldots, y).$$

It is straightforward to check that these maps do form a morphism of simplicial sets and we have:

$$H_n((x_0,\dots,x_n),\mathrm{const}_0) = H_n((x_0,\dots,x_n),k_{n+1}) = (x_0,\dots,x_n),$$
$$H_n((x_0,\dots,x_n),\mathrm{const}_1) = H_n((x_0,\dots,x_n),k_0) = (y,\dots,y).$$

□

Now let G be any group. Then G acts on itself by right translation. So for $g \in G$ we get a morphism $E_{rg} : EG \to EG$, where rg is right multiplication by g. This defines an action of G on EG by morphisms of simplicial set, i.e. it makes EG into a G-simplicial set.

A **G-simplicial set** is a functor $Y : \Delta^{\mathrm{op}} \to G$-Set. More explicitly, a G-simplicial set is the data of an underlying simplicial set Y and a G-action on Y_n for any $n \geqslant 0$ such that $\alpha^* : Y_n \to Y_m$ is G-equivariant for all $\alpha : [m] \to [n]$ in Δ.

The **simplicial orbit set** is the composite functor:

$$Y/G : \Delta^{\mathrm{op}} \xrightarrow{Y} G\text{-Set} \xrightarrow{-/G} \text{Set},$$

i.e. $(Y/G)_n = Y_n/G$, the set of G-orbits on Y_n.

$$\alpha^* : Y_n/G = (Y/G)_n \to (Y/G)_m = Y_m/G,\ yG \mapsto \alpha^*(y)G$$

This comes with a morphism of simplicial sets $Y \to Y/G$ with $p_n : Y_n \to Y_n/G$, $y \mapsto yG$.

IV.2. Theorem. — *Let G be a group and Y a free G-simplicial set, i.e. the G action on Y_n is free for all $n \geqslant 0$. Then G acts freely and properly discontinuously on $|Y|$ and the continuous map $|p| : |Y| \to |Y/G|$ is a covering space with deck transformation group G.*

Not sure about what he was trying to say... I guess he put it into the Exercise 11.3

Note: we did not prove the theorem, but the key part should be that $|-| :$ Set $\to$ Top is a left adjoint, so it preserves all colimits, giving (somehow) that $|Y|/G \cong |Y/G|$. If G happens to be finite, we only need the freeness of the action on $|Y|$ and the fact that $|Y|$ is Hausdorff.

We apply this theorem to $Y = EG$. The right translation action of G on itself is free, so the map:

$$|p| : |EG| \to |(EG)/G| \cong |EG|/G$$

is a covering space with deck transformation group G. Since $|EG|$ is contractible $|(EG)/G|$ is a $K(G,1)$.

Remark. — The **bar-construction** BG of a group G (introduced in AT1Sheet 4) is the simplicial set $(BG)_n = G^n$ with structure maps:

$$d_i^*(g_1,\dots,g_n) = \begin{cases} (g_2,\dots,g_n) & i=0 \\ (g_1,\dots,g_ig_{i+1},\dots,g_n) & 1 \leqslant i \leqslant n-1 \\ (g_1,\dots,g_{n-1}) & \end{cases}$$

$$s_i^*(g_1,\dots,g_n) = (g_1,\dots,g_{i-1},1,g_i,\dots,g_n).$$

IV.3. Lemma. — *The simplicial sets EG/G and BG are isomorphic. Therefore $|BG|$ is homeomorphic to $|EG/G|$, hence a $K(G,1)$.*

Proof. Consider $q : EG \to BG$ defined in dimension n by:

$$\begin{aligned} q_n : (EG)_n = G^{n+1} &\to G^n = (BG)_n \\ (g_0, \dots, g_n) &\mapsto (g_0 g_1^{-1}, g_1 g_2^{-1}, \dots, g_{n-1} g_n^{-1}). \end{aligned}$$

The map q_n is constant on G-orbits by the composite right translation action. Then q_n factors through a bijection $(EG)_n/G \xrightarrow{\cong} (BG)_n$. □

We also have explicit isomorphism (see AT1Sheet4-1):

$$\begin{aligned} G &\to \pi_1(|BG|, 1) \\ g &\mapsto \{[0,1] \to |BG|\} \\ t &\mapsto [(g, (t, 1-t))] \ni G \times \nabla^1 \to |BG|. \end{aligned}$$

What do classifying spaces classify? — A short digression on what the classifying space $BG = K(G, 1)$ actually classifies. Let G be a group. A ***G*-principal bundle** is a G-space X with the following property: for every $xG \in X/G$ there is an open neighbourhood U of xG in X/G and a G-equivariant homeomorphism $p^{-1}(U) \to U \times G$ (making the usual diagram commute) where $p : X \to X/G$ is the quotient map.

Example. — $|EG|$ with the G action is a principal G-bundle.

- The action of principal G-bundles are in particular free.
- Principal G-bundles admit base change. Consider a principal G-bundle and a continuous map $f : Y \to X/G$:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "f"] & X/G \end{tikzcd}$$

  Then $Y \times_{X/G} X$ comes with a continuous G-action by $(y, x)g = (y, xg)$, i.e. $Y \times_{X/G} X$ is another principal G-bundle with

$$(Y \times_{X/G} X)/G \xrightarrow{\cong} Y.$$

IV.4. Theorem. — *For all paracompact spaces Y and all groups G, the map:*

$$\begin{aligned} [Y, |BG|] &\to \mathrm{Prin}_G(Y) \\ [f] &\mapsto [f^*(|EG| \to \underbrace{|EG/G|}_{=|BG|})] \end{aligned}$$

is a bijection.

Construction of EM-Spaces

The first step towards construction of EM-spaces is to have a way to "kill homotopy groups".

IV.5. Theorem (Killing homotopy groups)**.** — *Let $n \geqslant 0$ and let (Y, y) be a based space. Then there is a relative CW-complex (X, Y) such that:*

(i) all relative cells have dimension $\geqslant n+1$,

(ii) The inclusion induces isomorphisms $\pi_i(Y, y) \to \pi_i(X, y)$ for all $0 \leqslant i < n$,

(iii) For all $i \geqslant n$, $\pi_i(X, y) = 0$.

Proof. We set $X^{(0)} = X^{(1)} = \cdots = X^{(n)} = Y$ and we construct the higher cells by induction, so that:

(a) $X^{(i+1)}$ is obtained from $X^{(i)}$ by attaching $(i+1)$-cells; in particular, the inclusion $X^{(i)} \hookrightarrow X^{(i+1)}$ induce isomorphisms of homotopy groups up to dimension $i-1$ (by cellar approximation),

(b) the homotopy groups $\pi_i(X^{(i+1)}, y)$ is trivial.

Then $X = \cup_{i \geqslant 0} X^{(i)}$ with the weak topology is the desired CW-complex relative to Y. Then we have $\pi_i(X^{(N)}, y) \cong \pi_i(X, y)$ for all large enough N, in particular $\pi_i(X, y) \cong 0$ for $i \geqslant n$ and $\pi_i(X, y) \cong \pi_i(Y, y)$ for $i < n$.

We describe the inductive procedure. Suppose that $X^{(i)}$ has already been constructed. Choose generators $\{x_j\}_{j \in J}$ for the group $\pi_i(X^{(i)}, y)$, choose representatives $f_j : S^i \to X^{(i)}$ of the classes x_j.

Define $X^{(i+1)} = X^{(i)} \cup_{S^i \times J} D^{i+1} \times J$ using the maps f_j as attaching maps.

We observe:

1. The map $\pi_i(X^{(i)}, y) \to \pi_i(X^{(i+1)}, y)$ is surjective by cellular approximation.
2. The map $\pi_i(X^{(i)}, y) \to \pi_i(X^{(i+1)}, y)$ is the zero map; it suffices to show that the set $\{x_j\}_{j \in J}$ of generators goes to zero. For all $j \in J$ the composite

$$S^i \xrightarrow{f_j} X^{(i)} \hookrightarrow X^{(i+1)}$$

extends to a continuous map on D^{i+1}, so it represents the trivial class in $\pi_i(X^{(i+1)}, y)$.

From this we get $\pi_i(X^{(i+1)}, y) = 0$. □

Lecture 14
1st Dec, 2021

We can now prove the existence of EM-spaces.

I missed this lecture because I thought on the Dies Academicus there were no lectures. Apparently the Dies Academicus starts at 10, though. -.-

IV.6. Theorem. — *Let $n \geqslant 2$ and let A be an abelian group. Then there is an EM-space of type (A, n) that is a CW-complex.*

Proof. We choose a free resolution of A as an abelian group:

$$0 \to \mathbb{Z}[I] \xrightarrow{d} \mathbb{Z}[J] \xrightarrow{\varepsilon} A \to 0,$$

i.e. a short exact sequence of abelian groups with I, J some sets.

We define a CW-complex with skeleta:

$$X^{(0)} = \cdots = X^{(n-1)} = \{x\},$$

$$X^{(n)} = \{x\} \cup_{J \times S^{n-1}} J \times D^n \cong \bigvee_J S^n.$$

By cellular approximation $X^{(n)}$ is $(n-1)$-connected. The Hurewicz theorem then provides an isomorphism
$$h : \pi_n(X^{(n)}, x) \xrightarrow{\cong} H_n(X^{(n)}; \mathbb{Z}) \cong \mathbb{Z}[J].$$
For every index $i \in I$ we chose a map $\alpha_i : S^n \to X^{(n)}$ such that $h([\alpha_i]) = d(i) \in \mathbb{Z}[J]$. We define
$$X^{(n+1)} = X^{(n)} \cup_{I \times S^n} I \times D^{n+1}$$
using the α_i's as attaching maps. Then $X^{(n+1)}$ is a CW-complex with one 0-cell, an n-cell for every element of J and an $(n+1)$-cell for every element of I. So the cellular chain complex of $X^{(n+1)}$ is concentrated in degrees 0, n and $n+1$ and in the relevant dimension it looks as follows:
$$0 \to C^{\text{cell}}_{n+1}(X^{(n+1)}; \mathbb{Z}) = \underbrace{H_{n+1}(X^{(n+1)}, X^{(n)}; \mathbb{Z})}_{\cong \mathbb{Z}[I]} \xrightarrow{\partial} C^{\text{cell}}_{n}(X^{(n+1)}; \mathbb{Z}) = \underbrace{H_{n}(X^{(n)}, \{x\}; \mathbb{Z})}_{\cong \mathbb{Z}[J]} \to 0.$$
Hence we have $H^{\text{cell}}_n(X^{(n+1)}; \mathbb{Z}) = \operatorname{coker}(\partial) \cong \operatorname{coker}(d : \mathbb{Z}[I] \to \mathbb{Z}[J]) \cong A$.

$X^{(n+1)}$ is again $(n-1)$-connected, so the Hurewicz theorem provides an isomorphism
$$\pi_n(X^{(n+1)}, x) \xrightarrow{\cong} H_n(X^{(n+1)}; \mathbb{Z}) \cong H^{\text{cell}}_n(X^{(n+1)}; \mathbb{Z}) \cong A.$$
Now we can use kill the homotopy groups as we have seen in Theorem IV.5, obtaining a relative CW-complex $(X, X^{(n+1)})$ with relative cells in dimensions $n+2$ and higher and such that:
$$\pi_i(X, x) \cong \pi_i(X^{(n+1)}, x) \cong \begin{cases} 0 & \text{for } 1 \le i < n \\ A & \text{for } i = n \end{cases}$$
and $\pi_i(X, x) = 0$ for $i \ge n+1$. So X is a CW-complex and a $K(A, n)$. □

There is also an alternative construction of $K(A, n)$ as the geometric realization of some simplicial set.

I am missing this part but I plan to add details eventually.

Uniqueness of EM-Spaces

We first prove an auxiliary lemma.

IV.7. Lemma. — *Let (X, Y) be a relative CW-complex, Z any space. Suppose that for all $m \geqslant 1$ such that (X, Y) has at least one relative m-cell, then $\pi_{m-1}(Z, z) = 0$ for all $z \in Z$. Then every continuous map $f : Y \to Z$ has a continuous extension to X.*

Proof. We construct inductively continuous maps $f^{(n)} : X^{(n)} \to Z$ on the relative skeleta, that successively extend each other. Then $g = \cup f^{(n)} : X = \cup X^{(n)} \to Z$ does the job.

Just to be sure: do not forget that this works because CW-complexes have the final topology with respect to the inclusions of their skeleta.

We start with $f = f^{(-1)} : X^{(-1)} = Y \to Z$. We extend $f^{(-1)}$ to $f^{(0)} : X^{(0)} = Y \amalg J_0 \to Z$ by mapping the relative 0-cells arbitrarily to Z.

Let now $m \geq 1$ and suppose that $f^{(m-1)}$ has been constructed. If there are no relative m-cells, we set $f^{(m)} = f^{(m-1)}$. Otherwise choose characteristic maps for the relative m-cells, call α_j the attaching maps and consider the following composite:
$$S^{m-1} \xrightarrow{\alpha_j} X^{(m-1)} \xrightarrow{f^{(m-1)}} Z.$$

Pick a basepoint $x \in S^{m-1}$. Then $f^{(m-1)} \circ \alpha_j$ represents an element in $\pi_{m-1}(Z, z)$, with $z = f^{(m-1)}(\alpha_j(x))$. By hypothesis we have $\pi_{m-1}(Z, z) = 0$, hence $f^{(m-1)} \circ \alpha_j$ all represent the trivial class, i.e. they can be extended to the disk D^m, which gives us a way to extend $f^{(m-1)}$ to a map $f^{(m)}$ defined on $X^{(m)}$. □

IV.8. Theorem. — *Every $(n-1)$-connected CW-complex Y is homotopy equivalent to a CW-complex whose $(n-1)$-skeleton is one 0-cell.*

Proof. Suppose that $n = 1$. We can choose a tree in Y (i.e. a 1-dimensional subcomplex containing all the 0-cells) and collapse it to obtain the desired CW-complex with just one cell (the quotient map will be an homotopy equivalence).

Now suppose that $n \geq 2$. Because Y is $(n-1)$-connected, its n-skeleton is still $(n-1)$-connected. Since the Hurewicz map is an isomorphism, considering

$$\pi_n(Y^{(n)}, y) \to H_n(Y^{(n)}; \mathbb{Z}) \cong H_n^{\mathrm{cell}}(Y^{(n)}; \mathbb{Z}) = \ker(\underbrace{C_n^{\mathrm{cell}}(Y^{(n)}; \mathbb{Z})}_{\text{free abelian}} \xrightarrow{d^{\mathrm{cell}}} C_{n-1}^{\mathrm{cell}}(Y^{(n)}; \mathbb{Z}))$$

we have that $H_n(Y^{(n)}; \mathbb{Z})$ and hence $\pi_n(Y^{(n)}, y)$ are free abelian groups.

We choose a basis I of the free abelian group $\pi_n(Y^{(n)}, y)$ and represent the basis elements by continuous maps $\alpha_i : S^n \to Y^{(n)}$. These α_i's together define a map

$$\alpha = \bigvee_{i \in I} \alpha_i : \bigvee_I S^n \to Y^{(n)}$$

with source and target $(n-1)$-connected CW-complexes. The map α induces an isomorphism on $H_n(-; \mathbb{Z})$ because it sends the natural basis of $\vee_I S^n$ to the chosen basis of $H_n(Y^{(n)}; \mathbb{Z})$. Note also that source and target of α have trivial homology above dimension n. Hence α is a homology isomorphism between simply-connected CW-complexes, hence a homotopy equivalence.

By cellular approximation we can assume that α is cellular for the CW-structure on $\vee_I S^n$ with one 0-cell and a n-cell for every $i \in I$. Then we form

$$Y' = Y \cup_{Y^{(n)}} \bigvee_I S^n$$

where the gluing is along a cellular homotopy inverse $g : Y^{(n)} \to \vee_I S^n$. The space Y' then comes with a CW-structure with one 0-cell and no cells in dimensions $1, \ldots, n-1$ and the map

$$Y \to Y \cup_{Y^{(n)}} \bigvee_I S^n = Y'$$

is a homotopy equivalence (using the homotopy extension property). □

LECTURE 15
6th Dec, 2021

There is one last preparatory (but really important and useful in itself) result needed to prove uniqueness of EM-spaces.

I missed both lectures this week...

IV.9. Theorem. — *Let $n \geqslant 1$ and Y an $(n-1)$-connected CW-complex. Let Z be a based space with $\pi_m(Z, z) = 0$ for $m > n$. Then for every group morphism $\Phi : \pi_n(Y, y) \to \pi_n(Z, z)$*

there is a continuous map $f : Y \to Z$ with $f(y) = z$ and $\pi_n(f) = \Phi$. Moreover, any two such realizations of Φ are based homotopic. Equivalently, the map:

$$\pi_n : [Y, Z]_* \to \mathrm{Hom}_{\mathrm{Grp}}(\pi_n(Y, y), \pi_n(Z, z))$$

is bijective.

Proof. If the theorem holds for some Y, then it holds for every homotopy equivalent Y. So without loss of generality we can assume (considering theorem IV.8), $Y^{(n-1)} = \{y\}$. For a given morphism $\Phi : \pi_n(Y, y) \to \pi_n(Z, z)$ we construct a continuous map $f^{(n)} : Y^{(n)} \to Z$ such that the following diagram of group morphisms commutes:

$$\begin{tikzcd} \pi_n(Y^{(n)}, y) \arrow[rr, "f^{(n)}_*"] \arrow[dr, "\mathrm{incl}_*"'] & & \pi_n(Z, z) \\ & \pi_n(Y, y) \arrow[ur, "\Phi"'] & \end{tikzcd} \tag{$*$}$$

Let I be an index set for the n-cells of Y. Let $\chi_i : D^n \to Y^{(n)}$, $i \in I$, be a characteristic map for the i-th n-cell. Because $Y^{(n-1)} = \{y\}$, we have $\chi_i(S^{n-1}) = y$, so this map factors over $D^n/\partial D^n$. Let $\bar{\chi}_i$ be the following composite:

$$\begin{tikzcd} S^n \arrow[r, "\cong"] \arrow[rr, bend left, "\bar{\chi}_i"] & D^n/\partial D^n \arrow[r] & Y^{(n)} \arrow[r, hook] & Y \\ & D^n \arrow[u, two heads] \arrow[ur, "\chi_i"'] & & \end{tikzcd}$$

The map $\bar{\chi}_i$ represents an element in $\pi_n(Y, y)$. Then $\Phi([\mathrm{incl} \circ \chi_i]) \in \pi_n(Z, z)$. We choose a representative ω_i for this class:

$$D^n \twoheadrightarrow D^n/\partial D^n \cong S^n \xrightarrow{\omega_i} Z.$$

We define:

$$f^{(n)} : Y^{(n)} \cong \frac{I \times D^n}{I \times \partial D^n} \xrightarrow{\amalg \bar{\omega}_i} Z.$$

Then the diagram $(*)$ commutes by construction because the class of the characteristic maps of the n-cells generate $\pi_n(Y^{(n)}, y)$.

Now we extend $f^{(n)}$ to the $(n+1)$-skeleton of Y. Let $\chi_j : D^{n+1} \to Y^{(n+1)}$, $j \in J$, be a characteristic map for the j-th $(n+1)$-cell. Then the attaching map is $\chi_j|_{S^n} : S^n \to Y^{(n)}$. Because $Y^{(n)}$ is path-connected, $\chi_j|_{S^n}$ is freely homotopic to a based map $\alpha_j : S^n \to Y^{(n)}$. Because (D^{n+1}, S^n) has the HEP and because $\chi_j|_{S^n}$ comes from a continuous map on D^{n+1}, also α_j admits a continuous extension to D^{n+1}. So the composite

$$g : S^n \xrightarrow{\alpha_j} Y^{(n)} \hookrightarrow Y$$

represents the zero element in $\pi_n(Y, y)$. Since $(*)$ commutes we have $0 = \Phi([g]) = [f^{(n)} \circ \alpha_j]$, so $f^{(n)} \circ \alpha_j : S^n \to Z$ has a continuous extension to D^{n+1}. Hence by the HEP also $f^{(n)} \circ \chi_j|_{S^n}$:

$S^n \to Z$ admits a continuous extension to D^{n+1}. Choose such an extension $g_j : D^{n+1} \to Z$ and define

$$f^{(n+1)} = f^{(n)} \cup \bigcup_{j \in J} g_j : Y^{(n+1)} = Y^{(n)} \cup_{J \times S^n} J \times D^{n+1} \to Z.$$

By theorem IV.7 $f^{(n+1)} : Y^{(n+1)} \to Z$ can be extended continuously to Y (consider the relative CW-complex $(Y, Y^{(n+1)})$ with relative cells of dimensions greater than $n+1$). Let $f : Y \to Z$ be any such extension.

We claim that $\pi_n(f) = \Phi$. This follows from $(*)$ because $\mathrm{incl}_* : \pi_n(Y^{(n)}, y) \to \pi_n(Y, y)$ is surjective by cellular approximation.

Now we can prove uniqueness up to homotopy of the map f. Let $f, f' : Y \to Z$ be two continuous based maps with $\pi_n(f) = \Phi = \pi_n(f')$. Since $Y^{(n-1)} = \{y\}$, f and f' are equal on the $(n-1)$-skeleton.

We choose characteristic maps $\chi_i : D^n \to Y^{(n)}$, $i \in I$, for all n-cells. Then consider:

$$\begin{tikzcd}
D^n/\partial D^n \arrow[r, "\bar{\chi}_i"] & Y^{(n)} \arrow[r, "f|_{Y^{(n)}}, f'|_{Y^{(n)}}"] \arrow[d] & Z \\
& Y^{(n+1)} \arrow[r, hook] & Y \arrow[u]
\end{tikzcd}$$

because f and f' have the same effect on π_n, $f \circ \bar{\chi}_i$ is based homotopic to $f' \circ \bar{\chi}_i$. We can then choose based homotopies $H_i : D^n \times [0,1] \to Z$ and we can glue them into a single homotopy

$$H = \bigcup_{i \in I} H_i : Y^{(n)} \times [0,1] \to Z$$

which shows that $f|_{Y^{(n)}}$ and $f'|_{Y^{(n)}}$ are based homotopic.

Now we apply the "extension theorem" IV.7 from last lecture to the relative CW-complex $(Y \times [0,1], Y \times \{0\} \cup Y^{(n)} \times [0,1] \cup Y \times \{1\})$ whose relative cells all have dimension greater than n.By the theorem we get a continuous extension $\bar{H} : Y \times [0,1] \to Z$ of the map

$$f \cup H \cup f' : Y \times 0 \cup Y^{(n)} \times [0,1] \cup Y \times 1 \to Z.$$

Then $\bar{H}$ is the desired based homotopy from f to f'. □

IV.10. Theorem. — *Let (X, φ) and (Y, ψ) be EM-spaces of type (A, n) and (B, n). Then if X is a CW-complex the map*

$$\begin{aligned} \pi_n : [X, Y]_* &\to \mathrm{Hom}_{\mathrm{Grp}}(A, B) \\ [f : X \to Y] &\mapsto \psi \circ \pi_n(f) \circ \varphi^{-1} \end{aligned}$$

is bijective.

Proof. By the previous theorem (IV.9) we have:

$$[X, Y]_* \xrightarrow[\cong]{\pi_n} \mathrm{Hom}_{\mathrm{Grp}}(\pi_n(X, x), \pi_n(Y, y)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Grp}}(A, B)$$

where the second map simply sends h to $\psi \circ h \circ \varphi^{-1}$. □

IV.11. Corollary. — *Let A be a group, abelian if $n \geq 2$, and let $(X, \varphi), (Y, \psi)$ be two EM-spaces of type (A, n) that are CW-complexes. Then there is a based homotopy equivalence $f : X \to Y$ that makes the following diagram commute:*

$$\begin{tikzcd} \pi_n(X,x) \arrow[rr, "\pi_n(f)"] \arrow[dr, "\varphi"', "\cong"] & & \pi_n(Y,y) \arrow[dl, "\psi", "\cong"'] \\ & A & \end{tikzcd}$$

Proof. The existence of a map f that makes the diagram commute is given by the previous theorem and f induces isomorphisms on all homotopy groups for all basepoints. Since X and Y are CW-complexes, f is a homotopy equivalence by Whitehead's theorem. $\square$

Example. — From the uniqueness of EM-space we have that $\mathbb{R}\mathbf{P}^\infty$ is homotopy equivalent to $|B\mathbb{Z}/2|$ and S^1 is homotopy equivalent to $|B\mathbb{Z}|$. Observe that S^1 is a finite one dimensional CW-complex, while $|B\mathbb{Z}|$ is an infinite dimensional one (we will return on this in chapter V).

Representability of Cohomology

Our goal is now to construct a natural isomorphism

$$H^n(X; A) \cong [X, K(A, n)]$$

for any CW-complex X.

We observe first that for an abelian group A and $n > 1$, an EM-space $K(A, n)$ can be seen as an "abelian group up to homotopy". Indeed, given a $K(A, n)$ which is a CW-complex, using theorem IV.10 we can define a "group structure up to homotopy" considering a based continuous map $\mu : K(A, n) \times K(A, n) \to K(A, n)$, which is unique up to homotopy, that realizes addition, i.e. makes the following diagram commute:

$$\begin{tikzcd} \pi_n(K(A,n) \times K(A,n), (*,*)) \arrow[r, "\mu_*"] \arrow[d, "\cong"] & \pi_n(K(A,n), *) \\ \pi_n(K(A,n), *) \times \pi_n(K(A,n), *) \arrow[d, "\varphi\times\varphi"', "\cong"] & A \arrow[u, "\varphi^{-1}"'] \\ A \times A \arrow[ur, "+"'] & \end{tikzcd}$$

Similarly we can construct a continuous based map ι which realizes the inverse map $a \mapsto a^{-1}$.

The map μ is "associative up to homotopy": this is because the two based continuous maps $\mu \circ (\mu \times \mathrm{id}), \mu \circ (\mathrm{id} \times \mu) : K(A, n)^3 \to K(A, n)$ realize the maps $(a, b, c) \mapsto (a + b) + c$ and $(a, b, c) \mapsto a + (b + c)$, hence $\pi_n(\mu \circ (\mu \times \mathrm{id})) = \pi_n(\mu \circ (\mathrm{id} \times \mu))$, since addition in A is associative, and therefore $\mu \circ (\mu \times \mathrm{id})$ and $\mu \circ (\mathrm{id} \times \mu)$ are homotopic as based maps.

Similarly, considering the map $\tau : K(A, n)^2 \to K(A, n)^2$, $(x, y) \mapsto (y, x)$, we have that μ and $\mu \circ \tau$ are based homotopic maps.

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7^{th} Dec, 2021

Given the above discussion, for all CW-complexes X, the set $[X, K(A,n)]$ becomes an abelian group by $[f]+[g] = [\mu \circ (f,g)]$.

I don't really understand this remark

Remark. — One can realize a $K(A,n)$ as a topological abelian group, e.g. $|\tilde{A}[\Delta^n/\partial]|$. $\tilde{A}[\Delta^n/\partial]$ is a simplicial abelian group, and $|-| : \mathrm{sSet} \to \mathrm{Top}_{\text{cpt. gen.}}$ commutes with products, which implies that $|\tilde{A}[\Delta^n/\partial]|$ is an abelian topological group with product given by:

$$|\tilde{A}[\Delta^n/\partial]|^2 \xrightarrow{\cong} |\tilde{A}[\Delta^n/\partial] \times \tilde{A}[\Delta^n/\partial]| \xrightarrow{|\mu|} |\tilde{A}[\Delta^n/\partial]|.$$

I really don't understand this example

Example. — We know that $\mathbb{R}\mathbf{P}^\infty = K(\mathbb{Z}/2,1)$ and $\mathbb{C}\mathbf{P}^\infty = K(\mathbb{Z},2)$. We also talked about how $\mathbb{R}\mathbf{P}^\infty$ classifies principal $\mathbb{Z}/2$-bundles (i.e. two-fold coverings). Then for every paracompact space X, we have:

$$\begin{tikzcd}
{[X, \mathbb{R}\mathbf{P}^\infty]} \arrow[r, "\cong"] & \mathrm{Prin}_{\mathbb{Z}/2}(X) \arrow[r, "\cong"] & \mathrm{Pic}_{\mathbb{R}}(X) \\
f \arrow[rr, mapsto] & & f^*(j) \\
& E \to X \arrow[r, mapsto] & E \times_{C_2} \mathbb{R} \to X \\
& \big(F - s_0(X)\big)/\mathbb{R}_{>0} \to X & F \to X \arrow[l, mapsto]
\end{tikzcd}$$

where j is the tautological line bundle on $\mathbb{R}\mathbf{P}^\infty$ and s_0 is the zero section. Under this bijection, the group operation on the left corresponds to the tensor product on the right (?).

Similarly we have:

$$[X, \mathbb{C}\mathbf{P}^\infty] \xrightarrow{\cong} \mathrm{Pic}_{\mathbb{C}}(X)$$

sending $[f : X \to \mathbb{C}\mathbf{P}^\infty]$ to pullback of the universal complex line bundle over $\mathbb{C}\mathbf{P}^\infty$ and again with addition corresponding to the tensor product of line bundles (?).

The Fundamental Cohomology Class

IV.12. Proposition/Definition (Fundamental cohomology class)**.** — *Let (X,φ) be an EM-space of type (A,n), $n \geq 1$, A abelian. Then there is a unique class $\iota \in H^n(X;A)$ such that the composite:*

$$\pi_n(X,x) \xrightarrow{\cong} H_n(X;\mathbb{Z}) \xrightarrow{\Phi(\iota)} A$$

is the isomorphism $\varphi : \pi_n(X,x) \to A$, where $\Phi : H^n(X;A) \to \mathrm{Hom}(H_n(X;\mathbb{Z}),A)$ is the homomorphism from the universal coefficient theorem.

Proof. Since X is $(n-1)$-connected, the Hurewicz homomorphism $h : \pi_n(X,x) \to H_n(X,\mathbb{Z})$ is an isomorphism; for $n=1$ this is true because A is abelian. Because $H_{n-1}(X,\mathbb{Z})$ is trivial (for $n \geq 2$) or free (if $n=1$), the Ext-term in the UCT vanishes, so:

$$\Phi : H^n(X;A) \xrightarrow{\cong} \mathrm{Hom}(H_n(X;\mathbb{Z}),A)$$

is an isomorphism. Then we can define $\iota = \Phi^{-1}(\varphi \circ h^{-1})$. $\square$

We want to see that the fundamental class $\iota \in H^n(K(A,n);A)$ gives rise to a natural isomorphism:

$$\mathrm{ev}_\iota : [Y, K(A,n)] \longrightarrow H^n(Y;A) \qquad [f : Y \to K(A,n)] \longmapsto f^*(\iota).$$

The convention for $n = 0$ is that $K(A,0)$ is just the group A with the discrete topology, the map $\mu : K(A,0)^2 \to K(A,0)$ is the addition on A and $i : K(A,0) \to K(A,0)$ is the inverse map, while $\iota \in \mathrm{Hom}(A,A)$ is represented by the identity cocycle.

IV.13. Theorem. — *For all $n \geq 0$ and all abelian groups A, the evaluation map ev_ι is a group morphism.*

Proof. Given the convention above, without loss of generality we can assume $n \geq 1$.

We start with a "universal example", from which the general case will easily follow. Let $Y = K(A,n) \times K(A,n)$ and $p_1, p_2 : K(A,n)^2 \to K(A,n)$ the projections. Observe that the sum $[p_1] + [p_2]$ is represented by the map μ realizing group addition:

$$K(A,n) \times K(A,n) \xrightarrow{(p_1,p_2)=\mathrm{id}} K(A,n) \times K(A,n) \xrightarrow{\mu} K(A,n),$$

$$\text{i.e. } [p_1] + [p_2] = [\mu \circ (p_1, p_2)] = [\mu].$$

Then we want to show that in the special case of $[\mu] = [p_1] = [p_2]$ evaluation at ι is additive:

$$\mu^*(\iota) = p_1^*(\iota) + p_2^*(\iota).$$

We will use the following isomorphism:

$$H^n(K(A,n)^2;A) \xrightarrow{\cong} \mathrm{Hom}(H_n(K(A,n)^2;\mathbb{Z}),A) \xrightarrow{\cong} \mathrm{Hom}(\pi_n(K(A,n)^2,*),A) \xrightarrow{\cong} \mathrm{Hom}(A \times A, A)$$

where the first isomorphism comes from the UCT and the second is the Hurewicz isomorphism. Under this isomorphism $\mu^*(\iota)$ corresponds to the addition $A \times A \to A$ and the projections to the corresponding projections $A \times A \to A$. But in $\mathrm{Hom}(A \times A, A)$ clearly we do have that the sum of the projections is the addition, hence the same holds in $H^n(K(A,n)^2;A)$.

Now for the general case, let Y be an arbitrary space and $f, g : Y \to K(A,n)$ continuous maps. Then we have:

$$\begin{aligned}(f+g)^*(\iota) = (\mu \circ (f,g))^*(\iota) = (f,g)^*(\mu^*(\iota)) &= (f,g)^*(p_1^*(\iota) + p_2^*(\iota)) \\ &= (f,g)^*(p_1^*(\iota)) + (f,g)^*(p_2^*(\iota)) \\ &= f^*(\iota) + g^*(\iota).\end{aligned}$$

$\square$

IV.14. Lemma. — *For all $n \geq 1$, all abelian groups A and all based CW-complexes Y, the forgetful map*

$$[Y, K(A,n)]_* \to [Y, K(A,n)]$$

is a bijection.

Proof. We start by showing surjectivity. We will use that for $n \geq 1$ any $K(A,n)$ is path-connected and that the inclusion of the basepoint $\{y\} \hookrightarrow Y$ has the HEP. In particular, given any continuous map $f : Y \to K(A,n)$, we can choose a path $w : [0,1] \to K(A,n)$ from $f(y)$ to $x \in K(A,n)$. The HEP for $(\{y\}, Y)$ lets us choose a homotopy $H : Y \times [0,1] \to K(A,n)$ starting with f and such that $H(y,-) = w$. Then $g = H(-,1)$ is freely homotopic to f and based, hence the forgetful map sends $[g]$ to $[f]$.

Note (for those who studied the HEP long ago): clearly not every inclusion of a point has the HEP, this works (as always in this course) because Y is a CW-complex.

To show injectivity, we first consider the case $n = 1$. Let $f, g : Y \to K(A,1)$ be based maps that are freely homotopic. Then we have:

$$f_* = g_* : H_1(Y;\mathbb{Z}) \to H_1(K(A,1);\mathbb{Z}).$$

Since A is abelian, the map $\pi_1(K(A,1),x) \to H_1(K(A,1),\mathbb{Z})$ is an isomorphism, hence we have:

$$f_* = g_* : \pi_1(Y,y) \to \pi_1(K(A,1),x).$$

By Theorem IV.9, f and g are then based homotopic.

Does the theorem not require Y to be connected?

Now let $n \geq 2$. We will exploit the fact that a $K(A,n)$ is then simply connected. Let Z be (more generally) any simply-connected space, $f, g : Y \to Z$ two based continuous maps that are freely homotopic. Let $H : Y \times [0,1] \to Z$ be a free homotopy from f to g. Then $w = H(y,-) : [0,1] \to Z$ is a loop at the basepoint of Z. Since Z is simply-connected, this loop is homotopic to the constant loop at the endpoint, relative to it, say by a homotopy $G : [0,1] \times [0,1] \to Z$. The HEP of the pair $(Y \times [0,1], Y \times \{0\} \cup \{y\} \times [0,1] \cup Y \times \{1\})$ applied to

$$K : (Y \times \{0\} \cup \{y\} \times [0,1] \cup Y \times \{1\}) \times [0,1] \xrightarrow{\mathrm{const}_f \cup G \cup \mathrm{const}\, g} Z$$

yields a map $\bar{K} : Y \times [0,1] \times [0,1] \to Z$ extending K and such that $\bar{K}(-,-,0) = H$. Then the map $H' = \bar{K}(-,-,1) : Y \times [0,1] \to Z$ is a based homotopy between f and g. □

We are now ready to prove the result we promised.

IV.15. Theorem. — *For all $n \geq 0$, all abelian groups A and all CW-complexes Y, the evaluation map*

$$\mathrm{ev}_\iota : [Y, K(A,n)] \longrightarrow H^n(Y;A) \qquad [f : Y \to K(A,n)] \longmapsto f^*(\iota).$$

is a group isomorphism.

Proof. If $n = 0$, we defined $K(A,0)$ as A with the discrete topology, so clearly

$$[Y, K(A,n)] = \mathrm{Hom}_{\mathsf{Set}}(\pi_0(Y), A) \cong H^0(Y;A)$$

with addition in $[Y, K(A,n)]$ corresponding to pointwise addition in $\mathrm{Hom}_{\mathsf{Set}}(\pi_0(Y), A)$.

Now let $n \geq 1$. The theorem holds for $Y = \varnothing$ (since both sides are then 0), so without loss of generality we can assume Y is non-empty. We choose a basepoint $y \in Y$. We will start by proving the theorem in a special case, which will then be used to conclude in generality.

Special case. Suppose Y is $(n-1)$-connected. Then $H_{n-1}(Y;\mathbb{Z})$ is trivial (if $n \geq 2$, by Hurewicz) or at least free (if $n = 1$), hence $\mathrm{Ext}^1(H_{n-1}(Y;\mathbb{Z}),\mathbb{Z})$ vanishes and the morphism $\Phi : H^n(Y;A) \to \mathrm{Hom}(H_n(Y;\mathbb{Z}), A)$ from the UCT is an isomorphism. We also have that for $n \geq 2$ the Hurewicz map $h : \pi_n(Y, y) \to H_n(Y; \mathbb{Z})$ is an isomorphism, while for $n = 1$ it is the universal homomorphism into the abelianization. In both cases precomposition yields an isomorphism

$$\mathrm{Hom}(h, A) : \mathrm{Hom}_{\mathrm{Grp}}(H_n(Y; \mathbb{Z}), A) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Grp}}(\pi_n(Y, y), A).$$

The composite

$$[Y, K(A, n)]_* \xrightarrow{\mathrm{ev}_\iota} H^n(Y; A) \underset{\cong}{\xrightarrow{\Phi}} \mathrm{Hom}(H_n(Y; \mathbb{Z}), A) \underset{\cong}{\xrightarrow{\mathrm{Hom}(h,A)}} \mathrm{Hom}_{\mathrm{Grp}}(\pi_n(Y, y), A)$$

sends $[f]$ to $\pi_n(f) : \pi_n(Y, y) \to \pi_n(K(A, n), *) \to A$, hence we know that it is a bijection by Theorem IV.9 (since Y is a $(n-1)$-connected CW-complex and $K(A, n)$ has vanishing homotopy groups for $k > n$). Since the composite, the second and the third maps are bijections, the valuation map ev_ι must also be. *To be continued...*

Lecture 17
14th Dec, 2021

General case. Induction on n. We consider the cone of the $(n-1)$-skeleton,

$$CY^{(n-1)} = (Y^{(n-1)} \times [0, 1])/(Y^{(n-1)} \times \{1\}),$$

and we form the CW-complex

$$Y \cup_{Y^{(n-1)}} CY^{(n-1)}.$$

Note that we have continuous maps

$$Y \xrightarrow{i} Y \cup_{Y^{(n-1)}} CY^{(n-1)} \xrightarrow{p} \Sigma Y^{(n-1)},$$

where i is the inclusion and p collapses Y.

We consider the commutative diagram of abelian groups:

$$\begin{tikzcd}
{[\Sigma Y^{(n-1)}, K(A,n)]} \arrow[r, "p^*"] \arrow[d] & {[Y \cup_{Y^{(n-1)}} CY^{(n-1)}, K(A,n)]} \arrow[r, "i^*"] \arrow[d] & {[Y, K(A,n)]} \arrow[r] \arrow[d, "?"] & 0 \\
H^n(\Sigma Y^{(n-1)}; A) \arrow[r, "p^*"] & H^n(Y \cup_{Y^{(n-1)}} CY^{(n-1)}; A) \arrow[r, "i^*"] & H^n(Y; A) \arrow[r] & 0
\end{tikzcd} \tag{$*$}$$

We want to use the 5-lemma to show that the map on the right is an isomorphism.

Step 1: Exactness of the upper row of $(*)$.
In the relative CW-complex $(Y \cup_{Y^{(n-1)}} CY^{(n-1)}, Y)$ all relative cells have dimension less or equal to n. Since the homotopy groups of $K(A, n)$ vanish up to dimension $n - 1$, every continuous map $Y \to K(A, n)$ admits a continuous extension to $Y \cup_{Y^{(n-1)}} CY^{(n-1)}$ by Lemma IV.7. So the upper map i^* is surjective.

Since $p \circ i$ is the constant map, we have that $i^* \circ p^* = (p \circ i)^*$ is the zero homomorphism. Let $f : Y \cup_{Y^{(n-1)}} CY^{(n-1)} \to K(A, n)$ represent an element in the kernel of i^*, i.e. $f|_Y$ is nullhomotopic. The HEP for the pair $(Y \cup_{Y^{(n-1)}} CY^{(n-1)}, Y)$ let us replace f by a homotopic map $g : Y \cup_{Y^{(n-1)}} CY^{(n-1)} \to K(A, n)$ such that $g|_Y$ is the constant map at the basepoint:

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[dr, "\mathrm{const}_*"'] & Y \cup_{Y^{(n-1)}} CY^{(n-1)} \arrow[r, "p"] \arrow[d, "g"] & \Sigma Y^{(n-1)} \arrow[dl, dashed, "h"] \\
& K(A, n) &
\end{tikzcd}$$

So there is a unique continuous map $h : \Sigma Y^{(n-1)} \to K(A, n)$ with $h \circ p = g$. In particular, we have that $p^*[h] = [g] = [f]$, so that $\ker(i^*) = \mathrm{im}(p^*)$.

Step 2: Exactness of the lower row of $(*)$.
This comes from the cohomology long exact sequence of the relative CW-complex $(Y \cup_{Y^{(n-1)}} CY^{(n-1)}, Y)$. In particular, observe that the sequence would continue with

$$H^n(Y; A) \xrightarrow{\partial} H^{n+1}(\Sigma Y^{(n-1)}; A)$$

and $H^{n+1}(\Sigma Y^{(n-1)}; A) = 0$, since $\Sigma Y^{(n-1)}$ is a CW-complex of dimension less or equal to n.

Step 3: The left vertical map in $(*)$ is surjective.
Since $\Sigma K(A, n-1)$ has trivial homotopy groups below the n-th and $K(A, n)$ above the n-th, using Theorem IV.9 we can choose a continuous map, unique up to homotopy, $\kappa_n : \Sigma K(A, n-1) \to K(A, n)$, that realizes the following homomorphism on homotopy groups:

$$\pi_n(\Sigma K(A, n-1), *) \underset{\text{Hurewicz}}{\xrightarrow{\cong}} H_n(\Sigma K(A, n-1); \mathbb{Z}) \underset{\text{Suspension}}{\xrightarrow{\cong}} H_{n-1}(K(A, n-1); \mathbb{Z})$$

$$\underset{\text{Hurewicz}}{\xrightarrow{\cong}} \pi_{n-1}(K(A, n-1), *) \xrightarrow{\cong} A \xrightarrow{\cong} \pi_n(K(A, n), *).$$

Then the following square commutes:

$$\begin{tikzcd}
{[Y^{(n-1)}, K(A, n-1)]} \arrow[r, "{[f] \mapsto [\kappa_n \circ \Sigma f]}"] \arrow[d, "{[f] \mapsto f^*(\iota_{n-1})}"', "\cong"] & {[\Sigma Y^{(n-1)}, K(A, n)]} \arrow[d, "{[f] \mapsto f^*(\iota_n)}"] \\
H^{n-1}(Y^{(n-1)}; A) \arrow[r, "\cong"] & H^n(\Sigma Y^{(n-1)}; A)
\end{tikzcd}$$

which gives us surjectivity of the map $[f] \mapsto f^*(\iota_n)$ (i.e. the left vertical map in $(*)$), since $[f] \mapsto f^*(\iota_{n-1})$ is an isomorphism by induction.

Step 4: The middle vertical map in $(*)$ is an isomorphism.
This follows from the previous special case, since the space $Y \cup_{Y^{(n-1)}} CY^{(n-1)}$ is $(n-1)$-connected: indeed, by cellular approximation any continuous map $f : S^k \to Y \cup_{Y^{(n-1)}} CY^{(n-1)}$, with $k \leq n-1$, is homotopic to a map with image the contractible space $CY^{(n-1)}$.

The 5-lemma then shows that the right vertical map in $(*)$ is an isomorphism. □

Example. — $\mathbb{R}\mathbf{P}^\infty$ is both an EM-space of type $(\mathbb{Z}/2, 1)$ and a classifying space for real line bundles. So for any CW-complex X we get two natural bijections

$$\mathrm{Pic}_{\mathbb{R}}(X) \xleftarrow{\cong} [X, \mathbb{R}\mathbf{P}^\infty] \xrightarrow{\cong} H^1(X, \mathbb{F}_2),$$

given by:

$$f^*(j) \leftarrow [f] \mapsto f^*(\iota)$$

where j is the universal/tautological line bundle on $\mathbb{R}\mathbf{P}^\infty$. We can combine the two bijections into a map $w_1 : \mathrm{Pic}_{\mathbb{R}}(X) \to H^1(X; \mathbb{F}_2)$ which is called the first **Stiefel-Whitney class**.

One can show that a real line bundle is completely determined (up to isomorphism) by its first Stiefel-Whitney class (it is possible to see that vector bundles of higher rank are in general *not* determined by their Stiefel-Whitney classes, though).

The complex version of this story uses $\mathbb{C}\mathbf{P}^\infty$:

$$\mathrm{Pic}_{\mathbb{C}}(X) \xleftarrow{\cong} [X, \mathbb{C}\mathbf{P}^\infty] \xrightarrow{\cong} H^2(X,\mathbb{Z}),$$

this gives a map $\mathrm{Pic}_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$ which is called the first **Chern class**. Again, one can show that a complex line bundle over a CW-complex is determined up to isomorphism by its first Chern class (and that the same is not true in general for vector bundles of higher rank and their Chern classes).

CW Approximation

Our aim: we want to show that every space Z admits a weak homotopy equivalence $X \to Z$ from a CW-complex X that is unique up to homotopy.

A continuous map $f : X \to Y$ is a **weak homotopy equivalence** if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is bijective and for all $n \geq 1$ and all $x \in X$, $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism.

Being "weakly homotopy equivalent" is the equivalence relation generated by the relation "there is a weak homotopy equivalence between x and y", which is neither symmetric nor transitive.

Note that every homotopy equivalence is a weak homotopy equivalence (note: sometimes people (e.g. me) forget that this is not entirely trivial... see [**Hatcher**] for a proof in the case of fundamental groups, which can be easily generalized to the higher homotopy groups).

For an example of a weak homotopy equivalence which is not an homotopy equivalence one could take the inclusion of a point into the long line or the Warsaw circle, or a map from a countable discrete space Y to $\{0\} \cup \{1/n \mid n \geq 1\} \subset \mathbb{R}$ with the subspace topology.

These pathological counterexamples can be somewhat subtle (even if perhaps ultimately not very interesting), I might add an appendix on them

IV.16. Theorem (Existence of CW-approximations)**.** — *Let Z be a path connected space, X a CW-complex, $x_0 \in X$ a 0-cell, $f : (X, x_0) \to (Z, z_0)$ a continuous map. Then there is a CW-complex Y that contains X as a subcomplex and a continuous extension $g : Y \to Z$ that is a weak homotopy equivalence.*

As a special case, $X = \{x_0\}$ gives the existence of CW-approximations.

Proof. This is similar to the proof of Theorem IV.5, the method for "killing homotopy groups". We construct inductively CW-complexes $X = Y^{(-1)} \subset Y^{(0)} \subset Y^{(1)} \subset \cdots$ such that $Y^{(n)}$ contains $Y^{(n-1)}$ as a subcomplex and a continuous map $g^{(n)} : Y^{(n)} \to Z$ such that $g^{(n)}|_{Y^{(n-1)}} = g^{(n-1)}$ and $\pi_k(g^{(n)}) = 0$ for all $1 \leq k \leq n$. Then $Y = \cup_{n\geq 0} Y^{(n)}$ with the weak topology is the desired CW-complex and $g = \cup_{n \geq 0} g^{(n)} : Y \to Z$ is the desired map because $\pi_k(g) = 0$ for all $k \geq 1$ by compactness, hence g is a weak homotopy equivalence by its associated long exact homotopy sequence.

A compact subspace of a CW complex is contained in a finite subcomplex. In case of doubt, see [**Hatcher**] for a proof.

Now, we proceed with the construction. Let $Y^{(-1)} = X$ and $g^{(-1)} = f : X \to Z$. Suppose that $Y^{(n-1)}$ and $g^{(n-1)} : Y^{(n-1)} \to Z$ with the desired properties have already been constructed. For each class $i \in \pi_n(g^{(n-1)})$ choose representing maps $\alpha_i : S^{n-1} \to Y^{(n-1)}$ and $\beta_i : D^n \to Z$ such that the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha_i"] \arrow[d, hook] & Y^{(n-1)} \arrow[d, "g^{(n-1)}"] \\ D^n \arrow[r, "\beta_i"] & Z \end{tikzcd}$$

commutes. By cellular approximation we can assume that the α_i are cellular maps.

We construct $Y^{(n)}$ by attaching n-cells to $Y^{(n-1)}$ using the α_i's as attaching maps. Similarly, we define $g^{(n)}$ taking the union of $g^{(n-1)}$ with the β_i's. Then $Y^{(n)}$ contains $Y^{(n-1)}$ as a subcomplex and $g^{(n)}$ extends $g^{(n-1)}$.

It remains to show that $\pi_k(g^{(n)}) = 0$ for $1 \leq k \leq n$. To this end we compare the long exact homotopy sequences of $g^{(n-1)}$ and $g^{(n)}$:

$$\begin{tikzcd}
\pi_k(Y^{(n-1)}, x_0) \arrow[r, "g_*^{(n-1)}"] \arrow[d, "\mathrm{incl}_*"] & \pi_k(Z, z_0) \arrow[r] \arrow[d, equal] & \pi_k(g^{(n-1)}) \arrow[r, "\partial"] \arrow[d, "\mathrm{incl}_*"] & \pi_{k-1}(Y^{(n-1)}, x_0) \arrow[r, "g_*^{(n-1)}"] \arrow[d, "\mathrm{incl}_*"] & \pi_{k-1}(Z, z_0) \arrow[d, equal] \\
\pi_k(Y^{(n)}, x_0) \arrow[r, "g_*^{(n)}"] & \pi_k(Z, z_0) \arrow[r] & \pi_k(g^{(n)}) \arrow[r, "\partial"] & \pi_{k-1}(Y^{(n)}, x_0) \arrow[r, "g_*^{(n)}"] & \pi_{k-1}(Z, z_0)
\end{tikzcd}$$

The fourth map is surjective for $k \leq n$ by cellular approximation, hence the middle map is surjective by the 5-lemma. Since $\pi_k(g^{(n-1)}) = 0$ for $1 \leq k < n$, we have that also $\pi_k(g^{(n)}) = 0$ for $1 \leq k < n$.

For $k = n$ the map $\mathrm{incl}_* : \pi_n(g^{(n-1)}) \to \pi_n(g^{(n)})$ sends all elements to zero by design: given an $i \in \pi_n(g^{(n-1)})$ represented by (β_i, α_i), we have the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\alpha_i"] \arrow[d, hook] & Y^{(n-1)} \arrow[r, hook] \arrow[d, "g^{(n-1)}"'] & Y^{(n)} \arrow[d, "g^{(n)}"] \\
D^n \arrow[r, "\beta_i"'] \arrow[urr, dashed] & Z \arrow[r, equal] & Z
\end{tikzcd}$$

where the dashed arrow is a characteristic map for the i-th cell. The existence of the diagonal filler means that the outer square represents the 0 element in $\pi_n(g^{(n)})$, hence incl_* is surjective and the zero homomorphism, so $\pi_n(g^{(n)}) = 0$. $\square$

This must be easy to show, but right now I don't know how (need to review lecture 12)

IV.17. Theorem (Uniqueness of CW-approximations). — *Let $f : X \to Z$ and $g : Y \to Z$ be two CW-approximations, i.e. weak homotopy equivalences to a path-connected space Z such that X and Y are CW-complexes. Then there is a homotopy equivalence $\psi : X \to Y$ such that the diagram:*

$$\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

commutes up to homotopy.

Proof. We consider the map $f \amalg g : X \amalg Y \to Z$. Then there is a CW-complex W that contains $X \amalg Y$ as a subcomplex and a weak homotopy equivalence $\Phi : W \to Z$ with $\Phi|_X = f$ and $\Phi|_Y = g$.

Because g and Φ are weak homotopy equivalences, so is the inclusion $i_Y : Y \hookrightarrow W$

$$\begin{tikzcd}
Y \arrow[rr, "i_Y"] \arrow[dr, "g"'] & & W \arrow[dl, "\Phi"] \\
& Z &
\end{tikzcd}$$

Since Y and W are path-connected, i_Y is a homotopy equivalence by Whitehead's theorem. Let $j : W \to Y$ be a homotopy inverse.

Interchanging the roles of X and Y shows that also $i_X : X \to W$ is a homotopy equivalence. So $\psi : j \circ i_X : X \to Y$ is a homotopy equivalence. Moreover:

$$g \circ \psi = g \circ j \circ i_X = \Phi \circ i_Y \circ j \circ i_X \simeq \Phi \circ i_X = f.$$

□

LECTURE 18
15th Dec, 2021

Addendum to last time: the results on CW-approximations hold for all spaces Z, not necessarily path-connected. Indeed, a space Z is the union of its path components, hence by choosing CW-approximations for each path component separately and taking the disjoint union we obtain a CW-approximation for Z (note: we might get some strange CW-complex).

Some Applications: Classifying Cohomology Operations

IV.18. Theorem. — *Let $f : X \to Y$ be a weak homotopy equivalence. Then f induces isomorphisms $f_* : H_n(X; A) \to H_n(Y; A)$ and $f^* : H^n(Y; A) \to H^n(X; A)$ for all abelian groups A.*

Proof. By the UCTs, it suffices to prove the homological case for $A = \mathbb{Z}$.

Because the simplices ∇^n are all path-connected, $H_n(X; A)$ decomposes as the direct sum of the homology of the path components:

$$\bigoplus_{i \in \pi_0(X)} H_n(X_i; A) \to H_n(X; A)$$

and similarly for Y. Since $f_* : \pi_0(X) \to \pi_0(Y)$ is bijective, it suffices to show the claim for each X_i. So we can assume without loss of generality that X and Y are path-connected.

We let $Z(f) = X \times [0, 1] \cup_f Y$ be the mapping cylinder of f. Then f factors as

$$X \xrightarrow{(-,0)} Z(f) \xrightarrow{p} Y$$

where the first map is a closed embedding and the second the projection. Since the homotopy equivalence p is a weak equivalence and induces isomorphisms on $H_n(-; \mathbb{Z})$, we can assume without loss of generality that X is a closed subspace of Y (by replacing Y with $Z(f)$).

Since $f_* : \pi_1(X, x) \to \pi_1(Y, x)$ and $f_* : \pi_n(X, x) \to \pi_n(Y, x)$ for $n \geq 2$ are isomorphisms, the Hurewicz theorem (in its relative version I.2) applies. By the long exact sequence we have that $\pi_k(Y, X, x) = 0$ for all $k \geq 1$, hence by Hurewicz $H_n(Y, X; \mathbb{Z}) = 0$ for all $n \geq 1$. So $f_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ is an isomorphism by the homology long exact sequence. □

Question: what are all natural transformations

$$H^2(-; \mathbb{Z}) \to H^6(-; \mathbb{Z})$$

as functors Top → Set?

Some examples are $x \mapsto x^3 = x \smile x \smile x$ (the cup product taken three times) or $x \mapsto kx^3$ for $k \in \mathbb{Z}$. Are these all of them?

IV.19. Theorem. — *Let A be an abelian group, $n \geq 0$. Let $F : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ be any functor that sends weak equivalences to isomorphisms. Then the map*

$$\begin{gathered}\mathrm{Nat}_{\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}}(H^n(-;A), F) \to F(K(A,n)) \\ \tau = \{\tau_X : H^n(X;A) \to F(X)\} \mapsto \tau_{K(A,n)}(\iota)\end{gathered}$$

is bijective.

Example. — We have a bijection

$$\mathrm{Nat}(H^2(-;\mathbb{Z}), H^6(-;\mathbb{Z})) \xrightarrow{\cong} H^6(K(\mathbb{Z},2);\mathbb{Z}) \cong H^6(\mathbb{C}\mathbf{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}\{\iota^3\}$$

sending $x \mapsto kx^3$ on the left and to $k\iota^3$ on the right.

In general, $\mathrm{Nat}(H^n(-;A), H^m(-;B)) \cong H^m(K(A,n);B)$ are called the cohomology operations of type (A,n,B,m).

Note: the cohomology operations of a given type are usually really difficult to compute. A couple of facts about this story: $H^*(K(\mathbb{F}_2,n);\mathbb{F}_2)$ for $n \geq 2$ is a polynomial algebra in infinitely many explicitly known generators (for $n=1$ on one generator) given by the "Steenrod operations"; for p an odd prime, $H^*(K(\mathbb{F}_p,n);\mathbb{F}_p)$ is a polynomial $\otimes$ exterior algebra. We know that $H^m(K(\mathbb{Z},n);\mathbb{Z})$ is finitely generated, so it can be decomposed into free and p-partition groups; $H^m(K(\mathbb{Z},n);\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = H^m(K(\mathbb{Z}_{(p)},n),\mathbb{Z}_{(p)})$ related to $A = B = \mathbb{F}_p$. Also:

⚠ I do not understand well these results, I might have written something dumb here and there.

$$H^*(K(\mathbb{Q},n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}[\iota] & \text{if } n \text{ is even} \\ \Lambda_{\mathbb{Q}}[\iota] & \text{if } n \text{ is odd} \end{cases}$$

Proof. The essential ingredients are the Yoneda lemma and CW-approximations, as expected.

Injectivity. Suppose that τ and μ are two natural transformations from $H^n(-;A)$ to F such that $\tau_{K(A,n)}(\iota) = \mu_{K(A,n)}(\iota)$.

If X is a CW-complex and $x \in H^n(X;A)$, there is a continuous map $f : X \to K(A,n)$ such that $f^*(\iota) = x$. So by naturality of τ and μ:

$$\tau_X(x) = \tau_X(f^*(\iota)) = f^*(\tau_{K(A,n)}(\iota)) = f^*(\mu_{K(A,n)}(\iota)) = \mu_X(f^*(\iota)) = \mu_X(x),$$

hence $\tau_X = \mu_X$ for all CW-complexes X.

If Y is any space, choose a CW-approximation $\alpha : X \xrightarrow{\sim} Y$. Let $y \in H^n(Y;A)$. Then by naturality:

$$\alpha^*(\mu_Y(y)) = \mu_X(\alpha^*(y)) = \tau_X(\alpha^*(y)) = \alpha^*(\tau_Y(y))$$

where the middle equality is by the previous special case. Since $\alpha^* : F(Y) \to F(X)$ is an isomorphism by hypothesis, this proves $\mu_Y(y) = \tau_Y(y)$, hence $\mu = \tau$.

Surjectivity. First we need to prove an intermediate result.

Claim. Let F be a functor that sends weak equivalences to isomorphisms. Then F takes homotopic maps to the same map (Warning: the converse is *not* true!).

Proof of the claim. Let $f, g : X \to Y$ be two homotopic continuous maps and choose an homotopy $H : X \times [0,1] \to Y$ from f to g. Let $i_0, i_1 : X \to X \times [0,1]$ be the two "extremal" inclusions, $p : X \times [0,1] \to X$ the projection. These maps are homotopy equivalences, hence weak equivalences. So $F(p) : F(X) \to F(X \times [0,1])$ is an isomorphism. We have

$$F(i_0) \circ F(p) = F(p \circ i_0) = F(\mathrm{id}_X) = F(p \circ i_1) = F(i_1) \circ F(p).$$

Because $F(p)$ is an isomorphism, we conclude that $F(i_0) = F(i_1)$. Now

$$F(f) = F(H \circ i_0) = F(i_0) \circ F(H) = F(i_1) \circ F(H) = F(H \circ i_1) = F(g).$$

■

To prove surjectivity, for any element $u \in F(K(A,n))$ we will construct a natural transformation $\Phi^u : H^n(-;A) \to F$ such that $\Phi^u_{K(A,n)}(\iota) = u$.

Let Y be a space and choose a CW-approximation $\alpha : X \xrightarrow{\sim} Y$. We let $y \in H^n(Y;A)$ be a class. Since X is a CW-complex, there is a continuous map $f : X \to K(A,n)$ such that $\alpha^*(y) = f^*(\iota)$ in $H^n(X;A)$. We define $\Phi^u_Y(y) = (\alpha^*)^{-1}(F(f)(u))$. Note that we have:

$$F(K(A,n)) \xrightarrow{F(f)} F(X) \underset{\cong}{\xleftarrow{\alpha^*}} F(Y)$$

Claim. $\Phi^u_Y(y)$ is independent of the choices of α and f.

Proof of the claim. Let $\beta : X' \xrightarrow{\sim} Y$ be another CW-approximation. Let $g : X' \to K(A,n)$ be another continuous map such that $\beta^*(y) = (f')^*(\iota)$.

By uniqueness of CW-approximations there is a homotopy equivalence $\psi : X \to X'$ such that

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\alpha"'] & & X' \arrow[dl, "\beta"] \\ & Y & \end{tikzcd}$$

commutes up to homotopy. Then:

$$(g \circ \psi)^*(\iota) = \psi^*((g)^*(\iota)) = \psi^*(\beta^*(y)) = (\beta \circ \psi)^*(y) = \alpha^*(y) = f^*(\iota).$$

Because X is a CW-complex, the maps $g \circ \psi, f : X \to K(A,n)$ are homotopic. Hence we have $F(f) = F(g \circ \psi) = F(\psi) \circ F(g)$. So the following diagram commutes:

$$\begin{tikzcd} & F(X) & \\ F(K(A,n)) \arrow[ur, "f^*"] \arrow[dr, "g^*"'] & & F(Y) \arrow[ul, "\alpha^*"', "\cong"] \arrow[dl, "\beta^*", "\cong"'] \\ & F(X') \arrow[uu, "\psi^*"', "\cong"] & \end{tikzcd}$$

So the potentially different definitions of $\Phi^u_Y(y)$ agree. ■

Clearly $\Phi^u_{K(A,n)}(\iota) = u$ since we can take both α and f to be $\mathrm{id} : K(A,n) \to K(A,n)$. Hence we are left to prove that Φ^u is indeed a natural transformation.

Claim. $\Phi^u = \{\Phi^u_Y\}$ is a natural transformation.

Proof of the claim. Let $h : Y \to Z$ be any continuous map, and let $z \in H^n(\mathbb{Z}; A)$. Let $\alpha : X \to Y$ be any CW-approximation to Y. We choose a CW-approximation for Z relative to the map $h \circ \alpha : X \to Z$ (which we can do by using theorem IV.16 in the general, i.e. relative to a map, version). This is a CW-complex $\bar{X}$ containing X as a subcomplex and a weak equivalence $\beta : \bar{X} \to Z$ such that $\beta|_X = h \circ \alpha$. Consider the following diagram:

$$\begin{tikzcd}
 & X \arrow[r, "\alpha", "\sim"'] \arrow[dl, "g"'] \arrow[d, hook, "i"] & Y \arrow[d, "h"] \\
K(A,n) & \bar{X} \arrow[l, "f"] \arrow[r, "\sim", "\beta"'] & Z
\end{tikzcd}$$

where $f : \bar{X} \to K(A,a)$ is any continuous map such that $f^*(\iota) = \beta^*(z)$ and $g = f|_X$. The map $g : X \to K(A,n)$ satisfies:

$$g^*(\iota) = (f \circ i)^*(\iota) = i^*(f^*(\iota)) = i^*(\beta^*(z)) = \alpha^*(h^*(z)),$$

hence we can use $\alpha : X \to Y$, $g : X \to K(A,n)$ to define $\Phi^u_Y(h^*(z))$. Then we have:

$$\Phi^u_Y(h^*(z)) = (\alpha^*)^{-1}(i^*(f^*(u))) = h^*((\beta^*)^{-1}(f^*(u))) = h^*(\Phi^u_Z(z))$$

i.e. Φ^u is a natural transformation. ■

□

What are all natural transformations, on paracompact spaces, of functors $\mathrm{Top}_{\mathrm{para}} \to \mathrm{Set}$, $\mathrm{Pic}_{\mathbb{R}} \to \mathrm{Pic}_{\mathbb{C}}$?

Surely we have:

$$[\xi : L \to X] \mapsto [\xi_{\mathbb{C}} : L \otimes_{\mathbb{R}} \mathbb{C} \to X],$$
$$[\xi : L \to X] \mapsto [X \times \mathbb{C} \xrightarrow{\mathrm{pr}} X].$$

These two turn out to be all natural operations:

$$\mathrm{Nat}(\mathrm{Pic}_{\mathbb{R}}, \mathrm{Pic}_{\mathbb{C}}) \cong \mathrm{Nat}(H^1(-, \mathbb{F}_2), \mathrm{Pic}_{\mathbb{C}}) \cong \mathrm{Pic}_{\mathbb{C}}(\mathbb{R}\mathbf{P}^\infty) \cong H^2(\mathbb{R}\mathbf{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}/2.$$

Chapter V.

V

Spaces and Simplicial Sets: an Introduction to Homotopy Theory (?)

Lecture 19
20^{th} Dec, 2021

The long term goal for the rest of the semester is to show that there is an equivalence of homotopy theories (whatever this means):

$$(\text{Top}, \text{weak eq.}) \simeq (\text{sSet}, \text{weak eq.}).$$

It is possible to prove (but it is beyond the scope of this course) that in fact there is an underlying equivalence of model categories/infinity categories (whatever this means): we will only show a "shadow" of this result, i.e. that there is an equivalence of homotopy categories (whatever this means).

I don't know what I'm talking about, I'll rewrite this introduction in some weeks

The short term goal for this lecture and the next one is to show the existence of a preferred CW-structure on $|X|$, the geometric realization of a simplicial set X.

In this course, we do not like the word "canonical".

The Preferred CW-structure on the Geometric Realization

Recall the basic theory of simplicial sets as exposed in chapter I.

The geometric realization of a simplicial set is the space:

$$|X| = \left(\coprod X_n \times \nabla^n \right) / \sim$$

where X_n is endowed with the discrete topology and the equivalence relation is generated by:

$$X_m \times \nabla^m \ni (x, \alpha_*(t)) \sim (\alpha^*(x), t) \in X_n \times \nabla^n \quad \text{for all } \alpha : [n] \to [m],\ x \in X_m,\ t \in \nabla^n.$$

Note that we did not give explicitly the equivalence relation $\sim$ on $|X|$, just a relation (not symmetric, in particular) that *generates* it, hence it might be difficult to tell when two points of $\coprod_{n \geq 0} X_n \times \nabla^n$ represent the same equivalence class in $|X|$. To solve this issue, we need to study the equivalence relation $\sim$ more in detail.

Remark*. — Given any relation R on a set X, the equivalence relation generated by R is the intersection of all the equivalence relations containing R, i.e. $a \sim b$ if and only if there exists $x_1, \ldots, x_n \in X$ such that $a = x_0$, $b = x_n$ and $x_i R x_{i-1}$ or $x_{i-1} R x_i$ for all $i = 1, \ldots, n$.

Minimal Representatives

An important feature of the relation $\sim$ is that classes have minimal representatives, as shown by the following proposition.

V.1. Proposition (Minimal representatives)**.** — *Let X be a simplicial set.*

(1) *Every equivalence class for $\sim$ has a unique representative $(x,t) \in X_l \times \nabla^l$ of minimal dimension l, called the* ***minimal representative****.*

(2) *A pair $(y,s) \in X_n \times \nabla^n$ is the minimal representative in its class if and only if:*
   - *y is a non-degenerate simplex,*
   - *s is an interior point of ∇^n.*

(3) *If $(x,t) \in X_l \times \nabla^l$ is the minimal representative in its class and $(y,s) \in X_n \times \nabla^n$ is equivalent to (x,t), then there is a unique triple (δ,σ,u) consisting of:*
   - *an injective morphism $\delta : [k] \hookrightarrow [n]$,*
   - *a surjective morphism $\sigma : [k] \twoheadrightarrow [l]$,*
   - *$u \in \mathring{\nabla}^k$,*

   such that $\delta^*(y) = \sigma^*(x)$, $s = \delta_*(u)$ and $t = \sigma_*(u)$.

Summing up: the first part of the proposition gives existence of minimal representatives, the second part a characterization of them and the third says that any element in an equivalence class is related to the minimal representative of the class by a chain of just two "elementary equivalences" (the relations by which the equivalence relation $\sim$ is generated), i.e. $(y,s) = (y,\delta_*(u)) \sim (\delta^*(y),u) = (\sigma^*(x),u) \sim (x,\sigma_*(u)) = (x,t)$.

Proof. We write X_l^{nd} for the set of non-degenerate l-simplices. We first define a map

$$\rho : \coprod_{n\geq 0} X_n \times \nabla^n \to \coprod_{l\geq 0} X_l^{\mathrm{nd}} \times \mathring{\nabla}^l$$

such that $\rho(y,s)$ is equivalent to (y,s).

Consider any $(y,s) \in X_n \times \nabla^n$. Suppose that $s = (s_0,\dots,s_n)$. Since $s_0 + \ldots + s_n = 1$ and all $s_i \geq 0$, there is at least a coordinate which is positive. Suppose that $k+1$ of the coordinates are positive. Define $u = (u_0,\dots,u_k)$ to be the coordinates of s with the zero entries deleted, in the same order. We let $\delta : [k] \to [n]$ be the unique injective morphism such that $\delta_*(u) = s$. The simplex $\delta^*(y) \in X_k$ can be written uniquely as $\delta^*(y) = \sigma^*(x)$ for a surjective morphism $\sigma : [k] \to [l]$ and a non-degenerate simplex $x \in X_l^{\mathrm{nd}}$, by proposition I.10. Then we set:

$$\rho(y,s) = (x,\sigma_*(u)).$$

Since $u \in \mathring{\nabla}^k$ and $\sigma_* : \nabla^k \to \nabla^l$ adds coordinates together (hence does not insert zeroes), also $\sigma_*(u) \in \mathring{\nabla}^l$. Hence the map ρ is well-defined and by construction $\rho(y,s) \sim (y,s)$.

Claim. If $(y,s) \in X_n \times \nabla^n$ and $(\bar{y},\bar{s}) \in X_{\bar{n}} \times \nabla^{\bar{n}}$ are equivalent pairs, then $\rho(y,s) = \rho(\bar{y},\bar{s})$.

Proof of the claim. It suffices to show this when (y,s) and $(\bar{y},\bar{s})$ are "elementary equivalent", i.e. there is a morphism $\alpha : [n] \to [\bar{n}]$ such that

$$(y,s) = (\alpha^*(\bar{y}),s) \sim (\bar{y},\alpha_*(s)) = (\bar{y},\bar{s}).$$

We let (δ, u, σ, x) be the data in the construction of $\rho(y,s)$ and we choose a factorization (which is necessarily unique) $\alpha \circ \delta = \bar{\delta} \circ \bar{\sigma}$ for a surjective morphism $\bar{\sigma} : [k] \to [\bar{k}]$ and an injective morphism $\bar{\delta} : [\bar{k}] \to [\bar{n}]$. Then

$$\bar{s} = \alpha_*(s) = \alpha_*(\delta_*(u)) = \bar{\delta}_*(\bar{\sigma}_*(u)).$$

Since $u \in \mathring{\nabla}^k$ and σ is surjective, $\bar{\sigma}_*(u) \in \mathring{\nabla}^{\bar{k}}$. Hence $\bar{s} = \bar{\delta}_*(\bar{\sigma}_*(u))$ must be the unique expression in the first step of the construction of $\rho(\bar{y}, \bar{s})$, since $\bar{\delta}_*$ is injective and $\bar{\sigma}_*(u)$ is an interior point. We write $\bar{\delta}^*(\bar{y}) = \hat{\sigma}^*(\hat{x})$ for a surjective morphism $\hat{\sigma} : [\bar{k}] \to [\hat{l}]$ and a non-degenerate element $\hat{x} \in X^{\mathrm{nd}}_{\hat{l}}$. Then we have:

$$\sigma^*(x) = \delta^*(y) = \delta^*(\alpha^*(\bar{y})) = \bar{\sigma}^*(\bar{\delta}^*(\bar{y})) = \bar{\sigma}^*(\hat{\sigma}^*(\hat{x})) = (\hat{\sigma} \circ \bar{\sigma})^*(\hat{x}).$$

By the uniqueness of pairs of degeneracies and non-degenerate simplices (of proposition I.10), we must have $l = \hat{l}$, $x = \hat{x}$, $\sigma = \hat{\sigma} \circ \bar{\sigma}$. So $\bar{\delta}^*(\bar{y}) = \hat{\sigma}^*(\hat{x}) = \hat{\sigma}^*(x)$ and $(\bar{\delta}, \bar{\sigma}_*(u), \hat{\sigma}, x)$ must be the data in the construction of $\rho(\bar{y}, \bar{s})$. Hence:

$$\rho(\bar{y}, \bar{s}) = (x, \hat{\sigma}_*(\bar{\sigma}_*(u))) = (x, \sigma_*(u)) = \rho(y, s).$$

■

(1) Suppose $(y,s) \in X_n \times \nabla^n$ is of minimal dimension in its equivalence class. In the construction of $\rho(y,s)$ we must have $n \geq k \geq l$, therefore $n = k = l$, since $(y,s) \sim \rho(y,s)$. Clearly the injective map $\delta : [k] = [n] \hookrightarrow [n]$ and the surjective map $\sigma : [k] = [l] \twoheadrightarrow [l]$ must be equal $\mathrm{id}_{[n]}$. Hence $\rho(y,s) = (y,s)$.

If $(\bar{y}, \bar{s})$ is another representative of the same minimal dimension, then

$$(y,s) = \rho(y,s) = \rho(\bar{y}, \bar{s}) = (\bar{y}, \bar{s}).$$

This also shows that $\rho(y,s)$ produces the unique minimal representative.

(2) The necessity falls out of the construction of ρ, which produces the minimal representative of a given class. Conversely, suppose y is non-degenerate and s an interior point. If $s = \delta_*(u)$ for another interior point u, then δ must be the identity, but then $\delta^*(y) = y = \sigma^*(x)$, hence also σ must be the identity, so $(y,s) = \rho(y,s)$ is the minimal representative.

(3) Suppose (x,t) minimal, and (y,s) equivalent (but not elementary equivalent) to it. We want to find a unique triple (δ, σ, u) of an injective map, a surjective map and an interior point such that $\delta^*(y) = \sigma^*(x)$, $s = \delta_*(u)$ and $t = \sigma_*(u)$. The existence comes from the construction of $\rho(y,s)$, while the uniqueness is because no choices were involved (every step is uniquely determined). □

While the previous proof is not exactly enlightening, having minimal representatives makes it easier to work with the geometric realization functor.

V.2. Corollary. — *Let $f : X \to Y$ be a morphism of simplicial sets such that $f_n : X_n \to Y_n$ is injective for all $n \geq 0$.*

(1) *For every non-degenerate $x \in X_n$, the simplex $f_n(x) \in Y_n$ is non-degenerate.*

(2) *The continuous map $|f| : |X| \to |Y|$ is injective.*

Note: if f has a retract, then (2) follows by functoriality, however, having a retract is much stronger than having injective components (the left inverses of the components might not assemble into a natural transformation).

Proof.

(1) Suppose that $f_n(x) = s_i^*(y)$ for some $0 \le i \le n-1$, $y \in Y_{n-1}$. Then (using that f is a morphism of simplicial sets) we have:

$$f_n(s_i^*(d_i^*(x))) = s_i^*(d_i^*(f_n(x))) = s_i^*(\underbrace{d_i^*(s_i^*(y))}_{\mathrm{id}_{[n]}^*(y)}) = s_i^*(y) = f_n(x).$$

Since f_n is injective, $s_i^*(d_i^*(x)) = x$, which contradicts the assumption that x is non-degenerate.

(2) Let $(x,t) \in X_n \times \nabla^n$ and $(x',t') \in X_m \times \nabla^m$ be minimal representatives of classes in $|X|$ such that $|f|[x,t] = |f|[x',t']$. By definition of the geometric realization functor we have $|f|[x,t] = [f_n(x), t]$ and $|f|[x',t'] = [f_m(x'), t']$, and by part (1) both $f_n(x)$ and $f_m(x')$ are non-degenerate. Then by uniqueness of minimal representatives we must have $(f_n(x), t) = (f_m(x'), t')$, hence in particular $n = m$, $t = t'$ and $x = x'$, i.e. $[x,t] = [x',t']$. $\square$

V.3. Corollary. — *Let X be any simplicial set.*

(1) *The composite*

$$\coprod_{n\ge 0} X_n^{nd} \times \nabla^n \hookrightarrow \coprod_{n\ge 0} X_n \times \nabla^n \twoheadrightarrow |X|$$

is surjective.

(2) *The composite:*

$$\coprod_{n\ge 0} X_n^{nd} \times \mathring{\nabla}^n \hookrightarrow \coprod_{n\ge 0} X_n \times \nabla^n \twoheadrightarrow |X|$$

is a continuous bijection.

We will see that the second statement gives us the decomposition into cells of the preferred CW-structure for the geometric realization of a simplicial set.

We also have a corollary of (1).

V.4. Corollary. — *Suppose that the total number of non-degenerate simplices in all dimensions is finite. Then $|X|$ is quasi-compact.*

The Skeleta filtration

The preferred CW-structure of $|X|$ arises from a "CW-like structure" that is intrinsic to the combinatorics of simplicial sets.

The m**-skeleton** $\mathrm{sk}^m X$ of a simplicial set X, for $m \ge 0$, is the simplicial set defined by:

$$(\mathrm{sk}^m X)_n = \{x \in X_n \mid x = \alpha^*(y) \text{ for some } \alpha : [n] \to [m] \text{ and some } y \in X_m\},$$

i.e. $\mathrm{sk}^m X$ is the smallest simplicial subset that contains X_m.

Clearly one has to convince himself that this is a simplicial set, which is not difficult.

Example. — Every constant simplicial set is zero-dimensional, i.e. $X = \operatorname{sk}^0 X$. Conversely if $X = \operatorname{sk}^0 X$, then X is isomorphic to the constant simplicial set on the zero-simplices X_0.

Example. — For $\Delta^m = \Delta(-, [m])$, the simplicial m-simplex, we have $\operatorname{sk}^m(\Delta^m) = \Delta^m$ and $\operatorname{sk}^{m-1}(\Delta^m) = \partial\Delta^m$, where $(\partial\Delta^m)_n = \{\alpha : [n] \to [m] \mid \alpha \text{ not surjective}\}$.

Note: these examples may not be necessarily obvious at first, one should diligently sit down and check the details.

Lecture 20
22nd Dec, 2021

The next proposition sheds some light on the definition of the skeleta of a simplicial set, showing that they are the right analogue of the skeleta of CW-complexes.

V.5. Proposition. — *let X be a simplicial set.*

(1) *For $n \leq m$, $(\operatorname{sk}^m X)_n = X_n$.*

(2) *For $n > m$, all simplices of $(\operatorname{sk}^m X)_n$ are degenerate.*

(3) $\operatorname{sk}^m X \subset \operatorname{sk}^{m+1} X$.

(4) *X is a colimit, in the category of simplicial sets, of the sequence:*

$$\operatorname{sk}^0 X \subset \operatorname{sk}^1 X \subset \cdots \subset \operatorname{sk}^m X \subset \cdots$$

(5) *For every morphism $f : X \to Y$ of simplicial sets, $f(\operatorname{sk}^m X) \subset \operatorname{sk}^m Y$.*

Proof.

(1) Let $n \leq m$. We choose an injective morphism $\alpha : [n] \to [m]$ and a surjective morphism $\sigma : [m] \to [n]$ such that $\sigma \circ \alpha = \mathrm{id}_{[n]}$. Then for all $x \in X_n$ we have:

$$x = \mathrm{id}_{[n]}^*(x) = \alpha^*(\underbrace{\sigma^*(x)}_{\in X_m}),$$

hence $x \in (\operatorname{sk}^m X)_n$.

(2) We suppose that $n > m$, $x \in (\operatorname{sk}^m X)_n$. We write $x = \alpha^*(y)$ for some $\alpha : [n] \to [m]$, $y \in X_m$. Since $n > m$, α cannot be injective, so $\alpha(i) = \alpha(i+1)$ for some $0 \leq i < n$. So $\alpha = \beta \circ s_i$ for some morphism $\beta : [n-1] \to [n]$. Then

$$x = \alpha^*(y) = s_i^*(\beta^*(y))$$

is degenerate.

(3) We consider $x \in (\operatorname{sk}^m X)_n$, so $x = \alpha^*(y)$ for some $\alpha : [n] \to [m]$, $y \in X_m$. Then

$$x = \alpha^*(y) = (s_0 \circ d_0 \circ \alpha)^*(y) = (d_0 \circ \alpha)^*(\underbrace{s_0^*(y)}_{\in X_{m+1}}),$$

hence $x \in (\operatorname{sk}^{m+1} X)_n$.

(4) Limits and colimits in functor categories are computed objectwise (a proof can be found in [**maclane:71**], or in the nLab as the statement that limits of presheaves are computed objectwise). Hence it suffices to show that in every fixed dimension n the set is a colimit of the sequence:

$$(\operatorname{sk}^0 X)_n \hookrightarrow (\operatorname{sk}^1 X)_n \hookrightarrow \cdots \hookrightarrow (\operatorname{sk}^m X)_n \hookrightarrow \cdots$$

but by (1) this diagram stabilizes from $(\operatorname{sk}^n X)_n = X_n$ upwards, so X_n is a colimit.

(5) Suppose $x = \alpha^*(y) \in (\operatorname{sk}^m X)_n$, for $\alpha : [n] \to [m]$, $y \in X_m$. Then

$$f_n(x) = f_n(\alpha^*(y)) = \alpha^*(\underbrace{f_m(y)}_{\in Y_m}),$$

hence $x \in (\operatorname{sk}^m Y)_n$. □

Remark. — Thanks to (5) of the previous proposition, we can make sk^m into an endofunctor of sSet by setting:

$$\operatorname{sk}^m f := f|_{\operatorname{sk}^m X} : \operatorname{sk}^m X \to \operatorname{sk}^m Y.$$

Then the inclusions $\operatorname{sk}^m \hookrightarrow X$ and $\operatorname{sk}^m X \hookrightarrow \operatorname{sk}^{m+1} X$ define natural transformations $\operatorname{sk}^m \to \mathrm{id}$ and $\operatorname{sk}^m \to \operatorname{sk}^{m+1}$ of endofunctors of sSet.

The following proposition (V.6) contains the technical work needed to show that the geometric realization of a simplicial set is a CW-complex with canonical CW-structure given by the realization of the simplicial skeleta (the statement is slightly more general and deals with generic simplicial subsets, we will specialize the result in theorem V.7).

Remark. — In order to state and prove proposition V.6 we need some preparation. First, we observe that there exists a natural transformation $x^\flat : \Delta^m \to X$, that we call the **characteristic morphism** of $x \in X_m$, determined via the Yoneda lemma by $x^\flat_m(\mathrm{id}_{[m]}) = x$ (hence $x^\flat_n(\alpha) = \alpha^*(x)$). Second, in the hypotheses of the proposition, we observe that the following square commutes (by naturality of the inclusion of the skeleton):

$$\begin{tikzcd}
\partial\Delta^m \arrow[r, equal] \arrow[d, "x^\flat|_{\partial\Delta^m}"] & \operatorname{sk}^{m-1}\Delta^m \arrow[r, hook] \arrow[d, "\operatorname{sk}^{m-1} x^\flat"] & \Delta^m \arrow[d, "x^\flat"] \\
\operatorname{sk}^{m-1} Y \arrow[r, equal] & \operatorname{sk}^{m-1} X \arrow[r, hook] & X
\end{tikzcd}$$

In other words: the characteristic morphism of every m-simplex of X sends $\partial\Delta^m$ into the simplicial subset Y.

V.6. Proposition. — *Let X be a simplicial set, $m \geq 0$. Let Y be a simplicial subset of X with $X_{m-1} = Y_{m-1}$. Suppose also that for $k > m$, every simplex in $X_k \smallsetminus Y_k$ is degenerate.*

(1) *The commutative square:*

$$\begin{tikzcd}
\coprod_{x \in X_m \smallsetminus Y_m} \partial\Delta^m \arrow[r, "\mathrm{incl}"] \arrow[d, "\amalg\, x^\flat|_{\partial\Delta^m}"'] & \coprod_{x \in X_m \smallsetminus Y_m} \Delta^m \arrow[d, "\amalg\, x^\flat"] \\
Y \arrow[r, "\mathrm{incl}"] & X
\end{tikzcd}$$

is a pushout of simplicial sets.

(2) *The commutative square of spaces:*

$$\begin{tikzcd}
\coprod_{x \in X_m \smallsetminus Y_m} |\partial\Delta^m| \arrow[r, "|\mathrm{incl}|"] \arrow[d, "\amalg\, |x^\flat||_{\partial\Delta^m}"'] & \coprod_{x \in X_m \smallsetminus Y_m} |\Delta^m| \arrow[d, "|\amalg\, x^\flat|"] \\
|Y| \arrow[r, "|\mathrm{incl}|"] & |X|
\end{tikzcd}$$

is a pushout of topological spaces.

(3) *The realization $|X|$ is obtained from $|Y|$ by attaching m-cells indexed by $X_m \smallsetminus Y_m$.*

Proof.

(1) Since limits and colimits in the category of simplicial sets are computed objectwise (as we already noted in the proof of proposition V.5), it suffices that for every fixed dimension n, the commutative square of sets

$$\begin{tikzcd} \coprod_{x \in X_m \smallsetminus Y_m} (\partial\Delta^m)_n \arrow[r, "\mathrm{incl}"] \arrow[d, "\amalg x_n^\flat|_{\partial\Delta^m}"'] & \coprod_{x \in X_m \smallsetminus Y_m} (\Delta^m)_n \arrow[d, "\amalg x_n^\flat"] \\ Y_n \arrow[r, "\mathrm{incl}"] & X_n \end{tikzcd}$$

is a pushout. Since the horizontal maps are injective, it suffices to show that the "complements" of the images of the horizontal maps are mapped bijectively by the right vertical map, i.e. that the right vertical map induces a bijection

$$\{X_m \smallsetminus Y_m\} \times ((\Delta^m)_n \smallsetminus (\partial\Delta^m)_n) \to X_n \smallsetminus Y_n, \quad (x, \alpha) \mapsto \alpha^*(x).$$

By definition the k-simplices of $\partial\Delta^m$ are all morphisms $\alpha : [n] \to [m]$ that are not surjective, hence we want to show bijectivity of the map

$$\{X_m \smallsetminus Y_m\} \times \{\alpha : [n] \to [m] \mid \alpha \text{ surjective}\} \to X_n \smallsetminus Y_n, \quad (x, \alpha) \mapsto \alpha^*(x).$$

If $n < m$, bijectivity is trivial because by hypothesis $X_n = Y_n$ (both sides are empty).

If $n = m$, bijectivity is immediate because we must have $\alpha = \mathrm{id}_{[n]}$.

If $n > m$, we can show bijectivity using proposition I.10 and the hypothesis that all simplices in $X_k \smallsetminus Y_k$ are degenerate for $k > m$. In particular, injectivity follows from uniqueness of the representation of a simplex as a degeneracy of a non-degenerate simplex, since elements of $(X_m \smallsetminus Y_m)$ are non-degenerate (because otherwise they would be images of a simplex of dimension smaller than m, but in those dimensions X and Y are equal). To show surjectivity, take any simplex $x \in X_n \smallsetminus Y_n$ and write $x = \alpha^*(\bar{x})$ for some surjective morphism $\alpha : [n] \to [k]$ and some non-degenerate simplex $\bar{x} \in X_k$. Since x does not belong to Y_n, $\bar{x}$ does not belong to Y_k, hence we must have $k \geq m$. But we must also have $k \leq m$, because otherwise all simplices in $X_k \smallsetminus Y_k$ would be degenerate. Hence $k = m$ and $(\bar{x}, \alpha)$ is an element of the domain with image x.

(2) The geometric realization functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ is a left adjoint (to the singular complex functor $\mathcal{S} : \mathrm{Top} \to \mathrm{sSet}$), hence it preserves colimits (in particular, finite ones, like coproducts and pushouts).

A proof of this fact on the nLab: adjoints preserve (co-)limits.

(3) From part (2) we know that $|X|$ can be obtained from $|Y|$ by attaching copies of $|\Delta^m|$ along $|\partial\Delta^m|$, indexed by $X_m \smallsetminus Y_m$. So it suffices to show that the pair $(|\Delta^m|, |\partial\Delta^m|)$ is homeomorphic to $(\nabla^m, \partial\nabla^m)$, hence to (D^m, S^{m-1}).

Recall that we have mutually inverse homeomorphisms:

$$\begin{aligned} &|\Delta^m| \to \nabla^m, \quad [\alpha : [n] \to [m], t \in \nabla^n] \mapsto \alpha_*(t), \\ &\nabla^m \to |\Delta^m|, \quad t \mapsto [\mathrm{id}_{[m]}, t]. \end{aligned}$$

We claim that these homeomorphisms map $|\partial\Delta^m|$ homeomorphically onto the boundary of the topological simplex $\partial\nabla^m = \{(t_0, \dots, t_m) \in \nabla^m \mid t_i = 0 \text{ for some } 0 \le i \le m\}$.

First, observe that the map $|\partial\Delta^m| \to |\Delta^m|$ is injective by corollary V.2 (which follows from the theory of minimal representatives). Secondly, since $\partial\Delta^m$ has only finitely many non-degenerate simplices, $|\partial\Delta^m|$ is compact by corollary V.4 (which is also a consequence of the theory of minimal representatives). since $|\Delta^m|$ Hausdorff (being homeomorphic to ∇^m), this is a closed map and hence a homeomorphism onto its image. Now, the non-degenerate simplices of $\partial\Delta^m$ are all injective morphisms $\alpha : [k] \to [m]$ except $\mathrm{id}_{[m]}$ (which is surjective); by corollary V.3, the image of $|\partial\Delta^m|$ is then the union of the images of the maps $\alpha_* : \nabla^k \to \nabla^m$ for all injective morphisms $\alpha : [k] \to [m]$ other than $\mathrm{id}_{[m]}$, hence it coincides with $\partial\nabla^m$, the boundary of ∇^m. □

We are now ready to introduce the preferred CW-structure on the realization of a simplicial set, and summarize all its key properties, which follow from all the theory we have developed so far.

V.7. Theorem. — *Let X be a simplicial set.*

(1) *The subspaces $|\operatorname{sk}^m X|$ for $m \ge 0$ form a CW-structure on $|X|$, we will refer to it as the* ***preferred CW-structure*** *on the geometric realization of X.*
(2) *The m-cells of the preferred CW-structure are in bijection with the set of the non-degenerate m-simplices of X.*
(3) *Suppose that for all $n > m$, all n-simplices of X are degenerate (i.e. $X = \operatorname{sk}^m X$). Then $|X|$ is an m-dimensional CW-complex.*
(4) *Suppose that the total number of non-degenerate simplices of X is finite. Then $|X|$ is a finite CW-complex. In particular, $|X|$ is compact.*
(5) *For every morphism of simplicial sets $f : X \to Y$, the induced continuous map $|f| : |X| \to |Y|$ is cellular with respect to the preferred CW-structure.*
(6) *If Y is a simplicial subset of X, then $|Y|$ is a sub-complex of $|X|$ in the preferred CW-structure.*

Note: another consequence of the theorem is that $|X|$ is Hausdorff. It might be not easy to show this directly.

Proof. We prove (1) and (2), the rest are immediate consequences.

Observe that we have $(\operatorname{sk}^{m-1} X)_{m-1} = X_{m-1} = (\operatorname{sk}^M X)_{m-1}$, and for every $n > m$, all simplices of $(\operatorname{sk}^n)_m$ are degenerate. So $(\operatorname{sk}^m X, \operatorname{sk}^{m-1} X)$ satisfies the hypothesis of proposition V.6. Hence $|\operatorname{sk}^m X|$ can be obtained from $|\operatorname{sk}^{m-1} X|$ by attaching m-cells indexed by $(\operatorname{sk}^m X)_m \setminus (\operatorname{sk}^{m-1} X)_m^{\mathrm{nd}}$, the set of non-degenerate m-simplices of X.

Since $|-|$ is a left adjoint, it preserves colimits. Since X is a colimit (in sSet) of the sequence $\{\operatorname{sk}^m X\}_{m \ge 0}$, $|X|$ is a colimit (in Top) of the sequence:

$$|\operatorname{sk}^0 X| \to |\operatorname{sk}^1 X| \to \cdots \to |\operatorname{sk}^m X| \to \cdots$$

Each of these maps is a cell attachment, hence in particular a closed embedding. So a colimit of the sequence is the union with the weak topology. □

Example. — We already know $|\Delta^m| \cong \nabla^m$ with the (obvious) CW-structure:

- $(\nabla^m)_0 = \{e_0, \dots, e_m\}$,
- $(\nabla^m)_1 =$ edges,
- $\cdots$,
- $(\nabla^m)_{m-1} = \partial\nabla^m$
- $(\nabla^m)_m = \nabla^m$

What about $|\Delta^m/\partial\Delta^m|$? Intuitively this should be S^m with the CW-structure consisting of one 0-cell and one m-cell. To prove it, observe that $(\Delta^m)_n^{\mathrm{nd}} = \{\alpha : [n] \to [m] \mid \alpha \text{ injective}\}$ and $(\partial\Delta^m)_n^{\mathrm{nd}} = \{\alpha : [n] \to [m] \mid \alpha \text{ injective, but not surjective}\}$. Then we have:

- for $0 \leq n < m$, $(\Delta^m/\partial\Delta^m)_n = \{*\}$,
- for $n = m$, $(\Delta^m/\partial\Delta^m)_m = \{\mathrm{id}_{[m]}, *\}$,
- for $n > m$, all simplices are degenerate.

This does not seem to give the CW-structure we predicted, but in fact it does: the only non-degenerate simplices are the one in $(\Delta^m/\partial\Delta^m)_0$ and $\mathrm{id}_{[m]} \in (\Delta^m/\partial\Delta^m)_m$.

Example. — An interesting fact: we have $|\partial\Delta^3| \cong S^2 \cong |\Delta^2/\partial\Delta^2|$ but as simplicial sets $\partial\Delta^3 \ncong \Delta^2/\partial\Delta^2$. Note that the CW-structure on S^2 resulting from realizing these two complexes is different, the one coming from $\partial\Delta^3$ being much bigger (a tetrahedron: four 0-cells, six 1-cells, four 2-cells).

Example. — We have $|B\mathbb{Z}/2| \cong K(\mathbb{Z}/2, 1) \cong \mathbb{R}\mathbf{P}^\infty$ (see the section in chapter IV, although we skipped many details about classifying spaces), but the preferred CW-structure on $|B\mathbb{Z}/2|$ is different than the usual CW-structure on $\mathbb{R}\mathbf{P}^\infty$, with one cell in every dimension. Indeed, remembering the bar construction of AT1Sheet4, we see that the non-degenerate simplices in dimension n are

$$(B\mathbb{Z}/2)_n^{\mathrm{nd}} = (\underbrace{1, \dots, 1}_{n\text{-times}}),$$

so the preferred CW-structure actually has one cell per dimension, but it is still not the same as the usual structure on $\mathbb{R}\mathbf{P}^\infty$ because the attaching maps behave differently (one can see this observing how attaching maps behave on the boundary of the cells, given the simplicial structure).

Not really sure about why the structures are different

Example. — We have $|B\mathbb{Z}| = K(\mathbb{Z}, 1) \cong S^1$. Following the same reasoning as in the last example, we see that the non-degenerate simplices of $B\mathbb{Z}$ in any dimension are infinitely many. We already noted this fact in chapter IV: S^1 and $|B\mathbb{Z}|$ are homeomorphic CW-complexes, but the former is one-dimensional, the latter infinite-dimensional.