

# Loose Ends from Tutorial 0

## Simplicial groups are Kan complexes

In the tutorial I had to rush through the proof of this result, so as promised here's a write-up. The proof I have written here is adapted from [\[May\]](#), and it proceeds by describing an algorithm to construct the horn fillings. In the end it really is the same proof I already gave at the tutorial, just in a slightly different vest (I guess I changed my mind and now I think this more explicit version is overall nicer than the slightly slicker one I presented, if you liked the other version more just ask at the next tutorial and you shall receive a better explanation). A different proof, more geometric and using a bit more technology (not sure it's really worth it, one way or another you have to get your hands dirty anyway) is given at page 14 of [this document](#) by Joyal and Tierney.

**Proposition.** — *Let  $X$  be a simplicial group. Then the underlying simplicial set of  $X$  is a Kan complex.*

*Proof.* We want to show that given any arbitrary  $n$ -horn in  $X$ , there exists a lift to a the  $n$ -simplex. Let  $\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n$  be the faces of an arbitrary  $n$ -horn in  $X$ ; we want to exhibit them as faces of a single  $n$ -simplex  $\theta$ . First, we will construct a simplex  $\eta$  with  $d_t \eta = \theta_t$  for all  $t < k$ .

If  $k = 0$ , there is nothing to do. Otherwise, we will inductively construct  $n$ -simplices  $\eta_0, \dots, \eta_m$  such that  $d_t \eta_m = \theta_t$  for all  $t \leq m$ . For  $m = 0$  we can set  $\eta_0 = s_0 \theta_0$ . Then assuming we have already constructed  $\eta_{m-1}$ , we define  $v_m := s_m(\theta_m \cdot d_m \eta_{m-1}^{-1})$  and  $\eta_m := v_m \cdot \eta_{m-1}$ . If now  $t < m$ , (using that the structure maps are group morphisms!)

$$\begin{aligned} d_t v_m &= d_t s_m(\theta_m \cdot d_m \eta_{m-1}^{-1}) \\ &= s_{m-1}(d_t \theta_m \cdot d_t d_m \eta_{m-1}^{-1}) \\ &= s_{m-1}(d_{m-1} \theta_t \cdot d_{m-1} d_t \eta_{m-1}^{-1}) \\ &= s_{m-1} d_{m-1}(\theta_t \cdot \theta_t^{-1}) = 1 \end{aligned}$$

while  $d_m v_m = \theta_m \cdot d_m \eta_{m-1}^{-1}$ . Thus  $d_t \eta_m = d_t \eta_{m-1} = \theta_t$  and

$$d_m \eta_m = \theta_m \cdot d_m \eta_{m-1}^{-1} \cdot d_m \eta_{m-1} = \theta_m$$

as desired.

To conclude, we will inductively construct  $\tau_n, \dots, \tau_k$  (note that we're moving downward this time!) such that  $d_t \tau_m = \tau_t$  for all  $t < k$  as well as all  $t > m$ ; then  $\theta := \tau_k$  will be the simplex filling the given horn. For  $m = n$  the simplex  $\eta$  constructed above already does the job. If now  $\tau_{m+1}$  has already been constructed, then we set  $\zeta_m = s_m(\tau_{m+1} \cdot d_{m+1} \tau_{m+1}^{-1})$ ,  $\tau_m = \zeta_m \cdot \tau_{m+1}$ . A similar computation to the one above then shows that  $\tau_m$  has the desired properties.  $\square$

## References

- [May] J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, 1967.