Introduction to Operator K-theory

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The K-theory of C*-algebras is a natural consequence of taking seriously the metaphor of C*-algebras as (functions on) noncommutative spaces stemming from Gelfand-Naimark duality, perhaps the first substantial such consequence. In this talk, I want to introduce topological K-theory, a topological invariant which the K-theory of C*-algebras generalizes, then explain the bridge from the commutative (topological) case to the noncommutative (operator-algebraic) one, and finally describe in some detail the K-theory of C*-algebras.

Topological K-Theory

Topological K-theory is an invariant of topological spaces built out of vector bundles. For a compact space X, consider the commutative monoid $V(X) = V_{\mathbb{C}}(X)$ of equivalence classes of complex vector bundles on X with addition given by Whitney sum (i.e. fiberwise direct sum), and neutral element the 0 trivial bundle. Similarly, define $V_{\mathbb{R}}(X)$ using real vector bundles instead.

A crucial property of $V(X)$ is functoriality: if we are given a map of compact spaces $\varphi : X \to Y$, then there is a map of monoids $V(Y) \to V(X)$ given by sending the class of a bundle $[E]$ to the class of the pullback bundle $[\varphi^* E]$. Moreover, it is easy to show that $V(X)$ is homotopy invariant, i.e. homotopic maps $X \to Y$ induce the same map of monoids $V(Y) \to V(X)$.

Examples. Throughout the talk, we will be little concerned with proofs and much more with understanding the construction of K-theory, along with its properties and significance. In particular, we will devote a lot of time to examples.

- If X is a point, every vector bundle on it has to be trivial, so $V(*) \cong \mathbb{N}$ with isomorphism given by the dimension of the trivial bundle.
- If X is contractible, by homotopy invariance we have $V(X) \cong V(*) \cong \mathbb{N}$; indeed, using homotopy invariance one can also easily show that over contractible spaces all vector bundles are trivial (and thus the isomorphism $V(X) \cong \mathbb{N}$ is again given by dimension).

- For the first nontrivial example, we have that $V_{\mathbb{R}}(S^1) \cong (\mathbb{N}_{>0} \times \mathbb{Z}/2) \cup \{0\}$. To give a proof would actually require some algebraic topology, such as the theory of classifying spaces or characteristic classes, but we can at least give an intuitive explanation of why this is the case: every real bundle on the circle has to be either a trivial bundle or the Whitney sum of a trivial bundle with a Möbius band, and the sum of two Möbius bands is easily seen to be trivial. Therefore, the $\mathbb{N}$ factor records the dimension of the bundle, while the $\mathbb{Z}/2$ factor records the presence of a Möbius strip.

- The theory of clutching functions gives a bijection

$$[S^{n-1}, \mathrm{GL}_k(\mathbb{C})] \to V_{\mathbb{C}}^k(S^n),$$

and a similar one in the real case. This shows for example that $V_{\mathbb{C}}(S^1) \cong \mathbb{N}$, since $\mathrm{GL}_k(\mathbb{C})$ is path-connected, but our computations end here: $[S^{n-1}, \mathrm{GL}_k(\mathbb{C})]$ is a complicated object and in fact even $V_{\mathbb{C}}(S^2) \cong (\mathbb{N}_{>0} \times \mathbb{Z}) \cup \{0\}$ is not entirely trivial to compute.

Remark. As always in algebraic topology, one can usually reduce computations for general spaces to computations for simple spaces. For example, with some work one can show that $V_{\mathbb{C}}(\mathbb{T}) \cong V_{\mathbb{C}}(S^2)$.

The commutative monoid $V(X)$ is an interesting object, but not easily computable. To fix this, we want to turn it into an abelian group, in order to access the computational techniques provided by homological algebra. The way to do this is the (*Grothendieck*) *enveloping group.*

Given a commutative monoid V, there is a monoid morphism to an abelian group K, the enveloping group, such that the following universal property is satisfied: for any abelian group A and any monoid morphism $V \to A$, there is a unique group morphism $A \to K$ making the relevant diagram commute.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & K \\ A \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Concretely, we can take the free abelian group on the elements of the monoid V, and quotient out by the relations given by the addition in V; even better, we can see K as being generated by formal differences $m - n$ of elements of V, subject to the equivalence relation $m - n \sim m' - n'$ if there is some $k \in V$ such that $m + m' + k = n + n' + k$. Note that the construction is clearly functorial.

Remark. One important observation: if the monoid V does not have the cancellation property (i.e. $m + n = m' + n$ implies that $m = m'$), then the map $V \to K$ will *not* be injective. This is relevant because in practice $V(X)$ often does not have the cancellation property: an easy example is given by the monoid $V_{\mathbb{R}}(S^2)$, which does not have the cancellation property since TS^2 plus a trivial line bundle is trivial.

In view of the preceding discussion, we define $K^{00}(X)$ as the enveloping group of $V_{\mathbb{C}}(X)$, from now on restricting to the complex case. Note that by functoriality, there is always a monomorphism

$$\mathbb{Z} \cong K^{00}(*) \hookrightarrow K^{00}(X),$$

therefore we can define the reduced K-theory group as

$$\tilde{K}^{00}(X) \coloneqq K^{00}(X)/\mathbb{Z}.$$

Examples. The enveloping group of the abelian monoid $(\mathbb{N}, +)$ is the group $\mathbb{Z}$, thus we get the enveloping group $K(X)$ of all the examples of $V(X)$ above by replacing $\mathbb{N}$ with $\mathbb{Z}$. In general, more interesting things might happen, e.g. if the monoid does not have the cancellation property.

We are now ready to define the (*real* or *complex*) *K-theory groups* of a more general locally compact space X. The zeroth K-theory group is in this case

$$K^0(X) \coloneqq \tilde{K}^{00}(X^+),$$

where X^+ is the one point compactification of X. Similarly, the negative K-theory groups are defined as

$$K^{-n}(X) \coloneqq K^0(\mathbb{R}^n \times X),$$

where $X \times \mathbb{R}^n$ is known as the n-th *reduced suspension* of X (this is because under compactification it is the reduced suspension of the compactification of X). As always, everything we defined is functorial.

Remark. The reason why we need to define K-theory in general using compactifications is that the naive definition of K-theory for a locally compact space would not be homotopy invariant (for homotopy invariance of $V(X)$ we need at least paracompactness)!

The main property of the K-theory groups, apart from homotopy invariance, is the following: given a closed subspace $Y \subset X$, there are exact sequences

$$K^{-n}(X \setminus Y) \to K^{-n}(X) \to K^{-n}(X/Y),$$

and they assemble in a long exact sequence via connecting maps

$$\delta : K^{-n}(X/Y) \to K^{-n+1}(X \setminus Y),$$

but by the celebrated Bott periodicity theorem, this long exact sequence is actually a six (or 24, in the real case) term cyclic exact sequence. We will expand on this later.

Remark One last thing before we go on: why is all this useful? Well, not only K-theory turns out to be a very powerful invariant, if sometimes difficult to compute, but many constructions in algebraic topology and geometry turn out to live in or be related to K-groups. As it turns out, the same is true of operator K-theory.

From Topological to Operator K-Theory

Now that we know what topological K-Theory is, we want to translate it in a language which can actually be reinterpreted in the noncommutative operator-algebraic case. The key for the translation is the following theorem.

Theorem (Serre-Swan)**.** *Let E be a vector bundle over a compact base space X. Then there is another vector bundle F over X such that the sum $E \oplus F$ is trivial.*

In view of the theorem, which is not too difficult to prove (use a partition of unity to find a surjection from a trivial bundle), it is easy to prove the following result, which provides the translation we are looking for:

Corollary. *Given a unital C*-algebra A, the following monoids are all isomorphic:*

(1) If A is commutative, the monoid $V(\hat{A})$ defined in the previous section, for $\hat{A}$ the Gelfand spectrum of A.

(2) The monoid $V_0(A)$ of isomorphism classes of finitely generated projective A-modules, with addition given by direct sum.

(3) The monoid $V(A)$ of equivalence classes of idempotents in $M_\infty(A)$, where $M_\infty(A)$ is the (non-complete) $*$-algebra defined as $\operatorname{colim}_n M_n(A)$, and two idempotents e and f are equivalent if there exist $x, y \in M_\infty(A)$ such that $e = xy$ and $f = yx$, with addition given by

$$[p] + [q] = [p' \oplus q']$$

where p' and q' are orthogonal representatives for the classes $[p]$ and $[q]$ (this is a well-defined equivalence relation because the sums of orthogonal projections are projections and respect the equivalence relation; moreover, one can easily show that we can always find orthogonal representatives).

Sketch of proof. The isomorphism $V(\hat{A}) \to V_0(A)$ is given by sending a vector bundle E to the $C(X)$ module of global sections $\Gamma(E)$. Indeed, it is easy to see that $\Gamma(E)$ is a module, and that isomorphic vector bundles have isomorphic modules of global section, while the theorem of Serre-Swan, along with the observation that $\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$, implies that $\Gamma(E)$ is finitely generated and projective. Thus $E \mapsto \Gamma(E)$ is a well-defined map of monoids. To show it is a bijection, one can construct an explicit inverse: this is done by considering the projection $e : C(X)^n \to C(X)^n$ onto M, which exists as M is projective, and interpreting e as an $n \times n$ matrix in $M_n(C(X)) \cong C(X \to M_n(\mathbb{C}))$ defining

$$\Xi(M) \coloneqq \{(x, \xi) \in X \times \mathbb{C}^n \mid \xi \in \operatorname{im} e(x)\}.$$

As for the isomorphism $V(A) \to V_0(A)$, this is given by sending an idempotent $p \in M_n(A)$ to its image $p(A^n)$. It is easy to show that the images of two idempotents are isomorphic if and only if the idempotents are equivalent, which shows that the map is well-defined and injective. Surjectivity is obvious by the definition of projective module. $\square$

Remarks. A couple of remarks about the different definitions of K-theory are in order.

- Perspective (2) is the one usually favored in algebra, where the Grothendieck group $K_0(R)$ for a ring R is an object of central interest, whose study leads one into the complicated world of algebraic K-theory. This is a theory of K-groups of rings for which Bott periodicity does not hold (at least on the nose), which in the case of stable C*-algebras (seen as mere rings) coincides with our operator K-theory (the proof being far from easy).
- Neither perspective (2) nor perspective (3) use that A is a C*-algebra, and in fact perspective (3) is used to define the K-theory of more general Banach algebras. Our definition of K-theory will be a slight modification of perspective (3), which takes into account the additional structure on A, although the two definitions turn out to be equivalent, with the importance for the modification becoming apparent only at a more advanced stage, mostly when considering the Fredholm picture.
- There is one fourth perspective on the K-theory of C*-algebras, given by considering the monoid of equivalence classes of projections in $\mathcal{K}(\mathcal{H}_A)$, where the equivalence is in $\mathcal{B}(\mathcal{H}_A)$ and the addition is defined as for perspective (3). This is a perspective which we will not explore in this talk, but which will be central for the talk on the relation between K-theory and Fredholm operators. In this case, there is an isomorphism with $V_0(A)$ given by sending a projection Q to $Q(\mathcal{H}_A)$; the proof is slightly more complicated and actually uses some analysis (e.g. polar decomposition). Finally, note that this perspective is remarkable in that it does not require considering matrices of arbitrary sizes: this makes sense because $\mathcal{K}(\mathcal{H}_A) \cong A \otimes \mathbb{K}$ is the completion of the $*$-algebra $M_\infty(A)$.
- The fourth perspective just mentioned will be used to define the generalized Fredholm index in a later on. This agrees with the intuition of K_0 of a ring as the "next best thing to dimension" for finitely generated projective module.

Projections and K_0 of a C*-Algebra

We are finally ready to discuss the K-theory of C*-algebras. A lot of the work leading to the definition of K-theory is actually already taken care of, since the parallels with the topological case are often quite literal: as we noted, one can take the point of view of perspective (3) in the previous section and simply apply it to the case of a unital Banach algebra A. The only slight modification we make, when A is a C*-algebra, is to consider the monoid $V(A)$ of equivalence classes of projections, i.e. self adjoint idempotents of A:

$$V(A) \coloneqq \{[p] \mid p = p^* = p^2 \in M_\infty(A)\},$$

where $p \sim q$ when there is a $v \in M_\infty(A)$ such that $p = v^*v$ and $q = vv^*$. The changes are not substantial: it is not difficult to show that every idempotent equivalence class contains a representative which is a projection (and in fact the difference is not really relevant for this talk), and the two notions of equivalence can be shown to be the same for the stabilized algebras $M_\infty(A)$. Finally, define addition as follows:

$$[p] + [q] = [\mathrm{diag}(p, q)],$$

where $\mathrm{diag}(p, q)$ is the matrix

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

To see that this is a good definition recovering the previous one, i.e. that there are orthogonal representatives p', q' of $[p]$ and $[q]$ for which we have $p' \oplus q' \sim \mathrm{diag}(p, q)$, we must take a short detour into different notions of equivalence for elements of a C*-algebra.

Intermezzo: equivalence relations for projections. There are three useful equivalence relation for elements of a C*-algebra A. Projections p and q in A are said to be:

- *equivalent* ($p \sim q$), when $p = v^*v$ and $q = vv^*$ for an element $v \in A$,
- *unitarily equivalent* ($p \sim_u q$), when $p = u^*qu$ for a unitary $u \in A$,
- *homotopic* ($p \sim_h q$), when p and q are connected by a norm continuous path of projections in A.

Now, it is not terribly difficult (the details are in Chapter 5 of [WO93]) to prove that

$$p \sim_h q \implies p \sim_u q \implies p \sim q,$$

and moreover that

$$p \sim q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p \sim_u q \implies \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix},$$

where there generalization of the equivalence relations to matrices is straightforward. Therefore, we see that in $M_\infty(A)$ all notions of equivalence coincide, and in particular we have that

$$p = \mathrm{diag}(p, 0) \sim_u \mathrm{diag}(0, p) \perp \mathrm{diag}(q, 0) = q$$

via the unitary $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, one can freely apply elementary matrix operations without leaving an equivalence class. In dealing with K-theory (at least in the standard picture), one often exploits the possibility of switching freely between the three notions of equivalence.

Note finally that $V(A)$ is *covariantly* functorial, in the obvious way (applying a morphism of C*-algebras to all the entries of a matrix). This is in contrast with topological K-theory, which was contravariant, but therefore in agreement with the contravariant duality between spaces and algebras.

Of course, the object we are actually interested in is the group $K_0(A)$, and not the bare monoid $V(A)$: mirroring the situation we had with spaces, we cannot define $K_0(A)$ to be just the enveloping group of $V(A)$, as this does not have good formal properties when

A is not unital (in particular the half-exactness property fails). Therefore, we make the following definition: denoting by $K_{00}(A)$ the enveloping group of $V(A)$, we define

$$K_0(A) := \ker(\pi_* : K_{00}(A^+) \to K_{00}(\mathbb{C}) \cong \mathbb{Z}) \subset K_{00}(A^+),$$

where $\pi : A^+ = A \oplus \mathbb{C} \to \mathbb{C}$ is the projection and the isomorphism $K_{00}(\mathbb{C}) \cong \mathbb{Z}$ follows from the topological theory, or by the observation that projections in $M_k(\mathbb{C})$ are equivalent precisely when their images have the same dimension. It is then easy to see that by functoriality and the split exact sequence $0 \to A \to A^+ \xrightarrow{\pi} \mathbb{C} \to 0$, we have

$$K_{00}(A^+) \cong K_0(A) \oplus \mathbb{Z},$$

so that if A has a unit $K_0(A)$ is indeed isomorphic to the enveloping group of $V(A)$, and that $K_0(A)$ thus defined is a functor (because unitisation is functorial). More in detail we have the following (Proposition 6.2.7 in [WO93]):

Theorem (Standard picture of $K_0(A)$)**.** *The assignment $A \mapsto K_0(A)$ is a functor from C^*-algebras (with $*$-morphisms) to abelian groups, and moreover we have the following descriptions $K_0(A)$.*

(1) The elements of $K_0(A)$ can be visualized as formal differences

$$[p] - [q]$$

where p and q are projections in $M_k(A^+)$ for some $k \in \mathbb{N}$ and $p - q \in M_k(A)$.

(2) Moreover, q can be taken to be p_n, the projection with n zeroes on the diagonal, for some $n \leqslant k$, and again with the same scalar part as p.

(3) If $[p] = [q]$ in $K_0(A)$, then for some $m \leqslant n$,

$$\operatorname{diag}(p, p_m) \sim_h \operatorname{diag}(q, p_m) \text{ in } M_{k+n}(A^+)$$

and vice versa.

Sketch of proof. Functoriality is clear, given that unitisation is also functorial. The rest is a lot of computations with matrices and equivalence relations. We sketch (2). If $x = [q_1] - [q] \in K_0(A)$, then for large enough p_n, the difference $p_n - q$ is a projection. By moving q_1 down the diagonal (e.g. using unitaries), we can find a projection $q_2 \in M_\infty(A^+)$ such that $q_2 \sim_u q_1$ and $q_2 \perp p_n$. Then we have $q_2 \perp (p_n - q) \perp q$ and setting $q_3 := q_2 \oplus (p_n - q)$, we get

$$\begin{aligned}
[q_3] - [p_n] &= [q_2 \oplus (p_n - q)] - [p_n] \\
&= [q_2] + [p_n - q] + [q] - [q] - [p_n] \\
&= [q_1] + [(p_n - q) \oplus q] - [q] - [p_n] \\
&= [q_1] - [q] = x.
\end{aligned}$$

Finally, we have $0 = \pi_*(x) = [\pi(q_3)] - [\pi(p_n)] = [\pi(q_3)] - [p_n]$, and one can show that in this case we can choose a $p \sim_u q_3$ with $p - p_n \in M_\infty(A)$ and $x = [p] - [p_n]$. In particular, one sees that having to deal with $K_0(A)$ for A non-unital is a bit of a pain. $\square$

Finally, we can mention some examples of K-theory!

Examples. We already mentioned that $K_0(\mathbb{C}) \cong \mathbb{Z}$, and by the observation that $M_n(M_k(\mathbb{C})) = M_{nk}(\mathbb{C})$ the same holds of $M_k(\mathbb{C})$. It is also essentially immediate that the stabilization maps induce $K_0(M_n(A) \cong K_0(A))$ for any C*-algebra A. Similarly, we have $K_{00}(\mathbb{K}) \cong \mathbb{Z}$, but $\mathbb{K}$ is not unital, thus we are not able to compute yet $K_0(\mathbb{K})$. Similarly, for $\mathbb{B}$ and the Calkin algebra $\mathbb{B}/\mathbb{K}$, the existence of infinite dimensional projections forces both K_0 groups to collapse. In the following, we will be able to compute a few more examples appealing to properties of K_0. In general, it is possible to show that any abelian group appears as K_0 of some C*-algebra.

Properties of $K_0(A)$

For lack of time, we will not be able to prove any of the fundamental properties of $K_0(A)$ (the proofs are all obvious in the unital case, but just sligthly annoying for A non-unital; all the details are in Chapter 6 of [WO93]), but we can at least list and comment them, to get a feeling of how the theory works.

Continuity of K_0. *The functor K_0 commutes with direct limits.*

This is best understood via an example: by continuity it is immediate that the CAR algebra, defined as the completion of the colimit

$$M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \cdots$$

where the maps are diagonal embeddings, has K_0 group isomorphic to the colimit

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \cdots$$

which is nothing but the group $\mathbb{Z}[2^{-1}]$ of dyadic integers.

Stability of K_0. *The morphism $A \to A \otimes \mathbb{K}$ sending $a \mapsto a \otimes e_{11}$, where e_{11} is a rank 1 projection in $\mathbb{K}$, induces an isomorphism $K_0(A) \cong K_0(A \otimes \mathbb{K})$. In particular, if A and B are stably isomorphic, i.e. $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, then $K_0(A) = K_0(B)$.*

A couple of remarks on this result:

- As a consequence of the theorem, we have that $K_0(\mathbb{K}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$.
- As we already noted, stability should not come as too big of a surprise, given that $A \otimes \mathbb{K}$ is just the completion of $M_\infty(A)$, and in fact the proof uses precisely this fact, combined with continuity.

- By the Brown-Green-Rieffel theorem we learned about in talk 3, Morita equivalent σ-unital C*-algebras also have the same K_0 (in analogy to Morita equivalence of rings inducing isomorphism of algebraic K-theory groups).
- There are stably non-isomorphic C*-algebras that have the same K_0 group. Indeed, as an invariant of C*-algebra the K_0 group is not very sharp. One workaround is to build more structure into it: in particular it is possible to turn K_0 into an ordered group, which then can be used to define a complete set of invariants for the classification of AF-algebras (Approximately Finite-dimensional C*-algebras, i.e. C*-algebras which are direct limits of finite dimensional algebras).

Half-exactness of K_0. *For J an ideal in A, the exact sequence $0 \to J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ induces an exact sequence of K_0 groups:*

$$K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

Note that the exact sequence need *not* be a short exact sequence: for example the sequence $0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0$ shows that the first map need not be injective, while the sequence $0 \to C_0(]0,1[) \to C([0,1]) \to \mathbb{C} \otimes \mathbb{C} \to 0$ shows that the second map need not be surjective. Note also that the same exact sequence would fail if we used K_{00} instead of K_0, as one can see by looking at the short exact sequence $0 \to C_0(\mathbb{R}^2) \to C(S^2) \to \mathbb{C} \to 0$.

Homotopy invariance. *Homotopic morphisms of C*-algebras induce the same morphisms of K_0 groups, where homotopy is defined by "dualizing" the definition of homotopy for maps of topological spaces. In particular, homotopy equivalences induce isomorphisms of K_0 groups, and contractible C*-algebras have vanishing K_0 group.*

Compared to the previous properties, homotopy invariance follows immediately from the definitions, even in the non-unital case. It is also easy to show that for a C*-algebra A the *cone*

$$CA := \{f \in C([0,1] \to A) \mid f(0) = 0\}$$

is contractible, and the *suspension*

$$SA := \{f \in CA \mid f(1) = 0\} \cong A \otimes C_0(\mathbb{R})$$

is contractible whenever A is.

Invertibles, Unitaries and $K_1(A)$

The K_1 group of a C*-algebra A is in some ways simpler than $K_0(A)$, for example having a single definition for the unital and non-unital cases. The group is built either from invertible or unitaries, the two approaches giving the same result: we have

$$\begin{aligned} K_1(A) &:= \mathrm{GL}^+_\infty(A)/\,\mathrm{GL}^+_\infty(A)_0 = \mathcal{U}^+_\infty(A)/\mathcal{U}^+_\infty(A)_0 \\ &= \mathrm{GL}_\infty(A^+)/\,\mathrm{GL}_\infty(A^+)_0 = \mathcal{U}_\infty(A^+)/\mathcal{U}_\infty(A^+)_0 \end{aligned}$$

where of course some theory goes into showing that these four definitions are all equivalent, and where some other theory goes into showing that the multiplication

$$[u][v] := [uv] = [\mathrm{diag}(u, v)]$$

is well-defined (for details, see Chapter 7 of [WO93]).

Another fact that makes $K_1(A)$ simpler than $K_0(A)$ is that

$$K_1(A) \cong \mathrm{colim}_n \, \mathrm{GL}_n^+(A)/\, \mathrm{GL}_n^+(A)_0$$

and similarly for the other three equivalent definitions. In particular, equality in $K_1(A)$ can be checked for some finite matrices of size n.

Finally, $K_1(A)$ satisfies the same properties as $K_0(A)$: continuity, stability, half-exactness and homotopy invariance (with proofs generally easier than those for $K_0(A)$). In fact, perhaps the most important result about $K_1(A)$ is that it is a K_0 group in disguise!

Theorem. *For any C*-algebra A there is a natural isomorphism*

$$\theta_A : K_1(A) \to K_0(SA).$$

Note that a priori it is not obvious how to even define the map! The crucial result is that elementary matrix operations preserve the homotopy class of unitaries, which gives a way to take a unitary and turn into a loop of projections (and once the map is defined, the complete proof of the theorem in [WO93] takes up another five pages!). We will see later the definition of the index map and the Bott map, which will give us an idea of how this kind of morphisms between K-groups are generally defined, mostly exploiting the equivalence between the different equivalence relations.

Before moving to the last part of the talk, where we will explore further the relations between K_0 and K_1, we give some examples of K_1-groups.

Examples. It is possible to show the following:

- The fact that $\mathrm{GL}_n +(\mathbb{C})$ is connected for all $n \in \mathbb{N}$ shows that $K_1(\mathbb{C})$ vanishes.
- By stability (or again analyzing connectedness), we have that $K_1(\mathbb{K})$ also vanishes.
- One can use the Borel functional calculus to show that $K_1(A)$ vanishes for any Von Neumann algebra A, and in particular for $\mathbb{B}$.
- By continuity, K_1 of any AF-algebra vanishes.
- Bott periodicity will provide us with the strongest computational tool for K-theory; in particular, we will show that $K_1(\mathbb{B}/\mathbb{K})$ has to vanish.

The Index Map and Bott Periodicity

To conclude this overview of the K-theory of C*-algebras, we will define an *index map* connecting the half exact sequences of K_0 and K_1 groups, and then state Bott periodicity.

For the index map $\delta : K_1(A/J) \to K_0(J)$, let $x \in K_1(A/J)$ and find:

- some $u \in \mathcal{U}_n^+(A/J)$ such that $x = [u]$,
- some $w \in \mathcal{U}_{2n}^+(A)$ which is a unitary lift of $\mathrm{diag}(u, u^*)$,

then define the map $K_1(A/J) \to K_0(J)$ by

$$\delta(x) := [wp_n w^*] - [p_n].$$

That the lift w can be found is a small technical result which explains the need for doubling the matrix. That $[wp_n w^*] - [p_n]$ is indeed in $K_0(J)$ follows from the fact that $\mathrm{diag}(u, u*)$ commutes with p_n, i.e. $wp_n w^*$ is clearly a projection and one shows that it belongs to $M_{2n}(J^+)$ and has p_n as its scalar part. Then one can show that the index map is a well-defined map (the choices we made do not matter) and group morphism, and that it connects the half-exact sequences of K_0 and K_1 groups in a six term exact sequence (doing all this takes up Chapter 8 in [WO93]).

Finally, let us discuss briefly the elephant in the room, Bott periodicity (which is the subject of Chapter 9 in [WO93]). We can define the *Bott map* β_A as follows: given a projection $p \in M_n(A^+)$, the map

$$f_p(z) = 1_n + p(z - 1)$$

is a loop of invertibles, and $f_p f_q = f_{p \oplus q}$ when p is orthogonal to q, thus we can extend the morphism $\beta'_A : V(A^+) \to K_1(SA),\ [p] \mapsto f_p$ to the Bott map:

$$\beta'_A : K_0(A) \to K_1(SA),\ [p] - [q] \mapsto [f_p f_q^*].$$

Bott periodicity then of course states:

Theorem (Bott periodicity)**.** *For any C*-algebra A the Bott map β_A is a natural isomorphism.*

Bott periodicity in particular implies that the six term exact sequence of K_0 and K_1 groups can be made into a cyclic six term exact sequence: defining the *exponential map*

$$\begin{tikzcd} K_0(A/J) \arrow[r, dashed, "\delta'"] \arrow[d, "\beta_{A/J}"] & K_1(J) \\ K_1(S(A/J)) \arrow[r, "\delta"'] & K_0(SJ) \arrow[u, "\theta_J^{-1}"'] \end{tikzcd}$$

we have the following cyclic exact sequence

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \delta \Big\uparrow & & & & \Big\downarrow \delta' \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array}$$

This sequence is one of the main computational tool for the K-theory of C*-algebras (for example it yields that $K_1(\mathbb{B}/\mathbb{K})$ vanishes). Note that in the real case there is a 24 term exact sequence (where higher K-groups are defined via suspension).

Other Exceptional Homology Theories?

We conclude the talk with the following interesting remark (see Chapter 11 of [WO93]): it is possible to show that K-theory as we defined it is essentially the only reasonable(?) homology theory for C*-algebras, as implied by the following:

Theorem. *Let K be a continuous Bott functor (i.e. continuous, homotopy invariant, half exact, stable) from* C*Alg *to* Ab. *If $K(\mathbb{C}) = \mathbb{Z}$ and $K(S\mathbb{C}) = 0$, then $K(A) = K_0(A)$ in a large subcategory of* C*Alg. *If $K(\mathbb{C}) = 0$ and $K(S\mathbb{C}) = \mathbb{Z}$, then $K(A) = K_1(A)$.*

What is the large subcategory mentioned in the theorem? We can at least say that it contains the smallest class of separable C*-algebras:

- containing the complex numbers $\mathbb{C}$,
- closed under direct limits,
- closed under extensions involving two elements of the class,
- every C*-algebra KK-equivalent not a C*-algebra in the class.

With some luck we may be able to see that in fact this is a very large class of C*-algebras in some of the later talks!

References

[WO93] N. E. Wegge-Olsen, *K-Theory and C*-algebras: A Friendly Approach*, Oxford University Press, 1993.