Graduate Seminar on Algebraic Geometry

Chow Groups and Motives

Organizers

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Talks by

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Chow Groups, Speaker: Thomas Manopulo

This talk will be devoted to the basics of Chow groups: after their definition, we will compute the Chow group of affine and projective space, all the while developing some basic features of the theory. As an application, we will discuss dual hypersurfaces.

Definition of Chow Groups, Basic Properties, Examples

The starting point for intersection theorem is Bézout's theorem.

Theorem. — *Let $X, Y \subset \mathbb{P}^2_k$ be two plane curves which share no irreducible component. The intersection $X \cap Y$ consists of $(\deg X)(\deg Y)$ many points (counted with multiplicity).*

Throughout this seminar char $k = 0$ and $k = \bar{k}$.

Remark. — We can make the following observations.

- If X intersects Y transversally (in the intuitive sense, we haven't defined this yet), then the topological space $X \cap Y$ has $d \cdot e$ components.
- The theorem can be read as saying that the "degree" of the algebraic cycle $X \cap Y$ in $\mathbb{P}^2_k$ does not depend on X or Y per se, but rather their "deformation type".
- The goal in introducing the Chow ring is to reinterpret the equality "$X \cap Y = d \cdot e$" as expressing what the "product" between elements X and Y is in some algebraic setting.

Definition. — Let X be a variety over k. The group of *algebraic cycles* on X is

$$Z(X) = \bigoplus_{Y \subset X} \mathbb{Z}\langle Y \rangle$$

where Y varies among the (closed) subvarieties of X, where by subvariety we mean integral (reduced and irreducible) subscheme. If $Y \subset X$ is a closed subscheme, then we associate to it the (effective) cycle

$$\langle Y \rangle = \sum_{i=1}^{m} \operatorname{length}_{\mathcal{O}_Y}(\mathcal{O}_{Y,Y_i}) \cdot \langle Y_i \rangle.$$

Cycles may be viewed as coarse approximations to subschemes.

Definition. — Let $\operatorname{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (X \times \{t_0\}) \rangle - \langle \Phi \cap (X \times \{t_1\}) \rangle,$$

where $t_0, t_1 \in \mathbb{P}^1_k(k)$ and Φ is a subvariety of $X \times \mathbb{P}^1_k$ not contained in any fiber $X \times \{t\}$. We say that two cycles are *rationally equivalent* if their difference is in $\operatorname{Rat}(X)$, and similarly for two subschemes with rationally equivalent associated cycles.

From now on, I will usually write just $\mathbb{P}^n$ for $\mathbb{P}^n_k$ and $\mathbb{A}^n$ for $\mathbb{A}^n_k$.

Example. — The reader is urged to construct some examples of rationally equivalent subschemes. The example Tom gave in the talk was the two rationally equivalent cycles of $X = \mathbb{A}^2$ given by $Y_1 = V(u^2 - v^2)$ and $Y_2 = V(u^2 - v^2 + 1)$, with the equivalence witnessed by $\Phi = V(u^2 - v^2 + t^2) \subset X \times \mathbb{P}^1$.

Definition. — The *Chow group* of X is defined as the quotient

$$A(X) := Z(X)/\operatorname{Rat}(X).$$

Remark. — The group of cycles $Z(X)$ is naturally graded by dimension; we denote by $Z_k(X)$ the group of k-cycles, i.e. formal linear combinations of subvarieties of dimension k. Since $\Phi \subset X \times \mathbb{P}^1$ is integral and not contained in any fiber over X, in an appropriate open affine $\Phi \cap (X \times \mathbb{A}^1) \subset \Phi$, the scheme $\Phi \cap (X \times \{t_i\})$ for $i = 1, 0$ is defined by the vanishing of the single regular section $t - t_i$. Then $\Phi \cap (X \times \{t_0\})$ and $\Phi \cap (X \times \{t_1\})$ have the same dimension (each of their components is of codimension 1 in Φ), thus the grading by dimension on $Z(X)$ factors through the quotient, i.e. $A(X)$ is also graded by dimension.

Definition. — We say that two subvarieties A and B of a variety X intersect *transversally* at a point $p \in A \cap B$ if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X

Clearly smoothness is necessary to make this definition work!

$$T_pA + T_pB = T_pX.$$

We say that A and B intersect *generically transversally* if they intersect transversally at the generic point of each component in $A \cap B$.

The following is a difficult theorem due to Fulton.

Theorem (Existence of the intersection product)**.** — *Suppose X is a smooth quasi-projective variety over k. Then there exists a unique ring structure on $A(X)$ satisfying:*

$$[A] \cdot [B] = [A \cap B]$$

for all $A, B \subset X$ which intersect generically transversally.

For a proof, see [Ful98, Section 8.3]. Note that the theorem rests on a key technical result known as the Moving lemma, which will be the subject of the second talk:

Theorem (Moving lemma)**.** — *Let X be a smooth quasi-projective variety.*

(1) *For all $\alpha, \beta \in A(X)$ there exist generically transverse algebraic cycles $A, B \in Z(X)$ such that $[A] = \alpha$ and $[B] = \beta$.*

(2) *The class $[A \cap B]$ is independent of the choice of A and B.*

We can now compute some first examples of Chow groups.

Proposition. — $A(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$.

Proof. Clearly we have $A_n(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$. If $Y \subset \mathbb{A}^n$ is a proper subvariety we can explicitly construct $\Phi \subset (\mathbb{A}^n \times \mathbb{P}^1)$ such that $\Phi \cap (\mathbb{A}^n \times \{1\}) = Y$ and $\Phi \cap (\mathbb{A}^n \times \{\infty\}) = \varnothing$. Suppose without loss of generality that $0 \notin Y$. Set

$$\Phi_0 := \{(ty, t) \in \mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\}) \mid t \in \mathbb{A}^1 \setminus \{0\},\ y \in Y\}$$

and let Φ be the scheme theoretic closure of the inclusion $\Phi_0 \hookrightarrow \mathbb{A}^n \times \mathbb{A}^1$. Evidently $\Phi \cap (\mathbb{A}^n \times \{1\}) = Y$ and to see why $\Phi \cap (\mathbb{A}^n \times \{\infty\}) = \varnothing$ note that if $0 \notin Y$ there exists $g \in k[z_1, \ldots, z_n]$ such that $Y \subset V(g)$ and $g(0, \ldots, 0) = a \neq 0$ (by the Nullstellensatz). Now if we set $G(z, t) = g(z/t)$, we have that G is a section on $\mathbb{A}^n \times (\mathbb{P}^1 \setminus \{0\})$ which vanishes on Φ_0 but not on the fiber $\mathbb{A}^n \times \{\infty\}$. □

Before seeing other examples we have a simple lemma, which shows that Chow groups behave much like singular homology groups:

Lemma. — *Let X/k be a scheme.*

(1) *Mayer-Vietoris. If $X = X_1 \cup X_2$, where $X_1, X_2 \subset X$ are subvarieties, then we have an exact sequence*

$$A(X_1 \cap X_2) \to A(X_1) \oplus A(X_2) \to A(X) \to 0.$$

(2) *Excision. For any open subset $U \subset X$ we have an exact sequence*

$$A(X \smallsetminus U) \to A(X) \to A(U) \to 0.$$

Proof. See [EH16, Proposition 1.14] □

We can now compute the Chow ring of projective space.

Proposition. — *The Chow ring of $\mathbb{P}^n$ is given by:*

$$A(\mathbb{P}^n) = \mathbb{Z}[\xi]/\xi^{n+1},$$

where $\xi \in A(\mathbb{P}^n)$ corresponds to the rational class of a hyperplane. Moreover if $[Y] \in A(\mathbb{P}^n)$ is the rational equivalence class of a subvariety of degree d and codimension m, then $[Y]$ corresponds to $d\xi^m$.

Proof. Note that projective spaces fit into a flag

$$* = \mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n,$$

hence applying excision to $U = \mathbb{P}^n \smallsetminus \{*\}$ yields:

$$A(*) \cong \mathbb{Z}[*] \cong \mathbb{Z}[\xi^n] \to A(\mathbb{P}^n) \to A(\mathbb{P}^n \smallsetminus \{*\}) \to 0.$$

Then reiterating

$$A(\mathbb{P}^1 \smallsetminus \{*\}) \cong \mathbb{Z}[\mathbb{A}^1] \cong \mathbb{Z}[\xi^{n-1}] \to A(\mathbb{P}^n \smallsetminus *) \to A(\mathbb{P}^n \smallsetminus \mathbb{P}^1) \to 0$$

and so on. In the end we get that $A(\mathbb{P}^n)$ is generated by $1, \xi, \dots, \xi^n$ (note that we use that $\mathbb{P}^i \smallsetminus \mathbb{P}^i - 1$ is $\mathbb{A}^i$ at each step). The second assertion is a consequence of the following theorem, whose (difficult) proof can be found in [Ful98, Section 1.4].

Theorem (Proper pushforward)**.** — *Let $f : X \to Y$ be a proper morphism of varieties over k, and set $f_* : Z(X) \to Z(Y)$ as*

$$f_*(\langle A \rangle) := \begin{cases} 0 & \text{if } \dim A > \dim f(A) \\ n \cdot \langle f(A) \rangle & \text{if } \dim A = \dim f(A), \text{ with } n = \deg(f|_A : A \to f(A)) \end{cases}$$

Then f_* induces a group morphism on Chow groups (i.e. $f_*(\mathrm{Rat}(X)) \subset \mathrm{Rat}(Y)$).

In particular, when X is proper over k we get a morphism $\deg : A(X) \to A(\operatorname{Spec} k) \cong \mathbb{Z}$. Moreover, if $f : X \to Y$ is flat, we get a pullback group morphism then $f^* : A(Y) \to A(X)$ defined by $\langle A \rangle \mapsto \langle f^{-1}(A) \rangle$.

Now, we can finish the proof: if $\dim Y = n - m$ and its degree is d, then Y intersects some $(n-m)$-plane in d points, which means that $\deg([Y] \cdot \xi^{n-m}) = d$, thus $[Y] = d\xi^m$. □

As a byproduct of the computation of the Chow ring of projective space, we get back (a generalization of) Bézout theorem.

Corollary. — *Let $X_1, \dots, X_r \subset \mathbb{P}^n$ be subvarieties of codimensions $e_1, \dots, e_r$ such that $\sum_i e_i \leqslant n$. If X_i intersects X_j generically transversally, then*

$$\deg X_i \cap \cdots \cap X_r = \deg X_i \cdot \cdots \cdot \deg X_r.$$

An Application: Dual Hypersurface

We can now describe a cool application of the machinery we developed so far.

Definition. — Let $X = V(F) \in \mathbb{P}^n$ be a smooth hypersurface of degree d. The *Gauss map* is defined by

$$\mathcal{G}_X : X \to \mathbb{P}^n = (\mathbb{P}^n)^*,\ p \mapsto [\partial_{z_0} F(p) : \cdots : \partial_{z_n} F(p)].$$

The scheme theoretic image $X^* \subset (\mathbb{P}^n)^*$ of the Gauss map is the *dual hypersurface* to X.

Remark. — The point $[\partial_{z_0} F(p) : \cdots : \partial_{z_n} F(p)]$ is the point in $(\mathbb{P}^n)^*$ defining the plane tangent to X at the point $p \in X$.

We will use the following theorem as a black box:

Theorem. — *$\mathcal{G}_X$ is a finite birational equivalence between X and X^*.*

Example. — Let $X = V(xz - y^2) \subset \mathbb{P}^3$, the Gauss map is then

$$[x : y : z] \mapsto [z : -2y : x]$$

and the dual hypersurface $X^* = V(xz - \frac{1}{4}y^2)$. The birational map in the other direction is

$$[x : y : z] \mapsto [z : -\frac{1}{2}y : x].$$

Proposition. — *Let X be a smooth hypersurface of degree $d > 1$. Then X^* is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof. We have

$$\deg X^* = \#X^* \cap H_1 \cap \cdots \cap H_{n-1} = \#X \cap \mathcal{G}_X^{-1}(H_1) \cap \cdots \cap \mathcal{G}_X^{-1}(H_{n-1}) = d(d-1)^{n-1},$$

where we have used that $\mathcal{G}_X$ is a birational map and that $\mathcal{G}_X^{-1}(H_i)$ is an hypersurface of degree $d-1$. $\square$

Examples. — If $X \subset \mathbb{P}^2$ is a smooth cubic, the dual curve $X^* \subset (\mathbb{P}^2)^*$ is a degree 6 curve. This means that there are 6 lines through a general point $p \in \mathbb{P}^2$ tangent to X.

Example. — If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $L \subset \mathbb{P}^3$ is a general line, how many planes contain L and are tangent to S? Note that the planes containing L are parameterized by a line in $(\mathbb{P}^3)^*$, and the answer is then given by the intersection of such a line with S^*, therefore this is $\deg S^* = 3 \cdot 2^2 = 12$.

Chow Motives (Speaker: Michele Lorenzi)

In the second half of the seminar, we will be talking about Chow motives. In particular, in this talk I will introduce pure motives, state the definition and some (very) basic properties and examples.

Motivating Motives (Ha!)

In the first talk of the seminar, Tom taught us about the Chow group of a quasi-projective variety X. These are objects which describe the intersection theory on the variety and whose definition is analogous in spirit to that of the singular homology (or even bordism) groups of a topological space: we consider the group

$$\mathrm{CH}(X) = \mathcal{Z}(X)/\mathcal{Z}_{\mathrm{rat}}(X)$$

where $\mathcal{Z}(X)$ is the free abelian group generated by the cycles on X and $\mathcal{Z}_{\mathrm{rat}}(X)$ the subgroup of cycles rationally equivalent to zero (that is, to the empty cycle).

Tom showed that rational equivalence respects the grading by dimension (or codimension) of cycles, and that assuming the Moving lemma, which was subsequently proved by Yang in the second talk, there is a well defined intersection product on $\mathrm{CH}(X)$, which gives it the structure of a graded ring. Moreover, he defined pullback and pushforward maps between Chow groups induced by (respectively) proper and flat morphisms of varieties.

Now, there are two motivations for introducing the theory of motives at this point. First, it is natural once we have defined an algebraic invariant for a class of spaces to ask for some geometric object encapsulating the information contained in the invariant. An example is the theory of Eilenberg-MacLane spaces in algebraic topology, which are representing objects for the singular cohomology of (nice enough) topological spaces. However, apart from $\mathrm{CH}^1(X)$, which is the Picard group of X, there is no hope for Chow groups to be representable in general, as it is suggested by the theorem of Mumford we have seen in the talk by Mauro. As a second motivation, in contrast to topological spaces, on which there is essentially one ordinary cohomology theory and different definitions coincide for reasonable spaces, algebraic varieties have a much richer structure and correspondingly there are, on top of Chow groups, many different Weil cohomology theories with various relations among them: singular, De Rham, étale (l-adic), crystalline... Indeed, this is what motivated the theory of motives in the first place: the work on the Weil conjectures by Weil, Grothendieck and collaborators hinted at deep connections between different cohomology theories.

First elephant to move out of the room here: the construction of pure motives and many results in the following talks are unconditional, although of course the Standard Conjectures will pop up quite often.

The theory of motives is an attempt at addressing these questions, providing a suitable category through which all Weil cohomology theories factor. Although the full success of the theory in unifying the cohomology of varieties rests on the infamous Standard Conjectures of Grothendieck (Grothendieck envisioned a $\mathbb{Q}$-linear semisimple abelian tensor category with "realization" functors to all Weil cohomology theories, the existence of which depends on one of the Standard Conjectures), the "philosopy" or "yoga" of motives is highly influential in algebraic geometry: in this talk we will construct Chow motives, and in the following talks we will see that these yield (unconditional!) information on the cohomology and the Chow groups of the associated varieties.

Adequate Equivalence Relations

Notation. — Throughout the talk, we work with the category $\mathrm{SmProj}(k)$, the category of smooth projective varieties over k, for k any field (but we might assume $k = \mathbb{C}$). A variety is not necessarily irreducible, we will write X_d to denote an irreducible variety of dimension d.

Second elephant to move out of the room: one might of course ask if there is a theory for more general varieties. Although there is yet no category of (so-called mixed) motives for general varieties, there are various candidates (i.e. triangulated categories) for what would be its derived category. The theory of mixed motives (and the related field of motivic homotopy theory) is an active area of research (making use of many sophisticated tools from homological algebra and homotopy theory) and is outside the scope of this seminar.

The construction of motives starts by considering cycles on smooth projective varieties, as we have done at the start of the seminar: for X a smooth projective variety over k, we denote by $\mathcal{Z}^i(X)$ the free abelian group on the codimension i irreducible subvarieties of X. (Correspondingly, we define $\mathcal{Z}_i(X)$ by considering dimension i subvarieties, then for X_d of pure dimension d we have $\mathcal{Z}^i(X) = \mathcal{Z}_{n-1}(X)$.)

Now, there are a few operations one wants to be able to carry out on cycles.

- Product: the obvious one, always defined.
- Pushforward: recall from the first talk that for a proper morphism $f : X \to Y$, we obtain a map $f_*\mathcal{Z}(X) \to \mathcal{Z}(Y)$ by extending linearly

$$f_*(\langle Z\rangle) := \begin{cases} 0 & \text{if } \dim Z > \dim f(Z) \\ n \cdot \langle f(Z)\rangle & \text{if } \dim Z = \dim f(Z), \text{ with } n := \deg(f|_Z : Z \to f(Z)) \end{cases}$$

  Note that since we are working in $\mathrm{SmProj}(k)$, every morphism is proper for us. In particular, pushforward is always defined.
- Intersection product: two subvarieties V, W of a smooth variety X of codimension i and j intersect each other in a union of subvarieties of codimension at most $i + j$. If all of them have codimension exactly $i + j$, we say that V and W intersect properly. Of course, varieties do not always intersect properly, which is why we need the moving lemma to prove that there is a well defined product on the Chow groups.
- Action by a correspondence: this was defined by Jake in his talk, but we recall the definition as it is central for the theory of motives. A correspondence between X and Y is a cycle in $X \times Y$. In particular, the graph of a morphism $X \to Y$ is a correspondence. Thus, correspondences generalize morphisms, and act on cycles in a similar way: given a correspondence $Z \in \mathcal{Z}^k(X \times Y)$, with $\dim(X) = d$, and a cycle $T \in \mathcal{Z}^i(X)$, we let

$$Z(T) = (\mathrm{pr}_Y)_*(Z \cdot (T \times Y)) \in \mathcal{Z}^{k+i-d}(Y)$$

  whenever the intersection product is defined. We call $k - d$ the degree of the correspondence. Note that correspondences of degree 0 preserve the codimension of the cycle. Moreover, observe that pushforward of cycles by a morphism f is just the action of the graph Γ_f (hence of degree $\dim(Y) - \dim(X)$).
- Pullback: we defined this one only for flat morphism of varieties, but in fact it is a special case of the previous operation. Indeed, for a morphism $f : X \to Y$ the pullback $f^* : \mathcal{Z}(Y) \to \mathcal{Z}(X)$ is just the action of the transpose of the graph of f, denoted ${}^t\Gamma_f$, where the transpose of a correspondence is its pushforward by the map $\tau : X \times Y \to Y \times X$ exchanging the factors in the product.

As we noted, these operations are not always defined in general, so we want to find a suitable equivalence relation on cycles such that considering equivalence classes of cycles gives a manageable theory (i.e. such that we can intersect and pullback cycles without worry).

Definition. — An *adequate* (or "*good*") *equivalence relation* is a family of equivalence relations on each $\mathcal{Z}(X)$ for $X \in \mathrm{SmProj}(k)$ satisfying the following properties:

(R1) Compatibility with grading and addition on each $\mathcal{Z}(X)$.

(R2) Moving lemma: Given $Z, W_1, \ldots, W_l \in \mathcal{Z}(X)$ there is $Z' \sim Z$ such that $Z' \cdot W_i$ is defined for $i = 1, \ldots, l$.

(R3) Compatibility with correspondences: if $T \sim 0$ in $\mathcal{Z}(X)$ and $Z \in \mathcal{Z}(X \times Y)$, then one has $Z(T) \sim 0$ in $\mathcal{Z}(Y)$. Note that this is equivalent to asking:
  - If $T \sim 0$ in $\mathcal{Z}(X)$, then for any variety Y one has $T \times Y \sim 0$ in $\mathcal{Z}(X \times Y)$.
  - If $T_1 \sim 0$ in $\mathcal{Z}(X)$, then for any cycle T_2 one has $T_1 \cdot T_2 \sim 0$.
  - If $T \sim 0$ in $\mathcal{Z}(X \times Y)$, then $(\mathrm{pr}_Y)_*(T) \sim 0$ in $\mathcal{Z}(Y)$.

Given $\sim$ any adequate equivalence relation, we write $Z_\sim(X)$ for the subgroup of cycles equivalent to 0 and $Z^i_\sim(X)$ for its subgroup of codimension i cycles, and

$$C^i_\sim(X) = Z^i(X)/Z^i_\sim(X),$$
$$C_\sim(X) = Z(X)/Z_\sim(X) = \bigoplus_i C^i_\sim(X).$$

Note that what we are asking in the definition of an adequate equivalence relation is precisely for intersection of cycles and consequently all the other operations to be always defined modulo the equivalence, and to respect the structure of $C_\sim(X)$. In fact, the following lemma now holds essentially by definition.

Lemma. — *For any adequate equivalence relation $\sim$ we have:*

1. *$C_\sim$ is a graded ring with product induced from intersection of cycles.*
2. *For any morphism $f : X \to Y$ in $\mathrm{SmProj}(k)$, pushforward of cycles by f induces a group morphism $f_* : C_\sim(X) \to C_\sim(Y)$ and pullback of cycles by f induces a graded ring morphism $f^* : C_\sim(Y) \to C_\sim(X)$. Moreover, the projection formula holds:*
$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$
for all $\alpha \in C_\sim(Y)$, $\beta \in C_\sim(B)$.
3. *A correspondence $Z \in \mathcal{Z}(X \times Y)$ of degree r induces a group morphism $Z_* : C^i_\sim(X) \to C^{i+r}_\sim(Y)$ and equivalent correspondences induce the same morphism.*

Examples of Adequate Equivalence Relations

Recall that $Z \in \mathcal{Z}^i(X)$ is rationally equivalent to zero if there exists a cycle $W \in \mathcal{Z}^i(\mathbb{P}^1 \times X)$ and $t_0, t_1 \in \mathbb{P}^1(k)$ such that, defining

$$W(t) := (\mathrm{pr}_X)_*(W \cdot (\{t\} \times X)),$$

we have $W(t_0) = 0$ and $W(t_1) = Z$. Of course, by what we have seen in the first two talks of the seminar (modulo some facts we did not prove which one can find in Fulton's book), rational equivalence is an adequate equivalence relation. One can prove (by showing that any two points of $\mathbb{P}^1$ are equivalent with respect to any adequate equivalence relation) that rational equivalence is the finest adequate equivalence relation. Moreover, it will be the most important equivalence for our purposes, as it is the one used in the construction of Chow motives. However, there are other equivalence relations of great interest.

Note that in the literature (e.g. in Fulton) rational equivalence is often defined slightly differently (as a generalization of linear equivalence).

Algebraic equivalence. — There is a slight modification we can make to the notion of rational equivalence to obtain another adequate equivalence relation: simply replace $\mathbb{P}^1$ in the definition of rational equivalence by any smooth irreducible curve C. Spelling out the change: $Z \in \mathcal{Z}^i(X)$ is algebraically equivalent to zero if there exists a cycle $W \in \mathcal{Z}^i(C \times X)$ for some smooth irreducible curve C and $t_0, t_1 \in C(k)$ such that, defining $W(t)$ as above, we have $W(t_0) = 0$ and $W(t_1) = Z$. One can prove that this defines an adequate equivalence relation. Clearly we have $\mathcal{Z}_{\mathrm{rat}}(X) \subset \mathcal{Z}_{\mathrm{alg}}$, but rational equivalence is strictly finer as an equivalence relation: for example, any two points on an elliptic curve C are obviously algebraically equivalent, but they cannot be rationally equivalent. Another striking example: it is a consequence of Bertini's theorem that any two points in a connected smooth variety can be joined by a smooth curve, but we have seen that $CH_0(X)$ can be extremely complicated..

Equivalently, we can take any smooth variety V instead of C.

It is not obvious that algebraic equivalence is in fact adequate, although at least the moving lemma is for free.

To define the next equivalence, we need the notion of Weil cohomology theory.

Definition. — Given a field F of characteristic 0, a *Weil cohomology theory* is a functor

$$H : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{GrVect}_F$$

satisfying the following axioms:

- There is a cup product.
- Poincaré duality holds.
- The Künneth formula holds.
- There are cycle class maps: these are maps

  $$\gamma_X : \mathrm{CH}^i(X) \to H^{2i}(X)$$

  - functorial,
  - compatible with intersection products,
  - compatible with points.

Note that often people include in the definition the following:

- Weak Lefschetz: If $j : Y_{d-1} \hookrightarrow X_d$ is a smooth hyperplane section, then

  $$j^* : H^i(X) \to H^i(Y)$$

  is an isomorphism for $i < d-1$ and injective for $i = d-1$.
- Hard Lefschetz: the Lefschetz operator $L(\alpha) = \alpha \smile \gamma_X(Y)$ induces isomorphisms

  $$L^{d-i} : H^{d-i}(X) \to H^{d+i}(X)$$

  for $0 \leqslant i \leqslant d$.

Examples. — Examples of Weil cohomology theories are the following:

- Betti cohomology: defined as $H^i_B(X) = H^i(X_{\mathrm{an}}; \mathbb{C})$ (or if the base field k is not $\mathbb{C}$ we can define it by choosing an embedding $k \hookrightarrow \mathbb{C}$, although of course this does not always exist and the end result depends on the embedding). It is not easy but classical to see that it is a Weil cohomology theory. The cycle map is given by pushforward on homology groups, using Poincaré duality.

- De Rham cohomology: similarly defined as $H_{\mathrm{dR}}(X_{\mathrm{an}};\mathbb{C})$, the de Rham cohomology of the analytification. There is also an algebraic version.
- l-adic cohomology: constructed via étale cohomology, there is an l-adic cohomology for every prime l different from the characteristic of the base field k. This cohomology will not concern us, but it is extremely important in number theory (and so is crystalline cohomology, yet another important Weil cohomology theory).

Remark. — There are deep relations between the cohomology we mentioned. In particular, we mention the de Rham and Artin comparison isomorphisms: for $k = \mathbb{C}$ one has

$$H_{\mathrm{dR}}(X_{\mathrm{an}};\mathbb{C}) \xrightarrow{\sim} H_B(X),$$
$$H_{\text{ét}}(X,\mathbb{Q}_l) \xrightarrow{\sim} H_B(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

Homological equivalence. — We can now define a cycle Z to be homologically equivalent to zero if $\gamma_X(Z) = 0$. This can be shown to be an adequate equivalence relation using the functorial and compatibility properties of the cycle map. Note that this definition depends a priori (i.e. not assuming the standard conjecture D) on the choice of Weil cohomology theory.

Numerical equivalence. — Let $X_d \in \mathrm{SmProj}(k)$, we define a cycle $Z \in \mathcal{Z}^i(X_d)$ to be numerically equivalent to 0 if for every $W \in \mathcal{Z}^{d-i}(X_d)$ such that $Z \cdot W$ is defined (and hence a zero cycle), we have

$$\deg(Z \cdot W) = 0.$$

It is not too difficult to show that this is an adequate equivalence relation.

Remark (Comparison of the various equivalences)**.** — For $X \in \mathrm{SmProj}(k)$, we have the following chain of inclusions:

$$\mathcal{Z}_{\mathrm{rat}}(X) \subset \mathcal{Z}_{\mathrm{alg}}(X) \subset \mathcal{Z}_{\mathrm{hom}}(X) \subset \mathcal{Z}_{\mathrm{num}}(X) \subsetneq \mathcal{Z}(X),$$

where the second and third inclusions can be proved using the properties of the cycle maps. A lot can be said about whether these inclusions are strict or not, we give just a few pointers. For divisors, algebraic and homological equivalence coincide, and they coincide up to torsion with numerical equivalence, but for higher codimensions Griffiths has shown that algebraic equivalence is strictly finer. Second, homological and numerical equivalence are conjectured to coincide (at least for algebraically closed fields), and they are proven to do in the cases of divisors (in arbitrary characteristic) and in characteristic zero for codimension 2, dimension 1 and abelian varieties (I am surely missing some other specific result, too). The equality of homological and numerical equivalence is of fundamental importance and is usually counted as one of the Standard Conjectures of Grothendieck: one can prove that the category of motives constructed considering numerical equivalence is the only one which is abelian semisimple (and almost Tannakian), thus possessing exceptionally good formal properties which would coronate Grothendieck vision of unifying Weil cohomology theories (of course, in the special case of smooth projective varieties).

The other standard conjectures are also related: they are questions about the existence of particular algebraic cycles inducing certain cohomology operations.

Construction of the Category of Pure (Chow) motives

Grothendieck's construction of the category of pure motives, with respect to a given adequate equivalence relation $\sim$, starts with the category $\mathrm{SmProj}(k)$ of smooth projective varieties over k and proceeds in three steps:

$$\mathrm{SmProj}(k)^{\mathrm{op}} \to C_{\sim}\,\mathrm{SmProj}(k) \to \mathrm{Mot}^{\mathrm{eff}}_{\sim}(k) \to \mathrm{Mot}_{\sim}(k).$$

The first step provides a contravariant functor from the category of smooth projective variety to a category with the same objects but a more general class of morphisms, while the second and third step are carried out in order to obtain a category with better formal properties (respectively, pseudo-abelian and rigid).

Category of correspondences. — In the following we consider equivalence classes of correspondences with rational coefficients:

$$\mathrm{Corr}_\sim(X,Y) := C_\sim(X \times Y) \otimes \mathbb{Q}.$$

We can compose two correspondences $f \in \mathrm{Corr}_\sim(X,Y)$ and $g \in \mathrm{Corr}_\sim(Y,Z)$ by the formula:

$$g \circ f := \mathrm{pr}_{XZ}\{(f \times Z) \cdot (X \times g)\}.$$

Note that the degree r correspondences are those that send codimension i cycles to codimension $i+r$ cycles, and composing a degree r correspondence with a degree s one yields a degree $r+s$ correspondence. Composition of correspondences makes $\mathrm{Corr}_\sim(X,X)$ into a ring, with the degree zero correspondences as a subring.

Example. — I can do a little drawing here, yay!

Note that given a morphism $f : X_d \to Y_e$, we have already noted that f_* and f^* are correspondences, respectively $\Gamma_f \in \mathrm{Corr}^{e-d}_\sim(X,Y)$ and ${}^t\Gamma_f \in \mathrm{Corr}^0_\sim(Y,X)$. In particular, we get a functor $\mathrm{SmProj}(k)^{\mathrm{op}} \to C_\sim \mathrm{SmProj}(k)$, where $C_\sim \mathrm{SmProj}(k)$ is the category which has smooth projective varieties as objects and degree zero correspondences as morphisms, with composition as described above.

The following is an useful result on correspondences, which we mention (as it will appear multiple times in the future) but do not prove, as the proof is not particularly interesting or illuminating.

Lemma (Lieberman). — *Given correspondences $f \in \mathrm{Corr}_\sim(X,Y)$, $\alpha \in \mathrm{Corr}_\sim(X,X')$ and $\beta \in \mathrm{Corr}_\sim(Y,Y')$, we have*

$$(\alpha \times \beta)_*(f) = \beta \circ f \circ {}^t\alpha.$$

Proof. Quite boring. $\square$

The category $C_\sim \mathrm{SmProj}(k)$ is a $\mathbb{Q}$-linear category, i.e. its Hom-sets are $\mathbb{Q}$-vector spaces and composition is bilinear (said otherwise: it is enriched over the category $\mathrm{Mod}_\mathbb{Q}$ of $\mathbb{Q}$ vector spaces). Moreover, $C_\sim \mathrm{SmProj}(k)$ has finite coproducts given by disjoint union of varieties, hence it is an additive category, and product of varieties give it a symmetric monoidal structure (with the point $\mathrm{Spec}\, k$ as the unit object). However, it does not have very good formal properties otherwise: in particular, it is quite far from being an abelian category, as one can see, for example, by noticing that there are idempotent morphisms which do not split (an idempotent morphism $e : X \to X$ is said to split if there are morphism $p : X \to Y$ and $q : Y \to X$, such that $q \circ p = e$ and $p \circ q = \mathrm{id}_Y$; in an abelian category we can take images and kernels of morphisms, thus all idempotents split). Indeed, one can take $e \times \mathbb{P}^1 \in \mathcal{Z}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ as an example: it is clearly idempotent and it is easily seen not to split. The next step in the construction of the category of pure motives addresses this problem.

Effective motives. — For the second step of our construction of pure motives, we consider the pseudo-abelian completion (or Karoubi envelope, or idempotent completion, even Cauchy completion, this thing has many names) of $C_\sim \operatorname{SmProj}(k)$.

Definition. — Given a preadditive category $\mathcal{C}$, its Karoubi envelope $\operatorname{Idem}(\mathcal{C})$ is the category which has for objects pairs (C,p) for C an object of $\mathcal{C}$ and $p : C \to C$ an idempotent morphism, and as morphisms $(C,p) \to (C',p')$ all morphisms $p' \circ f \circ p$ for $f \in \operatorname{Hom}_{\mathcal{C}}(C,C')$. The Karoubi envelope is easily shown to be pseudo-abelian (all idempotents split) and it is easy to see that the obvious embedding $\mathcal{C} \hookrightarrow \operatorname{Idem}(\mathcal{C})$ is universal among functors from $\mathcal{C}$ to pseudo-abelian categories (i.e. any other functor from $\mathcal{C}$ to a pseudo-abelian category factors through it).

Remark. — Clearly, the identity of an element (C,p) of $\operatorname{Idem}(\mathcal{C})$ is p. Moreover, there is an useful way to characterize the morphisms in the Karoubi envelope (usually this characterization is taken as the definition), that is,

$$\operatorname{Hom}_{\operatorname{Idem}(\mathcal{C})}((C,e),(C',e')) \cong \{f \in \operatorname{Hom}_{\mathcal{C}}(C,C') \mid f = e' \circ f \circ e\}.$$

This entails that for two objects (X,p) and (Y,q) to be isomorphic is equivalent to ask for the existence of maps f and g satisfying:

$$\begin{cases} p \circ g \circ q \circ f \circ p = p \\ q \circ f \circ p \circ g \circ q = q. \end{cases}$$

In some sense the idempotent completion is the cheapest modification we can make to an additive category to make it “a bit more abelian”, without modifying it too much. This is the best we can do in most cases to get something closer to an abelian category from an additive one, since finding a way to sensibly add images and kernels for every morphism is hopeless. For an example of a *not* sensible way to obtain an abelian category, one can embed any preadditive category in the category of **Ab**-valued presheaves on it: this is abelian but it destroys existing colimits and replaces them, so it is not a sensible modification if we are interested in the category we were starting with.

The category $\operatorname{Mot}^{\mathrm{eff}}_\sim(k)$ of effective motives is the idempotent completion, as defined above, of the category $C_\sim \operatorname{SmProj}(k)$. Explicitly, it is the category which has as objects pairs (X,p) where X is a smooth projective variety and $p : X \to X$ an idempotent correspondence, called a *projector*, and as morphisms sets

$$\operatorname{Hom}_{\operatorname{Mot}^{\mathrm{eff}}_\sim(k)}((X,p),(Y,q)) = p \circ \operatorname{Corr}^0_\sim(X,Y) \circ q.$$

Direct sums of effective motives are given by disjoint unions and tensor products by products. Finally, the embedding $C_\sim \operatorname{SmProj}(X) \hookrightarrow \operatorname{Mot}^{\mathrm{eff}}_\sim(k)$ is given by $X \mapsto (X,\Delta_X)$ and $f \mapsto f$.

Remark. — It is easy (and even easier with a drawing!) to see explicitly that idempotents split in the category of effective motives. To show this, observe first that for two orthogonal projectors $p,q \in \operatorname{Corr}^0_\sim(X,X)$, i.e. projectors such that $p \circ q = q \circ p = 0$, there is an isomorphism

$$(X, p+q) \cong (X \amalg X, p \amalg q)$$

given by

$$(\operatorname{id}_X \times i_1)_*(p) + (\operatorname{id}_X \times i_2)_*(q) \in \operatorname{Corr}^0_\sim(X, X \amalg X),$$

where $i_1, i_2 : X \to X \amalg X$ are the inclusions, and

$$(\mathrm{id}_{X \amalg X} \times \nabla)_*(p \amalg q) \in \mathrm{Corr}^0_\sim(X \amalg X, X),$$

where $\nabla : X \amalg X \to X$ is the fold map. It is easy to see that we are indeed working with morphisms of effective motives and that the two compositions are the identities. Now, if q is a projector on (X, p), then $p = q + (p - q)$ is a decomposition of $p = \mathrm{id}_{(X,p)}$ into orthogonal projectors: $q \circ (p - q) = q \circ p - q \circ q = 0$ and $(p - q) \circ q = 0$. Hence we obtain the splitting $(X, p) \cong (X, q) \oplus (X, p - q)$ where (X, q) is the image of q and $(X, p - q)$ is its kernel.

Category of pure motives. — The last step is also formal and consists in "inverting" the Lefschetz motive $\mathbf{L}_\sim$. This is done in order to obtain a *rigid* category, i.e. a monoidal category in which every object has a dual. The significance of this step is related to Tate twists in cohomology and also to the concept of Tannakian categories and ideas surrounding the motivic Galois group.

We define the category $\mathrm{Mot}_\sim(k)$ of pure motives as follows: the objects are triples (X, p, m), where X is a smooth projective variety, p a projector on X and $m \in \mathbb{Z}$, and the morphisms sets are

$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((X, p, m), (Y, q, n)) = p \circ \mathrm{Corr}^{n-m}_\sim(X, Y) \circ q.$$

The category of pure motives is a pseudo-abelian $\mathbb{Q}$-linear tensor category. Direct sums of effective motives are given by $(X, p, m) \oplus (Y, q, n) = (X \amalg Y, p \amalg q, m)$ whenever $n = m$ (the general case is more complicated, but it will become evident how to work it out from the special case once we talk about Tate twists), and tensor products are given by by $(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m + n)$. Of course, there is a fully faithful embedding $\mathrm{Mot}^{\mathrm{eff}}_\sim(k) \hookrightarrow \mathrm{Mot}_\sim(k)$, and altogether we get a functor $h_\sim : \mathrm{SmProj}(k)^{\mathrm{op}} \to \mathrm{Mot}_\sim(k)$.

Remark. — Intuitively, one should think of motives of the form $(X, p, 0)$ as a "piece" of X which is responsible for part of its geometric and/or cohomological properties. More precisely, the motive $(X, p, 0)$, for any projector p, is a direct summand of $h_\sim(X)$, as we have seen above, and one can prove that (X, p, m) is a direct summand of

$$h_\sim(X \times (\mathbb{P}^1)^m).$$

Later we will see this in practice in some examples.

Remark. — Note that the functor $h_\sim$ is not conservative: there exist non-isomorphic varieties with isomorphic motives (recall that two motives (X, p, m) and (Y, q, n) are isomorphic if there are correspondences f and g of degree $n - m$ and $m - n$ satisfying $p \circ g \circ q \circ f \circ p = p$ and $q \circ f \circ p \circ g \circ q = q$). This is expected behavior: motives are supposed to capture just some specific kind of information about varieties, the one seen by chow groups and Weil cohomologies.

First Examples of Motives

We can now give some interesting examples.

- There are three fundamental motives:
  - The motive of a point $\mathbf{1}_\sim = (\mathrm{Spec}\, k, \mathrm{id} = \Delta_{\mathrm{Spec}\, k}, 0) = h_\sim(*)$, also called the unit motive (as it is the unit of the monoidal structure).

  - The Lefschetz motive $\mathbf{L}_\sim = (\mathbb{P}^1, \mathbb{P}^1 \times e, 0)$, an effective motive.
  - The Tate motive $\mathbf{T}_\sim = (\operatorname{Spec} k, \mathrm{id}, 1)$, the dual of $\mathbf{L}_\sim$.

- Let $X_d \in \mathrm{SmProj}(k)$ and $e \in X(k)$ a rational point. The cycles $p_0(X) := e \times X$ and $p_{2d}(X) := X \times e$ define orthogonal projectors. By what we saw before, we obtain the decomposition
$$h_\sim(X) \cong h^0_\sim(X) \oplus h^+_\sim(X) \oplus h^{2d}_\sim(X)$$
where $h^i_\sim(X) := (X, p_i(X))$ for $i = 0, 2d, +$, with $p_+(X) = \mathrm{id}_X - p_0(X) - p_{2d}(X)$. In fact, it is easy to see that $h^0_\sim(X)$ is isomorphic to the unit motive $\mathbf{1}_\sim$ and we will prove that the "top dimensional part" $h^{2d}_\sim(X)$ only depends on the dimension d of X. It follows that the nontrivial information about $h_\sim(X)$ is concentrated in the middle part $h^+_\sim(X)$. If X is a curve, we will see in the following talk that this middle part $h^+_\sim(X)$ is related to its Jacobian.

- It is not difficult to see that the diagonal $\Delta_{\mathbb{P}^1} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is rationally equivalent to $\mathbb{P}^1 \times e + e \times \mathbb{P}^1$ (consider $H = V(\lambda_0 x_0 y_0 + \lambda_1(x_0 y_1 - x_1 y_0)) \subset (\mathbb{P}^1_x \times \mathbb{P}^1_y) \times \mathbb{P}^1_\lambda$) for the relevant homotopy). But then we have $h^+_\sim(\mathbb{P}^1) = 0$, thus
$$h_\sim(\mathbb{P}^1) = \mathbf{1}_\sim \oplus \mathbf{L}_\sim,$$
which is a funny equation, as it is saying that the projective line is the sum of a point and a line ("$\mathbb{P}^1$ =[point]+[line]"). Moreover, one can see with just a bit more effort (the details are in [EH16, Section 2.1.6]) that $\Delta_{\mathbb{P}^n}$ is rationally equivalent to $\sum_i \mathbb{P}^{n-i} \times \mathbb{P}^i$, a sum of pairwise orthogonal projectors, yielding a decomposition
$$(\mathbb{P}^n, \mathrm{id}, 0) = \bigoplus_i (\mathbb{P}^n, \mathbb{P}^{n-i} \times \mathbb{P}^i),$$
and it is not difficult to see that $(\mathbb{P}^n, \mathbb{P}^{n-i} \times \mathbb{P}^i) \cong (Y, Y \times e) \cong \mathbf{L}^{\otimes i}_\sim$, setting $Y = \mathbb{P}^i$, via the morphisms
$$\mathbb{P}^i \times e = (Y \times e) \circ (\mathbb{P}^i \times e) \circ (\mathbb{P}^i \times \mathbb{P}^{n-i}) \in (Y \times e) \circ \mathrm{Corr}^0_\sim(\mathbb{P}^n, Y) \circ (\mathbb{P}^i \times \mathbb{P}^{n-i})$$
and
$$Y \times \mathbb{P}^{n-i} = (\mathbb{P}^i \times \mathbb{P}^{n-i}) \circ (Y \times \mathbb{P}^{n-i}) \circ (Y \times e) \in (\mathbb{P}^i \times \mathbb{P}^{n-i}) \circ \mathrm{Corr}^0_\sim(Y, \mathbb{P}^n) \circ (Y \times e).$$
Therefore, the motive of projective space splits as follows
$$h_\sim(\mathbb{P}^n) = \mathbf{1}_\sim \oplus \mathbf{L}_\sim \oplus \mathbf{L}^{\otimes 2}_\sim \oplus \cdots \oplus \mathbf{L}^{\otimes n}_\sim.$$
Moreover, note that in the previous computation we did not use anything about Y other than its, dimension, which proves what we asserted about the "top dimensional part" of any motive. Lastly, note that any *cellular variety*, such as any Grassmannian, admit a similar decomposition, with one $\mathbf{L}^{\otimes d}_\sim$ for each cell of dimension d.

- For $f : X_d \to Y_d$ a surjective generically finite morphism of degree n, we can prove easily that $f_* \circ f^* = n\,\mathrm{id}$, so that $P := \frac{1}{n} f^* \circ f_*$ is a projector and $(Y, p) \cong h_\sim(X)$, so
$$h_\sim(Y) \cong h_\sim(X) \oplus h_\sim(Y, \mathrm{id} - p).$$
One can check that this decomposition works more generally whenever we have two motives $M = (X, p, 0), N = (Y, q, 0)$ and two morphisms $\alpha : N \to M, \beta : M \to N$ such that $\beta \circ \alpha = \mathrm{id}_N = q$.

- As promised, we can show that the Lefschetz and Tate motives are dual: this is evident as soon as one proves the isomorphism

  $$\mathbf{L}_\sim \cong (\operatorname{Spec} k, \mathrm{id}, -1)$$

  which is witnessed by the morphisms

  $$* \times e = (\mathbb{P}^1 \times e) \circ (* \times e) \in (\mathbb{P}^1 \times e) \circ \mathrm{Corr}^1_\sim(*, \mathbb{P}^1)$$

  and
  $$\mathbb{P}^1 \times * = (\mathbb{P}^1 \times *) \circ (\mathbb{P}^1 \times e) \in \mathrm{Corr}^{-1}_\sim(\mathbb{P}^1, *) \circ (\mathbb{P}^1 \times e).$$

  Moreover, in $\mathrm{Mot}_\sim(k)$ every object has a dual. We define the dual of $M = (X, p, m)$ as $D(M) = (X, {}^t p, d - m)$, where $d = \dim(X)$. One then has

  $$\begin{aligned} \mathrm{Hom}_{\mathrm{Mot}_\sim}(\mathbf{1}_\sim, M \otimes D(M)) &= p \times {}^t p \circ \mathrm{Corr}^d_\sim(\operatorname{Spec} k, X \times X) \circ \mathrm{id}_{\operatorname{Spec} k} \\ &= p \circ \mathrm{Corr}^d_\sim(X \times X) \circ p \\ &= \mathrm{Hom}_{\mathrm{Mot}_\sim}(M, M) \end{aligned}$$

  where we have used that for all $\Gamma \in \mathrm{Corr}^d_\sim(\operatorname{Spec} k \times X \times X)$

  $$(p \times {}^t p \circ \Gamma \circ \mathrm{id}_{\operatorname{Spec} k}) = (\mathrm{id} \times p \times {}^t p)_* \Gamma = (p \times {}^t p)_* \Gamma = p \circ \Gamma \circ p,$$

  applying Lieberman's lemma twice.

- One clearly has $(X, p, 0) \otimes \mathbf{T}^{\otimes m}_\sim = (X, p, m)$, and (X, p, m) is called the m-th Tate twist of $(X, p, 0)$.

Chow Groups and Cohomology Groups of Motives

We can define Chow groups and cohomology groups of certain motives.

Chow groups. — If we consider rational equivalence as our adequate equivalence relation of choice in the definition of $\mathrm{Mot}_\sim(k)$, we get the category of *Chow motives*, oftentimes denoted $\mathrm{CHM}(k)$, with a functor $\mathrm{ch} : \mathrm{SmProj}(k) \to \mathrm{CHM}(k)$ taking each variety to the corresponding effective Chow motive. For a Chow motive $M = (X, p, m)$, we define the i-th Chow group of the motive as

$$\mathrm{CH}^i(M) := \operatorname{im} p_* \subset \mathrm{CH}^{i+m}(X) \otimes \mathbb{Q}.$$

With this definition, we have

$$\mathrm{CH}(\mathrm{ch}(\mathbb{P}^1)) = \mathrm{CH}(\mathbf{1}) \oplus \mathrm{CH}(\mathbf{L})$$

and $\mathrm{CH}(\mathbf{1}) = \mathrm{CH}^0(\mathbb{P}^1)$, $\mathrm{CH}(\mathbf{L}) = \mathrm{CH}^1(\mathbb{P}^1)$, so the decomposition as a motive induces the decomposition of the Chow groups. This is a general pattern which will recur in the future.

In the category of Chow motives chow groups are now representable:

Proposition. — *In the category of Chow motives one has*

$$\mathrm{CH}^i(M) \cong \mathrm{Hom}_{\mathrm{CHM}(k)}(\mathbf{L}^{\otimes i}, M).$$

Proof. If $M = (X, p, m)$, then

$$\mathrm{Hom}_{\mathrm{CHM}(k)}(\mathbf{L}^{\otimes i}, M) = \{f = p \circ \Gamma \mid \Gamma \in \mathrm{CH}^{i+m}(X)\},$$

and by Lieberman's lemma $p \circ \Gamma = (\mathrm{id}_{\mathrm{Spec}\, k} \times p) * (\Gamma) = \mathrm{im}\, p_*$ in $CH^{i+m}(X)$. □

Of course everything that we said about Chow groups of motives holds for the various cycle groups $C_\sim(k)$ in the other categories of pure motives.

Cohomology groups. — If we choose any equivalence relation finer than homological equivalence, and in particular for Chow motives, we can define the cohomology groups of a motive $M = (X, p, m)$, with respect to a Weil cohomology theory H, as

$$H^i(M) := \mathrm{im}\, p_* \subset H^{i+2m}(X).$$

With this definition, considering for any smooth projective variety X of dimension d the decomposition $h_\sim(X) = h^0_\sim(X) \oplus h^+_\sim(X) \oplus h^{2d}_\sim(X)$ we get

$$H(h_\sim(X)) = H(h^0_\sim(X)) \oplus H(h^+_\sim(X)) \oplus H(h^{2d}_\sim(X)),$$

and $H(h^0_\sim(X)) = H^0(X)$, $H(h^{2d}_\sim(X)) = H^{2d}(X)$, thus the interesting cohomological information for the variety X is contained in the middle motive $h^+_\sim(X)$, as expected.

Relations Between Categories of Motives

The category of motives is a $\mathbb{Q}$-linear tensor category through which all Weil cohomology theory factor, but it is not abelian semisimple, thus falling short of the properties Grothendieck envisioned when for pure motives, while the category of numerical motives has been proven to be abelian semisimple (and assuming another Standard Conjecture, Tannakian, with a slight modification), but we cannot say that Weil cohomology theories factor through it unless the Standard Conjecture on the equivalence of numerical and homological equivalence holds. Moreover, we need the conjecture to speak of the category of $\mathrm{Mot}_{\mathrm{hom}}$, which otherwise depends a priori on the choice of the Weil cohomology theory.

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