Loose Ends from Tutorial 0

Simplicial groups are Kan complexes

In the tutorial I had to rush through the proof of this result, so as promised here's a write-up. The proof I have written here is adapted from [May], and it proceeds by describing an algorithm to construct the horn fillings. In the end it really is the same proof I already gave at the tutorial, just in a slightly different vest (I guess I changed my mind and now I think this more explicit version is overall nicer than the slightly slicker one I presented, if you liked the other version more just ask at the next tutorial and you shall receive a better explanation). A different proof, more geometric and using a bit more technology (not sure it's really worth it?) is given at page 14 of this document by Joyal and Tierney.

Proposition. — *Let X be a simplicial group. Then the underlying simplicial set of X is a Kan complex.*

Proof. We want to show that given any arbitrary n-horn in X, there exists a lift to a the n-simplex. Let $\theta_0, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_n$ be the faces of an arbitrary n-horn in X; we want to exhibit them as faces of a single n-simplex θ. First, we will construct a simplex η with $d_t \eta = \theta_t$ for all $t < k$.

If $k = 0$, there is nothing to do. Otherwise, we will inductively construct n-simplices $\eta_0, \ldots, \eta \coloneqq \eta_{k-1}$ such that $d_t \eta_m = \theta_t$ for all $t \leq m$. For $m = 0$ we can set $\eta_0 = s_0 \theta_0$. Then assuming we have already constructed η_{m-1}, we define $v_m \coloneqq s_m(\theta_m \cdot d_m \eta_{m-1}^{-1})$ and $\eta_m \coloneqq v_m \cdot \eta_{m-1}$. If now $t < m$, (using that the structure maps are group morphisms!)

$$
\begin{aligned}
d_t v_m &= d_t s_m(\theta_m \cdot d_m \eta_{m-1}^{-1}) \\
&= s_{m-1}(d_t \theta_m \cdot d_t d_m \eta_{m-1}^{-1}) \\
&= s_{m-1}(d_{m-1} \theta_t \cdot d_{m-1} d_t \eta_{m-1}^{-1}) \\
&= s_{m-1} d_{m-1}(\theta_t \cdot \theta_t^{-1}) = 1
\end{aligned}
$$

while $d_m v_m = \theta_m \cdot d_m \eta_{m-1}^{-1}$. Thus $d_t \eta_m = d_t \eta_{m-1} = \theta_t$ and

$$
d_m \eta_m = \theta_m \cdot d_m \eta_{m-1}^{-1} \cdot d_m \eta_{m-1} = \theta_m
$$

as desired.

To conclude, we will inductively construct $\tau_n, \ldots, \tau_k$ (note that we're moving downward this time!) such that $d_t \tau_m = \tau_t$ for all $t < k$ as well as all $t > m$; then $\theta := \tau_k$ will be the simplex filling the given horn. For $m = n$ the simplex η constructed above already does the job. If now τ_{m+1} has already been constructed, then we set $\zeta_m = s_m(\tau_{m+1} \cdot d_{m+1} \tau_{m+1}^{-1})$, $\tau_m = \zeta_m \cdot \tau_{m+1}$. A similar computation to the one above then shows that τ_m has the desired properties. $\qquad\square$

References

[May]   J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, 1967.