Deformations of Complex Manifolds

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The first two talks have been pragmatic and proved a lot of concrete facts specifically about K3 surfaces. In this talk instead we will rather be interested in general features of the deformation theory of complex manifolds, which are the basis for many of the future talks, and luckily for the audience, we will prove rather little! Ideally, the structure of the talk should be as follows:

- We start by introducing a couple of examples of smooth families,
- We define the Kodaira-Spencer map, state the fundamental results on the deformation theory of complex manifolds, and apply them in some cases,
- We reconsider the Kodaira-Spencer theory from the point of view of (complex) differential geometry, and try to give a brief overview of the proofs of the fundamental theorems (most likely not so much the required analysis).

Of course, we will only be able to scratch the surface of the topic, but hopefully we will understand enough to blackbox the results needed for the rest of the seminar.

Smooth Families of Complex Manifolds

Deformation theory could be described as the local study of moduli spaces. The starting point of the theory of moduli spaces is the observation that the very definition of many examples of compact complex manifolds makes use of a set of complex parameters, and that varying such parameters yields different complex manifolds. This simple fact leads to the observation, first made by Riemann himself (he used the term "modulus" for "parameter", hence the term moduli space), that perhaps those complex parameters are themselves the coordinates of some complex structure parametrizing a family of given complex manifolds, and in fact we will spend most of the seminar (some of us maybe most of their life?) expanding on this circle of ideas (in few of the many possible directions!).

Examples. — Let's begin by noting some well-known examples of families of spaces which come with "parameters attached".

(0) I said "many" examples of compact complex manifolds come with complex parameters attached, but this is not the case for projective space! We will see later that projective space is in fact rigid: its complex structure does not admit deformations.

(1) Universal elliptic curve. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ be the complex upper half-plane (our parameter space) and consider

$$\mathbb{H} \times \mathbb{C}/\mathbb{Z}^2,\ (\tau, z) \sim (\tau, z + m + n\tau)\ \forall m, n \in \mathbb{Z}$$

with the map given by projection to $\mathbb{H}$. Then it is the most classic of all classic facts about complex geometry that the fibers E_τ of this map, which are complex tori, correspond to all the elliptic curves over the complex numbers (and two fibers E_τ and $E_{\tau'}$ are isomorphic if and only τ and τ' are in the same orbit of the natural $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathbb{H}$), so we call the family $\mathbb{H} \times \mathbb{C}/\mathbb{Z}^2 \to \mathbb{H}$ the universal elliptic curve over the upper half-plane.

Of course, example (1) can be generalized to higher dimensional complex tori

$$A_w = \mathbb{C}^g/\Lambda_w,\ \Lambda_w = \langle e_1, \ldots, e_g, w_1 \ldots, w_g \rangle$$

where $e_1, \ldots, e_g$ is the standard basis of $\mathbb{C}^g$ and $e_1, \ldots, e_g, w_1 \ldots, w_g$ are linearly independent over $\mathbb{R}$ (which happens if and only if $\det \Im(w_1, \ldots, w_g) \neq 0$). As before considering

$$B = \{w \in M_{g\times g}(\mathbb{C}) \mid \det(\Im(w)) > 0\},$$

we can define

$$B \times \mathbb{C}^g/\mathbb{Z}^{2g},\ (w, z) \sim (w, z + \sum m_i e_i + \sum n_i w_i)\ \forall m, n \in \mathbb{Z}$$

and the map induced by projection to B, so that the fiber of w is the complex torus A_w.

(2) Universal degree d hypersurface in $\mathbb{P}^n$. Consider the vector space $V = H^0(\mathbb{P}^n, \mathcal{O}(d))$ and let $f_0, \ldots, f_N$ be a basis. Then consider

$$Z = \{([y_0 : \cdots : y_n], [x_0 : \cdots : x_n]) \mid \sum y_i f_i(x_0, \ldots, x_n) = 0\} \subset \mathbb{P}(V) \times \mathbb{P}^n$$

and the restriction to Z of the projection $\mathbb{P}(V) \times \mathbb{P}^n \to \mathbb{P}(V)$. Of course Z contains all sorts of singular stuff, so let $U \subset \mathbb{P}(V)$ be the locus of smooth surfaces, which is a Zariski open subset of $\mathbb{P}(V)$, and set $Z_U = \pi^{-1}(U)$. Then the fibers of the projection $Z_U \to U$ are all the smooth degree d hypersurfaces in $\mathbb{P}^n$ (so U is our parameter space), and we call this family the universal degree d hypersurface.

We can abstract these examples in the following definition:

Definition. — A *smooth family* of compact complex manifolds parametrized by B is a proper and submersive holomorphic map of complex manifolds $\pi : \mathcal{X} \to B$.

If we pick some distinguished point $0 \in B$ and an isomorphism $\varphi : X \to X_0$ from a compact complex manifold X to the distinguished (or special, or central) fiber $X_0 := \pi^{-1}(0)$, we say that $X_t := \pi^{-1}(t)$, which is another compact complex submanifold of $\mathcal{X}$ since π is a proper submersion, is a *deformation* of X, and all the data mentioned so far $(\pi, \varphi, 0)$ a *deformation family* of X.

As a first observation, note that from the viewpoint of differential geometry smooth families don't really deform much, because of the following:

Theorem (Ehresmann's Fibration Theorem). — *Any proper submersive morphism of differentiable manifolds $M \to N$ is (differentiably) locally trivial, i.e. for all $t \in N$ there exists an open subset $U_t \subset N$ containing t such that*

$$\pi^{-1}(U_t) \cong U_t \times X_t$$

as differentiable manifolds.

We give a sketch of the proof of this theorem, which is elementary (and so more often than not stated without proof) but underlies the theory of variations of complex structures.

Proof. We have to produce a trivializing chart at each $x_0 \in N$. By picking a chart (U, u) on N centered at x_0 such that $u(U) \cong \mathbb{R}^m$, we can assume $N = \mathbb{R}^m$. We will map the fibers in $\pi^{-1}(\mathbb{R}^m)$ to the central fiber $X_0 \coloneqq \pi^{-1}(0)$ in a diffeomorphic way, by moving them along the integral curves of a set of m vector fields.
Since π is a submersion (and so locally looks like a projection $\mathbb{R}^{m+n} \to \mathbb{R}^m$), with the help of a partition of unity on $\pi^{-1}(\mathbb{R}^m)$ we may construct vector fields $\xi_i \in \mathfrak{X}(\pi^{-1}(\mathbb{R}^m))$ which are π-related to $\partial_i \coloneqq \partial/\partial x^i$, i.e. such that $T\pi(\xi_i) = \partial_i \circ \pi$.
Now, it is a small lemma that if ξ_i is π-related to ∂_i, then $\pi \circ \Phi_t^{\xi_i} = \Phi_t^{\partial_i} \circ \pi$ whenever either of the sides of the equality is defined, and therefore, since π is proper and ∂_i is complete, ξ_i has a global flow too (essentially by Cauchy's theorem). Then the map

$$\pi^{-1}(\mathbb{R}^m) \to U \times X_0, \;\; z \mapsto (\pi(z), \Phi_{-t_n}^{\xi_n} \circ \cdots \circ \Phi_{-t_1}^{\xi_1}(z)),$$

where $\pi(z) = (t_1, \ldots, t_n)$, is well-defined (i.e. the image is really $U \times X_0$), a diffeomorphism (with inverse given by $(t_1, \ldots, t_n, y) \mapsto \Phi_{t_1}^{\xi_1} \circ \cdots \circ \Phi_{t_n}^{\xi_n}(y)$), and (clearly) a fiber chart. $\square$

Remark. — Of course if we ask that M is connected, then the fibers $p^{-1}(x)$ are all diffeomorphic (i.e. we get an actual fiber bundle).

In particular, Ehresmann's theorem shows that the fibers in the examples above are all diffeomorphic, while they are definitely not isomorphic as complex manifolds! This allows us to see smooth families as parametrizations of (some of) the complex structures on a given fixed differentiable manifold.

Remark (Transversely holomorphic trivialization). — Ehresmann's theorem does not use that smooth families are holomorphic, which is sometimes crucial. Indeed, one can show that given a smooth family we can choose the local trivializations to be transversely holomorphic: a transversely holomorphic local trivialization is a diffeomorphism

$$\varphi = (\varphi_U, \varphi_{X_0}) : \pi^{-1}(U) \to U \times X_0$$

such that the fibers of φ_{X_0} are complex manifolds. In a way, this could be taken to mean that the complex structure X_t varies holomorphically with $t \in U \subset B$. Having said this, we should not need this refined trivialization result in this talk.

Before we move on, we can give a couple more examples showing that the behavior of smooth families can be quite varied from the holomorphic perspective:

Examples. — (3) A boring example. Consider the projectivization $\mathbb{P}(E)$ of a rank r holomorphic vector bundle $E \to S$. Then of course $\pi : \mathbb{P}(E) \to S$ is a family with all fibers isomorphic to $\mathbb{P}^{r-1}$, and moreover it is a locally (*holomorphically*) trivial family, i.e. if $\{U_i\}_{i\in I}$ is a trivializing cover for E, we have that $\pi^{-1}(U_i)$ is isomorphic as a complex manifold to $S \times \mathbb{P}^{r-1}$, even though the family need not be globally trivial, e.g. take the Hirzebruch surfaces $\mathbb{F}_n$ which are the projectivization of $\mathcal{O} \otimes \mathcal{O}(n)$ over $\mathbb{P}^1$ (show that the n-th Hirzebruch surface is isomorphic to the hypersurface $V(x_0^n y_1 - x_1^n y_2) \subset \mathbb{P}^1 \times \mathbb{P}^2$ and use your favorite argument to argue that the blow-up of $\mathbb{P}^2$ at a point is not isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1$).

(4) Let $S \coloneqq V(x_0^2 y_1 - x_1^2 y_2 - t x_0 x_1 y_0) \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}$ and consider the map defined by the projection on the third factor, which is a smooth family. It is an exercise to show that for $t \neq 0$ we have $S_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, but for $t = 0$ we have $S_0 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)) = \mathbb{F}_2$. Note however that it is possible to show that if all the fibers are isomorphic as complex manifolds, then a family is indeed locally trivial.

The zoo of smooth families of complex manifolds is quite vast, we could go on forever with the examples and there would be a lot to say about smooth families in general (e.g. Kodaira has shown using hard analysis that “small” deformations of Kähler manifolds are still Kähler, while the famous example of Hironaka is a deformation of Kähler manifolds which is not Kähler, and the source of many interesting counterexamples; more easily one can show, as we most likely will in the following talk, that “small” deformations of K3 surfaces are still K3 surfaces), but in this talk we will be concerned only with the local aspects of this story, on which now we start focusing on.

The Kodaira-Spencer Map and General Results

There are multiple definitions of the Kodaira-Spencer map: given a family $\mathcal{X} \to B$, morally they can be interpreted as the differential at 0 of a map

$$B \to \{\text{isomorphism classes of complex manifolds}\},\ t \mapsto B_t$$

which comes from assigning the isomorphism class of the fiber of $t \in B$. In other words, the Kodaira-Spencer map should classify “infinitesimal first-order” deformations. We will make this intuition precise later.

The Kodaira-Spencer map. — There are various equivalent ways to define the Kodaira-Spencer map. We start with a sketchy "definition" which is meant mainly to provide motivation (in fact, up to some missing details, this was the original definition considered by Kodaira and Spencer, see Chapter 4 of [Kod], in which Kodaira recalls the steps they took to get to the right definitions), as it makes clear how we are "differentiating" the complex structure of the central fiber, then give a proper one.

If we have a smooth family $\pi : \mathcal{X} \to B$, for any point $x \in X = X_0$, by Ehresmann's Fibration Theorem there exist open subsets $V_x \subset B$, $U_x \subset X$, $W_x \subset \mathcal{X}$ such that

$$W_x \cap \pi^{-1}(V_x) \cong V_x \times U_x.$$

Since X is compact, there are $x_1, \ldots, x_m \in X$ such that $\mathcal{U} = \{U_{x_1}, \ldots, U_{x_m}\}$ covers X; we will write $U_i := U_{x_i}$. By replacing V_{x_i} by $V = V_{x_1} \cap \cdots \cap V_{x_m}$, we can assume $V_{x_i} = V$. Possibly shrinking V some more, we have:

$$\begin{aligned} \pi^{-1}(V) &= \bigcup_{i=1}^{m} V \times U_i \\ &= \coprod_{i=1}^{m} V \times U_i / \sim, \ (t, z_j) \sim (t, \varphi_{ij}(t, z_j) = z_i) \end{aligned}$$

Then X_t is determined by the holomorphic transition maps $\varphi_{ij}(t, -)$ for $t \in V$. Now, the maps φ_{ij} satisfy a cocycle condition: on $U_i \cap U_j \cap U_k$ we have

$$\varphi_{ik}(t, z_k) = \varphi_{ij}(t, \varphi_{jk}(t, z_k)),$$

so differentiating

$$\partial_{t_b} \varphi_{ik}(t, z_k) = \partial_{t_b} \varphi_{ij}(t, z_j) + (d\varphi_{ij}) \partial_{t_b} \varphi_{jk}(t, z_k). \tag{$*$}$$

Defining $\theta_{ij}^b := \frac{\partial \varphi_{ij}}{\partial t_b} \in H^0(U_i \cap U_j, T_{U_{ij}})$, in view of $(*)$ we have

$$-\theta_{ik}^b|_{U_{ijk}} + \theta_{ij}^b|_{U_{ijk}} + \theta_{jk}^b|_{U_{ijk}} = 0,$$

hence $\left(\theta_{ij}^b\right)_{ij} \in \check{H}^1(\mathcal{U}, T_X) \hookrightarrow H^1(X, T_X)$. We can then define the Kodaira-Spencer map:

$$KS_0 : T_0 B \to H^1(X, T_X), \ \textstyle\sum_b \lambda_b \partial_{t_b} \mapsto \left(\sum_b \lambda_b \theta_{ij}^b\right)_{ij}$$

although it is not so clear this is well-defined, so we turn to a more sound approach.

More properly, given a smooth family $\pi : \mathcal{X} \to B$, consider the short exact sequence:

$$0 \to T_X \to T_{\mathcal{X}}|_X \to \pi^* T_B|_X \to 0.$$

Taking the long exact sequence in cohomology, we get the Kodaira-Spencer map as the connecting morphism:

$$KS_0 : T_0 B = H^0(X, \pi^* T_B|_X) \to H^1(X, T_X).$$

One can show that this second definition recovers the first if one considers the exact sequence of Čech cohomology groups instead.

Remark (An alternative (perhaps better) approach). — The proper setting to talk about deformation of complex manifolds would be that of complex spaces: these are locally ringed spaces locally modeled on some analytic subset $Z \subset U \subset \mathbb{C}^n$ endowed with the sheaf $\mathcal{O}_U/\mathcal{I}$ where $\mathcal{I}$ is a sheaf of holomorphic functions such that $Z = Z(\mathcal{I})$. Then one might define a deformation family of X as a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, hook, "0"] & B \end{tikzcd}$$

where X is a complex manifold, $\mathcal{X}$ and B complex spaces, and $\mathcal{X}$ is flat over B. In this situation, there is a short exact sequence

$$0 \to T_X \to \Theta_{\mathcal{X}}|_X \to T_0 B \otimes \mathcal{O}_X \to 0,$$

where of course $\Theta_{\mathcal{X}}|_X$ has to be defined appropriately and $T_0 B \coloneqq \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_0/\mathfrak{m}_0^2, \mathbb{C})$. Then one obtains a Kodaira-Spencer map as above from a long exact sequence. Finally, it is a nice exercise to show that specializing to the case of $B = \operatorname{Spec}\mathbb{C}[\varepsilon]$, where $\mathbb{C}[\varepsilon]$ is a shorthand for $\mathbb{C}[t]/(t^2)$, one can repeat the argument we used for the first definition of the Kodaira-Spencer map (almost verbatim!) and obtain a natural bijection between classes in $H^1(X, T_X)$ and isomorphism classes of deformations $\mathcal{X} \to \operatorname{Spec}\mathbb{C}[\varepsilon]$: this gives an intrinsic way to define first-order infinitesimal deformations (we don't need a smooth family a priori anymore to run the argument).

Remark (Going Grothendieck). — Of course once deformation theory has been made more algebraic, such as in the definition above, one can consider deformations of schemes (there are again different definitions, using cocycles or short exact sequences of Kähler differentials), and even go one step further and consider deformation functors, i.e. functors from the category of Artin algebras over a field to sets (or groupoids and whatnot). The idea is that such functors classify the infinitesimal deformations of arbitrary order (this is why we consider Artin algebras, the motivating example is something like $\mathbb{C}[t]/t^{n+1}$), and then one can ask questions about representability of such a functor (usually pro-representability, in some form, and then the algebraization of a formal solution, again in some form, is a separated problem). The theory goes very deep in this direction, especially because this philosophy can be applied to all sorts of problems in algebraic geometry.

Remark (Obstructions). — Not every $\theta \in H^1(X, T_X)$ comes from a smooth family, i.e. is in the image of the Kodaira-Spencer map. We will see later (it is slightly easier using the differential geometric description of infinitesimal deformations rather than the definitions we gave so far) that there are *obstructions* to the existence of a deformation, which live in $H^2(X, T_X)$. It is an interesting exercise to find examples some compact complex manifold with $H^2(X, T_X) \neq 0$.

Before we go on, we need some definitions.

Warning ⚠. — From now on we will implicitly consider germs of $0 \in B$, i.e. everything we say will be "local" around 0, i.e. hold in some open neighborhood of 0. Less importantly, we will always assume that $X = X_0$ is the central fiber and forget about the isomorphism $\varphi : X \to X_0$ in the definition of deformation family.

Definition. — (1) A deformation family $\pi : \mathcal{X} \to B$ of the compact complex manifold X is called *complete* if any other deformation $\pi' : \mathcal{X}' \to B'$ of X is obtained by pullback under some map $f : B' \to B$.

(2) The deformation family π is *universal* if the map f in (1) is unique (i.e. for any two choices of maps $f_1 : U_1 \to B$ and $f_2 : U_2 \to B$, with U_1 and U_2 containing $0'$, there exists some $U \subset U_1 \cap U_2$ containing $0'$ such that $f_1|_U = f_2|_U$).

(3) The morphism π is called *versal* (lol) if the differential $T_0 f : T_0 B' \to T_0 B$ is unique.

The first important result about deformation families of compact complex manifolds is the following criterion for completeness (see [Kod, Theorem 6.1]):

Theorem (Kodaira-Spencer). — *Let $\pi : \mathcal{X} \to B$ be a deformation family of $X = X_0$ such that the Kodaira-Spencer map $KS_0 : T_0 B \to H^1(X, T_X)$ is surjective. Then π is a complete family.*

Remark. — It is easy to see that when we have a complete family and $B' \subset B$ a submanifold such that $KS_0 : T_0 B' \to H^1(X, T_X)$ is an isomorphism, then the pullback family $\mathcal{X}_{B'} \to B'$ is versal at 0.

The second fundamental result (about which we will try to say more in the last part of the talk) is a criterion for the existence of deformation families:

Theorem (Kodaira-Nirenberg-Spencer). — *Let X be a compact complex manifold for which $H^2(X, T_X) = 0$, in which case we say that X has unobstructed deformations. Then there exist a deformation family $\pi : \mathcal{X} \to B$ of X such that $KS_0 : T_0 B \to H^1(X, T_X)$ is an isomorphism, i.e. π is versal. Moreover, we have the following:*

(1) *if $H^0(X, T_X) = 0$, then π is universal (with respect to 0),*

(2) *if $h^1(X_t, T_{X_t})$ is locally constant, then the family $\pi : X \to B$ is versal for all of its fibers X_t in a neighborhood of $0 \in B$.*

There are a number of important comments to be made about this theorem: we will first go through them, then discuss some examples, and finally we will try to present as much of a proof as possible, after introducing the differential geometric picture.

(1) If X satisfies the hypothesis of the theorem, we write $\mathrm{Def}(X) := B$ for B in a universal deformation family $\mathcal{X} \to$ B of X. Since we have $H^0(X, T_X) = H^2(X, T_X) = 0$, any two

choices are uniquely isomorphic on an open neighborhood of the base point: we will remark later that this is the case for K3 surfaces.

(2) There is a parallel theorem (due to Kuranishi) which holds for *any* compact complex manifold X, but then the base B must be allowed to be singular. In any case, one still has an isomorphism

$$T_0B \to H^1(X, T_X),$$

and therefore the deformation family is at least versal. Then, whenever the hypothesis of the theorem of Kodaira-Spencer are satisfied, the theorem of Kuranishi reduces to the former (so in particular, one could say that having unobstructed deformations implies having a versal deformation family over a smooth base).

(3) The vanishing of $H^2(Y, T_Y)$ is not necessary for B to be smooth. A famous example are Calabi-Yau manifolds, which have a smooth deformation space. This is the content of the famous Bogomolov-Tian-Todorov theorem, about which (time permitting!) we will try to say something at the end of the talk.

Example. — Of course $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$, so $\mathbb{P}^n$ does not have infinitesimal deformations, confirming our hunch that it is a rigid space: indeed, a theorem of Kodaira-Spencer shows that a deformation family whose Kodaira-Spencer map is zero has to be a trivial family!

Example. — For the case of a curve C Riemann-Roch implies the following: for any linear bundle L on C

$$\chi(C, L) = h^0(C, L) - h^1(C, L) = \deg(L) + \chi(C, \mathcal{O}_C).$$

For any curve of genus g one always has $\chi(C, \mathcal{O}_C) = 1 - g$ and $\deg(TC) = 2 - 2g$. Therefore, we know that $h^0(TC) - h^1(TC) = 3 - 3g$, and so it is sufficient to find at least one of these numbers. Then:

- If g = 1, then C is an elliptic curve, TC is trivial and $h^0(TC) = h^0(\mathcal{O}_C) = 1$. Thus, $h^1(TC)$ also equals 1, of course agreeing with the first example of the talk!
- If $g > 1$ then $\deg(TC) < 0$. However, a divisor of negative degree can never have global sections. This means that $h^0(TC) = 0$ and $h^1(TC) = -(3 - 3g) = 3g - 3$. In fact, you might know that $3g - 3$ is the dimension of the moduli space (say the moduli stack) of genus g curves!

Example. — Consider the universal degree $d \geqslant 3$ hypersurface in $\mathbb{P}^n$ for $n \geqslant 2$. Given some hypersurface $X \subset \mathbb{P}^n$ of degree d, consider the normal bundle exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \to 0.$$

Then by the long exact sequence

$$0 \to H^0(X, T_X) \to H^0(X, T_{\mathbb{P}^n}|_X) \to H^0(X, N_{X/\mathbb{P}^n}) \xrightarrow{\delta} H^1(X, T_X) \to H^1(X, T_{\mathbb{P}^n}|_X) \to 0$$

where $H^0(X, T_X) = 0$ by a computation, and the last 0 is by Kodaira vanishing (or Serre duality in the case $n = 2$). Then one can show that under the identification

$$H^0(X, N_{X/\mathbb{P}^n}) = H^0(X, \mathcal{O}(d)|_X) \cong H^0(\mathbb{P}^n, \mathcal{O}(d)) \cong T_{[X]}\mathbb{P}(V)$$

for $V = H^0(\mathbb{P}^n, \mathcal{O}(d))$, the map δ becomes the Kodaira-Spencer map. Now, one can observe the following: if $n, d \geqslant 3$, we have

$$H^1(X, T_{\mathbb{P}^n}|_X) = \begin{cases} \mathbb{C} & \text{if } (n,d) = (3,4) \\ 0 & \text{else} \end{cases}$$

Thus we can say that if $(n, d) \neq (3, 4)$, then all infinitesimal deformations of X come from deformations of X as a hypersurface in $\mathbb{P}^n$. If $(n, d) = (3, 4)$, the only case where this does not work, then $X \subset \mathbb{P}^3$ is a quartic, hence a K3 surface! We will have something to say about this soon.

Example. — The case of the family with central fiber isomorphic to $\mathbb{F}_2$ is a fun example of a family which is versal for all fibers *but* the central one.

Example. — Let X be a K3 surface. Then the Kähler form induces an isomorphism between tangent and cotangent bundles, and therefore we have:

$$H^0(S, T_S) = 0, \;\; H^1(S, T_S) \cong \mathbb{C}^{20}, \;\; H^2(S, T_S) = 0.$$

Then there exists a universal 20-dimensional deformation family $X \to \mathrm{Def}(X)$ of X. The weird state of affairs is that no explicit construction of the family $S \to \mathrm{Def}(S)$ is known (apparently, algebraic K3 surfaces occur in 19-dimensional families). The local Torelli theorem which we are going to see next time provides a local isomorphism

$$\mathcal{P} : \mathrm{Def}(X) \to D \subset \mathbb{P}(H^2(X, \mathbb{C})),$$

where of course $\mathcal{P}$ is the period map.

In the rest of the seminar, we will use the theorems of Kodaira and Spencer more or less as a blackbox, as we did in the previous examples. However, to actually prove Kodaira and Spencer's theorems, one has to turn to more analytic methods.

Deformation Theory via Almost Complex Structures

There is a more intrinsic way to define deformations of complex structures, which makes use of the (difficult!) theorem of Newlander-Nirenberg, which says that a complex structure on a smooth manifold is the same as an integrable complex structure (if you vibe with hard analysis, see [Dem, Theorem 11.8] for a proof). We will show how this more analytic point of view on deformations leads to the existence theorem of Kodaira-Nirenberg-Spencer that we stated in the last section (we will just present some of the main ideas, for more details see [Sch, Theorem 2] or [Kod, Theorem 5.6]: in both sources the proof spans several pages and auxiliary results).

Remark. — Other than the one we sketch here, there are complex analytic and algebraic approaches to the existence of deformations of complex manifolds. While the complex analytic results are more general than the original ones by Kodaira-Spencer, the algebraic approach will fail to get the full 20-dimensional family of deformations of an algebraic K3 surface, as there are (many) non-algebraic K3 surfaces in the family. Note also that the original proof of the Bogomolov-Tian-Todorov theorem used this differential geometric setting (however in the case of BTT an algebraic proof exists, which is not crazy since all Calabi-Yau manifolds which are not K3 surfaces are algebraic (at least if you define your Calabi-Yaus wisely, I guess)).

The general setting is the following. The space of isomorphism classes of complex structures can be described in differential geometric terms: we start with the Fréchet manifold of almost complex structures on a given compact complex manifold X,

$$\mathcal{I}_{\mathrm{ac}}(X) \coloneqq \{I \mid I^2 = -1\} \subset \operatorname{End}(T_{\mathbb{C}}X),$$

then use the Newlander-Nirenberg theorem to describe the subspace $\mathcal{I}_{\mathrm{c}}(X) \subset \mathcal{I}_{\mathrm{ac}}(X)$ of integrable complex structures, and finally divide out by the action of the diffeomorphism group $\mathrm{Diff}(X)$, since two complex manifolds (X, I) and (X', I') are isomorphic if and only if there exists a diffeomorphism $F : X \to X'$ such that $dF \circ I = I' \circ dF$ (as this is the same as a biholomorphism). Now, to describe a deformation of complex structure, we can consider a continuous family $I(t)$ of integrable almost complex structures with $I(0) = I$, or in other words, a continuous family of decompositions $T_{\mathbb{C}}X = T_t^{1,0} \oplus T_t^{0,1}$, such that the integrability condition $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$ needed to apply the Newlander-Nirenberg theorem holds (and identifying isomorphic integrable complex structures).

A continuous family of almost complex structures $I(t)$ such as above, can be encoded for small t by maps

$$\varphi(t) : T^{0,1} \to T^{1,0}, \text{ with } v + \varphi(t)(v) \in T_t^{0,1}$$

which can be defined explicitly as the composition $-(\pi\iota)$ of the projection $\pi : T_{\mathbb{C}}X \to T_t^{1,0}$ and the inclusion $\iota : T^{0,1} \hookrightarrow T_{\mathbb{C}}X$ (the requirement that t be "small" is needed to ensure that $T_t^{1,0} \subset T_{\mathbb{C}}X \to T^{1,0}$, and for the converse $T_t^{0,1} \subset T_{\mathbb{C}}X \to T^{0,1}$, are isomorphisms, which for a compact manifold is an open condition; note that we are implicitly using these isomorphisms in the definitions we give in this paragraph). Conversely, if $\varphi(t)$ is given, then one defines, again for small t,

$$T_t^{0,1} \coloneqq (\mathrm{id} + \varphi(t))(T^{0,1}).$$

Therefore, we can see a deformation of almost complex structures as a continuous function from an open interval to $\mathcal{A}^{0,1}(T_X) \coloneqq \bigwedge^{0,1} X \otimes T_X$.

Now, the Lie bracket of vector fields can be extended to a bracket

$$[-,-] : \mathcal{A}^{0,p}(T_X) \times \mathcal{A}^{0,q}(T_X) \to \mathcal{A}^{0,p+q}(T_X)$$

by taking the Lie bracket in T_X and the exterior product in $\mathcal{A}^{0,\bullet}$ (note that this bracket is easily seen to extend to a bracket in cohomology). Similarly, we extend the $\bar{\partial}$ operator to an operator $\mathcal{A}^{0,1}(T_X) \to \mathcal{A}^{0,2}(T_X)$, as usual. Then one can show:

Lemma. — *The integrability condition* $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$ *is satisfied if and only if* $\varphi(t)$ *satisfies (one of the many incarnations of) the* Maurer-Cartan equation

$$\bar{\partial}\varphi(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0.$$

The proof of the lemma is one fairly ugly computation in local coordinates, not exactly insightful, see [Huy, Lemma 6.1.2] or [Sch, Theorem 1] (but beware that depending on conventions one might or not get the factor 1/2 in the Maurer-Cartan equation).

Consider now the power series expansion

$$\varphi(t) = \varphi_0 + \varphi_1 t + \varphi_2 t^2 + \cdots,$$

where the φ_i are elements of $\mathcal{A}^{0,1}(T_X)$ and $\varphi_0 = 0$. Replacing the power series expansion in the Maurer-Cartan equation, one obtains the equation

$$\bar{\partial}\left(\sum_{i=1}^{\infty} \varphi_i t^i\right) + \sum_{i,j=1}^{\infty} [\varphi_i, \varphi_j] t^{i+j}$$

which can be written as a recursive system of equations:

$$\begin{aligned} 0 &= \bar{\partial}\varphi_1 \\ 0 &= \bar{\varphi}_2 + [\varphi_1, \varphi_1] \\ &\vdots \\ 0 &= \bar{\varphi}_k + \sum_{0<i<k} [\varphi_i, \varphi_{k-i}] \end{aligned}$$

Focusing on the first equation, we see that a first-order infinitesimal deformation of complex structures is described by a $\bar{\partial}$-closed $(0,1)$-form φ_1 with values in the holomorphic tangent bundle T_X, and thus defines an element $[\varphi_1] \in H^1(X, T_X)$.

Remark. — To identify $H^1(X, T_X)$ with the first order infinitesimal deformations, we have to take into account the action of the diffeomorphism group $\mathrm{Diff}(X)$, which is an infinite dimensional Lie group (a Fréchet Lie group, but really we just need the smooth group structure), on the space of complex structures: easy arguments in local coordinates show that isomorphic first-order deformations of complex structures differ by a $\bar{\partial}$-exact form (for details see [Huy, pg. 259] or [GHJ, pg. 77]).

Now that we have defined infinitesimal deformations, the task is to integrate a given first-order deformation $[\varphi_1] \in H^1(X, T_X)$ to an actual one-parameter family of complex structures $I(t)$. What this entails in practice is the following: we need to find convergent(!) solutions $\sum_i \varphi_i t^i$ to the system of equations above. We can start by looking at formal (i.e. not necessarily convergent) solutions. First, note that from this perspective it is easy to see that there might be obstructions to the existence of deformations: already at order two (and the same could happen at any order) if we choose a representative φ_1 for a class in $H^1(X, T_X)$, we have to find a $\bar{\partial}$-exact form φ_2 such that $\bar{\partial}\varphi_2 = -[\varphi_1, \varphi_1]$, which will not be possible if the class of $[\varphi_1, \varphi_1]$ does not vanish in $H^2(X, T_X)$! On the other hand, suppose that $H^2(X, T_X)$ is trivial, as in the hypothesis of Kodaira-Nirenberg-Spencer theorem: then we are guaranteed formal solutions, so we need to find a way to turn them into convergent ones.

To find convergent solutions, one proceeds in two steps. First, one has to find a way to choose a specific formal solution which has good properties leading to convergence: this is the stuff of Hodge theory, which is a theory of "good" representatives for cohomology, the harmonic forms, which have minimal norm with respect to some chosen metric (note that we do not need compatibility of the metric with the complex structure, i.e. to work with Kähler manifolds). Once a specific solution has been found, the actual proof that such solution converges requires hard analysis, specifically Hölder estimates for elliptic operators. A nice exposition of the full proof (assuming some basic analysis results) is given in the fourth and fifth sections of [Sch].

Remark. — In the case of Calabi-Yau manifolds, as we already noted, we do not have vanishing $H^2(X, T_X)$, so for the Bogomolov-Tian-Todorov theorem we need another way to find formal solutions: luckily one can still find formal solutions thanks to the triviality of the canonical bundle (see [Huy, Proposition 6.1.11] for a very hands-on and elementary proof). Finding convergent solution requires Hodge theory (i.e. analysis), similarly to what one needs for the Kodaira-Nirenberg-Spencer theorem.

Remark. — Note that in this section we have not talked about smooth families, and in particular we haven't defined a Kodaira-Spencer map, although of course it can (and should, to develop the whole theory) be done.

DGLAs and Formal Deformation Theory. — There is one last obligatory remark about this Kodaira-Spencer story, which deserves some space for the importance it has had in various areas of mathematics in the last decades: the graded vector space

$$L = \bigoplus_{q \geqslant 0} \mathcal{A}^{0,q}(T_X)$$

is the prototypical example of a differential graded (or dg-)Lie algebra, which is the structure said to underlie any formal deformation problem.

Definition. — A *dg-Lie algebra* (or dgla for short) is the data of:

- a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_i \mathfrak{g}^i$,
- a linear map $\partial : \mathfrak{g} \to \mathfrak{g}$,
- a bracket, i.e. a bilinear map $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$,

such that (all conditions are expressed for homogeneously graded elements):

- ∂ is a differential that makes $(\mathfrak{g}, \partial)$ into a chain complex (with cohomological grading, i.e. $\partial(\mathfrak{g}^i) \subset \mathfrak{g}^{i+1}$),
- ∂ is a graded derivation of the bilinear pairing, i.e. $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ and

$$\partial[x_1, x_2] = [\partial x_1, x_2] + (-1)^{|x_1|}[x_1, \partial x_2],$$

- the bilinear pairing is graded skew-symmetric, i.e.

$$[x_1, x_2] = -(-1)^{|x_1||x_2|}[x_2, x_1],$$

- the bilinear pairing satisfies the graded Jacobi identity (saying that $[x, -]$ is a graded derivation), i.e.

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0.$$

There is a "philosophy" of formal deformation theory, introduced by Deligne, Quillen and others, which says that (at least in characteristic 0) every formal deformation problem is "controlled" by a dgla, with the corresponding Maurer-Cartan elements (modulo gauge action) being the formal deformations. This point of view is the start of much interesting work in geometry, algebra and mathematical physics (see e.g. the notes [Man]), and recently has been strengthened to a theorem by Pridham and Lurie.

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