Loose Ends from Tutorial 4

Careful with that Yoneda, Eugene

The motto of this week's exercise sheet could be *it just works*, with "it" being the theory of higher categories we have developed so far: it is quite remarkable how smoothly the language of ∞-categories (as provided by quasicategories) lets us talk about and prove results of formal nature in homotopical settings. Having said this, one should not get too careless with this technology, as exemplified by the third exercise: many people (correctly) appealed to Yoneda to solve this one, but without providing a full justification of why it is possible to do so (which I think should have been part of the solution).

The problem was to show that for I a simplicial set and $\mathcal{C}$ a pointed ∞-category with finite limits, there is an equivalence of ∞-categories:

$$\mathrm{Sp}(\mathrm{Fun}(I, \mathcal{C})) \simeq \mathrm{Fun}(I, \mathrm{Sp}(\mathcal{C}))$$

(thereby showing that the right hand side is stable).

The equivalence is immediately proven once one shows that for an arbitrary J-shaped limit of ∞-categories $\mathcal{D} = \lim_J \mathcal{D}_j$, there is an equivalence of ∞-categories.

$$\mathrm{Fun}(I, \mathcal{D}) \simeq \lim_J \mathrm{Fun}(I, \mathcal{D}_j)$$

This can be shown in different ways, e.g. by appealing to the fact (HTT, Corollary 5.1.2.3) that limits in functor ∞-categories are computed pointwise. Another proof using Yoneda goes as follows. First, note that we can find an ∞-category I' with a functor $I \to I'$ such that $\mathrm{Fun}(I, \mathcal{D}) \simeq \mathrm{Fun}(I', \mathcal{D})$ (this is a fibrant replacement in the Joyal model structure, essentially we are "adding all the missing compositions" to I), thus reducing the problem to the case of I being an ∞-category. Second, we compute

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D})) &\simeq \mathrm{Hom}_{\mathrm{sSet}}(\mathcal{C}, \mathrm{Fun}(I, \mathcal{D}))^{\mathrm{core}} \\
&\simeq \mathrm{Hom}_{\mathrm{sSet}}(\mathcal{C} \times I, \mathcal{D})^{\mathrm{core}} \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C} \times I, \mathcal{D}) \\
&\simeq \lim_J \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C} \times I, \mathcal{D}_j) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \lim_J \mathrm{Fun}(I, \mathcal{D}_j))
\end{aligned}$$

where we used in the first step that mapping spaces in Cat_∞ can be modeled as the core (biggest Kan subcomplex) of the mapping simplicial set (which is a quasicategory), in the second step the usual currying adjunction, in the fourth step that mapping spaces in ∞-categories commute with limits, and in the fifth step all these things at once. Now we can appeal to Yoneda to conclude.

At this point, one might complain that we are using a lot of things that were not mentioned in the course: while such complaint is understandable, it is not possible to cover every nook and cranny of the theory of higher categories in a course such as this one (for which higher categories are a language and a tool, but not the sole focus). Indeed, you should take this as an occasion to familiarize yourself with the literature: it is perfectly fine to quote or assume results in HTT or Land sometimes!