Algebraic Topology II - Sheet 3 Solutions

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Problem 8 (Mapping animæ in slices). Omitted for now. We tried, but we cannot really give an elementary proof of this exercise without introducing some technology that wasn't covered in the lectures: in particular, you shouldn't feel bad if you did not do this one, it wasn't a very fair homework problem. We will write a complete solution covering the relevant background later.

Problem 9 (Initial objects). Let's go point by point.

1. The nerve functor preserves limits, in particular

$$\mathrm{map}_{N\mathcal{C}}(X,Y) \cong N(\mathcal{C})^{\Delta^1} \times_{\Delta^0} \{(X,Y)\} \cong N(\mathcal{C}^{[1]}) \times_{\Delta^0} \{(X,Y)\} \cong N(\mathrm{hom}_{\mathcal{C}}(X,Y))$$

Since $\mathrm{hom}_{\mathcal{C}}(X,Y)$ is a set, its nerve is contractible if and only if $\mathrm{hom}_{\mathcal{C}}(X,Y)$ consists of a single point. Now, for all $Y \in \mathcal{C}$, X is initial in the 1-categorical sense if and only if $\mathrm{hom}_{\mathcal{C}}(X,Y)$ is the one point set, if and only if the nerve of $\mathrm{hom}_{\mathcal{C}}(X,Y)$ is contractible, if and only if $\mathrm{map}_{N\mathcal{C}}(X,Y)$ is contractible, i.e. X is initial in the ∞-categorical sense.

2. Consider $\mathcal{C}$ as pointed, with basepoint being a initial object $X : \Delta^0 \to \mathcal{C}$. Since geometric realization preserves finite limits, in the diagram

$$
\begin{array}{ccc}
|\mathrm{map}_{\mathcal{C}}(X,X)| & \longrightarrow & |\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \{X\}| \\
\downarrow & & \downarrow{\scriptstyle p} \\
\{X\} & \longrightarrow & |\mathcal{C}|
\end{array}
$$

the right vertical arrow is a trivial Serre fibration with contractible domain, hence the diagram is a homotopy pullback of spaces. Since $|\mathrm{map}_{\mathcal{C}}(X,X)|$ is contractible, then p is a weak homotopy equivalence (use the long exact fiber sequence for homotopy groups), hence $|\mathcal{C}|$ is contractible.

3. If X is initial, then $\mathrm{map}_{\mathcal{C}}(X,Y)$ is contractible, for all Y, in particular it is non-empty and every two morphisms are homotopic, hence equal in the homotopy category.

4. Take $\mathcal{C} = \mathrm{Sing}\, S^2$. Since this space is simply connected, every point is initial in $\mathrm{h}\mathcal{C}$. However, its loop space is not contractible, since the loop of the equator has two ways of being contracted to a point and these are not homotopic.

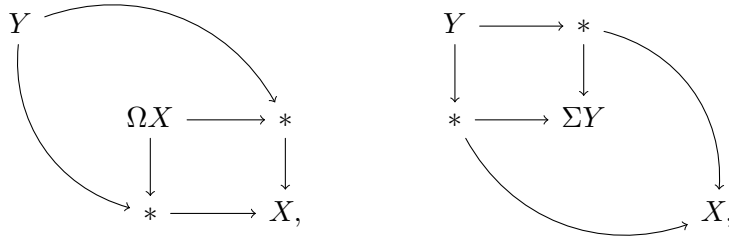**Problem 10 (Loops and homotopy groups).** For this one we give two solutions.

First solution. Since $\pi_n X$ is given by the set $[S^n, X]_*$ of homotopy classes of maps in pointed animæ, we see that

$$\begin{aligned}
\pi_n \Omega X &= [S^n, \Omega X]_* \\
&\cong [\Sigma S^n, X]_* \\
&\cong [S^{n+1}, X]_* \\
&= \pi_{n+1} X,
\end{aligned}$$

where the first isomorphism comes from the image under π_0 of the adjunction

$$\mathrm{map}(\Sigma Y, X) \simeq \mathrm{map}(Y, \Omega X).$$

This can be seen from the definition. From the diagrams

we see that (up to contractible choice) a map to the pullback $Y \to \Omega X$ is a map from Y to the cospan $* \to X \leftarrow *$, which is equivalent to a map from the span $* \leftarrow Y \to *$ to X, i.e. a map from the pushout $\Sigma Y \to X$.

Second solution. The loop space of a point $x : \Delta^0 \to X$ is calculated as the strict pullback of x with the fibration $PX \to X$, where $PX = X^{\Delta^1} \times_X \{x\}$. Geometric realization maps pullback to pullbacks and fibrations to Serre fibrations. Taking the long exact sequence associated to $|\Omega X| \to |PX| \to |X|$, we get

$$\cdots \to \pi_n(|\Omega X|) \to \pi_n(|PX|) \to \pi_n(|X|) \to \pi_{n-1}(|\Omega X|) \to \cdots$$

Since $|PX|$ is contractible, exactness of the sequence above implies $\pi_n(|\Omega X|) \simeq \pi_{n+1}(|X|)$.

Problem 11 (Building animæ out of points). We can use that taking the singular set and geometric realization induce an equivalence of ∞-categories between CW complexes and Kan complexes (i.e. animæ). Every sphere S^n is homeomorphic to the suspension of the previous sphere S^{n-1}, which is the homotopy pushout of $S^{n-1} \to *$ along itself. This way, we can start with S^0 and construct every sphere by homotopy pushouts. Then general CW complexes are constructed by pushout along coproducts of inclusions $S^{n-1} \to D^n$, which are cofibrations in the Quillen model structure of topological spaces. In particular, pushout along the inclusion of spheres into disks are also homotopy pushouts. With this, we proved that given a general CW complex X, its n-skeleton X_n is generated by finitely

many homotopy pushouts. Finally, each inclusion $X_n \subseteq X_{n+1}$ is a cofibration (being the pushout of a cofibration), hence the strict colimit $\operatorname{colim}_n X_n$, which is just X, is a model for the homotopy colimit of the sequence $X_0 \subseteq X_1 \subseteq \cdots$.

Of course, there is no reason to pass to CW complexes to solve this exercise, one can run the same argument using the skeletal filtration of simplicial sets, as we have seen in the previous sheet that the skeleta can be written as pushouts of boundary inclusions, which are homotopy pushouts. On the other hand, while higher level arguments using straightening or Yoneda also work (obviously), they rather miss the point of the exercise.