Lecture Notes for

Representation Theory II

Lecturer

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⚠ These are not official lecture notes!!! ⚠

Goes without saying, errors, annoying comments in the margins and bad taste are all mine (especially since I'm not always strictly following what the Lecturer did in class).
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About the course:

Representation theory has strong connections with both combinatorics and geometry. Classifications and calculations in representation theory often involve a great deal of explicit combinatorics; conversely, many interesting combinatorial formulas can be proved and given conceptual explanations via representation theory. Similarly, there are many constructions and examples in representation theory for which geometry provides a natural framework, while in geometry representation theory often underlies the structures studied and is useful as a tool for concrete calculations.

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Chapter I.

Grassmannians, Flag Varieties & Friends

I

The goal of this course is to explore some relations between representation theory, geometry and combinatorics. In this chapter we introduce the geometric objects on which such relations hinge: Grassmannians, flag varieties, Schubert cells and Schubert varieties.

Remark. — This course is essentially always over $k = \mathbb{C}$, although many things have more or less obvious generalizations to arbitrary fields.

Grassmannians

Lecture 1
12th Oct, 2022

The first object we want to study has a fairly simple definition.

Definition. — For $d, n \in \mathbb{N}$ with $n \geqslant 1$, the *Grassmannian* $\mathrm{Gr}(d, n)$ is the set of d dimensional subspaces of $\mathbb{C}^n$.

Examples. — If $d > n$ the Grassmannian is empty (so we usually assume $d \leqslant n$). We have $\mathrm{Gr}(\{0\}, n) = \{\{0\}\}$ and $\mathrm{Gr}(n, n) = \mathbb{C}^n$. More interestingly, Grassmannians generalize projective spaces, as $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$.

Definition. — We can define the *Plücker map*

$$\mathrm{Pl} : \mathrm{Gr}(d, n) \to \mathbb{P}(\textstyle\bigwedge^d \mathbb{C}^n)$$

by sending each the subspace $V = \langle v_1, \dots, v_d \rangle$ to the class $[v_1 \wedge \cdots \wedge v_d]$ (the assignment does not depend on the choice of basis vectors $v_1, \dots, v_d$, as any other choice will differ by the determinant of the base change matrix). Note that we will usually write $v_1 \wedge \cdots \wedge v_d$ instead of $[v_1 \wedge \cdots \wedge v_d]$.

Remark. — Throughout this course we fix a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ and call it the standard basis. This induces a standard basis on $\bigwedge^d \mathbb{C}^n$, given by $e_I = e_{i_1} \wedge \cdots \wedge e_{i_d}$, with the i_k a choice of d increasing indices in $[n] = \{1, \dots, n\}$ and $I = (i_1, \dots, i_d)$. Note that via this basis we have

Sometimes we may denote I as $(i_1 < \cdots < i_d)$ to emphasize that the indices are increasing.

$$\mathbb{P}(\textstyle\bigwedge^d \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{d}-1}.$$

Definition. — To $V \in \mathrm{Gr}(d, n)$ with basis $v_1, \dots, v_d$ we assign the *Plücker coordinates* p_I by setting

$$v_1 \wedge \cdots \wedge v_d = \sum_{I \in \mathcal{I}_{d,n}} p_I e_I,$$

where $\mathcal{I}_{d,n} = \{I \subset \{1, \dots, n\} \mid |I| = d\}$. Note that the p_I's are independent of the choice of basis for V up to multiplication with a scalar.

Example. — Take the point $V = \langle e_1 + e_2 = v_1, e_1 + e_3 = v_2\rangle \in \mathrm{Gr}(2,3)$. Then we have $v_1 \wedge v_2 = e_1 \wedge e_3 - e_1 \wedge e_2 + e_2 \wedge e_3$, so the Plücker coordinates are $p_{13} = 1$, $p_{12} = -1$, $p_{23} = 1$. Note that these are exactly the maximal minors of the matrix having v_1 and v_2 as first and second column: this holds in general, as shown in RT2Sheet1.1.

$(v_1\ v_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

Remark. — Plücker coordinates should be thought of as projective coordinates of the points of a Grassmannian. We can make this precise: Plücker coordinates can be seen as elements of $(\bigwedge^d \mathbb{C}^n)^*$ defined by $p_I(e_J) = \delta_{I,J}$ for $I, J \in \mathcal{I}_{d,n}$. These are homogeneous polynomials of degree 1, so they define elements in the coordinate ring of $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$.

So far we have described the Grassmannians just as sets, although the Plücker map suggests that they should be considered as subsets of projective space: indeed, we will now show that the Plücker map is injective, and even better, that the Grassmannians are actually projective varieties.

Definition. — A *projective variety* is a subset X of projective space $\mathbb{P}^n$ of the form $V(S)$, where $S \subset \mathbb{C}[x_0, \dots, x_n]$ is a set of homogeneous polynomials and

$$V(S) = \{x \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \in S\}.$$

The *coordinate ring* of X is defined as

$$\mathbb{C}[X] \coloneqq \mathbb{C}[x_0, \dots, x_n]/I(X),$$

where $I(X) = \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f(x) = 0 \ \forall x \in X\}$.

Note that projective varieties are the closed subsets of the Zariski topology on $\mathbb{P}^n$. The next theorem shows that Grassmannians are projective varieties.

I.1. Theorem (Plücker embedding). — *The Plücker map*

$$\mathrm{Pl} : \mathrm{Gr}(d,n) \to \mathbb{P}(\textstyle\bigwedge^d \mathbb{C}^n), \ \langle v_1, \dots, v_d\rangle \to [v_1 \wedge \cdots \wedge v_d]$$

is an embedding (i.e. it is injective) with image a Zariski closed subset of $\mathbb{P}(\bigwedge^d \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{d}-1}$. *In particular, this gives* $\mathrm{Gr}(d,n)$ *the structure of a projective variety.*

To prove the theorem, we need a little preparatory result, a way to “single out” pure wedges among the elements of $\bigwedge^d \mathbb{C}^n$:

I.2. Lemma. — *Let V be a finite dimensional vector space and $0 \neq x \in \bigwedge^d V$. Consider the linear map*

$$\varphi_x : V \to \textstyle\bigwedge^{d+1} V, \ v \mapsto x \wedge v.$$

Then $\dim\ker\varphi_x \leqslant d$. *Moreover,* $\dim\ker\varphi_x = d$ *if and only if x is a pure wedge, meaning that $x = v_1 \wedge \cdots \wedge v_d$ for some linearly independent vectors $v_i \in V$, in which case we have in particular* $\ker\varphi_x = \langle v_1, \dots, v_d\rangle$.

Proof. Pick a basis $e_1, \dots, e_n$ of V such that $e_1, \dots, e_s$ is a basis of $\ker\varphi_x$. With this choice of basis, e_I for $I \in \mathcal{I}_{d,n}$ is a basis of $\bigwedge^d V$, and in particular we have $x = \sum_{I \in \mathcal{I}_{d,n}} a_I e_I$ for some coefficients a_I. Then

$$\varphi_x(e_i) = \sum_{I \in \mathcal{I}_{d,n}} a_I e_I \wedge e_i,$$

meaning that for all $I \in \mathcal{I}_{d,n}$ such that $a_I \neq 0$, we must have $1, 2, \dots, s \in I$. Thus $s \leqslant d$, and equality holds if and only if the only I for which $a_I \neq 0$ is the one where all the first d indices occur, i.e. $x = v_1 \wedge \cdots \wedge v_d$ is a pure wedge. □

Proof of Theorem I.1. First, let us show that the Plücker map is injective. For $d = n$ this is clear, so let $d < n$ and $x = v_1 \wedge \cdots \wedge v_d$ be equal to $y = w_1 \wedge \cdots \wedge w_d$ in $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$. Then by Lemma I.2, $\ker \varphi_x = \ker \varphi_y$, thus $\langle v_1, \ldots, v_d \rangle = \langle w_1, \ldots, w_d \rangle$, which proves injectivity.

We are left to prove that the image of the Plücker map is (Zariski) closed. Again, this is clear if $d = n$. If $d < n$, consider a class $[x] \in \operatorname{im} \mathrm{Pl}$. Any representative x is a pure wedge, thus by Lemma I.2 we have $\dim \ker \varphi_x = d$, hence $\operatorname{rk} \varphi_x = n - d$, and $\operatorname{rk} \varphi_x$ equals the maximum r such that all $(r+1)$-minors in (any) matrix corresponding to φ_x vanish. But minors are homogeneous polynomials in the entries of a matrix, which in the case of a matrix corresponding to φ_x are coordinates of x, and we have just shown that $\operatorname{im} \mathrm{Pl}$ is defined by the vanishing of $n - d + 1$ of them: this shows that $\operatorname{im} \mathrm{Pl}$ is a projective variety. ☐

Example. — The proof of Theorem I.1 shows that $\mathrm{Gr}(2, 4)$ is the zero set of all 3-minors in 4×4 matrices: these give already 16 equations.

We managed to prove that Grassmannians are projective varieties, i.e. defined by homogeneous polynomials, but the previous example shows that already for a Grassmannian as simple as $\mathrm{Gr}(2, 4)$ the set of defining polynomials can be quite sizeable. On the other hand, given that as a variety $\mathrm{Gr}(2, 4)$ lives in $\mathbb{P}^5$, it is reasonable to expect that some better equations can be found for it. The goal of the next section is exactly this: to give a better description of the coordinate rings of Grassmannians.

Plücker Relations

The first step is the construction of an affine covering (i.e. a covering by affine spaces) for Grassmannians.

Note that there is a surjective map of sets

$$M_{n \times d}(\mathbb{C}) \smallsetminus Z \to \mathrm{Gr}(d, n),$$

where $Z = \{A \in M_{n \times d}(\mathbb{C}) \mid \operatorname{rk} A < d\}$, given by sending a matrix A to the subspace generated by its columns. This induces a bijection

$$\Phi : (M_{n \times d}(\mathbb{C}) \smallsetminus Z)/\sim \; \to \mathrm{Gr}(d, n)$$

where $A \sim B$ when $A = BC$ for some $C \in \mathrm{GL}_d(\mathbb{C})$. Note that in the case $d = 1$, this is the standard description of $\mathbb{P}^{n-1}$.

I.3. Lemma. — *There is an open covering* $\mathrm{Gr}(d, n) = \bigcup_{I \in \mathcal{I}_{d,n}} U_I$, *where*

$$U_I \coloneqq \{x \in \mathrm{Gr}(d, n) \mid p_I(x) \neq 0\},$$

and $U_I \cong \mathbb{A}^{d(n-d)}$ *as algebraic varieties.*

Proof. Clearly the sets U_I are open and cover $\mathrm{Gr}(d, n)$. We just need to prove that, as algebraic varieties, we have $U_I \cong \mathbb{A}^{d(n-d)}$. Assume first that $I = \{1, \ldots, d\}$, and consider the map (of sets):

$$f : \mathbb{A}^{d(n-d)} = M_{(n-d) \times d}(\mathbb{C}) \to U_I \subset \mathrm{Gr}(d, n), \; B \mapsto \Phi\left(\begin{pmatrix} \mathbb{1}_d \\ B \end{pmatrix}\right).$$

Clearly f is injective. The Plücker coordinates of the image of a matrix B are polynomials in the entries of B, thus f is a morphism of affine algebraic varieties. Moreover, the entries of B can be recovered from the Plücker coordinates of its image, thus f is bijective and its inverse is again a morphism of varieties, hence f is an isomorphism of varieties. For the general case, it suffices to permute rows. ☐

Example. — In the case of $\mathbb{P}^n$, we recover the usual covering by $n+1$ affine spaces.

I.4. Corollary. — *The dimension of* $\mathrm{Gr}(d,n)$ *is* $d(n-d)$.

Proof. Giving a rigorous definition of dimension takes time and we are not going to do it (although the result makes sense intuitively), thus we omit this proof. ☐

Lecture 2
14th Oct, 2022

Now we are ready to study more in detail (and hopefully reduce the number of) the defining relations for the coordinate ring $\mathbb{C}[\mathrm{Gr}(d,n)]$. First, notice the following facts about the Plücker coordinates of a Grassmannian (which we identify with their pullback via Φ):

- they are $\mathbb{C}$-linear forms on $\bigwedge^d \mathbb{C}^n$,
- they are multilinear and alternating in the columns of the matrix $\Phi^{-1}(x) \in M_{n\times d}(\mathbb{C})/\sim$ for any $x \in \mathrm{Gr}(d,n)$,
- the coordinate p_I is alternating in the rows corresponding to I (as it is the determinant of the minor corresponding to I),

thus $f = p_I p_J$ is alternating in the rows given by I and (separately!) in those given by J, but not in all the rows given by $I \cup J$. We want to modify f to a form $\tilde{f}$ which is alternating in the rows given by $I \cup \{j\}$ for some $j \in J \smallsetminus I$: this would give a form which is alternating of degree $d+1$, hence must be zero in $\mathbb{C}[\mathrm{Gr}(d,n)]$, since it has to vanish on d-dimensional vector spaces. This is our strategy to find "good" relations for the coordinate ring.

Example. — For $f = p_{12}p_{34}$, we can guess $\tilde{f}$ to be $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14}$: it is easy to check that $\tilde{f}$ is alternating in the indices $\{1,2,3\}$.

In the previous example we guessed the right modification to f, but of course we want a systematic way of constructing alternations. The next definition takes care of this.

Definition. — Given $\underline{i} = (i_1,\dots,i_d) \in [n]^d$, possibly with non-distinct entries, set

$$p_{\underline{i}} := \begin{cases} \mathrm{sgn}(w) p_I & \text{if } |I| = d, \text{ for any } w \in S_d \text{ such that } i_{w(1)} < \cdots < i_{w(d)} \\ 0 & \text{if } |I| \neq d \end{cases}$$

where $\mathrm{sgn}(w) = (-1)^{l(w)}$. Then write $p_{(\underline{i},\underline{j})} := p_{\underline{i}} p_{\underline{j}}$.

Now, given $\underline{i}, \underline{j} \in [n]^d$ with all entries distinct, set

$$\mathrm{Alt}_t \, p_{\underline{i}} p_{\underline{j}} := \sum_{w \in S_{d+1}/(S_t \times S_{d+1-t})} \mathrm{sgn}(w) \cdot w \cdot p_{(\underline{i},\underline{j})}$$

where $w \cdot p_{(\underline{i},\underline{j})} := p_{(\tilde{\underline{i}},\tilde{\underline{j}})}$ with

$$\tilde{\underline{i}} := (w(i_1),\dots,w(i_t),i_{t+1},\dots,i_d), \;\; \tilde{\underline{j}} := (j_1,\dots,j_{t-1},w(j_t),\dots,w(j_d))$$

where the action of w on $\{i_1,\dots,i_t,j_t,\dots,j_d\}$ is the only reasonable one.

Example. — For $f = p_{12}p_{34}$, we have

We write $p_{(12|34)}$ for $p_{((1,2),(3,4))}$.

$$\mathrm{Alt}_1(p_{12}p_{34}) = \sum_{w \in S_3/(S_1 \times S_2)} \mathrm{sgn}(w) \cdot p_{(12|34)}.$$

Now, as shown in general in RT2Sheet1.2, representatives of cosets of $S_3/(S_1 \times S_2)$ are e, s_1 and $s_2 s_1$, where s_i is the simple transposition $(i, i+1)$. In particular, s_1 acts on $\{1,3,4\}$ by swapping 1 and 3 and s_2 by swapping 3 and 4, thus we get

$$\mathrm{Alt}_1(p_{12}p_{34}) = p_{12}p_{34} - p_{32}p_{14} + p_{42}p_{13} = p_{12}p_{34} + p_{23}p_{14} - p_{24}p_{13},$$

which is what we had guessed before.

I.5. Lemma. — *Let* $1 \leqslant t \leqslant d$ *and* $\underline{i}, \underline{j} \in [n]^d$ *with all entries distinct. Then* $\mathrm{Alt}_t(p_{\underline{i}}\, p_{\underline{j}})$ *vanishes on* $\mathrm{Gr}(d,n)$.

Proof. Consider

$$\Psi : M_{n\times d} \to \bigwedge^d \mathbb{C}^n \xrightarrow{\mathrm{Alt}_t(p_{\underline{i}}\, p_{\underline{j}})} \mathbb{C},$$

where the first map sends a matrix A with column vectors $v_1, \ldots, v_d$ to the wedge $v_1 \wedge \cdots \wedge v_d$. Then Ψ is linear, alternating in the rows $i_1, \ldots, i_t, j_t, \ldots, j_d$. But the columns of A span a vector space of dimension less or equal than d, thus $\Psi = 0$ and hence $\mathrm{Alt}_t(p_{\underline{i}}\, p_{\underline{j}}) = 0$ as an element in $\mathbb{C}[\mathrm{Gr}(d,n)]$. □

So far we have defined $\mathrm{Alt}_t(p_{\underline{i}}\, p_{\underline{j}})$ only for $\underline{i}$ and $\underline{j}$ with all entries distinct, but we need to extend the definition so that this requirement can be dropped.

Example. — For $\underline{i} = (1,2,1,3)$ and $\underline{j} = (4,1,2,5)$, we modify the entries to $\underline{\tilde{i}} = (1,2,6,3)$ and $\underline{\tilde{j}} = (4,11,7,5)$ by adding $5, 10$ and 15 to the repeated entries. Now we can apply the previous definition of Alt_t to $p_{\underline{\tilde{i}}}\, p_{\underline{\tilde{j}}}$ and then change back the modified entries.

The example shows how to proceed. We set

$$\underline{\tilde{i}}_k = i_k + mn, \;\; \underline{\tilde{j}}_k = j_k + m'n$$

where n is the same as in $\mathrm{Gr}(d,n)$ and

$$m = |\{b \mid i_b = i_k,\; b < k\}|,\; m' = |\{b \mid j_b = j_k,\; b < k\}| + |\{b \mid i_b = j_k\}|,$$

one should then convince oneself that this works as intended (i.e. that the resulting indices $\underline{\tilde{i}}_k$ and $\underline{\tilde{j}}_k$ are all distinct).

I.6. Proposition. — *Let* $1 \leqslant t \leqslant d$ *and* $\underline{i}, \underline{j} \in [n]^d$ *without conditions on the entries. Then* $\mathrm{Alt}_t(p_{\underline{i}}\, p_{\underline{j}})$ *vanishes on* $\mathrm{Gr}(d,n)$. *We call the relations obtained in this way generalized Plücker relations.*

Proof. In the case where $\underline{i}$ and $\underline{j}$ have all entries distinct this is Lemma I.5. In the general case, the result is a consequence of the same proof: consider the map

$$\bar{\Psi} : M_{n\times d} \to M_{mn\times d} \to \mathbb{C}$$

where the second map is as Ψ in the previous proof with respect to $\mathrm{Alt}_t(p_{\tilde{\underline{i}}}p_{\tilde{\underline{j}}})$, and the first map is given by

$$B \mapsto \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix}$$

where B is repeated m times, with m depending on the number of multiple entries, as defined above. Then $\bar{\Psi}$ vanishes by the same arguments that proved the vanishing of Ψ, and agrees with $\mathrm{Alt}_t(p_{\underline{i}}p_{\underline{j}})$ considered as a form on $\mathrm{Gr}(d,n)$. $\square$

I.7. Theorem. — *The Grassmannian* $\mathrm{Gr}(d,n)$ *is the zero set of the homogeneous ideal in* $\mathbb{C}[\bigwedge^d \mathbb{C}^n]$ *generated by*

$$\sum_{r=1}^{d+1} (-1)^r p_{(i_1,\ldots,\hat{i}_r,\ldots,i_{d+1})} p_{(j_1,\ldots,j_{d-1},i_r)} \tag{PRel}$$

for any $\underline{i} \in [n]^{d+1}$ *and* $\underline{j} \in [n]^{d-1}$. *These polynomials are called the Plücker relations.*

⚠ I don't quite get the details of this proof!

Proof. The inclusion $\mathrm{Gr}(d,n) \subset V(\text{(PRel)})$ is given by Proposition I.6, since the Plücker relations are a special case of Alt_t, for $t=1$ (as seen in RT2Sheet1.2).

It remains to show the other inclusion. Let $x = \sum_{I \in \mathcal{I}_{d,n}} x_I e_I$ satisfying (PRel). Define $x_{\underline{i}}$ similarly to the definition of $p_{\underline{i}}$ and assume $x_K \neq 0$ for some $K = (k_1 < \cdots < k_d)$, without loss of generality with $x_K = 1$. Define

$$a_{rs} = x_{(k_1 k_2 \cdots k_{s-1} r k_{s+1} \cdots k_d)}.$$

as the entries of a matrix $A \in M_{n\times d}(\mathbb{C})$. By construction, for $1 \leqslant j,t \leqslant d$ we have

$$a_{k_j t} = \begin{cases} 1 & \text{if } j = t \\ 0 & \text{otherwise} \end{cases}$$

thus we have exactly one non-zero entry in each of the rows given by K. Hence $\mathrm{rk}(A) = d$, and the span of the columns of A, call it V, is d-dimensional. To conclude the proof, we just need to show that $Pl(V) = x$. In particular, it is enough to show that

$$p_I(A) = x_I$$

for all $I \in \mathcal{I}_{d,n}$. We check this by induction on $|I \cap K|$.

- For $I = K$, we have $p_I(A) = 1 = x_K = x_I$ by definition of A and assumption.
- For $|I \cap K| = d-1$, let $r \in I \setminus K$ and $k_s \in K \setminus I$. Now,

$$p_I(A) = \pm a_{rs} = \pm x_{(k_1,\ldots,k_{s-1},r,k_{s+1},\ldots,k_d)} = x_I.$$

- For $|I \cap K| < d-1$, there is $r \in I \setminus K$.

Without loss of generality assume $r = i_d$. By assumption and (PRel),

$$x_I x_K = \sum \pm x_{I'} x_{K'},$$

where I' and K' differ from I and respectively K in exactly one entry. If $x_{I'}x_{K'} = 0$, then $|K' \cap I| > |K \cap I|$, so by induction

$$x_{K'} = p_{K'}(A)$$

and $x_{I'} = p_{I'}(A)$. By Proposition I.6, the d-minors of A satisfy (PRel) as well. Thus

$$x_K(A)x_I(A) = \sum \pm p_{K'}(A)p_{I'}(A)$$

and $x_K(A) = 1$, hence $x_I(A) = p_I(A)$. □

Remark. — In general the Plücker relations are still not a minimal set of relations for the coordinate ring of the Grassmannian! Finding minimal sets of generators for homogeneous ideals is a non-trivial problem.

You have to use Gröbner basis and slightly evil stuff like that, I guess?

Remark. — Standard monomial theory gives a homogeneous basis of the coordinate ring of $\mathrm{Gr}(d, n)$ (or other varieties) in terms of monomials in the Plücker coordinates (respectively, generalizations of these). For $\mathrm{Gr}(d, n)$, there is a bijection between the semi-standard tableaux of shape $m^d = (m, \ldots, m)$ with fillings from $1, \ldots, n$ and the degree m elements in such a monomial basis of $\mathbb{C}[\mathrm{Gr}(d, n)]$, sending a semi-standard tableau to the product of the Plücker coordinates specified by the columns. Recall that a semi-standard tableau is a Young tableau filled with numbers weakly increasing to the right and strictly increasing downwards. Note that by convention there is one semi-standard tableau of shape $\varnothing$. For instance for $\mathrm{Gr}(2, 4)$ the tableau

1	1
2	3

corresponds to the monomial $p_{12}p_{13}$.

Remark. — The number of semi-standard tableaux of shape $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with the number i occurring μ_i times is called the *Kostka number* $K_{\lambda,\mu}$.

Schubert Cells

Lecture 3
19th Oct, 2022

Observe that $G = \mathrm{GL}_n(\mathbb{C})$ acts transitively on $X = \mathrm{Gr}(d, n)$, via the action on $\mathbb{C}^n$. Let $T \subset B \subset G$ be the (*standard*) *torus* of diagonal matrices and the (*standard*) *Borel* of upper triangular matrices. Denote by $U \subset B$ the group of upper triangular matrices with 1 on the diagonal, which is called the *unipotent radical*. We have $U \cong \mathbb{A}^{n(n-1)/2}$. Notice that T fixes the *coordinate subspaces* $V_I = \langle e_i \mid i \in I\rangle$ for $I \in \mathcal{I}_{d,n}$. We want to use this fact to construct a *cell decomposition* of the Grassmannian $X = \cup_{I\in\mathcal{I}_{d,n}} C_I$, where the C_I are disjoint affine spaces such that each contains exactly one of the V_I, called *Schubert cells*.

Note that this Schubert cell business mostly depends only on the root system of the group G and thus generalizes to other algebraic groups.

I.8. Lemma. — *The following holds.*

(1) *Let X^T be set of T-fixed points in X, then*

$$X^T = \{V_I \subset \mathbb{C}^n \mid I \in \mathcal{I}_{d,n}\}.$$

(2) *There is a bijection between $S_n/(S_d \times S_{n-d})$ and X^T defined by*

$$[\sigma] \mapsto V_{\sigma\{1,\ldots,d\}}.$$

Proof. (1) Clearly "$\supset$" holds. As for "$\subset$", we can show that every $V \in X^T$ is generated by some of the e_i's. In fact, take any vector $v = \sum_{i=1}^n a_i e_i \in V$ with some $a_{i_0} \neq 0$ and consider the diagonal matrix t with entry -1 in the i_0-th spot and 1 in the others. Then we have $2a_{i_0} e_{i_0} = v - t.v \in V$, and our claim follows.

(2) This is clear, as S_n acts transitively on the set of coordinate subspaces and $V_{\{1,\dots,d\}}$ has stabilizer $S_d \times S_{n-d}$. □

I.9. Proposition. — *Let $C_I = UV_I$. There is a cell decomposition*

$$\mathrm{Gr}(d,n) = \bigcup_{I \in \mathcal{I}_{d,n}} C_I$$

where the C_I's are called the Schubert cells. The Schubert cells satisfy:

- *$C_I \cong \mathbb{A}^{d_I}$, where $d_I = \sum_{k=1}^d (i_k - k)$ for $I = \{i_1, \dots, i_d\}$,*
- *the C_I's are the B-orbits in $\mathrm{Gr}(d,n)$,*
- *each C_I contains exactly one T-fixed point.*

Proof. First we show that $\cup_{I \in \mathcal{I}_{d,n}} C_I = \mathrm{Gr}(d,n)$. Consider $V \in \mathrm{Gr}(d,n)$ and choose a basis $\{v_1, \dots, v_d\}$ such that $v_k = e_{i_k} + \sum_{j<i_k} \beta_j e_j$. Then $V \in UV_I$ for $I = \{i_1, \dots, i_d\}$.

Thinking in terms of matrices is always helpful in these proofs.

Now we want to show that the C_I are disjoint and in doing so we will prove the remaining assertions. Let U_I be the stabilizer of V_I in U. Explicitly we have:

$$U_I = \{A \in U \mid a_{ij} = 0 \;\; \forall j \in I, i \notin I\}.$$

Let also

$$U^I = \{A \in U \mid a_{ij} = 0 \;\; \forall i, j (i \in I \vee j \notin I)\},$$

i.e. the complement of U_I in U. We have that $U = U^I U_I$ as groups (one should check). Now, the map $U^I \to \mathrm{Gr}(d,n), \;\; g \mapsto gV_I$ is an embedding with image

$$U^I V_I = U^I U_I V_I = UV_I = C_I,$$

hence $C_I \cong U^I \cong \mathbb{A}^{d_I}$, where $d_I = \dim U^I$, and this will be equal to $\sum_{k=1}^d (i_k - k)$, since for a matrix $(a_{ij}) \in U^I$ we must have $a_{ij} = 0$ for $k \notin I$ and $k \in I$ gives $i_k - k$ possible non-zero entries. Moreover, by the previous Lemma (I.8), we have that $BV_I = UTV_I = UV_I = C_I$ is a B-orbit, and different V_I lie in different orbits. These two observations show that the C_I are disjoint. □

I.10. Remark. — We can describe the points of C_I for $I = (i_1 < \dots < i_d) \in \mathcal{I}_{d,n}$ as matrices constructed as follows:

This is much easier to draw than it is to describe.

- put 1 in column k, row i_k,
- put 0's below and to the right of the 1's,
- the remaining entries are arbitrary (put $*$ on them).

The T fixed point in C_I is represented by the matrix with all arbitrary entries set to 0 (this represents the coordinate subspace V_I). Conversely, we can recover I from the matrices, by observing that i_k is the row with the last non-zero entry in the k-th column.

Remark. — Observe that $\dim C_I = l(\sigma)$, where σ is the unique shortest coset representative in $S_n/(S_d \times S_{n-d})$ with $\sigma(1,\dots,n) = I$, as shown in RT2Sheet2.1.

Remark (Combinatorics)**.** — There are bijections between the coordinate subspaces and partitions $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-d} \geqslant 0$ with $\lambda_i \leqslant d$, i.e. partitions fitting into a rectangle with $n-d$ rows and d columns, as we now describe.

- Given I, we draw a path in a $(n-d)\times d$ grid starting from the top left corner. In the i-th step, if $i \in I$ we go right and if $i \notin I$ we go down. Then the (mirrored) Young diagram (i.e. the partition) above the path determines the shape of the points of C_I using the description we gave in Remark I.10: if we take the matrices representing points in C_I and delete the d rows not containing any $*$'s, then the $*$'s in the remaining $(n-d)\times d$ matrix form precisely the partition constructed above. In particular, the number of boxes in the Young diagram determined by the partition equals the dimension of C_I.

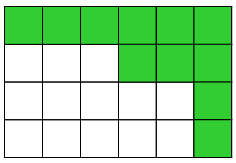
- Another bijection is given by sending V_I for $I = \{i_1 \dots, i_d\}$ to the partition λ_I with $(\lambda_I^t)_k = n - d - i_k + k$. This partition can be obtained by the same procedure as above, but taking the boxes below the path. Then the number of boxes of λ_I is the codimension of C_I.

Clearly, the partitions constructed in the two ways just described will determine complementary shapes in the $(n-d)\times d$ grid.

Example. — It is easy and quite illuminating to write down, using the notation described in Remark I.10, the Schubert cells of $\mathrm{Gr}(2,4)$, and associate them with corresponding Young diagrams and shortest coset representatives.

I won't judge you if you don't ever find the time, though.

Example. — Take $d = 6$, $n = 10$ and $I = \{2,3,4,6,7,10\}$. The path and partitions constructed from I determine the Schubert cell as follows:



$$\begin{pmatrix} * & * & * & * & * & * \\ 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & * & * & * \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & * \\ & & & & & * \\ & & & & & 1 \end{pmatrix}$$

Thus we associate C_I with the partition $(4,2,2,1,1,1)$.

We can say something more about Schubert cells, before moving on.

Definition. — The *standard flag* in $\mathbb{C}^n$ is $F^{\mathrm{st}} = (F_0^{\mathrm{st}} \subset F_1^{\mathrm{st}} \subset \cdots \subset F_n^{\mathrm{st}})$, where

$$F_r^{\mathrm{st}} = \langle e_i \mid 1 \leqslant i \leqslant r \rangle.$$

I.11. Proposition (Relative position)**.** — *Let $I \in \mathcal{I}_{d,n}$. Then*

$$C_I = \left\{ V \in \mathrm{Gr}(d,n) \;\middle|\; \begin{array}{l} \dim(V \cap F_{i_r}^{\mathrm{st}}) = r \ \forall r \leqslant d, \\ \dim(V \cap F_j^{\mathrm{st}}) < r \ \forall j < i_r \end{array} \right\}.$$

Proof. RT2Sheet2.2. $\square$

Schubert Varieties

Considering Schubert cells in projective space leads to the following definition:

Definition. — For $I \in \mathcal{I}_{d,n}$, the *Schubert variety* Ω_I is the Zariski closure of the Schubert cell C_I.

We want now to understand better the Ω_I.

Definition. — The *Bruhat order* on $\mathcal{I}_{d,n}$ is defined by

$$I \leqslant J \iff i_r \leqslant j_r \text{ for all } 1 \leqslant r \leqslant d.$$

This defines a partial order on $\mathcal{I}_{d,n}$.

I.12. Proposition. — *For $I \in \mathcal{I}_{d,n}$ we have:*

(1) $\Omega_I = \bigcup_{J \leqslant I} C_J$,

(2) $\Omega_I = \{V \in \mathrm{Gr}(d,n) \mid \dim(V \cap F^{\mathrm{st}}_{i_r}) \geqslant r \ \forall 1 \leqslant r \leqslant d\}$.

Proof. RT2Sheet2.4. $\square$

I.13. Lemma. — *We have*

$$\Omega_I = \{V \in \mathrm{Gr}(d,n) \mid p_J(V) = 0 \ \forall J \nleqslant I\}.$$

In particular, Ω_I is the intersection of finitely many hyperplanes with X (hence a projective variety).

Proof. ($\subset$) Let $V \in \Omega_I$. Then by Proposition I.12 there is some $L \leqslant I$ such that $V \in C_L$. Let $\{v_k = e_{l_k} + \sum_{s<l_k} \beta_s e_s\}$ be a basis of V, i.e. $v_k \equiv e_{l_k} \mod \langle e_s \mid s < l_k\rangle$. Then

$$v_1 \wedge \cdots \wedge v_d \equiv e_L \mod \langle e_K \mid K \leqslant I\rangle,$$

therefore $p_J(V) = 0$ for all $J \nleqslant I$.

($\supset$) Consider $V \in \mathrm{Gr}(d,n)$ such that $p_J(V) = 0$ for all $J \nleqslant I$. We know that V must be in C_H for some H, hence $p_H(V) \neq 0$ and thus $H \leqslant I$. By Proposition I.12 then $V \in C_H \subset \Omega_I$. $\square$

(Partial) Flag Varieties

An object of fundamental interesting in many parts of mathematics (and of course especially in representation theory) are flag varieties:

Definition. — Let $\underline{d} = (d_1, \dots, d_l) \in \mathbb{N}^l$ with $0 \leqslant d_1 < \cdots < d_l = n$. Set $d_0 = 0$ and $c_i = d_i - d_{i-1}$ for $1 \leqslant i \leqslant l$. The (*partial*) *flag variety* of flags in $\mathbb{C}^n$ with dimension vector $\underline{d}$ is the set

$$\mathrm{Fl}_{\underline{d}} = \{(\{0\} \subset F_{d_1} \subset \cdots \subset F_{d_{l-1}} \subset F_{d_l} = \mathbb{C}^n) \mid F_{d_i} = d_i\}.$$

The elements of a flag variety are called (*partial*) *flags*. The c_i are called *dimension jumps* and l the *length* of the flag.

Examples. — For $l = 2$ we recover the Grassmannians, since $\mathrm{Fl}_{(d_1,d_2)} = \mathrm{Gr}(d_1, d_2)$. If $c_i = 1$ for all i we get the *(full) flag variety*, the set of all the *full flags*.

I.14. Lemma. — *We have*

$$\mathrm{Fl}_{\underline{d}} = \{(F_1, \dots, F_l) \in \mathrm{Gr}(d_1, n) \times \cdots \times \mathrm{Gr}(d_l, n) \mid F_i \subset F_{i+1} \ \forall\, 1 \leqslant i \leqslant l-1\}.$$

In particular, flag varieties are projective varieties.

Proof. The equality is clear. As for the statement that flag varieties are projective varieties, note that products of projective varieties are projective varieties, and that we can encode the condition on the inclusions of the F_i as polynomial equations in the Plücker coordinates. $\square$

A more conceptual point of view on flag varieties is provided by homogeneous spaces. Let $\underline{d}$ as above and consider $B \subset P_{\underline{d}} \subset G = \mathrm{GL}_n(\mathbb{C})$, where $P_{\underline{d}}$ is the *standard parabolic subgroup* of type $\underline{d}$, i.e. $P_{\underline{d}}$ consists of upper block matrices of the form

$$\begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_l \end{pmatrix}$$

with the $A_i \in \mathrm{GL}_{c_i}(\mathbb{C})$. The subgroup $L_{\underline{d}}$ of $P_{\underline{d}}$ consisting of block diagonal matrices

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_l \end{pmatrix}$$

is called the *Levi subgroup* corresponding to $P_{\underline{d}}$. Note that $L_{\underline{d}} \cong \mathrm{GL}_{c_1} \times \cdots \times \mathrm{GL}_{c_l}$.

Remark. — More generally, the *parabolic subgroups* of G are by definition the conjugates of the standard parabolic subgroups $P_{\underline{d}}$. The *Borel subgroups* are conjugates of B.

Remark. — Passing to the Lie algebra $\mathfrak{gl}_n$, we get corresponding Lie subalgebras. Choosing a Borel subalgebra corresponds to choosing a basis of the roots of $\mathfrak{gl}_n$ and choosing a parabolic subalgebra corresponds to choosing a subset of simple roots.

Observe that there is a bijection $\mathrm{GL}_n / P_{\underline{d}} \to \mathrm{Fl}_{\underline{d}}$, $g \mapsto g.F^{\mathrm{st}}_{\underline{d}}$, where $F^{\mathrm{st}}_{\underline{d}}$ is the *standard partial flag* $(F^{\mathrm{st}}_{d_1} \subset \cdots \subset F^{\mathrm{st}}_{d_l})$, since G acts transitively on $\mathrm{Fl}_{\underline{d}}$ and the stabilizer of $F^{\mathrm{st}}_{\underline{d}}$ is precisely $P_{\underline{d}}$. In particular, one can prove:

I.15. Theorem. — $\mathrm{Fl}_{\underline{d}}$ *is isomorphic as a projective variety to the quotient of affine algebraic groups* $\mathrm{GL}_n / P_{\underline{d}}$.

To be able to prove this theorem, and to learn how to work with affine algebraic groups at all, we will now quickly review the general theory of (classical) algebraic varieties.

Chapter II.

(Classical) Algebraic Geometry and (Affine) Algebraic Groups

II

A Refresher on Algebraic Varieties

Let $(X, k[X])$ be an affine algebraic variety over an arbitrary algebraically closed field $k = \overline{k}$. By this we mean that X is a set and $k[X] \subset \mathrm{Maps}(X, k)$ is a finitely generated subalgebra such that there is a bijection $X \to \mathrm{Hom}_{k-\mathrm{Alg}}(k[X], k)$, $x \mapsto \mathrm{ev}_x$. Then it is a fundamental fact of life that:

This already looks a lot like schemes...

II.1. Theorem. — *$(X, k[X]) \cong (V(M), k[X_1, \dots, X_n]/\mathcal{I}(V(M)))$ for some $n \in \mathbb{N}$ and a set of polynomials $M \subset k[X_1, \dots, X_n]$. In particular X becomes a topological space with the Zariski topology.*

We are not defining everything, assuming that this is at least vaguely familiar to all. Maybe let's say some more: the Nullstellensatz tells us that closed subsets of $(X, k[X])$ correspond to radical ideals of its coordinate ring $k[X]$. Also, there is a bijection between affine varieties over a field $k = \bar{k}$ and *reduced* finitely generated k-algebras.

Definition. — Let $x \in U \subset X$, U open. Then $f : U \to k$ is called *regular at x_0* if there are an open $V \subset U \subset X$ containing x_0 and $g, h \in k[X]$ such that $h(x) \neq 0$ and $f(x) = g(x)/h(x)$ for all $x \in V$.

Definition. — For an open $U \subset X$, let

$$\mathcal{O}_X(U) = \{f : U \to k \mid f \text{ regular at all } x \in U\}$$

be the *ring of regular functions on U*.

II.2. Proposition. — $\mathcal{O}_X(X) = k[X]$.

Proof. The "$\supset$" inclusion is clear. As for "$\subset$", let $f \in \mathcal{O}_X(X)$, $x \in X$. Then we have $f = g_x/h_x$ locally around x (i.e. there is some open $U_x \subset X$ containing x such that h_x is nonvanishing and $f(y) = g_x(y)/h_x(y)$ for all $y \in U_x$). Without loss of generality we can assume that $U_x = U_{h_x} = U_{h_x^2}$ is a distinguished open subset (the complement of a vanishing locus, i.e. a subset of the form $X \setminus V(h_x)$). Note that $fh_x^2 = g_x h_x$ on all of X since h_x vanishes outside of $U_{h_x^2}$. Consider the ideal $I \lhd k[X]$ generated by all the h_x^2 for all $x \in X$. By Hilbert's Nullstellensatz we have $1 = \sum_{i=1}^r a_i h_{x_i}^2$ for some $a_i \in k[X]$ and $x_1, \dots, x_r \in X$, thus

$$f = f \cdot 1 = \sum_{i=1}^r a_i f h_{x_i}^2 = \sum_{i=1}^r a_i g_{x_i} h_{x_i} \in k[X],$$

which concludes our proof. □

Remark. — If X is irreducible (any non-empty open subset is dense), all the calculations in the previous proof just take place in $\mathrm{Frac}(k[X])$.

Lecture 4
21st Oct, 2022

Definition. — Let $(X, k[X])$ be an affine algebraic variety. The assignment $U \mapsto \mathcal{O}_X(U)$ defines a *sheaf of rings* (*or k-algebras*) *on X*. A sheaf is a *presheaf*, i.e. a functor from the opposite category of open sets of X ordered by inclusion to **Ring** (or $k-$ **Alg**), which is a fancy way to say that a sheaf associates *sections* (elements of a ring, such as the ring of regular functions) to open subsets and prescribes a way to restrict sections on an open U to smaller open subsets contained in U. In addition, a sheaf satisfies two more conditions:

One could synthesize the two conditions into just one by requiring that gluing of sections be unique, which leads quite naturally to restating the sheaf properties categorically, in terms of equalizers (this always looks like pointless mumbo jumbo at first but it's actually very handy).

(S1) Locality. If $s, t \in \mathcal{F}(U)$ are such that
$$s|_{U\alpha} = t|_{U\alpha}$$
for all open subsets $U_\alpha \subset U$, then $s = t$.

(S2) Gluing. For every open subset $U \subset X$ and any open covering $\{U_\alpha\}_{\alpha\in A}$ of U, and each family of sections $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha\in A}$ such that
$$s_\alpha|_{U_\alpha\cap U\beta} = s_\beta|_{U_\alpha\cap U\beta}$$
for all $\alpha, \beta \in A$, there exists a $s \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all $\alpha \in A$.

Definition. — A *ringed space* (over a field k) is a pair $(X, \mathcal{O}_X)$, where

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of k-algebras.

The sheaf $\mathcal{O}_X$ is called the *structure sheaf*, the ring $\mathcal{O}_X(U)$ for any open $U \subset X$ is called the *ring of regular functions on U*.

Definition. — A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for all open $U \subset X$ the map $f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, $g \mapsto g \circ f$ is a ring morphism.

In particular, the image has to actually land in $\mathcal{O}_X(f^{-1}(U))$

One easily sees that there is an obvious way to obtain a ringed space $(Y, \mathcal{O}_X|_Y)$ on a subset $Y \subset X$ from a ringed space on X, by restriction.

Definition. — A *prevariety* (over k) is a ringed space $(X, \mathcal{O}_X)$ such that:

- X is a Noetherian topological space,
- there exists an open cover $X = \bigcup_{i\in I} U_i$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is a ringed space isomorphic to an affine algebraic variety.

Example. — Any affine algebraic variety over a field is a prevariety. Note that affine algebraic varieties are Noetherian by the Nullstellensatz and the fact that $k[X_1, \dots, X_n]$ is a Noetherian ring.

Example. — Complex projective space $(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}})$, where we define $f : U \to k$ to be regular at x if $f = g/h$ locally around x for g, h homogeneous polynomials with $\deg g = \deg h$ (and h nonvanishing), is a ringed space and prevariety. To see this, let $\bigcup_{i=0}^n U_i$ be the standard open covering. We claim that $\mathcal{O}_X(U) \cong \mathcal{O}_{\mathbb{C}^n}(U)$ for all $U \subset U_i$, with the isomorphism given by $f \mapsto \bar{f}$ with
$$\bar{f}(y_1, \dots, y_n) = f([y_1 : \cdots : y_{i-1} : 1 : y_{i+1} : \cdots : y_n]).$$
Without loss of generality say $i = 0$. This is clearly a well-defined injective morphism of algebras. As for surjectivity, given any $\varphi = g/h \in \mathcal{O}_{\mathbb{C}^n}(U)$, a preimage is given by the element $\tilde{g}/\tilde{h}$, where $\tilde{g}$ and $\tilde{h}$ are the homogenizations of g and h. Finally, a finite union of Noetherian spaces is Noetherian.

Example. — The Grassmannians $\mathrm{Gr}(d,n)$ with induced structure sheaf from projective space via the Plücker embedding are prevarieties.

Remark. — If $X = \mathbb{P}^n_{\mathbb{C}}$ (or any irreducible projective variety over $\mathbb{C}$), then $\mathcal{O}_X(X) = \mathbb{C}$. In particular, $\mathbb{P}^n_{\mathbb{C}}$ is not an affine algebraic variety.

Products of prevarieties exist, extending products of affine algebraic varieties:

II.3. Proposition. — *The category of prevarieties has finite products.*

Sketch of proof.. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be prevarieties with corresponding open coverings $X = \bigcup_i U_i$ and $Y = \bigcup_j V_j$. It is not difficult to see that the products $U_i \times V_j$ are isomorphic to an affine algebraic variety with $k[U_i \times V_j] = k[U_i] \otimes k[V_j]$, see [Spr83, Section 1.5, Products]. Define $(X \times Y, \mathcal{O}_{X\times Y})$ by taking the set-theoretic product of X and Y with topology induced by declaring the affine algebraic varieties $U_i \times V_j$ to be an open covering, and regular functions as those which are regular in some $U_i \times V_j$. One can then prove that we have defined a product in the category of prevarieties. More details (but really nothing much more profound than this) can be found in [Spr83, Proposition 1.6.3]. □

We will call the topology on the product prevariety the Zariski topology on $X \times Y$.

Definition. — A prevariety $(X, \mathcal{O}_X)$ is a *variety* if is *separated*, i.e. if the diagonal

$$\Delta = \{(x,x) \in X \times X \mid x \in X\}$$

is (Zariski) closed in $X \times X$.

Remark. — Although usually somewhat pathological and not very interesting, there are prevarieties which are not varieties. The motivation behind the definition of separatedness is the following: recall that a topological space X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$ *with the product topology.* The Zariski topology on $X \times X$ is not the product topology (most varieties are not Hausdorff anyway), but having the diagonal be closed in the Zariski topology still turns out to rule out unpleasant pathologies.

Apropos this topic: affine space surely is *not* pathological, but the Zariski topology on $\mathbb{A}^2$ is not Hausdorff, hence the diagonal is not closed in the product topology (this latter is too coarse!). The point is that the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ is closed in the Zariski topology on $\mathbb{A}^2$.

Remark. — The product of two varieties (taken in the category of prevarieties) is a variety. In particular, products exist in the category of varieties.

II.4. Remark. — It can be easily seen that the separation axiom implies that the graph of a morphism $f : Y \to X$ of prevarieties is closed in case X is a variety (it is the preimage of the diagonal under the morphism $f \times \mathrm{id} : Y \times X \to X \times X$). Moreover for morphisms of prevarieties $\varphi, \psi : Y \to X$ with X a variety, if $\varphi = \psi$ on a dense open set, then $\varphi = \psi$ (use that the set where two morphisms agree is the inverse image under $\varphi \times \psi$ of the diagonal, so it is closed.). Both of these (very reasonable) properties fail for X a general prevariety.

Remark (Affine criterion)**.** — Let $(X, \mathcal{O}_X)$ be a prevariety and $X = \bigcup_i U_i$ an open covering by affine sets. Then $(X, \mathcal{O}_X)$ is a variety if and only if $U_i \cap U_j$ is an affine open for all i, j and the image of the maps $\mathrm{incl}^* : \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_i \cap U_j)$ generate $\mathcal{O}_X(U_i \cap U_j)$ as an algebra. Thus regular functions are restrictions of regular functions in $k[U \times V] \cong k[U] \otimes k[V]$. The latter is generated by $k[U] \otimes 1$ and $1 \otimes k[V]$.

The main idea to show the necessity of the affine criterion is that $U \cap V$ equals $\Delta_X \cap (U \times V)$ for any U, V open, see [Spr83, 1.6.12].

II.5. Lemma. — *Affine algebraic varieties are varieties.*

Proof. This is easy, as

$$\begin{aligned}\Delta &= \{(x,y) \in X \times X \mid \mathrm{ev}_x = \mathrm{ev}_y\} \\ &= \{(x,y) \in X \times X \mid f(x) = f(y)\ \forall f \in k[X]\} \\ &= V(\{f \otimes 1 - 1 \otimes f \mid f \in k[X]\}),\end{aligned}$$

where we identify $k[X \times X] = k[X] \otimes k[X]$. □

Example (Affine line with double origin)**.** — The easiest (and in a sense which could be made precise, prototypical) example of a non-separated prevariety is given by the prevariety which is constructed by gluing two affine lines along the identity of the open sets $\mathbb{A}^1 \smallsetminus \{0\}$, thus identifying everything but the two origins, which remain distinct. Then it is easy to see that such a prevariety, call it X, is not separated (i.e. not a variety): one can take the obvious morphisms $f_1, f_2 : \mathbb{A}^1 \to X$ to each of the two copies of $\mathbb{A}^1$ contained in X and see that the locus where they agree is not closed (it is $\mathbb{A}^1 \smallsetminus \{0\}$), contradicting what we proved in Remark II.4 about varieties.

How to glue prevarieties is kinda obvious, but the formalities of it can be confusing at first. For more details, this is [Spr83, 1.6.13].

From now on, we will write just X for a variety, omitting the structure sheaf $\mathcal{O}_X$ (but please, remember it is there!).

Affine Algebraic Groups

It is interesting to consider group objects in the category of varieties.

Definition. — An *affine algebraic group* is an affine algebraic variety $(G, k[G])$ together with morphisms of algebraic varieties $\mu : G \times G \to G$ and $\iota : G \to G$ turning G into a group.

It is funny to notice that for formal reasons (the Zariski topology on the product being different from the product topology), an algebraic group need not be a topological group.

Examples. — Important examples of affine algebraic groups are:

(1) The general linear group $(\mathrm{GL}_n(\mathbb{C}), \mathbb{C}[X_{11}, \dots, X_{nn}, \det^{-1}])$ is an affine algebraic group with the usual matrix multiplication and inversion, and with $\mathrm{GL}_n(\mathbb{C})$ seen as a Zariski closed subset of $\mathbb{A}^{n^2+1}$.

(2) Affine space is an affine algebraic group with addition of vectors.

II.6. Lemma. — *(Zariski) closed subgroups of affine algebraic groups are again affine algebraic groups.*

Proof. Let H be a closed subgroup of an affine algebraic group (G, μ, ι). We want to show that the induced (by restriction and corestriction) map of sets $\bar{\mu} : H \times H \to H$ is a morphism of varieties. Note that $H \times H$ is closed in the Zariski topology whenever H is, so the only thing that could go wrong is that we fail to have a diagram

Not a very slick proof this one, not gonna lie.

$$\begin{array}{ccc} k[H \times H] & \xleftarrow{\bar{\mu}^*} & k[H] \\ \uparrow & & \uparrow \\ k[G \times G] & \xleftarrow{\mu^*} & k[G] \end{array}$$

because $\bar{\mu}^*$ does not have image in $k[H \times H]$. To see that this is not the case, consider the general setup: let $\varphi : Z \to X$ be a morphism of varieties and $Y \subset X$ closed with $\varphi(Z) \subset Y$. Define $\bar{\varphi} : Z \to Y$ by corestriction of φ. Then we get a commutative diagram

$$\begin{tikzcd} k[X] \arrow[r, "\varphi^*"] \arrow[d, two heads] & k[Z] \\ k[Y] \arrow[ur, "\bar{\varphi}^*"'] & \end{tikzcd}$$

and in particular $\operatorname{im} \bar{\varphi}^* \subset \operatorname{im} \varphi^* \subset k[Z]$. In particular, $\bar{\mu}^*$ has image in $k[H \times H]$ and thus $\bar{\mu}$ is a morphism. The argument for the inverse map is the same. □

Examples. — Some examples of affine algebraic subgroups are $\mathbb{C}^\times$ and

$$T \subset B \subset P_{\underline{d}} \subset G = \mathrm{GL}_n(\mathbb{C}).$$

Remark. — If $(G, k[G])$ is an affine algebraic group, then $k[G]$ together with the maps induced by μ, the unit and ι is a Hopf algebra.

II.7. Corollary. — *Linear algebraic groups (i.e. closed subgroups of some* $\mathrm{GL}_n(k), k = \bar{k}$*) are affine algebraic groups.*

Definition. — Let G be an affine algebraic group. A *G-variety* is a variety X together with a morphism of (pre)varieties $G \times X \to X,\ (g, x) \mapsto g.x$ such that

(1) $e.x = x$ for all $x \in X$,

(2) $g.(h.x) = (gh).x$ for all $g, h \in G$, $x \in X$.

It is often useful to notice that for a fixed g the map $X \to X$, $x \mapsto g.x$ is an isomorphism, and in particular a homeomorphism. The same is true of translation in G by an element g and the inverse map ι.

We write $G \curvearrowright X$ if X is a G-variety, and we say that G *acts on* X. Note that morphisms between G-varieties are usually taken to be G-equivariant.

Lecture 5
26th Oct, 2022

Examples. — We give some examples of group actions.

(1) An affine algebraic group G acts on itself in at least three ways:
- by left multiplication $g.h = gh$ (*left regular action*),
- by right multiplication $g.h = hg^{-1}$ (*right regular action*),
- by conjugation $g.h = ghg^{-1}$ (*conjugation action*).

(2) $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n$ as usual.

(3) The previous action induces an action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{CP}^{n-1}$ (and more generally on the Grassmannians $\mathrm{Gr}(d, n)$ for all $0 \leqslant d \leqslant n$).

II.8. Lemma. — *Let G be an affine algebraic group acting on a variety X, $H \subset G$. Then the set of H-fixed points*

$$X^H := \{x \in X \mid h.x = x \ \forall h \in H\}$$

is closed in X.

Proof. We have $X^H = \cap_{h \in H} X^h$, where $X^h := \{x \in X \mid h.x = x\}$. Then it is enough to show that X^h is closed in X. Consider the morphism $\varphi : X \to X \times X,\ x \mapsto (x, h.x)$. We have $X^h = \varphi^{-1}(\Delta)$ and the diagonal Δ is closed since X is a variety, hence X^h is closed. □

Facts. — Here are some useful things to know about algebraic groups.

- Given any morphism of affine algebraic groups φ, both $\operatorname{im}\varphi$ and $\ker\varphi$ are closed subgroups, see [Hum75, Proposition 7.4B].
- Closures of G-orbits are unions of G-orbits, see [Do it as an exercise].
- In general one has to be careful because orbits may not be closed! It is possible though to prove that closed orbits always exist (using that if $f : X \to Y$ is a morphism of varieties, then $f(X)$ contains an open subset $U \in \overline{f(X)}$, see [Spr83, Theorem 1.9.5]), which we do in the following lemma.

II.9. Lemma. — *Let $G \curvearrowright X$ and $x_0 \in X$. Then $G.x_0$ is an open (dense) subset of its closure and there exist some $x \in G$ with $G.x$ closed.*

Proof. First we prove that the orbit $G.x_0$ is open in its closure $\overline{G.x_0}$. By the theorem we mentioned in the previous paragraph the image of the morphism $G \to X,\ g \mapsto g.x_0$ contains an open subset $U \subset \overline{G.x_0}$. But then

$$G.x_0 = \bigcup_{g\in G} g.U$$

is open in $\overline{G.x_0}$, since the map $X \to X,\ x \mapsto g.x$ for any fixed $g \in G$ is a homeomorphism. We are left to prove that there exist closed orbits. Set $S_x \coloneqq \overline{G.x} \smallsetminus G.x$ for any $x \in X$. Pick x with S_x minimal (we can do this since X is Noetherian).

- Case 1: $S_x = \varnothing$. This means that $\overline{G.x} = G.x$, i.e. the orbit of x is closed.
- Case 2: $S_x \neq \varnothing$. We know that S_x is a union of G-orbits, so there is some $y \in X$ such that $G.y \neq G.x$ and $G.y \subset S_x$. But this means S_y is strictly contained in S_x, which contradicts the minimality of S_x.

Therefore closed orbits exist. □

Definition. — A G-variety X is called a *homogeneous G-space* if the action of G is transitive.

Examples. — We give some examples of homogeneous spaces and one non-example.

- The action of $G = \mathrm{GL}(\mathbb{C})$ on $\mathrm{Gr}(d, n)$ and $\mathrm{Fl}_{\underline{d}}$ makes them homogeneous G-spaces. As a special case $\mathbb{P}^1_{\mathbb{C}}$ is a homogeneous $\mathrm{GL}_2(\mathbb{C})$-space.
- The action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ defined by $g.(x, y) = (gx, gy)$ is *not* homogeneous. Remember that the $\mathrm{GL}_2(\mathbb{C})$-orbits on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ are in bijections with the B-orbits on $\mathbb{P}^1_{\mathbb{C}}$ via the map $G.(h.[1 : 0], g.[1 : 0]) \mapsto B.h^{-1}g.[1 : 0]$ for $h, g \in \mathrm{GL}_2(\mathbb{C})$ which is a well-defined bijection on orbits. Similar ideas work for the general case of $\mathrm{GL}_n(\mathbb{C})$ acting on $\mathbb{P}^{n-1}_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}}$.

There are two B-orbits on $\mathbb{P}^1_{\mathbb{C}}$ corresponding to the two T-fixed points.

Definition. — Let G be an affine algebraic group (over $\mathbb{C}$). A *rational representation* of G is a morphism $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ of algebraic groups for some $n \in \mathbb{N}$.

Affine Algebraic Groups Are Linear

For the rest of the lecture, our aim is to show that affine algebraic groups are linear (i.e. isomorphic to closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ for some n).

Of course, by Lemma II.6 linear groups are affine, so affine and linear algebraic groups are the same.

Assume that G is an affine algebraic group, X an affine variety and $G \curvearrowright X$. Then $a : G \times X \to X$, $(g, x) \mapsto g.x$ induces a morphism of algebras

$$a^* : k[X] \to k[G \times X] \cong k[G] \otimes k[X],\ f \mapsto f \circ a.$$

So if G and X are affine any action is algebraic? It seems so by this remark, but I'm not sure what she was trying to say here...

Note that this is a morphism for any action a, as it is clear that a^* is linear and

$$\begin{aligned} a^*(f_1 f_2)(g, x) &= (f_1 f_2)(g.x) = f_1(g.x) f_2(g.x) \\ &= (a^*(f_1)(g, x))(a^*(f_2)(g, x)) = (a^*(f_1) a^*(f_2))(g, x) \end{aligned}$$

for all $f_1, f_2 \in k[X]$, $g \in G$, $x \in X$.

We have linear maps τ_g for $g \in G$

$$\tau_g : k[X] \to k[X],\ f \mapsto g.f$$

Careful about this! Without the -1 the maps τ_g would not be well defined.

where $g.f(x) \coloneqq f(g^{-1}.x)$ for all $x \in X$ and $f \in k[X]$, and these maps are such that

$$G \to \mathrm{GL}(k[X]),\ g \mapsto \tau_g$$

is an abstract (we say this to emphasize that we are disregarding the variety structures) representation of the group G (thanks to the G-action on X). The next result shows that one can use a left or right regular action and these τ_g maps to check membership in an affine algebraic group.

II.10. Lemma. — *Let $H < G$ be a closed subgroup of an affine algebraic group G. Let $I \lhd k[X]$ be a vanishing ideal for H. Then*

$$H = \{h \in G \mid \rho_h(I) \subset I\}$$

where $\rho_h = \tau_h$ for the right regular action (similarly for $\lambda_h = \tau_h$ and the left regular action).

Proof. For "$\subset$", let $h \in H$, $f \in I$. Then $\rho_h(f)(g) = f(gh) = 0$ for $g \in H$, so $\rho_h(f) \in I$. As for "$\supset$", let $\rho_h(I) \subset I$. Then $\rho_h(f)(e) = 0$ for all $f \in I$, hence $0 = \rho_h(f)(e) = f(eh) = f(h)$ for any $f \in I$, so $h \in H$. □

Remark. — Let G be an affine algebraic group and X an affine G-variety. Then the morphism $k[X] \to k[G] \otimes k[X]$ is a $k[G]$-comodule structure on $k[X]$. In fact, there is a bijection (extending the usual one) between affine G-varieties X and commutative finitely generated reduced algebras $k[X]$ with $k[G]$-comodule structure.

One last preparatory result and then we will be able to prove what we want (affine algebraic groups are linear), modulo one little fact.

II.11. Proposition. — *Let G be an affine algebraic group, X an affine G-variety. Assume that $V \subset k[X]$ is a finite dimensional vector space. Then the following holds.*

(1) *There exists a finite dimensional vector space W such that $V \subset W$ and W is stable under τ_g for all $g \in G$.*

Note that $k[X]$ is usually infinite dimensional as a k-vector space.

(2) *V is stable under τ_g for all $g \in G$ if and only if $\varphi^*(V) \subset k[G] \otimes V$, where*

$$\varphi : G \times X \to X,\ (g, x) \mapsto g^{-1}x.$$

φ is a morphism of varieties, since $(g, x) \mapsto g.x$ and taking the inverse are morphisms.

Proof. (1) Without loss of generality let V be one dimensional, i.e. $V = kf$. Consider

$$\varphi^* : k[X] \to k[G \times X] \cong k[G] \otimes k[X].$$

Let $\varphi^*(f) = \sum_{i \in I} u_i \otimes f_i$ for $u_i \in k[G]$, $f_i \in k[X]$, I some finite set. Then

$$\tau_g(f)(x) = f(g^{-1}.x) = \varphi^*(f)(g, x) = \sum_{i \in I} u_i(g) f_i(x). \qquad (*)$$

So $\tau_g(f) = \sum_{i \in I} u_i(g) f_i$, which means that $\tau_g(f)$ is in the subspace generated by the f_i for any $g \in G$. But then if we let W be the span of $\tau_g(f)$ for all $g \in G$, we have that W is finite dimensional and contains $\tau_e(f) = f$ and thus $V \leqslant W$.

(2) For the "if" part, note that in $(*)$ we would have $f_i \in V$ for all f_i, and then V would be stable under τ_g for all $g \in G$. For the converse, pick a basis $(f_i)_{i \in I}$ of V and extend it via some $(g_j)_{j \in J}$ to a basis of $k[X]$. If

$$\varphi^*(f) = \sum_{i \in I} \alpha_i \otimes f_i + \sum_{j \in J} \beta_j \otimes g_j$$

with almost all $\beta_i = 0$, then

$$\tau_g(f) = \sum \alpha_i(g) \otimes f_i + \sum \beta_j(g) \otimes g_j.$$

By assumption, $\beta_j(g) = 0$ for all $j \in J$ and $g \in G$. But this means $\beta_j = 0$ for all $j \in J$. Then

$$\varphi^*(f) = \sum_{i \in I} \alpha_i \otimes f_i \in k[G] \otimes V,$$

which is what we wanted. □

II.12. Theorem. — *Any affine algebraic group G is linear.*

Proof (up to one little fact). Pick algebra generators $f_1', \ldots, f_n'$ of $k[G]$ (we can take finitely many as $k[G]$ is finitely generated). Let V be the finite dimensional subspace of $k[G]$ generated by the the generators chosen. Appealing to the previous proposition (II.11), consider a finite dimensional vector space W containing V and stable under τ_g for all $g \in G$, where the τ_g are induced by the right regular action of G. Let $f_1, \ldots, f_m$ be a basis of W and note that it also generates $k[G]$ as an algebra. We will now compute $\tau_g|_W$ in terms of matrices in this basis. For this consider

$$\varphi : G \times G \to G,\ (x, y) \mapsto yx,$$

and note that we have

$$\varphi^*(f_i) = \sum_{j=1}^{m} a_{ij} \otimes f_j$$

for some $a_{ij} \in k[G]$. Then

$$\tau_g(f_i)(y) = f_i(yg) = \varphi^*(f_i)(g, y) = \sum_{j=1}^{m} a_{ij}(g) f_j(y),$$

thus

$$\tau_g(f_i) = \sum_{j=1}^{m} a_{ij}(g) f_j,$$

i.e. $(a_{ij}(g))_{1\leqslant i,j\leqslant m}$ is a matrix describing $\tau_g|_W$. In other words, we have a morphism of varieties

$$\Phi : G \to \mathrm{GL}(W),\ g \mapsto (a_{ij}(g)).$$

From this point onward the proof needs to be cleaned up...

Now we want to prove that Φ is an isomorphism onto its image.

Injectivity of Φ. We have

$$f_i(x) = f_i(ex) = \sum_{j=1}^{m} a_{ij}(x) f_j(e)$$

so that $f_i = \sum_{j=1}^{m} f_j(e) a_{ij}$. In particular, the a_{ij}'s generate $k[G]$ as an algebra, which means that Φ is injective.

Here rephrasing in the language of schemes would make the situation clearer (surjective ring morphisms correspond to closed immersions).

To prove that Φ is an isomorphism onto its image, note first that $K := \operatorname{im}\Phi \subset \mathrm{GL}(W)$ is closed (see [Hum75, Proposition 7.4-B]), hence an affine variety. Then we are just left to prove that Φ is in fact an isomorphism of varieties. But the restrictions to K of the coordinate functions $T_{ij} \in k[\mathrm{GL}_W] = k[T_{ij}, \det^{-1}]$ are sent by Φ^* to the respective a_{ij}, which were just shown to generate $k[G]$ as an algebra, so Φ^* is surjective, hence it identifies the coordinate rings $k[K]$ and $k[G]$. □

How to Take Quotients of Affine Algebraic Groups

Lecture 6
28th Oct, 2022

Given an affine algebraic group G and a closed subgroup $H < G$, we want to turn G/H into a variety satisfying something similar to the usual universal property of quotients: given a map $f : G \to Y$ with $\ker(f) \subset H$ there is a unique map $\bar{f} : G/H \to Y$ which makes the following diagram commute:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ G/H \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

There are two possible approaches:

- We can look for a "categorical" quotient, i.e. an object defined by an appropriate universal property (but which may not exist, for all we know). It is often the case with universal objects that even after existence is proven one is left with some not-so-explicit construction which is cumbersome to use.

- We can study the notion of Chevalley quotient, which has a useful explicit construction, but which is not a priori defined by a universal property.

What we will do is construct the Chevalley quotient first, then the categorical quotient, then show that the former is isomorphic to the latter and so is a categorical quotient itself.

Definition. — A *categorical quotient* of an affine algebraic group G by a closed subgroup H is a triple $(G/\!\!/H, x, \pi_x)$, where $G/\!\!/H$ is a homogeneous G-space, x a point in $G/\!\!/H$ such

What we are doing is considering a coequalizer in the category of pointed homogeneous G-spaces with basepoint preserving G-equivariant morphisms (note that quotients are coequalizers, and so when they exist they consist of an object *and* a morphism).

For a G-space X and $x \in X$, the subgroup $G_x \subset G$ of the elements of G which fix x is called the *isotropy subgroup* of x.

that $G_x = \{g \in G \mid gx = x\} = H$ and $\pi = \pi_x : G \to G/\!\!/H,\ g \mapsto g.x$ a (G-equivariant) morphism constant on cosets of H, such that the following universal property holds: given any other such triple (Y, y, π_y) with $G_y = \{g \in G \mid gy = y\} \supset H$, there is a unique morphism of G-varieties $\varphi : G/\!\!/H \to Y$ with $\varphi(x) = y$, such that the diagram

$$\begin{tikzcd} G \arrow[r, "\pi_y"] \arrow[d, "\pi"'] & Y \\ G/\!\!/H \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

is commutative.

Remark. — As usual, the categorical quotient is unique up to unique isomorphism.

Of course, the isomorphisms between any two universal objects can be many (and most of the times are), it's just that there is a unique one *coming from the universal property.*

Definition. — A *Chevalley quotient* of an affine algebraic group G by a closed subgroup H is a quasi-projective homogeneous G-space X with a basepoint $x_0 \in X$, such that

Observe that (ChQ1) is implied by (ChQ2), so is more of a reality check (I guess?).

(ChQ1) $G_{x_0} = \{g \in G \mid g.x_0 = x_0\} = H$,

(ChQ2) the fibers of $G \to X,\ g \mapsto g.x_0$ are the cosets gH, $g \in G$.

II.13. Lemma. — *Let G be an affine algebraic group and $H < G$ a closed subgroup. Then there is a finite-dimensional k-vector space $V \subset k[G]$ with a subspace $W \subset V$ such that*

(1) *V is stable under all right translations ρ_g for $g \in G$ (i.e. for all $f \in V$ and $g \in G$, we have $g.f \in V$, where $(g.f)(x) = f(xg)$ for $x \in G$),*

(2) *$H = \{g \in G \mid g.W \subset W\}$.*

Proof. Let I be a vanishing ideal for H, generated by $f_1, \dots, f_n \in k[G]$. Let $V \subset k[G]$ be a finitely dimensional vector space satisfying (1), which exists by Proposition II.11. Set $W = V \cap I$. We want to show that (2) holds for this choice.

($\supset$) Assume $g.f \in W$ for all $f \in W$. Then $g.f_i \in W \subset I$ for all i, and thus $g.I \subset I$ (chase the definitions to see it). Now, for $\varphi \in I$ we have $\varphi(g) = (g.\varphi)(e) = 0$, hence $g \in H$.

($\subset$) Let $f \in W$, $h \in H$. Then $(h.f)(x) = f(xh) = 0$ for all $x, h \in H$, hence $h.f \in I$ for all $h \in H$. By (1), $h.f \in V \cap I = W$. ☐

II.14. Lemma. — *Let G be an affine algebraic group, $V \subset k[G]$ a finitely dimensional k-vector space and $W \subset V$ a subspace with $\dim W = d$. Consider $L = \bigwedge^d W \subset \bigwedge^d V$ with natural action of $\mathrm{GL}(V)$. Then for any $g \in G$, $g.W \subset W$ if and only if $g.L = L$.*

Proof. ($\Longrightarrow$) If $g.W \subset W$, then $g.(\bigwedge^d W) = \bigwedge^d g.W \subset \bigwedge^d W$, i.e. $g.L \subset L$.

($\Longleftarrow$) Fix $g \in \mathrm{GL}(V)$ and consider $g.W \subset V$. Pick a basis $v_1, \dots, v_n$ of V, such that $v_1, \dots, v_d$ is a basis of W and $v_{l+1}, \dots, v_{l+d}$ for some $l \in \mathbb{N}$ with $l + d = n$ is a basis of $g.W$ (note that W and $g.W$ need not be disjoint). Let $e = v_1 \wedge \cdots \wedge v_d$, $e' = v_{l+1} \wedge \cdots \wedge v_{l+d}$. Then $g.e = \gamma e'$ for some $\gamma \in h^\times$. Now if $l > 0$, then $e \neq e'$, which contradicts the assumption that $g.L = L$. So we must have $l = 0$, and thus $g.W = W$. ☐

II.15. Corollary. — *Given any affine algebraic group G and $H < G$ a closed subgroup, there is a natural representation $G \to \mathrm{GL}(Z)$, where Z is a finite dimensional vector space with $v_0 \in Z$ such that H is the stabilizer of the line spanned by v_0:*

$$H = \{g \in G \mid g.v_0 \in \langle v_0 \rangle\}.$$

Proof. Take V and $W \subset V$ as in Lemma II.13, with $\dim W = d$. Let Z be defined as

$$Z \coloneqq \bigwedge^d V \supset \bigwedge^d W = \langle v_0 \rangle$$

for some $v_0 \neq 0$. We have a natural representation

$$\varphi : G \to \mathrm{GL}(Z), \; g \mapsto \rho_g$$

where

$$H = \{g \in G \mid g.W \subset W\} = \{g \in G \mid g.\langle v_0 \rangle = \langle v_0 \rangle\} = \{g \in G \mid g.v_0 \in \langle v_0 \rangle\}$$

using Lemma II.13 for the first equality and Lemma II.14 for the second. □

II.16. Corollary. — *Chevalley quotients exist.*

Proof. Consider Z as in Corollary II.15 with its G-action. Then G also acts on $\mathbb{P}(Z)$. Consider the class $[v_0]$ of v_0 in $\mathbb{P}(Z)$. Let $X \coloneqq G.[v_0] \subset \mathbb{P}(Z)$. Clearly X is an homogeneous G-space. Moreover, $X = G.[v_0]$ is open in its closure (Lemma II.9), hence X is quasi-projective. By Corollary II.15

$$G_{[v_0]} = \{g \in G \mid g.[v_0] = [v_0]\} = H,$$

so (ChQ1) holds for X.

Consider now $\psi : G \to X, \; g \mapsto g.[v_0]$. For any $b \in G$, the fibers of ψ are

$$\begin{aligned}\psi^{-1}(b.[v_0]) &= \{g \in G \mid g.[v_0] = b.[v_0]\}\\ &= \{g \in G \mid b^{-1}g.[v_0] = [v_0]\}\\ &= \{g \in G \mid b^{-1}g \in H\} = bH.\end{aligned}$$

In particular, the fibers are cosets of H, which shows that (ChQ2) holds. □

II.17. Theorem. — *Let G be an affine algebraic group, $H < G$ a closed subgroup.*

One could prove that every categorical quotient is in fact a Chevalley quotient, but we will not do that!

(1) *The categorical quotient $(X /\!/ G, a)$ exists (and so it is unique up to isomorphism).*

(2) *The Chevalley quotient constructed above is a categorical quotient.*

Proof. (1) We will construct explicitly a categorical quotient $(G/\!/H, a, \pi_a)$.

As a set. Set $G/\!/H \coloneqq G/H$ (i.e. the usual set of cosets of H in G). Set also $a \coloneqq H$ and take as π_a the canonical projection $\pi : G \to G/\!/H, \; g \mapsto gH$.

As a topological space. Define $U \subset G/\!/H$ to be open whenever $\pi^{-1}(U) \subset G$ is open (i.e. we give $G/\!/H$ the quotient topology). Then π is an open map, since if $U \subset G$ is open, $\pi^{-1}(\pi(U))$ is the union of all cosets gH intersecting U nontrivially, but this is the union of all H-translates of U, then a union of open sets, and thus open. Hence $\pi(U) \subset G/\!/H$ is open.

As a variety. We define the structure sheaf by

$$\mathcal{O}_{G/\!/H}(U) \coloneqq \{f : U \to k \mid f \circ \pi \in \mathcal{O}_G(\pi^{-1}(U))\}.$$

One has to check that this is indeed a sheaf of rings (using that $\mathcal{O}_G(\pi^{-1}(U))$ satisfies the sheaf axioms).

As a homogeneous G-space. We prove that

$$\varphi_x : G/\!/H \to G/\!/H, \; gH \mapsto xgH$$

is an isomorphism of ringed spaces. This is because for all opens $U \subset G/\!\!/H$,

$$\begin{aligned} f \in \mathcal{O}_{G/\!\!/H}(U) &\iff f \circ \pi \in \mathcal{O}_G(\pi^{-1}(U)) \\ &\iff f \circ \pi \circ \mu_x \in \mathcal{O}_G(x^{-1}\pi^{-1}(U)) \\ &\iff f \circ \varphi_x \circ \pi \in \mathcal{O}_G(\pi^{-1}(x^{-1}U)) \\ &\iff f \circ \varphi_x \in \mathcal{O}_{G/\!\!/X}(x^{-1}U) \end{aligned}$$

where the second implication holds because G is an affine algebraic group and the third holds because of π is G-equivariant, i.e. the diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\mu_x"'] & G/\!\!/H \arrow[d, "\varphi_x"] \\ G \arrow[r, "\pi"'] & G/\!\!/H \end{tikzcd}$$

commutes. Then φ_x is a morphism of ringed spaces with inverse $\varphi_{x^{-1}}$.

The action is clearly transitive, and it is also clear that $G_a = H$. So we are left only with proving that our $G/\!\!/H$ really has the expected universal property.

Universal property. If (Y, b) and $\pi_b : G \to Y$ are as in the universal property hypotheses, then there exists a unique (G-equivariant) map of sets ψ such that $\pi_b = \psi \circ \pi_a$

$$\begin{tikzcd} G \arrow[r, "\pi_b"] \arrow[d, "\pi_a=\pi"'] & Y \\ G/\!\!/H \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

and $\psi(ga) = \psi(gH) = gb$. We now want to prove that ψ is a morphism. Let $U \subset Y$ be open, $f \in \mathcal{O}_Y(U)$. Then $\mathcal{O}_G(\pi_b^{-1}(U)) \ni f \circ \pi_b = f \circ \psi \circ \pi_a$ if and only if $f \circ \psi \in \mathcal{O}_{G/\!\!/H}(\pi_a \pi_b^{-1}(U))$ by definition of $\mathcal{O}_{G/\!\!/H}$. But $\pi_a \pi_b^{-1}(U) = \psi^{-1}(U)$, so we are done.

(2) If (X, x) is the Chevalley quotient constructed in Corollary II.16, considering the (clearly G-equivariant) morphism

$$G \to X, \; g \mapsto g.x$$

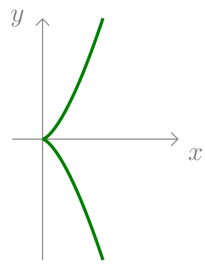
we get a morphism $\Phi : G/\!\!/H \to X$ by the universal property. We now want to show that this is an isomorphism of ringed spaces, which we will do assuming some nontrivial results we do not have time to prove. From (ChQ2) we have that Φ is a continuous bijection. It is also a homeomorphism (as it is a general fact that morphisms between homogeneous G-spaces are open, see [Spr83, Theorem 4.3.3 (i)]). Finally, it is an isomorphism by Zariski's main theorem (a version of which is: if $f : X \to Y$ is a bijective morphism of irreducible varieties, Y is smooth, and there exists $x \in X$ a smooth point such that $d_x f$ is surjective, then f is an isomorphism; see [Spr83, Theorem 5.2.8] for a proof). $\square$

Zariski's main theorem is far from an easy result, and we did not quote it in full generality. For another proof (if you like schemes better) which does not use too much technology, see [GW20, Cor 12.88].

From now on we will write G/H instead of $G/\!\!/H$ for the categorical quotient.

Remark. — As an example of a bijective morphism of varieties which is not an isomorphism, one can take the usual parametrization of the cuspidal curve,

$$\mathbb{R} \to Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}, \; t \mapsto (t^2, t^3).$$



The reason we do not get an isomorphism is that Zariski's main theorem requires Y to be smooth (more precisely, only the weaker condition of normality is required).

More about Quotients

Lecture 7
2nd Nov, 2022

We are building tools to study more deeply linear algebraic groups and the algebraic varieties we have constructed so far (Grassmannians, flag varieties).

II.18. Theorem. — *There is an isomorphism of varieties*

$$\mathrm{Gr}(d, n) \cong \mathrm{GL}_n(\mathbb{C})/P$$

where P is the closed subgroup of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with the zero block a $(n-d) \times d$ block and the other blocks arbitrary.

Proof. As usual, the (obvious) action of $G = \mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ induces an action on $\bigwedge^d \mathbb{C}^n$ (diagonally) and this in turn induces an action on $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$. Consider

$$\Phi : G \to \mathbb{P}(\textstyle\bigwedge^d \mathbb{C}^n),\ g \mapsto g.x_0,$$

where $x_0 = [e_1 \wedge \cdots \wedge e_d] \in \mathbb{P}(\bigwedge^d \mathbb{C}^n)$. We can corestrict Φ to its image to get

$$\Phi : G \to \operatorname{im} \Phi = \mathrm{Gr}(d, n) \subset \mathbb{P}(\textstyle\bigwedge^d \mathbb{C}^n).$$

Note that $\operatorname{im} \Phi$ is then closed. Moreover,

$$G_{x_0} = \{g \in G \mid gx_0 = x_0\} = P(= \Phi^{-1}(x_0))$$

and

$$\Phi^{-1}(g.x_0) = g\Phi^{-1}(x_0) = gP$$

is a coset of P. This shows that $\mathrm{Gr}(d, n)$ is a Chevalley quotient, and thus isomorphic to $\mathrm{GL}_n(\mathbb{C})/P$ by uniqueness of the quotient. □

So far we have been talking about quotients as in "quotients of a space by a group action", but we can also take quotients of affine algebraic groups in the group theory sense:

Proposition. — *Quotients of affine algebraic groups by closed normal subgroups are affine algebraic groups (with the induced multiplication).*

Warning ⚠. — Parabolic subgroups $P_{\underline{d}} \subset \mathrm{GL}_n(\mathbb{C})$ are usually not normal, so the proposition does not apply to flag varieties. For example, take $P_{\underline{d}} = B \subset \mathrm{GL}_2(\mathbb{C})$, then:

$$\begin{aligned} s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}),\ sBs^{-1} &= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{C},\ ac \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} 0 & c \\ a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid ac \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} c & 0 \\ b & a \end{pmatrix} \mid ac \neq 0 \right\} \not\subset B. \end{aligned}$$

So G/B is not a group in the usual way. Note also that $G/B \cong \mathbb{P}^1_{\mathbb{C}}$ is not affine.

We will need some preparation to prove the proposition. In particular, we need to introduce a notion of characters for algebraic groups.

Definition. — Let G be an algebraic group. A *character* of G is a morphism of algebraic groups $\chi : G \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$. Often $\mathbb{C}^\times$ is denoted $\mathbb{G}_m$ (and called the *multiplicative group*). The set $X^*(G)$ of the characters of G is an abelian group via

$$(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g),\ \chi_1, \chi_2 \in X^*(G),$$

where the multiplication in the right hand side is the one in $\mathbb{C}^\times$. We write the neutral element χ_e, where

$$\chi_e(g) = 1\ \forall g \in G.$$

II.19. Lemma (Dedekind lemma). — *Let G be an (abstract) group.*

(1) *The set*

$$C = \{\varphi : G \to \mathbb{C}^\times \text{ group morphisms}\} \subset \mathrm{Maps}(G, \mathbb{C})$$

is a linearly independent subset.

(2) *If G is moreover an affine algebraic group, then*

$$X^*(G) \subset C \subset \mathrm{Maps}(G, \mathbb{C})$$

and these characters form thus a linearly independent subset.

Proof. We just need to prove the first point, as the second clearly follows. Assume C is not linearly independent. In particular there are $\varphi_1, \dots, \varphi_n \in C$ such that there is a linear combination of them that vanishes $\sum_{i=1}^n a_i \varphi_i = 0$ (where the equation is in $\mathrm{Maps}(G, \mathbb{C})$ with complex coefficients). Choose φ_i's such that n is minimal (note that $n > 1$). Now for any $g, h \in G$, we have

$$\varphi_1(g)(\sum_{i=1}^n a_i \varphi_i(h)) = 0,$$

$$\sum_{i=1}^n a_i \varphi_i(gh) = \sum_{i=1}^n a_i \varphi_i(g)\varphi_i(h) = 0,$$

and so taking differences we get

$$\sum_{i=2}^{n} (\varphi_1(g) - \varphi_i(g)) a_i \varphi_i = 0$$

for all $g \in G$, and for an appropriate choice of g we can assume $\varphi_1(g) - \varphi_i(g) \neq 0$, as $\varphi_1 \neq \varphi_2$. Thus we have constructed a shorter linear combination with at least one nonzero coefficient, contradicting minimality of n. □

II.20. Lemma. — *The map*

$$\Theta : (\mathbb{Z}, +) \to X^*(\mathbb{G}_m),\ n \mapsto (t \mapsto t^n)$$

is an isomorphism of (abstract) groups.

Proof. Note that $k[\mathbb{G}_m] \cong k[t^{\pm 1}] = k[t, t^{-1}]$. The multiplication $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ corresponds to

$$\mu^* : k[\mathbb{G}_m] \cong k[t, t^{-1}] \to k[\mathbb{G}_m \times \mathbb{G}_m] \cong k[t^{\pm 1}] \otimes k[t^{\pm 1}],\ t \mapsto t \otimes t$$

(and thus $t^a \mapsto (t \otimes t)^a = t^a \otimes t^a$ for all $a \in \mathbb{Z}$, since μ^* is an algebra morphism) and the unit map $\varepsilon : \{*\} \to \mathbb{G}_m,\ * \mapsto e = 1 \in \mathbb{G}_m$ to

$$\varepsilon^* : k[\mathbb{G}_m] \cong k[t, t^{-1}] \to k[\{*\}] \cong k,\ t^a \mapsto 1\ \forall a \in \mathbb{Z}.$$

Claim 1. $\Theta(n) \in X^*(\mathbb{G}_m)$.

Proof of the claim. For $t_1, t_2 \in \mathbb{G}_m$, we have

$$\Theta(n)(t_1 t_2) = (t_1 t_2)^n = t_1^n t_2^n = \Theta(n)(t_1)\Theta(n)(t_2),$$

so Θ is a group morphism. Moreover

$$(\Theta(n)^*)(f) = f \circ \Theta(n) \in k[\mathbb{G}_m]$$

is a Laurent polynomials with entries in $\mathbb{G}_m$. Hence $\Theta(n)$ is a morphism of algebraic groups, which is our claim. ■

Claim 2. Θ is injective.

Proof of the claim. This is clear. ■

Claim 3. Θ is surjective.

Proof of the claim. Let $\varphi \in X^*(\mathbb{G}_m)$, i.e. φ is an automorphism of $\mathbb{G}_m$. Consider the corresponding morphism of algebras $\varphi^* : k[t^{\pm 1}] \to k[t^{\pm 1}]$. We have $\varphi^*(t) = \sum_{i \in \mathbb{Z}} a_i t^i$ with almost all the coefficients a_i zero. Then, since φ is a group morphism we have

$$\mu^*(\varphi^*(t)) = \sum_i a_i t^i \otimes t^i = \varphi^*(t) \otimes \varphi^*(t) = \sum_{i,j} a_i a_j t^i \otimes t^j$$

It is easy to check that $\mu^*(\varphi^*(t))$ and $\varphi^*(t) \otimes \varphi^*(t)$ are equal as elements of $k[\mathbb{G}_m \times \mathbb{G}_m]$ (but it was not obvious to me on first sight, not sure if it should be!).

and this can be the case only if just one coefficient a_i is nonzero, say a_n for some $n \in \mathbb{Z}$. Thus $\varphi^*(t) = a_n t^n$. Again thanks to φ being a group morphism, we have $\varphi \circ \varepsilon = \varepsilon$, i.e. $\varepsilon^* \circ \varphi^* = \varepsilon^*$, so the morphism

$$k[t^{\pm 1}] \xrightarrow{\varphi^*} k[t^{\pm 1}] \xrightarrow{\varepsilon^*} k,\ t \mapsto a_n t^n \mapsto a_n,$$

is the same as $\varepsilon^* : k[t^{\pm 1}] \to k$, $t \mapsto 1$, hence $a_n = 1$. Then $\varphi^*(t) = t^n$, meaning that φ is the morphism $t \mapsto t^n$, thus in the image of Θ. ■

□

II.21. Lemma. — *If G and G' are affine algebraic groups, then there is an isomorphism of groups*

$$\Phi : X^*(G \times G) \to X^*(G) \times X^*(G), \ \chi \mapsto (\chi|_G, \chi|_{G'}).$$

Proof. Note that $G \xrightarrow{\cong} G \times * \hookrightarrow G \times G' \xrightarrow{\chi} \mathbb{G}_m$ is by definition $\chi|_G$. Similarly for $\chi|_{G'}$. So $\chi|_G$ and $\chi|_{G'}$ are morphisms of varieties and obviously group morphisms. Then Φ is well defined. An inverse is given by $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$, where $\chi_1\chi_2$ is defined as

$$\chi_1\chi_2((g, g')) = \chi_1(g)\chi_2(g').$$

It is easy to check that the map $\chi_1\chi_2 : G \times G' \to \mathbb{G}_m$ is a morphism of varieties and we have

$$\chi_1\chi_2((g, g')(h, h')) = \chi_1(gh)\chi_2(g'h')$$

by definition, so $\chi_1\chi_2$ is a group morphism. It is clear that we defined an inverse, so Φ is an isomorphism. □

II.22. Corollary. — *Let $T \subset \mathrm{GL}_n(\mathbb{C})$ be the standard torus. Then $X^*(T) \cong \mathbb{Z}^n$.*

Proof. Note that $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ so applying the previous lemmas (II.21 and II.20) we get the result. □

Definition. — Let G be an affine algebraic group acting linearly on a vector space V, $\chi \in X^*(G)$. Then

$$V_\chi := \{v \in V \mid g.v = \chi(g)v \ \forall g \in G\}.$$

Note that if $\chi \neq \chi'$ then $V_\chi \cap V_{\chi'} = \{0\}$. In the case $G = T$, the (algebraic) torus $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ or equivalently the standard torus in $\mathrm{GL}_n(\mathbb{C})$, then V_χ is called the χ-weight space with respect to T and χ is called a weight of (or with respect to) V if $V_\chi \neq \{0\}$.

II.23. Proposition. — *Assume $G = T$ an algebraic torus acting linearly on a finite dimensional vector space V (i.e. $\varphi : G \to \mathrm{GL}(V)$ is a rational representation). Then*

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

is a weight space decomposition.

Proof. This is RT2Sheet4.3, but we give a sketch. Note that it is enough to consider the case $T = \mathbb{G}_m$ (since we can use simultaneous eigenspace decomposition). A key point of the proof is the following: the action map $a : \mathbb{G}_m \times V \to V$ gives $a^* : k[V] \to k[\mathbb{G}_m \times V] \cong k[\mathbb{G}_m] \otimes k[V]$ which induces

A highly important question: why is this map called \Delta?

$$\Delta := a^*|_V : V \to k[\mathbb{G}_m] \times V$$

where we are fixing an isomorphism $V \cong V^*$. Then the properties of the action imply that there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\Delta"] \arrow[d, "\Delta"] & k[\mathbb{G}_m] \otimes V \arrow[d, "m^* \otimes \mathrm{id}"] \\ k[G] \otimes V \arrow[r, "\mathrm{id} \otimes \Delta"] & k[\mathbb{G}_m] \otimes k[\mathbb{G}_m] \otimes V \end{tikzcd}$$

induced by the action property $(gh).v = g.(h.v)$.

Write $\Delta(v) = \sum_{n\in\mathbb{Z}} t^n \otimes f_n(v)$ with almost all $f_n = 0$ (remembering that $k[\mathbb{G}_m] \cong k[t^{\pm 1}]$). Then we can conclude the proof by showing that $f_n \in \mathrm{End}_{\mathbb{C}}(V)$ and they are pairwise orthogonal idempotents projecting onto $V_\chi = V_{t\mapsto t^n}$ (remembering that $X^*(\mathbb{G}_m) \cong \mathbb{Z}$). $\square$

Now we can prove the proposition we set out to prove:

II.24. Proposition. — *let G be an affine algebraic group, $H < G$ a closed subgroup. Assume moreover that $H \lhd G$. Then G/H with the induced multiplication from G is an affine algebraic group.*

Proof. We explain the structure of the proof, before carrying it out. Consider the Chevalley/categorical quotient G/H.

- As an ordinary set, G/H is a group with multiplication induced from the one on G, since $H \lhd G$.
- Claim 1. The inverse morphism $\iota : G \to G$ induces
$$\iota : G/H \to G/H,\ gH \mapsto g^{-1}H,$$
which is a morphism of varieties.
- Claim 2. The multiplication morphism $\mu : G \times G \to G$ induces
$$\mu : G/H \times G/H \to G/H,\ (xH, yH) \mapsto xyH,$$
which is a morphism of varieties.
- Claim 3. G/H is an *affine* variety.

To be continued...

Lecture 8
4th Nov, 2022

Consider the inverse map $\iota = \iota_{G/H} : G/H \to G/H$ induced on cosets in the usual way by the inverse morphism $\iota_G : G \to G$ of G.

$$\begin{tikzcd} G \arrow[r, "\iota_G"] \arrow[d, "\pi"] & G \arrow[r, "\pi"] & G/H \\ G/H \arrow[urr, dashed, "\iota_{G/H}"'] & & \end{tikzcd} \tag{$*$}$$

We want to show that $\iota_{G/H}$ is a morphism of varieties. This is true because for any open subset $U \subset G/H$ we have

$$\begin{aligned} f \in \mathcal{O}_{G/H}(U) &\iff f\circ\pi \in \mathcal{O}_G(\pi^{-1}(U)) \\ &\iff f\circ\pi\circ\iota_G \in \mathcal{O}_G(\iota_G^{-1}\circ\pi^{-1}(U)) \\ &\iff f\circ\iota_{G/H}\circ\pi \in \mathcal{O}_G(\pi^{-1}\circ\iota_{G/H}(U)) \\ &\iff f\circ\iota_{G/H} \in \mathcal{O}_{G/H}(\iota_{G/H}^{-1}(U)) \end{aligned}$$

where the second implication holds because ι_G is a morphism and the third comes from commutativity of $(*)$.

Now we want to show the multiplication map $\mu = \mu_{G/H} : G/H \times G/H \to G/H$ induced by multiplication of G is a morphism. Consider G/H as a $G \times G$-variety via $(x, y).gH = xgy^{-1}H$ (one has to check that this is an action). Now, the stabilizer of H in $G \times G$ contains $H \times H$, because $xy^{-1}H \in H$ if x and y are in H. Then we get a morphism

$$\begin{tikzcd} G \times G \arrow[r, "(x,y)\mapsto xy^{-1}H"] \arrow[d] & G/H \\ (G \times G)/(H \times H) \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

by the universal property of the quotient. By RT2Sheet3.4, we have

$$(G \times G)/(H \times H) \xrightarrow{\cong} G/H \times G/H,\ (x, y)H \times H \mapsto (xH, yH).$$

Then

$$\psi \circ \Phi^{-1} \circ (\mathrm{id} \times \iota_{G/H}) : G/H \times G/H \to G/H,\ (xH, yH) \mapsto xyH$$

is a morphism and equals μ.

We are left to show that G/H is affine. From the construction of the Chevalley quotient, in particular Corollary II.15, we find a finite dimensional vector space V together with a morphism of affine algebraic groups $G \to \mathrm{GL}(V)$ and a some point $v_0 \in V$ such that

$$H = \{g \in G \mid g.v_0 \in \langle v_0 \rangle\}. \tag{$*$}$$

Then

$$V' = \bigoplus_{\chi \in X^*(H)} V_\chi \subset V.$$

Note that $V' \neq \{0\}$, as it contains at least $\langle v_0 \rangle$, in view of $(*)$. Moreover, the action of G permutes the spaces V_χ, since for $v \in V_\chi$, $g \in G$ and $h \in H$ we have

$$h.(g.v) = gg^{-1}hg.v = g.(h'.v) = g.\chi(h')v = \chi(h')g.v$$

for some $h' \in H$ (remembering that H is normal), but the conjugate of a character is a character, so $g.v \in V_{\chi'}$ for some $\chi' \in X^*(H)$. Then G stabilizes V', thus without loss of generality we can assume $V' = V$.

Define

$$W := \{f \in \mathrm{End}_k(V) \mid f(V_\chi) \subset V_\chi\ \ \forall \chi \in X^*(H)\}$$

and a map

$$\Psi : G \to \mathrm{GL}(W),\ g \mapsto (f \mapsto g \cdot f),$$

where $(g \cdot f)(v) = gf(g^{-1}.v)$ (one should check that $g \cdot f$ is in W). Then we have

$$\Psi(g) = \mathrm{id} \iff g.(f.(g^{-1}.v)) = f(v)$$

for all $v \in V$ and $f \in W$, i.e. whenever the action of f on $V = \prod_\chi \mathrm{End}_k(V_\chi)$ commutes with the action of g for all $f \in W$. But this is equivalent as saying that g acts on each V_χ by a fixed scalar (depending on V_χ), and in particular implies that g acts by a scalar on the V_χ containing v_0, which means that g has to be in H. Thus we have proved that $\ker \Psi \subset H$, so Ψ induces an injective morphism

$$\bar{\Psi} : G/H \hookrightarrow \mathrm{GL}(W).$$

We will omit the last passage, which is the following: one proves that the image of $\bar{\Psi}$ is closed as a subgroup of $\mathrm{GL}(W)$ and that $\bar{\psi}$ is an isomorphism onto its image, thus proving that G/H is affine (in view of Lemma II.6). ☐

CHAPTER III.

Tits Systems and Bruhat Decompositions

III

Complete Varieties

There are some natural questions one might want to ask about a quotient G/H of algebraic groups:

- When is G/H "compact"?
- When is it projective?

First, note that asking for compactness in this setting is not reasonable (the Zariski topology is not even Hausdorff most of the time), but there is a property which is the analogue of compactness in algebro-geometric settings.

Definition. — A variety X is *complete* if for any variety Z, the projection morphism

This is properness for varieties.

$$\mathrm{pr}_2 : X \times Z \to Z, \ (x, z) \mapsto z$$

is closed (i.e. it sends closed sets to closed sets).

Remark. — There is one proposition which exemplifies why this is the correct analogue of compactness for algebraic varieties: X is compact if and only if for all spaces Z, the canonical projection $\mathrm{pr}_2 : X \times Z \to Z$ is closed.

Again, the topology on the product of varieties is not the product topology.

Example. — Let $X = \mathbb{A}^1$ and consider $Z = \mathbb{A}^1$. Then take

$$Y = \{(a, b) \in \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \mid ab = 1\}.$$

This is obviously closed in $\mathbb{A}^2 = X \times Z$. But

$$\mathrm{pr}_2(Y) = \{b \in Z = \mathbb{A}^1 \mid b \neq 0\} = \mathbb{A}^1 \setminus \{0\},$$

which is not closed. In particular $\mathbb{A}^1$ is not complete.

Example. — RT2Sheet4.1 asks to show that $\mathbb{P}^n_{\mathbb{C}}$ is complete.

Remark. — Let $(X, \mathcal{O}_X)$ be a variety, U a locally closed subset of X (i.e. the intersection of a closed and an open subset of X). Then $(U, \mathcal{O}_X|_U)$ is again a variety.

The next lemma shows that completeness behaves similarly to compactness.

III.1. Lemma. — *Let X be a complete variety.*

(1) *Any closed subset subvariety Y of X is complete.*

(2) *If Y is another complete variety, the product $X \times Y$ is also complete.*

(3) *If $\varphi : X \to Y$ is any morphism of varieties, then $\operatorname{im}\varphi$ is closed and complete.*

(4) *If X is a subvariety of Y, then X is closed in Y.*

Proof. (1) Given a variety Z, the morphism

$$Y \times Z \hookrightarrow X \times Z \xrightarrow{\mathrm{pr}_2} Z$$

is closed (as $Y \subset X$ is closed).

(2) Given a variety Z, we want to show that

$$(X \times Y) \times Z \xrightarrow{\mathrm{pr}_3} Z,\ ((x, y), z) \mapsto z$$

is closed. Rewrite the morphism as

$$X \times Y \times Z \xrightarrow{\mathrm{pr}_2 \times \mathrm{pr}_3} Y \times Z \xrightarrow{\mathrm{pr}_2} Z,\ (x, y, z) \mapsto (y, z) \mapsto z$$

and it becomes clear it is closed (it is the composite of two closed morphisms).

(3) Consider $\Gamma_\varphi = \{(x, \varphi(x)) \in X \times Y \mid x \in X\}$, the graph of φ. This equals $(\varphi \times \mathrm{id})^{-1}(\Delta_Y)$, which is closed, since Y is a variety. Hence Γ_φ is closed (we already remarked this before). But $\mathrm{pr}_2(\Gamma_\varphi) = \operatorname{im}\varphi$ is closed in Y, since X is complete. We are then left to prove that $\operatorname{im}\varphi$ is complete. Given a variety Z, consider

$$\begin{tikzcd} X \times Z \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\varphi \times \mathrm{id}"'] & Z \\ \operatorname{im}\varphi \times Z \arrow[ur, "\mathrm{pr}_2|_{\operatorname{im}\varphi \times Z}"'] & \end{tikzcd}$$

We want to show that the morphism $\mathrm{pr}_2|_{\operatorname{im}\varphi \times Z}$ is closed. If $V \subset \operatorname{im}\varphi \times Z$ is closed, then $(\varphi \times \mathrm{id})^{-1}(V) \subset X \times Z$ is closed. Since φ is a morphism, $\mathrm{pr}_2((\varphi \times \mathrm{id})^{-1}(V)) \subset Z$ is closed, as X is complete. But this agrees with $\mathrm{pr}_2|_{\operatorname{im}\varphi \times Z}(V)$. Hence $\operatorname{im}\varphi$ is complete.

(4) This follows from (3) using the inclusion morphism $X \hookrightarrow Y$. □

III.2. Lemma. — *Let G be an affine algebraic group, $H < G$ a closed subgroup. Then if G/H is complete, G/H is projective.*

Proof. G/H is isomorphic to the Chevalley quotient, a quasi-projective variety. More precisely, a quasi-projective variety of some $\mathbb{P}^n_{\mathbb{C}}$. By (3) of the previous lemma (III.1), we have that

$$\operatorname{im}(G/H \hookrightarrow \mathbb{P}^n_{\mathbb{C}})$$

is closed in $\mathbb{P}^n_{\mathbb{C}}$. Then G/H is projective. □

III.3. Proposition. — *Let G be an affine algebraic group, $Q < P < G$ closed subgroups. The if G/P and P/Q are complete, G/Q is also complete.*

Unchecked!

Proof. We have to show that given any variety Z, the morphism

This one is ugly!

$$G/Q \times Z \xrightarrow{\mathrm{pr}_2} Z$$

is closed. Equivalently, closed subsets in $G \times Z$ of the form $A = \bigcup_{(g,z)\in I}(gQ, z)$ for some set $I \subset G \times Z$ are sent to closed subsets in Z by $\mathrm{pr}_2 : G \times Z \to Z$.

Consider the morphism

$$\alpha : P \times G \times Z \to G \times Z,\ (p, g, z) \mapsto (gp, z).$$

Then $\alpha^{-1}(A) = \{(p, g, z) \mid (gp, z) \in A\}$. Consider now the *not commutative* diagram

$$\begin{tikzcd}
P \times G \times Z \arrow[r, "\pi \times \mathrm{id} \times \mathrm{id}"] \arrow[d, "\alpha"'] & P/Q \times G \times Z \arrow[dl, "\mathrm{pr}_2 \times \mathrm{pr}_3"] \\
G \times Z &
\end{tikzcd}$$

Then $\alpha^{-1}(A) \subset P \times G \times Z$ is closed. Then $\tilde{A} = \pi \times \mathrm{id} \times \mathrm{id}(\alpha^{-1}(A)) \subset P/Q \times G \times Z$ is closed. Since P/Q is complete, $B \coloneqq \mathrm{pr}_2(\tilde{A}) \subset G \times Z$ is closed.

$$\begin{tikzcd}
G \times Z \arrow[r, "\pi \times \mathrm{id} \times \mathrm{id}"] \arrow[d, "\mathrm{pr}_2"'] & G/P \times Z \arrow[dl, "\mathrm{pr}_2"] \\
Z &
\end{tikzcd}$$

We have that $\tilde{B} = \bigcup_{(g,z)\in B}(gP, Z)$ is a closed subset in $G \times Z$, so $\mathrm{pr}_2(\tilde{B}) = \mathrm{pr}_2(\pi \times \mathrm{id}(B))$ is closed, since G/P is complete. By construction (one should check that) $\mathrm{pr}_2(\tilde{B}) = \mathrm{pr}_2(A)$, hence $\mathrm{pr}_2(A)$ is closed and we have proved that G/Q is complete. $\square$

Maybe try to clean this one up a bit? Or look for a different approach?

III.4. Lemma. — *Let G be an affine algebraic group, $P < G$ a closed subgroup.*

(1) *If G/P is complete and $Q < G$ is a closed subgroup such that $P < Q < G$, then G/Q is complete.*

(2) *G/P is complete if and only if G_0/P_0 is complete, where G_0 and P_0 are the irreducible components of G and P containing the identity element e.*

Proof (rather the first part thereof). (1) By the universal property we have

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[d, "\pi"'] & G/Q \\
G/P \arrow[ur, dashed, "\exists!\varphi"'] &
\end{tikzcd}$$

where φ is $gP \mapsto gQ$. The image of φ is complete by (1) of Lemma III.1, since G/P is complete. Then G/Q is complete. *To be continued...*

Before we continue with the second point of the proof, we need to make a little detour on Noetherianity and its interaction with algebraic groups.

Remark (Noetherian decomposition). — We recall here some basic facts about irreducible and Noetherian spaces.

- A topological space Y is *reducible* if it is the disjoint union of two closed *proper* subsets. It is *irreducible* if it is not reducible.
- If X is a Noetherian topological space, then
$$X = \bigcup_{i \in I} X_i,$$
where the X_i are finitely many irreducible components (maximal irreducible subsets). This decomposition is unique (up to order, of course).
- If Y is irreducible, so is its closure (if we had $\overline{Y} = Y_1 \cup Y_2$ two disjoint proper closed subsets, we would have that the same holds for $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$, a contradiction). This also shows that irreducible components are closed.
- It is an easy exercise to show that if X is irreducible and $f : X \to Y$ a continuous map, then $f(X)$ is irreducible.
- It is also an easy exercise to show that if X and Y are irreducible spaces, so is their product $X \times Y$. However, this is cheating: it is not true in general that the product of two irreducible varieties is irreducible! Fortunately, it turns out this is still true when our varieties are defined over an algebraically closed field (such as in our case $k = \mathbb{C}$; with varieties over other fields (or schemes) one ought to be more careful).

Lecture 9
9^{th} Nov, 2022

The previous remark listed general facts about Noetherian varieties, but the next lemma shows that in the case of algebraic groups there is more to be said.

III.5. Lemma. — *Let G be an affine algebraic group.*

(1) *There exists an irreducible component $G_0 \subset G$ with $e \in G_0$.*

(2) *$G_0 \lhd G$ is a closed normal subgroup of G and is of finite index.*

(3) *The cosets gG_0 of G_0 are the irreducible and connected components of G.*

(4) *If $H < G$ is a closed subgroup of finite index, then $G_0 \leqslant H$*

Proof. (1) Assume that X_1, X_2 are irreducible components containing e. Then consider

$$\mu : X_1 \times X_2 \to G,\ (x_1, x_2) \mapsto x_1 x_2.$$

This is a morphism from $X_1 \times X_2$, which is irreducible, so its image is irreducible and contains e, X_1 and X_2. Thus $X_1, X_2 \subset \operatorname{im} \mu$. By maximality of components we must have $X_1 = \operatorname{im} \mu = X_2$. Hence there exist a unique irreducible component containing the identity.

(2) Pick $x \in G_0$. Then $x^{-1} G_0$ is an irreducible component containing the identity. By point (1) we have $G_0 = x^{-1} G_0$, so $x^{-1} \in G_0$, as G_0 contains the identity. Moreover, by varying x it must be that $G_0 G_0 \subset G_0$, so G_0 is a subgroup of G. Now for any $g \in G$, $gG_0 g^{-1}$ is again an irreducible component (since conjugation preserves irreducibility and inclusions). Obviously $e \in gG_0 g^{-1}$ implies that $gG_0 g^{-1} = G_0$, hence G_0 is a normal subgroup of G. Finally, G_0 is closed, since irreducible subsets have irreducible closure, but then $G_0 = \overline{G_0}$ by maximality. We have proved that G_0 is a closed normal subgroup; as for the index, note that

gG_0 is an irreducible component for all $g \in G$, but as there are only finitely many irreducible components (by Noetherianity), G_0 must be of finite index.

(3) From the previous discussion we see that these cosets are disjoint irreducible components, but then they are also connected components.

(4) Assume $H < G$ is a closed subgroup of finite index. We have that the cosets $gH \subset G$ for $g \in G$ are closed and then so is the union of such cosets excluding H itself (as there are finitely many by assumption). Hence H is open and closed. But then since $G_0 \cap H \neq 0$ and G_0 is irreducible, we must have that $G_0 \cap H = G_0$ (otherwise we would find a decomposition of G_0 in proper closed subsets), so $G_0 \leqslant H$. □

Remark. — If $G = G_0$, then G is said to be *connected*. Just a an additional remark: observe that irreducible spaces are connected, but being irreducible is stronger than being connected, hence the connected components of a general topological space may be unions of the irreducible ones: by the previous proposition this is clearly not true for affine algebraic groups.

Connected is the same as irreducible in all Noetherian spaces (in fact it is enough that the set of irreducible components is locally finite).

We can now finish the proof of Lemma III.4. We were left to prove that (given an affine algebraic group G and $P < G$ a closed subgroup) G/P is complete if and only if G_0/P_0 is complete, where G_0 and P_0 are the irreducible components of G and P containing the identity element e.

Proof of (2) of Lemma III.4. ($\Longrightarrow$) Consider $P_0 < P$, which is a closed normal subgroup of finite index by the previous lemma (III.5). The quotient P/P_0 is then of dimension 0, or in other words consists of finitely many points, and it is then easy to show that it is complete. By assumption G/P is complete, hence G/P_0 is complete by transitivity (Proposition III.3). Now, $G_0/P_0 \hookrightarrow G/P_0$ is a closed subvariety of a complete variety, thus complete itself.

($\Longleftarrow$) Assume G_0/P_0 is complete. As before, G/G_0 is complete and thus G/P_0 also is by transitivity. Then by (1) G/P is complete. □

Borel and Parabolic Subgroups

We now prove a theorem (Borel fixed point theorem) which is the analogue of the (concrete) Lie theorem for Lie algebras in the context of algebraic groups.

III.6. Proposition. — *Let G be an affine algebraic group $G = G_0$. Then there exists $P < G$ a closed proper subgroup such that G/P is complete if and only if G is not solvable.*

Proof. See [Spr83, Proposition 6.2.5]. □

III.7. Theorem (Borel fixed point theorem)**.** — *Let G be a connected solvable affine algebraic group acting on a complete variety $X \neq \varnothing$. Then $X^G \neq \varnothing$, i.e. there is a G-fixed point in X*

Proof. Take a closed orbit, say $G.x_0$ in X (by Lemma II.9 we know this exists). Consider G_{x_0}, the isotropy subgroup of x_0. This is a closed subgroup as it is the preimage of x_0 under the map $\bar{a} : G \to X,\ g \mapsto g.x_0$. Now the image of the morphism $G/G_{x_0} \hookrightarrow X$ induced by $\bar{a}$, is $G.x_0$, which is a closed subvariety of X. But X is complete and thus G/G_{x_0} is complete. Then by Proposition III.6, $G_{x_0} = G$, so $x_0 \in X^G$. □

Exercise. — Consider $G = \mathrm{GL}_n(\mathbb{C})$ acting on the Grassmannian $X = \mathrm{Gr}(d, n)$ in the usual way. Consider $B < G$ the standard Borel or $T < G$ the standard torus. Both B and T are closed solvable subgroups, hence their actions have fixed points. In the case of the standard torus, we proved (Lemma I.8) that these are the coordinate subspaces.

Definition. — A *Borel subgroup* of an affine algebraic group is a closed connected solvable subgroup and maximal with this property (with respect to inclusion).

Remark. — If $B < G$ is a Borel subgroup, $gBg^{-1} < G$ is a Borel subgroup for all $g \in G$.

Remark. — Borel subgroups always exist (we can always take solvable connected groups of maximal dimension).

Exercise. — In RT2Sheet5.2 we will see that the standard Borel in $\mathrm{GL}_n(\mathbb{C})$ is a Borel subgroup (so there is no clash in terminology, phew).

III.8. Theorem. — *Let G be an affine algebraic group, $P < G$ a closed subgroup.*

(1) *If $P = B$ is a Borel, then G/P is complete.*

(2) *G/P is complete if and only if P contains a Borel.*

(3) *All Borel subgroups are conjugate.*

Definition. — A closed subgroup P of an affine algebraic group G is *parabolic* if G/P is complete.

Exercise. — By Theorem III.8 and RT2Sheet5.2, the standard parabolics $P_{\underline{d}} < \mathrm{GL}_n(\mathbb{C})$ are parabolic subgroups of $\mathrm{GL}_n(\mathbb{C})$ (so again there is no clash of terminology, yay!)

Proof of Theorem III.8. Let $H < G$ be closed. By Lemma III.4 we know that G/H is complete if and only if G_0/H_0 is complete. Hence without loss of generality we can assume that G is connected.

(2) We first show "$\implies$". Let G/P be complete, $B < G$ a Borel with B acting on G/P via left multiplication. By Borel fixed point theorem (III.7) we have that $(G/P)^B \neq \varnothing$. Then there exists $g \in G$ such that $BgP \subset gP$, and so $g^{-1}BgP \subset P$, i.e. P contains $g^{-1}Bg$. But then $B' = g^{-1}Bg$ is a Borel contained in P.

The structure of the proof is a bit confusing, we first prove "$\implies$" of (2), then (1) *using* "$\implies$" of (2), then "$\impliedby$" of (2). The last point is a byproduct of the proof of (1).

(1) If G is solvable, i.e. $B = G$, then $G/B = \{*\}$ which is complete. In general we can use induction on $\dim G$, exploiting the general fact that proper closed subgroups have strictly smaller dimension.

- For the base case we use the following fact without proof: if $\dim G = 1$ and $G = G_0$, then $G \cong (\mathbb{C}, +)$ or $G = (\mathbb{C}^\times, \cdot) = \mathbb{G}_m$ (the proof is surprisingly hard, see for example this random blogpost or these notes). Both of these groups are solvable, so we are done.
- Consider now $\dim G > 1$ and assume that (1) holds for groups of dimension less than $\dim G$ (in particular all proper closed subgroups of G). Assume that G is not solvable, then by Proposition III.6 there exists a closed proper subgroup $P < G$ such that G/P is complete. Then by "$\implies$" of (2), P contains a Borel subgroup $B' \subset P$. Thus P/B' is complete by inductive hypothesis and then by transitivity G/B' also is. Now, B acts on G/B' by left multiplication, so by Borel fixed point theorem we get that $(G/B')^B \neq \varnothing$. Then there exists a $g \in G$ such that $BgB' = gB'$, so $B' = g^{-1}Bg$, thus B' is conjugate to B by g. Then conjugating by g^{-1} we get that G/B is complete.

Maybe try to fix the structure of this proof...

(2-bis) Now we can take care of " $\Longleftarrow$ ". Assume that P contains a Borel B, i.e. $B \leqslant P \leqslant G$. Then we have

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi"] & G/P \\ G/B \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

so we get a map $\varphi : G/B \to G/P,\ gB \mapsto gP$ by the universal property of the quotient. By point (1) G/B is complete. Hence the image of φ is closed and complete (by Lemma III.1), and the morphism is surjective. Thus G/P is complete.

(3) This was proved at the end of the proof of (1). □

Bruhat Decomposition (Concrete)

Lecture 10
11th Nov, 2022

There is a standard way of decomposing $\mathrm{GL}_n(\mathbb{C})$, which we will see can be generalized to other (abstract) groups.

III.9. Theorem ((Concrete) Bruhat decomposition)**.** — *Let $T < B < \mathrm{GL}_n(\mathbb{C}) = G$ be the standard torus and Borel. Then the following holds.*

(1) *The Weyl group of G (with respect to T) defined as $W \coloneqq N_G(T)/T$ is isomorphic (as a group) to S^n.*

(2) *If for each $w \in W$ we pick a representative $\bar{w} \in N_G(T)$, the set $B\bar{w}B \subset G$ is independent of the choice of $\bar{w}$. We denote $BwB = B\bar{w}B$ as $C(w)$. In fact we can take for $\bar{w}$ any permutation matrix in G (i.e. each column and each row has exactly one nonzero entry and this is equal to 1). The $C(w)$ are called the Bruhat cells.*

(3) *There is a decomposition into double B-cosets*

$$G = \bigcup_{w \in W} BwB = \bigcup_{w \in W} C(w)$$

where the union is disjoint. This is called the Bruhat decomposition of $\mathrm{GL}_n(\mathbb{C})$.

Proof. (1) Consider $N \coloneqq N_G(T) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid AtA^{-1} \in T\ \ \forall t \in T\}$. One can prove that N consists of the monomial matrices in $\mathrm{GL}_n(\mathbb{C})$, i.e. invertible matrices with exactly one nonzero entry in each row and each column. We sketch the proof by showing the $n = 2$ case. Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N,\ T_1 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, T_2 \begin{pmatrix} t'_1 & 0 \\ 0 & t'_2 \end{pmatrix} \in T$$

we have that $AT_1 = T_2A$. This means that the following set of equations hold

$$\begin{cases} at_1 = t'_1 a \\ bt_2 = t'_1 b \\ ct_1 = t'_2 c \\ dt_2 = t'_2 d \end{cases}$$

and it is clear that having both a and b or both c and d be nonzero leads to a contradiction. Thus A has to have at most one nonzero entry in each row, but at the same time, being an invertible matrix, it has to have at least one nonzero entry in each row, which means that A is a monomial matrix. Conversely, every monomial matrix $A \in \mathrm{GL}_n(\mathbb{C})$ satisfies $AtA^{-1} \in T$ for all $t \in T$. The general case works similarly (just with more unwieldy equations). Now one can prove that N is a group, and then clearly $T \lhd N$. Finally we claim that

$$S_n \to N/T, \ \pi \mapsto A^{\pi^{-1}}T$$

is an isomorphism of groups, where A^π is the permutation matrix for π, defined as

$$(A^\pi)_{ij} = \begin{cases} 1 & \text{if } j = \pi(i), \\ 0 & \text{otherwise} \end{cases}$$

Note that the map we defined is surjective, since right multiplication by T multiplies the entries in A^π by the respective entries in T. It is also clearly injective (as the element in S_n determines the position of the nonzero entries). The map is then a bijection of sets, so we are left to show that it is a morphism of groups. This is just a computation, however offensive to the eye:

$$(A^{\sigma^{-1}}A^{\pi^{-1}})_{rs} = \sum_k a^{\sigma^{-1}}_{rk}\, a^{\pi^{-1}}_{ks} = a^{\sigma^{-1}}_{r\sigma(r)} a^{\pi^{-1}}_{\sigma(r),\pi^{-1}(\sigma^{-1}(r))} = \begin{cases} 1 & \text{if } \pi^{-1}(\sigma^{-1}(r)) = s \\ 0 & \text{otherwise} \end{cases}$$

and this is precisely $(A^{(\sigma\pi)^{-1}})_{rs}$. Then we have also proved (modulo some sloppiness) that the representatives for $w \in W$ can be given by permutation matrices.

(2) We are left to prove that $B\bar{w}B$ is independent of the choice of representative $\bar{w}$, but this is clear, since $T \subset B$.

(3) This is by the Gauss elimination algorithm, apparently. References for this entire story are [Hum75, Section 28.3] and [Spr83, Section 8.3]. ☐

Remark. — The Bruhat decomposition also works for $\mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$. In RT2Sheet5.2 we will describe the Bruhat decomposition of $\mathrm{O}_n(\mathbb{C})$.

Remark. — The Bruhat cells of $\mathrm{GL}_n(\mathbb{C})$ induce the Schubert cells on $G/P_{\underline{d}}$.

III.10. Theorem. — *The following holds.*

(1) *Assume $P < \mathrm{GL}_n(\mathbb{C})$ is a closed subgroup containing some Borel B'. Then P is conjugate to some $P_{\underline{d}} < \mathrm{GL}_n(\mathbb{C})$, a standard parabolic. We call subgroups such as P parabolic.*

(2) *The parabolic subgroups in $\mathrm{GL}_n(\mathbb{C})$ correspond to the conjugates by $g \in G$ of the standard parabolics $P_{\underline{d}}$.*

⚠ Unchecked!

Proof. We have that (1) implies (2). Indeed for "$\subset$", consider a parabolic $P < \mathrm{GL}_n(\mathbb{C})$, i.e. P is closed and contains a Borel. Then by (1), there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gP_{\underline{d}}g^{-1} = P$. As for "$\supset$", $gP_{\underline{d}}g^{-1}$ contains gBg^{-1} which is a Borel, hence it is a parabolic.

We are left with proving (1). Since all Borels are conjugate, we can assume $B' = B$, the standard Borel. Consider

$$Z = \{W \leqslant \mathbb{C}^n \mid p(W) \subset W \ \ \forall p \in P\}$$
$$Z \subset \{W \leqslant \mathbb{C}^n \mid b(W) \subset W \ \ \forall b \in B\} = \{F_j^{\mathrm{st}} = \langle e_i, \ldots, e_j\rangle \mid 1 \leqslant j\}.$$

Then $Z = \{F^{\mathrm{st}}_{d_1}, \dots, F^{\mathrm{st}}_{d_l}\}$ with $1 \leqslant d_1 < d_2 < \dots < d_l \leqslant n$. We claim that

$$P = P_{\underline{d}} = \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_l \end{pmatrix}$$

where $A_i \in \mathrm{GL}_{c_i}(\mathbb{C})$, with $c_i = d_i - d_{i-1}$. Now the set $\{pe_i \mid p \in P\}$ spans the smallest P-invariant subspace containing e_i and this latter spans $F^{\mathrm{st}}_{d_i}$. Then there exists $p \in P$ such that

$$p(e_i) = e_{d_i} + \sum_{i=1}^{d_i} \beta_i e_i$$

for some $\beta_i \in \mathbb{C}$. By Bruhat decomposition there exists $w \in W$ such that $p \in BwB$. In particular, there exists a unique permutation matrix $\bar{w}$ such that $p \in B\bar{w}B$, and thus $\bar{w} \in BpB \subset BPB \subset P$. Now multiplication with B from the left does not change the position of the lowest nonzero entry in the first column. We have $\bar{w}(e_i) = e_d$. Let $\bar{w}^{-1}(e_1) = e_s$. Now consider (for $1 \leqslant i + j \leqslant n$) the maps

$$X_{ij} : \mathbb{C} \to G = \mathrm{GL}_n(\mathbb{C}),\ t \mapsto \mathbb{1} + tE_{ij}.$$

We have that X_{ij} is a group morphism $(\mathbb{C}, +) \to G$, since

$$(\mathbb{1} + tE_{ij})(\mathbb{1} + t'E_{ij}) = \mathbb{1} + t'E_{ij} + tE_{ij} = \mathbb{1} + (t + t')E_{ij}.$$

For $d \coloneqq d_1$, we have the following.

(i) $X_{d,1}(t) = \mathbb{1} + tE_{d,1} = \bar{w}^{-1}(\mathbb{1} + E_{1,s})\bar{w} \in P$.

(ii) $X_{1,i}(t) \in B \subset P$ for all $i \neq 1$ and $t \in \mathbb{C}$.

(iii) $X_{jd}(t) \in B \subset P$ for all $j < d$. Now, $X_{di}(t/2) = (X_{d,1}(1), X_{1,i}(t))$ for distinct d, i, both different from 1, where the brackets denote the group commutator in P. Indeed

$$\begin{aligned} X_{d1}(1)X_{1i}(t)X_{d1}(1)^{-1}X_{1i}(t)^{-1} &= (\mathbb{1} + E_{d1})(\mathbb{1} + tE_{1i})(\mathbb{1} - E_{d1})(\mathbb{1} - tE_{1i}) \\ &= (\mathbb{1} + tE_{1i} + E_{d1} + tE_{di})(\mathbb{1} - tE_{1i} - E_{d1} + tE_{di}) \\ &= \mathbb{1} - tE_{1i} - E_{d1} + tE_{di} + tE_{1i} + E_{d1} + tE_{di} \\ &= \mathbb{1} + 2tE_{di}. \end{aligned}$$

(iv) $X_{pi}(t) = (X_{pd}(1), X_{di}(t)) \in P$ for $p \neq i, d$, $d \neq i$ and $p < d$ (check if you dare!).

Then we have $X_{pi}(t) \in P$ for all $p \leqslant d$ and all i. We repeat the argument for 2 columns, 3 columns, etc. Then we get $P_{\underline{d}} < P$. Conversely $P < P_{\underline{d}}$ by the characterization using subspaces in $\mathbb{C}^n$. ☐

Remark. — The group morphisms $X_{ij} : (\mathbb{C}, +) \to \mathrm{GL}_n(\mathbb{C})$ we defined in the last proof are called 1-parameter subgroups.

We now want to define Bruhat decompositions of groups in a more general setting.

Bruhat Decomposition (Abstract)

In order to define Bruhat decompositions for arbitrary groups (in particular we are interested to do so for $\mathrm{GL}_n(k)$, for arbitrary k), we need to introduce some kind of structure on them, modeled on the standard structure of $\mathrm{GL}_n(\mathbb{C})$.

Definition. — A *Tits system* (sometimes called BN-pair) is a quadruple (G, B, N, S), consisting of a group G, subgroups B and N, and S a subset of $N/(B \cap N)$, satisfying the following properties.

(TS0) The group G is generated by B and N.

(TS1) The subgroup $T = B \cap N$ is normal in N.

(TS2) The subset S generates the group $W \coloneqq N/T$ and $s^2 = e$ for all $s \in S$.

(TS3) For all $w \in W$ and $s \in S$ it holds $wBs \subset BwsB \cup BwB$.

(TS4) For all $s \in S$ it holds $sBs \neq B$.

The quotient W is called the *Weyl group* of the Tits system, S the set of the *simple reflections*, $C(w) \coloneqq BwB$ for $w \in W$ are the *Bruhat cells*, and $l(w) = \min\{r \mid w = s_1 s_2 \cdots s_r,\ s_i \in S\}$ is the *length* of w.

Remark. — If we have $s \in S$, by (TS4) $s \neq e$. Also (TS3) is equivalent to the conditions

$$(BwB)(BsB) \subset (BwsB) \cup (BwB) \iff C(w)C(s) \subset C(ws) \cup C(w).$$

Note that G could be an affine algebraic group, but that does not play any role in the definition of Tits systems, which makes sense for any abstract group.

III.11. Theorem ((Abstract) Bruhat decomposition)**.** — *Let (G, B, N, S) be a Tits system with Weyl group W. Then*

$$G = \bigcup_{w \in W} BwB = \bigcup_{w \in W} C(w),$$

with the unions disjoint.

Proof. First we claim that $\bigcup_{w \in W} C(w)$ is a group, i.e. we have to show that it is closed under inverses and multiplication.

- By definition $x \in C(w)$ is of the form $x = b_1 w b_2$ for some $b_i \in B$ and $w \in W$. Then

$$x^{-1} = b_2^{-1} w^{-1} b_1^{-1} \in C(w).$$

- We will show that $C(w_1)C(w_2) \subset \bigcup_{w \in W} C(w)$, by induction on $l(w_2)$. If $l(w_2) = 0$, we have that $w_2 = e$, thus

$$C(w_1)C(W_2) = (Bw_1B)(BeB) = Bw_1B = C(w_1).$$

The inductive step is more delicate. Assume $l(w_2) > 0$ and write $w_2 = sy$ for some $y \in W$ with $l(y) < l(w_2)$ and $s \in S$. Then we have

$$\begin{aligned} C(w_1)C(w_2) &= C(w_1)C(sy) \\ &= Bw_1BBsyB \\ &= B(w_1Bs)yB \\ &\subset Bw_1sByB \cup Bw_1ByB \\ &= C(w_1s)C(y) \cup C(w_1)C(y) \subset \bigcup_{w\in W} C(w). \end{aligned}$$

where the first inclusion is by (TS3) and the last by inductive hypothesis.

Now we can show $G = \bigcup_{w\in W} C(w)$. We have $C(e) = BeB = B$, $W = N/T$ and $T \subset B$. Then $\bigcup_{w\in W} C(w)$ contains B and N, thus by (TS0) we have $\bigcup_{w\in W} C(w) = G$.

Lecture 11
16th Nov, 2022

Now we prove that the union of the Bruhat cells is disjoint. In particular, we will show that if the intersection $C(w) \cap C(w')$ is not empty, then $w = w'$. First, observe that if $C(w) \cap C(w') \neq \varnothing$, then $C(w) = C(w')$, as they are double cosets. To show that $w = w'$ we assume without loss of generality that $l(w) \leqslant l(w')$ and we proceed by induction on $l(w)$.

- For the base case, let $l(w) = 0$. Then $w = e$, so $C(w) = BeB = B$, thus $w' \in B$. Then if we consider a representative $\bar{w}'$ of w' we have $\bar{w}' \in N \cap B = T$, hence by definition of the Weyl group $w' = e$ in W.
- If $l(w) > 0$. Then write $w = ys$ with $l(y) = l(w) - 1$ and $s \in S$. Now $C(w) = C(w')$ implies $C(w) = BysB$, thus

$$y \in C(w)s = BwBs = Bw'Bs$$

where the last equality is our assumption. Then by (TS3) we get $y \in C(w's) \sup C(w')$. But now either $g = w'$ by induction, hence

$$w = ys = w'ss = w'$$

where we have used (TS2), or $y = w'$ also by induction, which is a contradiction as we assumed $l(y) < l(w) \leqslant l(w')$. □

A Tits System for $\mathrm{GL}_n(k)$

The most important Tits system for our purposes is the standard Tits system on $\mathrm{GL}_n(k)$, for k arbitrary.

III.12. Theorem. — *Let k be an arbitrary field. Then the data of $G = \mathrm{GL}_n(k)$, the standard Borel B, the group $N = N_G(T)$, where T is standard torus of diagonal matrices, and the set S of the simple transpositions in $S_n \cong W \coloneqq N_G(T)/T$, constitutes a Tits system.*

The proof of the theorem needs some preparation.

I quite dislike the way this whole section is organized...

Definition. — Let G be a linear algebraic group. A 1-parameter subgroup or cocharacter of G is a morphism of algebraic groups $\mathbb{C}^\times \to G$. We denote by $Y(G) = X_*(G)$ the set of cocharacters of G, which is a (generally *not* commutative) group with pointwise multiplication.

III.13. Lemma. — *Let G be a linear algebraic group.*

(1) *There is a pairing*

$$X^*(G) \times X_*(G) \to \mathbb{Z} = \mathrm{Hom}_{\mathbf{AlgGrp}}(\mathbb{C}^\times, \mathbb{C}^\times), \ (\chi, \varphi) \mapsto \chi \circ \varphi.$$

This should be put in context (root data & co)...

(2) *If $G = T$ is a torus in $\mathrm{GL}_n(\mathbb{C})$, then this pairing is nondegenerate.*

Proof. This is RT2Sheet5.4. □

Definition. — A *root* for $T \subset G = \mathrm{GL}_n(\mathbb{C})$ is a character $\alpha \in X^*(T)$ such that

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \coloneqq \mathfrak{gl}_n \mid txt^{-1} = \alpha(t)x \ \forall t \in T\} \neq \{0\}$$

and $\alpha \neq 1$. We denote by $R = R(T)$ the set of roots of T (or G), and $\mathfrak{g}_\alpha$ is the *root space* corresponding to α.

Example. — Let $1 \leqslant i, j \leqslant n$, $i \neq j$, and $\alpha = \alpha_{ij} : T \to \mathbb{C}^\times, \ t \mapsto t_i t_j^{-1}$. Then we have

$$tE_{ij}t^{-1} = t_i E_{ij} t_j^{-1} = t_i t_j^{-1} E_{ij} = \alpha(t) E_{ij}$$

so α is a root. One can show (using the root decomposition of $\mathfrak{g}$) that these are all the roots of $\mathrm{GL}_n(\mathbb{C})$.

Remark. — Observe that $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space containing $\alpha \otimes 1$ for any root α. One can show that they form the roots of an abstract root system. In general one might look at $\mathrm{GL}_n(k)$ for k an arbitrary field, but then one defines

But is it obvious that our arguments then work similarly in the general case?

$$X^*(T_k) \coloneqq X^*(T_{\bar{k}}).$$

Remark (Connection to Lie algebras)**.** — There are strong connections of what we are doing with Lie algebra theory: root spaces for algebraic (or Lie) groups correspond to root spaces for Lie algebras.

(1) For $g \in G \coloneqq \mathrm{GL}_n(\mathbb{C})$ the morphism

$$\mathrm{conj}(g) : G \to G, \ x \mapsto gxg^{-1}$$

is an isomorphism (in particular a homeomorphism).

(2) The differential of the morphism $\mathrm{conj}(g)$ is an invertible linear map

$$D\mathrm{conj}(g)_e : \mathrm{Lie}\, G = \mathfrak{g} \to T_e G = \mathfrak{g},$$

hence we get a map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), \ g \mapsto D\mathrm{conj}(g)_e,$$

and it is easy to check that $\mathrm{Ad}(g)(x) = gxg^{-1}$.

(3) Taking the differential of Ad we get a morphism

$$\mathrm{ad} \coloneqq D\mathrm{Ad}_e : T_e G = \mathfrak{g} \to T_e \,\mathrm{Aut}(\mathfrak{g}) = T_e \,\mathrm{End}_{\mathbb{C}}(\mathfrak{g}),$$

and it is easy to check that $\mathrm{ad}(x)(y) = xy - yx = [x, y]$, where the bracket is commutator of endomorphisms (this is RT2Sheet6.7).

Now we see that root spaces $\mathfrak{g}_\alpha$ for the conjugation action of T via Ad are the same as the root spaces $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ with respect to the adjoint action of $\mathfrak{h}$, the tangent space at the identity of T (which can be identified with the subalgebra of diagonal matrices in $\mathfrak{g}$), via ad.

Definition. — For $1 \leqslant i, j \leqslant n$ consider $\alpha = \alpha_{ij} \in X^*(T)$ as in the example above, i.e. α_{ij} is $t \mapsto t_i t_j^{-1}$, then:

- if $i < j$ we say α_{ij} is a *positive root*,
- if $i > j$ we say α_{ij} is a *negative root*,
- if $j = i + 1$ we say α_{ij} is a *simple root*.

Let $R = R(T)$ and denote R^+ the positive roots, R^- the negative roots and Π the simple roots. Note that $R = R^+ \cup R^-$. Moreover, the positive roots are in correspondence with the transpositions and the simple roots with the simple transpositions.

Definition. — For $\alpha = \alpha_{ij} \in R$ we have the associated (additive) 1-parameter subgroup

$$X_\alpha : (\mathbb{C}, +) \to G = \mathrm{GL}_n(\mathbb{C}), \ \lambda \mapsto \mathbb{1} + \lambda E_{ij}.$$

By definition we have $tX_\alpha(\lambda)t^{-1} = X_\alpha(\alpha(t)\lambda)$ for all $t \in T$ and $\lambda \in \mathbb{C}$.

Remark. — The previous definition makes sense over any field, given a root α_{ij}.

Definition. — The obvious action of S_n on $R = R(T)$ satisfies

$$wX_\alpha(\lambda)w^{-1} = X_{w(\alpha)}(\lambda)$$

for all $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

III.14. Lemma. — *Under the hypothesis of Theorem III.12, the set $C(e) \cup C(s)$ for any $s \in S$ is a parabolic subgroup of $G = \mathrm{GL}_n(\mathbb{C})$, and it is a minimal proper (that is, different from a Borel) parabolic subgroup.*

What is the significance of this lemma really?

Proof. RT2Sheet5.3 asks to show this. □

Definition. — Consider $\alpha = \alpha_{ij} \in \Pi$.

- We denote by L_α the subgroup of matrices in $G = \mathrm{GL}_n(\mathbb{C})$ consisting of diagonal matrices except that the positions $(i, i+1)$ and $(i+1, i)$ might also be nonzero. We call L_α the *Levi associated to* α.
- We denote by P_α the subgroup of matrices in $G = \mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices except that the positions $(i, i+1)$ and $(i+1, i)$ might also be nonzero. We call L_α the *parabolic associated to* α. Note that this is the star of the previous lemma.
- We denote by U_α the subgroup of matrices in $G = \mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices except that the position $(i, i+1)$ is zero and the position $(i+1, i)$ might be nonzero. We call U_α the *unipotent associated to* α. Note that this is the star of the previous lemma.

Note that clearly $P_\alpha = L_\alpha U_\alpha$.

Remark. — Of course, all of this has analogues for Lie algebras...

By now this doesn't come as a big surprise.

We will need a few lemmas about symmetric groups, which are the Weyl groups of the coveted Tits system for $\mathrm{GL}_n(k)$. In fact, these are really results about general Coxeter groups, of which Weyl groups of Tits systems are a particular case.

I wonder, how much more difficult would it be to do everything in greater generality, i.e. for abstract Weyl groups or Coxeter groups?

III.15. Lemma. — *Let S_n act on $R = R(T)$ in the obvious way. For $w \in S_n$ define*

$$l'(w) := |R^+ \cap w^{-1}(R^{-1})| = |w(R^+) \cap R^-|.$$

If $s = (i, i+1)$ is a simple transposition and $\alpha = \alpha_{i,i+1}$ the corresponding simple root, then

$$l'(sw) = \begin{cases} l'(w)+1 & \text{if } w^{-1}(\alpha) \in R^+ \\ l'(w)-1 & \text{if } w^{-1}(\alpha) \in R^- \end{cases} \qquad l'(ws) = \begin{cases} l'(w)+1 & \text{if } w(\alpha) \in R^+ \\ l'(w)-1 & \text{if } w(\alpha) \in R^- \end{cases}$$

Proof. We claim that if $s = (r, r+1)$, then $s(R^+)$ is R^+ without $\alpha_{r,r+1}$ but with $\alpha_{r+1,r}$, and $s(R^-)$ is R^- without $\alpha_{r+1,r}$ but with $\alpha_{r,r+1}$. This is because (for $i < j$):

$$s\alpha_{ij}s^{-1} = \alpha_{s(i)s(j)} = \begin{cases} \alpha_{ij} & \text{if } \{i,j\} \cap \{r, r+1\} = \varnothing \\ \alpha_{r+1,r} & \text{if } i = r, j = r+1 \\ \alpha_{i\pm1,j} \text{ or } \alpha_{i,j\pm1} & \text{if } |\{i,j\} \cap \{r,r+1\}| = 1 \end{cases}$$

Then $(sw)^{-1}(R^-) = w^{-1}(s(R^-))$, as $w^{-1}s^{-1} = w^{-1}s$. Thus $(sw)^{-1}(R^-)$ is $w^{-1}(R^-)$ with $w^{-1}(\alpha_{r+1,r})$ removed and $w^{-1}(\alpha_{r,r+1})$ added. Now,

$$l'(sw) = |R^+ \cap (sw)^{-1}(R^-)| = l'(w) \pm 1$$

if $w^{-1}(\alpha) \in R^\pm$.

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As for $l'(ws)$, we have

$$l'(ws) = |R^+ \cap (ws)^{-1}(R^-)| = |s(R^+) \cap w^{-1}(R^-)| = |R^+ \cap w^{-1}(R^-)| + a - b,$$

where

$$a = \begin{cases} 1 & \text{if } \alpha' \in w^{-1}(R^-), \text{ i.e. } w(\alpha) \in R^+ \\ 0 & \text{otherwise} \end{cases} \qquad b = \begin{cases} 1 & \text{if } \alpha \in w^{-1}(R^-), \text{ i.e. } w(\alpha) \in R^- \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha' = \alpha_{r+1,r}$ if $\alpha = \alpha_{r,r+1}$. ☐

III.16. Lemma. — *Let S be the set of simple transpositions, $s = s_\alpha \in S$. Assume $s_1, \dots, s_r$ are simple transpositions with $s_1 \cdots s_r(\alpha) \in R^-$. Then there exists an index $1 \leqslant j \leqslant r$ such that*

$$s_1 \cdots s_r = s_1 \cdots s_{j-1} \hat{s}_j s_{j+1} \cdots s_r s.$$

Proof. Given $\alpha \in \Pi \subset R^+$ as in Lemma III.15, there exist some index $1 \leqslant j \leqslant r$ which is the minimal one for which $s_{j+1} \cdots s_r(\alpha) \in R^+$ (in particular we have $s_j s_{j+1} \cdots s_r(\alpha) \in R^-$). Since $s_j(R^+) \cap R^- = \{\alpha_j\}$, with α_j the simple root attached to s_j, we have

$$s_{j+1} \cdots s_r(\alpha) = \alpha_j.$$

Now, setting $y = s_{j+1} \cdots s_r$, we have $ys_{\alpha_{i,j}}y^{-1} = s_{\alpha_{y(i),y(j)}} = s_{y(\alpha_{i,j})}$, which implies

$$ys_\alpha y^{-1} = s_{y(\alpha)} = s_{\alpha_j} = s_j \iff s_j y = ys \iff s_j s_{j+1} \cdots s_r = s_{j+1} \cdots s_r s$$

which means that $s_1 \cdots s_r = s_1 \cdots s_{j-1} \hat{s}_j s_{j+1} \cdots s_r s$, as claimed. □

Remark. — If $w = s_1, \cdots s_r$ is reduced (i.e. $s_i \in S$ for all i) and $l(w) = r$, then for $s = s_\alpha \in S$ we have $ws = s_1 \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r$ for some $1 \leqslant j \leqslant r$ if $s_1 \cdots s_r(\alpha) \in R^-$. Then $l(ws) < l(w)$.

III.17. Lemma. — *For $w \in S_n$, we have $l'(w) = l(w)$.*

Proof. RT2Sheet6.2 (b). □

There are a bunch of useful facts about S_n (and again, more generally about Coxeter groups) which are consequences of Lemmas III.15 and III.16, starting from:

- For $w \in S_n$, we have $l'(w) = l(w)$. This is Lemma III.17 above.
- If $w \in S_n$ is such that $w(R^+) = R^+$, then $l'(w) = l'(w) = 0$, thus $w = e$.
- If $\alpha \in R$, then there exists $w \in S_n$ such that $w(\alpha) \in \Pi$.

Moreover, we have the following lemmas:

III.18. Lemma. — *Assume $w \in S_n$ with $l(w) = r$. Let $w = s_1 \cdots s_r$ be a reduced expression. Then*

$$\{\alpha \in R^+ \mid w(\alpha) \in R^-\} = \{\alpha_r, s_r(\alpha_{r-1}), s_r s_{r-1}(\alpha_{r-2}), \ldots, s_r \cdots s_2(\alpha_1)\}.$$

Proof. Note that the cardinality of the left hand side is $l'(w) = l(w) = r$, which is the same as the cardinality of the right hand side, hence it is enough to show "$\supset$" and that the elements of the right hand side are distinct. We proceed by induction on $l(w)$.

If $l(w) = 0$, then $w = e$, so both sides of the equations are empty, the right hand side by construction.

Assume now $l(w) > 0$. Write $w = s_i y$ for $y \in S_n$ with $l(y) = l(w) - 1$. By induction,

$$\{\alpha \in R^+ \mid w(\alpha) \in R^-\} = \{\alpha_r, s_r(\alpha_{r-1}), s_r s_{r-1}(\alpha_{r-2}), \ldots, s_r \cdots s_3(\alpha_3)\}$$

where the elements of the set in the right hand side are distinct.

Claim 1. We have

$$\{\alpha \in R^+ \mid y(\alpha) \in R^-\} \subset \{\alpha \in R^+ \mid s_1 y(\alpha) \in R^-\} = \{\alpha \in R^+ \mid w(\alpha) \in R^-\}.$$

Proof of the claim. Assume $\alpha \in R^+$ with $y(\alpha) \notin R^-$. Now, as $y(\alpha) \in R^-$ and $s_1 y(\alpha) \in R^+$, we have $y(\alpha) = \alpha_{j+1,j}$ and $s_1 = s_{\alpha_{j+1,j}} = (j+1, j)$. Since $l(s_1 y) = l(w) = r = l(y) + 1$, by Lemma III.16 it must be that $\alpha = y^{-1}(\alpha_{j,j+1})$, a contradiction. ■

Claim 2. Set $\beta = y(\alpha) \in R^-$ for α as in the previous paragraph. Then

$$\beta = s_r s_{r-1} \cdots s_2(\alpha_1) \in \{\alpha \in R^+ \mid w(\alpha) \in R^-\},\ \beta \notin \{\alpha \in R^+ \mid y(\alpha) \in R^-\}.$$

Proof of the claim. We have $y(\beta) = \alpha_1 \in R^+$ by definition of y and α_1, thus $y(\beta) \notin R^-$ and $w(\beta) = s_1 y(\beta) = s_1(\alpha_1) \in R^-$. ■

Applying the two claims then we are done. □

III.19. Lemma (Exchange condition). — *Let $w = s_1 \cdots s_r \in S_n$ for $s_i \in S$, with S the set of simple transpositions. Let $s \in S$ such that $l(ws) < l(w)$. Then there exists an index $1 \leqslant j \leqslant r$ such that $ws = s_1, \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r$. Moreover if $l(w) = r$, the index j is unique.*

Proof. We have $l(ws) < l(w)$, so $l'(ws) < l'(w)$, thus $w(\alpha) \in R^-$ for α such that $s = s_\alpha$, by Lemma III.15. This then proves the existence of j by Lemma III.16.

Now we prove uniqueness. Assume $w = s_1, \cdots s_r$ is reduced and

$$s_1 \cdots s_{i-1}\hat{s}_i s_{i+1} \cdots s_r = ws = s_1, \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r$$

without loss of generality with $i < j$. Then

$$s_{i+1} \cdots s_r = s_i \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r$$

and so

$$s_{i+1} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}.$$

Then

$$s_i \cdots s_j = s_{i+1} \cdots s_{j-1}$$

which contradicts the assumption that $w = s_1 \cdots s_r$ is reduced. □

III.20. Lemma (Strong exchange condition). — *Same as Lemma III.19, but with $s = s_\alpha$ a (not necessarily simple) transposition corresponding to the root α.*

Proof. Exercise (but it was not in the exercise sheets, unless I missed it). □

III.21. Proposition (Deletion property). — *Let $w = s_1 \cdots s_r \in S_n$, with $s_i \in S$ for all i, with $l(w) < r$. Then there exist indices $1 \leqslant i, j \leqslant r$ such that*

$$w = s_1 \cdots s_{i-1}\hat{s}_i s_{i+1} \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r.$$

Proof. By assumption there exist an index $1 \leqslant j \leqslant r$ such that $l(s_1 \cdots s_r) < l(s_1 \cdots s_{j-1})$. Then by the Exchange condition III.19 there exist and index $1 \leqslant i \leqslant j$ such that

$$s_1 \cdots s_j = s_1 \cdots s_{i-1}\hat{s}_i s_{i+1} \cdots s_{j-1},$$

thus

$$w = s_1 \cdots s_{i-1}\hat{s}_i s_{i+1} \cdots s_{j-1}\hat{s}_j s_{j+1} \cdots s_r,$$

as claimed. □

Remark. — *Coxeter groups* are exactly the groups generated by a finite set of involutions (i.e. elements that square to the identity), satisfying the Deletion property (or equivalently, the Strong Exchange condition). Note that there are a number of other equivalent definitions of Coxeter groups, such as Definition V.1 in the next chapter, see [Hum90] or [Bou75] for more on this story.

Remark. — Weyl groups of Tits system are Coxeter groups. If you really must, you can consult [Bou75, IV.2.4 Theorem 2] for a proof (not terribly difficult, but far from illuminating).

Paralleling Lie algebra theory, roots provide a way of studying the structure of algebraic groups, as shown by the following propositions leading to the proof of Theorem III.12.

III.22. Proposition. — *Let $G = \mathrm{GL}_n(k)$, for k an arbitrary field, and $\alpha = \alpha_{i,i+1} \in \Pi$, with $s = s_\alpha = (i, i+1) \in W \cong S_n$. Let $w \in W$ be such that $w(\alpha) \in R^+$. Then*

$$C(w)C(s) = C(ws).$$

Proof. It is enough to show $wBs \subset BwsB$. Let $b \in B$ and consider $wbs \in wBs$. Then $b = tX_\alpha(\lambda)u$ for some $\lambda \in k$, $t \in T$, $u \in U$. Now, we have

$$wbs = wtw^{-1}wX_\alpha(\lambda)w^{-1}wss^{-1}tus$$

and if one is not used to German words, they might want to rewrite this as

$$wbs = (wtw^{-1})(wX_\alpha(\lambda)w^{-1})ws(s^{-1}tus)$$

and notice that $wtw^{-1} \in T \subset B$, $wX_\alpha(\lambda)w^{-1} \in X_\alpha(k) \subset B$ and $s^{-1}tus \in s^{-1}TU_\alpha s \subset B$ (this last one by definition of U_α for $s = s_\alpha$). But then

$$wbs \in BwsB = C(ws),$$

which is what we set out to prove. □

III.23. Proposition. — *Let $G = \mathrm{GL}_n(k)$ for k any field. Assume $w, w' \in W := N_G(T)/T$ satisfy $l(ww') = l(w) + l(w')$. Then $C(w)C(w') = C(ww')$.*

Proof. It is enough to show that for $w = s_1 \cdots s_r$ a reduced expression, we have

$$C(w) = C(s_1) \cdots C(s_r).$$

Let $y = s_1 \cdots s_{r-1}$, $s = s_\alpha \in S$. Then $l(y) = l(w) - 1$, thus $l'(y) = l'(w) - 1$ and so $y(\alpha) \in R^+$ by Lemma III.15. Hence $C(w) = C(y)C(s)$ by Proposition III.22. The claim then follows by induction. □

We can now prove Theorem III.12, on the standard Tits system for $\mathrm{GL}_n(k)$.

Proof of Theorem III.12. It is not difficult to show that (TS0) is verified, and we leave it as an exercise (see RT2Sheet6.4), while (TS1) and (TS2) are immediately clear.

As for (TS3), we need to prove that

$$wBs \subset BwsB \cup BwB$$

for all $w \in W$ and $s = s_\alpha \in S$. We proceed by cases.

- Suppose $w(\alpha) \in R^+$. Then $C(w)C(s) = C(ws)$ by Proposition III.22, which already proves the claim (since $e \in B$).
- Suppose instead $w(\alpha) \in R^-$. Then $ws(\alpha) \in R^+$, hence by the previous case we have

$$wsBs \subset BwssB \cup BwsB = BwB \cup BwsB.$$

Now, the set $B \cup BsB$ is a group by Lemma III.14 (in particular, a minimal proper parabolic) and contains s, so

$$B \cup BsB = sB \cup sBsB,$$

hence $Bs \subset sB \cup sBsB$. But then

$$wBs \subset wsB \cup wsBsB \subset BwsB \cup BwB,$$

as we claim that $wsBsB \subset BwB$. Indeed, we have

$$wsBsB \subset (BwsB)(BsB) = C(ws)C(s) \subset BwB$$

It should be possible to clean up this proof...

by Lemma III.15, since $l(ws) + 1 = l(w)$ and thus $l(wss) = l(w) = l(ws) + l(s)$.

Finally for (TS4) we prove that $sBs \neq B$. Let $s = s_\alpha$ for $\alpha \in \pi$. Then for all $s \in S$ we have

$$sX_\alpha(\lambda)s = sX_\alpha(\lambda)s^{-1} = X_{s(\alpha)}(\lambda)$$

and $X_{s(\alpha)}(\lambda) \notin B$ for $\lambda \neq 0$. □

Remark. — Note that it is not too difficult to prove Theorem III.12 without developing this much amount of preliminary (in fact, RT2Sheet6.4 asks to prove the theorem directly, and even our proof did not use a lot anyway), but the technology of roots is extremely important beyond this particular theorem.

Remark (Tits systems for non-archimedean weirdos)**.** — Consider $G = \mathrm{GL}_n(\mathbb{Q}_p)$. Then there is a Tits system $(G, I, N_G(T), S)$, for I the Iwahori subgroup and T some analogue of the torus. The Iwahori subgroup is defined as follows. There is a canonical map

$$\varphi : \mathrm{GL}_n(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}_p/\mathfrak{m}) = \mathrm{GL}_n(\mathbb{F}_p)$$

where $\mathfrak{m}$ is the maximal ideal in $\mathbb{Z}_p$ (note that $\mathbb{Z}_p$ is a local ring). The Iwahori subgroup (or subalgebra) is then the inclusion in $\mathrm{GL}_n(\mathbb{Q}_p)$ of $\varphi^{-1}(B)$, where B is the standard Borel in $\mathrm{GL}_n(\mathbb{F}_p)$. The Weyl group for this Tits system is an infinite group called the affine Weyl group.

Warning ⚠. — The set S is part of the data of a Tits System! In fact, Mühlherr constructed Tits systems for the same group with equal B and N, but different S. This is why we do not refer to Tits systems as (B, N)-pairs, as some people do.

Humphreys and Bourbaki prove in their books that S is uniquely determined by (G, B, N), though! I guess I misheard this?

Chapter IV.

IV

Hecke Algebras and Kazhdan-Lusztig Polynomials

Hecke Algebras

Lecture 13
23rd Nov, 2022

This chapter is essentially an "application" of the theory we developed in the previous one, i.e. Tits systems and Bruhat decompositions.

Definition. — Fix a finite field $\mathbb{F}_q$ and an arbitrary field k. The *Hecke algebra* $H_n(q)$ associated to $G := \mathrm{GL}_n(\mathbb{F}_q)$, where $B < G$ is the standard Borel, is the k-algebra with underlying vector space

$$\begin{aligned} H_n(g) &= \{f : B\backslash G/B \to k\} \\ &= \{f : G \to k \mid f(gb) = f(g) = f(bg)\ \forall g \in G,\ \forall b \in B\} \\ &= \{f : G \to k \mid f \text{ is } B\text{-biinvariant}\} \end{aligned}$$

with the vector space structure induced by k, and multiplication given by convolution:

$$f * h(x) := \frac{1}{|B|}\sum_{y\in G} f(y)h(y^{-1}x) = \frac{1}{|B|}\sum_{y\in G} f(xy^{-1})h(y)$$

for all $f, h \in H_n(q)$.

It is easy to show that multiplication is indeed associative: we have

$$(f * g) * h(x) = \frac{1}{|B|}\sum_{y\in G}(f * g)(y)h(y^{-1}x) = \frac{1}{|B|^2}\sum_{y,z\in G} f(z)g(z^{-1}y)h(y^{-1}x) \tag{1}$$

and on the other hand

$$f * (g * h)(x) = \frac{1}{|B|^2}\sum_{a,b\in G} f(xa^{-1})g(b)h(b^{-1}a), \tag{2}$$

but now setting $z := xa^{-1}$ we have $z^{-1}y = b$, thus $y = xa^{-1}b$ and $y^{-1}x = b^{-1}ax^{-1}x = b^{-1}a$, which implies that expressions (1) and (2) are the same.

We also want to show that there is a unit. Setting

$$\mathbb{1}_e := \begin{cases} 1 & \text{if } x \in B = BeB \\ 0 & \text{otherwise} \end{cases}$$

this is an element of $H_n(q)$, and we can easily check that $f * \mathbb{1}_e = \mathbb{1}_e * f$. Indeed

$$f * \mathbb{1}_e(x) = \frac{1}{|B|}\sum_{y\in G} f(xy^{-1})\mathbb{1}_e(y) = \frac{1}{|B|}\sum_{y\in B} f(xy^{-1})\mathbb{1}_e y = \frac{1}{|B|}\sum_{y\in B} f(x) = f(x),$$

where we notice that $f(xy^{-1}) = f(x)$ if $y \in B$. Similarly one proves that $\mathbb{1}_e * f = f$.

Remark. — The construction works for any finite group G with subgroup B. Moreover, we could consider compact topological groups or even general topological groups, in the second case adding some conditions on the elements of the Hecke algebra (e.g. compact support).

Remark. — One could define Hecke algebras for arbitrary commutative rings instead of fields (considering modules instead of vector spaces).

IV.1. Lemma. — *The Hecke algebra $H_n(q)$ has a k-basis*

$$\mathcal{B} = \{\mathbb{1}_w \mid w \in S_n\}$$

where we define

$$\mathbb{1}_w(x) = \begin{cases} 1 & \text{if } x \in BwB \\ 0 & \text{otherwise} \end{cases}$$

Proof. This follows directly from the definition of $H_n(q)$ and the Bruhat decomposition. □

Remark. — The previous lemma holds for an Hecke algebra defined for any finite group with Tits system.

IV.2. Proposition. — *Let $W = S_n$.*

(1) *If $w, w' \in S_n$ are such that $l(w) + l(w') = l(ww')$, then*

$$\mathbb{1}_w * \mathbb{1}_{w'} = \mathbb{1}_{ww'}.$$

(2) *For all simple transpositions $s \in S \subset W$, we have*

$$\mathbb{1}_s * \mathbb{1}_s = q\mathbb{1}_e + (q-1)\mathbb{1}_s.$$

(3) *For all $s, t \in S$ such that $st = ts$ we have*

$$\mathbb{1}_s * \mathbb{1}_t = \mathbb{1}_t * \mathbb{1}_s.$$

To prove the proposition we need a lemma.

IV.3. Lemma. — *Define $\varepsilon : H_n(q) \to k$ by*

$$\varepsilon(f) = \frac{1}{|B|}\sum_{y\in G} f(y).$$

Then ε is an algebra morphism with $\varepsilon(\mathbb{1}_e) = 1$.

Proof. We have

$$\varepsilon(f*h)=\frac{1}{|B|}\sum_{y\in G}(f*h)(y)=\frac{1}{|B^2|}\sum_{y\in G}\sum_{x\in G}f(x)h(x^{-1}y)=\varepsilon(f)*\varepsilon(h)$$

so ε is an algebra morphism, and

$$\varepsilon(\mathbb{1}_e)=\frac{1}{|B|}\sum_{y\in G}\mathbb{1}_e(y)=\frac{1}{|B|}\sum_{y\in B}1=1$$

as claimed. □

Remark. — As a consequence of Proposition IV.2, if $w\in W$, $s\in S$ and $l(ws)<l(w)$, then

$$\mathbb{1}_w*\mathbb{1}_s=q\mathbb{1}_{ws}+(q-1)\mathbb{1}_w.$$

Indeed, write $w=ys$ for some y with $l(y)=l(w)-1$ and $l(ys)=l(y)+1$, then

$$\begin{aligned}\mathbb{1}_w*\mathbb{1}_s=(\mathbb{1}_y*\mathbb{1}_s)*\mathbb{1}_s&=\mathbb{1}_y*(\mathbb{1}_s*\mathbb{1}_s)\\&=q\mathbb{1}_y*\mathbb{1}_e+(q-1)\mathbb{1}_y*\mathbb{1}_s\\&=q\mathbb{1}_y+(q-1)\mathbb{1}_{ys}\\&=q\mathbb{1}_{ws}+(q-1)\mathbb{1}_w.\end{aligned}$$

where we have used Proposition IV.2, given that $l(ys)=l(y)+l(s)$.

Proof of Proposition IV.2. (1) Consider

$$\mathbb{1}_w*\mathbb{1}_{w'}(x)=\frac{1}{B}\sum_{y\in G}\mathbb{1}_w(y)\mathbb{1}_{w'}(y^{-1}x).$$

If $y\in BwB=C(w)$ and $y^{-1}x\in Bw'B=C(w')$, then $x=y(y^{-1}x)\in C(w)C(w')=C(ww')$ by Proposition III.23, as $l(w)+l(w')=l(ww')$, thus

$$\mathbb{1}_w*\mathbb{1}_{w'}=\gamma\mathbb{1}_{ww'}$$

for some $\gamma\in h$. Applying ε to both sides of the equation we get

$$\varepsilon(\mathbb{1}_w*\mathbb{1}_{w'})=\varepsilon(\mathbb{1}_w)*\varepsilon(\mathbb{1}_{w'})=\varepsilon(\gamma\mathbb{1}_{ww'})=\gamma\varepsilon(\mathbb{1}_{ww'}).$$

We use that

$$\varepsilon(\mathbb{1}_w)=q^{l(w)}\tag{$*$}$$

I think we did not prove this in the end, but it shouldn't be difficult.

for all $w\in S_n$. Armed with this fact, we have

$$q^{l(w)}q^{l(w')}=\gamma q^{l(ww')},$$

thus $\gamma=1$, as $l(w)+l(w')=l(ww')$. Hence

$$\mathbb{1}_w*\mathbb{1}_{w'}=\mathbb{1}_{ww'}.$$

(2) Consider

$$\mathbb{1}_s*\mathbb{1}_s(x)=\frac{1}{|B|}\sum_{y\in G}\mathbb{1}_s(y)\mathbb{1}_s(y^{-1}x).$$

Note that $x = y(y^{-1}x)$, hence the support of $\mathbb{1}_s * \mathbb{1}_s$ is contained in

$$C(s)C(s) \subset C(e) \cup C(s)$$

using (TS3). Then

$$\mathbb{1}_s * \mathbb{1}_s = a\mathbb{1}_e + b\mathbb{1}_s$$

for some $a, b \in k$. Evaluating at e we get

$$\frac{1}{|B|} \sum_{y \in G} \mathbb{1}_s(y)\mathbb{1}_s(y^{-1}) = a$$

thus

$$\frac{1}{|B|}|BsB| = a$$

as $y \in BsB$ if and only if $y^{-1}BsB$. Now $\varepsilon(\mathbb{1}_s) = \frac{|BsB|}{|B|}$ by definition of ε and $\mathbb{1}_s$, so

$$a = \varepsilon(\mathbb{1}_s) = q$$

by $(*)$. We can determine b by applying ε to the equation $\mathbb{1}_s * \mathbb{1}_s = a\mathbb{1}_e + b\mathbb{1}_s$. We have

$$q \cdot q = q + qb$$

so $b = q - 1$.

(3) Assuming $s \neq t$, we have

$$\mathbb{1}_s * \mathbb{1}_t = \mathbb{1}_{st} = \mathbb{1}_{ts} = \mathbb{1}_t * \mathbb{1}_s,$$

which is already the claim. □

Definition. — The *generic* (*finite*) *Hecke algebra* $\mathcal{H}_v(S_n)$ associated to S_n (and S, the set of simple transposition), is the $\mathbb{Z}[v, v^{-1}]$-algebra with generators $H_i = H_{s_i}$ for $1 \leqslant i \leqslant n-1$ (or $s_i = (i, i+1) \in S$), and relations

(H1) $H_i^2 = 1 + (v^{-1} - v)H_i$,

(H2) $H_iH_j = H_jH_i$ for all $1 \leqslant i, j \leqslant n-1$, if $|i-j| > 1$, or equivalently $s_is_j = s_js_1$,

(H3) $H_iH_jH_i = H_jH_iH_j$ for all $1 \leqslant i, j \leqslant n-1$, if $|i-j| = 1$, or equivalently $s_is_js_i = s_js_is_j$.

For $w \in S_n$ with reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ with $s_{i_j} \in S$, set

$$H_w \coloneqq H_{i_1}H_{i_2}\cdots H_{i_r} \in \mathcal{H}_v(S_n),$$

which is well defined by the Lemma of Matsumoto V.3, and $H_e \coloneqq 1 \in \mathcal{H}_v(S_n)$.

IV.4. Lemma. — *Set $T_i \coloneqq v^{-1}H_i$. There is a $\mathbb{Z}$-algebra morphism*

$$\begin{aligned} \Phi_q : \mathcal{H}_v(S_n) &\to H_n(q) \\ T_i &\mapsto \mathbb{1}_{s_i} \\ v &\mapsto q^{-1/2} \end{aligned}$$

assuming $q^{-1/2} \in k$, for any finite field $\mathbb{F}_q$.

We will always assume $q^{-1/2} \in k$ in the following, without mention.

Proof. By definition $\mathcal{H}_v(S_n)$ is generated as a $\mathbb{Z}[v, v^{-1}]$-algebra by H_i for $1 \leqslant i \leqslant n-1$, so it is generated as a $\mathbb{Z}$-algebra by $v, v^{-1}, \{H_i\}_i$, and thus also by $v, v^{-1}, \{T_i\}_i$. We are left to check that the assignment is well defined.

(H1) In $\mathcal{H}_v(S_n)$ we have

$$H_i^2 = 1 + (v^{-1} - v)H_i \iff v^{-2}H_i^2 = v^{-2} + (v^{-2} - 1)v^{-1}H_i \iff T_i^2 = v^{-2} + (v^{-2} - 1)T_i$$

and applying Φ_q to the last equation we get

$$\mathbb{1}_{s_i} * \mathbb{1}_{s_i} = q\mathbb{1}_e + (q-1)\mathbb{1}_{s_i}$$

which we know to be true by Proposition IV.2.

(H2) We have

$$\Phi_q(H_iH_j) = q\mathbb{1}_{s_i}\mathbb{1}_{s_j} = q\mathbb{1}_{s_j}\mathbb{1}_{s_i} = \Phi_q(H_jH_i),$$

where we have used Proposition IV.2.

(H3) For $|i-j| = 1$ we have

$$\begin{aligned}\Phi_q(H_iH_jH_i) = q^{3/2}\mathbb{1}_{s_i} * \mathbb{1}_{s_j} * \mathbb{1}_{s_i} &= q^{3/2}\mathbb{1}_{s_is_js_i}\\ &= q^{3/2}\mathbb{1}_{s_js_is_j} = q^{3/2}\mathbb{1}_{s_j} * \mathbb{1}_{s_i} * \mathbb{1}_{s_j} = \Phi_q(H_jH_iH_j),\end{aligned}$$

where use that $s_is_js_i$ and $s_js_is_j$ are reduced if $|i-j| = 1$ and again Proposition IV.2.

Thus we have proved that Φ_q is well defined. □

IV.5. Theorem. — *The generic Hecke algebra $\mathcal{H}_v(S_n)$ is a free $\mathbb{Z}[v, v^{-1}]$-module with a basis given by $\{H_w \mid w \in W = S_n\}$.*

Proof. Let $\mathcal{B}$ be the proposed basis, i.e. $\mathcal{B} = \{H_w \mid w \in S_n\}$. We first show that $\mathcal{B}$ generates $\mathcal{H}_v(S_n)$ as a $\mathbb{Z}[v, v^{-1}]$-module. Note that $1 = H_e \in \mathcal{B}$ and the algebra generators H_i are in $\mathcal{B}$, so it is enough to show $\operatorname{span}_{\mathbb{Z}[v,v^{-1}]} \mathcal{B}H_i \subset \operatorname{span}_{\mathbb{Z}[v,v^{-1}]} \mathcal{B}$. To this end let

$$h = \sum_{y \in W} a_yH_y \in \operatorname{span}_{\mathbb{Z}[v,v^{-1}]} \mathcal{B}$$

with $a_y \in \mathbb{Z}[v, v^{-1}]$. Then

$$\begin{aligned}hH_i = (\sum_{y\in W} a_yH_y)H_i &= \sum_{l(ys_i)>l(y)} a_yH_yH_{s_i} + \sum_{l(ys_i)<l(y)} a_yH_yH_{s_i}\\ &= \sum_{l(ys_i)>l(y)} a_yH_{ys_i} + \sum_{l(ys_i)<l(y)} a_yH_{ys_i}H_{s_i}H_{s_i}\\ &= \sum_{l(ys_i)>l(y)} a_yH_{ys_i} + \sum_{l(ys_i)<l(y)} a_yH_{ys_i} + \gamma\sum_{l(ys_i)<l(y)} a_yH_y,\end{aligned}$$

where we have used $(H1)$ in the last passage. Thus $hH_i \in \operatorname{span}_{\mathbb{Z}[v,v^{-1}]} \mathcal{B}$.

Lecture 14
30th Nov, 2022

We are left to prove linear independence of the elements of $\mathcal{B}$. Assume $\sum_{w\in W} a_wH_w = 0$ with $a_w \in \mathbb{Z}[v, v^{-1}]$. Without loss of generality we can assume $a_w \in \mathbb{Z}[v]$ (multiplying with big enough powers of v). Applying the $\mathbb{Z}$-algebra morphism Φ_q we get

$$0 = \Phi_q(\sum_{w\in W} a_wH_w) = \sum_{w\in W} a_w(q^{-\frac{1}{2}})q^{\frac{l(w)}{2}}\mathbb{1}_w,$$

and since the elements $\mathbb{1}_w$ form a basis of $H_q(S_n)$, we must have $a_w(q^{-\frac{1}{2}}) = 0$ for all w. Now varying q over all finite fields $\mathbb{F}_q$, we get $a_w = 0$ for all $w \in W$, since it is a polynomial which vanishes at infinitely many points, thus proving linear independence of the elements of $\mathcal{B}$.

In other words, for $f \in \mathbb{Z}[v, v^{-1}]$, $f \cdot x = f(\xi, \xi^{-1})x$

IV.6. Corollary. — *Let $\xi \in \mathbb{C}^\times$ and let $\mathbb{C}_\xi$ be the $\mathbb{Z}[v, v^{-1}]$-module which is $\mathbb{C}$ as a vector space, with $v \cdot x = \xi x$ and $v^{-1} \cdot x = \xi^{-1} x$. Then*

$$\mathcal{H}_\xi(S_n) \coloneqq \mathbb{C}_\xi \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{H}_v(S_n)$$

is a $\mathbb{C}$-algebra with basis $1 \otimes H_w$ for $w \in S_n$. We call $\mathcal{H}_\xi(S)$ the specialized (at ξ) Hecke algebra.

Remark. — Instead of $\mathbb{C}_\xi$ one can also take, for example, $\mathbb{Q}[v, v^{-1}]$, $\mathbb{C}[v, v^{-1}]$ or $\mathbb{Q}(v)$, $\mathbb{C}(v)$ with the obvious action of $\mathbb{Z}[v, v^{-1}]$.

Remark. — Note that $\mathcal{H}_1(S_n) \cong \mathbb{C}[S_n]$, the group algebra of $W = S_n$, with the isomorphism witnessed by the algebra morphism

$$\Phi : H_s \mapsto s$$

which is obviously surjective, as it hits all the generators, and sends the $\mathbb{C}$-basis given by elements $1 \otimes H_w$ of $\mathcal{H}_1(S_n)$ to the $\mathbb{C}$-basis $w \in W$ for $\mathbb{C}[S_n]$.

Kazhdan-Lusztig Basis

Before we continue, notice the following: in the algebra $\mathcal{H}_v(S_n)$, the element H_s is invertible with inverse $H_s^{-1} = H_s + (v - v^{-1})$. Indeed,

$$H_s H_s^{-1} = H_s^2 + (v - v^{-1})H_s = 1 + (v^{-1} - v)H_s + (v - v^{-1})H_s = 1.$$

Moreover, this implies that for all $w \in W = S_n$ with $w = s_1 \cdots s_r$ a reduced expression, we have $(H_w)^{-1} = H_{s_r}^{-1} \cdots H_{s_1}^{-1}$.

IV.7. Definition. — There is a unique ring morphism

$$\mathcal{H}_v(S_n) \to \mathcal{H}_v(S_n), \ h \mapsto \bar{h},$$

given by $v \mapsto v^{-1}$, $v^{-1} \mapsto v$ and $H_w \mapsto (H_{w^{-1}})^{-1}$, called *bar-involution*. Uniqueness of the morphism is clear, but we need to check that it is well defined: the relations (H2) and (H3) are evidently respected, and so is (H1) since H_s^2 is sent to

$$H_s^{-1} H_s^{-1} = H_s^2 + 2(v - v^{-1})H_s + (v - v^{-1})^2 = 1 + (v - v^{-1})H_s + (v - v^{-1})^2,$$

and on the other hand $1 + (v^{-1} - v)H_s$ is sent to

$$1 + (v - v^{-1})H_s^{-1} = 1 + (v - v^{-1})H_s + (v - v^{-1})^2.$$

Definition. — An element $h \in \mathcal{H}_v(S_n)$ is *selfdual* if it is *bar-invariant*, i.e. $\bar{h} = h$.

Example. — Of course, $1 \in \mathcal{H}_v(S_n)$ is selfdual. More interestingly, elements of the form $C_s \coloneqq H_s + v$, for s a simple transposition, are selfdual. Indeed,

$$\overline{C_s} = H_s^{-1} + v^{-1} = H_s + (v - v^{-1}) + v^{-1} = H_s + v = C_s.$$

The C_s are then selfdual generators for the generic Hecke algebra.

Remark. — We have $C_s^2 = (v + v^{-1})C_s$. Indeed,

$$C_s^2 = (H_s + v)^2 = 1 + (v^{-1} - v)H_s + 2vH_s + v^2.$$

IV.8. Theorem (Kazhdan-Lusztig). — *For any $w \in W = S_n$ there is a unique selfdual element of the form*

$$\underline{H}_w = H_w + \sum_{\substack{y \in W \\ l(y) < l(w)}} h_{y,w}(v) H_y, \tag{KL}$$

with $h_{y,w} \in \mathbb{Z}[v]$ without constant terms. The coefficients $h_{y,w}$ are the very famous Kazhdan-Lusztig polynomials (and (KL) the Kazhdan-Lusztig condition).

Remark. — We define $h_{w,w}(v) = 1$ for all $w \in W$ and $h_{y,w}(x) = 0$ if $l(y) \nleqslant l(w)$. Note the important formula:

$$H_w C_s = \begin{cases} H_{ws} + vH_w & l(ws) > l(w) \\ H_{ws} + v^{-1}H_w & l(ws) < l(w) \end{cases} \tag{$\star$}$$

for all $w \in W$ and $s \in S$. This follows from

$$H_w H_s = \begin{cases} H_{ws} & l(ws) > l(w) \\ H_{ws} + (v^{-1} - v)H_w & l(ws) < l(w) \end{cases}$$

which is clear from (H1).

Proof. We set $\underline{H}_e \coloneqq H_e$ and $\underline{H}_s \coloneqq C_s = H_s + vH_e$ for $s \in S$. Assume $\underline{H}_x$ is defined for all $x \in W$ with $l(x) < l(w)$. Let x be such that $xs = w$ for some $s \in S$ and $l(w) = l(x) + 1$, and let

$$\underline{H}_x \coloneqq H_x + \sum_{l(z) < l(x)} a_z H_z,$$

with $a_z \in v\mathbb{Z}[v]$. Consider $\underline{H}_x C_s$. This is selfdual, being a product of selfdual elements (and bar-involution being an algebra morphism). Moreover, we have

$$\begin{aligned} \underline{H}_x C_s &= H_x(H_s + v) + \sum_{\substack{l(z) < l(x) \\ l(zs) > l(z)}} a_z H_z C_s + \sum_{\substack{l(z) < l(x) \\ l(zs) < l(z)}} a_z H_z C_s \\ &= H_{xs} + vH_x + \sum_{l(z) < l(x)} a_z H_{zs} + \sum_{\substack{l(z) < l(x) \\ l(zs) > l(z)}} a_z v H_z + \sum_{\substack{l(z) < l(x) \\ l(zs) < l(z)}} a_z v^{-1} H_z \\ &= H_w + \sum_{l(y) < l(w)} \beta_y H_y \end{aligned}$$

for some $\beta_y \in \mathbb{Z}[v]$, noting that $l(z) < l(x) < l(xs)$ and $l(zs) \leqslant l(z) + 1 < l(xs)$. If we consider now

$$\underline{H}_w = \underline{H}_x C_s - \sum_{l(y) < l(w)} \beta_y(0) \, \underline{H}_y,$$

this satisfies (KL), in particular we get coefficients in $\mathbb{Z}[v]$ without constant terms, and is selfdual, thus we have proved existence.

As for uniqueness, assume that $\underline{H}_w$ and $\underline{H}'_w$ are two selfdual elements satisfying (KL) and let $H := \underline{H}_w - \underline{H}'_w$. Then H is selfdual and

$$H = \sum_{l(y)<l(w)} \gamma_y H_y$$

for some $\gamma_y \in v\mathbb{Z}[v]$. We claim that $H = 0$. Let z be of maximal length with $\gamma_z \neq 0$. Then the coefficient of H_z in $\bar{H}$ is $\bar{\gamma}_z$ (since $\underline{H}_z = (H_{z^{-1}})^{-1}$ and $H_s^{-1} = H_s + (v - v^{-1})$). As $H = \bar{H}$, we get $\gamma_z = \bar{\gamma}_z$, thus $\gamma_z = 0$ (since $\gamma \in v\mathbb{Z}[v]$), which is a contradiction. $\square$

Remark. — The elements $\underline{H}_w$ for $w \in W$ form a basis of $\mathcal{H}_v(S_n)$ as a $\mathbb{Z}[v, v^{-1}]$-module, the *Kazhdan-Lusztig* (or *KL*) *basis* (after a good choice of ordering for the basis vectors, the base change matrix is an upper triangular matrix with 1's on the diagonal). The elements H_w for $w \in W$ form the *standard basis*. Note that the KL polynomials are the $\mathbb{Z}[v]$-polynomials appearing in the base change from the standard basis to the KL basis

Example. — We compute the KL basis for $W = S_3$. Let $s = (1,2)$, $t = (2,3)$ and $w_0 = (1,3) = sts = tst$. We have:

- $\underline{H}_e = H_e$,
- $\underline{H}_s = C_s = H_s + v$,
- $\underline{H}_s \underline{H}_t = \underline{H}_s C_t = (H_s + v)(H_t + v) = H_{st} + vH_s + vH_t + v^2$,
- $\underline{H}_t \underline{H}_s = H_{ts} + vH_t + vH_s + v^2$.

Finally we need to compute $\underline{H}_{w_0}$. We have

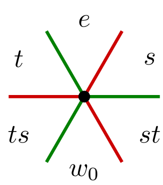
$$\begin{aligned} \underline{H}_{st} C_s &= (H_{st} + vH_s + vH_t + v^2)(H_s + v) \\ &= H_{w_0} + vH_{st} + vH_s^2 + vH_{ts} + v^2 H_t + v^2 H_s + v^3, \end{aligned}$$

hence

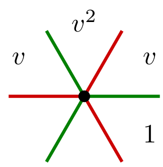
$$\underline{H}_{w_0} = \underline{H}_{st} C_s - \underline{H}_s = H_{w_0} + v(H_{st} + H_{ts}) + v^2(H_t + H_s) + v^3.$$

Remark. — There is a graphical way to compute the KL basis. We describe it here in the case of $W = S_3$, but of course this can be adapted to any Coxeter group. Start by identifying W with the Weyl chambers for $\mathfrak{sl}_3(\mathbb{C})$.

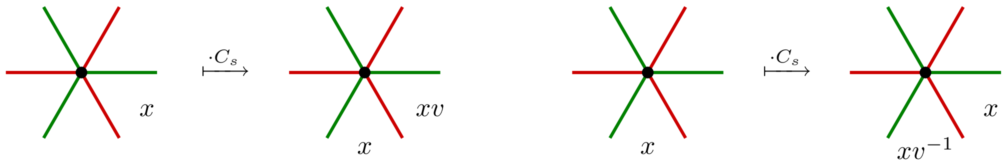
Note that the Weyl group elements indexing adjacent chambers differ by a simple transposition. Therefore we can distinguish between s-hyperplanes (here in red) and t-hyperplanes (here in green).



This identification allows us to associate a diagram to any element in the KL basis, by letting the coefficient of H_w occupy the chamber labelled by w. For instance, the basis element $\underline{H}_{st} = H_{st} + vH_s + vH_t + v^2$ corresponds to the following picture:

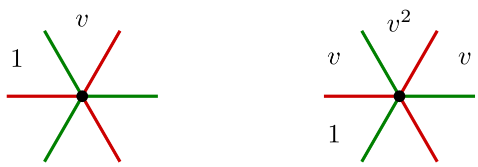


Consider the element xH_w. We know by the formula ($\star$) above that multiplying by C_s amounts to “reflecting” the coefficient x with respect to the s-hyperplane and adding it in the original chamber, multiplied by v if $l(ws) > w$, or by v^{-1} if $l(ws) < w$. For example:



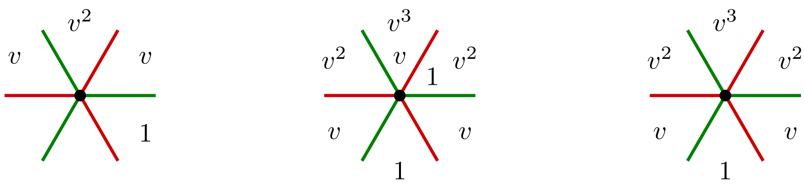
xH_{st} $\qquad$ $xH_{w_0} + xvH_{st}$ $\qquad$ xH_{w_0} $\qquad$ $xv^{-1}H_{w_0} + xH_{st}$

Doing this for each coordinate in the standard basis, we can multiply any element of the Hecke algebra by C_s. For example:



$\underline{H}_t$ $\qquad$ $\underline{H}_t C_s = \underline{H}_{ts}$

Similarly for C_t. In this manner, we can compute $\underline{H}_e$, $\underline{H}_v$, $\underline{H}_t$, $\underline{H}_{st}$ and $\underline{H}_{ts}$. When computing $\underline{H}_{st}C_s$, we identify $\underline{H}_s$ as a subdiagram. Since we do not want KL polynomials to have constant terms, we remove it to get $\underline{H}_{w_0}$.



$\underline{H}_{st}$ $\qquad$ $\underline{H}_{st}C_s$ $\qquad$ $\underline{H}_{w_0}$

Conjectures and Results about Kazhdan-Lusztig Polynomials

Lecture 15
2nd Dec, 2022

Looking at computations of KL polynomials, one can conjecture the following:

IV.9. Conjecture (Positivity conjecture). — *It always holds that* $h_{y,w}(v) \in \mathbb{N}_0[v]$.

Remark. — The conjecture is known to be true in some cases, and we will observe below that it is indeed true for $\mathcal{H}_v(S_n)$, the case we are most concerned with. More generally, the conjecture holds for Weyl groups. However, it is still open for general Coxeter systems.

Remark. — It was proved by Polo in [Pol99] that any polynomial $p \in v\mathbb{N}_0[v]$ appears as a KL polynomial of some S_n.

IV.10. Theorem (Kazhdan-Lusztig conjecture). — *Let* $\mathfrak{g} = \mathrm{sl}_n(\mathbb{C})$ *with* $\mathfrak{b}$ *and* $\mathfrak{h}$ *the standard Borel and Cartan. Let* $w \in W = S_n$ *be an element in the Weyl group. Consider the Verma module*

$$M(w.0) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{w.0}$$

where $w.0$ *is* w *applied to* 0 *via dot action. Then defining the multiplicity* $[M(w.0) : L(y.0)]$ *as the times the irreducible highest weight module* $L(y.0)$ *occurs as a subfactor in a Jordan-Hölder series, we have*

$$[M(w.0) : L(y.0)] = h_{w,y}(1)$$

with $h_{w,y}$ *the KL polynomials with respect to* $\mathcal{H}_v(S_n)$*..*

Example. — Let $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$ and $M(0) = M(e.0)$. We have a short exact sequence

$$M(s.0) \hookrightarrow M(0) \twoheadrightarrow L(0)$$

and $M(s.0) = L(s.0)$, thus

$$[M(0) : L(0)] = 1,\ [M(0) : L(s.0)] = 1.$$

On the other hand, considering

$$\begin{aligned} \underline{H}_e &= H_e = h_{e,e}H_e + h_{s,e}H_s, \\ \underline{H}_s &= C_s = H_s + vH_e = h_{e,s}H_e + h_{s,s}H_s, \end{aligned}$$

we see that the KL polynomials for $\mathcal{H}_v(S_2)$ are $h_{e,e} = 1$, $h_{s,e} = 0$, $h_{e,s} = v$ and $h_{s,s} = 1$, thus

$$\begin{gathered} [M(0) : L(0)] = h_{e,e}(1) = 1,\ [M(s.0) : L(0)] = h_{s,e}(1) = 0, \\ [M(0) : L(s.0)] = h_{e,s}(1) = 1,\ [M(s.0) : L(s.0)] = h_{s,s}(1) = 1, \end{gathered}$$

which is correct, as $M(s.0) = L(s.0)$.

Remark. — Knowing the multiplicities referred to in the theorem allows one to compute characters of the $L(y.0)$'s (using that we know $\operatorname{char} M(w.0)$ for all $w \in W$):

$$\operatorname{char} L(y.0) = \sum_{\lambda \in \mathfrak{h}^*} \dim L(y.0)_\lambda e^\lambda.$$

Remark. — The first proof of the Kazhdan-Lusztig conjecture IV.10 was given by Beilinson-Brylinski-Kashiwara, using the category of perverse sheaves on $\mathrm{GL}_n(\mathbb{C})/B$. A second proof was given by Elias-Williamson in 2014 using Soergel bimodules (which are essentially cohomology rings of $\mathrm{GL}_n(\mathbb{C})/B$) and resolutions of Schubert varieties.

The Kazhdan-Lusztig conjecture give a meaning to the value $h_{w,y}(1)$, but how can we interpret the polynomial $h_{w,y}(v)$? Assume that M is an R-module for some ring R. Assume M has finite length (i.e. a finite Jordan Hölder series). Then $\mathrm{soc}^1(M) = \mathrm{soc}(M)$, called the *socle* of M, is the largest semisimple submodule of M. We want to define $\mathrm{soc}^j(M)$ for $j \in \mathbb{N}$ inductively, so assume $\mathrm{soc}^{j-1}(M)$ is defined. Let $\pi : M \to M/\mathrm{soc}^{j-1}(M)$ be the canonical projection. Then

$$\mathrm{soc}^j(M) \coloneqq \mathrm{can}^{-1}(\mathrm{soc}(M/\mathrm{soc}^{j-1}(M)))$$

is a submodule of M. Clearly we get a filtration of M (i.e. an increasing sequence of submodules $\{0\} \subset \mathrm{soc}^1(M) \subset \mathrm{soc}^2(M) \subset \cdots \subset \mathrm{soc}^r(M) = M$ for some $r \in \mathbb{N}$). This is called the *socle filtration* of M. One can check that by construction the subquotients of this filtration are semisimple.

IV.11. Theorem (Refined Kazhdan-Lusztig conjecture)**.** — *In the setting of the Kazhdan-Lusztig conjectures IV.10, we have that*

$$[\operatorname{soc}^j(M(w.0))/\operatorname{soc}^{j+1}(M(w.0)) : L(y.0)]$$

equals the coefficient of v^{r-j} in $h_{w,y}(v)$, where r is the length of the socle filtration.

Remark. — The proof is based on work of Beilinson-Ginzburg-Soergel, using perverse sheaves and the category $\mathcal{O}$ around Verma modules, and work of Irving, equating the socle filtration of Verma modules and the radical filtration (the two filtrations are in some sense dual). The theorem was proved by our beloved Lecturer (yay!).

Remark. — As a consequence of Theorem IV.11, we get the positivity conjecture in the case of $\mathcal{H}_v(S_n)$.

Remark. — Theorems IV.10 and IV.11 hold for any complex semisimple Lie algebra with $\mathcal{H}_v(S_n)$ replaced by $\mathcal{H}_v((W,S))$ with W the Weyl group of $\mathfrak{g}$ and S the simple reflections. Here $\mathcal{H}_v((W,S))$ is the Hecke algebra attached to (W,S) viewed as a Coxeter group (we already defined Coxeter group in a previous remark, but see the next section).

A first question to ask about KL polynomials is: when is $h_{y,x}(v) \neq 0$? We know already that if $h_{y,x}(v) \neq 0$ we have $l(y) < l(x)$. We want something more, and our quest will lead us later to consider some connections with geometry. However, first we need to introduce Coxeter systems, a generalization of symmetric and Weyl groups which is essential to study Hecke algebras in general.

Chapter V. Coxeter Groups

V

Coxeter Systems

V.1. Definition. — A *Coxeter system* is a pair (W, S), where W is a group and $S \subset W$ is a set of generators of W such that the relations between them are of the form:

(Cox1) $s^2 = e$ for all $s \in S$,

(Cox2) $st \cdots st \cdots = ts \cdots ts \cdots$, where the products have m_{st} factors for some $m_{st} \in \mathbb{N}$, for $s, t \in S$.

Of course, W is called a *Coxeter group* (but a Coxeter system is strictly more data, as a single Coxeter group can have different subsets giving rise to Coxeter systems).

Definition. — We define $\mathcal{H}_v((W, S))$ as the $\mathbb{Z}[v, v^{-1}]$-algebra generated by H_s for $s \in S$, modulo the relations:

(H1) the same as before,

(H2) $H_s H_t \cdots H_s H_t = H_t H_s \cdots H_t H_s \cdots$ where as above the products have m_{st} factors for some $m_{st} \in \mathbb{N}$.

Remark. — As we already remarked, Weyl groups with simple reflections give Coxeter systems. This is of course particularly evident if we consider Weyl groups of root systems (as opposed to, more abstractly, Weyl groups of Tits systems).

We want to show that S_n and the subset of simple transpositions form a Coxeter system.

V.2. Theorem (Coxeter presentation of $\mathbb{C}[S_n]$)**.** — *The group algebra $\mathbb{C}[S_n]$ is isomorphic to the $\mathbb{C}$-algebra generated by the simple transposition s_i for $1 \leqslant i \leqslant n-1$, modulo relations:*

(1) $s_i^2 = e = 1$,

(2) $s_i s_j = s_j s_i$ *if* $|i - j| > 1$,

(3) $s_i s_j s_i = s_j s_i s_j$ *if* $|i - j| = 1$.

Remark. — Note that (2) and (3) can be formulated as in (Cox2) with

$$m_{ij} = \begin{cases} 2 & |i-j| > 1 \\ 3 & |i-j| = 1 \end{cases}$$

Proof. The relations hold for simple transpositions, thus we get a well-defined $\mathbb{C}$-algebra morphism

$$\mathcal{H}_1(S_n) \to \mathbb{C}[S_n]$$

which is clearly an isomorphism (we had already observed this before). $\square$

Remark. — Clearly (Cox1) and (Cox2) are equivalent to the condition that for any two elements $s, t \in S$, one either has no relations between them or $(st)^{m_{s,t}} = e$ for some integer $m_{s,t} \in \mathbb{N}$, greater than 2 if $s \neq t$ and equal to 1 otherwise (and these relations define the group). One can also equivalently require that $m_{s,t} = m_{t,s}$.

Lecture 16
6th Dec, 2022

A fundamental result on Coxeter groups is the Lemma of Matsumoto. We will prove it for the symmetric group $W = S_n$ and S the subset of simple transpositions (and this in turn will give a proof that (S_n, S) is a Coxeter system). The setup is as follows: if w and w' are words in letters from S, they are called equivalent if they differ by applying finitely many times the following rewritings:

- $s_i s_j \sim s_j s_i$ if $|i-j| > 1$,
- $s_i s_j s_i \sim s_j s_i s_j$ if $|i-j| = 1$.

The Lemma of Matsumoto says that different reduced expressions of the same element are equivalent, i.e. differ only by the two rewritings above.

V.3. Theorem (Lemma of Matsumoto)**.** — *Let $w \in S_n$ and assume we are given two reduced expressions $w = s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$. Then*

$$s_{i_1} \cdots s_{i_r} \sim s_{j_1} \cdots s_{j_r}.$$

Remark. — The theorem is used (as we mentioned), to prove that $\mathcal{H}_v(S_n)$ is well defined.

Proof. We prove the claim by induction on $l(w)$. For $l(w) \leqslant 1$ it is clear. Assume then that $l(w) \geqslant 2$. We know that $l(s_{i_1} w) < l(w)$, so $s_{i_1} s_{j_1} \cdots s_{j_r}$ is not reduced. By the Exchange condition III.19, we have

$$s_{i_1} s_{j_1} \cdots s_{j_r} = s_{j_1} \cdots s_{j_{a-1}} \hat{s}_{j_a} s_{j_{a+1}} \cdots s_{j_r}$$

for some $1 \leqslant a \leqslant r$. We get the following two formulas:

$$s_{i_2} s_{i_3} \cdots s_{i_r} = s_{j_1} \cdots s_{j_{a-1}} \hat{s}_{j_a} s_{j_{a+1}} \cdots s_{j_r} \tag{1}$$
$$s_{i_1} s_{j_1} \cdots s_{j_{a-1}} = s_{j_1} \cdots s_{j_{a-1}} s_{j_a} \tag{2}$$

where all the expressions are reduced. Now we proceed by cases.

- Assume $a < r$, then we can apply the inductive hypothesis on (1) and (2), getting

$$(i_1, \ldots, i_r) \overset{(1)}{\sim} (i_1, j_1, \ldots, j_{a-1}, \hat{j}_a, j_{a+1}, \ldots, j_r) \overset{(2)}{\sim} (j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_r).$$

- If instead $a = r$, we have further subcases. If $i_1 = j_1$ the claim follows by induction. If instead $i_1 \neq j_1$, then we can apply induction on (1) to get

$$(i_2, \ldots, i_r) \sim (j_1, \ldots, j_{r-1}).$$

Moreover by symmetry we have $(j_2, \ldots, j_r) \sim (i_1, \ldots, i_{r-1})$ (otherwise we fall in the first case). Hence setting

$$A \coloneqq (i_1, i_2, \ldots, i_r) \sim (i_1, j_1, \ldots, j_{r-1}),$$
$$B \coloneqq (j_1, j_2, \ldots, j_r) \sim (j_1, i_1, \ldots, i_{r-1}),$$

it is enough to show that $A \sim B$.

– If $|i_i - j_1| > 1$, then using (2) two times we have

$$s_{i_1} s_{j_1} \cdots s_{j_{r-1}} = s_{j_1} \cdots s_{j_r} = w = s_{i_1} \cdots s_{i_r} = s_{j_1} s_{i_1} \cdots s_{j_{r-1}},$$

hence multiplying with $s_{i_1} s_{j_1}$ we get

$$s_{j_2} \cdots s_{j_{r-1}} = s_{i_2} \cdots s_{i_{r-1}},$$

and by induction

$$(j_2, \ldots, j_{r-1}) \sim (i_2, \ldots, i_{r-1})$$

so $A \sim B$, since $(i_1, j_1) \sim (j_1, i_1)$.

– If instead $|i_1 - j_1| = 1$ we proceed as follows. Observe that by the above to prove

$$(i_1, \ldots, i_r) \sim (j_1, \ldots, j_r),$$

it is enough to show that

$$(i_1, j_1, \ldots, j_{r-1}) \sim (j_1, i_1, \ldots, i_{r-1}),$$

and by repeating this line of argument, we can also show that

$$(j_1, i_1, j_1, j_2, \ldots, j_{r-2}) \sim (i_1, j_1, i_1, i_2, \ldots, i_{r-2}).$$

But $(i_1, j_1, i_1) \sim (j_1, i_1, j_1)$, therefore it is enough to show that

$$(j_2, \ldots, j_{r-2}) \sim (i_2, \ldots, i_{r-2}),$$

thus the claim follows by induction. □

Remark. — The analogue of the Lemma of Matsumoto for Coxeter system also holds and is called Tits theorem, see the book of Bourbaki [Bou75]. As one can imagine reading the proof above, the proof of the general statement is not very pleasant.

Remark. — Consider (Artin's) braid group

$$\mathrm{Br}_n = \langle \beta_i \mid 1 \leqslant i \leqslant n-1 \rangle / (\beta_i \beta_j - \beta_j \beta_i \text{ if } |i-j| > 1,\ \beta_i \beta_j \beta_i - \beta_j \beta_i \beta_j \text{ if } |i-j| = 1).$$

The Lemma of Matsumoto gives a split map

$$\mathrm{Br}_n \twoheadrightarrow S_n,\ \beta_i \mapsto s_i = (i, i+1),$$

which means that we can always lift elements of S_n to Br_n.

Remark. — The Lemma of Matsumoto also implies that

$$S_n = \langle s_i \mid 1 \leqslant i \leqslant n-1 \rangle / (s_i s_j - s_j s_i \text{ if } |i-j| > 1,\ s_i s_j s_i - s_j s_i s_j \text{ if } |i-j| = 1,\ s_i^2 = e).$$

which shows that (S_n, S) is a Coxeter system.

Bruhat Ordering and Schubert Varieties

In the following it will be essential to consider the notion of Bruhat ordering.

Definition. — Recall that in a Tits system we have

$$(BwB)(BsB) \subset BwsB \cup BwB.$$

Let $w \in W = S_n$ and define $x \leqslant y$ if there exists a reduced expression of w, say $s_1 \cdots s_r$, such that y is equal to a subexpression of it, i.e. $y = s_{i_1} \cdots s_{i_n}$ for some $1 \leqslant i_1 < \cdots < i_n \leqslant r$. Clearly this defines a partial order, called the *Bruhat ordering*, on $W = S_n$.

Definition. — Let $y, w \in W = S_n$ and $t \in \bigcup_{z \in W} zSz^{-1} = J$, the set of transpositions. Then we write $y \xrightarrow{t} w$ if $w = yt$ and $l(w) > l(y)$ and $y \to w$ if there exists $t \in J$ such that $y \xrightarrow{t} w$. Define then $y \preceq w$ if there exist $y_0, \ldots, y_l \in W$ and $t_1, \ldots, t_l \in J$ such that $y = y_0 \xrightarrow{t_1} y_1 \xrightarrow{t_2} \cdots \xrightarrow{t_l} y_l = w$. Clearly this defines a partial ordering on $W = S_n$.

Note that clearly $y \preceq w$ implies $l(y) \leqslant l(w)$, and for all $w \in W$ and $t \in J$, $w \preceq wt$ if and only if $l(w) \leqslant l(wt)$. The same holds for $\leqslant$. But beware, in general clearly $l(w) \leqslant l(y)$ does not imply $w \leqslant y$ (for an example, consider distinct $s, u, v \in S$).

V.4. Lemma. — *Let $y, w \in W = S_n$, $y \neq$. Let $w = s_1 \cdots s_r$ be a reduced expression and suppose there exist a superexpression equal to y. Then there exists some $v \in W$ such that:*

(1) $y \preceq v$,

(2) $l(v) = l(y) + 1$,

(3) *there exists some subexpression of $s_1 \cdots s_r$ which equals v.*

Proof. RT2Sheet7.3. ☐

V.5. Proposition. — *Let $w \in W = S_n$ and $s_1 \cdots s_r$ a reduced for w. Then $y \preceq w$ if and only if there exists a reduced expression $s_{i_1} \cdots s_{i_k}$ for y with $1 \leqslant i_1 < \cdots < i_k \leqslant r$.*

Proof. ($\Longrightarrow$) Suppose that $y = y_0 \xrightarrow{t_1} y_1 \xrightarrow{t_2} \cdots \xrightarrow{t_l} y_l = w$. Then $w = y_{l-1}t_l$ with $t_l \in J$, so $y_{l-1} = wt_l$ and $l(y_{l-1}) < l(w)$. Then by the Strong Exchange Property we get

$$y_{l-1} = s_1 \cdots \hat{s}_a \cdots s_r$$

for some $1 \leqslant a \leqslant r$. By the Deletion Property we can find a reduced expression for y_{l-1} as a subexpression, i.e. $y_{l-1} = s_{j_1} \cdots s_{j_m}$ for some $1 \leqslant j_1 < \cdots < j_m \leqslant r$. Then repeating the argument we get that y has a reduced expression of the form $s_{i_1} \cdots s_{i_k}$ for some $1 \leqslant i_1 < \cdots < i_k \leqslant r$.

($\Longleftarrow$) Just apply Lemma V.4 multiple times. ☐

The following corollary is an immediate consequence of the previous proposition, and gives a more satisfying characterization of the Bruhat ordering.

V.6. Corollary. — *For $y, w \in W = S^n$, the following are equivalent:*

(1) $y \preceq w$,

(2) *some reduced expression of w contains a subexpression equal to y,*

(3) *any reduced expression of w contains a subexpression equal to y.*

Remark. — The corollary above also tells us that $y \leqslant w$ if and only if all reduced expressions of w contain a subexpression equal to y.

Remark. — Of course, all we have said so far about Bruhat orderings applies more generally to Coxeter groups. However, beware that the rest of this section will be specifically about S_n!

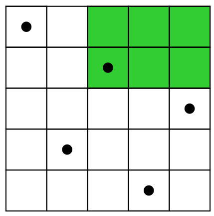
Now we have a good definition for an order on S_n, but how can we concretely check whether $y \leqslant w$?

Definition. — For $w \in S_n$ and $1 \leqslant r, s, n$, define

$$w[r,s] := \#\{1 \leqslant i \leqslant r \mid w(i) \geqslant s\}.$$

In words, $w[r,s]$ counts the number of 1's in the transpose of the permutation matrix of w that are above and to the right of the entry in position (r,s).

Example. — Let $w = 13524 \in S_5$. In this case, $w[2,3] = 1$, as shown in the following picture.

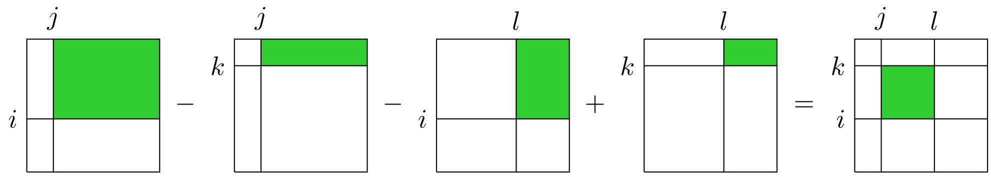


⚠ This is probably obvious to most people but I had never seen it: the element 13524 is the permutation $\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{matrix}$ in particular one should not mistake it for the element (13524) in cycle notation, which is $\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$

Remark. — Assume $1 \leqslant j < l \leqslant n$ and $1 \leqslant k < i \leqslant n$. Then one can prove the formula

$$w[i,j] - w[k,j] - w[i,l] + w[k,l] = \#\{k+1 \leqslant a \leqslant i \mid j \leqslant w(a) < l\}.$$

The formula might look quite complicated at first, but once translated in terms of pictures it becomes obvious:



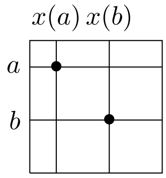
V.7. Proposition. — *Let $x, y \in S$. We have*

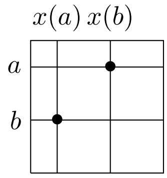
$$x \to y \iff x[r,s] \leqslant y[r,s] \ \forall r, s.$$

Proof. ($\Longrightarrow$) If $x = y$ there is nothing to prove, so assume $x \xrightarrow{t} y$ with $t = (a,b)$ and without loss of generality $a < b$. Then

$$y[r,s] = \begin{cases} x[r,s] + 1 & \text{if } 1 \leqslant a \leqslant r \leqslant b \text{ and } x(a) < x(s) \leqslant x(b) \\ x[r,s] & \text{otherwise} \end{cases}$$

Using this formula repeatedly the claim follows. To prove the formula observe the following. We have assumed $x \xrightarrow{t} y$. In particular, $l(x) < l(y)$. Since the length of a permutation is the number of inversions, x and y coincide everywhere except for a and b, and we can furthermore deduce $x(b) > x(a)$.

$x =$ 

$y =$ 

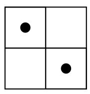
Showing the formula is now a matter of counting and comparing the number of 1's in a top-right justified box with bottom-left corner (r, s).

($\Longleftarrow$) This is RT2Sheet8.3. A hint is to define

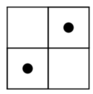
$$N[r, s] \coloneqq y[r, s] - x[r, s] \geqslant 0.$$

$\square$

Example. — Consider S_2 (trying really hard today). For each $w \in S_2$, we record below the values of $w[r, s]$ in a table. We see that $s[r, s] \geq e[r, s]$ for all (r, s), as expected.

$e =$ 

	1	2
1	1	0
2	2	1

$s =$ 

	1	2
1	1	1
2	2	1

Lecture 17
14^{th} Dec, 2022

Our goal for the rest of this section is to give a geometric meaning to the Bruhat ordering of S_n. With this in mind, recall that the full flag variety $\mathrm{Fl} = \mathrm{GL}_n(\mathbb{C})/B$, where B is the Borel subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, is the variety with points corresponding to the full flags in $\mathbb{C}^n$, with the correspondence given by $gB \mapsto gF^{\mathrm{st}}$, where the standard flag F^{st} is the full flag given by the subspaces $F_j = \langle e_1, \ldots, e_j \rangle$ for all $1 \leqslant j \leqslant n$.

V.8. Lemma. — *For T the torus of diagonal matrices in $B \subset \mathrm{GL}_n(\mathbb{C})$, we have*

$$\mathrm{Fl}^T \coloneqq \{T\text{-fixed points in } \mathrm{Fl}\} = \{\text{coordinate flags}\} \xleftrightarrow{1:1} S_n$$

with the bijection given by sending $w \in S_n$ to F^w, where F^w is the full flag given by the subspaces $F^w_j \coloneqq \langle e_{w(1)}, \ldots, e_{w(j)} \rangle$ for $1 \leqslant j \leqslant n$ (and of course $F^e = F^{\mathrm{st}}$).

Proof. The bijection between coordinate flags and elements of S_n is obvious. As for the equality, we have that $F_\bullet \in \mathrm{Fl}^T$ is a fixed point if and only if $F_j \in \mathrm{Gr}_j(n)^T$ for all $i \leqslant d \leqslant n$ and $F_{j-1} \subset F_j$, which clearly (as we already noted in Lemma I.8) is the same as saying that the subspaces F_j are coordinate subspaces for all j and $F_{j-1} \subset F_j$, or in other words $F_\bullet$ is a coordinate flag. $\square$

Definition. — The *Schubert cells* of Fl are defined as the B-orbits in Fl, which are the Bruhat cells of $\mathrm{GL}_n(\mathbb{C})/B$ under the identification $gB \mapsto gF^{\mathrm{st}}$.

Definition. — The *Schubert varieties* $\Omega(w)$ for $w \in S_n$ are defined as the closure in the Zariski topology on the Bruhat cells $C(w)$.

Remark. — We have $C(w) \cong \mathbb{C}^{l(w)}$.

Example. — For $w = 23154 \in S_5$, we can identify F^w as the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where F_j^w is spanned by the first j columns, and the Bruhat cell $C(w)$ as the subset of $\mathrm{GL}_n(\mathbb{C})/B$ of matrices of the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where the entries marked with asterisks are arbitrary, as we already noted in Remark I.10. The number of stars in this description of the Bruhat cell $C(w)$ is easily seen to equal the number of inversions of w, i.e. the length $l(w)$.

V.9. Lemma. — *Pick the standard flag $F^e = F^{\mathrm{st}}$ and $w \in S_n$.*

(1) *We have*
$$\dim(F_j^w \cap F_s^e) = \#\{1 \leqslant i \leqslant j \mid w(i) \leqslant s\} = j - w[j, s+1]$$
for all j and s. This dimension is called relative position of F^w (with respect to F^e).

(2) *We have*
$$F_\bullet \in C(w) \iff \dim(F_j \cap F_s^e) = j - w[j, s+1]$$
for all j and s.

(3) *We have*
$$F_\bullet \in \Omega(w) \iff \dim(F_j \cap F_s^e) \geqslant j - w[j, s+1]$$
for all j and s.

Proof. Clearly (1) follows from the definition. We prove (2). First observe that $F^w \in C(w)$ satisfies the dimension formulas given by (1). Multiplication by B doesn't change the relative position, thus any flag in $C(w)$ satisfies the dimension formulas. Conversely, every $F_\bullet \in \mathrm{Fl}$ is contained in some $C(w)$ for some unique $w \in S_n$, therefore it satisfies the dimension formulas of (1) precisely for that w. As for (3), it is clear, as taking the closure of $C(w)$ means that we can turn the 1's in the matrices into 0's. ☐

Maybe I should explain why this is the case?

Example. — In the case of $\mathbb{P}^1 \cong \mathrm{GL}_2(\mathbb{C})/B$ (under the identification $g[1:0] \mapsto gB$), we have:

- $C(e) = \{[1:0]\} \cong \{B/B\}$,
- $C(s) = \{[a:b] \mid b \neq 0\} \cong \{\langle ae_1 + be_2 \rangle \subset \mathbb{C}^2 \mid b \neq 0\}$,
- $\overline{C(e)} = C(e) = \{[1:0]\}$,

- $\overline{C(s)} = \mathbb{P}^1 = C(e) \cup C(s)$.

The relative positions are as follows:

There should have been pretty tables here.

V.10. Theorem. — *Given any $w \in S_n$ and the corresponding Schubert variety $\Omega(w)$, we have*

$$\Omega(w) = \bigcup_{y \leqslant w} C(y).$$

Proof. By Lemma V.9, we have that $F \in \Omega(w)$ if and only if

$$\dim(F_j \cap F_s^e) \geqslant j - w[j, s+1]$$

for all j and s. Then $C(y) \subset \Omega(w)$ is equivalent to the condition that

$$j - y[j, s+1] \geqslant j - w[j, s+1] \ \forall j, s,$$

i.e. $y[j, s+1] \leqslant w[j, s+1]$ for all j and s, or in other words $y \leqslant w$. □

So far we have been working with the full flag variety Fl, but one could ask whether analogous results hold for the partial flag varieties $\mathrm{Fl}_{\underline{d}}$. In the next section we answer in the affirmative, but to do so we need to introduce some more theory.

Parabolic Subgroups and General Schubert Varieties

We want to study more in detail the flag varieties $\mathrm{Fl}_{\underline{d}}$, such as $\mathrm{Gr}(d, n)$. In particular, the notion of parabolic subgroups of a Coxeter system will be fundamental.

Bad notation!

Definition. — Given a Coxeter system (W, S) and $S_p \subset S$ any subset of the generators, we call

$$W_p := \langle s \mid s \in S_p \rangle < W$$

a *parabolic subgroup* of W.

V.11. Proposition. — *Let (W, S) be a Coxeter system and $S_p, S_{p'} \subset S$.*

(1) *(W_p, S_p) is a Coxeter group; we denote by l_p its length function.*

(2) *$l_p(w) = l(w)$ for all $w \in W_p$.*

(3) *$W_p \cap W_{p'} = W_{p \cap p'}$ (where $S_{p \cap p'}$ is defined as $S_p \cap S_{p'}$).*

(4) *$\langle W_p \cup W_{p'} \rangle = W_{p \cup p'}$ (where $S_{p \cup p'} = S_p \cup S_{p'}$).*

(5) *$W_p = W_{p'}$ implies $S_p = S_{p'}$.*

Proof. (2) Let $w \in W_p$, so that $w = s_1 \cdots s_r$ for some $s_i \in S_p \subset S$. Without loss of generality we can assume $s_1 \cdots s_r$ is a reduced expression for $w \in W$ (because otherwise we can pass to a reduced subexpression by removing some s_i's, since (W, S) is a Coxeter system). Then a fortiori it is a reduced expression for $w \in W_p$.

(1) Assume $w \in W_p$ has reduced expression $w = s_1 \cdots s_r$ for some $s_i \in S_p$. Let $s \in S_p$ with $l_p(ws) < l_p(w)$. Then $l(ws) < l(w)$ by (2). Then by the Exchange Condition for W, we have

$$ws = s_1 \cdots \hat{s}_j \cdots s_r.$$

Thus the Exchange Condition holds for (W_p, S_p), hence it is a Coxeter system (beware that we are cheating here: we are assuming that the Exchange Condition characterizes Coxeter systems, which we haven't proved).

(3) The "$\supset$" inclusion is clear. As for "$\subset$", it follows immediately from Matsumoto's lemma for S_n and for general Coxeter groups from Tits Theorem. We recall the latter: if $w \in W$ has reduced expressions $s_{i_1} \cdots s_{i_r} = w = s_{j_1} \cdots s_{j_r}$. Then $(i_1, \dots, i_r) \sim (j_1, \dots, j_r)$ or we can get the second expression from the first by the first Coxeter relation (Cox1); in particular, $\{s_{i_1} \cdots s_{i_r}\} = \{s_{j_1} \cdots s_{j_r}\}$. Now if $w \in W_p \cap W_{p'}$, then the elements in a reduced expression must be in $S_p \cap S_{p'}$, thus $w \in W_{p \cap p'}$.

(4) This is clear.

(5) First we show that given any Coxeter system (W, S), the set S is a minimal set of generators. Indeed, assume this is not the case, so that there exists some $s \in S$ such that $s = s_1 \cdots s_r$, with $s_j \in S \setminus \{s\}$. If this is a reduced expression, then $r = 1$ and thus $s = s_1$, a contradiction. If instead it is not a reduced expression, then we can find a reduced subexpression which equals s, also a contradiction. Thus our claim about the minimality of S is proved. Now let $W_p = W_{p'}$. Assume there exists some $s \in S_p$ such that $s \notin S_{p'}$. We can find a reduced expression $s = s_{i_1} \cdots s_{i_r}$ for some $s_{i_k} \in S_{p'}$, as s is an element of (W, S), which contradicts the fact we have just proved. Hence $S_p = S_{p'}$. □

Not sure why?

V.12. Proposition. — *Let (W, S) be a Coxeter system and (W_p, S_p) a parabolic subsystem. Then any $w \in W$ can be written uniquely as $w = xy$ with $x \in W_p$ and $y \in {}^pW$, where*

$${}^pW := \{z \in W \mid l(sz) > l(z) \ \forall s \in S_p\} = \{z \in W \mid sz > z \ \forall s \in S_p\}.$$

Elements in ${}^pW$ are called shortest/minimal (length) coset representatives with respect to W_p, and one often denotes y as ${}^pw$.

Proof. We first prove existence of the factorization described. Pick $s_1 \in S_p$ such that $s_1 w < w$. If this is not possible then just set $x := e$ and $y := w$. Pick as many as possible $s_2, \dots, s_r \in S_p$ with $\cdots < s_2 s_1 w < s_1 < w$. We must stop at latest after $l(w)$ steps. Let $y := s_r \cdots s_1 w$. By construction we have $y \in {}^pW$. Then $w = xy$ with $x := s_1 \cdots s_r \in W_p$.

As for uniqueness, assume that $w = x_1 y_1 = x_2 y_2$ with $x_1, x_2 \in W_p$ and $y_1, y_2 \in {}^pW$. Write $s_{i_1} \cdots s_{i_r}$ a reduced expression for $y_2 \in {}^pW$ and $s_{j_1} \cdots s_{j_t}$ with $s_{j_k} \in S_p$ a reduced expression for $x_1^{-1} x_2 \in W_p$. Then considering

$$y_1 = x_1^{-1} x_2 y_2 = s_{j_1} \cdots s_{j_t} s_{i_1} \cdots s_{i_r} \tag{$*$}$$

we can obtain a reduced expression of y_1 by passing to a subexpression of $(*)$. Such an expression cannot start with any $s \in S_p$, since y_1 is an element of ${}^pW$, so it must be a subexpression of $s_{i_1} \cdots s_{i_r} = y_2$, yielding $y_1 \leqslant y_2$. Similarly we have $y_2 \leqslant y_1$, thus $y_1 = y_2$, and consequently also $x_1 = x_2$, showing that the factorization is unique. □

Remark. — Similarly we have $w = y'x'$ for unique $x' \in W_p$ and $y' \in W^p$, where

$$W^p := \{z \in W \mid l(zs) > l(z) \ \forall s \in S_p\} = \{z \in W \mid zs > z \ \forall s \in S_p\},$$

and one denotes y' as w^p. Moreover, we have $l(w) = l(x) + l(y) = l(x') + l(y')$.

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V.13. Corollary. — *Each coset in $W_p \backslash W$ has a unique representative $y \in {}^pW$ of minimal length.*

Proof. Take an element w of a given coset in $W_p \backslash W$. Then by Proposition V.12, there is a unique factorization $w = xy$ with $x \in W_p$, $y \in {}^pW$ and $l(w) = l(x) + l(y)$; in particular, $l(y) \leqslant l(w)$. Thus y is a minimal length coset representative of the given coset $W_p y$. Now take another representative of the same coset, say $z \in W_p y$. Then $z = \tilde{x}y$ for some $\tilde{x} \in W_p$, and again by Proposition V.12, this factorization of z is unique, so $l(z) = l(\tilde{x}) + l(y) \geqslant l(y)$, and $l(z) = l(y)$ if and only if $z = y$. Thus y is a unique representative of minimal length. □

We now want to study Schubert cells and Schubert varieties of $\mathrm{Fl}_{\underline{d}}$ (defined in the obvious way), via the map

$$\Phi : G/B \to G/P_{\underline{d}}\,, \ gB \mapsto gP_{\underline{d}}\,.$$

Note that the results we will obtain in this way will recover results for Grassmannians.

For the next proposition, recall that in G/B, we have (Theorem V.10):

$$\Omega^B(w) = \bigcup_{y \leqslant w} C^B(y).$$

V.14. Proposition. — *Let $P \coloneqq P_{\underline{d}}$ and Φ the map defined above.*

(1) *The B-orbits in G/P are in bijection with the sets W^p with bijection given by sending $y \in W^p$ to the Schubert cell in G/P associated to y, i.e. $C^P(w) = ByP/P$.*

(2) *Define $\Omega^P(y) \coloneqq \overline{C^p(y)}$, the Schubert variety in G/P associated to y. We have*

$$\Omega^P(y) \coloneqq \bigcup_{\substack{y \in W^p \\ y \leqslant w}} C^P(y).$$

Proof (modulo details). For (1), first we prove that Φ is B-equivariant and surjective, and

$$\Phi(C^B(w)) = C^P(w^p) \tag{$*$}$$

for all $w \in W$. It is clear that Φ is B-equivariant and surjective, moreover, for $w \in W$ we have

$$\Phi(C^B(w)) = \Phi(BwB/B) = BwP/P = Bw^p xP/P = Bw^p P/P,$$

where $x \in W_p$ and we have used equivariance in the second equality, Proposition V.12 in the third, and that the permutation matrix corresponding to $x \in W_p$ is contained in P in the fourth. One can (and should) check that $BwP/P \neq Bw'P/P$ if $w \neq w'$, for $w, w' \in W^p$, which combined with the above proves (1).

Now for (2), we claim first that for $w \in W^p$, we have

$$\bigcup_{\substack{y \in W^p \\ y \leqslant w}} C^P(y) \subset \Omega^P(w).$$

Note that $C^P(w) \subset \overline{C^P(w)} = \Omega^P(w)$, but we proved above that $\Phi(C^B(w)) = C^P(w)$, so $\Phi(C^B(w)) \subset \Omega^P(w)$, thus $C^B(w) \subset \Phi^{-1}(\Omega^P(w))$. Note that $\Phi^{-1}(\Omega^P(w))$ is closed, so $\Omega^B(w) \subset \Phi^{-1}(\Omega^P(w))$. Now $(*)$ gives

$$\bigcup_{y \leqslant w} C^B(y) \subset \Phi^{-1}(\Omega^P(w)) \implies \Phi(\bigcup_{y \leqslant w} C^B(y)) \subset \Omega^P(w) \implies \bigcup_{\substack{y \in W^p \\ y \leqslant w}} C^P(y) \subset \Omega^P(w),$$

which proves our claim.

Moreover, we use without proof that

$$X = \bigcup_{\substack{y \in W^P \\ y \leqslant w}} C^P(y)$$

is closed. Once we know this, it is easy to prove that $X = \Omega^P(w)$: the "$\subset$" inclusion is the previous claim, while "$\supset$" holds because $C^P(w) \subset X$ and X is closed, thus $\Omega^P(w) \subset X$. $\square$

What's the significance of this remark? Is it related to the fact that the union of the Schubert cells is closed?

Remark. — In the parabolic setting we have the Schubert conditions:

$$C^P(w) = \{F_\bullet \in \mathrm{Fl} \mid \dim(F_{d_j} \cap F_s^{\mathrm{st}}) = \#\{1 \leqslant i \leqslant d_j \mid w(i) \leqslant s\} \ \forall 1 \leqslant j \leqslant l, 1 \leqslant s \leqslant n\}.$$

A Geometric Representation of Coxeter Groups

In order to understand Coxeter groups, we turn to their representations: given a Coxeter group (W, S) with $|S| < \infty$, we want to construct a representation of it, ideally a faithful one. This would generalize the action of the Weyl group W on the Cartan $\mathfrak{h}^\times$ of a semisimple complex Lie algebra.

First, recall the (alternative) definition of Coxeter systems.

It is easy to see, as we already noted earlier, that this is equivalent to our original definition

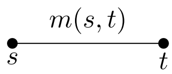
Definition. — A Coxeter system is a pair (W, S) with relations

$$(st)^{m(s,t)} = e,$$

where $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function such that $m(s, s) = 1$ and $m(s, t) = m(t, s) \geqslant 2$ for $s \neq t$, and by convention if $m(s, t) = \infty$ there are no relations between s and t.

To any Coxeter system we can associate a Coxeter graph.

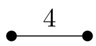
Definition. — The *Coxeter graph* attached to (W, S) is a graph with the elements of S as vertices and edges labeled by $m(s, t)$ if $m(s, t) \geqslant 3$ (we usually don't write the edge label when $m(s, t) = 3$).



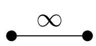
Examples. — If we take $(W, S) = (S_n, \{\text{simple transpositions}\})$, the corresponding Coxeter graph is the Dynkin diagram of type A_{n-1}:

with $n - 1$ vertices.

If $(W, S) = (D_4, S)$, where S is the subset of D_4 given by a reflection and a reflection composed with a rotation, this is a Coxeter system (as one sees easily considering the usual presentation of D_4) and the corresponding Coxeter graph is:



If we consider the Coxeter graph



we get the universal Coxeter group, the quotient of the free group generated by two elements s and t by the relations $s^2 = e$ and $t^2 = e$. This is known as the affine Weyl group of type A_1.

Remark. — Coxeter groups are not trivial since the map

$$\varepsilon : W \twoheadrightarrow \{\pm 1\}$$

given by $s \mapsto -1$ for all $s \in S$ (i.e. the map $w \mapsto (-1)^{l(w)}$) is a well-defined group morphism such that $\varepsilon(s) = -1$ and

$$\varepsilon(e) = \varepsilon(s^2) = (\varepsilon(s))^2 = (-1)^2 = 1,$$

thus $|w| \geqslant 2$.

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Definition. — A *reflection* (in the classical sense) is an endomorphism s of some finite dimensional $\mathbb{R}$-vector space V which fixes (pointwise) a hyperplane $H_s \leqslant V$, and such that there exist a nonzero $v \in V$ with $s(v) = -v$.

Remark. — Any reflection s squares to the identity, since v and the points of H_s are fixed by s^2, and together they span V.

Definition. — Let (W, S) be a Coxeter system with $|S| < \infty$.

- Let $V = V_{\mathrm{geom}}$ be the $\mathbb{R}$-vector space spanned by basis vectors α_s for $s \in S$.
- Define a bilinear form on V by setting

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m(s,t)}\right) & \text{if } m(s,t) \neq \infty \\ -1 & \text{if } m(s,t) = \infty \end{cases}$$

- For all $s \in S$, define the hyperplane

$$H_s := \{v \in V \mid (v, \alpha_s) = 0\}.$$

  Note that this is indeed a hyperplane, as it is a vector subspace of dimension at least one less than the dimension of V, and precisely one less, as $(\alpha_s, \alpha_s) = -\cos(\pi) = 1 \neq 0$.

- For all $s \in S$ define a linear map

$$R_s : V \to V,\ v \mapsto v - 2(v, \alpha_s)\alpha_s.$$

  Note that R_s keeps H_s pointwise fixed and $R_s(\alpha_s) = \alpha_s - 2\alpha_s = -\alpha_s$. Thus R_s is a reflection.

As we already noted, $(\alpha_s, \alpha_s) = 1$ for all $s \in S$. Moreover, the bilinear form $(-,-)$ is symmetric and $(\alpha_s, \alpha_t) \leqslant 0$ for all $s \neq t$ (both facts following immediately from the properties of $m(s,t)$). Less obvious is the invariance property:

$$(v, w) = (R_s(v), R_s(w))$$

for all $s \in S$ and $v, w \in V$. Indeed,

$$\begin{aligned}(R_s(v), R_s(w)) &= (v - 2(v, \alpha_s)\alpha_s, w - 2(w, \alpha_s)\alpha_s) \\ &= (v, w) - 2(w, \alpha_s)(v, \alpha_s) - 2(v, \alpha_s)(\alpha_s, w) + 4(v, \alpha_s)(w, \alpha_s)(\alpha_s, \alpha_s) \\ &= (v, w).\end{aligned}$$

We can now state the main theorem of this section: the reflections R_s give us a way to construct a faithful representation of the Coxeter group W.

V.15. Theorem (Geometric representation of Coxeter groups)**.** — *Let (W, S) be a Coxeter system with $|S| < \infty$.*

(1) *There exists a unique group morphism*

$$\Phi = \Phi_{(W,S)} : W \to \mathrm{GL}(V) = \mathrm{GL}(V_{\mathrm{geom}})$$

defined by sending $s \in S$ to R_s.

(2) *The morphism Φ is a faithful representation of W, i.e. it is injective.*

Before we prove the theorem, we need some preparation.

V.16. Lemma. — *Let (W, S) be a Coxeter system, and $s, t \in S$ two distinct reflections. Consider*

$$U = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t \subset V_{\mathrm{geom}}.$$

If $m(s,t) < \infty$, then $(-,-)|_{U\times U}$ is (nondegenerate) positive definite.

Proof. Let $\lambda = a\alpha_s + b\alpha_t$ for $a, b \in \mathbb{R}$. Then

$$(\lambda, \lambda) = a^2 + 2ab(\alpha_s, \alpha_t) + b^2 = \left(a - b\cos\frac{\pi}{m(s,t)}\right)^2 + b^2\left(\sin\frac{\pi}{m(s,t)}\right)^2 \geqslant 0$$

and unless both a and b are zero, there is always a strictly positive term in the sum. □

Remark. — If $m(s,t) = \infty$, then $(\alpha_s + \alpha_t, \alpha_s + \alpha_t) = 1 - 2 + 1 = 0$, i.e. $\alpha_s + \alpha_t$ is isotropic.

V.17. Lemma. — *Let (W, S) be a Coxeter system, and $s, t \in S$ two distinct reflections with $m(s,t) = \infty$. Then for all $k \in \mathbb{N}_0$ we have:*

(1) $(R_sR_t)^k(\alpha_s) = 2k(\alpha_s + \alpha_t) + \alpha_s$,

(2) $R_t(R_sR_t)^k(\alpha_s) = (2k+1)(\alpha_s + \alpha_t) + \alpha_t$.

In particular, $R_sR_t \in \mathrm{GL}(V_{\mathrm{geom}})$ has infinite order.

Proof. If $k = 0$, (1) holds trivially and (2) holds since

$$R_t(\alpha_s) = \alpha_s - 2(\alpha_s, \alpha_t)\alpha_t = \alpha_s + 2\alpha_t.$$

If $k = 1$, moreover, (1) holds since

$$R_sR_t(\alpha_s) = R_s(\alpha_s + 2\alpha_t) = -\alpha_s + 2\alpha_t + 4\alpha_s = 2(\alpha_t + \alpha_s) + \alpha_s.$$

The whole result now follows by induction. □

V.18. Lemma. — *The assignment*

$$s \mapsto R_s$$

defines a group morphism $\Phi : W \to \mathrm{GL}(V_{\mathrm{geom}})$.

I don't understand, surely we need this to have a faithful representation, but not here?

Proof. We need to check that $(R_s R_t)^{m(s,t)} = \mathrm{id} \in \mathrm{GL}(V_{\mathrm{geom}})$ for all $s, t \in S$ with $m(s,t) < \infty$, and $R_s R_t$ has infinite order if $m(s,t) = \infty$. Set $m := m(s,t)$ and $U = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$. First, if $s = t$ we have $R_s R_s = \mathrm{id}$, since R_s is a reflection. If instead $s \neq t$ with $m < \infty$, observe that by Lemma V.16 $(-,-)$ is positive definite. And where there is a positive definite bilinear form, we can do Euclidean geometry: we know that R_s and R_t are orthogonal reflections with respect to two hyperplanes H_s and H_t, and these latter intersect at an angle of π/m, since the cosine of the angle between $\mathbb{R}_{\geqslant 0}\alpha_s$ and $\mathbb{R}_{\geqslant 0}\alpha_t$ equals

$$(\alpha_s, \alpha_t) = -\cos(\pi/m) = \cos(\pi - \pi/m).$$

This implies that $R_s R_t \in \mathrm{GL}(U)$ is a rotation by $2\pi/m$, which has order m (on U). This in turn implies that $R_s R_t \in \mathrm{GL}(V_{\mathrm{geom}})$ has order m (on V_{geom}), since $R_s R_t|_{H_s \cap H_t} = \mathrm{id}$ and $(H_s \cap H_t) + U = V_{\mathrm{geom}}$, hence proving that $(R_s R_t)^m = \mathrm{id}$. Finally, if $s \neq t$ and $m = \infty$, then by Lemma V.17 the element $R_s R_t$ has infinite order in $\mathrm{GL}(V_{\mathrm{geom}})$. $\square$

Remark. — Note that for distinct $s, t \in S$ we have $\Phi(s) \neq \Phi(t)$ (since $m(s,t) \geqslant 2$).

Definition. — Let (W, S) be a Coxeter system with $|S| < \infty$ and define

$$R := \{v \in V_{\mathrm{geom}} \mid v = \Phi(w)(\alpha_s) \text{ for some } w \in W, s \in S\}.$$

The elements of R are called the *roots* of (W, S).

Remark. — For all $\beta \in R$ we have $(\beta, \beta) = 1$ by invariance. Moreover we have $R = -R$, because $R_s(\alpha_s) = -\alpha_s$, thus

$$-\Phi(w)(\alpha_s) = \Phi(w)(-\alpha_s) = \Phi(w)(R_s(\alpha_s)) = \Phi(ws)(\alpha_s).$$

Definition. — We say that $v \in R$ is a *positive root* if $v \in \sum_{s \in S} \mathbb{R}_{\geqslant 0}\alpha_s$ and a *negative root* if $v \in \sum_{s \in S} \mathbb{R}_{\leqslant 0}\alpha_s$. We call R^+ the set of positive roots and R^- the set of the negative ones.

Note that the sets R^+ and R^- are clearly disjoint (the α_s's constituting a basis of V_{geom}). As an exercise, one should prove by hand that $R = R^+ \cup R^-$, but we can also use the following proposition (whose proof, given below, is more complicated than one might expect):

V.19. Proposition. — *Let (W, S) be a Coxeter system with $|S| < \infty$, $w \in W$ and $s \in S$. Then,*

(1) *$l(ws) > l(w)$ implies $\Phi(w)(\alpha_s) \in R^+$,*

(2) *$l(ws) < l(w)$ implies $\Phi(w)(\alpha_s) \in R^-$.*

Assuming this result, it is easy to prove:

V.20. Corollary. — *We have $R = R^+ \cup R^-$.*

Proof. By definition for $\beta \in R$ we have $\Phi(w)(\alpha_s)$ for some $w \in W$ and $s \in S$. Then either $l(ws) > l(w)$ or $l(ws) < l(w)$ so by Proposition V.19 either $\beta \in R^+$ or $\beta \in R^-$. $\square$

Proof of Proposition V.19. We first prove (1), i.e. that $l(ws) > l(w)$ implies $\Phi(w)(\alpha_s) \in R^+$. We will prove the claim by induction on $l(w)$. If $l(w) = 0$, we have $w = e$, thus $l(ws) > l(w)$ for all $s \in S$ and $\Phi(w)(\alpha_s) = \alpha_s$ for all $s \in S$ and $\alpha_s \in R^+$ by definition.

Assume now $l(w) > 0$ and pick $t \in S$ such that $l(wt) < l(w)$. In particular, since by assumption $l(ws) > l(w)$, we have $t \neq s$. Set $S_p = \{s, t\}$, so that $W_p = \langle s, t \rangle$ is a parabolic subgroup of W. The idea for the rest of the proof is to consider $v \in W^p$ a minimal coset representative in wW_p, reduce the problem to this case and then use the inductive hypothesis. We will need a number of preparatory observations.

First, observe the following:

- $l(v) < l(w)$. This is because we have $w = (wt)t$ with $t \in W_p$, thus

$$l(v) \leqslant l(wt) \leqslant l(w) - 1.$$

- $l(vs) > l(v)$ and $l(vt) > l(v)$, since $v \in W^p$.
- Write $w = vx$ for $x \in W_p$ with $l(w) = l(v) + l(x)$, then $l(xs) > l(x)$. Indeed, if we assume this is not the case, we must have

$$l(ws) = l(vv^{-1}ws) \leqslant l(v) + l(v^{-1}ws) = l(v) + l(xs) < l(v) + l(x) = l(w),$$

  a contradiction. Thus, any reduced expression of $x \in W_p = \langle s, t \rangle$ ends with t.

Now we need the following claim: given x as above, we have

$$\Phi(x)(\alpha_s) \in \mathbb{R}_{\geqslant 0}\alpha_s + \mathbb{R}_{\geqslant 0}\alpha_t.$$

In particular, $\Phi(x)(\alpha_s)$ is a positive root.

Proof of the claim. We distinguish two cases.

- Suppose $m(s,t) = \infty$. Then $(R_sR_t)^k(\alpha_s)$ and $R_t(R_sR_t)^k(\alpha_s)$ are of the form $k_1\alpha_s + k_2\alpha_t$ for some $k_i \in \mathbb{N}_0$, by Lemma V.17. Now, we have $x = stst\cdots t$ or $x = tsts\cdots t$, so by applying Φ we get one of the two expressions just mentioned.
- Now suppose instead that $m(s,t) = m < \infty$. Then $l(x) \leqslant m$, since W_p is the dihedral group of order $2m$ with m elements of the form $e, s, st, sts, \ldots$ and other m of the form $t, ts, tst, \ldots$, and longest element the product of m factors $stst\cdots = tsts\cdots$. Moreover, $l(x) < m$ as the reduced expressions of x have to end with t. Now, we have two cases: either $\Phi(x) = (R_sR_t)^k$ or $\Phi(x) = R_t(R_sR_t)^k$, for some $0 \leqslant k < m/2$. In the first case, observe that R_sR_t is a rotation by $2\pi/m$, thus the angle between α_s and $\Phi(x)(\alpha_s)$ is less or equal to $|\pi - 2\pi/m|$ which is less than the angle between α_s and α_t, this latter being $\pi - \pi/m$. The second case is similar and we can derive the same conclusion. Hence it must be that $\Phi(x)(\alpha_s) = a\alpha_s + b\alpha_t$ is contained in $\mathbb{R}_{\geqslant 0}\alpha_s + \mathbb{R}_{\geq 0}\alpha_t$.

Thus the claim is proven. ■

Given the claim, we get that $\Phi(w)(\alpha_s) \in R^+$, for $w = vx$, by applying $\Phi(v)$ to $\Phi(x)(\alpha_s)$, using the fact that $l(vs) < l(s)$ and $l(vt) < l(v)$, and thus applying the inductive hypothesis.

Finally, we can prove (2) in the following way. By hypothesis $l(ws) < l(v)$, hence taking $\tilde{w} \coloneqq ws$, we have $l(\tilde{w}s) > l(\tilde{w})$, thus by (1) $\Phi(\tilde{w})(\alpha_s) \in R^+$. But then

$$\Phi(w)(\alpha_s) = \Phi(\tilde{w})\Phi(s)(\alpha_s) = \Phi(\tilde{w})(R_s(\alpha_s)) = \Phi(\tilde{w})(-\alpha_s) = -\Phi(\tilde{w})(\alpha_s) \in -R^+ = R^-,$$

which concludes the proof. □

We can finally prove that the representation Φ we constructed is faithful.

Proof of Theorem V.15. We already proved as Lemma V.18 that Φ is a well defined representation of W, so we just need to prove that it is faithful. Let $w \in \ker \Phi$. If $w \neq e$, then there exists $s \in S$ such that $l(ws) < l(w)$. Hence $\alpha_s = \Phi(w)(\alpha_s) \in R^-$, whereas clearly $\alpha_s \in R^+$, a contradiction. Thus $w = e$, which proves that Φ is faithful. $\square$

Now we can prove the least surprising result ever.

V.21. Corollary. — *Let (W, S) be a Coxeter system with $|S| < \infty$ and $S_p \subset S$ an arbitrary subset. Then (W_p, S_p) is a Coxeter group with defining relations $(st)^{m(s,t)} = e$ for all $s, t \in S_p$.*

Proof. Let $\tilde{W}_p$ be the group generated by S_p with relations $(st)^{m(s,t)} = e$ for all $s, t \in S_p$. Then there is a surjective morphism $\pi : \tilde{W}_p \to W_p$ given by $s \mapsto s$ and $\tilde{W}_p$ has a faithful representation $\tilde{\Phi}$ on $\tilde{V}_{\text{geom}} \subset V_{\text{geom}}$, i.e. the vector space generated by α_s for $s \in S_p$. Moreover, if $(-,-)$ is the bilinear form on V_{geom}, the bilinear form $(-,-)^\sim$ on $\tilde{V}_{\text{geom}}$ equals the restriction $(-,-)|_{\tilde{V}_{\text{geom}}}$. In particular, we get that $\tilde{R}_s = R_s|_{\tilde{V}_{\text{geom}}}$ for all $s \in S_p$, and thus a commutative diagram

$$\begin{tikzcd}
\tilde{W}_p \arrow[r, hook, "\tilde{\Phi}"] \arrow[d, two heads, "\pi"'] & \mathrm{GL}(\tilde{V}_{\text{geom}}) \arrow[d, hook] \\
W_p \arrow[r, hook, "\Phi|_{W_p}"] & \mathrm{GL}(V_{\text{geom}})
\end{tikzcd}$$

which shows that π is injective, hence an isomorphism. $\square$

Parabolic Hecke Modules

Lecture 20
23rd Dec, 2022

One last goal for this chapter is to construct for every parabolic subgroup (W_p, S_p) of (W, S) a (right) module for $\mathcal{H}_v(W, S)$. As a motivation, parabolic Hecke modules are used to classify the irreducible representations of generic Hecke algebras, in a way which is analogous to the classifications of irreducible representations of symmetric groups (of which Hecke algebras are a deformation).

Recall first the defining relations for the Hecke algebra $\mathcal{H}_v(W, S)$ (actually, this is the first time we write them down in the case of a general Coxeter group):

(H1) $H_s^2 = 1 + (v^{-1} - v)H_s$ for all $s \in S$,

(H2) $H_s H_t \cdots = H_t H_s \cdots$ for any corresponding Coxeter relation $st \cdots = ts \cdots$.

V.22. Lemma. — *The maps Φ_{triv}, defined by $\Phi_{\text{triv}}(H_s) = v^{-1}$ for $s \in S$, and Φ_{sgn}, defined by $\Phi_{\text{sgn}}(H_s) = -v$ for $s \in S$, are both algebra morphisms $\mathcal{H}_v(W, S) \to \mathbb{Z}[v, v^{-1}]$.*

Proof. We just have to check that the two maps are well defined: that (H2) is respected is obvious (in both cases), while for (H1) we have

$$(v^{-1})^2 = 1 + (v^{-1} - v)v^{-1} \iff v^{-2} = 1 + v^{-2} - 1,$$

which proves that the relation is respected by Φ_{triv}, and

$$(-v)^2 = 1 + (v^{-1} - v)(-v) \iff v^2 = 1 - 1 + v^2$$

which proves that the relation is respected by Φ_{sgn}. $\square$

By pulling back (via Φ_{triv} and Φ_{sgn}) the obvious $\mathbb{Z}[v, v^{-1}]$ action on $\mathbb{Z}[v, v^{-1}]$ we obtain two (right) $\mathcal{H}_v(W, S)$-module structures on $\mathbb{Z}[v, v^{-1}]$, called the *trivial representation* and the *signed representation*. We denote them by $\mathrm{triv}_{(W,S)}$ and $\mathrm{sgn}_{(W,S)}$, respectively.

V.23. Lemma. — *$\mathcal{H}_v(W, S)$ is a free left (respectively, right) $\mathcal{H}_v(W_p, S_p)$ module with basis $\{H_x \mid x \in {}^pW\}$ (respectively, $\{H_x \mid x \in W^p\}$), and action given by multiplication.*

Proof. Exercise. $\square$

If we set ${}^p\mathcal{H} = \mathcal{H}_v(W_p, S_p)$ and $\mathcal{H} = \mathcal{H}_v(W, S)$, we have that $\mathcal{H}$ is a $({}^p\mathcal{H}, \mathcal{H})$-bimodule.

Definition. — For (W_p, S_p) as above, define right $\mathcal{H}$-modules $\mathcal{M}^p$ and $\mathcal{N}^p$ by

$$\mathcal{M}^p = \mathrm{triv}_{(W_p,S_p)} \otimes_{\mathcal{H}_v(W_p,S_p)} \mathcal{H}_v(W, S),\ \ \mathcal{N}^p = \mathrm{sgn}_{(W_p,S_p)} \otimes_{\mathcal{H}_v(W_p,S_p)} \mathcal{H}_v(W, S).$$

$\mathcal{M}^p$ and $\mathcal{N}^p$ are called *parabolic Hecke modules.*

Example. — If $S_p = S$, then ${}^p\mathcal{H} = \mathcal{H}$, thus $\mathcal{N}^p \cong \mathrm{sgn}_{(W,S)}$ as a right $\mathcal{H}$-module. If instead $S_p = \varnothing$, then ${}^p\mathcal{H} = \mathbb{Z}[v, v^{-1}]$, thus $\mathcal{N}^p \cong \mathcal{H}$ again as a right $\mathcal{H}$-module.

Remark. — We have that $\mathcal{M}^p$ and $\mathcal{N}^p$ are free $\mathbb{Z}[v, v^{-1}]$-modules of rank $|W_p\backslash W| = |{}^pW|$. Basis are given by the elements $M_y = 1 \otimes H_y \in \mathcal{M}^p$ and $N_y = 1 \otimes H_y \in \mathcal{N}^p$ for $y \in {}^pW$.

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11^{th} Jan, 2022

Our goal for the rest of this section is to show that for parabolic Hecke modules there exists an analogous of the bar-involution (which we have introduced in the previous chapter for Hecke algebras, see Definition IV.7), and correspondingly an analogous of the Kazhdan-Lusztig basis (i.e. of Theorem IV.8).

V.24. Lemma. — *There is a $\mathbb{Z}$-linear endomorphism on $\mathcal{M}^p$ given by*

$$f \otimes H \mapsto \overline{f \otimes H} := \overline{f} \otimes \overline{H},$$

where the bar is the usual bar-involution on $\mathcal{H}$. Similarly for $\mathcal{N}^p$.

Proof. We have to check well-definedness. For $s \in S^p$, we have

$$\begin{aligned}\overline{f \otimes H_s H} = \overline{f} \otimes \overline{H_s H} = \overline{f} \otimes \overline{H_s H} &= \overline{f} \otimes (H_s + (v - v^{-1}))\overline{H} \\ &= \overline{f}(H_s + (v - v^{-1})) \otimes \overline{H} = \overline{f}\gamma \otimes \overline{H}\end{aligned}$$

where

$$\gamma = \begin{cases} v^{-1} + v - v^{-1} = v & \text{for } \mathcal{M}^p \\ -v + v - v^{-1} = -v^{-1} & \text{for } \mathcal{N}^p \end{cases}$$

On the other hand, we have

$$\overline{f.H_s \otimes H} = \overline{f.H_s \otimes H} = \begin{cases} \overline{fv^{-1}} \otimes \overline{H} & \text{for } \mathcal{M}^p \\ \overline{-fv} \otimes \overline{H} & \text{for } \mathcal{N}^p \end{cases} = \overline{f}\gamma' \otimes \overline{H}$$

where

$$\gamma = \begin{cases} v & \text{for } \mathcal{M}^p \\ -v^{-1} & \text{for } \mathcal{N}^p \end{cases}$$

which concludes the check. $\square$

V.25. Lemma. — *The bar-involution is compatible with the $\mathcal{H}$-module structure, i.e. we have $\overline{mH} = \overline{m}.\overline{H}$ for any $m \in \mathcal{M}^p$ or $\mathcal{N}^p$ and $H \in \mathcal{H}$.*

Proof. This is obvious from the definitions. □

V.26. Proposition. — *For $x \in {}^pW$, the following formula holds in $\mathcal{M}^p$:*

$$M_x \underline{H}_s = \begin{cases} M_{xs} + vM_x & \text{if } xs > x,\ xs \in {}^pW \\ M_{xs} + v^{-1}M_x & \text{if } xs < x,\ xs \in {}^pW \\ (v+v^{-1})M_x & \text{if } xs \notin {}^pW \end{cases}$$

Correspondingly, in $\mathcal{N}^p$ we have:

$$M_x \underline{H}_s = \begin{cases} N_{xs} + vN_x & \text{if } xs > x,\ xs \in {}^pW \\ N_{xs} + v^{-1}N_x & \text{if } xs < x,\ xs \in {}^pW \\ 0 & \text{if } xs \notin {}^pW \end{cases}$$

Proof. We have

$$\begin{aligned} M_x \underline{H}_s = (1 \otimes H_x)(H_s + v) &= \begin{cases} 1 \otimes H_{xs} + v(1 \otimes H_x) & \text{if } xs > x \\ 1 \otimes H_{xs} + (v^{-1} - v)(1 \otimes H_x) + v(1 \otimes H_x) & \text{if } xs < x \end{cases} \\ &= \begin{cases} M_{xs} + vM_x & \text{if } xs > x,\ xs \in {}^pW \\ M_{xs} + v^{-1}M_x & \text{if } xs < x,\ xs \in {}^pW \end{cases} \end{aligned}$$

which proves the first two cases of the formula for $\mathcal{M}^p$. The first two cases of the formula for $\mathcal{N}^p$ are proven analogously.

Assume now that $xs \notin {}^pW$. We will soon prove (Lemma V.30 below) that if $x \in {}^pW$ and $xs \notin {}^pW$, then $xs > x$ and there exists some $t \in S_p$ with $xs = tx$. Assuming this, we have

$$\begin{aligned} M_x \underline{H}_s = 1 \otimes H_{xs} + v(1 \otimes H_x) &= 1 \otimes H_{tx} + v(1 \otimes H_x) \\ &= 1 \otimes H_t H_x + v(1 \otimes H_x) \\ &= v^{-1}(1 \otimes H_x) + v(1 \otimes H_x) = (v + v^{-1})M_x \end{aligned}$$

which proves the third case of the formula for $\mathcal{M}^p$. Similarly, we have

$$N_x \underline{H}_s = -v(1 \otimes H_x) + v(1 \otimes H_x) = 0,$$

which proves the third case of the formula for $\mathcal{N}^p$. □

V.27. Lemma. — *Assume $x \in {}^pW$, $s \in S$, $xs < x$. Then $xs \in {}^pW$. In particular, $xs > s$.*

Proof. We need to show that $l(txs) > l(xs)$ for all $t \in S_p$. Indeed, we have

$$l(txs) \geqslant l(tx) - 1 = l(x) + 1 - 1 = l(x) > l(xs)$$

where we use that $x \in {}^pW$ in the second equality and that $x > xs$ in the last inequality. □

V.28. Lemma (Lifting lemma)**.** — *Let (W,S) be a Coxeter system, $w,z \in W$ and $s \in S$. Assume $z \leqslant w$ and that $l(sw) < l(w)$ and $l(sz) > l(z)$. Then $z \leqslant sw$ and $sz \leqslant w$. Graphically, this means we can fill the dashed arrows in the diagram*

$$\begin{tikzcd}
& w \arrow[dl] \arrow[dd] \arrow[dr, dashed] & \\
sw \arrow[dr, dashed] & & sz \arrow[dl] \\
& z &
\end{tikzcd}$$

Proof. Let $sw = s_1 \cdots s_r$ be a reduced expression. Then $s_0 s_1 \cdots s_r$ is a reduced expression for w (with $s_0 \coloneqq s$). We can find a subexpression for z in this expression, i.e.

$$z = s_{i_1} \cdots s_{i_k}.$$

Since $sz > z$ we have $s_{i_1} \neq s = s_0$. Thus, $z < sw$. Moreover, $sz = ss_{i_1} \cdots s_{i_k}$ is a subexpression of $s_0 s_1 \cdots s_r = w$, hence $sz < w$. □

Remark. — The previous lemma is often “used” without mention in the literature (or rather mistakenly taken for obvious), but it is not trivial!

V.29. Lemma. — *Let (W,S) be a Coxeter system, (W_p, S_p) a parabolic subsystem. Consider elements $w_1, w_2 \in W$. Then $w_1 \leqslant w_2$ implies that ${}^p w_1 \leqslant {}^p w_2$ (and similarly $w_1^p \leqslant w_2^p$).*

Proof. By induction on $l(w_2)$.

- If $l(w_2) = 0$, then $w_2 = e$, thus $w_1 = e$, so ${}^p w_1 = e \leqslant e = p_2^w$.
- If $l(w_2) > 1$, we have ${}^p w_1 \leqslant w_1$. Then if ${}^p w_2 = w_2$ we have ${}^p w_1 \leqslant w_1 \leqslant w_2 = {}^p w_2$. If instead ${}^p w_2 \neq w_2$, then there exists some $s \in S_p$ such that $sw_2 < w_2$. Applying the Lifting lemma V.28 we get

$$\begin{tikzcd}
& w \coloneqq w_2 \arrow[dl] \arrow[dd] \arrow[dr, dashed] & \\
sw_2 \arrow[dr, dashed] & & s^p w_1 \arrow[dl] \\
& z \coloneqq {}^p w_1 &
\end{tikzcd}$$

  thus $p_1^w \leqslant sw_2$. By induction we are done.

□

From the previous lemmas, we get the claim we needed in the proof of Proposition V.26.

V.30. Lemma. — *Let $x \in {}^p W$, $s \in S$, $xs \notin {}^p W$. Then $xs > x$, and there exists $t \in S_p$ such that $xs = tx$.*

Proof. We know already that $xs > x$, by Lemma V.27. We need to show the existence of t. By Lemma V.29 we have
$$x = {}^p x \leqslant {}^p(xs) \leqslant xs,$$
thus x has a reduced expression which is a subexpression of ${}^p(xs)$. But $xs \notin {}^pW$ implies $l(xs) > l({}^p(xs))$ and $l(xs) = l({}^p(xs)) + l(y)$, where $xs = y^p(xs)$ for some $y \in W_p$ not equal to the identity. Then we have $l({}^p(xs)) \geqslant l(x)$ and $l(xs) = l(x) + 1$, thus
$$l(y) = l(xs) - l({}^p(xs)) = l(x) + 1 - l(x) = 1$$
(the second equality because $y \neq e$) which implies $y \in W_p$ and $l(y) = 1$, thus $y = t \in S_p$. Moreover
$$xs = t^p(xs) = tx,$$
since $x < {}^p(xs)$ and $l(xs) = l(x) + 1 = l(tx)$. □

Now we can state the analogous of Theorem IV.8 for parabolic Hecke modules.

V.31. Theorem (Kahzdan-Lusztig basis). — *There exists a unique $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{M}^p$, denoted $\{\underline{M}_y \mid y \in {}^pW\}$, such that:*

(1) $\overline{\underline{M}_y} = \underline{M}_y$ *for all* $y \in {}^pW$,

(2) $\underline{M}_y = M_y + \sum_{x<y} m_{y,x} M_x$ *for some* $m_{x,y} \in v\mathbb{Z}[v]$.

Analogously, there exists a unique $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{N}^p$, denoted $\{\underline{N}_y \mid y \in {}^pW\}$, such that:

(1) $\overline{\underline{N}_y} = \underline{N}_y$ *for all* $y \in {}^pW$,

(2) $\underline{N}_y = N_y + \sum_{x<y} n_{y,x} N_x$ *for some* $n_{x,y} \in v\mathbb{Z}[v]$.

Of course, the coefficients $m_{x,y}, n_{x,y} \in v\mathbb{Z}[v]$ introduced in the theorem are called the *parabolic Kazhdan-Lusztig polynomials.*

Proof. Let $\underline{M}_e \coloneqq M_e = 1 \otimes H_e = 1 \otimes 1$. This obviously satisfies the (KL) conditions. Let $x \in {}^pW$ and assume $\underline{M}_y$ is defined for all $y \in {}^pW$ and $y < x$. Pick $s \in S$ such that $xs < x$. Note that $xs \in {}^pW$ and so by induction $\underline{M}_{xs}$ is defined. Now consider $\underline{M}_{xs}\underline{M}_s$. This is selfdual (i.e. $\overline{\underline{M}_{xs}\underline{M}_s} = \underline{M}_{xs}\underline{M}_s$) and
$$\underline{M}_{xs}\underline{H}_s = \underline{M}_{xs}(H_s + v) = M_{xs}H_s + vM_{xs} + \sum_{y<xs} a_y M_y H_s + \sum_{y<xs<x} v a_y M_y$$
for some polynomials $a_y \in v\mathbb{Z}[v]$. Now the last term in the previous equation is equal to
$$M_x + vM_{xs} + \sum_{\substack{y<xs \\ ys \in {}^pW}} \beta_y M_y + \sum_{y<xs<x} \gamma_y M_y$$
for some $\beta_y \in v\mathbb{Z}[v]$ and $\gamma_y \in \mathbb{Z}[v]$. Hence if we take
$$\underline{M}_x \coloneqq \underline{M}_{xs}\underline{H}_s - \sum_{y<x} \gamma_y(0)\, \underline{M}_y$$
this is the element we are looking for.

The proof of uniqueness is the same as in the Hecke algebra case, and the proof for $\mathcal{N}^p$ is completely analogous. □

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As an application of the preceding theorem, we can check smoothness of Schubert varieties.

Example. — Consider a Schubert cell $C_{w_0} \subset G/B$ for w_0 a longest element in S_n. Then the corresponding Schubert variety $\Omega_{w_0} = \overline{C}_{w_0} = G/B$ is smooth. At the other end of triviality, $C_e \subset G/B$ is just a (closed) point, thus Ω_e is also smooth. In general Ω_w is not smooth, as seen in RT2Sheet10.4 and RT2Sheet10.5.

V.32. Proposition (Lakshmibai-Seshadri). — *The Schubert variety* $\Omega_w^p \subset G/P$ *for* $w \in W^p$ *is smooth if and only if* $\dim T_{P/P}\Omega_w = l(w)$.

Proof. Omitted (it is quite involved!). □

Remark. — We never properly defined smoothness, but one can use the Jacobian criterion. In particular, the latter yields an explicit description of the Zariski tangent space.

Definition. — Let $w \in S_n$, then w is called 3412-avoiding if there are no indices $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant n$ such that $1 \leqslant w(i_3) < w(i_4) < w(i_1) < w(i_2) \leqslant n$. Similarly, we can define 4231-avoiding permutations.

Example. — The element

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

is not 3412-avoiding.

V.33. Theorem (Smoothness criterion). — *For* $w \in S_n$ *and* $B < G = \mathrm{GL}_n(\mathbb{C})$ *as usual, the following are equivalent:*

(1) *the Schubert variety* $\Omega_w \subset G/B$ *is smooth,*

(2) *the element* w *is 3412- and 4231-avoiding,*

(3) *the (KL) polynomials* $h_{y,w}(v)$ *for* $y \leqslant w$ *(i.e. the non-trivial ones) satisfy*

$$v^{l(w)-l(y)}.$$

Proof. Also omitted. We mentioned the theorem to give an idea of the connections between geometry and KL polynomials. For more about this stuff, see the monograph [BL00]. □

Chapter VI.

Borel-Weil Theorem

VI

The final part of the course will be concerned with the representation theory of $\mathrm{GL}_n(\mathbb{C})$. In particular, we will prove the Borel-Weil theorem, which characterizes irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ (or really any reductive algebraic group).

Vector Bundles

To even state the Borel-Weil theorem, we need to study vector bundles first. We will take the more concrete approach, i.e. without mentioning locally free sheaves and whatnot.

Definition. — A *vector bundle* of rank r over an algebraic variety X is a variety E together with a morphism $\pi : E \to X$ such that there exists an open covering $\{U\}_{i\in I}$ of X such that:

(1) The open sets U_i are trivializing, i.e. there exists an isomorphism $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$ such there is a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{A}^r \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

(2) For all $i, j \in I$ there exists an $(r \times r)$ matrix A_{ij}, called the *transition matrix*, with entries in regular functions $U_i \cap U_j$ such that

$$\varphi_{ij} \coloneqq \psi_j \circ \psi_i^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{A}^r$$

is of the form $\varphi_{ij}(x, v) = (x, A_{ij}(x)v)$.

The variety E is called *total space*, X the *base space*. For $x \in X$, we denote by $E_x \coloneqq \pi^{-1}(x)$ the fiber over x. Note that E_x is a vector space of dimension r.

A *morphism of vector bundles* $\pi : E \to X$ and $\pi' : E' \to X'$ is a pair of morphisms of varieties $\varphi : E \to E'$ and $\bar{\varphi} : X \to X'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "\bar{\varphi}"] & X' \end{tikzcd}$$

commutes and induces linear maps between the fibers.

Remark. — Conditions (1) and (2) stay true under refinement of open covering.

Remark. — It is easy to show that the data of $(r \times r)$-matrices A_{ij} for each pair of open sets U_i and U_j in a cover of X, with entries in regular functions on $U_i \cap U_j$, and satisfying the conditions $A_{ii} = \mathbb{1}$ and $A_{ik} = A_{jk}A_{ij}$, define a vector bundle.

Definition. — A vector bundle of rank r is said to be *trivial* if it is isomorphic to $X \times \mathbb{A}^r$.

Definition. — A vector bundle of rank 1 is called a *line bundle*.

Example. — For $X = \mathbb{P}^1$, it is not difficult to show that the set of line bundles on X modulo isomorphism is in bijection with $\mathbb{Z}$, with the bijection given by sending $n \to \mathcal{O}_{\mathbb{P}^1}(n)$.

One can also upgrade this bijection to an isomorphism of groups.

Definition. — A *section* of a vector bundle $\pi : E \to X$ is a morphism of varieties $s : X \to E$ such that $\pi \circ s = \mathrm{id}$. Denote by $H^0(X, E)$ the set of sections of $\pi : E \to X$. Note that the set $H^0(X, E)$ is never empty, since we always have the 0-section:

$$U_i \to U_i \times \mathbb{A}^r, \ x \mapsto (x, 0).$$

Moreover, $H^0(X, E)$ is a vector space.

VI.1. Proposition. — *If X is a smooth projective variety, then for any vector bundle $\pi : E \to X$, the vector space $H^0(X, E)$ is finite dimensional.*

Proof. See [Har77, Theorem II.5.19] □

Remark. — For $X = \mathbb{P}^1$, the sections of $\mathcal{O}(n)$ are

$$\begin{cases} \{0\} & \text{if } n < 0 \\ \mathbb{C}[x_0, x_1]_n & \text{if } n \geqslant 0 \end{cases}$$

where $\mathbb{C}[x_0, x_1]_n$ is the degree n part (the vector space of degree n homogeneous polynomials).

Definition. — Let G be an (affine) algebraic group acting on a variety X. A *G-equivariant vector bundle* is a vector bundle $\pi : E \to X$ such that:

(1) G acts on E and π is G-equivariant,

(2) the induced map on fibers $E_x \to E_{gx}, \ v \mapsto gv$, is linear.

Note that the second condition makes sense, since E_x and E_{gx} are vector spaces and by equivariance of π we have $\pi(g.v) = g.\pi(v) = g.x$ for all $v \in E_x$.

A *morphism of pointed equivariant vector bundles* is a morphism of vector bundles $(\varphi, \bar{\varphi})$ such that φ and $\bar{\varphi}$ are G-equivariant and $\bar{\varphi} : X \to X'$ sends the base point of X to the base point of X'.

Example (Tautological vector bundle)**.** — Consider $X = \mathrm{Gr}(d, n)$. Define

$$E \coloneqq \{(W, v) \in \mathrm{Gr}(d, n) \times \mathbb{C}^n \mid v \in W\}$$

and consider $\pi : E \to \mathrm{Gr}(d, n)$ induced by the projection $\mathrm{Gr}(d, n) \times \mathbb{C}^n \to \mathrm{Gr}(d, n)$. This is (one should check) a $\mathrm{GL}_n(\mathbb{C})$-equivariant vector bundle of rank d, with action on E given by

$$g.(W, v) \coloneqq (gW, gv)$$

for all $g \in G$ and on X in the obvious way, i.e. $W \mapsto gW$. Similarly, there is the i-th tautological vector bundle for partial flag varieties, with total space

$$E \coloneqq \{(F, v) \in \mathrm{Fl}_{\underline{d}} \times \mathbb{C}^n \mid v \in F_{d_i}\}$$

and π induced by projection on $\mathrm{Fl}_{\underline{d}}$, which is again $\mathrm{GL}_n(\mathbb{C})$-equivariant.

Definition. — For G an affine algebraic group, $H < G$ a closed subgroup and V a representation of H, we define

$$G \times_H V \coloneqq (G \times V)/H$$

where H acts from the right on $G \times V$ via $(g, v).h = (gh, h^{-1}v)$ for all $g \in G$ and $v \in V$. Hence $G \times_H V$ as a set equals $(G \times V)/\sim$, where

$$(g, v) \sim (gh, h^{-1}v).$$

In particular, $(gh, v) \sim (g, hv)$, so that we can "move around" elements of H, as with tensor products. Note also that $G \times_H V$ is an algebraic variety.

As a special case for $V = \{0\}$, we have

$$G \times_H V = G \times_H \{0\} \cong G/H$$

by sending gH to the class $[(g, 0)]$.

Definition. — For G an affine algebraic group, $H < G$ a closed subgroup and V a representation of H, the construction $G \times_H V$ of the previous paragraph defines an *associated vector bundle* $\mathcal{V} = \mathcal{V}(G, H)$ with base G/H as follows:

$$\begin{tikzcd} G \times_H V \arrow[d, "\mathrm{id} \times 0"] \\ G \times_H \{0\} = G/H \end{tikzcd}$$

This is G-equivariant with the obvious left G-action on E and X.

We can make the following somewhat trivial but important observation: if $\pi : E \to X$ is a G-equivariant vector bundle, then $H^0(X, E)$ is a representation of G via

$$(g.s)(x) = g.(s(g^{-1}x))$$

for $s \in H^0(X, E)$, $g \in G$ and $x \in X$. But this is not the only way to produce representations of G given a G-equivariant vector bundles. To see this, we need the following general result:

VI.2. Proposition. — *Let G be an affine algebraic group, $H < G$ a closed subgroup. Then isoclasses of finite dimensional representations of H correspond to isoclasses of pointed G-equivariant vector bundles on G/H with base point H/H, by sending a finite dimensional representation of V to $\mathcal{V}$, as defined above, and conversely a G-equivariant vector bundle $\pi : E \to G/H$ to $\pi^{-1}(H/H)$. This correspondence is actually an equivalence of categories.*

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Proof. First we need to show that the assignments are well defined. We start with the assignment $V \mapsto \mathcal{V}$. In particular, we want to show that for V and V' isomorphic finite dimensional representations of H, we have $\mathcal{V} \cong \mathcal{V}'$ as pointed G-equivariant vector bundles. Let $\psi : V \to V'$ be an isomorphism of representations. Then assuming that $\mathrm{id} \times \psi$ is a well-defined morphism of G-varieties, we would have a morphism of G-equivariant pointed vector bundles

$$\begin{tikzcd} G \times_H V \arrow[rr, "\mathrm{id} \times \psi"] \arrow[dr, "\mathrm{id} \times 0"'] & & G \times_H V' \arrow[dl, "\mathrm{id} \times 0"] \\ & G \times_H \{0\} = G/H & \end{tikzcd}$$

To check that $\mathrm{id}\times\psi$ is well defined, first observe that it is well defined as a set theoretic map, since we have

$$\mathrm{id}\times\psi([(g,v)]) = [(g,\psi(v))] = [(gh, h^{-1}\psi(v))] = [(gh, \psi(h^{-1}v))] = \mathrm{id}\times\psi([(gh, h^{-1}v)])$$

for all $g \in G$, $h \in H$ and $v \in V$. We also have to check that $\mathrm{id}\times\psi$ is a morphism of G-varieties, but this is quite clear and we omit it. Moreover, $\mathrm{id}\times\psi^{-1}$ is clearly the inverse morphism.

For the other assignment, consider isomorphic G-equivariant vector bundles $\pi : E \to G/H$ and $\tilde{\pi} : \tilde{E} \to G/H$, with base point H/H, which we will denote just by H. We want first to show that $\pi^{-1}(H) = \tilde{\pi}^{-1}(H)$ as vector spaces. An isomorphism between π and $\tilde{\pi}$ comes with an (equivariant) isomorphism of varieties $\varphi : E \to E$ which then induces a linear map on the fibers

$$\pi^{-1}(H) = E_H \to \tilde{E}_{\bar{\varphi}(H)} = \tilde{E}_H = \tilde{\pi}^{-1}(H)$$

which is an isomorphism of vector spaces. Moreover, we claim that $\pi^{-1}(H)$ is a representation and the map induced by φ is an isomorphism of representations. By G-equivariance, the action of G induces linear maps $E_x \to E_{gx}$ for all $x \in X = G/H$. In particular, we get for all $h \in H$ a linear map

$$\pi^{-1}(H) = E_H \to E_{h.H} = E_H = \pi^{-1}(H)$$

which is invertible with inverse given by the action of h^{-1}. Thus we have proved that $\pi^{-1}(H)$ is a representation of H and by definition φ is G-equivariant, hence H-equivariant.

Finally, we prove that the assignments are mutually inverse. First, consider

$$V \mapsto \mathcal{V} = (G \times_H V \xrightarrow{\pi} G \times_H \{0\}) \mapsto \pi^{-1}(H).$$

By chasing the definitions, one gets that $\pi^{-1}(H) = \{[(g,v) \mid g \in H, v \in V]\}$, which is isomorphic (as a representation) to V via the maps $v \mapsto [e,v]$ and $[h,v] \mapsto h.v$, all checks being really easy. For the converse, observe that given

$$(E \xrightarrow{\pi} G/H) \mapsto \pi^{-1}(H) \mapsto (G \times_H \pi^{-1}(H) \xrightarrow{\mathrm{id}\times 0} G \times_H \{0\})$$

it is easy to check that the map $G \times_H \pi^{-1}(H) \to E,\ [(g,v)] \mapsto g.v$ defines an isomorphism of pointed G-equivariant pointed vector bundles. □

As anticipated, we now have two different ways to construct representations of G (an affine algebraic group with $H < G$ a closed subgroup) given some G-equivariant vector bundle $\pi : E \to G/H$: we can take global sections $H^0(G/H, E)$, or consider $\mathrm{ind}_H^G \pi^{-1}(H)$ where ind_H^G is induction from H to G. The next result characterizes induction in the case of finite groups, and is interesting for us more as an analogy that points in the right direction than in itself.

VI.3. Lemma. — *For finite groups $H < G$, given a representation V of H there is an isomorphism of representations of G*

$$\mathbb{C}G \otimes_{\mathbb{C}H} V \cong \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V),$$

with $\mathbb{C}H$ and $\mathbb{C}G$ the group algebras, and where $g \in G$ acts on $f \in \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$ by

$$(g.f)(m) = f(g^{-1}m)$$

for any $m \in \mathbb{C}G$, and $h \in H$ acts on $m \in \mathbb{C}G$ by

$$h.m = mh^{-1}.$$

Proof. Pick representatives $\{g_i\}_{i\in I}$ for the cosets of G/H (so that $g_i \otimes v_j$, for v_j the vectors of a basis of V, form a basis of $\mathbb{C}G \otimes_{\mathbb{C}H} V$). Define a map

$$\mathbb{C}G \otimes_{\mathbb{C}H} V \to \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V),\ \ g_i h \otimes v \mapsto f_{i,h,v}$$

where $f_{i,h,v}(g_j k) = \delta_{ij} k^{-1} hv$, and for $f \in \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V)$ define a map

$$\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V) \to \mathbb{C}G \otimes_{\mathbb{C}H} V,\ \ f \mapsto \sum_{i\in I} g_i \otimes f(g_i).$$

Note that the last assignment does not depend on the choice of the representatives g_i. Indeed, given $f \in \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, X)$ one has:

$$\begin{aligned}
\sum_{i\in I} g_i \otimes f(g_i) &= \sum_{i\in I} g_i h_i h_i^{-1} \otimes f(g_i) \\
&= \sum_{i\in I} g_i h_i \otimes h_i^{-1}.(f(g_i)) \\
&= \sum_{i\in I} g_i h_i \otimes f(h_i^{-1}.g_i) \\
&= \sum_{i\in I} g_i h_i \otimes f(g_i h_i).
\end{aligned}$$

We have to check that these two assignments are mutually inverse, which is quite boring (and checking well-definedness was already boring enough), so we omit it. □

Note that

$$\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V) = \{f : G \to V \mid f(gh^{-1}) = h.f(g)\ \ \forall g \in G, h \in H\}$$

and this makes sense for any affine algebraic group G with closed subgroup $H < G$ and V a representation of H. We can then give the following definition.

Definition. — Given an affine algebraic group G with a closed subgroup $H < G$, define

$$\mathrm{ind}_H^G V := \{f : G \to V \text{ morphism of varieties} \mid f(gh^{-1}) = h.f(g)\}.$$

The action of $g \in G$ on $f \in \mathrm{ind}_H^G V$ is given by

$$g.f(x) = f(g^{-1}x).$$

VI.4. Proposition. — *Let G be an affine algebraic group, $H < G$ a closed subgroup and V a finite dimensional representation of H with associated vector bundle $\mathcal{V}$ given by $\pi : E \to G/H$. Then there is an isomorphism of representations of G*

$$H^0(G/H, E) \cong \mathrm{ind}_H^G V := \{f : G \to V \mid f(gh^{-1}) = h.f(g)\ \ \forall g \in G, h \in H\},$$

with $V = \pi^{-1}(H)$.

Proof. Let $s \in H^0(G/H, E)$ be a section, i.e. a morphism $s : G/H \to E = G \times_H V$ such that $s(gH) = [(g, f(g))]$ for some morphism $f : G \to V$. For s to be well defined we need

$$s(gH) = s(gh^{-1}H) = [(gh^{-1}, f(gh^{-1}))] = [(g, h^{-1}.f(gh^{-1}))] = [(g, f(g))],$$

so $f(g) = h^{-1}.f(gh^{-1})$, thus $h.f(g) = f(gh^{-1})$ for all $h \in H$ and $g \in G$. Conversely, this condition defines a section. □

Our goal is to understand the representations arising this way (at least for $G = \mathrm{GL}_n(\mathbb{C})$ and an appropriate H). One important ingredient is the following theorem, an analogue of Lie's theorem in the theory of Lie algebras.

VI.5. Theorem (Lie-Kolchin). — *Assume W is a finite dimensional $\mathbb{C}$-vector space and H a connected solvable closed subgroup of $\mathrm{GL}(W)$. Then*

1. *H has a common eigenvector $v \in W$,*
2. *Every finite dimensional representation of H is 1-dimensional.*

Proof. See [Hum75], [Bor91] or [Spr83]. □

Example. — Any Borel subgroup $B' < \mathrm{GL}_n(\mathbb{C})$ satisfies the conditions of the theorem. If V is a 1-dimensional representation of B', then for $v \in V$ and $b \in B'$ we have $b.v = \lambda(b)v$ for some character $\lambda : B' \to \mathbb{C}^\times$.

In the previous example we can take $B' = B$ the standard Borel, with T the standard torus and U the unipotent radical (so that $U = [B, B] = \langle aba^{-1}b^{-1} \mid a, b \in B\rangle$ and $B = TU$). Then we have the following:

VI.6. Lemma. — *For B the standard Borel, T the standard torus and U the unipotent radical, we have $X(B) \cong X(T)$ with bijections $\lambda \mapsto \lambda|_T$ and $\lambda \mapsto \hat{\lambda}$, where*

$$\hat{\lambda}(tu) \coloneqq \lambda(t)$$

for all $t \in T$ and $u \in U$.

Proof. We have that the composition

$$\lambda \mapsto \lambda|_T \mapsto \hat{\lambda}$$

is the identity, since $\lambda(u) = 1$ (because $u \in [B, B]$ and λ is a group morphism). Conversely, clearly the composition

$$\lambda \mapsto \hat{\lambda} \mapsto \hat{\lambda}|_T = \lambda$$

is the identity. □

Some Facts about $\mathrm{GL}_n(\mathbb{C})$

Lecture 24
20th Jan, 2023

Recall the definition of roots: a root is a morphism $\beta : T \to \mathbb{C}$, such as

$$\beta\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right) = t_i t_j^{-1}$$

for some $i \neq j$. To β we associate an additive cocharacter

$$X_\beta : \mathbb{C} \to G,\ X_\beta(\lambda) = \mathbb{1} + \lambda E_{ij}.$$

The image of X_β is a closed subgroup U_β of G (called the *root subgroup* corresponding to β).

As a fact, the (upper triangular) unipotent group U is generated by the positive root subgroups; that is, $U = \prod_{\beta \in R^+} U_\beta$. Similarly, $U^- = \prod_{\beta \in R^-} U_\beta$. Recall also that

$$X(T) \cong \mathbb{Z}^n,\ \lambda_{\underline{c}} \leftarrow\!\shortmid \underline{c} = (c_1, \ldots, c_n),$$

where $\lambda_{\underline{c}}$ takes the diagonal matrix with entries $t_1, \ldots, t_n$ to the product $\prod_{i=1}^n t_i^{c_i}$.

VI.7. Theorem (Finite dimensional representations of $\mathrm{GL}_n(\mathbb{C})$)**.** — *Let V be a finite dimensional complex representation of $G = \mathrm{GL}_n(\mathbb{C})$.*

1. *We have*
$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$
where $V_\lambda = \{v \in V \mid t.v = \lambda(t)v \ \forall t \in T\}$ is called a (λ-)weight space. If $V_\lambda \neq \{0\}$ we call λ a weight of V.

2. *We have $\dim V_\lambda = \dim V_{w(\lambda)}$ where $w \in W \cong S_n$ and $w(\lambda)(t) = \lambda(\dot{w}^{-1}t\dot{w})$ for $\dot{w}$ a permutation matrix lifting w.*

3. *We have*
$$U_\beta V_\lambda \subset \sum_{r \in \mathbb{Z}_{\geqslant 0}} V_{\lambda + r\beta}.$$

4. *V has a maximal vector, i.e. there exists $0 \neq v \in V$ such that $U_\beta v = v$ for all β in R^+.*

5. *Assume V has maximal vector $v \in V_\lambda$. Then we can consider the subrepresentation $\tilde{W}$ of V generated by v. This has a unique maximal proper subrepresentation U, so $\tilde{W}/U$ is an irreducible representation with maximal vector of weight λ.*

6. *If V is irreducible and $v \in V_\lambda$ is a maximal vector, then λ is dominant, i.e. for any simple transposition $s \in W \cong S_n$ we have*
$$\lambda - s(\lambda) = \sum_{\beta \in R^+} \mathbb{Z}_{\geqslant 0}\beta.$$

Proof. We omit (1), as it is a standard argument.

(2) We claim that $\dot{w}V_\lambda \subset V_{w(\lambda)}$. This would prove (2) since the action by $\dot{w}$ is invertible. Let $v \in V_\lambda$, $t \in T$, then (given that $\dot{w} \in N_G(T)$)
$$t.\dot{w}.v = \dot{w}(\dot{w}^{-1}t\dot{w})v = \lambda(\dot{w}^{-1}t\dot{w})\dot{w}v,$$
which shows that $\dot{w}v \in V_{w(\lambda)}$.

(3) Let $v \in V_\lambda$. Them for $r \in \mathbb{C}$, $t \in T$ and β the root $t \mapsto t_i t_j^{-1}$, we have
$$t.rE_{ij}v = rtE_{ij}t^{-1}tv = rt_it_j^{-1}E_{ij}tv = r\lambda(t)t_it_j^{-1}E_{ij}v = r\lambda(t)\beta(t)E_{ij}v = r(\lambda+\beta)(t)E_{ij}v$$
in the last passage using additive notation for $X(T)$. This proves the claim.

(4) Either by Lie-Kolchin theorem (VI.5) or Borel's fixed point theorem (III.7) applied to the action of B on $\mathbb{P}^1(V)$, we obtain $v \in V$ such that $bv = \mu(b)v$ for all $b \in B$. Now, we have $u.v = v$ for all $u \in U$, since $U = [B,B]$. Then $u.v = v$ for all $u \in U_\beta$ for any $\beta \in R^+$.

(5) Recall that $B = TU$ and that B^-, the subgroup of lower diagonal matrices, is equal to U^-T. Take the maximal weight vector $v \in V_\lambda$. We have (using (3) as the only non-trivial input, in the second to last passage)
$$B^-Bv = B^-TUv = B^-Tv = B^-v = U^-Tv \subset \mathbb{C}U^-v \subset \sum_{\substack{r \in \mathbb{Z}_{\geqslant 0} \\ \beta \in R^-}} V_{\lambda + r\beta} = \sum_{\substack{r \in \mathbb{Z}_{\geqslant 0} \\ \beta \in R^+}} V_{\lambda - r\beta}.$$

Now, $B^- B = \dot{w}_0 B \dot{w}_0 B \subset G$, where

$$w_0 = \begin{pmatrix} 1 & \cdots & n \\ r & \cdots & 1 \end{pmatrix}$$

is the longest elements in S_n. We claim that $\dot{w}_0 B \dot{w}_0 B \subset G$ is dense. We know that the Schubert cell $Bw_0B/B \subset G/B$ is dense, thus $Bw_0B \subset G$ is dense and so is $w_0Bw_0B \subset G$, since w_0 is an invertible continuous map. Thus we have proved that $B^-B \subset G$ is dense, hence

$$\tilde{W} \subset V_\lambda + \sum_{\gamma \neq 0} V_{\lambda - \gamma}.$$

In particular $\tilde{W}$ and any of its subrepresentations inherit a weight space decomposition and thus the sum of all proper subrepresentations is proper, which proves the claim.

(6) If V is irreducible and $v \in V_\lambda$, then $V_{s(\lambda)} \neq \{0\}$ by (2) for any simple reflection s and thus by (3) we have

$$s(\lambda) = \lambda - \sum_{\substack{\beta \in R^+ \\ c^s_\beta \in \mathbb{Z}_{\geqslant 0}}} c^s_\beta \beta$$

since $\tilde{W}$ equals V by irreducibility. This already proves that λ is dominant. □

Remark. — For $\lambda_{\underline{c}} \in X(T)$ where $\underline{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, we have that λ is dominant if and only if $c_i - c_j \geqslant 0$ for all $i < j$.

VI.8. Corollary. — *We have an embedding of the set of irreducible finite dimensional representations of $G = \mathrm{GL}_n(\mathbb{C})$ modulo isomorphism into $X^+(T)$, the set of dominant weights, i.e. dominant elements in $X(T)$.*

Proof. First we need to prove that the assignment is well defined. By Theorem VI.7 we know that V has a weight space decomposition and there exists a maximal vector $v \in V$ (i.e. such that $U_\beta v = v$ for all $\beta \in R^+$). Without loss of generality, we can take $v \in V_\lambda$ for some λ to be the maximal weight vector. Assume v and v' are maximal weight vectors of weight respectively λ and λ'. Then, since V is irreducible and thus generated by v and v', we have (by (6) of Theorem VI.7)

$$\lambda' = \lambda - \sum_{\alpha \in R^+} c_\alpha \alpha$$

for some $\alpha \in \mathbb{Z}_{\geqslant 0}$, and also

$$\lambda = \lambda' - \sum_{\alpha \in R^+} c'_\alpha \alpha$$

for some $c'\alpha \in \mathbb{Z}_{\geqslant 0}$. Thus

$$\lambda' = \lambda' - \sum_{\alpha \in R+} c_\alpha \alpha - \sum_{\alpha \in R+} c'_\alpha \alpha$$

so we must have $c_\alpha = 0$, $c'_\alpha = 0$ for all $\alpha \in R^+$, which means that $\lambda = \lambda'$. We have thus proven that the assignment is well defined, we are left to prove that it is injective.

Assume now that V and V' are two irreducible representations with maximal vectors respectively v and v' of weight λ. We claim that $V \cong V'$. Consider $Z \coloneqq V \otimes V'$. Then Z has maximal vector $v + v'$. Let U be the subrepresentation generated by $v + v'$. If we denote by incl the inclusion of U in Z and by pr_i the projection to V or V', then $\mathrm{pr}_i \circ \mathrm{incl}$ is surjective (since V and V' are irreducible). Thus

$$K = \ker(\mathrm{pr}_1 \circ \mathrm{incl}) \subset V' \cap U,$$

but it doesn't contain v' by uniqueness of maximal vectors. Then K is a proper subrepresentation, so it must be $\{0\}$, as V is irreducible. This shows that $U \cong V$, and similarly $V' \cong U$, thus $V \cong V'$. □

Lecture 25
25th Jan, 2023

VI.9. Corollary. — *There is an embedding of the set of isoclasses of finite dimensional representation of* $\mathrm{GL}_n(\mathbb{C})$ *into* $X^+(T)$*, by sending a finite dimensional representation* V *to the weight of its maximal vector.*

Borel-Weil

We want to upgrade the injection of the previous corollary to a bijection.

Definition. — Given $\lambda \in X(T)$, consider $\mathbb{C}_\lambda$, the 1-dimensional representation of T corresponding to λ. So $t.v = \lambda(t)v$ for $t \in T$ and $v \in \mathbb{C}_\lambda$. Let $B^- < G = \mathrm{GL}_n(\mathbb{C})$ be the opposite (or negative) of the standard Borel, i.e. the lower diagonal matrices. Extend $\mathbb{C}_\lambda$ to a 1-dimensional representation of B^- via $t.v = \lambda(t)v$ and $u.v = v$ for $t \in T$ and $u \in U^-$. Let $\mathcal{L}_\lambda$ be the corresponding vector bundle, i.e. the vector bundle

$$\mathrm{id} \times 0 : G \times_{B^-} \mathbb{C}_\lambda \to G \times_{B^-} \{0\} = G/B^-.$$

Then define

$$H^0(\mathcal{L}_\lambda) \coloneqq H^0(G/B^-, G \times_{B^-} \mathbb{C}_\lambda).$$

We know already that

(1) $H^0(\mathcal{L}_\lambda)$ is a finite dimensional repesentation of G,

(2) we have an isomorphism of representations

$$H^0(\mathcal{L}_\lambda) \cong \{f : G \to \mathbb{C}_\lambda \mid h.f(g) = f(gh^{-1}) \ \forall g \in G, h \in B^-\},$$

with $(g.f)(x) = f(g^{-1}x)$ for $x, g \in G$.

VI.10. Theorem (Borel-Weil). — *Let* $G = \mathrm{GL}_n(\mathbb{C})$ *with the opposite Borel and the standard Torus, as above. Given* $\lambda \in X(T)$*, we have:*

(1) $H^0(\mathcal{L}_\lambda)$ *is either an irreducible representation of* G *of highest weight* λ *or zero,*

(2) $H^0(\mathcal{L}_\lambda) \neq 0$ *if and only if* $\lambda \in X^+(T)$*.*

VI.11. Corollary (Classification Theorem). — *There is a bijection between isoclasses of finite dimensional irreducible representations of* $\mathrm{GL}_n(\mathbb{C})$ *and* $X^+(T)$*, by sending a finite dimensional representation* V *to the weight of its maximal vector (as in Corollary VI.9).*

Proof. We already proved as Corollary VI.8 that the assignment is injective. For surjectivity, every $\lambda \in X^+(T)$ gives by Borel-Weil an irreducible finite dimensional representation $H^0(\mathcal{L}_\lambda) \neq 0$ of highest weight λ. □

We will not prove Borel-Weil in full detail, which is usually done using Lie algebra cohomology (see [A88]), but we will give three sketches of the proof:

- the algebraic one, which uses complete reducibility, the Peter-Weyl theorem and (after giving an independent proof) Corollary VI.11,
- the geometric one, which uses complete reducibility and Hartog's lemma,
- the proof in [Ful91], which uses a mix of algebraic and geometric ideas.

VI.12. Theorem (Complete reducibility)**.** — *Every finite dimensional representation of $G = \mathrm{GL}_n(\mathbb{C})$ is completely reducible (or semisimple), i.e. a direct sum of irreducible representations. The same holds even for rational representations of G, except for possibly having to consider infinite direct sums.*

Proof. Maybe later. □

Definition. — Let $(G, k[G])$ be an affine algebraic group and V a vector space with a group action
$$\alpha : G \times V \to V.$$
We call V a *rational representation* of G if $V = \bigcup_{i\in I} V_i$, where V_i for $i \in I$ is a finite dimensional vector space such that V_i is stable under the G-action with
$$\alpha|_{G\times V_i} : G \times V_i \to V_i$$
a morphism of algebraic varieties.

The definition is a bit strange looking at first, but the next example should clarify why it is the right one.

Example. — Recall the left regular and right regular actions of G on $k[G]$. Since we know that every $f \in k[G]$ is contained in a finite dimensional vector subspace $E_f \subset k[G]$ invariant under the action of G, we have
$$k[G] = \bigcup_{f\in k[G]} E_f$$
and this turns the left or right representations into rational representations.

Remark. — One can show that the following are equivalent:

(1) Every finite dimensional representation of $G = \mathrm{GL}_n(\mathbb{C})$ is completely reducible,

(2) Every finite dimensional rational representation of $G = \mathrm{GL}_n(\mathbb{C})$ is completely reducible,

(3) $k[G]$ is a completely reducible representation for the left or right regular action.

From now on $k[G]$ will be always viewed with the right regular action if we see it as a representation of G.

Remark. — Complete reducibility of a representation means that
$$M = \bigoplus_{i\in I} M_i$$

for some irreducible representations M_i, and we call for a fixed irreducible representation V, the subrepresentation

$$\mathrm{Iso}_V(M) := \bigoplus_{\substack{i\in I\\ M_i\in V}} M_i$$

the *isotypical component* of M with respect to V. Note that

$$M \cong \bigoplus_{V\in \mathrm{Irr}} \mathrm{Iso}_V(M)$$

with Irr a system of representatives of isoclasses of irreducible representations.

An ingredient in our proof of Borel-Weil is the classification theorem, which we deduced as a corollary of Borel-Weil, hence to avoid circularity we now prove the classification theorem in an algebraic way.

Proof of Corollary VI.11 independent from Theorem VI.10. The proof will essentially consist of guessing a list of finite dimensional irreducible representations, and then showing that they exhaust all such representations. Given $G = \mathrm{GL}_n(\mathbb{C})$ with natural action on $V := \mathbb{C}^n$, we can consider $\bigwedge^k V$ for $0 \leqslant k \leqslant n$. We have that $\bigwedge^1 V = V$ has as maximal vector the first basis vector e_1, with weight ε_1, where

$$\varepsilon_1(t) = t_1$$

for $t \in T$ with diagonal coefficients t_i. We have by convention that $\bigwedge^0 V = \mathbb{C}$ is the trivial representation and moreover $\bigwedge^n V$ is the determinant representation $\mathbb{C}_{\det}$. Indeed, we act on a vector $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$ by

$$g.(v_1 \wedge \cdots \wedge v_k) := gv_1 \wedge \cdots \wedge gv_k$$

for $g \in G$, and then a calculation shows that $\bigwedge^n V \cong \mathbb{C}_{\det}$.

Consider now the vector $z_k = e_1 \wedge \cdots \wedge e_k \in \bigwedge^k V$, where $e_i, \ldots, e_n$ is the standard basis of V. Note that for $t \in T$,

$$t.z_k = t.(e_1 \wedge \cdots \wedge e_k) = \varepsilon_1(t)\cdots\varepsilon_k(t)e_1 \wedge \cdots \wedge e_k,$$

thus z_k generates a subrepresentation of $\bigwedge^k V$ with maximal vector z_k of weight

$$\varepsilon_1 \cdots \varepsilon_k \in X(T)$$

(correspondingly $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ with k entries equal to 1). Then by Theorem VI.7 this subrepresentation has an irreducible quotient L of the same highest weight (and by direct calculation one can in fact show that $L \cong \bigwedge^k V$).

Note that if W_1 and W_2 are finite dimensional irreducible representations of highest weight respectively λ and μ and highest weight vector respectively w_1 and w_2, then $w_1 \otimes w_2 \in W_1 \otimes W_2$ is a highest weight vector with

$$t(w_1 \otimes w_2) = \lambda(t)\mu(t) w_1 \otimes w_2$$

for $t \in T$. Note also that if $\lambda = \lambda_{\underline{c}}$ and $\mu = \mu_{\underline{d}}$ for $\underline{c}, \underline{d} \in \mathbb{Z}^n$, then $\lambda\mu$ corresponds to $\underline{c} + \underline{d} \in \mathbb{Z}^n$, thus we can construct irreducible representations of highest weight equal to

$$\sum_{k=0}^{n} n_k w_k$$

with $n_k \in \mathbb{Z}_{\geqslant 0}$.

But we also have a 1-dimensional representation $\det^{-1}$ such that $g.v \coloneqq (\det(g))^{-1}v$. It is an irreducible representation of highest weight equal to $(-1, \dots, -1)$. Hence any $\underline{c} \in \mathbb{Z}^n$ is such that $\lambda_{\underline{c}} \in X^+(T)$ can be written as a sum of highest weight of $\bigwedge^k V$, det, $\det^{-1}$, for some $1 \leqslant k \leqslant n$. Thus by complete reducibility, any irreducible representation of highest weight sits inside a tensor product of $\bigwedge^k V$, det, $\det^{-1}$, with a given highest weight $\lambda_{\underline{c}} \in X^+(T)$. $\square$

VI.13. Proposition. — *Let $G = \mathrm{GL}_n(\mathbb{C})$ and V a finite dimensional irreducible representation of G.*

(1) *The morphism*

$$\mathrm{incl}_V : V^* \otimes V \to \mathbb{C}[G], \ \varphi \otimes v \mapsto f_{\varphi,v}$$

where $f_{\varphi,v}(g) = \varphi(gv)$ for $g \in G$ defines an inclusion of representations of $G \times G$, where $((g_1, g_2).f)(x) = f(g_1^{-1}, xg_2)$, and $(g_1, g_2).(\varphi \otimes v) = g_1.\varphi \otimes g_2 v$ with $(g_1.\varphi)(v) = \varphi(g_1^{-1}v)$ (in less words: all the actions are the reasonable ones).

(2) *The image of incl_V equals $\mathrm{Iso}_V(\mathbb{C}[G])$ as representations of $G \cong \{0\} \times G$.*

Proof. First, we prove $G \times G$-equivariance. In particular, we have to show that

$$\varphi \otimes v \mapsto (g_1, g_2).(\varphi \otimes v) = g_1\varphi \otimes g_2 v \mapsto f_{g_1\varphi, g_2 v}$$

equals

$$\varphi \otimes v \mapsto f_{\varphi,v} \mapsto (g_1, g_2).f_{\varphi,v}.$$

This is indeed the case, as

$$((g_1, g_2).f_{\varphi,v})(x) = f_{\varphi,v}(g_1^{-1}xg_2) = \varphi(g_1^{-1}xg_2v) = f_{g_1\varphi,g_2v}(x) = (g_1\varphi)(xg_2v).$$

Now for injectivity, assume $\varphi \neq 0$. Then there exists $w \in V$ such that $\varphi(w) \neq 0$, and thus there exists $g \in G$ and a basis vector $v \in V$ such that $\varphi(gv) \neq 0$, since V is irreducible. Hence $f_{\varphi,v} \neq 0$, which shows that incl_V is also not zero. But $V^* \otimes V$ is irreducible as a representation of $G \times G$, because V is irreducible as a representation of G, thus incl_V is injective, since its kernel is a $G \times G$-subrepresentation of $V^* \otimes V$ not equal to $V^* \otimes V$. This concludes the proof of (1).

For (2) assume that V is a G-subrepresentation of $\mathbb{C}[G]$ which is irreducible. Then consider an inclusion $i : V \hookrightarrow \mathbb{C}[G]$ and let $\varphi \in V^*$ be such that $\varphi(v) = i(v)(e)$. Then for $g \in G$,

$$i(v)(g) = i(v)(eg) = i(gv)(e) = \varphi(gv) = \mathrm{incl}_V(\varphi \otimes v)(g)$$

which concludes the proof, since it proves that the image of any inclusion $i : V \hookrightarrow \mathbb{C}[G]$ is in the image of incl_V. $\square$

LECTURE 26
27th Jan, 2023

VI.14. Theorem. — *Let $G = \mathrm{GL}_n(\mathbb{C})$. There is an isomorphism of $G \times G$-representations*

$$\mathbb{C}[G] \xrightarrow{\cong} \bigoplus_{L(\mu) \in \mathrm{Irr}(G)} L(\mu)^* \otimes L(\mu),$$

given by sending $f \otimes v \in L(\mu)^* \otimes L(\mu)$ to $\mathrm{incl}_{L(\mu)}(f \otimes v)$.

Proof. The map we defined is a morphism of $G \times G$-representations by Proposition VI.13. It is surjective, since $L(\mu)^* \otimes L(\mu)$ maps surjectively onto $\mathrm{Iso}_{L(\mu)}(\mathbb{C}[G])$ and using that $\mathbb{C}[G]$ is completely reducible. It is also injective, since any summand $L(\mu)^* \otimes L(\mu)$ maps isomorphically (as a G-representation) onto $\mathrm{Iso}_{L(\mu)}(\mathbb{C}[G])$ (note that $\mathrm{Iso}_{L(\mu)}(\mathbb{C}[G]) \cap \mathrm{Iso}_{L(\lambda)}(\mathbb{C}[G]) = \{0\}$ if $L(\lambda) \neq L(\mu)$). □

Remark. — Analogously, we have that

$$M_{n\times n}(\mathbb{C}) \cong V^* \otimes V,\ E_{i,j} \mapsto e_i^* \otimes e_j$$

where $V = \mathbb{C}^n$ with basis $e_1, \ldots, e_n$, is an isomorphism of $\mathrm{GL}_n(\mathbb{C})$-representations where the action on $M_{n\times n}(\mathbb{C})$ is the natural one and the action on $V^* \otimes V$ is on the second factor (and even of $G \times G$-representations, extending the action on $V^* \otimes V$ to be the tensor action on V^* and V).

Proof of Theorem VI.10 (Borel-Weyl). Consider the following vector bundles depending on $\lambda \in X(B^-) \cong X(T)$:

$$\mathcal{L}_\lambda = (\mathrm{id} \times 0 : G \times_{B^-} \mathbb{C}_\lambda \to G \times_{B^-} \{0\} = G/B^-),\ \hat{\mathcal{L}}_\lambda = (\mathrm{pr}_1 : G \times \mathbb{C}_\lambda \to G).$$

Now, $\hat{\mathcal{L}}_\lambda$ is B^--equivariant, where $b.g := gb^{-1}$ and $b.(g,v) = (gb^{-1}, bv)$ for $b \in B^-$, $g \in G$ and $v \in \mathbb{C}_\lambda$. Moreover $\mathcal{L}_\lambda$ and $\hat{\mathcal{L}}_\lambda$ are G-equivariant, where the G-action is given by left multiplication of G on G. Moreover $(\mathrm{can}, \mathrm{can})$ is clearly a morphism of G-equivariant vector bundles. Compare $H^0(\mathcal{L}_\lambda)$ with $H^0(\hat{\mathcal{L}}_\lambda)$. We have

$$H^0(\hat{\mathcal{L}}_\lambda) \cong \{f : G \to \mathbb{C}_\lambda \mid f \text{ is a morphism of affine algebraic varieties}\}$$

by sending a morphism of varieties $f : G \to \mathbb{C}_\lambda$ to $\hat{f}$, defined as $\hat{f}(g) := (g, f(g))$. Consider the action of B^- on $\mathbb{C}[G]$ induced by restriction of the right regular action of G, i.e. $(b.f)(g) = f(gb)$ for $b \in B^-$, $f \in \mathbb{C}[G]$ and $g \in G$. Then B^- acts on $\mathbb{C}[G] \otimes \mathbb{C}_\lambda$ via the usual action on tensor products.

We claim that $\hat{f}$ as defined above is a section of $\mathcal{L}_\lambda$ if and only if $f \otimes v \in (\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^{B^-}$ for all $v \in \mathbb{C}_\lambda$. Observe that for $b \in B^-$, $f \in \mathbb{C}[G]$ and $v \in \mathbb{C}_\lambda$, we have

$$b.(f \otimes v) = b.f \otimes b.v = \lambda(b)(b.f \otimes v).$$

Then we see that

$$\begin{aligned} f \otimes v \in (\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^{B^-} &\iff b.(f \otimes v) = f \otimes v \ \forall b \in B^- \\ &\iff (b.f) \otimes v = \lambda(b)^{-1} f \otimes v \ \forall b \in B^- \\ &\iff f(gb) = \lambda(b)^{-1} \ \forall b \in B^- \\ &\iff f \in H^0(\mathcal{L}_\lambda) \cong \{f : G \to \mathbb{C} \mid f(gb) = b^{-1} f(g) \ \forall b \in B^-\}. \end{aligned}$$

Now we claim that

$$(\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^{B^-} \cong \begin{cases} L(\lambda) & \text{if } \lambda \in X^+(T) \\ 0 & \text{otherwise} \end{cases}$$

where $L(\lambda)$ is an irreducible finite dimensional representation of G of highest weight λ. By Peter-Weyl (VI.14) we have that

$$\mathbb{C}[G] \otimes \mathbb{C}_\lambda \cong \bigoplus_{\mu \in X^+(T)} L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda$$

is an isomorphism of $B \times G$-representations. Then

$$(\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^{B^-} \cong \left(\bigoplus_{\mu \in X^+(T)} L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda \right)^{B^-} \cong \bigoplus_{\mu \in X^+(T)} (L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda)^{B^-}.$$

Now, observe that we have

$$\begin{aligned} f \otimes v \otimes 1 \in (L(\mu)^* \otimes L(\mu) \otimes \mathbb{C}_\lambda)^{B^-} &\iff b.(f \otimes v \otimes 1) = f \otimes v \otimes 1 \ \forall b \in B^- \\ &\iff \begin{cases} t.(f \otimes v \otimes 1) = f \otimes v \otimes 1 \ \forall t \in T \\ u.(f \otimes v \otimes 1) = f \otimes v \otimes 1 \ \forall u \in U^- \end{cases} \end{aligned}$$

where we wrote b as tu for $t \in T$ and $u \in U^-$, and the last condition is also equivalent to

$$\begin{cases} t.f = \lambda(t)^{-1} f \ \forall t \in T \\ u.f = f \ \forall u \in U^- \end{cases}$$

Let $z \in L(\mu)$ be the dual vector of $f \in L(\mu)^*$, i.e. some $f : W \to \mathbb{C}$ with $(g.f)(x) = f(g^{-1}x)$. Then we have that for some permutation matrix $w_0 \in N_G(T)$,

$$\begin{cases} t.z = \lambda(t) z \ \forall t \in T \\ u.z = z \ \forall u \in U^- \end{cases} \iff \begin{cases} t.w_0 z = \lambda(t) w_0 z \ \forall t \in T \\ u'.w_0 z = w_0 z \ \forall u' \in U \end{cases}$$

Indeed, since $U^- = w_0 U w_0$ (where $w_0 \in S_n$ is the longest element) and

$$w_0 w_0^{-1} t w_0 z = \lambda(w_0^{-1} t w_0) w_0 z,$$

we have $t.w_0 z = \lambda(t) w_0 z$ and $u.w_0 z = w_0 z$ for all $t \in T$, $u \in U$. But this is equivalent to $w_0 z$ being a highest weight vector of weight λ, i.e. an element of $L(\mu)_\lambda$, and therefore $\lambda = \mu$. Thus

$$(\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^{B^-} \cong ((L(\lambda)^* \otimes \mathbb{C}_\lambda) \otimes L(\lambda))^{B^-} = ((L(\lambda)^* \otimes \mathbb{C}_\lambda)^{B^-} \otimes L(\lambda)) \cong L(\lambda)$$

as representations of G. □

A Geometric Proof of Borel-Weil

We conclude the chapter (and the course) with a sketch of a geometric proof of the Borel-Weil theorem, and a mention to a generalization of it.

Geometric proof of Theorem VI.10. Assume $H^0(\mathcal{L}_\lambda) \neq 0$. We have to show that $\lambda \in X^+(T)$. By Lie-Kolchin or Borel's fixed point theorem, there exist $L \subset H^0(\mathcal{L}_\lambda)$ a 1-dimensional subspace such that $b.f = \mu(b) f$ for all $b \in B$, $f \in L$ and some $\mu \in X(B)$. Now, let $b \in B$, $c \in B^-$. Then

$$f(bc) = c^{-1}.(f(b)) = c^{-1}.(b^{-1}.f)(e) = \lambda(c)^{-1}(b^{-1}.f)(e) = \lambda(c)^{-1}\mu(b)^{-1} f(e) \qquad (*)$$

where the we used that $f \in H^0(\mathcal{L}_\lambda)$, the definition of $\mathcal{L}_\lambda$ and the previous observation. Thus $f|_{BB^-}$ is determined by $f(e)$, but $BB^- \subset G$ is dense and thus f is determined by $f(e)$. Now, $f(e) = [(e, v)]$ for some $v \in \mathbb{C}_\lambda$, hence L is unique.

We claim that $\lambda = \mu$. If $f \in T$, $f \in H^0(\mathcal{L}_\lambda)$, $f(e) \neq 0$, then by $(*)$

$$\mu(t)^{-1} f(e) = f(t) = \lambda(t)^{-1} f(e),$$

hence $\mu(t) = \lambda(t)$ for all $t \in T$.

Finally, by the Classification Theorem VI.11 we have that $H^0(\mathcal{L}_\lambda)$ is a finite dimensional representation of G of highest weight λ, since $H^0(\mathcal{L}_\lambda)$ is completely reducible, and $L(\lambda)$ occurs and is unique (and $H^0(\mathcal{L}_\lambda) \neq 0$). By our argument above, only one summand occurs, which means that $H^0(\mathcal{L}_\lambda) \cong L(\lambda)$, or the assumption doesn't hold and then $H^0(\mathcal{L}_\lambda) = \{0\}$.

We are left to show that $\lambda \in X^+(T)$ implies that $H^0(\mathcal{L}_\lambda)$ is not empty. This is non-trivial, and we only sketch it. It is enough to construct a section of $\mathcal{L}_\lambda$, and we can do this as follows:

- Step 1. Construct a section on BB^-/B^-. Let $f = f_\lambda$ be such that $f(u_1 t u_2) = \lambda(t^{-1})$ for $u_1, u_2 \in U$ and $t \in T$. We claim that f_λ is a section on BB^-/B^-. To show this, write $b = ru^-$ for $r \in T$ and $u \in U^-$. Then

  $$f(u_1 t u_2 r u) = f(u_1 t r r^{-1} u_2 r u) = \lambda(tr)^{-1},$$

  where we used that $tr \in T$ and $r^{-1} u_2 r \in U^-$. On the other hand

  $$\lambda(b^{-1}) f(u_1 t u_2) = \lambda(b^{-1}) \lambda(t)^{-1} = \lambda(r)^{-1} \lambda(t)^{-1} = \lambda(rt)^{-1},$$

  hence f is a section on BB^-/B^-.

- Step 2. Extend f to G/B. A sketch of this is the following.
  - Step 2a. Extend f by hands (via explicit formulas) to $\bigcup_{l(s)=1} BsB^-/B^-$.
  - Step 2b. Apply Hartog's lemma. This says that if X is a normal irreducible variety, $\mathcal{L}$ a line bundle over X and $U \subset X$ an open subset such that

    $$\dim(X \smallsetminus U) \leqslant \dim X - 2,$$

    then a section $s \in H^0(U, \mathcal{L})$ extends to a section of X, i.e. the restriction map

    $$H^0(X, \mathcal{L}) \to H^0(U, \mathcal{L})$$

    is a surjection. We can apply Hartog's lemma to

    $$U := BB^-/B^- \cup \bigcup_{l(s)=1} BsB^-/B^- \subset G/B^-,$$

    after checking that it satisfies the hypothesis on the dimension.

$\square$

Remark. — For a group G and a subgroup H, group cohomology can be defined as the right derived functor of the left exact functor F_H from the category of H-representations to the category of G-representations sending an H-representation M to $(\mathbb{C}[G] \otimes M)^H$, where H acts by

$$h.(f \otimes m) = h.f \otimes hm, \ \ (h.f)(g) = f(gh).$$

Example. — It is easy to show that the functor defined in the previous remark is left exact, but in general it is not exact: consider $B \subset \mathrm{GL}_2(\mathbb{C})$ and the natural representation $\mathbb{C}^2 = \langle e_1, e_2 \rangle$. We have a short exact sequence of B-representations

$$0 \to \langle e_1 \rangle \hookrightarrow \mathbb{C}_2 \twoheadrightarrow \mathbb{C}^2 / \langle e_1 \rangle \to 0.$$

But taking fixed points gives the sequence

$$0 \to \mathbb{C} \hookrightarrow \mathbb{C} \to \mathbb{C} \to 0,$$

which is only left exact.

VI.15. Theorem (Borel-Weil-Bott)**.** — *Let $\lambda \in X(T)$ and consider $\mathcal{L}_\lambda$. Pick $w \in S_n$ with $w.\lambda$ dominant (if this exists). Then*

$$H^i(\mathcal{L}_\lambda) = R^i F_{B^-}(\mathbb{C}_\lambda) \cong \begin{cases} L(w.\lambda) & \textit{if } i = l(w) \\ 0 & \textit{otherwise} \end{cases}$$

Clearly for $\lambda \in X^+(T)$, this has as a special case the Borel-Weil theorem.

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