

Loose Ends from Tutorial 0

Simplicial groups are Kan complexes

In the tutorial I had to rush through the proof of this result, so as promised here's a write-up. The proof I have written here is adapted from [\[May\]](#), and it proceeds by describing an algorithm to construct the horn fillings. In the end it really is the same proof I already gave at the tutorial, just in a slightly different vest (I guess I changed my mind and now I think this more explicit version is overall nicer than the slightly slicker one I presented, if you liked the other version more just ask at the next tutorial and you shall receive a better explanation).

Proposition. — *Let X be a simplicial group. Then the underlying simplicial set of X is a Kan complex.*

Proof. We want to show that given any arbitrary n -horn in X , there exists a lift to a the n -simplex. Let $\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n$ be the faces of an arbitrary n -horn in X ; we want to exhibit them as faces of a single n -simplex θ . First, we will construct a simplex η with $d_t \eta = \theta_t$ for all $t < k$.

If $k = 0$, there is nothing to do. Otherwise, we will inductively construct n -simplices $\eta_0, \dots, \eta := \eta_{k-1}$ such that $d_t \eta_m = \theta_t$ for all $t \leq m$. For $m = 0$ we can set $\eta_0 = s_0 \theta_0$. Then assuming we have already constructed η_{m-1} , we define $v_m := s_m(\theta_m \cdot d_m \eta_{m-1}^{-1})$ and $\eta_m := v_m \cdot \eta_{m-1}$. If now $t < m$, (using that the structure maps are group morphisms!)

$$\begin{aligned} d_t v_m &= d_t s_m(\theta_m \cdot d_m \eta_{m-1}^{-1}) \\ &= s_{m-1}(d_t \theta_m \cdot d_t d_m \eta_{m-1}^{-1}) \\ &= s_{m-1}(d_{m-1} \theta_t \cdot d_{m-1} d_t \eta_{m-1}^{-1}) \\ &= s_{m-1} d_{m-1}(\theta_t \cdot \theta_t^{-1}) = 1 \end{aligned}$$

while $d_m v_m = \theta_m \cdot d_m \eta_{m-1}^{-1}$. Thus $d_t \eta_m = d_t \eta_{m-1} = \theta_t$ and

$$d_m \eta_m = \theta_m \cdot d_m \eta_{m-1}^{-1} \cdot d_m \eta_{m-1} = \theta_m$$

as desired.

To conclude, we will inductively construct τ_n, \dots, τ_k (note that we're moving downward this time!) such that $d_t \tau_m = \tau_t$ for all $t < k$ as well as all $t > m$; then $\theta := \tau_k$ will be the

simplex filling the given horn. For $m = n$ the simplex η constructed above already does the job. If now τ_{m+1} has already been constructed, then we set $\zeta_m = s_m(\tau_{m+1} \cdot d_{m+1} \tau_{m+1}^{-1})$, $\tau_m = \zeta_m \cdot \tau_{m+1}$. A similar computation to the one above then shows that τ_m has the desired properties. \square

A different proof, more geometric and using a bit more technology (not sure it's really worth it, one way or another you have to get your hands dirty anyway) is given at page 14 of [this document](#) by Joyal and Tierney, and yet another proof which seems worth reading is given in Higher Algebra (Proposition 1.3.2.11): at a first glance I don't get it, but it's probably sleek.

References

- [May] J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, 1967.