Lecture Notes for

Algebraic Topology II

Lecturer

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Some notes for the course *Algebraic Topology II* on the basics of stable homotopy theory given by Prof. Stefan Schwede at Bonn University during the Summer Semester 2021/22. Note that Professor Schwede himself has written lecture notes for this course (which appeared at some point during the semester) and also for the follow up course.

A good chunk of these notes (especially for the second half of the course) are shamelessly stolen from Qi Zhu, so credit to him (but as always, I might have introduced some errors, and the credit for those is mine). Perhaps it would make sense to read his notes instead (he also has notes for the follow up course, which I ditched).

Last update: A long time ago

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Some Motivation

Lecture 1
11^{th} Apr, 2022

The main object we want to study in this course are spectra. Spectra are "representing objects for (co)homology theories", a fact known as Brown's representability theorem. Another motivation for the introduction of spectra is given by the following well-known theorem.

Theorem (Freudenthal's suspension theorem)**.** — *Let X be an n-connected CW-complex. Then the suspension morphism*

$$\Sigma : \pi_k(X, *) \to \pi_{k+1}(\Sigma X, *), \ [S^k \xrightarrow{f} X]_* \mapsto [S^{k+1} \cong \Sigma S^k \xrightarrow{\Sigma f} \Sigma X]$$

is an isomorphism for $1 \leqslant k \leqslant 2n$ and an epimorphism for $k = 2n+1$.

We are not going to prove the theorem. It follows easily from the Blackers-Massey excision theorem (which we also will not see but which is a fairly elementary, if hard to prove, result), see [Hat02, Theorem 4.23, Corollary 4.24]; moreover, it can be generalized to any n-connected space. A remarkable consequence of the Freudenthal suspension theorem is the following: the homotopy groups of the iterated suspension $\Sigma^m X$ of a CW-complex (or space) X stabilize for large enough m (as the bound given in the theorem for the suspension morphism to be an isomorphism grows faster than the number of applications of the suspension functor).

A different proof is given in [Tom08] and in this document by Schwede.

Example. — In the case of spheres, we have

$$\underbrace{\pi_1(S^1, *)}_{\cong \mathbb{Z}} \xrightarrow{\Sigma} \underbrace{\pi_2(S^2, *)}_{\cong \mathbb{Z}} \xrightarrow{\cong} \pi_3(S^3, *) \xrightarrow{\cong} \cdots$$

and also

$$\underbrace{\pi_2(S^1, *)}_{\cong 0} \xrightarrow{\Sigma} \underbrace{\pi_3(S^2, *)}_{\cong \mathbb{Z}\{\eta\}} \xrightarrow{\Sigma} \underbrace{\pi_4(S^3, *)}_{\cong \mathbb{Z}/2\{\Sigma\eta\}} \xrightarrow{\cong} \pi_5(S^4, *) \xrightarrow{\cong} \cdots$$

at least if one is able to compute $\pi_4(S^3)$... we do this in AT2Sheet1.2.

Spectra are somehow the "universal home for stable phenomena". This can be restated in fancy terms: the ∞-category of spectra is the free presentable stable ∞-category on one object (the sphere spectrum).

Sadly, all we will be getting on the ∞-category side of this story will be cryptic remarks...

Sequential Spectra

We now give a first (rather quick and dirty) definition of spectra as sequential spectra (this will be later superseded by orthogonal spectra).

I.1. Definition. — A *(sequential) spectrum* X consists of:

- based spaces $(X_n, *)$ for $n \geqslant 0$,
- continuous based maps $\sigma_n : \Sigma X_n \to X_{n+1}$.

Given that the basepoint is part of the structure, we will usually omit it in $\pi_k(X_n, *)$.

A *morphism of spectra* $f : X \to Y$ consists of a sequence $f_n : X_n \to Y_n$ of based continuous maps such that the following square commutes:

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma_n^X"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma_n^Y"] & Y_{n+1} \end{tikzcd}$$

I.2. Remark. — This class will take place in the category of compactly generated spaces.

Let X be a topological space. A subset A of X is compactly closed if for any continuous map $f : K \to X$ from a compact space K, $f^{-1}(A)$ is closed. The space X is called a *k-space* (Kelley space) if every compactly closed subset is closed, it is called *weak Hausdorff* if for every compact space K and any continuous map $f : K \to X$, the image of f is closed in X. A *compactly generated*[1] space is a weak Hausdorff k-space.

Let $\mathcal{T}$ denote the full subcategory of Top consisting of compactly generated spaces. We collect some of the properties of $\mathcal{T}$.

- $\mathcal{T}$ is cartesian closed, complete and cocomplete.
- The inclusion of the subcategory of k-spaces in Top has a right adjoint, usually called "Kelleyfication". The inclusion of $\mathcal{T}$ in the subcategory of k-spaces has a left adjoint, called (sic) "weak Hausdorffification".

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook] & \text{k-spaces} \arrow[r, hook] \arrow[l, bend right, "\text{weak Hausdorffification}"'] & \text{Top} \arrow[l, bend left, "\text{Kelleyfication}"] \end{tikzcd}$$

  Given the situation, there is no hope for limits and colimits in $\mathcal{T}$ to be just limits and colimits taken in Top: one has to take the limit or colimit in Top and then apply one of the two functors going towards $\mathcal{T}$ (just one, as the other one will be the "good" adjoint). Even worse: in general the forgetful functor $\mathcal{T} \to \text{Set}$ *does not* preserve colimits.

  Implying that it does preserve limits, at least...

- Every closed subset of a compactly generated space is compactly generated.
- Every locally compact Hausdorff space, CW-complex, metric space, manifold is compactly generated.
- Any disjoint union of compactly generated spaces is compactly generated.
- The ordinary product of a compactly generated space and a locally compact Hausdorff space is compactly generated.

[1] In the literature our compactly generated spaces are sometimes called "compactly generated weak Hausdorff" (or CGWH) spaces. It is more convenient to include "weak Hausdorff" in "compactly generated" as compactly generated but not weak Hausdorff spaces are not as common or as useful (and in fact, the first definition of compactly generated spaces followed our convention).

The *k-th homotopy group*, for $k \in \mathbb{Z}$, of a spectrum X is

$$\pi_k(X) = \operatorname*{colim}_{n, n+k>0} \pi_{n+k}(X_n),$$

with colimit taken over the diagram

$$\pi_{n+k}(X_n) \xrightarrow{\Sigma} \pi_{1+n+k}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{1+n+k}(X_{1+n}).$$

It will become apparent in the future that here is a good reason for writing $1+n$ instead of $n+1$, i.e. keeping track of coordinates (or in other terms if we smash on the left or the right).

Eventually, all maps in the diagram are morphisms of abelian groups, so $\pi_k(X)$ inherits an abelian group structure. The construction of $\pi_k(X)$ is clearly functorial for morphisms of spectra, so we have defined functors $\pi_k : \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Ab}$ for $k \in \mathbb{Z}$.

A morphism $f : X \to Y$ of spectra is a *stable equivalence* if

$$\pi_k(f) : \pi_k(X) \to \pi_k(Y)$$

is an isomorphism for all $k \in \mathbb{Z}$. The *stable homotopy category* is the 1-categorical localization

$$\mathcal{SH} := \mathrm{Sp}^{\mathbb{N}}[\mathrm{steq}^{-1}],$$

the ∞-*category of spectra* is the ∞-categorical localization

$$\mathrm{Sp}_\infty := \mathrm{Sp}^{\mathbb{N}}[\mathrm{steq}^{-1}]_\infty.$$

The rest of this lecture and most of the following will be occupied with remarkable examples of spectra, often citing results without proof, mainly for motivation.

Remarkable Classes of Spectra

Suspension Spectra

Let K be a based space. The *suspension spectra* $\Sigma^\infty K$ is the spectrum with $(\Sigma^\infty K)_n = S^n \wedge K$, where the *smash product* $X \wedge Y$ is defined as

$$X \wedge Y := \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

To define the structure maps it is convenient to consider $S^n = \mathbb{R}^n \cup \{\infty\}$ with the one-point compactification topology, and we will do so throughout the course. In particular we have

$$S^m \wedge S^n \cong S^{m+n}$$

with homeomorphism given by the obvious map

$$\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n},\ ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Note that we also have

$$\Sigma X = \frac{X \times [0,1]}{X \times \{0,1\} \cup \{x_0\} \times [0,1]} \cong \frac{X \times S^1}{X \times \{z_0\} \cup \{x_0\} \times S^1} = X \wedge S^1.$$

The smash product is associative since we are working with compactly generated spaces (in general, i.e. in Top, it is not). This is because in cartesian closed categories finite products commute with (finite) colimits.

Then the structure maps of $\Sigma^\infty K$ are

$$\sigma_n : \Sigma(\Sigma^\infty K)_n = S^1 \wedge (S^n \wedge K) \cong (S^1 \wedge S^n) \wedge K \cong S^{1+n} \wedge K.$$

We call the k-th homotopy group of $\Sigma^\infty K$,

$$\pi_k(\Sigma^\infty K) = \operatorname*{colim}_n \pi_{n+k}(S^n \wedge K),$$

the *k-th stable homotopy group* of K. If the base point is non-degenerate, then $S^n \wedge K$ is $(n-1)$-connected, hence for $k < 0$ we have

$$\pi_k(\Sigma^\infty K) = 0,$$

which we express by saying that the $\Sigma^\infty K$ is *connective*, and

$$\pi_0(\Sigma^\infty K) \cong \tilde{H}_0(K;\mathbb{Z}) = \mathbb{Z}[\pi_0(K)].$$

As a special (very important) case, we have the *sphere spectrum*

This is the "real" initial ring, the brave new integers.

$$\mathbb{S} := \Sigma^\infty S^0 = \{S^n, \sigma_n : S^1 \wedge S^n \xrightarrow{\cong} S^{1+n}\}.$$

The group $\pi_k(\mathbb{S}) = \operatorname{colim}_n \pi_{n+k}(S^n)$ is called the *k-th stable homotopy group of spheres* or the *k-th stable stem*. We have that $\pi_k(\mathbb{S}) = 0$ for $k < 0$ and $\pi_0(\mathbb{S}) \cong \mathbb{Z}$. Moreover we have the following results (which we will not prove).

Theorem (Serre)**.** — *For $k, n \geqslant 1$, the group $\pi_k(S^n, *)$ is finite, unless $k = n$ or n is even and $k = 2n-1$.*

Corollary. — *$\pi_k(\mathbb{S})$ is finite for $k > 0$.*

We list here the first stable stems (up until the first non-cyclic one), in terms of the first, second and third Hopf maps (whose definition is recalled below).

k	0	1	2	3	4	5	6	7	8
$\pi_k(\mathbb{S})$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
Generators	id_{S^1}	η	η^2	ν			ν^2	σ	$\eta\sigma, \varepsilon$

- $\eta : S(\mathbb{C}^2) \cong S^3 \to \mathbb{C}P^1 \cong S^2,\ (x,y) \mapsto [(x,y)]$,
- $\nu : S(\mathbb{H}^2) \cong S^7 \to \mathbb{H}P^1 \cong S^4,\ (x,y) \mapsto [(x,y)]$,
- $\sigma : S(\mathbb{O}^2) \cong S^{15} \to \mathbb{O} \cup \{\infty\} \cong S^8,\ (x,y) \mapsto xy^{-1}$.

One has to be a bit careful when dealing with the octonions, as they are not associative.

Eilenberg-Maclane Spectra

Lecture 2
13th Apr, 2022

Let A be an abelian group. Choose EM-spaces $(K(A,n), \varphi_n)$ of type (A,n) for all $n \geqslant 0$. Then $\Omega K(A,n+1)$ is an EM-space of type (A,n), so (by [AT1, Theorem IV.10]) there is a based continuous map $\tilde{\sigma}_n : K(A,n) \to \Omega K(A,n+1)$, unique up to homotopy, such that the following triangle of abelian groups commute:

$\Omega Y = \mathrm{map}_*(S^1, Y)$

$$\begin{tikzcd} \pi_n(K(A,n),*) \arrow[rr, "(\tilde{\sigma}_n)_*"] \arrow[dr, "\varphi_n"'] & & \pi_n(\Omega K(A,n+1),*) \arrow[dl, "\varphi_{n+1}"] \\ & A & \end{tikzcd}$$

Moreover, $\tilde{\sigma}_n$ is a weak homotopy equivalence. The *Eilenberg-MacLane spectrum HA* is then given by

$$(HA)_n = K(A,n),\ \sigma_n : S^1 \wedge K(A,n) \to K(A,n+1),$$

where σ_n is the adjoint of $\tilde{\sigma}_n$.

Remark. — It will (probably) be an exercise at some point that any two sets of choices in the definition of HA give stably equivalent spectra.

We have

$$\pi_k(HA) = \operatorname*{colim}_n \pi_{n+k}(K(A,n)) \cong \begin{cases} 0 & k \neq 0 \\ A & k = 0 \end{cases}$$

Moreover, there is a bijection (which hopefully we will prove later)

$$\mathcal{SH}(HA, HB) \xrightarrow{\pi_0} \mathrm{Hom}_{\mathrm{Ab}}(A, B),$$

which is a glorified version of [AT1, theorem IV.10].

Bordism Spectra/Thom Spectra

Let Gr_n be the Grassmannian of n-planes in $\mathbb{R}^\infty$

$$Gr_n = \bigcup_{k \geqslant 0} Gr_n(\mathbb{R}^k)$$

equipped with weak topology. On Gr_n sits a "universal" n-plane bundle γ_n with total space

$$E_n = \{(x, L) \in \mathbb{R}^\infty \times Gr_n : x \in L\},$$

i.e. $\gamma_n : E_n \to Gr_n$, $(x, L) \mapsto L$.

I.3. Remark (Universality of γ_n)**.** — For any paracompact space X, the map

$$[X, Gr_n] \to \mathrm{Vect}_n(X), \; [f] \mapsto [f^*(\gamma_n)]$$

is bijective, where Vect_n is the set of isomorphism classes of rank n vector bundles on X.

This is the usual universal G-bundle story, thinly disguised by the equivalence between principal GL_n-bundles and n-dimensional vector bundles. In other words, Gr_n is a classifying space for GL_n.

The *Thom Space* of γ_n is

$$(MO)_n = \frac{D(\gamma_n)}{S(\gamma_n)} = \frac{\text{disc bundle}}{\text{sphere bundle}}$$

with

$$D(\gamma_n) = \{(x, L) \mid |x| \leqslant 1\}, \; S(\gamma_n) = \{(x, L) \mid |x| = 1\}.$$

The $(n+1)$-plane bundle $\gamma_n \oplus \underline{\mathbb{R}}$ is classified (via Remark I.3) by a map $c : Gr_n \to Gr_{n+1}$, unique up to homotopy, such that

$$c^*(\gamma_{n+1}) \cong \gamma_n \oplus \underline{\mathbb{R}}$$

and this isomorphism gives a commutative square

$$\begin{array}{ccc} \mathbb{R} \times E_n & \xrightarrow{\bar{c}} & E_{n+1} \\ \big\downarrow {\scriptstyle \underline{\mathbb{R}} \oplus \gamma_n} & & \big\downarrow {\scriptstyle \gamma_{n+1}} \\ Gr_n & \xrightarrow{c} & Gr_{n+1} \end{array}$$

which defines a map of Thom spaces. The structure maps of a spectrum MO are then

$$S^1 \wedge MO_n = S^1 \wedge \frac{D(\gamma_n)}{S(\gamma_n)} \cong \frac{D(\underline{\mathbb{R}} \oplus \gamma_n)}{S(\underline{\mathbb{R}} \oplus \gamma_n)} \xrightarrow{\bar{c}} \frac{D(\gamma_{n+1})}{S(\gamma_{n+1})} = MO_{n+1}.$$

Theorem (Thom)**.** — *The Thom-Pontryagin construction induces an isomorphism of graded rings between Ω_*, the ring of bordism classes of smooth closed manifolds (or Ω_k, the abelian group of bordism classes of smooth closed k-manifolds) and $\pi_*(MO)$.*

Topological K-theory Spectra

Let $U = \operatorname{colim}_{n \geqslant 0} U(n)$ be the infinite unitary group, along

$$U(n) \to U(n+1),\ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Bott periodicity provides a homotopy equivalence $U \xrightarrow{\simeq} \Omega^2 U$.

The *spectra of topological K-theory KU* is defined by

$$(KU)_n = \begin{cases} U & \text{for } n \text{ odd} \\ \Omega U & \text{for } n \text{ even} \end{cases}$$

with structure maps

$$\begin{cases} \Sigma\Omega U \to U \text{ adjoint to } \mathrm{id}_{\Omega U} & \text{for } n \text{ even} \\ \Sigma U \to \Omega U \text{ adjoint to the equivalence } U \xrightarrow{\simeq} \Omega^2 U & \text{for } n \text{ odd} \end{cases}$$

$U(n)$ is path-connected (as one can easily see considering diagonalized) and one can show that $\det_* : \pi_1(U(n), *) \to \pi_1(U(1) \simeq S^1, *) \cong \mathbb{Z}$ is an isomorphism. Moreover, the inclusion $U(n) \to U(n+1)$ induces isomorphisms on π_0 and π_1 for all $n \geqslant 1$. Then $\pi_0(U) \cong 0$ and $\pi_1(U) \cong \mathbb{Z}$ implies

$$\pi_k(KU) \cong \begin{cases} \mathbb{Z} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

through Bott periodicity.

Orthogonal Spectra

The "motivation" part is over.

Sequential spectra are fine for many purposes, such as introducing the the stable homotopy category as a triangulated category or even the ∞-category of spectra. But sequential spectra are not ideal for multiplicative properties (ring spectra, smash product) and equivariant generalizations. Other possible theories are symmetric spectra and unitary spectra, but it is the opinion of the Professor that orthogonal spectra are the "lesser evil".

I.4. Definition. — A *(coordinatized) orthogonal spectra* consists of

- a based space X_n for $n \geqslant 0$,
- a continuous and based action of $O(n)$ on X_n for all $n \geqslant 0$,
- structure maps $\sigma_n : S^1 \wedge X_n \to X_{1+n}$ such that the iterated structure map

  $$\sigma_n^m : S^m \wedge X_n \to X_{m+n}$$

  defined as the composite

  $$S^m \wedge X_n \cong S^{m-1} \wedge S^1 \wedge X_n \xrightarrow{S^{m-1} \wedge \sigma_n} S^{m-1} \wedge X_{1+n} \to S^1 \wedge X_{m+n-1} \xrightarrow{\sigma_{m+n-1}} X_{m+n}$$

  is $O(m) \times O(n)$-equivariant.

As $O(n)$ is compact, the product is the usual, so there is no need to worry about strange topologies.

The action on S^m is the one-point compactification of the tautological action on $\mathbb{R}^m$. The action on X_{m+n} via the block sum embedding

One could spend some more words on how this all works, but eh...

$$O(m) \times O(n) \to O(m+n),\ (A, B) \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

A *morphism of orthogonal spectra* $f : X \to Y$ consists of $O(n)$-equivariant based continuous maps $f_n : X_n \to Y_n$ such that the square

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "\sigma_n^X"] \arrow[d, "S^1 \wedge f_n"] & X_{1+n} \arrow[d, "f_{1+n}"] \\ S^1 \wedge Y_n \arrow[r, "\sigma_n^Y"] & Y_{1+n} \end{tikzcd}$$

commutes. Hence we get a category Sp of orthogonal spectra, with a forgetful functor

$$U : \mathrm{Sp} \to \mathrm{Sp}^{\mathbb{N}}$$

defined by forgetting the action of the orthogonal groups. Homotopy groups and stable equivalences of orthogonal spectra are defined after forgetting to sequential spectra. The stable homotopy category could equivalently be defined as $\mathcal{SH} = \mathrm{Sp}[\mathrm{steq}^{-1}]$.

Suspension spectra, revisited. — The suspension spectra $\Sigma^\infty K$ of a based space K is

$$(\Sigma^\infty K)_n = S^n \wedge K$$

with tautological $O(n)$-action of S^n and structure maps $\sigma_n : S^1 \wedge S^n \wedge K \xrightarrow{\mathrm{can} \wedge K} S^{1+n} \wedge K$.

Equivariance of the structure maps boils down to the map

$$\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$$

$$((x_1, \dots, x_m), (y_1, \dots, y_n)) \mapsto (x_1, \dots, x_m, y_1, \dots, y_n)$$

being $O(m) \times O(n)$-equivariant by construction.

Eilenberg-MacLane spectra, revisited. — Let K be a based space. Let $A[K]$ denote the *reduced A-linearisation of* K, $A\{K\}/A\{*\}$. We endow $A[K]$ with the quotient topology induced by the surjective map

$$\coprod_{n \geqslant 0} A^n \times K^n \to A[K]$$

$$(a_1, \dots, a_n, k_1, \dots, k_n) \mapsto a_1 k_1 + \dots + a_n k_n.$$

Theorem (Dold-Thom)**.** — *For all $n \geqslant 0$, the space $A[S^n]$ is a $K(A, n)$. More generally, for every based CW-complex K, there is a natural isomorphism*

$$\pi_k(A[K], 0) \cong \tilde{H}_k(K; A)$$

This extends to a continuous endofunctor of based spaces $A[-] : T_* \to T_*$. The $O(n)$-action on S^n induces a continuous $O(n)$-action on $A[S^n]$. The orthogonal spectrum HA is then defined by

Here continuous means functor of $\mathcal{T}$-enriched categories (not limit preserving).

$$(HA)_n = A[S^n]$$

with (continuous and equivariant!) structure maps

$$S^1 \wedge HA_n = S^1 \wedge A[S^n] \to A[S^{1+n}] = HA_{1+n}$$

$$x \wedge \sum a_i y_i \mapsto \sum a_i (x \wedge y_i).$$

Orthogonal Spectra, Coordinate-free

Lecture 3
20^{th} Apr, 2022

For the sake of (?), we want now to go coordinate-free and define orthogonal spectra without referencing $\mathbb{R}_n$, replacing it everywhere with an inner product space V.

AT2Sheet2.2 shows that there is an equivalence of categories between orthogonal spectra and their coordinatized version, so it is not entirely obvious to me why this definition is better... I guess we get a nicer theory by seeing spectra as functors?

Definition. — An *inner product space* is just a finite-dimensional Euclidean vector space.

The idea is to make the following changes to the definition of orthogonal spectra:

- X_n becomes $X(V)$ for an inner product space V,
- the action of $O(n)$ on X_n is replaced by the action of $O(V)$ on $X(V)$,
- the structure maps $\sigma^m : S^m \wedge X_n \to X_{m+n}$ become maps $S^V \wedge X(W) \to X(V \oplus W)$, where $S^V = V \cup \{\infty\}$ with the one-point compactification topology.

More generally, we want structure maps for all linear isometric embeddings $\varphi : V \hookrightarrow W$, given as maps

$$\varphi_* : S^{W-\varphi(V)} \wedge X(V) \to X(W).$$

Now we carry out the actual construction. Given inner product spaces V, W, let $L(V, W)$ denote the space of linear isometric embeddings. If $(v_1, \dots, v_n)$ is an orthonormal basis of V, then the evaluation

$$L(V, W) \to V_n(W) \subset W^n,\ (\varphi : V \to W) \mapsto (\varphi(v_1), \dots, \varphi(v_n))$$

is a bijection. We topologize $L(V, W)$ by requiring this map to be an homeomorphism. During some idle afternoon, one should check that the topology does not depend on the choice of the basis. For this topology, we have that the composition

$$L(V, W) \times L(U, V) \to L(U, W),\ (\varphi, \psi) \mapsto \varphi \circ \psi$$

is continuous. Hence we get a $\mathcal{T}$-enriched category.

The *orthogonal complement bundle* $\xi(V, W)$ over $L(V, W)$ is

$$\xi(V, W) = \{(w, \varphi) \in W \times L(V, W) \mid w \perp \varphi(W)\} \to L(V, W),\ (w, \varphi) \mapsto \varphi.$$

During the same idle afternoon, one should check that this is an Euclidean vector bundle of rank $\dim W - \dim V$.

We define

$$\mathbf{O}(V, W) = \operatorname{Thom}(\xi(V, W)) = \xi(V, W) \cup \{\infty\}$$

with the one-point compactification topology. Note that the base space $L(V, W)$ is compact, which is why the Thom space coincides with the one-point compactification.

The map

$$\xi(V, W) \times \xi(U, V) \to \xi(U, W),\ ((w, \varphi), (v, \psi)) \mapsto (w + \varphi(v), \varphi \circ \psi)$$

cover the composition map $L(V, W) \times L(U, V) \to L(U, W)$. If one wants his idle afternoon to turn into a non-idle one, they should check that the map we defined is continuous. This extends continuously (one checks) to based maps

What should go wrong anyway?

$$\mathbf{O}(V, W) \wedge \mathbf{O}(U, V) \to \mathbf{O}(U, W)$$

hence one gets a based topological category $\mathbf{O}$, the category of inner product spaces. The identity of V in this category is $\mathrm{id}_V = (0, \mathrm{id}_V) \in \mathbf{O}(V, V)$. Note also that

$$\mathbf{O}(V,V) = \xi(V,V) \cup \{\infty\} = \{(0,A) \mid A \in O(V)\} \cup \{\infty\} = O(V)_+$$

using that $O(V)$ is already compact.

This remark (although sort of intuitive already) is explained better on the official notes

Remark. — If $\dim V = m$ and $\dim W = n$, then

$$L(V,W) \cong V_m(\mathbb{R}^n) \cong O(n)/O(n-m).$$

This is a remark from the future (fourth lecture).

Moreover, we obtain

$$\xi(V,W) \cong V_m(\mathbb{R}^n) \times_{O(n-m)} \mathbb{R}^{n-m}$$

and

$$\mathbf{O}(V,W) \cong V_m(\mathbb{R}^n)_+ \wedge_{O(n-m)} S^{n-m} = O(n)_+ \wedge_{O(n-m)} S^{n-m}.$$

I.5. Definition. — A *(coordinate-free) orthogonal spectra* is a continuous ($\mathcal{T}_*$-enriched) based functor $X : \mathbf{O} \to \mathcal{T}_*$.

Explicitly:

- for all inner product spaces V we get a based space $X(V)$,
- we have based continuous maps

  By adjunction.

  $$\mathbf{O}(V,W) \wedge X(V) \to X(W)$$

  for all $V, W \in \mathbf{O}$,
- we have commutative diagrams

  $$\begin{tikzcd} \mathbf{O}(V,W) \wedge \mathbf{O}(U,V) \wedge X(U) \arrow[r, "\mathbf{O}(V,W)\wedge X"] \arrow[d, "\circ \wedge X(U)"] & \mathbf{O}(V,W) \wedge X(V) \arrow[d, "X"] \\ \mathbf{O}(U,W) \wedge X(U) \arrow[r, "X"] & X(W) \end{tikzcd}$$

  for all $U, V, W \in \mathbf{O}$.

Then, $O(V)$ acts on $X(V)$ by the composite

$$\begin{aligned} O(V) \times X(V) \to \mathbf{O}(V,V) \wedge X(V) &\xrightarrow{X} X(V) \\ (A, x) \mapsto ((0,A), x) & \end{aligned}$$

Given two inner product spaces V, W, we write

$$i_V : S^V \to \mathbf{O}(W, V \oplus W), \;\; v \mapsto ((v,0),(0,\mathrm{id}_W)).$$

For every orthogonal spectrum X, the structure map $\sigma_{V,W} : S^V \wedge X(W) \to X(V \oplus W)$ is the composite

$$S^V \wedge X(W) \xrightarrow{i_V \wedge \mathrm{id}_{X(W)}} \mathbf{O}(W, V \oplus W) \wedge X(W) \xrightarrow{X} X(V \oplus W).$$

The *opposite structure map* $\sigma^{\mathrm{op}}_{V,W} : X(V) \wedge S^W \to X(V \oplus W)$ is the composite

$$X(V) \wedge S^W \cong S^W \wedge X(V) \xrightarrow{\sigma_{W,V}} X(W \oplus V) \xrightarrow{X(\tau_{V,W})} X(V \oplus W)$$

where $\tau_{V,W} : V \oplus W \to W \oplus V$ is the obvious isomorphism.

The forgetful functor

$$U : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{coord}}$$

is defined on objects by $(UX)_n = X(\mathbb{R}^n)$ (with an $O(n)$-action) and $\sigma_n : S^1 \wedge X_n \to X_{1+n}$ given by

$$S^{\mathbb{R}} \wedge X(\mathbb{R}^n) \to X(\mathbb{R} \oplus \mathbb{R}^n) \cong X(\mathbb{R}^{1+n}).$$

I.6. Theorem (AT2sheet2.2). — *The forgetful functor* $\mathrm{Sp} \to \mathrm{Sp}^{\mathrm{coord}}$ *is an equivalence of categories.*

We will often specify a functor $X : \mathbf{O} \to \mathcal{T}_*$ by specifying $X(V)$ with $O(V)$ actions along with $\sigma_{V,W} : S^V \wedge X(W) \to X(V \oplus W)$ and leave the rest implicit.

Suspension Spectra, Re-revisited. — The suspension spectrum $\Sigma^\infty K$ of a based space

Check stuff!

K has values $(\Sigma^\infty)(V) = S^V \wedge K$ with functoriality

$$\mathbf{O}(V, W) \wedge S^V \wedge K \to S^W \wedge K, \; (w, \varphi) \wedge v \wedge k \mapsto (w + \varphi(v)) \wedge k.$$

Action and structure maps are given by:

- $O(V) \times S^V \wedge K \to S^V \wedge K, \; A(v \wedge k) = A(v) \wedge k,$
- $\sigma_{V,W} : S^V \wedge X(W) \to X(V \oplus W), \; \sigma_{V,W}(v \wedge w \wedge k) = ((v, 0) + (0, w)) \wedge k = (v, w) \wedge k$

similarly to the coordinatized version of suspension spectra.

Eilenberg-MacLane spectra, re-revisited. — Let A be an abelian group. The Eilenberg-MacLane spectrum HA has values $(HA)(V) = A[S^V]$ with functoriality

$$\mathbf{O}(V, W) \wedge A[S^v] \to A[S^W], \; (w, \varphi) \wedge \sum_i a_i v_i \mapsto \sum_i a_i(w + \varphi(v_i)).$$

Thom spectra, re-revisited. — For an inner product space V, let $Gr_{|V|}(V^\infty)$ denote the Grassmannian of $\dim(V)$-planes (of $V^\infty = \bigoplus_{n \geqslant 0} V$) with the weak topology. For $V \neq 0$ we have $V^\infty \cong \mathbb{R}^\infty$. The space $Gr_{|V|}(V^\infty)$ comes with a continuous $O(V)$-action from the coordinatized action on V^∞, i.e. by $A(v_0, v_1, \dots) = (Av_0, Av_1, dots)$. Let

$$\gamma_V : L(V, V^\infty) \to Gr_{|V|}(V^\infty), \; \varphi \mapsto \varphi(V)$$

denote the tautological principal $O(\dim V)$-bundle where we again give the weak topology to

check stuff!

$L(V, V^\infty)$. We get $L(V, V^\infty) \times_{O(V)} V \to Gr_{|V|}(V^\infty)$ which is the tautological vector bundle. Then we define

$$\begin{aligned}(MO)(V) &= L(V, V^\infty) = L(V, V^\infty)_+ \wedge_{O(V)} S^V \\ &= (L(V, V^\infty)_+ \wedge S^V)/((\varphi A, v) \sim (\varphi, Av) \iff \varphi : V \hookrightarrow V^\infty, \; A \in O(V), \; v \in S^V)\end{aligned}$$

Quotients by compact topological groups do not pose topology issues, so this has the usual quotient topology.

We can see that $MO(V)$ is isomorphic to MO_n:

$$\begin{aligned}
L(V,V^\infty)_+ \wedge_{O(V)} S^V &\cong L(V,V^\infty)_+ \wedge_{O(V)} (D(V)/S(V)) \\
&= \left(\frac{L(V,V^\infty)\times D(V)}{L(V,V^\infty)\times S(V)}\right)/\sim \\
&= \frac{L(V,V^\infty)\times_{O(V)} D(V)}{L(V,V^\infty)\times_{O(V)} S(V)} \\
&= \frac{D(L(V,V^\infty)\times_{O(V)} V)}{S(L(V,V^\infty)\times_{O(V)} V)} \cong D(\gamma_n/S(\gamma_n)).
\end{aligned}$$

Here, the action of $O(V)$ on $MO(V)$ is defined through the coordinate-wise action on V^∞ given by $A[\varphi,v]=[A^\infty\circ\varphi,v]$. We get structure maps

$$\sigma_{V,W}: S^V\wedge MO(W)\to MO(V\oplus W),\ v\wedge[\varphi:W\hookrightarrow W^\infty,w]\mapsto[i_0\oplus\varphi,(v,w)]$$

where we define $i_+\oplus\varphi$ as the composite

$$V\oplus W\to V^\infty\oplus W^\infty\to(V\oplus W)^\infty$$

with

$$\begin{aligned}
V\oplus W\to V^\infty\oplus W^\infty,&\ (v,w)\mapsto((v,0,0,\dots),w)\\
V^\infty\oplus W^\infty\to(V\oplus W)^\infty,&\ ((v_0,v_1,\dots),(w_0,w_1,\dots))\mapsto((v_0,w_0),(v_1,w_1),\dots).
\end{aligned}$$

Constructions in the Category of Spectra

Limits and colimits in the category of spectra. — The category of orthogonal spectra has limits and colimits and they are constructed objectwise in $\mathcal{T}_*$.

Let J be a small category and $F: J\to \mathrm{Sp}$ a functor. We define the colimit in Sp of F by

$$(\operatorname*{colim}_J F)(V)=\operatorname*{colim}_J F(j)(V)=\operatorname*{colim}_J(\mathrm{ev}_V\circ F)$$

where $\mathrm{ev}_V:\mathrm{Sp}\to\mathcal{T}_*,\ X\mapsto X(V)$. This inherits structure maps as follows:

$$\begin{tikzcd}
S^V\wedge(\operatorname{colim}_J F(W)) \arrow[r, dashed, "\sigma^J_{V,W}"] & \operatorname{colim}_J F(V\oplus W) \\
\operatorname{colim}_J(S^V\wedge F(W)) \arrow[u, "\cong"] \arrow[ur, "\operatorname{colim}_{j\in J}\sigma^{F(j)}_{V,W}"'] &
\end{tikzcd}$$

A limit of $F: J\to\mathrm{Sp}$ can be constructed by

$$(\lim_J F)(V)=\lim_J F(j)(V)=\lim_J(\mathrm{ev}_V\circ F)$$

with structure maps $\sigma^J_{V,W}: S^V\wedge\lim_J F(W)\to\lim_J F(V\oplus W)$ adjoint to

$$\begin{tikzcd}
\lim_J F(W) \arrow[r, dashed, "\tilde\sigma^J_{V,W}"] \arrow[dr, "\lim_{j\in J}\tilde\sigma^{F(j)}_{V,W}"'] & \operatorname{map}_*(S^V,\lim_J F(V\oplus W)) \arrow[d, "\cong"] \\
& \lim_J \operatorname{map}_*(S^V, F(V\oplus W))
\end{tikzcd}$$

where for an orthogonal spectrum X, the map $\tilde\sigma_{V,W}: X(W)\to\operatorname{map}_*(S^V,X(V\oplus W))$ is the adjoint of the structure map $\sigma_{V,W}: S^V\wedge X(W)\to X(V\oplus W)$.

Suspension and Loop

Lecture 4
25th Apr, 2022

Let $F : \mathcal{T}_* \to \mathcal{T}_*$ be a continuous based functor. For every orthogonal spectrum X, the composite

$$\mathcal{O} \xrightarrow{X} \mathcal{T}_* \xrightarrow{F} \mathcal{T}_*$$

is another orthogonal spectrum. The same can be done on morphisms (which are natural transformations), so we obtain a functor $F \circ - : \mathrm{Sp} \to \mathrm{Sp}$.

Given a based space A, smashing with $- \wedge A$ and maps from $\mathrm{map}_*(A, -)$ are an adjoint functor pair

$$\mathcal{T}_* \underset{\mathrm{map}_*(A,-)}{\overset{-\wedge A}{\rightleftarrows}} \mathcal{T}_*$$

which extend to an adjoint functor pair

$$\mathrm{Sp} \underset{\mathrm{map}_*(A,-)}{\overset{-\wedge A}{\rightleftarrows}} \mathrm{Sp}.$$

To check that these are indeed adjoint functors, notice that we can define unit and counit levelwise using the unit and counit of the original adjunction.

More concretely, this means that for an orthogonal spectrum X we get an orthogonal spectrum

$$(X \wedge A)(V) = X(V) \wedge A$$

with structure maps

$$S^V \wedge (X \wedge A)(W) = S^V \wedge X(W) \wedge A \xrightarrow{\sigma^X_{V,W} \wedge A} X(V \oplus W) \wedge A = (X \wedge A)(V \oplus W)$$

We will be sporty with parentheses (because thanks to the symmetric monoidal structure on the category $\mathcal{T}_*$ we can pretend we have strict associativity).

and an orthogonal spectrum

$$\mathrm{map}_*(A, X)(V) = \mathrm{map}_*(A, X(V))$$

with structure maps

$$\begin{aligned} S^V \wedge \mathrm{map}_*(A, X(W)) &\to \mathrm{map}_*(A, S^V \wedge X(W)) \xrightarrow{\mathrm{map}_*(A, \sigma^X_{V,W})} \mathrm{map}_*(A, X(V \oplus W)) \\ v \wedge f &\mapsto \{a \mapsto v \wedge f(a)\} \end{aligned}$$

Now we will study the special case $A = S^1$ in more detail, a new version of the good old loop-suspension adjunction

$$\mathrm{Sp} \underset{\mathrm{map}_*(S^1,-)}{\overset{-\wedge S^1}{\rightleftarrows}} \mathrm{Sp}.$$

Unstably, we have that in some sense suspension is good with (co-)homology and loop is good with homotopy groups. We will see that stably they behave the same.

We will see that stably (i.e. for spectra), loop and suspension are “inverse up to homotopy”.

Note that this part of the story works in general for sequential spectra. So let X be a sequential spectrum. The adjunction isomorphism is

$$\begin{aligned} \alpha : \pi_{n+k}(\Omega X_n) = \pi_{n+k}(\mathrm{map}_*(S^1, X_n)) &\cong \pi_{n+k+1}(X_n) \\ [f : S^{n+k} \to \mathrm{map}_*(S^1, X_n)] &\mapsto [\hat{f} : S^{n+k+1} \to X_n] \end{aligned}$$

where $\hat{f}(x \wedge t) = f(x)(t)$ for $x \in S^{n+k},\ t \in S^1$. It is easy to check that these bijections are compatible with stabilization. Moreover, for large enough $n+k$ they are group isomorphisms. Hence in the colimit over n, these bijections induce an isomorphism of groups

$$\alpha : \pi_k(\Omega X) \xrightarrow{\cong} \pi_{k+1}(X).$$

Checking compatibility (in this and the next paragraph) is just a matter of unravelling definitions and carefully keeping track of the side we suspend on.

The maps

$$- \wedge S^1 : \pi_{n+k}(X_n) \to \pi_{n+k+1}(X_n \wedge S^1)$$
$$[f : S^{n+k} \to X_n] \mapsto [f \wedge S^1 : S^{n+k+1} \to X_n \wedge S^1]$$

Note that we smash with S^1 *on the left.* This is often crucial and it is useful to keep track of it by writing the $+1$ on the right.

are also compatible with stabilization and they induce a morphism

$$- \wedge S^1 : \pi_k(X) \to \pi_{k+1}(X \wedge S^1),$$

the suspension morphism.

The next theorem is a fundamental one, possibly the "real" beginning of stable homotopy theory: we will show that we are "inverting" the suspension functor, i.e. making it into an equivalence of categories (contrast this with spaces, where for example there are no nontrivial maps $S^1 \to S^0$, but suspending twice leads us to consider maps $S^3 \to S^2$, of which the Hopf fibration is a nontrivial example).

I.7. Theorem. — *Let X be a sequential spectrum.*

i) The loop and suspension morphisms

$$\alpha : \pi_k(\Omega X) \to \pi_{k+1}(X) \quad \textit{and} \quad - \wedge S^1 : \pi_k(X) \to \pi_{k+1}(X \wedge S^1)$$

are isomorphisms.

ii) The unit $\eta : X \to \Omega(X \wedge S^1)$ and counit $\varepsilon : (\Omega X) \wedge S^1 \to X$ of the adjunction are stable equivalences.

iii) For all $m \geqslant 1$, the morphism of sequential spectra $X \wedge S^m \to X \wedge S^m$ induced by the $O(m)$-action on S^m induces multiplication by the determinant on all homotopy groups.

Proof. (i) For α the statement is immediate (as it is induced by bijections), so we just need to prove it for $- \wedge S^1$. To show injectivity, let $f : S^{n+k} \to X_n$ represent a class in $\pi_k(X)$ in the kernel of the suspension morphism. By increasing n if necessary, we can assume without loss of generality that $f \wedge S^1 : S^{n+k+1} \to X_n \wedge S^1$ is null-homotopic. Then considering

The crux of many of these first arguments is seeing in what way $S^1 \wedge -$ and $- \wedge S^1$ are compatible.

$$\begin{tikzcd}
S^{n+k+1} \arrow[r, "f \wedge S^1"] \arrow[d, "\chi_{n+k,1}"', "\cong"] & X_n \wedge S^1 \arrow[r, "\tau", "\cong"'] & S^1 \wedge X_n \arrow[r, "\sigma_n"] & X_{1+n} \\
S^{1+n+k} \arrow[urr, bend right, "S^1 \wedge f"] \arrow[urrr, bend right, "\text{stabilization of } f"'] & & &
\end{tikzcd}$$

where $\chi_{n+k,1}$ is the homeomorphism which swaps the last coordinate with the first $n+k$ (and thus has degree $(-1)^{n+k}$), we see that the stabilization of f, i.e. the composite $\sigma_n \circ (S^1 \wedge f)$, is also null-homotopic, hence f represents the trivial element in $\pi_k(X)$.

We now show surjectivity. Let $g : S^{n+k+1} \to X_n \wedge S^1$ be any map, representing a generic class in $\pi_{k+1}(X \wedge S^1)$. We define $f := \sigma_n \circ \tau \circ g$.

$$\begin{tikzcd}
S^{n+k+1} \arrow[r, "f"] \arrow[d, "g"'] & X_{1+n} \\
X_n \wedge S^1 \arrow[r, "\tau", "\cong"'] & S^1 \wedge X_n \arrow[u, "\sigma_n"']
\end{tikzcd}$$

Then $[f] \in \pi_k(X)$ and we claim that $[f] \wedge S^1 = (-1)^{k+n}[g]$. We consider the following diagram

This might look obvious at first (and it sort of is) but formally one has to be careful...

$$\begin{tikzcd}
S^{1+n+k+1} \arrow[d, "\chi_{1,n+k}\wedge S^1"'] \arrow[rrr, bend left, "S^1\wedge g"] & & & \\
S^{n+k+1+1} \arrow[r, "g\wedge S^1"] \arrow[rrrd, bend right, "f\wedge S^1"'] & X_n \wedge S^1 \wedge S^1 \arrow[rr, "\tau\wedge S^1"] & & S^1 \wedge X_n \wedge S^1 \arrow[d, "\sigma_n\wedge S^1"] \\
& & & X_{1+n} \wedge S^1
\end{tikzcd}$$

"A diagram does not commute just because it is a diagram" – W.Lück

Beware that the upper half of the diagram *does not* commute. To remedy the failure of commutativity, we need two interchanges of the two S^1's in the source and target in the upper triangle. Now, the two automorphisms involved induce (-1) in the source and *after suspension* (-1) on the target when taking homotopy groups (see AT2Sheet1.1). Altogether this shows that the upper triangle commutes up to homotopy after suspension, and so the suspension map on homotopy groups is also surjective.

AT2Sheet1.1 also shows that this in general will induce (-1) on homotopy groups *only* after suspension!

(ii) This follows from the commutative triangles (induced by naturality of the adjunction)

$$\begin{tikzcd}
\pi_k(\Omega X) \arrow[rr, "\alpha"] \arrow[dr, "-\wedge S^1"'] & & \pi_{k+1}(X) & \pi_k(X) \arrow[rr, "-\wedge S^1"] \arrow[dr, "\eta_*"'] & & \pi_{k+1}(X\wedge S^1) \arrow[dl, "\alpha"] \\
& \pi_{k+1}((\Omega X)\wedge S^1) \arrow[ur, "\varepsilon_*"'] & & & \pi_k(\Omega(X\wedge S^1)) &
\end{tikzcd}$$

which imply respectively that ε and η are stable equivalences.

(iii) Let $A \in O(m)$ and denote $S^A : S^m \to S^m$ the map induced by A on the sphere S^m, with $\deg(S^A) = \det(A) \in \{\pm 1\}$. By (i), the map $- \wedge S^m : \pi_k(X) \to \pi_{k+m}(X \wedge S^m)$ is an isomorphism. So any class in $\pi_{k+m}(X \wedge S^m)$ has a representative of the form

$$f \wedge S^m : S^{n+k+m} \to X_n \wedge S^m$$

for some $f : S^{n+k} \to X_n$. Then

$$\begin{aligned}
(X \wedge S^A)_*[f \wedge S^m] &= [(X_n \wedge S^A) \circ (f \wedge S^m)] \\
&= [(f \wedge S^m) \circ (S^{n+k} \wedge S^A)] \\
&= \det(A)[f \wedge S^m]
\end{aligned}$$

since precomposition with a degree $\det(A)$ map induces multiplication by $\det(A)$. □

In particular, the theorem makes precise the statement that in the stable case loop and suspension are "inverse up to homotopy". Moreover, both loop and suspension induce isomorphisms of the stable homotopy groups. This is an instance of homology and homotopy "converging" in the stable setting (recall that unstably, suspension induces isomorphisms on homology, loop on homotopy). We will see another example later with the mapping cone and homotopy fiber sequences.

The following easy corollary will be updated later.

I.8. Corollary. — *For every morphism of orthogonal or sequential spectra $f : X \to Y$, the following are equivalent:*

On the official notes there is something more: the adjunction is a bijection on stable equivalences.

i) The morphism $f : X \to Y$ is a stable equivalence,

ii) The morphism $\Omega f : \Omega X \to \Omega Y$ is a stable equivalence,

iii) The morphism $f \wedge S^1 : X \wedge S^1 \to Y \wedge S^1$ is a stable equivalence.

Shift. — Let V be an inner product space. We can define a (based and continuous) functor $- \oplus V : \mathbf{O} \to \mathbf{O}$ by $U \mapsto U \oplus V$ on objects and on morphisms by

$$\begin{gathered}\mathbf{O}(U, W) \to \mathbf{O}(U \oplus V, W \oplus V)\\ (w, \varphi) \mapsto ((w, 0), \varphi \oplus V).\end{gathered}$$

One should check $(w, 0) \perp \mathrm{im}(\varphi \oplus V)$. It doesn't take more than the time it takes to notice we have to check something...

The V-th shift of an orthogonal spectrum X is the composite

$$\mathrm{sh}^V X : \mathbf{O} \xrightarrow{-\oplus V} \mathbf{O} \xrightarrow{X} \mathcal{T}_*$$

as a functor $\mathrm{sh}^V : \mathrm{Sp} \to \mathrm{Sp}$. More explicitly

$$(\mathrm{sh}^V X)(U) = X(U \oplus V)$$

with structure maps

$$\begin{array}{ccc} S^U \wedge (\mathrm{sh}^V X)(W) & \xrightarrow{\sigma^{\mathrm{sh}^V X}_{U,W}} & (\mathrm{sh}^V X)(U \oplus V) \\ \| & & \| \\ S^U \wedge X(W \otimes V) & \xrightarrow{\sigma^X_{U, W\otimes V}} & X(U \otimes W \otimes V) \end{array}$$

Shift commutes on the nose with all constructions of the form $F \circ -$, for $F : \mathcal{T}_* \to \mathcal{T}_*$. For example we have

$$(\mathrm{sh}^V X) \wedge A = \mathrm{sh}^V (X \wedge A), \ \mathrm{map}_*(A, \mathrm{sh}^V X) = \mathrm{sh}^V (\mathrm{map}_*(A, X)).$$

There is a canonical isomorphism $\mathrm{sh}^V (\mathrm{sh}^W X) \cong \mathrm{sh}^{V \oplus W} X$ with component at U given by the composite

$$(\mathrm{sh}^V (\mathrm{sh}^W X))(U) = (\mathrm{sh}^W X)(U \oplus V) = X((U \oplus V) \oplus W) \cong X(U \oplus (V \oplus W)) = (\mathrm{sh}^{V \oplus W} X)U.$$

There is a natural morphism $\lambda^V_X : X \wedge S^V \to \mathrm{sh}^V X$ with components $(\lambda^V_X)_U$ given by the opposite structure maps of X (with V fixed)

$$\begin{array}{ccc} X(U) \wedge S^V & \xrightarrow{\sigma^{\mathrm{op}}_{UV}} & X(U \oplus V) \\ {\scriptstyle\tau}\downarrow & & \uparrow{\scriptstyle X(\tau_{UV})} \\ S^V \wedge X(U) & \xrightarrow[\sigma_{VU}]{} & X(V \oplus U) \end{array}$$

Later we will show that the morphism λ^V_X is a stable equivalence! As a special case, for $V = \mathbb{R}$, we get $\mathrm{sh}\, X = \mathrm{sh}^{\mathbb{R}} X$, $\lambda_X = \lambda^{\mathbb{R}}_X : X \wedge S^1 \to \mathrm{sh}\, X$.

Warning ⚠. — There are forgetful functors $\mathrm{Sp} \to \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\mathbb{N}$. On these three categories, the notion of λ-maps behaves drastically different and this is one reason to prefer working with orthogonal spectra. In particular:

- in Sp the λ-maps exists and are π_*-isomorphism,
- in Sp^Σ the λ-maps exist but are not π_*-isomorphisms, AT2Sheet3.2 asks to produce a counterexample.
- in $\mathrm{Sp}^\mathbb{N}$ there are no natural λ-maps at all.

Related to this, there is a popular mistake in the literature. Let $X \in \mathrm{Sp}^\mathbb{N}$ and consider $X \wedge S^1$ and $\mathrm{sh}\, X$, which are well defined in the category of sequential spectra. We have $(\mathrm{sh}\, X)_n = X_{n+1}$, hence we get

$$\begin{tikzcd} S^1 \wedge (\mathrm{sh}\, X)_n \arrow[r, "\sigma_n^{\mathrm{sh}\, X}"] \arrow[d, equal] & (\mathrm{sh}\, X)_{1+n} \arrow[d, equal] \\ S^1 \wedge X_{n+1} \arrow[r, "\sigma_{n+1}"'] & X_{1+n+1} \end{tikzcd}$$

The mistake (which the professor has seen even in published papers) is thinking that the maps

$$\sigma_n : (X \wedge S^1)_n = X_n \wedge S^1 \xrightarrow{\tau} S^1 \wedge X_n \xrightarrow{\sigma_n} X_{1+n} = (\mathrm{sh}\, X)_n$$

assemble into a morphism of sequential spectra: they do not as the the naturality squares do not commute in general!

Lecture 5
27th Apr, 2022

Now we prove what we claimed last time: that for orthogonal spectra the λ-maps are stable equivalences.

I.9. Theorem. — *Let X be an orthogonal spectrum.*

i) For every $h \in \mathbb{Z}$, the map

$$\pi_{k+1}(X \wedge S^1) \xrightarrow{(\lambda_X)_*} \pi_{1+k}(\mathrm{sh}\, X) = \pi_k(X)$$

is inverse to the suspension isomorphism up to $(-1)^k$.

ii) The morphism $\lambda_X : X \wedge S^1 \to \mathrm{sh}\, X$ and its adjoint $\tilde{\lambda}_X : X \to \Omega\, \mathrm{sh}\, X$ are stable equivalences.

Proof. (i) The composite

$$\pi_k(X) \xrightarrow{-\wedge S^1} \pi_{k+1}(X \wedge S^1) \xrightarrow{(\lambda_X)_*} \pi_{1+k}(\mathrm{sh}\, X) = \pi_k(X)$$

sends the class of $f : S^{n+k} \to X_n$ to the composite

$$S^{n+k+1} \xrightarrow{f \wedge S^1} X_n \wedge S^1 \cong S^1 \wedge X_n \xrightarrow{\sigma_n} X_{1+n} \xrightarrow{X(\chi_{1,n})} X_{n+1}$$

and this is equal to the composite

$$S^{n+k+1} \xrightarrow{\chi_{1,n+k}} S^{1+n+k} \xrightarrow{S^1 \wedge f} S^1 \wedge X_n \xrightarrow{\sigma_n} X_{1+n} \xrightarrow{X(\chi_{1,n})} X_{n+1}$$

which by AT2Sheet2.1 represents

$$(-1)^{n+k}[X(\chi_{1,n})\circ\sigma_n\circ(S^1\wedge f)] = (-1)^{n+k}\det(\chi_{1,n})[\sigma_n\circ(S^1\wedge f)] = (-1)^k[f].$$

(ii) The map $(\lambda_X)_*$ is an isomorphism by (i) because $-\wedge S^1$ is an isomorphism. The adjoint of λ_X equals the composite

$$X \xrightarrow{\eta} \Omega(X\wedge S^1) \xrightarrow{\Omega\lambda_X} \Omega\,\mathrm{sh}\,X$$

and both of these maps are stable equivalences by Theorem I.7 and Corollary I.8. □

Mapping Cone and Homotopy Fiber

We already noted last semester that given a map $f : A \to B$ there are strong analogies between the mapping cone Cf and the homotopy fiber Ff. Notably, the mapping cone construction yields a long exact sequence of homology groups, while the homotopy fiber yields a long exact sequence of homotopy groups. We will see that, as with the loop-suspension adjunction, in the stable case there is some sort of convergence of homological and homotopical concepts. In particular, both the mapping cone and the homotopy fiber yield a long exact sequence of homotopy groups.

What does this mean, *really?*

The Mapping Cone Sequence

Construction. — Let $A \to B$ be a based map of based spaces. The *mapping cone* is

$$Cf = (A\wedge[0,1])\cup_{A\times 1,f} B,$$

This is homotopy equivalent to the unreduced cone in nice enough cases.

where $[0,1]$ is based at 0. We define

$$t:[0,1]\to S^1=\mathbb{R}\cup\{\infty\},\ x\mapsto\frac{2x-1}{x(1-x)}$$

which descends to a homeomorphism $[0,1]/(0\sim 1)\to S^1$. The mapping cone comes with two natural continuous maps

$$B\xrightarrow{i} Cf \xrightarrow{p} A\wedge S^1$$

where p is defined by $a\wedge X\mapsto a\wedge t(x)$ and $b\mapsto *$.

We know that the mapping cone construction (for spaces) induces a long exact sequence of homology groups: our present goal is to prove that we have a long exact sequence of stable homotopy groups of spectra analogous to the old one. In order to prove the statement, we first need to prove a (if you ask me, quite terrifying) lemma which will serve as the main technical tool.

I.10. Lemma. — *Let $f : A \to B$ be a continuous based map.*

i) The collapsing map

$$*\cup p : Ci = (B\wedge[0,1])\cup_{B\times 1,i} Cf \to A\wedge S^1$$

is a based homotopy equivalence.

ii) The square

$$\begin{tikzcd} Ci \arrow[r, "p\cup *"] \arrow[d, "*\cup p"'] & B\wedge S^1 \arrow[d, "p\wedge\tau"] \\ A\wedge S^1 \arrow[r, "f\wedge S^1"] & B\wedge S^1 \end{tikzcd}$$

(where $\tau(x) = -x$) commutes up to based homotopy equivalences.

iii) Let $\beta : Z \to B$ be a based continuous map such that $i \circ \beta : Z \to Cf$ is null-homotopic. Then there is a map $h : Z \wedge S^1 \to A \wedge S^1$ such that

$$(f \wedge S^1) \circ h : Z \wedge S^1 \to B \wedge S^1$$

is homotopic to $\beta \wedge S^1$.

Proof. Statements (i) and (ii) follow from explicit formulas of homotopies and can be found in [Sch18]. For example, for (i), a homotopy inverse to $* \cup p$ is $r : A \wedge S^1 \to (B \wedge [0, 1]) \cup_i Cf$ is the map

$$r(a \wedge x) \begin{cases} a \wedge 2x & \text{in } Cf \text{ for } 0 \leqslant x \leqslant 1/2 \\ f(a) \wedge (2 - 2x) & \text{in } B \wedge [0,1] \text{ for } 1/2 \leqslant x \leqslant 1 \end{cases}.$$

There are explicit homotopies between the composites of $x \cup p$ and r to the respective identities, we give one:

We are allowed (even encouraged) not to read these formulas. Instead, it would be useful to try and come up with a homotopy by yourself.

$$[0,1] \times (B \wedge [0,1]) \to Ci, \ (t, b \wedge x) \mapsto b \wedge (1-t)x \quad \text{in } B \wedge [0,1] \subset Ci$$

$$[0,1] \times (A \wedge [0,1]) \to Ci, \ (t, a \wedge x) \mapsto \begin{cases} a \wedge (1+t)x & \text{in } Cf \text{ for } 0 \leqslant x \leqslant 1/(1+t) \\ f(a) \wedge (2 - x(1+t)) & \text{in } B \wedge [0,1] \text{ for } 1/(1+t) \leqslant x \leqslant 1 \end{cases}$$

For (iii), let $H : Z \wedge [0, 1] \to Cf$ be a continuous map that witnesses that $\beta \circ i$ is null-homotopic, i.e $H(z \wedge 1) = i(\beta(z))$. Then we have

$$\begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d, "Z\wedge 1"'] & B \arrow[d, "i"] \\ Z\wedge[0,1] \arrow[r, "H"] \arrow[d, "p_Z"'] & Cf \arrow[d, "p_A"] \\ Z\wedge S^1 \arrow[r, "\exists! h"] & A\wedge S^1 \end{tikzcd}$$

so that the composite $p_A \circ H$ factors uniquely as $h \circ p_Z$ for $h : Z \wedge S^1 \to A \wedge S^1$. We claim that $(f \wedge S^1) \circ h$ is homotopic to $\beta \wedge S^1$. This would follow from

$$(f \wedge S^1) \circ h \circ (* \cup p_Z) \simeq (\beta \wedge S^1) \circ (* \cup p_Z)$$

since $* \cup p : CZ \cup_Z CZ \to Z \wedge S^1$ is a homotopy equivalence. Now

$$\begin{aligned} (f \wedge S^1) \circ h \circ (* \cup p_Z) &= (f \wedge S^1) \circ (* \cup p_A) \circ ((\beta \wedge [0,1]) \cup H) \\ &\simeq (B \wedge \tau) \circ (p_B \cup *) \circ ((\beta \wedge [0,1]) \cup H) \\ &= (B \wedge \tau) \circ (\beta \wedge S^1) \circ (p_Z \cup *) \\ &= (\beta \wedge S^1) \circ (Z \wedge \tau) \circ (p_Z \cup *) \\ &\simeq (\beta \wedge S^1) \circ (* \cup p_Z) \end{aligned}$$

where we have used (ii) two times. □

Construction. — Now let $f : X \to Y$ be a morphism of orthogonal (or sequential) spectra. The *mapping cone* Cf is a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "-\wedge 1"'] & Y \arrow[d, "i"] \\ X \wedge [0,1] \arrow[r] & Cf \end{tikzcd}$$

i.e. the mapping cone levelwise. The unstable i and p maps, taken in every level, provide morphisms of orthogonal (or sequential) spectra

It is quite clear that the levelwise maps commute with stabilization (later, we will not even mention this most of the time).

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{p} X \wedge S^1.$$

The *connecting morphisms* $\delta : \pi_{k+1}(Cf) \to \pi_k(X)$ is defined to be the composite

$$\pi_{k+1}(Cf) \xrightarrow{p_*} \pi_{k+1}(X \wedge S^1) \xrightarrow{-\wedge S^1} \pi_k(X).$$

Remark. — Note that for commutative square of spectrum morphisms

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[d, "b"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

the following also commutes

$$\begin{tikzcd} \pi_{k+1}(Cf) \arrow[r, "\delta"] \arrow[d] & \pi_k(X) \arrow[d, "a_*"] \\ \pi_{k+1}(Cf') \arrow[r, "\delta"] & \pi_k(X') \end{tikzcd}$$

I.11. Proposition. — *For every morphism of sequential spectra $f : X \to Y$ the sequence*

$$\cdots \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf) \xrightarrow{\delta} \pi_{k-1}(X) \to \cdots$$

is exact.

Proof. Exactness at $\pi_k(Y)$. The composite $i_* \circ f_* = (i \circ f)_*$ is the zero morphism, because $i \circ f$ is levelwise constant at the basepoint. Now let $\beta : S^{n+k} \to Y_n$ represent an element in the kernel of $i_* : \pi_k(Y) \to \pi_k(Cf)$. Without loss of generality, $i_n \circ \beta : S^{n+k} \to Y_n \to Cf_n$ is null-homotopic. Part (iii) of Lemma I.10 provides a continuous based map $h : S^{n+k+1} \to X_n \wedge S^1$ such that $(f_n \wedge S^1) \circ h \simeq \beta \wedge S^1$. Hence the composite

$$\tilde{h} : S^{1+n+k} \xrightarrow{\chi_{1,n+k}} S^{n+k+1} \xrightarrow{h} X_n \wedge S^1 \xrightarrow{\cong} S^1 \wedge X_n$$

is such that $(S^1 \wedge f_n) \circ \tilde{h} \simeq S^1 \wedge \beta$. Then $\sigma_n \circ h : S^{1+n+k} \to X_{1+n}$ represents a class in $\pi_k(X)$ such that

$$f_*[\sigma_n \circ \tilde{h}] = [f_{1+n} \circ \sigma_n \circ \tilde{h}] = [\sigma_n \circ (S^1 \wedge f_n) \circ \tilde{h}] = [\sigma_n \circ (S^1 \wedge \beta)] = [\beta]$$

so $\ker(i_*) = \mathrm{im}(f_*)$.

Exactness at $\pi_k(Cf)$. We compare the long exact sequence for f to the long exact sequence for $i_f : Y \to Cf$ with a rotation. The collapse maps provide the map

$$* \cup p : Ci : CY \cup Cf \to X \wedge S^1,$$

which is a levelwise homotopy equivalence (by Lemma I.10.i), so f induces isomorphisms of all homotopy groups. From the diagram

$$\begin{tikzcd} Cf \arrow[r, "i_{i_f}"] \arrow[dr, "p_f"'] & Ci_f \arrow[r, "p_{i_f}"] \arrow[d, "*\cup p_f"] & Y \wedge S^1 \arrow[d, "Y\wedge\tau"] \\ & X \wedge S^1 \arrow[r, "f\wedge S^1"'] & Y \wedge S^1 \end{tikzcd}$$

where the right square commutes up to homotopy (by Lemma I.10.ii) we get

$$\begin{tikzcd} \pi_k(Y) \arrow[r, "(i_f)_*"] \arrow[d, equal] & \pi_k(Cf) \arrow[r, "(i_{i_f})_*"] \arrow[d, equal] & \pi_k(Ci) \arrow[r, "\delta"] \arrow[d, "(-\wedge S^1)\circ(*\cup p)", "\cong"'] & \pi_{k-1}(Y) \arrow[d, "-1", "\cong"'] \\ \pi_k(Y) \arrow[r, "(i_f)_*"'] & \pi_k(Cf) \arrow[r, "\delta"'] & \pi_{k-1}(X) \arrow[r, "f_*"'] & \pi_{k-1}(Y) \end{tikzcd}$$

where the upper row is exact at $\pi_k(Cf)$ by the previous part applied to $i : Y \to Cf$. Hence the lower row is exact at $\pi_k(Cf)$. But this means the upper row is exact at $\pi_k(C_i)$, so in turn the lower row is also exact at $\pi_{k-1}(X)$. □

Lecture 6
2nd May, 2022

We conclude the story of the mapping cone with a familiar (from back when we first learned about homology) twist.

Fyi, in $\mathcal{T}_*$ one doesn't have to worry about ndr pairs and similar amenities: every h-cofibration is a closed embedding!

Definition. — A continuous map $f : A \to B$ is a *h-cofibration* if it has the homotopy extension property (HEP). As a consequence, the quotient map

$$Cf = A \wedge [0,1] \cup_f B \xrightarrow{*\cup \mathrm{id}} B/A$$

is a homotopy equivalence.

Let $f : X \to Y$ be a morphism of spectra that is levelwise a h-cofibration. Then the quotient morphism $Cf \to Y/X$ is levelwise a homotopy equivalence, hence a stable equivalence (this works for sequential or orthogonal spectra, indifferently). We define a connecting morphism $\pi_{k+1}(Y/X) \to \pi_k(X)$ via

$$\begin{tikzcd} \pi_{k+1}(Y/X) \arrow[r, dashed] & \pi_k(X) \\ \pi_{k+1}(Cf) \arrow[u, "\cong"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

and this buys us the following corollary (a "strict" version of the previous proposition).

The quotient is the strict cofiber, the mapping cone the homotopy cofiber.

I.12. Corollary. — *Let $f : X \to Y$ be a morphism of spectra that is levelwise a h-fibration. Then the sequence*

$$\cdots \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(X/Y) \xrightarrow{\delta} \pi_{k-1}(X) \to \cdots$$

is exact.

The Homotopy Fiber Sequence

Construction. — The *homotopy fiber* of a based continuous map $f : A \to B$ is the pullback

$$Ff = * \times_B B^{[0,1]} \times_B A = \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\ \lambda(1) = f(a)\}.$$

This comes with natural maps

$$\mathrm{map}_*(S^1, B) = \Omega B \to Ff \xrightarrow{q} A$$

where q is the projection and the first map is $(\mu : S^1 \to B) \mapsto (\mu \circ t, *)$, where t is the map defined at the start of the section, winding the interval to a circle (or any similar map, really).

Construction. — Let $f : X \to Y$ be a morphism of sequential/orthogonal spectra, its *homotopy fiber* is the pullback

$$\begin{tikzcd} \Omega Y \arrow[r, "i"] & Ff \arrow[r, "q"] \arrow[d] & X \arrow[d, "(*,f)"] \\ & Y^{[0,1]} \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & Y \times Y \end{tikzcd}$$

The connecting morphisms $\delta : \pi_{1+k}(Y) \to \pi_k(Ff)$ is defined as the composite

$$\pi_{1+k}(Y) \xrightarrow[\cong]{\alpha^{-1}} \pi_k(\Omega Y) \xrightarrow{i_*} \pi_k(Ff)$$

Moreover, we can define a comparison morphism $(Ff) \wedge S^1 \to Cf$. Let $f : A \to B$ be a continuous map of based spaces. We define

$$\begin{aligned} \bar{h} : Ff \times [0,1] &\to (A \wedge [0,1]) \cup_f B \\ (\lambda, a, t) &\mapsto \begin{cases} a \wedge 2t & 0 \leqslant t \leqslant 1/2 \\ \lambda(2-2t) & 1/2 \leqslant t \leqslant 1 \end{cases} \end{aligned}$$

and clearly $\bar{h}(\lambda, a, 0) = \bar{h}(\lambda, a, 1) = *$, so $\bar{h}$ factors uniquely as $\bar{h} = h \circ p$

$$\begin{tikzcd} Ff \times [0,1] \arrow[rr, "p"] \arrow[dr, "\bar{h}"'] & & Ff \wedge S^1 \arrow[dl, dashed, "h"] \\ & Cf & \end{tikzcd}$$

giving us a comparison morphism $h : (Ff) \wedge S^1 \to Cf$.

I.13. Proposition. — *For every morphism $f : X \to Y$ of sequential spectra, the sequence*

$$\cdots \to \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\delta} \pi_{k-1}(Ff) \xrightarrow{q_*} \pi_{k-1}(X) \to \cdots$$

is exact, and the morphism

$$h : (Ff) \wedge S^1 \to Cf$$

is a stable equivalence.

Proof. The long exact homotopy group sequence

$$\cdots \to \pi_{n+k}(X_n) \xrightarrow{(f_n)_*} \pi_{k+n}(Y_n) \xrightarrow{\delta} \pi_{n+k-1}(Ff_n) \xrightarrow{(q_n)_*} \pi_{n+k-1}(X_n) \to \cdots$$

Last time we had to establish a long exact sequence from scratch, as the mapping cone does not already yield one in the unstable case.

exists and is exact for sufficiently large n, depending on k. Since filtered (hence in particular, sequential) colimits of modules are exact, we get an exact sequence on colimits.

To show that $h : (Ff) \wedge S^1 \to Cf$ is a stable equivalence, it suffices to show that the map $h_* \circ (- \wedge S^1) : \pi_k(Ff) \to \pi_{k+1}(cf)$ is an isomorphism for all $k \in \mathbb{Z}$.

Claim. The following diagram commutes

$$\begin{array}{ccccc} \pi_{k+1}(Y) & \xrightarrow{\delta} & \pi_k(Ff) & \xrightarrow{q_*} & \pi_k(X) \\ \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle h_* \circ (- \wedge S^1)} & & \| \\ \pi_{k+1}(Y) & \xrightarrow[i_*]{} & \pi_{k+1}(Cf) & \xrightarrow[\delta]{} & \pi_k(X) \end{array}$$

Proof of the claim. To show that the right square commutes, observe that the composite

$$(Ff) \wedge S^1 \xrightarrow{h} Cf \xrightarrow{p} X \wedge S^1$$

is homotopic to $q \wedge S^1$ via the homotopy

$$[0,1] \times ((Ff) \wedge S^1) \to X \wedge S^1$$
$$(t, (\lambda, a), s) \mapsto \begin{cases} a \wedge \frac{2s}{2-t} & 0 \leqslant s \leqslant 1 - t/2 \\ * & 1 - t/2 \leqslant s \leqslant 1 \end{cases},$$

Not sure I understand what's happening

hence

$$\begin{aligned} \delta \circ h_*(- \wedge S^1) &= (- \wedge S^{-1}) \circ p_* \circ h_* \circ (- \wedge S^1) \\ &= (- \wedge S^1) \circ (q \wedge S^1)_* \circ (- \wedge S^1) \\ &= (- \wedge S^{-1}) \circ (- \wedge S^1) \circ q_* = q_*. \end{aligned}$$

For the left half of the diagram, we will need that the following diagram commutes up to homotopy

$$\begin{array}{ccc} (\Omega Y) \wedge S^1 & \xrightarrow{i \wedge \tau} & Ff \wedge S^1 \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{i} & Cf \end{array}$$

The formula for the homotopy could be wrong

as witnessed by

$$[0,1] \times ((\Omega Y) \wedge S^1) \to Cf$$
$$(t, \omega \wedge x) \mapsto \begin{cases} * & 0 \leqslant x \leqslant t/2 \\ \omega\left(\frac{2(1-t)}{2-x}\right) & t/2 \leqslant x \leqslant 1 \end{cases}$$

This gives

$$\begin{aligned} h_*(\delta(y) \wedge S^1) = h_*(i_*(\alpha^{-1}(y)) \wedge S^1) &= h_*((i \wedge S^1)_*(\alpha^{-1}(y) \wedge S^1)) \\ &= -i_*(\varepsilon(\alpha^{-1}(y) \wedge S^1)) \\ &= -i_*(y), \end{aligned}$$

considering

$$\begin{tikzcd} \pi_k(\Omega Y) \arrow[rr, "(-\wedge S^1)_*"] \arrow[dr, "\alpha^{-1}"', "\cong"] & & \pi_{k+1}((\Omega Y)\wedge S^1) \arrow[dl, "\varepsilon_*"] \\ & \pi_{k+1}(Y) & \end{tikzcd}$$

for $y \in \pi_{k+1}(Y)$. ■

Thus we can conclude via five lemma. □

Let $f : X \to Y$ be a map of spectra that is levelwise a Serre fibration. Then the strict fiber $F_n = f_n^{-1}(*_{Y_n}) \subset X_n$ maps by a weak equivalence to the homotopy fibre $(Ff)_n = F(f_n)$, via $x \mapsto (\mathrm{const}_*, x)$. We get a new connecting morphism

This is dual to the homotopy equivalence $Cf \simeq X/Y$ for h-cofibrations.

$$\begin{tikzcd} \pi_k(Y) \arrow[r, dashed, "\delta"] \arrow[dr, "\delta"'] & \pi_{k-1}(F) \arrow[d, "\cong"] \\ & \pi_{k-1}(Ff) \end{tikzcd}$$

where F is the levelwise strict fiber over the basepoint.

I.14. Corollary. — *Let $f : X \to Y$ be a morphism of spectra that is levelwise a Serre fibration. Then the sequence*

$$\cdots \to \pi_k(F) \xrightarrow{\mathrm{incl}_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{\delta} \pi_{k-1}(F) \to \cdots$$

is exact.

Now that we have the mapping cone and homotopy fiber sequences we can prove some basic facts on spectra.

Important (Albeit Annoyingly Technical) Spectra Facts

I.15. Proposition. — *The following holds.*

The first point is a standard fact for homology which fails horribly for unstable homotopy groups, conversely for the second point. Somehow in the stable world homotopy behaves often much like homology (again: what does this mean, *really?*).

i) For every family $\{X^i\}_{i\in I}$ of sequential spectra, the natural map

$$\bigoplus_{i\in I} \pi_k(X^i) \to \pi_k\left(\bigvee_{i\in I} X^i\right)$$

is an isomorphism.

ii) For every finite index set I, the canonical map

$$\pi_k\left(\prod_{i\in I} X^i\right) \to \prod_{i\in I} \pi_k(X^i)$$

is an isomorphism.

Note that the wedge product is the coproduct in $\mathcal{T}_*$ (and so in Sp).

iii) If I is finite, the natural map

$$\bigvee_{i\in I} X^i \to \prod_{i\in I} X^i$$

is a stable equivalence.

Proof. (i) We prove first the special case of two summands A and B. Consider the long exact mapping cone sequence associated to the inclusion $i_A : A \to A \vee B$

$$\cdots \to \pi_k(A) \xrightarrow{(i_A)_*} \pi_k(A \vee B) \xrightarrow{(i_{i_A})_*} \pi_k(C(i_A)) \xrightarrow{\delta} \pi_{k-1}(A) \to \cdots .$$

The retraction $r : A \vee B \to A$ to i_A decomposes the long exact sequence into short exact sequences, and there are homotopy equivalences $C(i_A) \cong (CA) \vee B \cong B$, thus we get short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \pi_k(A) \arrow[r, "(i_A)_*"] & \pi_k(A \vee B) \arrow[r, "(\mathrm{proj}_B)_*"] \arrow[l, bend left, "r_*"] & \pi_k(B) \arrow[r] \arrow[l, bend left, "(i_B)_*"] & 0 \end{tikzcd} .$$

Now let I be arbitrary. We consider

$$\begin{tikzcd} \bigoplus_{i\in I} \pi_k(X^i) \arrow[r] \arrow[rr, bend right, "\text{canonical}"'] & \pi_k\left(\bigvee_{i\in I} X^i\right) \arrow[r] & \prod_i \pi_k(X^i) \end{tikzcd} .$$

In Ab the canonical map is always injective. Thus, also $\bigoplus_{i\in I} \pi_k(X^i) \hookrightarrow \pi_k\left(\bigvee_{i\in I} X^i\right)$ is.

To prove surjectivity, let $f : S^{n+k} \to \bigvee_{i\in I} X^i_n$ represent an element in $\pi_k\left(\bigvee_{i\in I} X^i\right)$. Since S^{n+k} is compact, by [Sch18, Proposition A.18] there is a finite subset $J \subseteq I$ such that $\operatorname{im} f \subseteq \bigvee_{i\in J} X^i_n$, hence surjectivity follows from the finite index case by considering

This is a standard argument, but one has to take care of some point-set subtleties to make sure that it works in this (possibly not Hausdorff) setting.

$$\begin{tikzcd} \bigoplus_{i\in J} \pi_k(X^i) \arrow[r] \arrow[d] & \pi_k\left(\bigvee_{i\in J} X^i\right) \arrow[d] \\ \bigoplus_{i\in I} \pi_k(X^i) \arrow[r] & \pi_k\left(\bigvee_{i\in I} X^i\right) \end{tikzcd}$$

(ii) Fix a finite index set I. For all k, n such that $k + n \geqslant 0$, the natural map

$$\pi_{n+k}\left(\prod_{i\in I} X^i_n\right) \to \prod_{i\in I} \pi_{n+k}(X^i_n)$$

is bijective. Since finite products commute with filtered colimits, we also get a product decomposition in the colimit as $n \to \infty$.

(iii) Again, let I be finite. Since in Ab finite direct sums agree with finite products, we have an isomorphism

$$\bigoplus_{i\in I} \pi_k(X^i) \xrightarrow{\cong} \pi_k(\bigvee_{i\in I} X^i) \to \pi_k(\prod_{i\in I} X^i) \xrightarrow{\cong} \prod_{i\in I} \pi_n(X^i)$$

where the first map is an isomorphism by (i) and the third by (ii). Hence the central morphism is also an isomorphism. □

Example. — The finiteness hypotheses in the last two points of last proposition cannot be dropped, as shown in the following example. Let $\mathbb{S}^{\leqslant i}$ be the *truncated sphere spectrum*

$$(\mathbb{S}^{\leqslant i})_n = \begin{cases} S^n & n \leqslant i \\ * & n > 1 \end{cases}$$

with $\pi_k(\mathbb{S}^{\leqslant i}) = 0$ for all k, i. Hence we have $\prod_{i\geqslant 1} \pi_k(\mathbb{S}^{\leqslant i}) = 0$. On the other hand the group $\pi_0(\prod_{i\geqslant 1} \mathbb{S}^{\leqslant i})$ is the colimit of the sequence of maps

$$\prod_{i\geqslant n} \pi_n(S^n) \to \prod_{i\geqslant n+1} \pi_{n+1}(S^{n+1})$$

which are the composition of the projection away from the first factor and then the product of the suspension morphisms $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$, so

$$\pi_0(\textstyle\prod_{i\geqslant 1} \mathbb{S}^{\leqslant i}) = \operatorname{colim}_n \pi_n(\textstyle\prod_{i\geqslant 1} \mathbb{S}^{\leqslant i}_n) = \dfrac{\prod_{k\in\mathbb{N}} \mathbb{Z}}{\bigoplus_{k\in\mathbb{N}} \mathbb{Z}} \neq 0.$$

Now that we have the long exact sequences associated to the mapping cone and the homotopy fiber constructions we can also update Corollary I.8 as follows.

I.16. Corollary. — *For a morphism $f : X \to Y$ of orthogonal spectra, the following are equivalent:*

(i) the morphism f is a stable equivalence,

(ii) the mapping cone Cf has trivial homotopy groups,

(iii) the suspension $f \wedge S^1$ is a stable equivalence,

(iv) the shift $\operatorname{sh} f$ is a stable equivalence,

(v) the loop Ωf is a stable equivalence,

(vi) the homotopy fibre Ff has trivial homotopy groups.

Point (iv) follows from point (iii) since the λ-maps are stable equivalences.

LECTURE 7
4th May, 2022

The next result is similar to the one that holds for the mapping telescope (AT1Sheet2.3 or Proposition III.2 for generalized homology theories).

I.17. Proposition. — *The following holds.*

i) Let

$$X^0 \xrightarrow{e^0} X^1 \xrightarrow{e^1} \cdots \xrightarrow{e^{n-1}} X^n \xrightarrow{e^{n+1}} \cdots$$

be a sequence of morphisms of sequential spectra that are levelwise closed embeddings. Then the canonical map

$$\operatorname*{colim}_{m\geqslant 0} \pi_k(X^m) \to \pi_k(\operatorname*{colim}_{m\geqslant 0} X^m)$$

is an isomorphism.

Colimits in $\mathcal{T}$ of sequences of closed embeddings coincide with the colimit in Top.

ii) Let $e^m : X^m \to X^{m+1}$ and $f^m : Y^m \to Y^{m+1}$ be morphisms of sequential spectra that are levelwise closed embeddings and $\psi^m : X^m \to Y^m$ stable equivalences which fit in commutative diagrams

$$\begin{tikzcd} X^m \arrow[r, "e^m"] \arrow[d, "\psi^m"'] & X^{m+1} \arrow[d, "\psi^{m+1}"] \\ Y^m \arrow[r, "f^m"] & Y^{m+1} \end{tikzcd}$$

then the induced morphism

$$\psi^\infty := \operatorname*{colim}_m \psi^m : \operatorname*{colim}_m X^m \to \operatorname*{colim}_m Y^m$$

is a stable equivalence.

iii) Let $e^m : X^m \to X^{m+1}$ be stable equivalences that are levelwise closed embeddings. Then the canonical map

$$X^0 \to \operatorname*{colim}_{m \geqslant 0} X^m$$

is a stable equivalence.

Proof. We just need to prove the first point, as the other two are immediate consequences. Let $f : S^{n+k} \to X_n^\infty = \operatorname{colim}_m X_n^m$ be a continuous based map representing a class in $\pi_k(X^\infty)$. Since the maps $e_n^m : X_n^m \to X_n^{m+1}$ are closed embeddings, by [Sch18, Proposition A.15] the map f factors through X_n^m for some m. So f is in the image $\pi_k(X^m) \to \pi_k(X^\infty)$. Injectivity follows from the same argument with homotopies. □

This standard fact has to be checked anew as we are working with weak Hausdorff spaces.

The next long proposition is a collection of reasonable results. However useful this might be, it is undeniably a bit dull, but some of these fact will play an important role later on: they hint towards a model structure on the category of spectra.

I.18. Proposition. — *The following holds.*

i) A coproduct (wedge) of stable equivalences is a stable equivalence.

ii) A finite product of stable equivalences is a stable equivalence.

iii) Consider a commutative square of sequential spectra

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "j"] & D \end{tikzcd} \tag{$*$}$$

and let $h = Cf \cup g : Ci \to Cj$ be the induced morphism of mapping cones. If two out of f, g and h are stable equivalences, then so is the third.

iv) In $(*)$, let $e_i : Fi \to Fj$ be the induced morphism of homotopy fibres. If two out of e, f and g are stable equivalences, then so is the third.

v) In $(*)$, suppose that one of the following conditions hold:

(a) *the square is a pushout and i or f is levelwise an h-cofibration,*

(b) *the square is a pullback and j or g is levelwise a Serre fibration.*

Then f is a stable equivalence if and only if g is.

vi) Let K be a based space that admits a CW-structure. Then $- \wedge K$ preserves stable equivalences.

vii) Let K be a based space that admits a finite CW-structure. Then $\mathrm{map}_*(K, -)$ preserves stable equivalences.

Proof. The first two points are immediate consequences of Proposition I.15. The third point follows by applying the 5-lemma to the long exact sequences of the mapping cones of i and j. The fourth point is dual to the third, it suffices to use the homotopy fiber in place of the mapping cone.

(v) It suffices to prove (a), as (b) follows from the dual argument. The square (∗) is a pushout. So the morphisms

$$j/i : C/A \to D/B$$

$$g/f : B/A \to D/C$$

This is easy to prove for spaces but is a general nonsense fact.

are isomorphisms. We split the proof in two cases

Case 1. Suppose f is levelwise an h-cofibration (hence so is g). The long exact sequence for the homotopy groups of f (the one in Corollary I.12) shows that

Cofibrations are closed under pushouts.

$$\pi_*(f) : \pi_*(A) \to \pi_*(C)$$

is an isomorphism if and only if

$$\pi_*(C/A) = 0 \iff \pi_*(D/B) = 0 \iff \pi_*(g) : \pi_*(B) \to \pi_*(D) \text{ is an isomorphism.}$$

Case 2. If i is an h-cofibration (and hence also j), we compare the long exact sequence of homotopy groups of the quotient in the horizontal direction

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & \pi_k(B) \arrow[r, "\mathrm{proj}_*"] \arrow[d, "g_*"] & \pi_k(B/A) \arrow[r, "\delta"] \arrow[d, "(g/f)_*"] & \cdots \\
\cdots \arrow[r] & \pi_k(C) \arrow[r, "j_*"'] & \pi_k(D) \arrow[r, "\mathrm{proj}_*"'] & \pi_k(D/C) \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

we have that $(g/f)_*$ is an isomorphism, so we can conclude via five Lemma.

(vi) The functor $-\wedge K$ commutes with mapping cones

$$(Cf) \wedge K \cong C(f \wedge K)$$

so the long exact sequence for the mapping cone reduces the claim to the following special case: if X has trivial homotopy groups, so does $X \wedge K$.

We choose a CW-structure in which the base point is a 0-cell with skeleta K_n. We show by induction that $X \wedge K_n$ has trivial homotopy groups. The case $n = -1$ is clear because $X \wedge * = *$. Now let $n \geqslant 0$. Then $K_n/K_{n-1} \cong \bigvee_I S^n$, so

$$\begin{aligned}
\pi_k(X \wedge K_n / X \wedge K_{n-1}) &\cong \pi_k(X \wedge (K_n/K_{n-1})) \\
&\cong \pi_k(X \wedge \textstyle\bigvee_I S^n) \\
&\cong \pi_k(\textstyle\bigvee_I (X \wedge S^n)) \\
&\cong \textstyle\bigoplus_I \pi_k(X \wedge S^n) \\
&\cong \textstyle\bigoplus_I \pi_{k-n}(X) = 0.
\end{aligned}$$

The inclusion $K_{n-1} \to K_n$ is an h-cofibration. So $X \wedge K_{n-1} \to X \wedge K_n$ is levelwise an h-cofibration. Hence the long exact sequence for the homotopy groups of the strict cofiber and the inductive hypothesis show that $X \wedge K_n$ has trivial homotopy groups.

In the general case the morphism

$$X \wedge K_0 \to X \wedge K_1 \to \cdots$$

are levelwise h-cofibration, hence levelwise closed embeddings. Thus

$$\pi_k(X \wedge K) \cong \operatorname*{colim}_m \pi_k(X \wedge K_m) = 0.$$

(vii) Let K be a finite based CW-complex. Then $\mathrm{map}_*(K,-)$ commutes with homotopy fibers

$$\mathrm{map}_*(K, Ff) \cong F(\mathrm{map}_*(K,f))$$

so once again the long exact sequence for the homotopy fiber reduces the statement to the special case: if X has trivial homotopy groups, so does $\mathrm{map}_*(K,X)$.

We argue by induction over the number of cells in K. If $K = \{k_0\}$ consists only of the basepoint, then $\mathrm{map}_*(*,X) = *$ has trivial homotopy groups. Otherwise suppose that L is obtained from K by attaching an n-cell and assume the claim for K. Then the restriction $\mathrm{map}_*(L,X) \to \mathrm{map}_*(K,X)$ is levelwise a Serre fibration (see [AT1, theorem III.7]), so we get a long exact sequence

Here is where we use the finiteness assumption and we need it because the pies do not preserve limits (such as the fiber)!

Secretly we are using that a self adjoint on the right functor sends colimits (the quotient) to limits (the fiber).

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(\text{fiber of } \mathrm{map}_*(L,X) \to \mathrm{map}_*(K,X)) \arrow[r] & \pi_k(\mathrm{map}_*(L,X)) \arrow[r] & \pi_k(\mathrm{map}_*(K,X)) = 0 \\
& \pi_k(\mathrm{map}_*(L/K,X)) \arrow[u, "\cong"'] & & \\
& \pi_k(\mathrm{map}_*(S^n,X)) = \pi_k(\Omega^n X) \arrow[u, "\cong"'] & & \\
& \pi_k(\mathrm{map}_{k+n}(X)) = 0 \arrow[u, "\cong"'] & &
\end{tikzcd}$$

and thus $\pi_k(\mathrm{map}_*(L,X)) = 0$. □

Ring and Module Spectra

As we already mentioned, sequential spectra are fine and dandy for many purposes and orthogonal spectra really become fundamental only in some situations, most notably equivariant stable homotopy theory. Another theory where orthogonal spectra become crucial is the theory of ring and module spectra. The issue with sequential spectra is that $\mathrm{Sp}^{\mathbb{N}}$ does not admit a smash product that yields a symmetric monoidal category (according to the nLab, there are smash products that yield such a structure *only* after passage to the stable homotopy category). Orthogonal spectra solve this problem (as do symmetric spectra, which are instead problematic because their π_*-isomorphisms are not suitable weak equivalences).

Definition. — An *orthogonal ring spectrum* is an orthogonal spectrum R equipped with $(O(V) \times O(W))$-equivariant multiplication maps

$$\mu_{V,W} : R(V) \wedge R(W) \to R(V \oplus W)$$

and a unit $\iota \in R(0)$ that satisfy the following.

- Associativity. For all inner product spaces U, V, W the following diagram commutes

$$\begin{tikzcd}
R(U) \wedge R(V) \wedge R(W) \arrow[r, "R(U)\wedge\mu_{V,W}"] \arrow[d, "\mu_{U,V}\wedge R(W)"'] & R(U) \wedge R(V \oplus W) \arrow[d, "\mu_{U,V\oplus W}"] \\
R(U \oplus V) \wedge R(W) \arrow[r, "\mu_{U\oplus V,W}"'] & R(U \oplus V \oplus W)
\end{tikzcd}$$

- Unitality. For all inner product spaces V, W, the composite

$$S^V \wedge R(W) \xrightarrow{-\wedge\iota\wedge -} S^V \wedge R(0) \wedge R(W) \xrightarrow{\sigma_{V,0}\wedge R(W)} R(V) \wedge R(W) \xrightarrow{\mu_{V,W}} R(V \oplus W)$$

  equals $\sigma_{V,W}$. Moreover,

$$R(V) \wedge S^W \xrightarrow{-\wedge\iota\wedge -} R(V) \wedge R(0) \wedge S^W \xrightarrow{R(V)\wedge\sigma^{\mathrm{op}}_{0,W}} R(V) \wedge R(W) \xrightarrow{\mu_{V,W}} R(V \oplus W)$$

  equals $\sigma^{\mathrm{op}}_{V,W}$.

An orthogonal ring spectrum is *commutative* if also the following square commutes

$$\begin{tikzcd}
R(V) \wedge R(W) \arrow[r, "\mathrm{twist}"] \arrow[d, "\mu_{VW}"'] & R(W) \wedge R(V) \arrow[d, "\mu_{WV}"] \\
R(V \oplus W) \arrow[r, "R(\tau_{VW})"'] & R(V \oplus W)
\end{tikzcd}$$

Remark. — Some first observations about the notion of orthogonal ring spectrum.

1. As special cases (for $V = 0$ or $W = 0$) of the unit condition

$$R(W) \xrightarrow{\iota\wedge -} R(0) \wedge R(W) \xrightarrow{\mu_{0,W}} R(0 \oplus W) \xrightarrow{R(\mathrm{proj}_W)} R(W)$$

   is the identity, and similarly for the other composite.

2. If the multiplication maps are commutative, then the two unit conditions are equivalent.

3. The two maps

$$S^V \xrightarrow{-\wedge\iota} S^V \wedge R(0) \xrightarrow{\sigma_{V,0}} R(V \oplus 0) \cong R(V),$$
$$S^V \xrightarrow{\iota\wedge -} R(0) \wedge S^V \xrightarrow{\sigma^{\mathrm{op}}_{0,V}} R(0 \oplus V) \cong R(V),$$

   are equal. We will denote them by

$$\iota_V : S^V \to R(V)$$

   and call them the *generalized unit map.*

4. Later we will introduce the smash product of orthogonal spectra $\wedge : \mathrm{Sp}\times\mathrm{Sp} \to \mathrm{Sp}$ which is a symmetric monoidal structure with unit object $\mathbb{S}$. We will see that the category of orthogonal ring spectra is equivalent to the category of monoids in $(\mathrm{Sp}, \wedge, \mathbb{S})$.

Definition. — A *morphisms of orthogonal ring spectra* is a morphism of orthogonal spectra $f : R \to S$ such that

$$\begin{tikzcd}
R(V) \wedge R(W) \arrow[r, "\mu_{VW}"] \arrow[d, "f(V)\wedge f(W)"'] & R(V \oplus W) \arrow[d, "f(V\oplus W)"] \\
S(V) \wedge S(W) \arrow[r, "\mu_{VW}"'] & S(V \oplus W)
\end{tikzcd}$$

and

$$\begin{tikzcd} & R(V) \arrow[dd, "f(V)"] \\ S^V \arrow[ur, "\iota_V"] \arrow[dr, "\iota_V"'] & \\ & S(V) \end{tikzcd}$$

meaning $f(V)(\iota) = \iota$.

Definition. — A *left module* over an orthogonal ring spectra R is an orthogonal spectrum M equipped with $O(V) \times O(W)$-equivariant action map

$$\alpha_{V,W} : R(V) \wedge M(W) \to M(V \oplus W)$$

that satisfies the following properties.

- Associativity. The following diagram commutes

$$\begin{tikzcd} R(U) \wedge R(V) \wedge M(W) \arrow[r, "R(U)\wedge\alpha_{VW}"] \arrow[d, "\mu_{UV}\wedge M(W)"'] & R(U) \wedge M(V \oplus W) \arrow[d, "\alpha_{U,V\oplus W}"] \\ R(U \oplus V) \wedge M(W) \arrow[r, "\alpha_{U\oplus V,W}"'] & M(U \oplus V \oplus W). \end{tikzcd}$$

- Unitality. The composite

$$S^V \wedge M(W) \xrightarrow{\iota_V \wedge M(W)} R(V) \wedge M(W) \xrightarrow{\alpha_{VW}} M(V \oplus W)$$

  equals σ_{VW}.

A *morphism of left R-modules* is a morphism of orthogonal spectra $f : M \to N$ such that the following commutes

$$\begin{tikzcd} R(V) \wedge M(W) \arrow[r, "\alpha_{VW}"] \arrow[d, "R(V)\wedge f(W)"'] & M(V \oplus W) \arrow[d, "f(V\oplus W)"] \\ R(V) \wedge M(W) \arrow[r, "\alpha_{VW}"'] & N(V \oplus W) \end{tikzcd}$$

Remark. — The category of R-modules has limits and colimits and they are computed in the category of orthogonal spectra. Let $F : I \to {}_R\mathrm{Mod}(\mathrm{Sp})$ be any functor, $M \in \mathrm{Sp}$ a colimit of the composite with the forgetful functor U,

$$I \xrightarrow{F} {}_R\mathrm{Mod}(\mathrm{Sp}) \xrightarrow{U} \mathrm{Sp}.$$

This inherits a canonical R-module structure with action maps

$$\begin{aligned} \alpha_{VW} : R(V) \wedge M(W) &= R(V) \wedge \operatorname*{colim}_{i\in I} F(i)(W) \\ &\xleftarrow{\cong} \operatorname*{colim}_{i\in I}(R(V) \wedge F(i)(W)) \\ &\xrightarrow{\operatorname{colim}_{i\in I} \alpha^{F(i)}_{VW}} \operatorname*{colim}_{i\in I} F(i)(V \oplus W) \\ &= M(V \oplus W). \end{aligned}$$

where we use that $\wedge$ and colimits commute.

Remark. — The adjoint functor pair

$$\mathrm{Sp} \underset{\mathrm{map}_*(K,-)}{\overset{-\wedge K}{\rightleftarrows}} \mathrm{Sp}$$

lifts to R-modules

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, shift left, "-\wedge K"] & \mathrm{Sp} \arrow[l, shift left, "\mathrm{map}_*(K,-)"] \\ {}_R\mathrm{Mod}(\mathrm{Sp}) \arrow[u, "U"] \arrow[r, shift left, "-\wedge K"] & {}_R\mathrm{Mod}(\mathrm{Sp}) \arrow[u, "U"'] \arrow[l, shift left, "\mathrm{map}_*(K,-)"] \end{tikzcd}$$

Multiplication on Homotopy Groups

Lecture 8
9th May, 2022

Let M be left module over the ring spectrum R. We define a pairing

$$- \cdot - : \pi_k(R) \times \pi_l(M) \to \pi_{k+l}(M)$$

for $k, l \in \mathbb{Z}$ as follows. Let $f : S^{m+k} \to R_m$, $g : S^{n+l} \to M_n$ represent homotopy classes of elements in $\pi_k(R)$ and $\pi_l(M)$ respectively. We define $f \cdot g$ to be the composite

$$S^{m+k+n+l} \xrightarrow{f\wedge g} R_m \wedge M_n \xrightarrow{\alpha_{m,n}} M_{m+n}$$

and we set

$$[f] \cdot [g] := (-1)^{kn}[f \cdot g].$$

Morally, adding a $(-1)^{kn}$ switches the $m+k+n+l$ coordinates around, to have the $k+l$ at the end.

Most of this lecture will be devoted to study the properties of this pairing (starting from well-definedness), which we collect in the next theorem.

I.19. Theorem. — *Let R be an orthogonal ring spectrum and M a left R-module.*

i) The pairing $-\cdot-$ is well-defined and biadditive.

ii) Let $1 \in \pi_0(R)$ denote the class of the unit $\iota_0 : S^0 \to R_0$, i.e. the map with $0 \mapsto \iota$ and $\infty \mapsto *$. Then, $1 \cdot x = x$ for all $\pi_l(M)$.

iii) For all $x \in \pi_i(R)$, $y \in \pi_k(R)$ and $z \in \pi_m(M)$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

iv) For $M = R$ the pairing $-\cdot-$ makes $\pi_*(R) = \{\pi_k(R)\}_{k\in\mathbb{Z}}$ into a graded ring. If R is commutative, then the graded multiplication satisfies $x \cdot y = (-1)^{kl} y \cdot x$ with $x \in \pi_k(R)$, $y \in \pi_l(R)$.

v) The pairing $-\cdot-$ make $\pi_*(M) = \{\pi_k(M)\}_{k\in\mathbb{Z}}$ into a graded left module over $\pi_*(R)$.

vi) For every morphism $\varphi : M \to N$ of left modules, the induced map $\varphi_* : \pi_k(M) \to \pi_k(N)$ for $k \in \mathbb{Z}$ form a morphism of graded $\pi_*(R)$-modules. In other words, we have enhanced the functor $\pi_* : \mathrm{Sp} \to \mathrm{Ab}$ to a functor

$$\pi_* : {}_R\mathrm{Mod}(\mathrm{Sp}) \to {}_{\pi_*(R)}\mathrm{GrMod}\,.$$

vii) The suspension and loop isomorphisms

$$- \wedge S^1 : \pi_k(M) \to \pi_{k+1}(M \wedge S^1) \text{ and } \alpha : \pi_k(\Omega M) \to \pi_{k+1}(M)$$

are $\pi_*(R)$-linear.

viii) For every morphism $f : M \to N$ of left R-modules, the connecting morphism

$$\delta : \pi_{k+1}(Cf) \to \pi_k(M) \text{ and } \delta : \pi_{k+1}(N) \to \pi_k(Ff)$$

are $\pi_*(R)$-linear.

Proof. (i) We start with well-definedness in $x = [f] \in \pi_k(R)$. The following diagram commutes:

I stole this proof from Qi, credit to him for writing it down, the diagrams are gnarly! It is also a pretty boring proof and I think that once one gets the idea it can safely be skipped.

$$\begin{tikzcd}
S^1 \wedge R_m \wedge M_n \arrow[r, "S^1 \wedge \alpha_{m,n}"] \arrow[d, "\iota_1 \wedge R_m \wedge M_n"] \arrow[dd, bend right=60, "\sigma_m \wedge M_n"'] & S^1 \wedge M_{m+n} \arrow[d, "\iota_1 \wedge M_{m+n}"] \arrow[dd, bend left=60, "\sigma_{m+n}"] \\
R_1 \wedge R_m \wedge M_n \arrow[r, "R_1 \wedge \alpha_{m,n}"] \arrow[d, "\mu_{1,m} \wedge M_n"] & R_1 \wedge M_{m+n} \arrow[d, "\alpha_{1,m+n}"] \\
R_{1+m} \wedge M_n \arrow[r, "\alpha_{1+m,n}"'] & M_{1+m+n}
\end{tikzcd}$$

The upper square commutes on the nose by construction. The lower square is by compatibility of R with M. The left part commutes by unitality in R. The right part commutes by unitality in M. Now, if we replace $f : S^{m+k} \to R_m$ by its stabilization $\sigma_m \circ (S^1 \wedge f) : S^{1+m+k} \to R_{1+m}$ we get

$$\begin{aligned}
(\sigma_m \circ (S^1 \wedge f)) \cdot g &= \alpha_{1+m,n} \circ ((\sigma_m \circ (S^1 \wedge f)) \wedge g) \\
&= \alpha_{1+m,n} \circ (\sigma_m \wedge M_n) \circ (S^1 \wedge f \wedge g) \\
&= \sigma_{m+n} \circ (S^1 \wedge (\alpha_{m,n} \circ (f \wedge g))) \\
&= \sigma_{m+n}(S^1 \wedge (f \cdot g))
\end{aligned}$$

The third equality is by the above diagram. In the homotopy group this also yields $[f \cdot g]$, so it depends only on $[f]$ and not on the representative.

Now we show well-definedness in $[g] \in \pi_\ell(M)$. Consider the diagram:

$$\begin{tikzcd}
S^1 \wedge R_m \wedge M_n \arrow[rr, "\sigma_m \wedge M_n"] \arrow[dd, "\tau \wedge M_n"'] & & R_{1+m} \wedge M_n \arrow[r, "\alpha_{1+m,n}"] \arrow[d, "R(\chi_{1,m}) \wedge M_n"] & M_{1+m+n} \arrow[dd, "M(\chi_{1,m} \oplus \mathbb{R}^n)"] \\
& & R_{m+1} \wedge M_n \arrow[dr, "\alpha_{m+1,n}"] & \\
R_m \wedge S^1 \wedge M_n \arrow[r, "R_m \wedge \iota_1 \wedge M_n"'] \arrow[urr, bend left, "\sigma_m^{\mathrm{op}}"] \arrow[drr, bend right, "R_m \wedge \sigma_n"'] & R_m \wedge R_1 \wedge M_n \arrow[ur, "\mu_{m,1} \wedge M_n"] \arrow[dr, "R_m \wedge \alpha_{1,m}"'] & & M_{m+1+n} \\
& & R_m \wedge M_{1+n} \arrow[ur, "\alpha_{m,1+n}"'] &
\end{tikzcd}$$

The "quadrilateral" on the top left commutes by definition of σ_m^{op}. The quadrilateral on the top right commutes by $(O(1+m) \times O(m))$-equivariance of $\alpha_{1+m,n}$. The quadrilateral on the bottom right is by compatibility of R with M. The bended "triangles" are by unitality in R.

Now, if we replace $g : S^{n+\ell} \to M_n$ by its stabilization $\sigma_n^M \circ (S^1 \wedge g) : S^{1+n+\ell} \to M_{1+n}$ we obtain

$$\begin{aligned} f \cdot (\sigma_n \circ (S^1 \wedge g)) &= \alpha_{m,1+n} \circ (f \wedge (\sigma_n \circ (S^1 \wedge g))) \\ &= \alpha_{m,1+n} \circ (R_m \wedge \sigma_n) \circ (f \wedge S^1 \wedge g) \\ &= M(\chi_{1,m} \oplus \mathbb{R}^n) \circ \alpha_{1+m,n} \circ (\sigma_m \wedge M_n) \circ (\tau_{R_m,S^1} \wedge M_n) \circ (f \wedge S^1 \wedge g) \\ &= M(\chi_{1,m} \oplus \mathbb{R}^n) \circ \alpha_{1+m,n} \circ (\sigma_m \wedge M_n) \circ (S^1 \wedge f \wedge g) \circ (\chi_{m+k,1} \wedge S^{n+\ell}). \end{aligned}$$

The third equality is the diagram and the fourth equality is naturality of the flip map. Finally, on homotopy classes

$$\begin{aligned} [f] \cdot [\sigma_m \circ (S^1 \wedge g)] &= (-1)^{k(1+n)}[f \cdot (\sigma_m \circ (S^1 \wedge g))] \\ &= (-1)^{k(1+n)} \cdot (-1)^m \cdot (-1)^{m+k} \cdot [(\sigma_m \circ (S^1 \wedge f)) \cdot g] \\ &= (-1)^{kn}[f \cdot g] \\ &= [f] \cdot [g] \end{aligned}$$

using AT2Sheet1.1 and AT2Sheet2.1 in the second equality and independency of the choice of f in the third equality.

(ii) For $g : S^{n+\ell} \to M_n$ the composite

$$S^{n+\ell} \cong S^0 \wedge S^{n+\ell} \xrightarrow{\iota_0 \wedge g} R_0 \wedge M_n \xrightarrow{\alpha_{0,n}} M_n$$

is g. So $1 \cdot [g] = [\iota] \cdot [g] = [g]$.

(iii) Let $e : S^{p+j} \to R_p,\ f : S^{m+k} \to R_m,\ g : S^{n+\ell} \to M_n$ be representatives of classes in $\pi_j(R),\ \pi_k(R),\ \pi_\ell(M)$. Then, the following diagram commutes:

$$\begin{tikzcd}
S^{p+j+m+k+n+\ell} \arrow[dr, "e\wedge f\wedge g"] \arrow[drr, bend left, "e\wedge(f\cdot g)"] \arrow[ddrrr, bend left, "e\cdot(f\cdot g)"] \arrow[ddrr, bend right, "(e\cdot f)\wedge g"'] \arrow[ddrrr, bend right=60, "(e\cdot f)\cdot g"'] & & & \\
& R_p \wedge R_m \wedge M_n \arrow[r, "R_p\wedge\alpha_{m,n}"] \arrow[dr, "\mu_{p,m}\wedge M_n"] & R_p \wedge M_{m+n} \arrow[dr, "\alpha_{p,m+n}"] & \\
& & R_{p+m} \wedge M_n \arrow[r, "\alpha_{p+m,n}"'] & M_{p+m+n}
\end{tikzcd}$$

(iv) Graded-commutativity. Let $f : S^{m+k} \to R_m$ and $g : S^{n+\ell} \to R_n$. The following commutes:

$$\begin{tikzcd} S^{m+k+n+\ell} \arrow[r, "f\wedge g"] \arrow[d, "\chi_{m+k,n+\ell}"] \arrow[rr, bend left, "f\cdot g"] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"] \arrow[d, "\tau"] & R_{m+n} \arrow[d, "R(\chi_{m,n})"] \\ S^{n+\ell+m+k} \arrow[r, "g\wedge f"'] \arrow[rr, bend right, "g\cdot f"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

Here, the right square commutes by commutativity of R. So

$$\begin{aligned} [f]\cdot[g] &= (-1)^{kn}[f\cdot g] \\ &= (-1)^{kn}(-1)^{(m+k)(n+\ell)}(-1)^{mn}[g\cdot f] \\ &= (-1)^{m\ell+k\ell}[g\cdot f] \\ &= (-1)^{k\ell}[g]\cdot[f]. \end{aligned}$$

(v) By (iv) we know that $\pi_*(R)$ is a graded ring. In points (i)–(iii) we verified that $\pi_*(M)$ is a graded left module over it.

(vi) Let $\psi : M \to N$ be a homomorphism of left R-modules and consider $f : S^{m+k} \to R_m$ and $g : S^{n+\ell} \to M_n$. Then,

$$\begin{aligned} f\cdot(\psi_n\circ g) &= \alpha_{m,n}\circ(f\wedge(\psi_n\circ g)) \\ &= \alpha_{m,n}\circ(R_m\wedge\psi_n)\circ(f\wedge g) \\ &= \psi_{m+n}\circ\alpha_{m,n}\circ(f\wedge g) \\ &= \psi_{m+n}\circ(f\cdot g) \end{aligned}$$

where the third equality uses that ψ is a homomorphism of R-modules. On homotopy classes

$$[f]\cdot\psi_*[g] = [f]\cdot[\psi_n\circ g] = (-1)^{kn}[f\cdot(\psi_n\circ g)] = (-1)^{kn}[\psi_{m+n}(f\cdot g)] = \psi_*([f]\cdot[g]),$$

which shows linearity.

(vii) Let $f : S^{m+k} \to R_m$ and $g : S^{n+\ell} \to M_n$. Then,

$$\begin{aligned} (f\cdot g)\wedge S^1 &= (\alpha_{m,n}\circ(f\wedge g))\wedge S^1 \\ &= (\alpha^M_{m,n}\wedge S^1)\circ(f\wedge g\wedge S^1) \\ &= \alpha^{M\wedge S^1}_{m,n}\circ(f\wedge(g\wedge S^1)) \\ &= f\cdot(g\wedge S^1) \end{aligned}$$

Note $\alpha^{M\wedge S^1}_{m,n} = \alpha^M_{m,n}\wedge S^1$. On the level of homotopy classes, we get $([f]\cdot[g])\wedge S^1 = [f]\cdot([g]\wedge S^1)$, which is the claim for $-\wedge S^1$. As for α, observe that it factors as the composite

$$\pi_k(\Omega M) \xrightarrow{-\wedge S^1} \pi_{k+1}((\Omega M)\wedge S^1) \xrightarrow{\varepsilon_*} \pi_{k+1}(M)$$

so we can deduce the result for α using (vi).

But we need to check that ε is a module morphism!

(viii) The connecting homomorphism is defined as the composite

$$\pi_{k+1}(Cf) \xrightarrow{p_*} \pi_{k+1}(M\wedge S^1) \xrightarrow{-\wedge S^{-1}} \pi_k(M)$$

check linearity

with $p : Cf = (M \wedge [0,1]) \cup_f N \to M \wedge S^1$. By expanding all definitions, p is R-linear, so p_* is $\pi_*(R)$-linear by (vi). On the other hand, we have seen in (vii) that $- \wedge S^{-1}$ is also $\pi_*(R)$-linear. $\square$

Examples of Ring Spectra

Example (Sphere spectrum)**.** — Let $\mu_{V,W} : S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W}$ be the canonical isomorphism. In AT2Sheet4.4 we prove that the functor ${}_{\mathbb{S}}\mathrm{Mod}(\mathrm{Sp}) \to \mathrm{Sp}$ is an isomorphism of categories. The sphere spectrum $\mathbb{S}$ is a commutative orthogonal ring spectrum and it is the initial ring spectrum.

Example. — Let M be a topological monoid (e.g. any monoid with the discrete topology) and R a ring spectrum. The *monoid ring spectrum* RM is $R \wedge M_+$ with multiplication maps

$$\mu^{RM}_{V,W} : (RM)(V) \wedge (RM)(W) \to RM(V \oplus W) = R(V \oplus W) \wedge M_+$$

given by

$$\begin{tikzcd}
(RM)(V) \wedge (RM)(W) \arrow[r, equal] \arrow[drr, dashed, bend right, "\mu^{RM}_{V,W}"'] & (R(V) \wedge M_+) \wedge (R(W) \wedge M_+) \arrow[r, "\cong"] & (R(V) \wedge R(W)) \wedge (M \times M)_+ \arrow[d, "\mu_{V,W} \wedge \mathrm{mult}_+"] \\
& & R(V \oplus W) \wedge M_+.
\end{tikzcd}$$

Here, observing that $K_+ \wedge L_+ \cong (K \times L)_+$,

$$\begin{gathered}
(R(V) \wedge M_+) \wedge (R(W) \wedge M_+) \xrightarrow{\cong} (R(V) \wedge R(W)) \wedge (M \times M)_+, \\
(r \wedge m) \wedge (\overline{r} \wedge \overline{m}) \mapsto (r \wedge \overline{r}) \wedge (m, \overline{m}).
\end{gathered}$$

If ι is the unit map for R, then the unit maps for RM are given by

$$S^V \to (RM)(V) = R(V) \wedge M_+, \ v \mapsto \iota_V(v) \wedge 1.$$

Special case: Set $R = \mathbb{S}$, then $\mathbb{S}M$ is the *spherical monoid ring*. Its underlying orthogonal spectrum is $\Sigma^\infty_+ M = \Sigma^\infty(M_+)$.

Exercise. — There is an isomorphism of categories

$${}_{\mathbb{S}M}\mathrm{Mod}(\mathrm{Sp}) \cong (\mathrm{Sp}, \text{continuous action of } M \text{ by endomorphisms}).$$

Example (Eilenberg-MacLane spectra)**.** — Let A be a ring and M be a left A-module (in the cowardly old) sense. Then, HA acts on HM by

$$\alpha_{V,W} : (HA)(V) \wedge (HM)(W) \to (HM)(V \oplus W)$$

given by

$$A\left[S^V\right] \wedge M\left[S^W\right] \to M\left[S^{V \oplus W}\right], \ \sum_i a_i v_i \wedge \sum_j m_j w_j \to \sum_{i,j} (a_i \cdot m_j) \cdot (v_i \wedge w_j)$$

and unit maps

$$\iota_V : S^V \to (HA)(V) = A\left[S^V\right],\ v \to 1 \cdot v.$$

This gives a functor

$$H : \mathrm{Ring} \to \mathrm{Sp},\ H_A : {}_A\mathrm{Mod} \to {}_{HA}\mathrm{Mod}(\mathrm{Sp}).$$

We sneaked in all of algebra in homotopy theory.

Lecture 9
11th May, 2022

Example. — The *opposite ring spectrum* R^{op} of an orthogonal ring spectrum R has the same underlying orthogonal spectrum and unit maps, but we define a new multiplication

$$\mu^{\mathrm{op}}_{V,W} R^{\mathrm{op}}(V) \wedge R^{\mathrm{op}}(W) \to R^{\mathrm{op}}(V \oplus W)$$

by the composite

$$R(V) \wedge R(W) \cong R(W) \wedge R(V) \xrightarrow{\mu^R_{W,V}} R(W \oplus V) \xrightarrow[R(\tau_{W,V})]{\cong} R(V \oplus W)$$

Observe that R is commutative if and only if $R = R^{\mathrm{op}}$. In AT2Sheet5.3 we will show that $\pi_*(R^{\mathrm{op}}) = (\pi_*(R))^{\mathrm{op}}$, where the opposite multiplication for graded ring is defined by

$$x \cdot_{\mathrm{op}} y = (-1)^{|x||y|} y \cdot x.$$

Matrix ring spectra. — Let R be an orthogonal ring spectra. We define an orthogonal spectrum

$$M_m(R) = \mathrm{map}_*(m_+, R \wedge m_+),$$

where $m_+ = \{0, 1, \ldots, m\}$, 0 is the basepoint and

$$\mathrm{map}_*(m_+, R \wedge m_+) \cong \prod_{i=1}^{m} \bigvee_{j=1}^{m} R$$

The Professor says this example is related to the first constructions one usually makes in algebraic K-theory, which are typically based on $\mathrm{GL}_n(k)$.

The multiplication maps $M_m(R)(V) \wedge M_m(R)(W) \to M_m(R)(V \oplus W)$ are given by

$$f \wedge g \mapsto \left(m_+ \xrightarrow{f} R(V) \wedge m_+ \xrightarrow{R(V) \wedge g} R(V) \wedge R(W) \wedge m_+ \xrightarrow{\mu_{V,W} \wedge m_+} R(V \oplus W) \wedge m_+ \right).$$

We will see in AT2Sheet5.4 that $\pi_*(M_m(R)) \cong M_m(\pi_*(R))$.

Remark. — Ideals and quotients do not generally translate easily to spectra.

Now an (important) example from geometric topology.

Example (MO and MOP)**.** — Recall that the Thom spectrum MO was defined by

$$\begin{aligned}\mathrm{MO}(V) &= \mathrm{Th}\left(\text{tautological } \dim(V)\text{-plane bundle over } \mathrm{Gr}_{\dim V}(V^\infty)\right) \\ &\cong L(\mathbb{R}^k, V^\infty)_+ \wedge_{O(k)} S^k\end{aligned}$$

with $k = \dim V$. This becomes a commutative orthogonal ring spectrum via the multiplication maps

$$\mathrm{MO}(V) \wedge \mathrm{MO}(W) \to \mathrm{MO}(V \oplus W),\ (v, L) \wedge (w, L') \mapsto (\kappa_{V,W}(v, w), \kappa_{V,W}(L \oplus L')),$$

where

$$\kappa_{V,W} : V^\infty \oplus W^\infty \to (V \oplus W)^\infty,\ ((v_1, v_2, \cdots), (w_1, w_2, \cdots)) \mapsto ((v_1, w_1), (v_2, w_2), \cdots).$$

The unit map is

$$S^V \to \mathrm{MO}(V),\ v \mapsto ((v, 0, 0, \cdots), V \oplus 0^\infty).$$

There would be a bunch of stuff to check now but we'll have faith. Hopefully later we will show:

Theorem. — *There is an isomorphism of graded rings $\pi_*(\mathrm{MO}) = \mathrm{MO}_* \cong \Omega_*$ where Ω_* is the ring of bordism classes of smooth closed manifolds.*

The *periodic Thom spectrum* MOP is a $\mathbb{Z}$-graded commutative orthogonal ring spectrum defined as follows:

$$\begin{aligned}
\mathrm{MOP}(V) &= \mathrm{Th}\left(\text{tautological vector bundle over } \mathrm{Gr}(V^\infty) = \coprod_{n\geqslant 0} \mathrm{Gr}_k(V^\infty)\right) \\
&= \bigvee_{k\geqslant 0} \mathrm{Th}\left(\text{tautological bundle over } \mathrm{Gr}_k(V^\infty)\right) \\
&= \bigvee_{k\geqslant 0} L(\mathbb{R}^k, V^\infty)_+ \wedge_{O(k)} S^k.
\end{aligned}$$

The multiplication and unit maps make just as much sense for MOP as for MO. The spectrum MOP is $\mathbb{Z}$-graded as follows: a subspectrum $\mathrm{MOP}^{[m]} \subseteq \mathrm{MOP}$ is defined by

$$\mathrm{MOP}^{[m]}(V) = L(\mathbb{R}^{\dim(V)+m}, V^\infty)_+ \wedge_{O(\dim(V)+m)} S^{\dim(V)+m}.$$

In particular, $\mathrm{MO} = \mathrm{MOP}^{[0]}$. Then, $\mathrm{MOP} \cong \bigvee_{m\in\mathbb{Z}} \mathrm{MOP}^{[m]}$. The multiplication is graded

$$\begin{tikzcd}
\mathrm{MOP}^{[m]}(V) \wedge \mathrm{MOP}^{[n]}(W) \arrow[r, hook] \arrow[d] & \mathrm{MOP}(V) \wedge \mathrm{MOP}(W) \arrow[d, "\mu_{V,W}"] \\
\mathrm{MOP}^{[m+n]}(V \oplus W) \arrow[r, hook] & \mathrm{MOP}(V \oplus W)
\end{tikzcd}$$

We define $t \in \pi_{-1}\left(\mathrm{MOP}^{[-1]}\right) \subseteq \pi_{-1}(\mathrm{MOP})$ as the class represented by

$$\{0, \infty\} = S^0 \to \mathrm{MOP}^{[-1]}(\mathbb{R}) = \left(\mathrm{MOP}^{[-1]}\right)_1,\ 0 \mapsto (0, \{0\}).$$

We define $\sigma \in \pi_1\left(\mathrm{MOP}^{[1]}\right) \subseteq \pi_1(\mathrm{MOP})$ represented by

$$S^2 \to \mathrm{MOP}^{[1]}(\mathbb{R}) = \left(\mathrm{MOP}^{[1]}\right)_1,\ x \mapsto \left((x, 0, 0, \cdots), \mathbb{R}^2 \oplus 0^\infty\right).$$

I.20. Theorem. — *Let t, σ be defined as above.*

(i) In $\pi_0(\mathrm{MO}) \subseteq \pi_0(\mathrm{MOP})$ we have $t \cdot \sigma = 1$.

(ii) The relation $2 = 0$ holds in $\pi_*(\mathrm{MO})$ and $\pi_*(\mathrm{MOP})$ and all these groups are $\mathbb{F}_2$-vector spaces.

(iii) Let $m \in \mathbb{Z}$.

(a) For $m \geqslant 0$, the spectrum $\mathrm{MOP}^{[m]}$ is stably equivalent to $\mathrm{MO} \wedge S^m$.

(b) For $m \leqslant 0$, the spectrum $\mathrm{MOP}^{[m]}$ is stably equivalent to $\Omega^{-m}(\mathrm{MO})$.

Proof. (i) The class $t \cdot \sigma$ is represented by the composite

$$S^2 = S^{0+2} \to \mathrm{MOP}(\mathbb{R}) \wedge \mathrm{MOP}(\mathbb{R}) \xrightarrow{\mu_{1,1}} \mathrm{MOP}(\mathbb{R}^2) = \mathrm{Th}(\mathrm{Gr}((\mathbb{R} \oplus \mathbb{R})^\infty))$$

given by

$$(x_1, x_2) = x \mapsto (0, \{0\}) \wedge (x, \mathbb{R}^2) \mapsto (((0, x_1), (0, x_2), (0, 0), \cdots), \kappa_{\mathbb{R},\mathbb{R}}(0^\infty \oplus (\mathbb{R}^2 \oplus 0^\infty)).$$

The unit is represented by the map

$$\iota_2 : S^2 \to \mathrm{MOP}(\mathbb{R}^2),\ (x_1, x_2) \mapsto (((x_1, x_2), (0, 0), (0, 0, \cdots)), \mathbb{R}^2 \oplus 0^\infty).$$

Choose a path $\omega : [0, 1] \to L(\mathbb{R}^2, (\mathbb{R}^2)^\infty)$ between the two linear isometric embeddings. This induces a homotopy between the representatives for $t \cdot \sigma$ and ι_2 by

$$f_t : S^2 \to \mathrm{MOP}(\mathbb{R}^2),\ x \mapsto (\omega_t(x), \omega_t(\mathbb{R}^2 \oplus 0^\infty)).$$

So $t \cdot \sigma = 1$ in $\pi_0(\mathrm{MOP})$.

(ii) Let R be any commutative ring spectrum and $t \in \pi_k(R)$ be invertible with k odd. Such t exists by (i). Then,

$$t^2 = (-1)^{k \cdot k} t^2 = -t^2 \implies 2t^2 = 0 \implies 2 = 0$$

by graded-commutativity.

(iii) Since $t \in \pi_{-1}\left(\mathrm{MOP}^{[-1]}\right)$ is invertible, the map

$$t \cdot - : \pi_{k+1}\left(\mathrm{MOP}^{[m+1]}\right) \to \pi_k\left(\mathrm{MOP}^{[m]}\right)$$

is bijective. We now show that this map is realized by a stable equivalence of orthogonal spectra. Consider the following map:

$$\begin{tikzcd}
\mathrm{MOP} \arrow[r, "j"] & \mathrm{sh}(\mathrm{MOP}) \\
\mathrm{MOP}^{[m+1]} \arrow[u, hook] \arrow[r] & \mathrm{sh}(\mathrm{MOP}^{[m]}) \arrow[u, hook]
\end{tikzcd}$$

The value of j at V is

$$\begin{aligned} j_V : \mathrm{MOP}(V) &\to (\mathrm{sh}\,\mathrm{MOP})(V) = \mathrm{MOP}(V \oplus \mathbb{R}) \\ (v, L) &\mapsto (i_V^\infty(v), i_V^\infty(L)) \end{aligned}$$

where $\iota_V : V \to V \oplus \mathbb{R},\ v \mapsto (v, 0)$. Now, by direct inspection the map

$$\mathrm{MOP}(V) \xrightarrow{- \wedge (0, \{0\})} \mathrm{MOP}(V) \wedge \mathrm{MOP}(\mathbb{R}) \xrightarrow{\mu_{V,\mathbb{R}}} \mathrm{MOP}(V \oplus \mathbb{R})$$

equals j_V. Thus,

$$j_* = (t \cdot -) : \pi_{k+1}\left(\mathrm{MOP}^{[m+1]}\right) \to \pi_{k+1}\left(\mathrm{sh}\,\mathrm{MOP}^{[m]}\right) = \pi_k\left(\mathrm{MOP}^{[m]}\right),$$

so j is a stable equivalence because $t \cdot -$ is bijective. We get

$$\mathrm{MOP}^{[m+1]} \xrightarrow{j} \mathrm{sh}\,\mathrm{MOP}^{[m]} \xleftarrow{\lambda_{\mathrm{MOP}^{[m]}}} \mathrm{MOP}^{[m]} \wedge S^1$$

which is a zig-zag of stable equivalences (Theorem I.9). The m-fold application of this yields the stable equivalence of $\mathrm{MOP}^{[m]}$ and $\mathrm{MO} \wedge S^m$.

For $m \leqslant 0$ we have that $\mathrm{MO} = \mathrm{MOP}^{[0]}$ is stably equivalent to $\mathrm{MOP}^{[-m]} \wedge S^{-m}$, so

$$\Omega^{-m}(\mathrm{MO}) \simeq \Omega^{-m}\left(\mathrm{MOP}^{[-m]} \wedge S^{-m}\right) \simeq \mathrm{MOP}^{[-m]}.$$

because Ω preserves stable equivalences (Corollary I.8) and because Ω and Σ shift homotopy groups in opposite directions (Theorem I.7). $\square$

Chapter II.

The Stable Homotopy Category

II

Now that we have a suitable class of spectra (as we already mentioned, there are others, but they all give rise to equivalent stable homotopy categories) we can turn to the construction of the stable homotopy category. This is the localization of the class of spectra at the stable equivalences, mimicking the construction of the classical homotopy category (and in fact it should be thought of as the "stabilization" of the latter), but in the previous years teaching this course the Professor noticed that this construction can be easily adapted to the more general setting of derived categories of ring spectra and in this class we will take this approach.

Definition. — The *derived category* $D(R)$ of an orthogonal ring spectrum R is the localization of the category of R-modules at the class of weak equivalences

$$_R\mathrm{Mod}(\mathrm{Sp})[(\text{stable equivalence})^{-1}].$$

Remark. — As we already remarked last semester localizations of categories always exists, but in principle only after choosing a set theoretic framework to work within (usually this would be ZFC with Grothendieck universes) and then fiddling a little with it.

Definition. — The *stable homotopy category* is

$$\mathcal{SH} = D(\mathbb{S}) = \mathrm{Sp}[(\text{stable equivalences})^{-1}].$$

The approach we will follow is illustrated by the following diagram

$$\begin{tikzcd}
\mathrm{Ring} \arrow[r, "H"] \arrow[dddr, bend right, "A\mapsto D(A)\cong D(HA)"'] & \mathrm{Ring}(\mathrm{Sp}) \arrow[d, "R\mapsto {}_R\mathrm{Mod}(\mathrm{Sp})"] \\
 & (\text{stable cofibration categories}) \arrow[d, dotted] \arrow[dd, bend left=60] \\
 & (\text{stable } \infty\text{-categories}) \arrow[d, dotted] \\
 & (\text{triangulated categories})
\end{tikzcd}$$

In particular, the passage from ring spectra to triangulated categories is by taking the derived category of a ring spectrum the way we defined it above. This process usually involves introducing model categories (and their homotopy categories). Instead of model categories, we take the shortcut of cofibration categories, a weaker notion which is easier to set up. It is probably not possible not to include any model category type nonsense, though.

We are skipping the stable ∞-category part but we should keep in mind that it is there.

Cofibration Categories

Lecture 10
16th May, 2022

Definition. — A *cofibration category* is a triple $(\mathcal{C}, \mathrm{Cof}, W)$ consisting of a category $\mathcal{C}$ and two distinguished classes of morphisms the cofibrations Cof and the weak equivalences W, satisfying the following axioms.

This notion is clearly modeled after the one of model category, and was introduced (as fibration categories, by Ken Brown) when model categories had already an established theory; we consider cofibration categories because they are simpler, but the notion is of interest in its own right.

(C1) Isomorphisms are cofibrations and weak equivalences. Cofibrations are closed under composition. There is an initial object $\varnothing$ and all initial morphisms are cofibrations.

(C2) If g and f are composable and two out of g, f and gf are weak equivalences, then so is the third.

(C3) Given a cofibration $i : A \to B$ and any morphism $f : A \to C$, there is a pushout in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & C \arrow[d, "j"] \\ B \arrow[r] & D \end{tikzcd}$$

and j is again a cofibration. If in addition, i is a weak equivalence, then so is j.

(C4) Every morphism in $\mathcal{C}$ can be factored as a composite of a cofibration followed by a weak equivalence.

An *acyclic cofibration* (or *trivial* cofibration) is a morphism that is both a cofibration and a weak equivalence. Cofibration are denoted by $a \rightarrowtail b$, weak equivalences by $a \xrightarrow{\sim} b$ and acyclic cofibrations by $a \overset{\sim}{\rightarrowtail} b$.

An immediate consequence of the definitions (in particular, of (C1) and (C3)) is that finite coproducts exist in $\mathcal{C}$. Moreover, the two canonical morphisms

$$A \to A \amalg B \leftarrow B$$

are cofibrations. We can see that the coproduct of two cofibrations $i : A \to B$ and $i' : A' \to B'$ is a cofibration, by applying (C3) twice and then (C2):

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ A \amalg A' \arrow[r, "i \amalg A'"] & B \amalg A' \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "i'"] \arrow[d] & B' \arrow[d] \\ B \amalg A' \arrow[r, "B \amalg i'"] & B \amalg B' \end{tikzcd}$$

$$\begin{tikzcd} A \amalg A' \arrow[rr, "i \amalg i'"] \arrow[dr, "i \amalg A'"'] & & B \amalg B' \\ & B \amalg A' \arrow[ur, "B \amalg i'"'] & \end{tikzcd}$$

Similarly, the coproduct of two acyclic cofibrations is an acyclic cofibration.

The *homotopy category* of a cofibration is the localization

Note that this depends only on W and not on Cof.

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$$

at the class W.

Definition. — A morphism of orthogonal spectra $f : A \to B$ is a h-cofibration (of spectra) if it has the homotopy extension property: for every morphism of orthogonal spectra $\varphi : B \to X$ and every homotopy $H : A \wedge [0,1]_+ \to X$ starting with $\varphi \circ f : A \to X$, there is a homotopy $\bar{H} : B \wedge [0,1]_+ \to X$ that extends H and begins with φ, i.e. such that

$$\bar{H} \circ (f \wedge [0,1]_+) = H \text{ and } \bar{H} \circ (- \wedge 0) = \varphi.$$

In particular, h-cofibrations of orthogonal spectra are levelwise h-cofibrations of spaces.

There is a universal test case, i.e. when $X = M_f = (A \wedge [0,1]_+) \cup_f B$, the mapping cone of f. In this case we have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "-\wedge 0"'] & B \arrow[d] \arrow[ddrr, bend left, "\varphi"] & & \\
A \wedge [0,1]_+ \arrow[r] \arrow[drrr, bend right] & (A \wedge [0,1]_+) \cup_f B = M_f \arrow[dr] & & \\
& & B \wedge [0,1]_+ \arrow[r, dashed, "\bar{H}"] & X
\end{tikzcd}$$

so f is a h-cofibration if and only if the canonical morphism

$$A \wedge [0,1]_+ \cup_f B \to B \wedge [0,1]_+$$

has a retraction.

There is an adjoint form of the HEP. Any given homotopy extension data (φ, H) adjoins to a commutative square

$$\begin{tikzcd}
A \arrow[r, "\tilde{H}"] \arrow[d, "f"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\
B \arrow[r, "\varphi"'] \arrow[ur, dashed, "\tilde{H}'"] & X
\end{tikzcd}$$

The h-cofibrations are precisely the class of morphism that have the left lifting property against all evaluations $\mathrm{ev}_0 : X^{[0,1]} \to X$ for all $X \in \mathrm{Sp}$.

Exercise (AT2Sheet6.1)**.** — There is an important general nonsense fact: let $\mathcal{E}$ be a class of morphism in a category $\mathcal{C}$. Let ${}^{\perp}\mathcal{E}$ be the class of morphism with LLP against all morphisms in $\mathcal{E}$. Then ${}^{\perp}\mathcal{E}$ is closed under

- cobase change,
- sequential composition,
- composition,
- retract.

In particular, the h-cofibrations of orthogonal spectra have all these closure properties.

We can now put a cofibration structure on modules over a ring spectra.

II.1. Theorem. — *The following data defines a cofibration structure on the category of left modules over an orthogonal ring spectrum:*

- *weak equivalences are stable equivalences of underlying orthogonal spectra,*
- *cofibrations are h-cofibrations of underlying orthogonal spectra.*

Proof. We just have to go diligently through the axioms.

See AT2Sheet6.2 (and 6.3) for other (sometimes) interesting examples of (co)fibration categories.

(C1) All isomorphisms are clearly cofibrations and weak equivalences. There is an initial object and it is easy to see that the morphism $* \to X$ is an h-cofibration. Finally, h-cofibrations are stable under composition by AT2Sheet6.1.

(C2) Clearly holds.

(C3) Cofibrations are stable under cobase change again by AT2Sheet6.1 and acyclic cofibrations are stable under cobase change by proposition I.18.

The factorization property is usually much more difficult to prove for model structures and generally relies on the small object argument.

(C4) Let $f : X \to Y$ be a morphism of R-modules. We factor it through the mapping cylinder as the composite of the mapping cylinder inclusion $- \wedge 0 : x \to (X \wedge [0,1]_+) \cup_f Y$ and the projection $(X \wedge [0,1]_+) \cup_f Y \to Y$. This projection is a homotopy equivalence of orthogonal spectra, hence a stable equivalence. We consider now the pushout square

$$\begin{tikzcd}
X \coprod X \arrow[r, tail, "\mathrm{id}_X \amalg f"] \arrow[d, tail, "(-\wedge 0)\amalg(-\wedge 1)"'] & X \coprod Y \arrow[d, tail] \\
X \wedge [0,1]_+ \arrow[r] & (X \wedge [0,1]_+) \cup_f Y
\end{tikzcd}$$

The left vertical morphism is an h-cofibration, hence so is the right vertical morphism. Since the inclusion $X \mapsto X \coprod Y$ is a cofibration, we can conclude that the mapping cylinder inclusion also is $- \wedge 0 : x \to (X \wedge [0,1]_+) \cup_f Y$. □

Remark. — The stable equivalences of R-modules can be complemented by various different classes of cofibrations into cofibration categories, e.g.

- HEP internal to R-modules,
- cofibrations in some model category structure.

Overall, Professor Schwede is of the opinion that the approach we are taking is the quickest.

II.2. Proposition (Gluing lemma). — *Let $\mathcal{C}$ be a cofibration category and*

$$\begin{tikzcd}
A \arrow[d, "\sim"] & B \arrow[l, tail, "i"'] \arrow[r, "f"] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\
A' & B' \arrow[l, tail, "i'"] \arrow[r, "f'"'] & C
\end{tikzcd}$$

such that i and i' are cofibrations and all the vertical morphisms are weak equivalences. Then the induced morphism

$$A \cup_B C \to A' \cup_{B'} C'$$

is a weak equivalence.

As a special case the diagram

$$\begin{tikzcd} A \arrow[d, equal] & B \arrow[l, tail, "i"'] \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "f"', "\sim"] \\ A & B \arrow[l, tail, "i"] \arrow[r, "f", "\sim"'] & C \end{tikzcd}$$

yields

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, "f"', "\sim"] & A \arrow[d, "\sim"] \\ C \arrow[r, tail] & A \cup_B C \end{tikzcd}$$

(Not quite a) Proof. This is an important technical lemma, related to homotopy pushouts, but we do not present a proof here (it is a rather dry proof, not very illuminating), a reference is [Rad09, Lemma 1.4.3]. Note that the special case implies that our cofibration categories are always left proper. □

The homotopy relation

Definition. — Let $\mathcal{C}$ be a cofibration category. A *cylinder object* for an object A is a quadruple (I, i_0, i_1, p) where I is a $\mathcal{C}$-object, the diagram

The cylinder object is far from unique!

$$\begin{tikzcd} A \arrow[dr, "i_0"'] \arrow[drr, bend left, "\mathrm{id}"] & & \\ & I \arrow[r, "p"] & A \\ A \arrow[ur, "i_1"] \arrow[urr, bend right, "\mathrm{id}"'] & & \end{tikzcd}$$

commutes, p is a weak equivalence and $i_0 + i_1 : A \amalg A \to I$ is a cofibration (note that i_0 and i_1 are weak equivalences by 2-out-of-3).

Cylinder objects always exist: it suffices to use (C4) to factor the fold map $\nabla = \mathrm{id} + \mathrm{id} : A \amalg A \to A$ into

$$A \amalg A \xrightarrow{i_0+i_1} I \xrightarrow{\sim} A.$$

Definition. — Two morphisms $f, g : A \to Z$ are homotopic if there is a cylinder object (I, i_0, i_1, p) for A and a morphism (the homotopy):

This definition of homotopy has a degree of flexibility because cylinder objects are not unique.

$$H : I \to Z$$

such that $H \circ i_0 = f$ and $H \circ i_1 = g$. In this case we write $f \simeq g$.

By the usual argument, the localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ takes the same value on homotopic morphisms.

Example. — Considering the cofibration structure on ${}_R\mathrm{Mod}(\mathrm{Sp})$ given in theorem II.1, the following is a cylinder object

$$M \vee M = M \wedge \{0,1\}_+ \rightarrowtail M \wedge [0,1]_+ \xrightarrow{p} M$$

where the first map is a h-cofibration and p is the map collapsing $[0,1)$ to $*$, which is a stable equivalence. So the morphisms in ${}_R\mathrm{Mod}(\mathrm{Sp})$ that are homotopic in the classical/concrete sense (via $M \wedge [0,1]_+$) are also homotopic in the abstract sense. But note that the converse is not true! Abstract homotopy is strictly more general than concrete homotopy.

II.3. Proposition. — *Let A and Z be objects in a cofibration category $\mathcal{C}$.*

i) "Being homotopic" is an equivalence relation on $\mathcal{C}(A,Z)$.

ii) Postcomposition with any morphism $\varphi : Z \to Z'$ preserves the homotopy relation.

iii) Let $f,g : A \to Z$ be homotopic and let $\varphi : \bar{A} \to A$ be any morphism. Then there is an acyclic cofibration $s : Z \xrightarrow{\sim} Z'$ such that $sf\varphi \simeq sg\varphi$ are homotopic.

iv) Let $f,g : A \to Z$ and $\tau : \bar{A} \to A$ a weak equivalence. If $f\tau \simeq g\tau : \bar{A} \to Z$, then $f \sim g$.

Proof. (i) Reflexivity. Let (I, i_0, i_1, p) be a cylinder object for A. Then set

$$H = f \circ p.$$

We have

$$Hi_0 = fpi_0 = f = fpi_1 = Hi_1,$$

so H is a homotopy from f to f.

Symmetry. Let $H : I \to Z$ be a homotopy from f to g based on (I, i_0, i_1, p). Then the same $H : I \to Z$ is a homotopy from g to f based on (I, i_1, i_0, p).

Transitivity. This one is quite annoying (but the ideas are the "obvious ones"). Let (I, i_0, i_1, p) and (J, j_0, j_1, q) be two cylinder objects for A.

Claim. We choose a pushout (C3)

$$\begin{tikzcd}
A \arrow[r, tail, "j_0"] \arrow[d, tail, "i_1"'] & J \arrow[d, "b"] \\
I \arrow[r, "a"'] & I \cup_A J
\end{tikzcd}$$

and put $l_0 = a \circ i_0$ and $l_1 = b \circ j_1$. Moreover, there is a unique morphism $r : I \cup_A J \to A$ such that $ra = p$ and $rb = q$.

$$\begin{tikzcd}
A \arrow[r, "j_0"] \arrow[d, "i_1"'] & J \arrow[d, "b"] \arrow[ddr, bend left, "q"] & \\
I \arrow[r, "a"'] \arrow[drr, bend right, "p"'] & I \cup_A J \arrow[dr, dashed, "r"] & \\
& & A
\end{tikzcd}$$

Then, $(I \cup_A J, l_0, l_1, r)$ is a cylinder object.

Proof of the claim. We first compute

$$rl_0 = rai_0 = pi_0 = \mathrm{id}_A$$

and similarly $rl_1 = \mathrm{id}_A$. Next, we note that j_0, i_1 are weak equivalences by 2-out-of-3. So the remaining arrows a, b are also acyclic cofibrations by (C3). Since $a : I \to I \cup_A J$ and $p : I \to A$ are weak equivalences, also r is a weak equivalence by 2-out-of-3. Furthermore, the following is a pushout:

$$\begin{tikzcd}
 & A \amalg A \amalg A \amalg A \arrow[r, tail, "i_0+i_1+j_0+j_1"] \arrow[d, "A\amalg\nabla\amalg A"'] & I \amalg J \arrow[d] \\
A \amalg A \arrow[r, tail, "0\&1"'] \arrow[rr, bend right, "l_0+l_1"'] & A \amalg A \amalg A \arrow[r, "l_0+l_{1/2}+l_1"'] & I \cup_A J
\end{tikzcd}$$

Here, $l_{1/2}$ is some map that we don't further care about. Note that the top map and the lower left map are cofibrations by (C3) and the fact that coproducts of cofibrations are cofibrations. By (C3) also the right lower map is a cofibration and thus so is the composition $l_0 + l_1$. We have verified that our construction is indeed a cylinder. ∎

Now let $H : I \to Z$ based on (I, i_0, i_1, p) be a homotopy from f to g and let $K : J \to Z$ based on (J, j_0, j_1, q) be a homotopy from g to h. Then, there is a unique morphism

$$\begin{tikzcd}
A \arrow[r, "j_0"] \arrow[d, "i_1"'] & J \arrow[d, "b"] \arrow[ddr, bend left, "K"] & \\
I \arrow[r, "a"'] \arrow[drr, bend right, "H"'] & I \cup_A J \arrow[dr, dashed, "H\cup K"'] & \\
 & & Z
\end{tikzcd}$$

The pushout condition is satisfied because H ends at the same map that K starts in. We get

$$(H \cup K)l_0 = (H \cup K) \circ a \circ i_0 = H \circ i_0 = f$$

and

$$(H \cup K)l_1 = (H \cup K) \circ b \circ j_1 = K \circ j_1 = h,$$

so $f \simeq h$. *To be continued...*

Lecture 11
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(ii) Let $H : I \to Z$ be a homotopy based on (I, i_0, i_1, p) from f to g. Then $\psi \circ H : I \to Z'$ is a homotopy based on the same cylinder from ψf to ψg.

(iii) This one is also pretty annoying. let (I, i_0, i_1, p) and (J, j_0, j_1, q) be any cylinder objects for A and $\overline{A}$. The left vertical morphism in

$$\begin{tikzcd}
\overline{A} \amalg \overline{A} \arrow[r, "i_0\varphi+i_1\varphi"] \arrow[d, tail, "j_0+j_1"'] & I \arrow[d, "p"] \\
J \arrow[r, "\varphi q"'] & A
\end{tikzcd}$$

is a cofibration by definition of cylinder objects. So there is the pushout $J \cup_{\overline{A}\amalg\overline{A}} I$ exists by (C3) and we get a map

$$(\varphi q) \cup p : J \cup_{\overline{A}\amalg\overline{A}} I \to A.$$

We factor this through a cofibration

$$\overline{\varphi} \cup t : J \cup_{\overline{A}\amalg\overline{A}} I \rightarrowtail I'$$

and a weak equivalence

$$p' : I' \xrightarrow{\sim} A.$$

via (C4). We set

$$i'_0 = ti_0, i'_1 = ti_1 : A \to I'.$$

Then, we have a commutative diagram

$$\begin{tikzcd}
\overline{A} \amalg \overline{A} \arrow[rr, "\varphi \amalg \varphi"] \arrow[d, tail, "j_0+j_1"'] & & A \amalg A \arrow[dl, tail, "i'_0+i'_1"'] \arrow[dr, "i_0+i_1"] & \\
J \arrow[r, "\overline{\varphi}"] \arrow[d, "q{,}\sim"'] & I' \arrow[dr, "p'{,}\sim"'] & & I \arrow[ll, tail, "t"'] \arrow[dl, "p{,}\sim"] \\
\overline{A} \arrow[rr, "\varphi"'] & & A &
\end{tikzcd}$$

Claim. The quadruple (I', i'_0, i'_1, p') is another cylinder object for A.

$$A \amalg A \xrightarrow{i'_0+i'_1} I' \xrightarrow{p'} A$$

Proof of the claim. Because p and p' are weak equivalences, so is t by 2-out-of-3. Because $j_0 + j_1$ is a cofibration, so is its cobase change $I \to J \cup_{\overline{A}\amalg\overline{A}} I$ by (C3). Since $\overline{\varphi} \cup t$ is a cofibration by definition, so is the composite

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[rr, bend right, "t"'] & J \cup_{\overline{A}\amalg\overline{A}} I \arrow[r, tail, "\overline{\varphi}\cup t"] & I',
\end{tikzcd}$$

so t is a cofibration. Thus,

$$\begin{tikzcd}
A \amalg A \arrow[r, tail, "i_0+i_1"] \arrow[rr, tail, bend right, "i'_0+i'_1"'] & I \arrow[r, tail, "t"] & I'
\end{tikzcd}$$

is also a cofibration. Moreover, $p'i'_k = pi_k = \mathrm{id}_A$ by the commutativity of our diagram. ∎

Our situation is

$$\begin{tikzcd}
\overline{A} \arrow[r, "\varphi"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z \arrow[r, tail, dashed, "\exists s{,}\sim"] & Z'.
\end{tikzcd}$$

Let $H : I \to Z$ be a homotopy from f to g, so $Hi_0 = f$ and $Hi_1 = g$. We choose a pushout

$$\begin{tikzcd}
I \arrow[r, tail, "t{,}\sim"] \arrow[d, "H"'] & I' \arrow[d, "K"] \\
Z \arrow[r, tail, "s{,}\sim"'] & Z'.
\end{tikzcd}$$

Here, s is an acyclic cofibration as the cobase change of an acyclic cofibration (C3). Take our $\overline{\varphi} : J \to I'$ from above. Then,

$$K\overline{\varphi}j_0 = Ki'_0\varphi = Kti_0\varphi = sHi_0\varphi = sf\varphi.$$

Similarly $K\overline{\varphi}j_1 = sg\varphi$, so $K\overline{\varphi} : J \to Z'$ is the desired homotopy.

(iv) This one is a bit less annoying than the previous, but still annoying. We are given $f, g : A \to Z$ and a weak equivalence $\tau : \overline{A} \to A$ such that $f\tau \simeq g\tau$. Let (J, j_0, j_1, q) be a cylinder for $\overline{A}$ and $H : J \to Z$ be a homotopy from $f\tau = Hj_0$ to $g\tau = Hj_1$. We choose a pushout

$$\begin{tikzcd} \overline{A} \amalg \overline{A} \arrow[r, tail, "j_0+j_1"] \arrow[d, "\tau\amalg\tau"'] & J \arrow[d, "\overline{\tau}"] \\ A \amalg A \arrow[r, "i_0+i_1"'] & I \end{tikzcd}$$

where $\tau \amalg \tau$ is a weak equivalence by the Gluing Lemma II.2 and also $\overline{\tau}$ is by the Gluing Lemma. Now, consider the diagram

$$\begin{tikzcd} \overline{A} \amalg \overline{A} \arrow[r, tail, "j_0+j_1"] \arrow[d, "\tau\amalg\tau"'] & J \arrow[d, "{\overline{\tau},\sim}"] \arrow[ddr, bend left, "{\tau q,\sim}"] & \\ A \amalg A \arrow[r, "i_0+i_1"'] \arrow[drr, bend right, "\nabla"'] & I \arrow[dr, dashed, "p"] & \\ & & A \end{tikzcd}$$

which shows that (I, i_0, i_1, p) is a cylinder object for A. We are using $qj_k = \mathrm{id}_{\overline{A}}$ to see that this is a cocone. Since $j_0 + j_1$ is a cofibration, so is its cobase change $i_0 + i_1$ by (C3). Moreover, p is a weak equivalence by 2-out-of-3. Then, the diagram

$$\begin{tikzcd} \overline{A} \amalg \overline{A} \arrow[r, tail, "j_0+j_1"] \arrow[d, "\tau\amalg\tau"'] & J \arrow[d, "{\overline{\tau},\sim}"] \arrow[ddr, bend left, "H"] & \\ A \amalg A \arrow[r, tail, "i_0+i_1"'] \arrow[drr, bend right, "f+g"'] & I \arrow[dr, dashed, "K"] & \\ & & Z \end{tikzcd}$$

shows that K is a homotopy from f to g. The commutativity of the cocone steps from the definition of H. □

Localization of a Cofibration Category

Construction. — Let $\mathcal{C}$ be a cofibration category. Fix $A, B \in \mathrm{Ob}(\mathcal{C})$. We consider a relation on the "set" of pairs (f, τ) with

Quote-unquote because we most likely have to pass to a bigger Grothendieck universe.

- $f : A \to Z$ any morphism of $\mathcal{C}$,
- $\tau : B \xrightarrow{\sim} Z$ an acyclic cofibration,

namely $(f, \tau) \approx (f', \tau')$ when there are acyclic cofibrations

$$a : Z \overset{\sim}{\rightarrowtail} \bar{Z},\ b : Z' \overset{\sim}{\rightarrowtail} \bar{Z}$$

such that $af \simeq bf'$ and $a\tau \simeq b\tau$.

$$\begin{tikzcd} & Z \arrow[d, tail, "a", "\sim"'] & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & \bar{Z} & B \arrow[ul, tail, "\tau"', "\sim"] \arrow[dl, tail, "\tau'", "\sim"'] \\ & Z' \arrow[u, tail, "b"', "\sim"] & \end{tikzcd}$$

II.4. Proposition. — *The relation $\approx$ is an equivalence relation.*

Proof. Reflexivity and symmetry are ok. For transitivity let $(f,\tau) \overset{(a,b)}{\approx} (f',\tau') \overset{(a',b')}{\approx} (f'',\tau'')$. Consider the diagram

$$\begin{tikzcd}
 & Z \arrow[d, tail, "{a,\sim}"'] & \\
 & \overline{Z} & \\
A \arrow[uur, bend left, "f"] \arrow[r, "f'"] \arrow[ddr, bend right, "f''"'] & Z' \arrow[u, tail, "{b,\sim}"] \arrow[d, tail, "{a',\sim}"'] & B \arrow[l, "\tau'"'] \arrow[uul, bend right, "\tau"'] \arrow[ddl, bend left, "\tau''"] \\
 & \overline{Z}' & \\
 & Z'' \arrow[u, tail, "{b',\sim}"] &
\end{tikzcd}$$

Here, the triangles commute up to homotopy. Then, we choose a pushout

$$\begin{tikzcd}
Z' \arrow[r, tail, "{a',\sim}"] \arrow[d, tail, "{b,\sim}"'] & \overline{Z}' \arrow[d, tail, "{\beta,\sim}"] \\
\overline{Z} \arrow[r, tail, "{\alpha,\sim}"'] & E
\end{tikzcd}$$

Here, α and β are acyclic cofibrations by (C3). Homotopy is compatible with postcomposition. Therefore

$$\alpha a f \simeq \alpha b f' = \beta a' f' \simeq \beta b' f''$$

and

$$\alpha a \tau \simeq \alpha b \tau' = \beta a' \tau' \simeq \beta b' \tau''.$$

So the acyclic cofibrations $\alpha a : Z \to E$ and $\beta b' : Z'' \to E$ witness that $(\tau, f) \approx (\tau'', f'')$.

$$\begin{tikzcd}
 & Z \arrow[d, "\alpha a"'] & \\
A \arrow[ur, "f"] \arrow[dr, "f''"'] & E & B \arrow[ul, "\tau"'] \arrow[dl, "\tau''"] \\
 & Z'' \arrow[u, "\beta b'"] &
\end{tikzcd}$$

$\square$

Construction. — Let $\mathcal{C}$ be a cofibration category. We define a category $\mathrm{Ho}(\mathcal{C})$ with the same objects as $\mathcal{C}$. Morphisms $\mathrm{Ho}\,\mathcal{C}(A,B)$ are $\approx$ classes of pairs $(f : A \to B, \tau : B \overset{\sim}{\rightarrowtail} Z)$. We write

$$\tau\backslash f := \gamma(\tau)^{-1} \circ \gamma(f) : A \to B$$

for the equivalence class of $(f\tau)$.

We define composition as follows. Let (f,τ) and (g,σ) represent morphisms $\tau\backslash f : A \to B$ and $\sigma\backslash g : B \to C$. We choose a pushout

$$\begin{tikzcd} & & C \arrow[d, tail, "\sim"', "\sigma"] \\ & B \arrow[r, "g"] \arrow[d, tail, "\sim"', "\tau"] & Y \arrow[d, tail, "\sim"', "\psi"] \\ A \arrow[r, "f"'] & Z \arrow[r, "\varphi"'] & W \end{tikzcd}$$

where ψ is an acyclic cofibration by C3, and we define

$$(\sigma\backslash g) \circ (\tau\backslash f) = (\psi\sigma)\backslash(\varphi f).$$

One of the main motivations for having a theory of model or cofibration categories is that the localization at the class W of the weak equivalences is well-behaved, i.e. it is a category of fractions, meaning that we can express morphisms in the localized category in the simplest possible way (we don't have to deal with zigzags of weak equivalences). The next theorem shows exactly this: the category $\mathrm{Ho}(\mathcal{C})$ we defined is the localization of $\mathcal{C}$ at the weak equivalences, hence the latter is a category of fractions.

II.5. Theorem. — *Let $\mathcal{C}$ be a cofibration category.*

i) Composition is well-defined and makes $\mathrm{Ho}(\mathcal{C})$ *into a category.*

ii) The assignments $\gamma(A) = A$ and $\gamma(f) = \mathrm{id}\backslash f$ define a functor

$$\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C}).$$

iii) For every acyclic cofibration $\tau : B \xrightarrow{\sim} Z$ the morphism $\gamma(\tau)$ is invertible and its inverse is $\tau\backslash \mathrm{id}_Z$. Moreover $\tau\backslash f = \gamma(\tau)^{-1} \circ \gamma(f)$.

iv) The functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ takes weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathrm{Ho}(\mathcal{C})$.

v) The functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is a localization at the weak equivalences.

Proof. Take a deep breath. Let's go.

(i) We start by noticing that if in the definition of composition we choose another pushout, it will be isomorphic to the first one, so

$$(\varphi f, \psi\sigma) \approx (\varphi' f, \psi'\sigma)$$

via an isomorphism and the identity.

The equivalence relation $\approx$ is generated by three elementary relations:

(1) For all acyclic cofibrations $a : Z \xrightarrow{\sim} \bar{Z}$, $(f,\tau) \approx (af, a\tau)$,

(2) For all pairs of homotopic morphisms $f, f' : A \to Z$, $(f,\tau) \approx (f',\tau)$,

(3) For all pairs of homotopic acyclic cofibrations $\tau, \tau' : B \to Z$, $(f,\tau) \approx (f,\tau')$.

$$
\begin{tikzcd}
 & Z \arrow[d, "\sim"', "a"] & \\
A \arrow[ur, "f"] \arrow[dr, "af"'] & \bar{Z} \arrow[d, equal] & B \arrow[ul, "\tau"'] \arrow[dl, "a\tau"] \\
 & \bar{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[d, equal] & \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & Z \arrow[d, equal] & B \arrow[ul, "\tau"'] \arrow[dl, "\tau"] \\
 & Z &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Z \arrow[d, equal] & \\
A \arrow[ur, "f"] \arrow[dr, "f"'] & Z \arrow[d, equal] & B \arrow[ul, "\tau"'] \arrow[dl, "\tau'"] \\
 & Z &
\end{tikzcd}
$$

We will show that postcomposition with a pair $(g : B \to Y, \sigma : C \rightarrowtail Y)$ is compatible with these three elementary relations. Precomposition is similar (and even slightly easier for (3)).

(1) We choose two pushouts (using that pushouts stack)

$$
\begin{tikzcd}
 & & C \arrow[d, tail, "\sigma{,}\sim"] \\
 & B \arrow[r, "g"] \arrow[d, tail, "\tau{,}\sim"'] & Y \arrow[d, tail, "\psi{,}\sim"] \\
A \arrow[r, "f"] \arrow[dr, "af"'] & Z \arrow[r, "\varphi"] \arrow[d, tail, "a{,}\sim"] & W \arrow[d, tail, "\lambda{,}\sim"] \\
 & \overline{Z} \arrow[r, "\kappa"] & V
\end{tikzcd}
$$

Then,

$$
\begin{aligned}
(\sigma \setminus g) \circ ((a\tau) \setminus (af)) &= (\lambda\psi\sigma) \setminus (\kappa(af)) \\
&= (\lambda\psi\sigma) \setminus (\lambda\varphi f) \\
&\overset{(1)}{\sim} (\psi\sigma) \setminus (\varphi f) \\
&= (\sigma \setminus g) \circ (\tau \setminus f).
\end{aligned}
$$

(2) If $f \simeq f'$ with the diagram

$$
\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[d, tail, "\tau{,}\sim"'] & Y \arrow[d, tail, "\psi{,}\sim"] \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Z \arrow[r, "\varphi"'] & W
\end{tikzcd}
$$

then $\varphi f \simeq \varphi f'$ by Proposition II.3, so $(\varphi f, \psi\sigma) \overset{(2)}{\approx} (\varphi f', \psi\sigma)$.

(3) We choose three pushouts

$$
\begin{tikzcd}
 & B \arrow[r, tail, "\tau{,}\sim"] \arrow[d, "g"'] & Z \arrow[d, "\varphi"] \\
B \arrow[r, "g"] \arrow[d, tail, "\tau'"'] & Y \arrow[r, tail, "\psi{,}\sim"] \arrow[d, tail, "\psi'{,}\sim"] & W \arrow[d, tail, "\alpha{,}\sim"] \\
Z \arrow[r, "\varphi'"'] & W' \arrow[r, tail, "\beta{,}\sim"'] & V
\end{tikzcd}
$$

So by the commutativity of the diagram and Proposition II.3 we get

$$\alpha\varphi\tau = \alpha\psi g = \beta\psi' g = \beta\varphi'\tau' \simeq \beta\varphi'\tau.$$

Therefore, $\alpha\varphi \simeq \beta\varphi'$ again by Proposition II.3. By the same Proposition, there is an acyclic cofibration $s : V \to V'$ such that

$$s\alpha\varphi f \simeq s\beta\varphi' f.$$

Moreover, $s\alpha\psi\sigma = s\beta\psi'\sigma : C \to V'$ by the commutativity of the bottom right square. So the acyclic cofibrations $s\alpha : W \to V'$ and $s\beta : W' \to V'$ witness

$$(\varphi f, \psi\sigma) \approx (\varphi' f, \psi'\sigma).$$

To be continued...

Lecture 12
25th May, 2022

Recall that we wanted to show that $\mathrm{Ho}(\mathcal{C})$ is a category. Unitality is clear, so we are left to prove associativity. Let $(e,\nu), (f,\tau)$ and (g,σ) be representatives of three composable morphisms. We choose three pushouts

$$\begin{tikzcd}
 & & & C \arrow[d, tail, "\sigma,\sim"] \\
 & & B \arrow[r, "g"] \arrow[d, tail, "\tau,\sim"'] & Y \arrow[d, tail, "\psi,\sim"] \\
 & A \arrow[r, "f"] \arrow[d, tail, "\nu,\sim"'] & Z \arrow[r, "\varphi"] \arrow[d, tail, "\xi,\sim"'] & W \arrow[d, tail, "\lambda,\sim"] \\
E \arrow[r, "e"'] & X \arrow[r, "\chi"'] & U \arrow[r, "\mu"'] & V
\end{tikzcd}$$

Therefore:

$$\begin{aligned}
(\sigma \setminus g) \circ ((\tau \setminus f) \circ (\nu \setminus e)) &= (\sigma \setminus g) \circ ((\xi\tau) \setminus (\chi e)) \\
&= (\lambda\psi\sigma) \setminus (\mu\chi e) \\
&= ((\psi\sigma) \setminus (\varphi f)) \circ (\nu \setminus e) \\
&= ((\sigma \setminus g) \circ (\tau \setminus f)) \circ (\nu \setminus e).
\end{aligned}$$

(ii) This one is easy. Let f, g be composable morphisms. Write

$$\begin{tikzcd}
 & & Y \arrow[d, equal] \\
 & Z \arrow[r, "g"] \arrow[d, equal] & Y \arrow[d, equal] \\
A \arrow[r, "f"'] & Z \arrow[r, "g"'] & Y
\end{tikzcd}$$

so that

$$\gamma(g) \circ \gamma(f) = (\mathrm{id} \setminus g) \circ (\mathrm{id} \setminus f) = \mathrm{id} \setminus (gf) = \gamma(gf).$$

Moreover, $\gamma(\mathrm{id}) = \mathrm{id} \setminus \mathrm{id} = \mathrm{id}$ by construction.

(iii) This one is a bit longer but not more difficult. Write

$$\begin{tikzcd}
& & B \arrow[d, "\tau"] \\
& Z \arrow[r, equal] \arrow[d, equal] & Z \arrow[d, equal] \\
B \arrow[r, "\tau"'] & Z \arrow[r, equal] & Z
\end{tikzcd}$$

so that

$$(\tau \setminus \mathrm{id}) \circ \gamma(\tau) = (\tau \setminus \mathrm{id}) \circ (\mathrm{id} \setminus \tau) = (\tau \setminus \tau) = \mathrm{id} \setminus \mathrm{id} = \mathrm{id}\,.$$

Moreover,

$$\begin{tikzcd}
& & Z \arrow[d, equal] \\
& B \arrow[r, tail, "{\tau,\sim}"] \arrow[d, tail, "{\tau,\sim}"'] & Z \arrow[d, "\psi_1"] \\
Z \arrow[r, equal] & Z \arrow[r, "\psi_0"'] & Z \cup_B Z
\end{tikzcd}$$

so

$$\gamma(\tau) \circ (\tau \setminus \mathrm{id}) = (\mathrm{id} \setminus \tau) \circ (\tau \setminus \mathrm{id}) = \psi_1 \setminus \psi_0 = \mathrm{id} \setminus \mathrm{id}$$

where $\psi_1\tau = \psi_0\tau$ implies $\psi_1 \simeq \psi_0$ by Proposition II.3, so $(\psi_0, \psi_1) \approx (\mathrm{id}, \mathrm{id})$. We use here that ψ_0 or ψ_1 is an acyclic cofibration by (C3) and can essentially apply (2) and (3) of the elementary instances (or write out another diagram).

(iv) Essentially by definition.

(v) We will use the following fact.

Claim. For all $f : A \to B$ there is a factorization $f = qj$ such that $j : A \to Z$ is a cofibration and $q : Z \to B$ is left inverse to an acyclic cofibration.

Proof of the claim. We mimic the construction of the the mapping cylinder in this more general context. Let (I, i_0, i_1, p) be a cylinder object for A. We chose a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "{i_1,\sim}"'] & B \arrow[d, "r"] \\
I \arrow[r] & I \cup_f B = Z
\end{tikzcd}$$

Note that r is also an acyclic cofibration by (C3). Put

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "{i_1,\sim}"'] & B \arrow[d, "r"] \arrow[ddr, equal] & \\
I \arrow[r, "\varphi"'] \arrow[drr, bend right, "f \circ p"'] & I \cup_f B = Z \arrow[dr, dashed, "q"] & \\
& & B
\end{tikzcd}$$

Put $j = \varphi \circ i_0$, then

$$qj = q\varphi i_0 = fpi_0 = f.$$

We choose another pushout

$$\begin{tikzcd} A \amalg A \arrow[r, "\mathrm{id} \amalg f"] \arrow[d, tail, "i_0+i_1"'] & A \amalg B \arrow[d, "j+r"] & A \arrow[l, tail] \arrow[dl, "j"] \\ I \arrow[r] & I \cup_f B = Z & \end{tikzcd}$$

So the composition

$$\begin{tikzcd} A \arrow[r, tail] \arrow[rr, bend right, "j"'] & A \amalg B \arrow[r, tail, "j+r"] & I \cup_f B \end{tikzcd}$$

is a cofibration as the composition of two cofibrations. The first map is a cofibration being a coproduct of cofibrations and the second map is a cofibration as the cobase change of a cofibration (C3). ■

Now we let $f : A \xrightarrow{\sim}$ be any weak equivalence. Factor it as above

$$f = q \circ j$$

with $j : A \rightarrowtail Z$, $q : Z \overset{\sim}{\rightarrowtail} B$, $r, B \overset{\sim}{\rightarrowtail} Z$, $qr = \mathrm{id}_B$. Here q is a weak equivalence by 2-out-of-3. Then j is an acyclic cofibration by 2-out-of-3, so by (iii) the maps $\gamma(j)$ and $\gamma(r)$ are isomorphisms. Now

$$\gamma(q) \circ \gamma(r) = \gamma(qr) = \gamma(\mathrm{id}) = \mathrm{id},$$

so $\gamma(q)$ is an isomorphism with $\gamma(q) = \gamma(r)^{-1}$. Moreover,

$$\gamma(f) = \gamma(qj) = \gamma(q) \circ \gamma(j)$$

is an isomorphism which also equals $\gamma(r)^{-1} \circ \gamma(j) = r\backslash j$.

(v) This is very similar to proofs we have already seen last semester. Let $F : \mathcal{C} \to \mathcal{D}$ be any functor that takes weak equivalences to isomorphisms. We need to show

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[dr, "F"'] & \mathrm{Ho}(\mathcal{C}) \arrow[d, dashed, "\exists! G"] \\ & \mathcal{D}. \end{tikzcd}$$

The following fact is proven as usual: the functor F takes the same values of homotopic morphisms. Indeed, let $f, g : A \to B$ with a cylinder (I, i_0, i_1, p) and a homotopy $H : I \to B$ with $Hi_0 = f$ and $Hi_1 = g$. Then, $F(p)$ is an isomorphism. So

$$F(p) \circ F(i_0) = F(pi_0) = F(\mathrm{id}) = F(p \circ i_1) = F(p) \circ F(i_1)$$

showing $F(i_0) = F(i_1)$. Thus,

$$F(f) = F(Hi_0) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(Hi_1) = F(g).$$

We now show uniqueness of the factorization. Let $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ with $G\gamma = F$. On objects, γ is the identity, so $G(A) = F(A)$ on objects. On morphisms,

$$G(\tau \setminus f) = G(\gamma(\tau)^{-1} \circ \gamma(f))) = G(\gamma(\tau))^{-1} \circ G(\gamma(f)) = F(\tau)^{-1} \circ F(f),$$

which forces uniqueness.

For existence we put $G(A) = F(A)$ on objects and $G(\tau \setminus f) = F(\tau)^{-1} \circ F(f)$. For well-definedness recall that $\sim$ is generated by three elementary instances. The parts (2) and (3) are fine because F is constant on homotopy classes. For (1) put

$$F(a\tau)^{-1} \circ F(af) = F(\tau)^{-1} \circ F(a)^{-1} \circ F(a) \circ F(f) = F(\tau)^{-1} \circ F(f).$$

We are left to show that G is actually a functor. For compatibility with the identity

$$G(\mathrm{id}) = G(\mathrm{id} \setminus \mathrm{id}) = F(\mathrm{id})^{-1} \circ F(\mathrm{id}) = \mathrm{id}\,.$$

For compatibility with composition let us choose a pushout first:

$$\begin{tikzcd}
 & & C \arrow[d, tail, "{\sigma,\sim}"] \\
 & B \arrow[r, "g"] \arrow[d, tail, "{\tau,\sim}"'] & Y \arrow[d, tail, "{\psi,\sim}"] \\
A \arrow[r, "f"'] & Z \arrow[r, "\varphi"'] & W
\end{tikzcd}$$

So

$$F(\varphi) \circ F(\tau) = F(\psi) \circ F(g) \implies F(\psi)^{-1} \circ F(\varphi) = F(g) \circ F(\tau)^{-1}.$$

Hence,

$$\begin{aligned}
G((\sigma \setminus g) \circ (\tau \setminus f)) &= G((\psi\sigma) \setminus (\varphi f)) \\
&= F(\psi\sigma)^{-1} \circ F(\varphi f) \\
&= F(\sigma)^{-1} \circ F(\psi)^{-1} \circ F(\varphi) \circ F(f) \\
&= F(\sigma)^{-1} \circ F(g) \circ F(\tau)^{-1} \circ F(f) \\
&= G(\sigma \setminus g) \circ G(\tau \setminus f),
\end{aligned}$$

which conludes the proof (thank goodness). □

II.6. Corollary. — *Let $\mathcal{C}$ be a cofibration category and $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ a localization at the class of weak equivalences.*

- *i) Every morphism in $\mathrm{Ho}(\mathcal{C})$ is a "left fraction", i.e. of the form $\gamma(\tau)^{-1} \circ \gamma(f)$ for two $\mathcal{C}$-morphisms f, τ with common target and τ an acyclic cofibration.*
- *ii) Given the two morphisms $f, g : A \to B$ in $\mathcal{C}$, $\gamma(f) = \gamma(g)$ if and only if there is an acyclic cofibration $s : B \overset{\sim}{\rightarrowtail} B'$ auch that $sf \simeq sg$.*

Proof. We prove (ii), as (i) was already proved in theorem II.5.

($\Longleftarrow$) If s is an acyclic cofibration with $sf \sim sg$, then

$$\gamma(s) \circ \gamma(f) = \gamma(sf) = \gamma(sg) = \gamma(s) \circ \gamma(g),$$

hence $\gamma(s)$ being invertible implies $\gamma(f) = \gamma(g)$.

($\Longrightarrow$) Suppose that $\gamma(f) = \gamma(g)$, i.e. $(f, \mathrm{id}_B) \approx (g, \mathrm{id}_B)$. In other words, there are acyclic cofibrations $a : B \overset{\sim}{\rightarrowtail} \bar{Z}$, $b : B \overset{\sim}{\rightarrowtail} \bar{Z}$ such that in the diagram

$$\begin{tikzcd}
 & B \arrow[d, tail, "a"', "\sim"] \arrow[dr, equal] & \\
A \arrow[ur, "f"] \arrow[dr, "g"'] & Z & B \\
 & B \arrow[u, tail, "b", "\sim"'] \arrow[ur, equal] &
\end{tikzcd}$$

the triangles commute up to homotopy, i.e. $fa \simeq gb$ and $a \simeq b$. We choose a pushout

$$\begin{tikzcd}
B \arrow[r, tail, "b", "\sim"'] \arrow[d, tail, "a"', "\sim"] & Z \arrow[d, tail, "\psi", "\sim"'] \\
Z \arrow[r, tail, "\sim", "\varphi"'] & W
\end{tikzcd}$$

Postcomposition preservers homotopy, so $\varphi a = \psi b \simeq \psi a$. Since a is a weak equivalence, we get $\varphi \simeq \psi$. There is an acyclic cofibration $t : W \to \bar{B}$ such that $t\varphi af \simeq t\psi af$. Since the homotopy relation is transitive we get

$$t\varphi af \simeq t\psi af \simeq t\psi bg = t\varphi ag.$$

So the acyclic cofibration $t\varphi a : B \to \bar{B}$ has the desired property. $\square$

Some simple (co)limits

In general homotopy categories have very few limits and colimits (look up homotopy (co)limits). Still, we can prove that we do at least have products and coproducts (which may sound like not much but is actually not at all a given nor very easy to prove!).

II.7. Proposition. — *Let $\mathcal{C}$ be a cofibration category.*

i) The localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ preserves finite coproducts. In particular, $\mathrm{Ho}(\mathcal{C})$ has finite coproducts.

ii) Suppose that $\mathcal{C}$ has I-indexed coproducts, and that the classes of weak equivalences and cofibrations are closed under I-indexed coproducts. Then $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ preserves I-indexed coproducts. In particular, $\mathrm{Ho}(\mathcal{C})$ has I-indexed coproducts.

Proof. The first point is an immediate consequence of the first. For the second point, let I be any set and $\{X_i\}_{i \in I}$ a family of $\mathcal{C}$-objects with coproduct $\coprod_{i \in I} X_i$ and morphisms $\kappa_j : X_j \to \coprod_{i \in I} X_i$. We show that the image of this coproduct under $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ has the universal property of a coproduct, i.e. for every object $Y \in \mathcal{C}$ (or $\mathrm{Ho}(\mathcal{C})$) the map

$$\mathrm{Ho}(\mathcal{C})(\textstyle\coprod_{i \in I} X_i, Y) \to \prod_{i \in I} \mathrm{Ho}(\mathcal{C})(X_i, Y),\ \psi \mapsto (\psi \circ \gamma(\kappa_j))_{j \in J}$$

is bijective.

Surjectivity. Let $(\psi_j : X_j \to Y)_{j \in I}$ be an I-indexed family of morphisms in $\mathrm{Ho}(\mathcal{C})$. By the calculus of fractions, there are $\mathcal{C}$-morphisms $f_j : X_j \to W_j$ and acyclic cofibrations $s_j : Y \overset{\sim}{\rightarrowtail} W_j$ such that

$$\psi_j = \gamma(s_j)^{-1} \circ \gamma(f_j).$$

Consider now

$$\textstyle\coprod_{i\in I} X_i \xrightarrow{\amalg_i f_i} \coprod_{i\in I} W_i \xleftarrow{\amalg_i s_i} \coprod_{i\in I} Y \xrightarrow{\nabla} Y.$$

This yields a morphism

$$\gamma(\nabla)\circ\gamma(\amalg s_i)^{-1}\circ\gamma(\amalg_i f_i) : \textstyle\coprod_{i\in I} X_i \to Y$$

in $\mathrm{Ho}(\mathcal{C})$ which is a preimage of $\{\psi_j\}_{j\in I}$.

Injectivity. Let $\psi,\psi' : \coprod_{i\in I} X_i \to Y$ be morphisms in $\mathrm{Ho}(\mathcal{C})$ such that

$$\psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j)$$

for all $j\in J$.

Special case. $\psi=\gamma(f)$ and $\psi'=\gamma(f')$ for two $\mathcal{C}$-morphisms $f,f' : \coprod_{i\in I} X_i \to Y$. Then

$$\gamma(f\kappa_j) = \gamma(f)\circ\gamma(\kappa_j) = \psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j) = \gamma(f')\circ\gamma(\kappa_j) = \gamma(f'\kappa_j).$$

By the calculus of fractions, there exist a family of acyclic cofibrations $t_j : Y \overset{\sim}{\rightarrowtail} \bar{Y}_j$ such that $t_j f\kappa_j \sim t_j f'\kappa_j$. So there are cylinder objects for all the X_j's and homotopies $H_j : I_j \to \bar{Y}_j$ that witness this. We choose a pushout

$$\begin{tikzcd} \coprod_{i\in I} Y \arrow[r, "\nabla"] \arrow[d, tail, "\amalg_j t_j"', "\sim"] & Y \arrow[d, tail, "t"', "\sim"] \\ \coprod_{i\in I} \bar{Y}_i \arrow[r, "\nabla'"'] & Y' \end{tikzcd}$$

The coproduct of the cylinder objects of all the X_j's is a cylinder object for $\coprod_{j\in I} X_j$. Then the morphism

$$\textstyle\coprod_{i\in I} I_j \xrightarrow{\amalg_{i\in I} H_i} \coprod_{i\in I} \bar{Y}_i \xrightarrow{\nabla'} Y'$$

is a homotopy from tf to tf'. So $\gamma(tf)=\gamma(tf')$ in $\mathrm{Ho}(\mathcal{C})$. Since t is an equivalence, we conclude $\gamma(f)=\gamma(f')$.

General case. Let $\psi,\psi' : \coprod_{i\in I} X_i \to Y$ be a morphism in $\mathrm{Ho}(\mathcal{C})$ such that

$$\psi\circ\gamma(\kappa_j) = \psi'\circ\gamma(\kappa_j)$$

for all $j\in I$. Given $f : \coprod_{i\in I} X_i \to W,\ f' : \coprod_{i\in I} X_i \to W'$, the calculus of fractions produces

$$s : Y \overset{\sim}{\rightarrowtail} W,\ s' : Y \overset{\sim}{\rightarrowtail} W'$$

such that $\psi=\gamma(s)^{-1}\circ\gamma(f)$ and $\gamma(s')^{-1}\circ\gamma(f')$. We choose a pushout

$$\begin{tikzcd} Y \arrow[r, tail, "s", "\sim"'] \arrow[d, "s'"', "\sim"] & W \arrow[d, "t"] \\ W' \arrow[r, "t'"'] & V \end{tikzcd}$$

then we have

$$\gamma(tf)\circ\gamma(\kappa_j) = \gamma(t)\circ\gamma(s)\circ\psi\circ\gamma(\kappa_j) = \gamma(t')\circ\gamma(s')\circ\psi'\circ\gamma(\kappa_j) = \gamma(t'f')\circ\gamma(k_j)$$

for all $j\in J$. By the special case we get $\gamma(tf)=\gamma(t'f')$. Thus,

$$\gamma(ts)\circ\psi = \gamma(tf) = \gamma(t'f') = \gamma(t's')\circ\psi'.$$

Moreover, $\gamma(ts)=\gamma(t's')$ is invertible, so $\psi=\psi'$. $\square$

Lecture 13
30th May, 2022

The next theorem looks a lot like the dualization of the previous one (theorem II.7), but as the axioms for cofibration categories are not self dual (contrary to the ones for model categories) we have to prove everything again (although the proof is surprisingly similar).

II.8. Theorem. — *Let $\mathcal{C}$ be a cofibration category.*

i) The localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ preserves terminal objects.

ii) Let I be any set. Suppose that $\mathcal{C}$ has I-indexed products and weak equivalences are closed under I-indexed products. Then $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ preserves I-indexed products. In particular $\mathrm{Ho}(\mathcal{C})$ has I-indexed products.

Proof. (i) Let $* \in \mathcal{C}$ be a terminal object. Call $p_A : A \to *$ the unique $\mathcal{C}$-morphism from an object A. Any morphism from A to $*$ in $\mathrm{Ho}(\mathcal{C})$ is of the form $\gamma(\tau)^{-1} \circ \gamma(f)$ for some morphism $f : A \to Z$ and $\tau : * \xrightarrow{\sim} Z$ in $\mathcal{C}$ with τ a weak equivalence. The unique $\mathcal{C}$-morphism $p_Z : Z \to *$ is left inverse to τ, so p_Z is a weak equivalence by 2-out-of-3 and $\gamma(p_Z) = \gamma(\tau)^{-1}$. Then

We could of course just prove (ii).

$$\gamma(\tau)^{-1} \circ \gamma(f) = \gamma(p_Z) \circ \gamma(f) = \gamma(p_Z f) = \gamma(p_A),$$

hence there is exactly one morphism from A to $*$ in $\mathrm{Ho}(\mathcal{C})$.

(ii) Let $\{X_i\}_{i \in I}$ be a I-family of $\mathcal{C}$-products. Let $\prod_{i \in I} X_i$ be a product in $\mathcal{C}$, with $p_j : \prod_{i \in I} X_i \to X_i$ the canonical morphisms. We want to show that for all $A \in \mathrm{ob}(\mathcal{C})$, the map

$$\begin{aligned} \mathrm{Ho}(\mathcal{C})(A, \textstyle\prod_{i \in I} X_i) &\to \textstyle\prod_{j \in I} \mathrm{Ho}(\mathcal{C})(A, Xj) \\ \psi &\mapsto (\gamma(p_j) \circ \psi)_{j \in I} \end{aligned}$$

is bijective.

Surjectivity. We consider an I-family of $\mathrm{Ho}(\mathcal{C})$-morphisms $\psi_j : A \to X_j$ for $j \in I$. The calculus of fractions provides $\mathcal{C}$-morphisms $f_j : A \to Z_j$ and acyclic cofibrations $s_j : X_j \overset{\sim}{\rightarrowtail} Z_j$ such that

$$\psi_j = \gamma(s_j)^{-1} \circ \gamma(f_j).$$

Then

$$\textstyle\prod s_j : \prod_{j \in I} X_j \xrightarrow{\sim} \prod_{j \in I} Z_j$$

is another weak equivalence by assumption.

We claim that the morphism $\gamma(\prod s_j)^{-1} \circ \gamma((f_i)_{i \in I}) : A \to \prod_{i \in I} X_i$ is a preimage of the family $(\psi_j)_{j \in I}$. In other words, the following is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "(f_i)_{i \in I}"] \arrow[dr, "f_j"'] & \prod_{i \in I} Z_i \arrow[d, "p_j"] & & \prod_{i \in I} X_i \arrow[ll, "\Pi_j s_j", "\sim"'] \arrow[d, "p_j"] \\ & Z_j & & X_j \arrow[ll, "\sim", "s_j"'] \end{tikzcd}$$

Indeed,

$$\gamma(p_j) \circ \gamma(\textstyle\prod s_j)^{-1} \circ \gamma((f_i)_{i \in I}) = \gamma(s_j)^{-1} \circ \gamma(p_j) \circ \gamma((f_i)_{i \in I}) = \gamma(s_j)^{-1} \circ \gamma(f_j) = \psi_j.$$

Injectivity. While surjectivity is (surprisingly) similar to the proof of surjectivity in theorem II.7, injectivity is more difficult! Let $\varphi, \psi : A \to \prod_{i \in I} X_i$ be two morphisms in $\mathrm{Ho}(\mathcal{C})$ such that

$$\gamma(p_j) \circ \varphi = \gamma(p_j) \circ \psi$$

for all $j \in I$. We will only do a special case, namely $\varphi = \gamma(f)$ and $\psi = \gamma(g)$ for some $\mathcal{C}$-morphisms $f, g : A \to \prod_{i \in I} X_i$. Then,

$$\gamma(p_j \circ f) = \gamma(p_j) \circ \varphi = \gamma(p_j) \circ \psi = \gamma(p_j \circ g).$$

The calculus of fractions (Corollary II.6) provides weak equivalences $s_j : X_j \xrightarrow{\sim} \overline{X}_j$ such that $s_j p_j f \simeq s_j p_j g$. So there are cylinder objects $(I^j, \iota_0^j, \iota_1^j, p_j)_{j \in I}$ for A and homotopies

$$H^j : I^j \to \overline{X}_j \quad \text{from } s_j p_j f \text{ to } s_j p_j g.$$

Each of the morphisms $p^j : I^j \to A$ is a weak equivalence by definition of cylinder objects. So also $\prod_{j \in I} p^j : \prod_{j \in I} I^j \to \prod_{j \in I} A$ is also a weak equivalence by assumption. So the two morphisms $\prod_j \iota_0^j, \prod_j \iota_1^j : \prod_{j \in I} A \to \prod_{j \in I} I^j$ are both right inverse to the equivalence $\prod_{j \in I} p^j$. Thus, $\gamma\left(\prod_{j \in I} \iota_0^j\right), \gamma\left(\prod_{j \in I} \iota_1^j\right)$ are both right inverses of the isomorphism $\gamma\left(\prod_{j \in I} p^j\right)$ which means that they must agree.

We let $\Delta : A \to \prod_{i \in I} A$ be the diagonal, i.e. $p_j \circ \Delta = \mathrm{id}_A$ for all $j \in I$. Then,

$$\begin{aligned}
\gamma\left(\prod_{i \in I} s_i\right) \circ \varphi &= \gamma\left(\prod_{i \in I} s_i\right) \circ \gamma(f) \\
&= \gamma\left(\prod_{i \in I} s_i p_i f\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} H^i \circ \iota_0^i\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} H^i\right) \circ \gamma\left(\prod_{i \in I} \iota_0^i\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} H^i\right) \circ \gamma\left(\prod_{i \in I} \iota_1^i\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} (H^i \circ \iota_1^i)\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} (s_i p_i g)\right) \circ \gamma(\Delta) \\
&= \gamma\left(\prod_{i \in I} s_i\right) \circ \gamma(g) \\
&= \gamma\left(\prod_{i \in I} s_i\right) \circ \psi.
\end{aligned}$$

Since $\gamma\left(\prod_{i \in I} s_i\right)$ is an isomorphism, we deduce $\varphi = \psi$. □

Additive Categories

A fundamental and basic structure (although not really a structure, see below) that one often considers when dealing with some kind of homological setting is that of additive categories. For most people, these are stepping stones to abelian categories, but in our case they will be stepping stones to a generalization of the latter, triangulated categories.

Definition. — A category $\mathcal{C}$ is *preadditive* if it has a zero object, finite coproducts and for all pair of objects X, Y the morphisms

$$p_1 = \mathrm{id} + 0 : X \coprod Y \to X,\ \ p_2 = 0 + \mathrm{id} : X \coprod Y \to Y$$

make $X \coprod Y$ into a product of X and Y, i.e. for all $\mathcal{C}$-objects A, the map

$$\begin{aligned} \mathcal{C}(A, X \textstyle\coprod Y) &\to \mathcal{C}(A, X) \times \mathcal{C}(A, Y) \\ f &\mapsto (p_1 f, p_2 f) \end{aligned}$$

is bijective.

Examples. — Examples are of course abelian groups, modules over a ring, abelian monoids, but also $D(R)$ for R a ring spectrum.

Remark. — Our definition of preadditive categories is a bit different than the usual definition, i.e. an Ab-enriched category with a zero object. In fact it is strictly related, as the next proposition shows (although not the same, and indeed our preadditive categories are usually called *semiadditive categories* instead). It could be argued that our definition is the "right" one to give, as it makes clear that being an additive category is a property and does not come from having additional structure.

Construction. — Let $\mathcal{C}$ be preadditive. Let $a, b : A \to X$ be two $\mathcal{C}$-morphisms. We denote by $a \perp b$ the unique morphism $A \to X \coprod X$ such that $p_1 \circ (a \perp b) = a$ and $p_2 \circ (a \perp b) = b$. The *sum* of a and b is defined by

$$a + b = \nabla(a \perp b) = (\mathrm{id} + \mathrm{id})(a \perp b).$$

II.9. Proposition. — *Let $\mathcal{C}$ be a preadditive category.*

i) *For all $\mathcal{C}$-objects A, X the binary operation $+$ makes $\mathcal{C}(A, X)$ into an abelian monoid. Composition is biadditive.*

ii) *If the shear map $\nabla \perp p_1 : X \coprod X \to X \coprod X$ is an isomorphism, then $\mathcal{C}(A, X)$ is an abelian group.*

iii) *Let $F : \mathcal{C} \to \mathrm{AbMon}$ be a functor that preserves zero objects and finite coproducts. Then for all $\mathcal{C}$-objects A, X and all $a \in F(a)$, the map*

$$\mathcal{C}(A, X) \to F(X),\ \ f \mapsto f_*(a) = F(f)(a)$$

is a monoid morphism

Proof. This proof is quite easy and fairly boring, but one should at least once try to do parts of it, to get an idea of how this additive categories stuff works. Therefore, let's do some of it.

(i) First we will show associativity of $+$. Let $a, b, c : A \to X$. Consider the following diagram:

$$\begin{tikzcd} & & X \amalg X \arrow[dr, "\nabla"] & \\ A \arrow[r] & X \amalg X \amalg X \arrow[ur, "\mathrm{id}_X \amalg \nabla"] \arrow[dr, "\nabla \amalg \mathrm{id}_X"'] & & X \\ & & X \amalg X \arrow[ur, "\nabla"'] & \end{tikzcd}$$

which commutes by the universal property of coproducts. Going through this diagram in both ways amounts to associativity.

For commutativity we first note $b \perp a = (a \perp b) \circ \tau$ where $\tau = i_2 + i_1 : X \amalg X \to X \amalg X$ is the flip morphism. So $p_1 \tau = p_2$ and $p_2 \tau = p_1$. We compute

$$p_1(b \perp a) = b \quad \text{and} \quad p_1 \tau (a \perp b) = p_2(a \perp b) = b.$$

Thus,

$$a + b = \nabla \circ (a \perp b) = \nabla \circ \tau \circ (a \perp b) = \nabla \circ (b \perp a) = b + a.$$

We omit to show that the 0-morphism is the neutral element for addition in $\mathcal{C}(A, X)$.

For biadditivity let $A \xrightarrow{a,b} X \xrightarrow{c} Y$. Consider

$$\begin{tikzcd}
A \arrow[r, "a\perp b"] \arrow[rr, bend left, "ca\perp cb"] \arrow[d, "a\perp b"'] & X \amalg X \arrow[r, "c\amalg c"] \arrow[d, "\nabla"] & Y \amalg Y \arrow[d, "\nabla"] \\
X \amalg X \arrow[r, "\nabla"'] & X \arrow[r, "c"'] & Y
\end{tikzcd}$$

For this note

$$\begin{tikzcd}
X \amalg X \arrow[r, "c\amalg c"] \arrow[d, "p_1 = \mathrm{id}_X + 0"'] & Y \amalg Y \arrow[d, "p_1 = \mathrm{id}_Y + 0"] \\
X \arrow[r, "c"'] & Y
\end{tikzcd}$$

So that

$$p_1(c \amalg c)(a \perp b) = c p_1(a \perp b) = ca = p_1(ca \perp cb)$$

and

$$p_2(c \amalg c)(a \perp b) = p_2(ca \perp cb),$$

which shows the commutativity of the upper part. The commutativity of the entire diagram amounts to $c(a + b) = (ca) + (cb)$.

(ii) We assume that $\nabla \perp p_1 : X \amalg X \to X \amalg X$ is an isomorphism. Note:

- An abelian monoid M is a group if and only if the map

  $$M^2 \to M^2, \ (x, y) \mapsto (x, x + y)$$

  is bijective. Indeed, the preimage of $(x, 0)$ would give $(x, -x)$, so it gives us the inverse of x.

- Every morphism $f : A \to X \amalg X$ satisfies $f = (p_1 f) \perp (p_2 f)$. Hence,

  $$(p_1 f) + (p_2 f) = \nabla((p_1 f) \perp (p_2 f)) = \nabla f.$$

So the following square commutes:

$$\begin{tikzcd}
\mathcal{C}(A, X \amalg X) \arrow[r, "(\nabla \perp p_1)\circ -"] \arrow[d, "f \mapsto (p_1 f, p_2 f)"'] & \mathcal{C}(A, X \amalg X) \arrow[d, "f \mapsto (p_1 f, p_2 f)"] \\
\mathcal{C}(A, X)^2 \arrow[r, "(a,b) \mapsto (a, a+b)"'] & \mathcal{C}(A, X)^2
\end{tikzcd}$$

The upper three maps are isomorphisms, so also the lower map is an isomorphism. So $\mathcal{C}(A,X)$ is an abelian group.

(iii) Let $F:\mathcal{C}\to \mathrm{AbMon}$ be a finite coproduct-preserving functor (with $F(0)=0$). Then, for all $a\in F(A)$ we show that the map

$$\mathcal{C}(A,X)\to F(X),\ f\mapsto f_*(a)$$

is additive.

Claim. For all $X\in\mathcal{C}$ we have

$$F(p_1)+F(p_2)=F(\nabla).$$

Proof of the claim. Let $i_1,i_2: X\to X\amalg X$ be the universal morphisms of the coproduct. Then,

$$\begin{aligned}F(\nabla)\circ F(i_1)&=F(\nabla\circ i_1)\\&=F(\mathrm{id}_X)\\&=F(\mathrm{id}_X)+F(0)\\&=F(p_1\circ i_1)+F(p_2\circ i_1)\\&=(F(p_1)+F(p_2))\circ F(i_1)\end{aligned}$$

where the latter $+$ is from abelian monoids and in particular biadditive. Similarly,

$$F(\nabla)\circ F(i_2)=(F(p_1)+F(p_2))\circ F(i_2).$$

The map

$$F(i_1)+F(i_2): F(X)\oplus F(X)\to F(X\amalg X),\ (x,y)\mapsto F(i_1)(x)+F(i_2)(x)$$

is bijective because F preserves coproducts. This implies $F(\nabla)=F(p_1)+F(p_2)$. ■

So for two $\mathcal{C}$-morphisms $f,g:A\to X$ we get

$$\begin{aligned}F(f+g)&=F(\nabla\circ(f\perp g))\\&=F(\nabla)\circ F(f\perp g)\\&=(F(p_1)+F(p_2))\circ F(f\perp g)\\&=(F(p_1)\circ F(f\perp g))+(F(p_2)\circ F(f\perp g))\\&=F(f)+F(g)\end{aligned}$$

where we use that the latter composition is the composition in CMon which is biadditive. □

Definition. — An *additive* category $\mathcal{C}$ is a preadditive category such that for all objects X the morphism $\nabla\perp p_1: X\prod X\to X\prod X$ is an isomorphism (in particular, it is Ab-enriched by proposition II.9).

II.10. Theorem. — *Let R be a ring spectrum.*

i) The localization functor $\gamma: {}_R\mathrm{Mod}(\mathrm{Sp})\to D(R)$ preserves arbitrary coproducts and finite products. In particular, $D(R)$ has arbitrary coproducts and finite products.

Actually, arbitrary products are also preserved, but we don't show that.

ii) The derived category $D(R)$ is additive.

Proof. Luckily, we have already done most of the work.

(i) Localization of nice enough cofibration categories, such as ${}_R\mathrm{Mod}(\mathrm{Sp})$, preserves arbitrary coproducts (Proposition II.7) and finite products (Theorem II.8).

(ii) For all R-modules M, N the canonical map $M \vee N \to M \times N$ in ${}_R\mathrm{Mod}$ is a stable equivalence (Proposition I.15), so it becomes an isomorphism in $D(R)$. Because the localization $\gamma : {}_R\mathrm{Mod}(\mathrm{Sp}) \to D(R)$ preserves finite coproducts and finite products by (i), this shows that finite coproducts in $D(R)$ are products. So $D(R)$ is preadditive.

The composite

$$\begin{tikzcd} \pi_k(M) \oplus \pi_k(M) \arrow[r, "\sim"] \arrow[rrr, bend right, "(x,y)\mapsto(x+y,y)"'] & \pi_k(M \vee M) \arrow[r, "\nabla \perp p_1"] & \pi_k(M \times M) \arrow[r, "\sim"] & \pi_k(M) \times \pi_k(M) \end{tikzcd}$$

commutes in $D(R)$. Note that the bottom map is the shear map which is an isomorphism because we are dealing with abelian groups. Thus, also the shear map $\nabla \perp p_1$ is an isomorphism in $D(R)$. So $D(R)$ is additive.

$\square$

An Explicit Description of Hom-sets in the Stable Homotopy Category

Lecture 14
1st Jun, 2022

It is usually really difficult to describe Hom-sets of derived categories, but in our case we can say something meaningful.

Construction. — Let E be a sequential spectrum and A a based space. Set

$$E\{A\} = \operatorname*{colim}_n [S^n \wedge A, E_n]_*$$

with the colimit taken over the maps

$$[S^n \wedge A, E_n] \xrightarrow{S^1 \wedge -} [S^1 \wedge S^n \wedge A, S^1 \wedge E_n] \xrightarrow{(\sigma_n)_*} [S^{1+n} \wedge A, E_{1+n}].$$

Remark. — For $n \geqslant 2$, $[S^n \wedge A, E_n]$ has an abelian group structure by "pinching", i.e. given $f, g : S^n \wedge A \to E_n$ we add them via

$$S^n \wedge A \xrightarrow{\text{pinch}} (S^n \vee S^n) \wedge A \cong (S^n \wedge A) \vee (S^n \wedge A) \xrightarrow{f+g} E_n.$$

For $n \geqslant 2$, the stabilization maps are group homomorphisms, hence $E\{A\}$ is an abelian group.

Remark. — If $A = S^k$, we have $S^n \wedge S^k \cong S^{n+k}$ so $[S^n \wedge S^k, E_n]_* \cong \pi_{n+k}(E_k)$, hence

$$E\{S^k\} \cong \pi_k(E).$$

More generally

$$\begin{aligned} E\{A\} = \operatorname*{colim}_n [S^n \wedge A, E_n]_* &\cong \operatorname{colim}[S^n, \operatorname{map}_*(A, E_n)]_* \\ &= \operatorname*{colim}_n \pi_n(\operatorname{map}_*(A, E_n)) \\ &= \pi_0(\operatorname{map}_*(A, E)). \end{aligned}$$

If A admits the structure of a finite CW-complex, then $\operatorname{map}_*(A, -)$ preserves stable equivalences (Proposition I.18), hence

$$\pi_0(\operatorname{map}_*(A, -)) \cong (-)\{A\} : \mathrm{Sp}^{\mathbb{N}} \to \mathrm{Ab}$$

takes stable equivalences to isomorphisms.

Definition. — Let R be an orthogonal ring spectrum and A a based space. The *tautological class* $\iota_A \in (R \wedge A)\{A\}$ is the class represented by

$$\iota_0 \wedge A : S^0 \wedge A \to R(0) \wedge A.$$

II.11. Theorem. — *Let R be an orthogonal ring spectrum and A a based space that admits the structure of a finite CW-complex.*

i) The functor $(-)\{A\} : {}_R\mathrm{Mod} \to \mathrm{Ab}$ takes stable equivalences to isomorphisms.

ii) Let $\Phi : {}_R\mathrm{Mod} \to \mathrm{Set}$ be any functor that takes stable equivalences to isomorphisms. Then evaluation at ι_A is a bijection

$$\mathrm{Nat}_{{}_R\mathrm{Mod} \to \mathrm{Set}}((-)\{A\}, \Phi) \cong \Phi(R, A)$$

$$\tau : (-)\{A\} \to \Phi \mapsto \tau_{R \wedge A}(\iota_A)$$

iii) The pair $(R \wedge A, \iota_A)$ represents the functor

$$\mathcal{D}(R) \to \mathrm{Set},\ M \mapsto M(A)$$

Examples. — We have

$$\mathcal{D}(R)(R \wedge A, M) \cong M\{A\} = \operatorname*{colim}_m [S^n \wedge A, M_n]$$

$$\mathcal{SH}(\Sigma^\infty A, M) \cong \operatorname*{colim}_n [S^n \wedge A, M_n]$$

and for $A = S^k$

$$\mathcal{D}(R)(R \wedge S^k, M) \cong \pi_k(M)$$

$$\mathcal{SH}(\Sigma^\infty S^k, M) \cong \pi_k(M).$$

Proof. We saw (i) in the remark above, so we start with (ii). We will show injectivity and surjectivity separately.

Injectivity. Let $\tau : (-)\{A\} \to \Phi$ be any natural transformation. Let M be any R-module and let $f : S^n \wedge A \to M_n$ be any based continuous map, representing a class in $M\{A\}$. Let $f^\flat : A \to \Omega^n M_n$ be the adjoint of f. Then there is a unique morphism of R-modules

$$f^\# : R \wedge A \to \Omega^n \operatorname{sh}^n M$$

such that the composite

$$A \xrightarrow{\iota \wedge -} R(0) \wedge A \xrightarrow{f^{\#}(0)} (\Omega^n \operatorname{sh}^n M)(0) = \Omega^n M(\mathbb{R}^n) = \Omega^n M_n$$

is $f^\flat$. The value of $f^{\#}$ at an inner product space V is

$$R(V) \wedge A \xrightarrow{R(V)\wedge f^\flat} R(V) \wedge \Omega^n M(\mathbb{R}^n) \xrightarrow{\alpha_{V,\mathbb{R}^n}} (\Omega^n M)(V \oplus \mathbb{R}^n) = (\Omega^n \operatorname{sh}^n M)(V).$$

The map satisfies

$$f^{\#}\{A\}(\iota_A) = \tilde{\lambda}^n\{A\}[f]$$

where $\tilde{\lambda}^n_M : M \xrightarrow{\sim} \Omega^n \operatorname{sh}^n M$ is the adjoint of $\lambda^n_M : M \wedge S^n \xrightarrow{\sim} \operatorname{sh}^n M$.

$$\begin{tikzcd}
(R \wedge A)\{A\} \arrow[r, "f^{\#}\{A\}"] \arrow[d, "\tau_{R\wedge A}"'] & (\Omega^n \operatorname{sh}^n M)\{A\} \arrow[d, "\tau_{\Omega^n \operatorname{sh}^n M}"] & M\{A\} \arrow[l, "\cong", "\tilde{\lambda}^n_M\{A\}"'] \arrow[d, "\tau_M"] \\
\Phi(R \wedge A) \arrow[r, "\Phi(f^{\#})"'] & \Phi(\Omega^n \operatorname{sh}^n M) & \Phi(M) \arrow[l, "\Phi(\tilde{\lambda}^n_M)", "\cong"']
\end{tikzcd}$$

So

$$\Phi(f^{\#})(\tau_{R\wedge A}(\iota_A)) = \tau_{\Omega^n \operatorname{sh}^n M}(f^{\#}\{A\}(\iota_A)) = \tau_{\Omega^n \operatorname{sh}^n M}(\tilde{\lambda}^n_M\{A\}[f]) = \Phi(\tilde{\lambda}^n_M)(\tau_M[f])$$

hence

$$\tau_M[f] = \Phi(\tilde{\lambda}^n_M)^{-1}(\Phi(f^{\#})(\tau_{R\wedge A}(\iota_A))).$$

Surjectivity. let $y \in \Phi(R \wedge A)$ be any element. We want to construct a natural transformation $\tau : (-)\{A\} \to \Phi$ such that $\tau_{R\wedge A} = y$. Let M be any R-module and let $f : S^n \wedge A \to M_n$ represent any given class in $M\{A\}$. Then we define

$$\tau_M[f] := \Phi(\tilde{\lambda}^n_M)^{-1}(\Phi(f^{\#})(y)).$$

This is well defined as it does not change if we change f by a homotopic f' or if we stabilize f to $\sigma_n \circ (S^1 \wedge f)$. We show the second claim. The following diagram commutes

$$\begin{tikzcd}
R \wedge A \arrow[r, "f^\sharp"] \arrow[dr, bend right, "(\sigma_n\circ(S^1\wedge f))^\sharp"'] & \Omega^n \operatorname{sh}^n M \arrow[d, "\tilde{\lambda}_{\Omega^n \operatorname{sh}^n M}, \approx"] & M \arrow[l, "\tilde{\lambda}^n_M"'] \arrow[dl, bend left, "\tilde{\lambda}^{1+n}_M, \approx"] \\
& \Omega^{1+n} \operatorname{sh}^{1+n} M &
\end{tikzcd}$$

Thus,

$$\begin{aligned}
\Phi(\tilde{\lambda})^{-1}(\Phi(f^{\#})(y)) &= (\Phi(\tilde{\lambda}^{1+n}_M)^{-1} \circ \Phi(\lambda_{\Omega^n \operatorname{sh}^n M}) \circ \Phi(f^{\#}))(y) \\
&= (\Phi(\tilde{\lambda}^{1+n}_M)^{-1} \circ \Phi((\sigma_n(S^1 \wedge f))^{\#}))(y).
\end{aligned}$$

Now we show naturality of the transformation τ we defined. Let $\psi : M \to N$ be a morphism of R-modules, $f : S^n \wedge A \to M_n$. The $\psi_*[f]$ is represented by $\psi_n \circ f : S^n \wedge A \to N_n$. The following diagram in ${}_M\mathrm{Mod}$ commutes:

$$\begin{tikzcd}
R \wedge A \arrow[r, "f^\sharp"] \arrow[dr, bend right, "(\psi_n\circ f)^\sharp"'] & \Omega^n \operatorname{sh}^n M \arrow[d, "\Omega^n \operatorname{sh}^n \psi"] & M \arrow[l, "\tilde{\lambda}^n_M, \approx"'] \arrow[d, "\psi"] \\
& \Omega^n \operatorname{sh}^n N & N \arrow[l, "\tilde{\lambda}^n_N, \approx"]
\end{tikzcd}$$

Therefore

$$\begin{aligned}\tau_N(\psi\{A\}[f]) &= (\Psi(\tilde{\lambda}^n_N)^{-1}\circ\Phi((\psi_n\circ f)^{\#}))(y)\\ &= (\Phi(\tilde{\lambda}^n_N)^{-1}\circ\Phi(\Omega^n \operatorname{sh}^n\psi)\circ\Phi(f^{\#}))(y)\\ &= (\Phi(\psi)\circ\Psi(\tilde{\lambda}^n_M)^{-1}\circ\Phi(f^{\#}))(y) = \Phi(f)(\tau_M[f]).\end{aligned}$$

(iii) By part (ii) applied to the functor $\mathcal{D}(R)(R\wedge A,-)\circ\gamma : {}_R\mathrm{Mod}\to\mathrm{Set}$ there is a unique natural transformation $\tau : (-)\{A\}\to\mathcal{D}(R)(R\wedge A,-)$ such that

$$\tau_{R\wedge A}(\iota_A) = \mathrm{id}_{R\wedge A}\,.$$

The Yoneda lemma provides a unique natural transformation $j : \mathcal{D}(R)(R\wedge A,-)\to(-)\{A\}$ such that $j_{R\wedge A}(\mathrm{id}_{R\wedge A}) = \iota_A$. Then, $j\circ\tau : (-)\{A\}\to(-)\{A\}$ sends ι_A to ι_A. So $j\circ\tau = \mathrm{id}_{(-)\{A\}}$ by uniqueness in (ii). Moreover, $\tau\circ j : D(R\wedge A,-)\to D(R\wedge A,-)$ takes $\mathrm{id}_{R\wedge A}$ to $\mathrm{id}_{R\wedge A}$, so $\tau\circ j = \mathrm{id}$ by the uniqueness of the Yoneda Lemma. In particular, j is a natural isomorphism

$$j_M : D(R)(R\wedge A, M)\xrightarrow{\sim} M\{A\}.$$

$\square$

Remark. — We already remarked that as special case of the theorem we have

$$\mathcal{D}(R)(R\wedge S^n, M)\cong M\{A\} = \pi_n(M)$$

$$\mathcal{SH}(\Sigma^\infty S^n, A)\cong\pi_n(M)$$

In particular,

$$\begin{aligned}\mathcal{SH}(\Sigma^\infty S^k\wedge A, HB)\cong HB\{S^k\wedge A\} &= \operatorname*{colim}_n [S^{n+k}\wedge A,(HB)_n]_*\\ &\cong \operatorname*{colim}_n \tilde{H}^{n+k}(S^n\wedge A, B)\cong H^k(A;B)\end{aligned}$$

where we consider representability of cohomology and the suspension isomorphism.

Triangulated Categories

Let $\mathcal{T}$ be a category equipped with an autoequivalence $\Sigma : \mathcal{T}\to\mathcal{T}$. A *triangle* in J is a triple (f,g,h) of composable morphisms

$$A\xrightarrow{f} B\xrightarrow{g} C\xrightarrow{h}\Sigma A.$$

A *morphism of triangles* $(f,g,h)\to(f',g',h')$ is a triple of morphisms $a : A\to A'$, $b : B\to B'$, $c : C\to C'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'. \end{tikzcd}$$

commutes.

Definition. — A *triangulated category* consists of an additive category $\mathcal{T}$, an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of distinguished triangles (exact triangles) that satisfy the following axioms:

(T0) The class of distinguished triangles is closed under isomorphisms.

(T1) Every morphism of f occurs in a distinguished triangle (f, g, h).

(T2) For every object X, the triangle $(0, \mathrm{id}_X, 0)$ is distinguished.

(T3) Rotation axiom. If (f, g, h) is distinguished, so is $(g, h, -\Sigma f)$.

(T4) Completion of triangles. Given maps between objects of two distinguished triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "\exists c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A' \end{tikzcd}$$

such that $b \circ f = f' \circ a$ (so only the left square needs to be commutative!) we can fill in the dashed arrow to obtain a morphism of triangles.

(T5) Octahedral axiom. For every composable pair of morphisms $f : A \to B$ and $f' : B \to D$ there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f'f"] & D \arrow[r, "g'"] \arrow[d, "g''"] & E \arrow[r, "h''"] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\ & F \arrow[r, equal] \arrow[d, "h'"] & F \arrow[r, "h'"] \arrow[d, "(\Sigma g) \circ h'"] & \Sigma B \\ & \Sigma B \arrow[r, "\Sigma g"] & \Sigma C & \end{tikzcd}$$

such that the two long rows and the two long columns are distinguished.

Remark. — There are some remarks that it is customary to make about this definition.

- This is probably the weakest formulation of triangulated categories. Historically, the definition of triangulated categories came up almost simultaneously in algebraic geometry and algebraic topology. Grothendieck was interested in derived categories of abelian categories, while topologists were interested in stable homotopy theory. Puppe gave axioms which were slightly weaker than Verdier's, as he omits the octahedral axiom. But the Professor is not even sure whether people have constructed examples which only satisfy Puppe's but not Verdier's axiom up to this day.

- The name octahedral axiom stems from the fact that the diagram looks like an octahedron if drawn in a specific why (not ours, evidently). Prof. Schwede says that this axiom is pretty hard to motivate other than saying that it is useful, but he will show us that the axiom is some generalization of Noether's third isomorphism theorem.

- Algebraists sometimes assume that Σ is an automorphism instead of just an autoequivalence and would denote it by [1]. This is because the main example for algebraists are the chain complexes with the shift functor [1] which is an automorphism. On the other hand, topologists care about the suspension Σ which is really just an autoequivalence and not an automorphism. At the end of the day it doesn't really matter: There is a 2-category Cat and we really only care about the equivalences in the end.

- The obligatory ∞-remark: it's pretty hard to construct triangulated categories that don't come from stable ∞-categories. It's possible (Schwede did) but basically all naturally occuring triangulated structures come from stable ∞-categories, and really the latter is the more natural notion.

Triangulated Structure on the Homotopy Category

Lecture 15
13^{th} Jun, 2022

Definition. — Let $\mathcal{C}$ be a pointed cofibration category. A *functorial cone* is the data of a functor $C : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\iota : \mathrm{id}_{\mathcal{C}} \to C$ such that for all $X \in \mathcal{C}$, the morphism $\iota_X : X \to CX$ is a cofibration and the unique morphism $CX \to *$ is a weak equivalence.

Remark. — Not all cofibration categories have *functorial* cones (of course you always get cones by factoring $X \to *$) and in fact they are not necessary to have a triangulated structure on the homotopy category, as derived categories turn out to always have functorial cones anyway. The requirement will just save us some time and the cofibration categories we are interested in do have functorial cones anyway. In particular, if $\mathcal{C} = {}_R\mathrm{Mod}(\mathrm{Sp})$ for and orthogonal ring spectrum R, we can take

$$CM = M \wedge [0,1],\ \ \iota : - \wedge 1 : M \to M \wedge [0,1].$$

Construction. — Let $\mathcal{C}$ be a pointed (i.e. with zero object) cofibration category with functorial cones. The *suspension functor* $\Sigma : \mathcal{C} \to \mathcal{C}$ is defined on object by $\Sigma X = CX/X$, i.e. the pushout

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, tail, "\iota_X"] & * \arrow[d] \\ CX \arrow[r, "q_X"] & \Sigma X \end{tikzcd}$$

and, given that we are considering functorial cones, we extend the assignment to morphism in the natural way. Any morphism $CX \to CY$ is a weak equivalence, because source and target are weakly equivalent to $*$. Given a weak equivalence $f : X \to Y$, by the Gluing Lemma II.2 for

$$\begin{tikzcd} CX \arrow[d, "Cf"'] & X \arrow[l, tail, "\iota_X"'] \arrow[r] \arrow[d, "f"] & * \arrow[d, equal] \\ CY & Y \arrow[l, tail, "\iota_Y"] \arrow[r] & * \end{tikzcd}$$

the morphism $\Sigma f : \Sigma X \to \Sigma Y$ is also a weak equivalence.

Suppose that $C' : \mathcal{C} \to \mathcal{C}$ is another functorial cone. Then by the gluing lemma again, the following two morphisms are natural weak equivalences

$$\Sigma X \xrightarrow{q_X \cup 0} CX \cup_X C'X \xrightarrow{0 \cup q'_X} \Sigma' X.$$

The composite $\gamma \circ \Sigma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ takes weak equivalences to isomorphisms, so there is a unique functor $\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ such that the following commutes

This is a fairly convoluted way to explain a harmless abuse of notation.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Sigma"] \arrow[d, "\gamma"] & \mathcal{C} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathcal{C}) \arrow[r, dashed, "\Sigma"] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

In particular, $\Sigma X = \Sigma X$ on objects and

$$\Sigma(\gamma(s)^{-1} \circ \gamma(f)) = (\Sigma\gamma(s))^{-1} \circ (\Sigma\gamma(f)) = \gamma(\Sigma s)^{-1} \circ \gamma(\Sigma f).$$

Construction. — Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. The *elementary distinguished triangle* of a $\mathcal{C}$-morphism $\psi : X \to Y$

$$X \xrightarrow{\gamma(\psi)} Y \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} \Sigma X$$

where $C\psi = CX \cup_\psi Y$ is the *mapping cone* of ψ, i.e. a pushout

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\iota_X"] & Y \arrow[d, "i"] \\ CX \arrow[r] & C\psi \end{tikzcd}$$

and p is the map induced by $0 : Y \to \Sigma X$ and $q_X : CX \to \Sigma X$.

A triangle (f, g, h) in $\mathrm{Ho}(\mathcal{C})$ is *distinguished* if it is isomorphic to some elementary distinguished triangle, i.e. there is a $\mathcal{C}$-morphism $\psi : X \to Y$ and isomorphisms

$$a : A \xrightarrow{\cong} X$$

$$b : B \xrightarrow{\cong} Y$$

$$c : C \xrightarrow{\cong} C\psi$$

in $\mathrm{Ho}(\mathcal{C})$ such that the following commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"', "\cong"] & B \arrow[r, "g"] \arrow[d, "b"', "\cong"] & C \arrow[r, "h"] \arrow[d, "c"', "\cong"] & \Sigma A \arrow[d, "\Sigma a"', "\cong"] \\ X \arrow[r, "\gamma(f)"] & Y \arrow[r, "\gamma(i)"] & Cf \arrow[r, "\gamma(p)"] & \Sigma X \end{tikzcd}$$

Construction (Distinguished triangles from cofibrations)**.** — Let $j : A \rightarrowtail B$ be a cofibration in $\mathcal{C}$, write B/A for a cokernel of j and $q : B \to B/A$ the "quotient" morphism. The Gluing Lemma for

$$\begin{tikzcd} CA \arrow[d, "{0,\sim}"'] & A \arrow[l, tail, "\iota_A"'] \arrow[r, tail, "j"] \arrow[d, equal] & B \arrow[d, equal] \\ * & A \arrow[l, "0"] \arrow[r, tail, "j"'] & B \end{tikzcd}$$

shows that $0 \cup \mathrm{id}_B : Cj \to B/A$ is also a weak equivalence (do you recognize this?).

We define the *connecting morphism* in $\mathrm{Ho}(\mathcal{C})$ as

$$\delta(j) = \gamma(p) \circ \gamma(0 \cup \mathrm{id}_B)^{-1} : B/A \to \Sigma A.$$

More suggestively, we write

$$B/A \xleftarrow{0\cup\mathrm{id}_B} Cj \xrightarrow{p=q_A\cup 0} \Sigma A.$$

One should draw some pictures of this for themself.

The following commutes by construction in $\mathrm{Ho}(\mathcal{C})$:

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(i)"] \arrow[d, equal] & Cj \arrow[r, "\gamma(p)"] \arrow[d, "\gamma(0\cup\mathrm{id}_B)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j)"'] & B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma A
\end{tikzcd}$$

The middle square commutes because both compositions are the universal map $B \to B/A$ of the pushout $B/A = B \amalg_A *$ (think: the top-right composition embeds B in Cj and then collapses the A-part, the bottom map immediately collapses the A-part). Therefore, the triangle $(\gamma(j), \gamma(q), \delta(j))$ is distinguished.

II.12. Proposition. — *Let $\mathcal{C}$ be a pointed cofibration category with functorial cones. Then a triangle in* $\mathrm{Ho}(\mathcal{C})$ *is distinguished if and only if it is isomorphic to* $(\gamma(j), \gamma(q), \delta(j))$ *for some cofibration $j \in \mathcal{C}$.*

Proof. We haveve just seen that these triangles formed with cofibrations are distinguished. Conversely, let $\psi : X \to Y$ be any morphism. We must show that the elementary distinguished triangle $(\gamma(\psi), \gamma(i), \gamma(p))$ is isomorphic to some $(\gamma(j), \gamma(q), \delta(j))$ for some cofibration j. For this, we choose a factorization

$$\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, tail, "j"'] & & Y \\
& Z \arrow[ur, "\pi{,}\sim"'] &
\end{tikzcd}$$

All vertical morphisms in the following commutative diagram are weak equivalences:

$$\begin{tikzcd}
X \arrow[r, tail, "j"] \arrow[d, equal] & Z \arrow[r, "q"] \arrow[d, equal] & Z/X & \\
X \arrow[r, tail, "j"] \arrow[d, equal] & Z \arrow[r, "i_j"] \arrow[d, "\pi"] & Cj \arrow[r, "p_j"] \arrow[u, "0\cup\mathrm{id}_Z"'] \arrow[d, "\mathrm{id}_{CX}\cup\pi"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\psi"'] & Y \arrow[r, "i_\psi"'] & C\psi \arrow[r, "p\psi"'] & \Sigma X
\end{tikzcd}$$

Applying the localization functor yields a commutative diagram in the homotopy category:

$$\begin{tikzcd}
X \arrow[r, "\gamma(j)"] \arrow[d, equal] & Z \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\pi)"] & Z/X \arrow[r, "\delta(j)"] \arrow[d, "\gamma(\mathrm{id}\cup\pi)\circ\gamma(0\cup\mathrm{id}_Z)^{-1}"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\gamma(\psi)"'] & Y \arrow[r, "\gamma(i_\psi)"'] & C\psi \arrow[r, "\gamma(p_\psi)"'] & \Sigma X.
\end{tikzcd}$$

which ends the proof. $\square$

II.13. Theorem. — *Let $\mathcal{C}$ be a pointed cofibration category with functorial cone. Suppose that* $\mathrm{Ho}(\mathcal{C})$ *is additive and* $\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ *is an equivalence. Then* $(\mathcal{C}, \Sigma, \textit{dist. triangles})$ *forms a triangulated category.*

Remark. — The hypothesis are slightly redundant: we don't really need functorial cones and more surprisingly $\mathrm{Ho}(\mathcal{C})$ is already forced to be additive by the suspension functor being an equivalence (although proving this carefully requires some time and doesn't add much to our understanding, also considered that most of the cofibration categories we work with have functorial cones and their homotopy categories are clearly additive anyway).

Proof. (T0) The class of distinguished triangles is closed under isomorphism. This is fine.

(T1) Every morphism f in $\mathrm{Ho}(\mathcal{C})$ is part of a distinguished triangle (f, g, h). We write $f : A \to B$ in $\mathrm{Ho}(\mathcal{C})$ as $f = \gamma(s)^{-1} \circ \gamma(\psi)$ for two $\mathcal{C}$-morphisms $\psi : A \to D$ and $s : B \to D$ and such that s is a weak equivalence. Then, the following diagram commutes in $\mathrm{Ho}(\mathcal{C})$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\gamma(is)"] \arrow[d, "{\gamma(s), \cong}"] & C\psi \arrow[r, "\gamma(p)"] \arrow[d, equal] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(\psi)"'] & D \arrow[r, "\gamma(i)"'] & C\psi \arrow[r, "\gamma(p)"'] & \Sigma A
\end{tikzcd}$$

but the lower triangle is distinguished.

(T2) For every object X of $\mathrm{Ho}(\mathcal{C})$, the triangle

$$* \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow *$$

is distinguished. The unique morphism $u : * \rightarrowtail X$ in $\mathcal{C}$ is a cofibration by (C1), so the following is distinguished

$$* \longrightarrow X \xrightarrow{\gamma(q) = \mathrm{id}_X} X \longrightarrow \Sigma * .$$

(T3) If (f, g, h) is distinguished, then so is $(g, h, -\Sigma f)$. Without loss of generality, $(f, g, h) = (\gamma(\psi), \gamma(i), \gamma(p))$ for some $\mathcal{C}$-morphism $\psi : X \to Y$. The morphism $\iota_Y : Y \to CY$ is a cofibration by definition of functorial cones and $p = q_X \cup 0 : C\psi \to \Sigma X$ models the quotient of $C\psi/Y$, so the triangle $(\gamma(i), \gamma(p), \delta(i))$ is distinguished.

We now need a couple of claims.

Claim 1. The two morphisms

$$q_Y \cup 0, 0 \cup q_Y : CY \cup_Y CY \to \Sigma Y$$

are additive inverses in $\mathrm{Ho}(\mathcal{C})$.

Proof of the claim. The two morphisms $q_Y \cup 0, 0 \cup q_Y : CY \cup_Y CY \to \Sigma Y$ become additive inverses in $\mathrm{Ho}(\mathcal{C})$: We consider the morphism

$$\xi : CY \cup_Y CY \to \Sigma Y \vee \Sigma Y$$

which intuitively amounts to pinching a suspension in the equator to get the wedge of two suspensions (draw it) and formally is the morphism induced on pushouts by

$$\begin{tikzcd}
CY \vee CY \arrow[d, equal] & Y \vee Y \arrow[l, "\iota_Y \vee \iota_Y"'] \arrow[r, "\nabla"] \arrow[d, equal] & Y \arrow[d] \\
CY \vee CY & Y \vee Y \arrow[l, "\iota_Y \vee \iota_Y"] \arrow[r] & *
\end{tikzcd}$$

So we have

$$(\mathrm{id}_{\Sigma Y} + 0) \circ \xi = q_Y \cup 0 \quad \text{and} \quad (0 + \mathrm{id}_{\Sigma Y}) \circ \xi = 0 \cup q_Y.$$

Thus, $\gamma(\xi) = \gamma(q_Y \cup 0) \perp \gamma(0 \cup q_Y)$. The following commutes in $\mathcal{C}$:

$$\begin{tikzcd}
CY \cup_Y CY \arrow[r, "\xi"] \arrow[d, "\mathrm{id}_Y \cup \mathrm{id}_Y"'] & \Sigma Y \vee \Sigma Y \arrow[d, "\nabla = \mathrm{id} + \mathrm{id}"] \\
CY \arrow[r, "q_Y"'] & \Sigma Y
\end{tikzcd}$$

Since $CY \sim *$ we obtain

$$\gamma(\nabla) \circ \gamma(\xi) = \gamma(q_Y) \circ \gamma(\mathrm{id}_Y \cup \mathrm{id}_Y) = 0$$

in $\mathrm{Ho}(\mathcal{C})$, so

$$\begin{aligned}
\gamma(0 \cup q_Y) + \gamma(q_Y \cup 0) &= \gamma(\nabla) \circ (\gamma(0 \cup q_Y) \perp \gamma(q_Y \cup 0)) \\
&= \gamma(\nabla) \circ \gamma(\xi) \\
&= 0.
\end{aligned}$$

■

This first claim let us prove a second claim.

Claim 2. $\delta(i) = -\Sigma\gamma(\psi)$.

Proof of the claim. We let $\zeta : CY \cup_i C\psi = Ci \to CY \cup_Y CY$ be the morphism induced on pushouts given by

$$\begin{tikzcd}
CY \vee CX \arrow[d, "\mathrm{id}_{CY} \vee C(\psi)"'] & X \vee X \arrow[l, "(\iota_Y \circ \psi) \vee \iota_X"'] \arrow[r, "\nabla"] \arrow[d, "\psi \vee \psi"] & X \arrow[d, "\psi"] \\
CY \vee CY & Y \vee Y \arrow[l, "\iota_Y \vee \iota_Y"] \arrow[r, "\nabla"'] & Y
\end{tikzcd}$$

Then,

$$(q_Y \cup 0) \circ \zeta = q_Y \cup 0 \quad \text{and} \quad (0 \cup q_Y) \circ \zeta = (\Sigma\psi) \circ (0 \cup p).$$

We have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "i"] & C\psi \arrow[r, "p_\psi = 0 \cup q_X"] & \Sigma X \arrow[rr, dashed, "\delta(i)"] & & \Sigma Y \\
& & & CY \cup C\psi \arrow[ul, "0 \cup p, \sim"] \arrow[ur, "q_Y \cup 0"'] &
\end{tikzcd}$$

Thus,

$$\begin{aligned}\delta(i) &= \gamma(q_Y \cup 0) \circ \gamma(0 \cup p)^{-1}\\ &= \gamma(q_Y \cup 0) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1}\\ &= -\gamma(0 \cup q_Y) \circ \gamma(\zeta) \circ \gamma(0 \cup p)^{-1}\\ &= -(\Sigma\psi) \circ (\gamma(0 \cup p) \circ \gamma(0 \cup p)^{-1})\\ &= -\Sigma\psi.\end{aligned}$$

■

Therefore, $(\gamma(i), \gamma(p), \delta(i)) = (\gamma(i), \gamma(p), -\Sigma\gamma(\psi))$ is distinguished as desired.

(T4) Given distinguished triangles $(f, g, h), (f', g', h')$ we can complete

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A' \end{tikzcd}$$

Without loss of generality $(f, g, h) = (\gamma(j), \gamma(q), \delta(j))$ for some cofibration $j : A \rightarrowtail B$ and also with no more loss of generality $(f', g', h') = (\gamma(j'), \gamma(q'), \delta(j'))$ for some cofibration $j' : A' \rightarrowtail B'$. We have

$$\begin{tikzcd} A \arrow[r, "\gamma(j)"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "\gamma(j')"'] & B' \end{tikzcd}$$

and we can focus on the special case $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for some $\mathcal{C}$-morphisms $\alpha : A \to A'$ and $\beta : B \to B'$. Because $\gamma(\beta j) = \gamma(j'\alpha)$, there exists $s : B' \overset{\sim}{\rightarrowtail} \overline{B}$ an acyclic cofibration, a cylinder object (I, i_0, i_1, p) for A, and a homotopy $H : I \to \overline{B}$ from $Hi_0 = sj'\alpha$ to $Hi_1 = s\beta j$, by the calculus of fractions II.6. Consider the diagram

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[d, equal] & B \\ A \arrow[r, tail, "i_0"] \arrow[d, "\alpha"'] & I \cup_{i_1} B = \mathrm{Zyl}(j) \arrow[u, "jp \cup \mathrm{id}_B, \sim"'] \arrow[d, "H \cup s\beta"] \\ A' \arrow[r, tail, "sj'"] \arrow[d, equal] & \overline{B} \\ A' \arrow[r, tail, "j'"'] & B' \arrow[u, tail, "s, \sim"'] \end{tikzcd}$$

Therefore the following diagram also commutes:

$$
\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q)"] & B/A \arrow[r, "\delta(j)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(i_0)"'] \arrow[d, "\gamma(\alpha)"'] & I \cup_{i_1} B \arrow[r, "\gamma(q)"'] \arrow[u, "\gamma(jp \cup B)"'] \arrow[d, "\gamma(H \cup s\beta)"] & (I \cup_{i_1} A)/A = Cj \arrow[r, "\delta(i_0)"'] \arrow[u] \arrow[d, "\gamma((H \cup s\beta)/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
A' \arrow[r, "\gamma(sj')"'] \arrow[d, equal] & \overline{B} \arrow[r, "\gamma(\overline{q})"'] & \overline{B}/A' \arrow[r, "\delta(sj')"'] & \Sigma A' \arrow[d, equal] \\
A' \arrow[r, "\gamma(j')"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "{\gamma(s), \cong}"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "{\gamma(s/A'), \cong}"'] & \Sigma A'
\end{tikzcd}
$$

We have $\gamma(s)^{-1} \circ \gamma(H \cup s\beta) \circ \gamma(jp \cup \beta)^{-1} = b$ and

$$C = \gamma(s/A')^{-1} \circ \gamma((H \cup s\beta)/\alpha) \circ \gamma(jp \cup B/A)^{-1}$$

does the job. We omit the general case.

To be continued...

Lecture 16
15th Jun, 2022

(T5) We prove a seemingly weaker version of the octahedral axiom, which turns out to imply the stronger version (it was not weaker after all).

We start with a special case. Suppose $f = \gamma(j)$ and $f' = \gamma(j')$ for cofibrations $j : A \rightarrowtail B$ and $j' : B \rightarrowtail D$. Then, $f' \circ f = \gamma(j' \circ j)$ where $j' \circ j : A \rightarrowtail D$ is another cofibration. So there is a diagram in $\mathcal{C}$

$$
\begin{tikzcd}
A \arrow[r, tail, "j"] \arrow[d, equal] & B \arrow[r, "q_j"] \arrow[d, tail, "j'"] & B/A \arrow[d, tail, "j'/A"] \\
A \arrow[r, tail, "j' \circ j"] & D \arrow[r, "q_{j'j}"] \arrow[d] & D/A \arrow[d, "D/j"] \\
 & D/B \arrow[r, equal] & D/B
\end{tikzcd}
$$

where the map D/j is induced by

$$
\begin{tikzcd}
* \arrow[d, equal] & A \arrow[l] \arrow[r, "j'j"] \arrow[d, "j"] & D \arrow[d, equal] \\
* & B \arrow[l] \arrow[r, "j'"'] & D
\end{tikzcd}
$$

Applying γ yields a diagram in $\mathrm{Ho}(\mathcal{C})$

$$
\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_j)"] \arrow[d, "\gamma(j')"] & B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(j'/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j')\circ\gamma(j)"] & D \arrow[r] \arrow[d, "\gamma(q)"] & D/A \arrow[r, "\delta(j'\circ j)"] \arrow[d, "\gamma(D/j)"] & \Sigma A \arrow[d, "\Sigma(\gamma(j))"] \\
& D/B \arrow[r, equal] \arrow[d, "\delta(j')"] & D/B \arrow[r, "\delta(j')"] \arrow[d, "\delta(j'/A)=\Sigma\gamma(q_j)\circ\delta(j')"] & \Sigma B \\
& \Sigma B \arrow[r, "\Sigma\gamma(q_j)"'] & \Sigma(B/A) &
\end{tikzcd}
$$

Now onto the general case. Let $f = \gamma(s)^{-1} \circ \gamma(a)$ for a $\mathcal{C}$-morphism $a : A \to B'$ and a weak equivalence $s : B \xrightarrow{\sim} B'$ by the calculus of fractions II.6. We factor $a = pj$ for a cofibration $j : A \rightarrowtail \overline{B}$ and a weak equivalence $p : \overline{B} \xrightarrow{\sim} B'$. Then, $f = \varphi \circ \gamma(j)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p)$ is an isomorphism. We apply the same reasoning to $f' \circ \varphi : \overline{B} \to D$ as $f' \circ \varphi = \psi \circ \gamma(j')$ for some cofibration $j' : \overline{B} \rightarrowtail \overline{D}$ and an isomorphism ψ. Then, j and j' are composable cofibrations in $\mathcal{C}$, so we get the claimed data for $\gamma(j)$ and $\gamma(j')$ by the special case. Since f and f' differ from $\gamma(j)$ and $\gamma(j')$ by isomorphisms, we complete the data from the special case to get the data for f and f'.

$\square$

Example. — Let R be an orthogonal ring spectrum (e.g. the sphere spectrum). A functorial cone on ${}_R\mathrm{Mod}(\mathrm{Sp})$ is given by $M \wedge [0,1]$ and

$$\iota_M = - \wedge 1 : M \to M \wedge [0,1].$$

A cokernel of ι_M is the morphism

$$M \wedge t : M \wedge [0,1] \to M \wedge S^1.$$

So for this choice of functorial cone and cokernel, we obtain $\Sigma M = M \wedge S^1$. The derived category $D(R)$ is triangulated with suspension functor $M \wedge S^1$ and distinguished triangles isomorphic to

$$M \xrightarrow{\gamma(f)} N \xrightarrow{\gamma(i)} (M \wedge [0,1]) \cup_\psi N \xrightarrow{\gamma(p)} M \wedge S^1.$$

Example. — Let $\mathcal{A}$ be an additive category and consider $\mathrm{Ch}(\mathcal{A})$, the category of $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$. The category $\mathcal{K}(\mathcal{A})$ is the homotopy category of $\mathcal{A}$, with objects $\mathbb{Z}$-graded chain complexes in $\mathcal{A}$ and morphisms the chain maps modulo chain homotopy. We show in AT2Sheet7.1 that $\mathrm{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is a localization at the class of chain homotopy equivalences. Moreover clearly $\mathcal{K}(\mathcal{A})$ is additive.

The cone of a complex $A \in \mathrm{Ch}(\mathcal{A})$ is the complex CA with

$$(CA)_n = A_n \oplus A_{n-1},\ d(a, a') = (da + (-1)^n a', da')$$

and $\iota_A : A \to CA$ the inclusion of the first summand. We have that CA is chain contractible and $A \to CA$ is a cofibration. A chain map $f : A \to B$ is a cofibration if there are objects C_n for $n \in \mathbb{Z}$ and isomorphisms $A_n \oplus C_n \cong B_n$ that restrict to $f_n : A_n \to B_n$. The cofibration and the chain homotopy equivalences are a cofibration category with functorial cone.

The shift $A[1]$ of $A \in \mathrm{Ch}(\mathcal{A})$ is given by $(A[1])_n = A_{n-1}$ with reindexed differential. The projection to the second summand
$$CA \to A[1]$$
exhibits $A[1]$ as a cokernel of $\iota_A : A \to CA$, so $\Sigma A = A[1]$.

Then we have that $\mathcal{K}(\mathcal{A})$ is a triangulated category with respect to the shift functor and the triangles that are isomorphic to
$$A \xrightarrow{\gamma(\psi)} B \xrightarrow{\gamma(i)} C\psi \xrightarrow{\gamma(p)} A[1]$$
where $(C\psi)_n = B_n \oplus A_{n-1}$.

Example. — Let $\mathcal{A}$ be an abelian category. As we saw in AT2Sheet6.2, there is another cofibration structure on $\mathrm{Ch}(\mathcal{A})$ with the same cofibrations and quasi-isomorphisms as weak equivalences. Let $\gamma : \mathrm{Ch}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ be a localization at the class of quasi-isomorphisms; the target $\mathcal{D}(\mathcal{A})$ is called the derived category of $\mathcal{A}$.

The derived category $\mathcal{D}(\mathcal{A})$ is triangulated with shift as the suspension functor and distinguished triangle the images of the distinguished triangles in $\mathcal{K}(\mathcal{A})$.

Algebraists who do not know much homotopy theory (i.e. most of them), would construct $\mathcal{D}(A)$ as a localization of $\mathcal{K}(A)$, but knowing homotopy theory makes the story clearer (in the opinion of the homotopy inclined writer).

The proof of the following theorem would take too much time to include in this course, but we record it here, to show one of the many instances of algebra and homotopy theory intertwining.

II.14. Theorem. — *For every ring S, there is an equivalences of triangulated categories*
$$\mathcal{D}({}_S\mathrm{Mod}) \cong \mathcal{D}(HS)$$

The following proposition contains some of the tools that allow one to do "homological algebra" with triangulated categories.

II.15. Proposition. — *Let $\mathcal{T}$ be a triangulated category.*

i) For every distinguished triangle (f, g, h) and every $\mathcal{T}$-object X, the following sequences are exact
$$\mathcal{T}(\Sigma A, X) \xrightarrow{\mathcal{T}(h,X)} \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X),$$
$$\mathcal{T}(X, A) \xrightarrow{\mathcal{T}(X,f)} \mathcal{T}(X, B) \xrightarrow{\mathcal{T}(X,g)} \mathcal{T}(X, C) \xrightarrow{\mathcal{T}(X,h)} \mathcal{T}(X, \Sigma A).$$

ii) Awakened five lemma. Let (a, b, c) be a morphism between distinguished triangles. Then if two of a, b and c are isomorphisms, so is the third.

iii) Let (f, g, h) be a triangle such that $(g, h, -\Sigma f)$ is distinguished. Then (f, g, h) is also distinguished.

iv) Let (f_1, g_1, h_1), (f_2, g_2, h_2) and (f_3, g_3, h_3) be distinguished triangles such that $f_3 = f_2 \circ f_1$. Then there are morphisms $\bar{x}$ and $\bar{y}$ such that $(\bar{x}, \bar{y}, (\Sigma g_1) \circ h_2)$ is distinguished and the following commutes

somebigdiagram

v) For every distinguished triangle, the following are equivalent.

- *The morphism $f : A \to B$ has a retraction.*
- *The morphism $g : B \to C$ has a retraction.*

- *The morphism $h : C \to \Sigma A$ is the 0-morphism.*

vi) If (f, g, h) is distinguished and $s : C \to B$ is such that $gs = \mathrm{id}_C$, then the morphisms $f : A \to B$ and $s : C \to B$ present B as a coproduct of A and C (the familiar $B = A \oplus C$).

vii) Let I be a set and $\{(f_i, g_i h_i)\}_{i\in I}$ be a family of distinguished triangles, then the following triangles are also distinguished

$$\bigoplus_{i\in I} A_i \xrightarrow{\oplus_i f_i} \bigoplus_{i\in I} B_i \xrightarrow{\oplus_i g_i} \bigoplus_{i\in I} C_i \xrightarrow{\mathrm{can}\circ(\oplus_i h_i)} \Sigma(\bigoplus_{i\in I} A_i),$$

$$\prod_{i\in I} A_i \xrightarrow{\Pi_i f_i} \prod_{i\in I} B_i \xrightarrow{\Pi_i g_i} \prod_{i\in I} C_i \xrightarrow{\mathrm{can}^{-1}\circ(\Pi_i h_i)} \Sigma(\prod_{i\in I} A_i).$$

viii) If $A \oplus B$ is a coproduct of A and B in $\mathcal{T}$ with canonical morphisms $i_A : A \to A \oplus B$, $i_B : B \to A \oplus B$, then the following triangle is distinguished

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

where p_B is characterized by $p_B i_A = 0$ and $P_B i_B = \mathrm{id}_B$.

Proof. (i) Exactness at $\mathcal{T}(B, X)$. If (f, g, h) is distinguished, then $gf = 0$. We have

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed, "c"', "(\mathrm{T4})"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \end{tikzcd}$$

where the first row is distinguished by (T2). To conclude, let $\psi : B \to X$ be in the kernel of f^*, i.e. $\psi \circ f = 0 : A \to X$. Applying (T4) to

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[r, "h"] \arrow[d, dashed, "\psi"] & \Sigma A \arrow[d] \\ 0 \arrow[r] & X \arrow[r, equal] & X \arrow[r] & 0 \end{tikzcd}$$

we get that $\varphi g = g^*(\varphi) = \psi$. So $\psi \in \mathrm{im}(g^*)$.

(ii) The proof of the five lemma in an abelian category should come in handy.

(iii) Suppose that $(g, h, -\Sigma f)$ is distinguished. Rotating twice (T3), also $(-\Sigma f, -\Sigma g, -\Sigma h)$ is distinguished. We choose a distinguished triangle $(f, \overline{g}, \overline{h})$ that starts with f. Then, the triangle $(-\Sigma f, -\Sigma \overline{g}, -\Sigma \overline{h})$ is distinguished. Consider the diagram

$$\begin{tikzcd} \Sigma A \arrow[r, "-\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[r, "-\Sigma g"] \arrow[d, equal] & \Sigma C \arrow[r, "-\Sigma h"] \arrow[d, dashed, "\overline{c}"] & \Sigma^2 C \arrow[d, equal] \\ \Sigma A \arrow[r, "\Sigma f"'] & \Sigma B \arrow[r, "-\Sigma \overline{g}"'] & \Sigma \overline{C} \arrow[r, "-\Sigma \overline{h}"'] & \Sigma^2 A \end{tikzcd}$$

There is an isomorphism $\overline{c} : \Sigma C \to \Sigma \overline{C}$ that makes the above commute by (T4) and the Five Lemma (ii). Because Σ is fully faithful, there is a unique isomorphism $c : C \to \overline{C}$ such that $\Sigma c = \overline{c}$. Then, we also have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, "\overline{g}"'] & \overline{C} \arrow[r, "\overline{h}"'] & \Sigma A
\end{tikzcd}$$

so (f, g, h) is isomorphic to a distinguished triangle, so it itself is distinguished.

Wait a minute... Doesn't it feel like we have only seen formal stuff all this course? Maybe it's better if we leave this proof and turn to something more concrete...

Chapter III.

Bordism Spectra and Thom's Theorem

III

Lecture 17
20^{th} Jun, 2022

Let's go!

Little did I know that Prof. Schwede was about to catch COVID and that we would skip three lectures after this one...

Generalized Homology Theories

Definition. — A *generalized homology theory* consists of functors $E_k \to \mathrm{Ab}$ for all $k \in \mathbb{Z}$ and natural morphisms (connecting morphisms)

$$\partial : E_{k+1}(Cf) \to E_k(A)$$

for all continuous based maps $f : A \to B$. This data has to satisfy the following axioms.

i) (Additivity) For all families of based spaces $\{A_i\}_{i \in I}$, the canonical map

$$\bigoplus_{i \in I} E_k(A_i) \to E_k(\bigvee_{i \in I} A_i)$$

is an isomorphism.

ii) (Homotopy invariance) If $f, g : A \to B$ are based homotopic, then $E_k(f) = E_k(g)$.

iii) (Exactness) For all $f : A \to B$ based continuous maps, the following sequence is exact:

$$\cdots \to E_{k+1}(Cf) \xrightarrow{\partial} E_k(A) \xrightarrow{E_k(f)} E_k(B) \xrightarrow{E_k(i)} E_k(Cf) \xrightarrow{\partial} \cdots .$$

III.1. Example. — Let E be a sequential spectrum. Then the functors

$$E_k(A) := \pi_k(E \wedge A)$$

define a generalized homology theory with respect to the connecting morphism ∂ defined as

$$\pi_{k+1}(E \wedge Cf) \xrightarrow{\pi_k(E \wedge p)} \pi_{k+1}(E \wedge A \wedge S^1) \xrightarrow[\cong]{(- \wedge S^1)^{-1}} \pi_k(E \wedge A)$$

where $(- \wedge S^1)^{-1}$ is the inverse of the suspension morphism.

Remark. — Every general homology theory $\{E_k, \partial\}$ satisfies the following basic properties.

(a) Additivity for $X = Y = *$ shows that the map

$$E_k(*) \oplus E_k(*) \to E_k(* \vee *) \cong E_k(*),\ (x, y) \mapsto x + y$$

is an isomorphism, hence $E_k(*) = 0$.

(b) If A is contractible to the base point, then $E_k(A) = 0$.

(c) The unique map $t_A : A \to *$ has a long exact sequence

$$\cdots \to E_{k+1}(Ct_A) \xrightarrow{\partial} E_k A \to E_k(*) \to E_k(Ct_A) \to \cdots$$

and the map $Ct_A = CA \cup_A * \xrightarrow{p} A \wedge S^1$ is a homeomorphism, hence we can redefine the suspension isomorphism (of singular homology fame):

$$E_k(A) \xrightarrow[\cong]{\partial^{-1}} E_{k+1}(Ct_A) \xrightarrow[\cong]{E_{k+1}(p)} E_{k+1}(A \wedge S^1).$$

(d) Let (B, A) be a pair of spaces, with $A \hookrightarrow B$ a h-cofibration. Then, as usual, the quotient map $0 \cup \mathrm{id}_B : A \wedge [0,1] \cup_A B = C(\mathrm{incl}) \to B/A$ is a based homotopy equivalence, so we get a long exact sequence

$$\cdots \to E_{k+1}(B/A) \xrightarrow{\delta} E_k(A) \xrightarrow{\mathrm{incl}_*} E_k(B) \xrightarrow{\mathrm{proj}_*} E_k(B/A) \to \cdots$$

with $\delta = \partial \circ E_{k+1}(0 \cup \mathrm{id}_B)^{-1}$.

Remark (Unreduced theories). — Let X be a space, $E = \{E_k, \partial\}$ a generalized homology theory. The corresponding *unreduced E-homology* is given by

$$E_k^+(X) = E_k(X_+),$$

i.e. the original generalized homology theory of X with a disjoint basepoint added. Given that the functor $(-)_+ : \mathcal{T} \to \mathcal{T}_*$ takes disjoint unions to wedges, the additivity property for the original theory gives an isomorphism

$$\textstyle\bigoplus_{i\in I} E_k^+(A_i) \to E_k^+(\coprod_{i\in I} A_i).$$

Unreduced generalized homology theories give a slightly different version of what a "generalized homology theory" should be, and often people denote them with E and the reduced (or based) version with $\tilde{E}$, but the Professor thinks that reduced homology theories are more fundamental, or at least that they tend to appear more.

Remark (Relative theories). — We can easily get a relative version of generalized homology theories. For a pair of spaces (X, Y) we write

$$CY \cup_Y X = \frac{Y \times [0,1] \cup_Y X}{Y \times \{0\}}$$

for the *unreduced cone*, based at the tip, the class of $Y \times \{0\}$. Then, given a generalized homology theory E, the corresponding *relative E-homology* is given by

$$E_k(X, Y) = E_k(CY \cup_Y X).$$

The unreduced cone is homeomorphic to the reduced cone of $i_+ : Y_+ \to X_+$. Hnece we get a long exact sequence

$$\cdots \to E_{k+1}(X, Y) \xrightarrow{\partial} E_k^+(Y) \xrightarrow{(\mathrm{incl}_+)_*} E_k^+(X) \to E_k(X, Y) \to \cdots.$$

Note that for X based by x_0 we have $E_k(X, \{x_0\}) \cong E_k(X)$.

III.2. Proposition. — *Let $\{E_k, \partial\}$ be a generalized homology theory. Let $\mathrm{tel}_{i\geqslant 0} X_i$ be the mapping telescope of a sequence of continuous maps*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots .$$

Then the canonical maps $j_m : X_m \to \mathrm{tel}_{i\geqslant 0} X_i$ induces an isomorphism

$$\operatorname*{colim}_n E_k^+(X_n) \xrightarrow{\cong} E_k^+(\mathrm{tel}_{i\geqslant 0} X_i).$$

Proof. Write the mapping telescope as the pushout of the appropriate span

$$\coprod_{i\geqslant 0}(X_i \times [0,1]) \leftarrow \coprod_{i\geqslant 0}(X_i \times \{0,1\}) \to \coprod_{i\geqslant 0} X_i.$$

The map $\coprod_{i\geqslant 0}(X_i \times \{0,1\}) \to \coprod_{i\geqslant 0}(X_i \times [0,1])$ is a cofibration, because cofibrations are closed under product with an identity and coproducts (both facts are easy to prove although not necessarily obvious), so its cobase change is also a cofibration. With this data we obtain a long exact sequence

$$\cdots \to E_{k+1}(\bigvee_{i\geqslant 0}(X_i)_+ \wedge S^1) \to E_k^+(\coprod_{i\geqslant 0} X_i) \to E_k^+(\mathrm{tel}_{i\geqslant 0} X_i) \to E_k(\bigvee_{i\geqslant 0}(X_i)_+ \wedge S^1) \to \cdots .$$

Using additivity and the suspension isomorphism we modify the sequence to get

$$\cdots \to \bigoplus_{i\geqslant 0} E_k^+(X_i) \xrightarrow{\Delta} \bigoplus_{i\geqslant 0} E_k^+(\coprod_{i\geqslant 0} X_i) \to E_k^+(\mathrm{tel}_{i\geqslant 0} X_i) \to \bigoplus_{i\geqslant 0} E_{k-1}^+(X_i) \to \cdots .$$

If one tries really hard, they can show that the map Δ is $x \mapsto x - (f_i)_*(x)$, which is injective, hence

$$E_k^+(\mathrm{tel}_{i\geqslant 0} X_i) \cong \operatorname{coker} \Delta$$

which is a presentation of $\mathrm{colim}_n\, E_k^+(X_n)$. □

If X is a CW-complex with skeleta X_i, then the quotient map collapsing the intervals

$$\mathrm{tel}_{i\geqslant 0} X_i \to \cup X_i = X$$

is a homotopy equivalence. Hence we get the following corollary.

III.3. Corollary. — *For a CW-complex X with skeleta X_i and a generalized homology theory $\{E_k, \partial\}$, the morphism*

$$\operatorname*{colim}_{n\geqslant 0} E_k(X_n) \xrightarrow{\cong} E_k(X)$$

is an isomorphism.

We have seen in Example III.1 that sequential spectra are enough to represent generalized homology theories, but if instead we have orthogonal ring spectra we obtain additional structure, as we now describe.

III.4. Construction (Exterior products). — Let E be an orthogonal ring spectrum. Then the homology theory represented by E supports exterior products

$$\times : E_k(A) \times E_l(B) \to E_{k+l}(A \wedge B)$$

defined as we now describe. Let $f : S^{n+k} \to E_m \wedge A$ and $g : S^{n+l} \to E_n \wedge B$ represent classes in $E_k(A) = \pi_k(E \wedge A)$ and $E_l(B) = \pi_l(E \wedge B)$, we write $f \cdot g$ for the composite

$$S^{m+k+n+l} \xrightarrow{f \wedge g} E_m \wedge A \wedge E_n \wedge B \xrightarrow{\cong} E_m \wedge E_n \wedge A \wedge B \xrightarrow{\mu_{m,n} \wedge A \wedge B} E_{m+n} \wedge A \wedge B.$$

Then we define

$$[f] \times [g] = (-1)^{kn}[f \cdot g]$$

which is an element of $\pi_{k+l}(E \wedge A \wedge B) = E_{k+l}(A \wedge B)$. If $A = S^0$ the definition has to be modified slightly (in the obvious way).

III.5. Proposition. — *If E is an orthogonal ring spectrum, the pairings defined in the previous paragraph (Construction III.4) are associative, unital and graded-commutative and as such they make the E-homology groups $E_*(B) = \{E_k(B)\}_{b \in \mathbb{Z}}$ into a graded module over the graded ring $\pi_*(E)$.*

Bordism

Definition. — A *singular manifold* of dimension k over a space X is a pair (M, h) consisting of a smooth closed manifold M and a continuous map $h : M \to X$. Two singular k-manifolds (M, h) and (M', h') over X are *bordant* if there is a *bordism* between them, where the latter is a triple (B, H, ψ) consisting of

- a compact smooth $(k+1)$-manifold B,
- a continuous map $H : B \to X$,
- a diffeomorphism $M \amalg M' \underset{\cong}{\xrightarrow{\psi}} \partial B$ such that

$$\begin{tikzcd} M \arrow[r, "\psi|_M"] \arrow[dr, bend right, "h"'] & B \arrow[d, "H"] & M' \arrow[l, "\psi|_{M'}"'] \arrow[dl, bend left, "h'"] \\ & X & \end{tikzcd}$$

Bordism of singular k-manifolds over X is an equivalence relation. Reflexivity and symmetry are clear, transitivity is more subtle: we need to glue bordisms, but the smooth structure at the gluing point is not well-defined. To circumvent this one needs to talk about collar neighborhoods a bit.

We denote by $\mathcal{N}_k(X)$ the set of bordism classes of singular k-manifold over X, which becomes a group if we endow it with the operation of disjoint union in M (so $\mathcal{N}_k(-)$ is a functor $\mathcal{T} \to \mathrm{Ab}$ by postcomposition with continuous maps $X \to Y$).

Observe that for all $x \in \mathcal{N}_k(X)$ we have the relation $2x = 0$, as one can (and should!) see by drawing a small picture. In particular, this means that $\mathcal{N}_k(X)$ is an $\mathbb{F}_2$-vector space.

III.6. Proposition. — *The following holds.*

i) Let $\varphi, \varphi' : X \to Y$ be homotopic maps, then

$$\varphi_* = \varphi'_* : \mathcal{N}_k(X) \to \mathcal{N}_k(Y).$$

ii) For every weak equivalence $\varphi : X \to Y$, the induced map $\varphi_* : \mathcal{N}_k(X) \to \mathcal{N}_k(Y)$ is an isomorphism.

iii) For all families of spaces $\{X_i\}_{i\in I}$, the canonical map

$$\bigoplus_{i\in I} \mathcal{N}_k(X_i) \to \mathcal{N}_k(\coprod_{i\in I} X_i)$$

is an isomorphism.

Proof. (i) Let $H : X \times [0,1] \to Y$ be a homotopy from φ to φ' and let (M, h) be a singular k-manifold over X. Then $(M \times [0,1], H \circ (h \times [0,1]), \psi)$, where $\psi : M \amalg M \to M \times [0,1]$ is the inclusion of the M-summands at both ends, is a bordism from $(M, \varphi \circ h)$ to $(M, \varphi' \circ h)$.

$$\begin{tikzcd}
M \arrow[r] \arrow[ddr, bend right] & M \times [0,1] \arrow[d, "h\times[0,1]"] & M \arrow[l] \arrow[ddl, bend left] \\
 & X \times [0,1] \arrow[d, "H"] & \\
 & X &
\end{tikzcd}$$

Therefore,

$$\varphi_*[M, h] = [M, \varphi h] = [M, \varphi' h] = \varphi'_*[M, h],$$

as we claimed.

(ii) Let $\varphi : X \to Y$ be a weak equivalence. First we show surjectivity of φ_*. Let (M, g) be a singular k-manifold over Y. A strong result from differential topology says that smooth manifolds can be triangulated, and so these admit CW structures. Since φ is a weak homotopy equivalence, the lifting problem

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & X \arrow[d, "\varphi{,}\sim"] \\
M \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

can be solved such that the lower triangle commutes up to homotopy (see AT1Sheet10.2). Then,

$$\begin{aligned}
\varphi_*[M, h] &= [M, \varphi h] \\
&= (\varphi h)_*[M, \mathrm{id}_M] \\
&= g_*[M, \mathrm{id}_M] \\
&= [M, g].
\end{aligned}$$

using (i).

Now for injectivity, let (M, h) be a singular k-manifold over X such that

$$[M, \varphi h] = \varphi_*[M, h] = 0$$

in $\mathcal{N}_k(Y)$. Then, there is a nullbordism $(B, H : B \to Y, \psi)$ of $\varphi h : M \to Y$.

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, bend right, "\varphi h"'] & B \arrow[d, "H"] & \varnothing \arrow[l, hook'] \arrow[dl, bend left] \\ & Y & \end{tikzcd}$$

The triangulation result from differential topology can even endow B with a triangulation such that ∂B is a subcomplex. So ∂B is CW subcomplex of B. Thus, we get

$$\begin{tikzcd} \partial B \overset{\psi}{\cong} M \arrow[r, "h"] \arrow[d, hook] & X \arrow[d, "\varphi, \sim"] \\ B \arrow[r, "H"'] \arrow[ur, dashed, "K"] & Y \end{tikzcd}$$

where the lower triangle is up to homotopy again (using again AT1Sheet10.2). Therefore (B, K, ψ) is a nullbordism for (M, h).

(iii) All manifolds and bordisms that we are considering are compact, so there are finitely many path components. Thus, all continuous maps $h : M \to \coprod_{i\in I} X_i$ take values in $\coprod_{i\in J} X_i$ for some finite subset $J \subseteq I$, and similarly for bordisms. For $j \in J$ we take $M_j = h^{-1}(X_j) \subseteq M$. Then,

$$[M, h] = \sum_{j\in J} [M_j, h|_{M_j}].$$

□

Lecture 18
7th Apr, 2022

Given the lack of time, from now on we will focus on definitions and constructions more than proofs.

Construction (Boundary map). — Let $X = A \cup B$ be an open cover of a space X. We will define a homomorphism $\partial : \mathcal{N}_k(X) \to \mathcal{N}_{k-1}(A \cap B)$. Let $(M, h : M \to X)$ be a singular k-manifold. Then, the two sets $h^{-1}(X - A)$ and $h^{-1}(X - B)$ are disjoint closed subsets of the compact manifold M. The Tietze extension theorem provides a continuous map $r : M \to \mathbb{R}$ such that

One really really should make some drawings here!

$$h^{-1}(X - A) \subseteq r^{-1}(0) \quad \text{and} \quad h^{-1}(X - B) \subseteq r^{-1}(1),$$

and we can assume r to be smooth. Let $t \in (0, 1)$ be a regular value of r via Sard's Theorem. By the Regular Value Theorem $r^{-1}(t)$ is a smooth closed regularly embedded submanifold of M of dimension $k - 1$. Moreover, $h(r^{-1}(t)) \subseteq A \cap B$. So we may set

$$\partial([M, h]) = \left[r^{-1}(t), h|_{r^{-1}(t)}\right].$$

III.7. Proposition. — *Let $X = A \cup B$ be an open cover, and let (M, h) be a singular k-manifold over X.*

(i) The class $[r^{-1}(t), h|_{r^{-1}(t)}] \in \mathcal{N}_{k-1}(A \cap B)$ is independent of the choices.

(ii) The following sequence is exact:

$$\cdots \longrightarrow \mathcal{N}_n(A \cap B) \longrightarrow \mathcal{N}_n(A) \oplus \mathcal{N}_n(B) \longrightarrow \mathcal{N}_k(X) \xrightarrow{\partial} \mathcal{N}_{k-1}(A \cap B) \longrightarrow \cdots$$

Construction. — Let X be a based space. The *reduced k-th bordism group* is defined as

$$\tilde{\mathcal{N}}_k(X) = \operatorname{coker}\left(\mathcal{N}_k = \mathcal{N}_k(*) \xrightarrow{\mathcal{N}_k(\mathrm{incl})} \mathcal{N}_k(X)\right).$$

This is not the most common definition, but the classical one can be fiddly when one has to deal with manifolds with corners. We write $[\![M, h]\!]$ for the class of (M, h) in $\tilde{\mathcal{N}}_k(X)$. The map

$$(\mathrm{proj}, u_*) : \mathcal{N}_k(X) \to \tilde{\mathcal{N}}_k(X) \oplus \mathcal{N}_k(*)$$

is an isomorphism. If Y is an unbased space, then the composite

$$\mathcal{N}_k(Y) \xrightarrow{\mathrm{incl}_*} \mathcal{N}_k(Y_+) \twoheadrightarrow \tilde{\mathcal{N}}_k(Y_+)$$

is an isomorphism.

Construction. — Let $f : X \to Y$ be a continuous map and let Cf be its unreduced mapping cone. The cone is covered by two open subsets:

$$A = (X \times [0,1))/(X \times 0) \quad \text{and} \quad B = Cf \setminus \text{cone point}.$$

Then,

$$A \cap B \cong X \times (0,1) \xrightarrow{\mathrm{proj}_1, \simeq} X$$

is a homotopy equivalence. The connecting homomorphism is

$$\begin{tikzcd}
\mathcal{N}_k(Cf) \arrow[rr, "\partial"] \arrow[dr, two heads] & & \mathcal{N}_{k-1}(X \times (0,1)) \arrow[r, "\mathrm{proj}_*"] & \mathcal{N}_{k-1}(X) \\
& \tilde{\mathcal{N}}_k(Cf) \arrow[urr, bend right, "\overline{\partial}"'] & &
\end{tikzcd}$$

We consider the diagrams

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x \mapsto (x,1/2)"'] & Y \arrow[r, "i"] \arrow[d, "\mathrm{incl}"] & Cf \arrow[d, equal] \\
A \cap B \arrow[r, hook, "\mathrm{incl}"'] & B \arrow[r, "\mathrm{incl}"'] & Cf
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{N}_k(X) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathcal{N}_k(Y) \arrow[r, "i_*"] \arrow[d, "\cong"] & \mathcal{N}_k(Cf) \arrow[d, equal] \\
\mathcal{N}_k(A \cap B) \arrow[r] & \mathcal{N}_k(B) \arrow[r] & \mathcal{N}_k(Cf)
\end{tikzcd}$$

III.8. Proposition. — *For every continuous map $f : X \to Y$ the following sequence is exact:*

$$\cdots \longrightarrow \mathcal{N}_k(X) \xrightarrow{f_*} \mathcal{N}_k(Y) \xrightarrow{i_*} \tilde{\mathcal{N}}_k(Cf) \xrightarrow{\overline{\partial}} \mathcal{N}_{k-1}(X) \longrightarrow \cdots$$

Remark (Products in bordisms)**.** — For spaces X, Y the map

$$\begin{aligned} \mathcal{N}_k(X) \times \mathcal{N}_\ell(Y) &\to \mathcal{N}_{k+\ell}(X \times Y), \\ ([M,f],[N,g]) &\mapsto [M \times N, f \times g] \end{aligned}$$

is well-defined, natural and biadditive. If X, Y are based spaces, the composite

$$\begin{tikzcd}
\mathcal{N}_k(X) \times \mathcal{N}_\ell(Y) \arrow[r, "\times"] \arrow[d] & \mathcal{N}_{k+\ell}(X \times Y) \arrow[d, "q_*"] \\
\tilde{\mathcal{N}}_k(X) \times \tilde{\mathcal{N}}_k(Y) \arrow[r, dashed, "\exists! \wedge"'] & \mathcal{N}_{k+\ell}(X \wedge Y) \arrow[d] \\
& \tilde{\mathcal{N}}_{k+\ell}(X \wedge Y)
\end{tikzcd}$$

is well-defined, natural and biadditive.

We define $d = [\![S^1, \mathrm{id}_{S^1}]\!] \in \tilde{\mathcal{N}}_1(S^1)$.

III.9. Proposition. — *Let X be a based CW complex. Then, the exterior product map*

$$- \wedge d : \tilde{\mathcal{N}}_k(X) \to \tilde{\mathcal{N}}_{k+1}(X \wedge S^1)$$

is an isomorphism. Moreover, for based continuous maps $f : X \to Y$ the map

$$p_* : \tilde{\mathcal{N}}_k(Cf) \to \tilde{\mathcal{N}}_k(X \wedge S^1)$$

equals the composite

$$\tilde{\mathcal{N}}_k(Cf) \xrightarrow{\overline{\partial}} \mathcal{N}_{k-1}(X) \xrightarrow{- \wedge d} \tilde{\mathcal{N}}_k(X \wedge S^1).$$

The Thom-Pontryagin Consruction

We want to construct a map

$$\Theta : \mathcal{N}_k(X) \to \mathrm{MO}_k(X_+) = \pi_k(\mathrm{MO} \wedge X_+).$$

To fix our notation let us recall some stuff about MO first.

We will work with the sequential spectrum given by $\mathrm{MO}_n = \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^\infty))$. Every linear monomorphism $\alpha : V \hookrightarrow W$ induces a based map

$$\alpha_* : \mathrm{Th}(\mathrm{Gr}_n(V)) \to \mathrm{Th}(\mathrm{Gr}_n(W)),\ [v, L] \mapsto [\alpha(v), \alpha(L)]$$

where we take a monomorphism so that the dimension under $L \mapsto \alpha(L)$ does not drop. The structure maps are given by

$$\begin{aligned} S^1 \wedge \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^\infty)) &\to \mathrm{Th}(\mathrm{Gr}_{1+n}(\mathbb{R}^\infty)), \\ x \wedge [v, L] &\mapsto [(x, v), \mathbb{R} \oplus L]. \end{aligned}$$

Let M be a compact smooth k-manifold, possibly with boundary. The *fundamental class* $\langle M \rangle \in \mathrm{MO}_k(M/\partial M)$ is defined as follows. We choose a smooth embedding $i : M \hookrightarrow \mathbb{R}^{n+k}$. The *normal bundle* ν has as total space

$$\nu = \{(v, m) \in \mathbb{R}^{n+k} \times M : v \perp (Ti)(T_m M)\}.$$

We can assume that the *exponential map*

$$D(\nu) = \{(v, m) \in \mathbb{R}^{n+k} \times M : v \perp (Ti)(T_m M),\ |v| \leqslant 1\} \to \mathbb{R}^{n+k},\ (v, m) \mapsto v + i(m)$$

is an embedding. The *collapse map* is the continuous map

$$\begin{aligned}
\mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{n+k})) \wedge M/\partial M \leftarrow S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\} : c(i),\\
* \leftarrow\!\shortmid S^{n+k} \setminus \mathring{D}(\nu)\\
\left(\frac{v}{1-|v|},\ (Ti)(T_mM)^{\perp}\right) \wedge m \leftarrow\!\shortmid v + i(m)
\end{aligned}$$

for $v \in \mathbb{R}^{n+k}, |v| < 1, m \in M$. Then, $\langle M \rangle$ is represented by the composite

$$S^{n+k} \xrightarrow{c(i)} \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{n+k})) \wedge M/\partial M \xrightarrow{\kappa_*^{n+k}} \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{\infty})) \wedge M/\partial M = \mathrm{MO}_n \wedge M/\partial M$$

in $\mathrm{MO}_k(M/\partial M)$ where

$$\kappa^{n+k} : \mathbb{R}^{n+k} \to \mathbb{R}^{\infty}, (x_1, \dots, x_{n+k}) \mapsto (x_1, \dots, x_{n+k}, 0, 0, \dots).$$

III.10. Proposition. — *The class $\langle M \rangle \in \mathrm{MO}_K(M/\partial M)$ is independent of the choice of the embedding i.*

Proof sketch.. If we replace $i : M \hookrightarrow \mathbb{R}^{n+k}$ by $\alpha \circ i : M \hookrightarrow \mathbb{R}^{1+n+k}$ with

$$\alpha : \mathbb{R}^{n+k} \to \mathbb{R}^{1+n+k},\ (x_1, \dots, x_{n+k}) \mapsto (0, x_1, \dots, x_{n+k}),$$

then $c(\alpha \circ i)$ is homotopic to the composite

$$\begin{tikzcd}
S^{1+n+k} \arrow[r, "S^1 \wedge c(i)"] & S^1 \wedge \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{n+k})) \wedge M/\partial M \arrow[r, "S^1 \wedge \kappa_*^{n+k}"] & S^1 \wedge \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{\infty})) \wedge M/\partial M \arrow[d, equal] \\
& & S^1 \wedge \mathrm{MO}_n \wedge M/\partial M \arrow[d, "\sigma_n"] \\
& & \mathrm{MO}_{1+n} \wedge M/\partial M.
\end{tikzcd}$$

Now let $i : M \hookrightarrow \mathbb{R}^{n+k}$ and $j : M \hookrightarrow \mathbb{R}^{\overline{n}+k}$ be two smooth embeddings. Without loss of generality $n = \overline{n}$. Consider

$$\begin{aligned}
M \times [0,1] &\to \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}\\
(m,t) &\mapsto (ti(m), (1-t)i(m))
\end{aligned}$$

which gives a homotopy $c(\mathrm{emb}_1 \circ i) \simeq c(\mathrm{emb}_2 \circ i)$. Moreover, there is

$$\begin{aligned}
M \times [0,1] &\to \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}\\
(m,t) &\mapsto (ti(m), (1-t)j(m)).
\end{aligned}$$

Together, these gives a homotopy $c(\mathrm{emb}_2 \circ i) \simeq c(\mathrm{emb}_2 \circ j)$. □

Example. — Consider $\langle S^1 \rangle \in \mathrm{MO}_1(S^1_+)$ along with $q : S^1_+ \to S^1,\ + \mapsto \infty$ and the induced map $q_* : \mathrm{MO}_1(S^1_+) \to \mathrm{MO}_1(S^1)$. Then,

$$q_* \langle S^1 \rangle = 1 \wedge S^1.$$

III.11. Proposition. — *Let M and N be smooth closed k-manifolds.*

(i) In $\mathrm{MO}_k(M/\partial M \wedge N/\partial N) \cong \mathrm{MO}_k\left(\frac{M \amalg N}{\partial(M \amalg N)}\right)$ *we have*

$$\langle M \cup N \rangle = i^1_* \langle M \rangle + i^2_* \langle N \rangle$$

where $i^1 : M/\partial M \to M/\partial M \vee N/\partial N$ *and* $i^2 : N/\partial N \to M/\partial M \vee N/\partial N$.

(ii) For $\delta : \mathrm{MO}_k(M/\partial M) \to \mathrm{MO}_{k-1}(\partial M)$ *we have*

$$\delta\langle M \rangle = \langle \partial M \rangle$$

is $\mathrm{MO}_{k-1}(\partial M_+)$.

Proof. We prove (i). Consider the projections

$$N/\partial N \xleftarrow{p_2} \frac{M \amalg N}{\partial(M \amalg N)} \xrightarrow{p_1} M/\partial M$$

Let $i : M \amalg N \hookrightarrow \mathbb{R}^{n+k}$ be a smooth embedding. Then, the collapse map of $i|_M : M \to \mathbb{R}^{n+k}$ equals the composite

$$S^{n+k} \xrightarrow{c(i)} \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{n+k})) \wedge \tfrac{M \amalg N}{\partial(M \amalg N)} \xrightarrow{\mathrm{id} \wedge p^1} \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^{n+k})) \wedge M/\partial M.$$

So $p^1_* \langle M \amalg N \rangle = \langle M \rangle$ in $\mathrm{MO}_k(M/\partial M)$. The map

$$(p^1_*, p^2_*) : \mathrm{MO}_k\left(\frac{M \amalg N}{\partial(M \amalg N)}\right) \to \mathrm{MO}_k(M/\partial M) \times \mathrm{MO}_k(N/\partial N)$$

is an isomorphism. Under this map we have

$$p^1_* \langle M \amalg N \rangle = \langle M \rangle = p^1_*(i^1_* \langle M \rangle + i^2_* \langle N \rangle).$$

□

Definition. — We define

$$\Theta : \mathcal{N}_k(X) \to \mathrm{MO}_k(X),\ [M, h] \mapsto h_* \langle M \rangle$$

as the *Thom-Pontryagin map.*

Remark. — Observe that the manifold M has a (mod 2)-fundamental class $[M] \in H_k(M, \partial M, \mathbb{F}_2)$ which satisfies $\partial[M] = [\partial M]$. There is a unique non-trivial morphism $\vartheta : \mathrm{MO} \to H\mathbb{F}_2$ in $\mathcal{SH}$ which satisfies

$$\vartheta_* \langle M \rangle = [M] \in (H\mathbb{F}_2)_k(M/\partial M) \cong H_k(M, \partial M; \mathbb{F}_2).$$

Lecture 19, 6th Jul, 2022

We still have to show that the Thom-Pontryagin map is well-defined.

III.12. Proposition. — *The class $\Theta[M, h]$ only depends on the bordism class of (M, h).*

Proof. Let (M^k, h) be a singular k-manifold over X and let $(B^{k+1}, H, \psi : M \xrightarrow{\sim} \partial B)$ be a null-bordism. Write $\iota : \partial B \hookrightarrow B$ for the inclusion. Then, $h = H \circ \iota \circ \psi$, so we obtain

$$h_*\langle M\rangle = (H \circ \iota \circ \psi)_*\langle M\rangle = H_*(\iota_*\langle \partial B\rangle) = H_*(\iota_*(\delta\langle B\rangle)) = 0$$

which uses that ι_* and δ are adjacent in a long exact sequence.

Because $\langle M\rangle$ is also additive for disjoint unions, we deduce $h_*\langle M\rangle = g_*\langle N\rangle$ if (M, h) is bordant to $\langle N, g\rangle$. ☐

So the map is well-defined. Moreover, we have the following.

III.13. Theorem. — *Let X be a space.*

(i) The Thom-Pontryagin map $\Theta : \mathcal{N}_k(X) \to \mathrm{MO}_k(X)$ is additive.

(ii) The Thom-Pontryagin assemble into a natural transformation of homology theories.

Proof. (i) is straightforward. For (ii), let $f : X \to Y$ be a continuous map. Consider

$$\begin{tikzcd}
\tilde{\mathcal{N}}_{k+1}(Cf) \arrow[r, "p_*"] \arrow[d, "\Theta"'] \arrow[rr, bend left, "\delta"] & \tilde{N}_{k+1}(X_+ \wedge S^1) \arrow[d, "\Theta"] & \mathcal{N}_k(X) \arrow[l, "{-\wedge d, \cong}"'] \arrow[d, "\Theta"] \\
\mathrm{MO}_{k+1}(Cf) \arrow[r, "p_*"'] \arrow[rr, bend right, "\delta"'] & \mathrm{MO}_{k+1}(X_+ \wedge S^1) & \mathrm{MO}_k(X) \arrow[l, "{-\wedge S^1, \cong}"]
\end{tikzcd}$$

The left square commutes by naturality. We check that the right square commutes. Let $x \in \tilde{\mathcal{N}}_k(X_+)$. Then

$$\Theta(x \wedge d) = \Theta(x) \times \Theta(d) = \Theta(x) \times (1 \wedge S^1) = \Theta(x) \wedge S^1.$$

☐

III.14. Theorem (Thom)**.** — *For every space X the map $\Theta : \mathcal{N}_k(X) \to \mathrm{MO}_k(X_+)$ is an isomorphism.*

Proof. We start with the special case $X = *$. In this case we will equivalently write

$$\Theta : \mathcal{N}_k \to \pi_k(\mathrm{MO}).$$

We discuss surjectivity. Let $f : S^{n+k} \to \mathrm{MO}_n = \mathrm{Th}(\mathrm{Gr}_n(\mathbb{R}^\infty))$ be a continuous based map that represents a class in $\pi_k(\mathrm{MO})$. Because S^{n+k} is compact, f has image in $\mathrm{Th}(\mathrm{Gr}_n(V))$ for some finite-dimensional linear subspace $V \subseteq \mathbb{R}^\infty$. We get

$$\mathrm{Th}(\mathrm{Gr}_n(V)) = \gamma_n(\mathrm{Gr}_n(V)) \cup \{\infty\}$$

where $\gamma_n(\mathrm{Gr}_n(V))$ is a smooth manifold again as the total space over a smooth manifold. The zero section

$$s : \mathrm{Gr}_n(V) \to \mathrm{Th}(\mathrm{Gr}_n(V)),\ L \mapsto (0, L)$$

is smooth. The map f is based homotopic to a map $g : S^{n+k} \to \mathrm{Th}(\mathrm{Gr}_n(V))$ that is transverse to the zero section. Thus, $g^{-1}(s(\mathrm{Gr}_n(V)))$ is a smooth submanifold of $\mathbb{R}^{n+k}$ of codimension n. So $M = g^{-1}(s(\mathrm{Gr}_n(V)))$ is a smooth closed k-manifold inside $\mathbb{R}^{n+k}$. Then,

The zero section does not hit ∞, so the preimage also will not hit $\infty \in S^{n+k}$.

$$\Theta[M] = [g] = [f].$$

We use the following fact: every natural transformation of homology theories that is an isomorphism of coefficients is an isomorphism on all based CW complexes (this can be proven using the suspension isomorphism and induction over the CW structure). Thus, Θ is an isomorphism for all CW complexes. Because both sides are invariant under weak equivalences, this is an isomorphism in general after a CW-approximation. □

Example. — Recall that 0-dimensional smooth closed manifolds are finite sets of points. The bordism classes correspond to the parity of cardinalities, so $\mathcal{N}_0 \cong \mathbb{F}_2$. Recall also that AT2Sheet10.2 yields

$$\mathcal{N}_0(X) \cong H_0(X; \mathbb{F}_2) = \mathbb{F}_2[\pi_0(X)].$$

Steenrod asked whether every homology class can be represented by a smooth manifold. It is not true for integral homology but Thom proved it for $\mathbb{F}_2$-homology.

III.15. Theorem (Thom). — *The map $\mathcal{N}_k(X) \to H_k(X, \mathbb{F}_2)$ is surjective.*

The following proposition is the goal of AT2Sheet10.3.

III.16. Proposition. — *For even-dimensional manifolds the Euler characteristic (mod 2) is a bordism invariant.*

Another interesting question is the following: for even k we get $[\mathbb{R}\mathrm{P}^k] \neq 0$ in $\mathcal{N}_k$, is $[\mathbb{R}\mathrm{P}^4] = [\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2]$ in $\mathcal{N}_k$? Unsurprisingly the answer was found by Thom.

III.17. Theorem (Thom). — *The bordism ring $\mathcal{N}_*$ is a polynomial algebra over $\mathbb{F}_2$ with a generator for all dimensions $i \geqslant 2$ which is not of the form $2^k - 1$. In other words,*

$$\mathcal{N}_* = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots].$$

Moreover, for even k one can take $x_k = [\mathbb{R}\mathrm{P}^k]$. The Dold manifolds can be taken as odd-dimensional generators.

Stiefel-Whitney Classes

Let $\xi : E \to X$ be a real vector bundle over a paracompact space X. Its *Stiefel-Whitney classses* are elements $w_i(\xi) \in H^i(X; \mathbb{F}_2)$ for $i \geqslant 0$ characterized by the following properties:

(i) Compatibility with pullbacks: $f^*(w_i(\xi)) = w_i(f^*\xi))$ for continuous $f : Y \to X$.

(ii) Effect of direct sum: $w_i(\xi \oplus \eta) = \sum_{a+b=i} w_a(\xi) \smile w_b(\eta)$.

(iii) Special classes: $w_0(\xi) = 1,\ w_1(\gamma_1 \downarrow \mathbb{R}\mathrm{P}^\infty) \neq 0 \in H^1(\mathbb{R}\mathrm{P}^\infty, \mathbb{F}_2)$.

(iv) Condition on dimensions: $w_i(\xi) = 0$ for $i > \mathrm{rank}(\xi)$.

The last property actually follows from the others.

Construction. — Let $r_1, \cdots, r_k \geqslant 0$ be natural numbers such that $r_1 + 2r_2 + \cdots + kr_k = k$. Let M be a smooth closed k-manifold. We write $w_i(M) = w_i(\tau_M) \in H^i(M; \mathbb{F}_2)$ for the i-th Stiefel-Whitney class of the tangent bundle TM of M. The *Stiefel-Whitney number* associated to $(r_1, \cdots, r_k)$ is

$$\underbrace{(w_1(M)^{r_1} \smile w_2(M)^{r_2} \smile \cdots \smile w_k(M)^{r_k})}_{\in H^k(M;\mathbb{F}_2)} \frown [M] \in \{0, 1\}.$$

If M is connected, then this number is 1 if and only if $w_1(M)^{r_1} \smile \cdots \smile w_k(M)^k \neq 0$.

III.18. Proposition. — *The Stiefel-Whitney numbers of any sequence of natural numbers are bordism invariants of smooth closed k-manifolds.*

Remark. — Thom also proved the converse!

Proof. First, homology and cohomology take disjoint unions to products, so Stiefel-Whitney numbers are additive. Therefore it suffices to show that all Stiefel-Whitney numbers vanish on nullbordant manifolds. Let B be a compact smooth $(k+1)$-manifold. The tangent bundle of B and ∂B are related by

$$\tau_B|_{\partial B} \cong \tau_{\partial B} \oplus \underline{\mathbb{R}}.$$

Hence,

$$w_i(\tau_B|_{\partial B}) = w_i(\tau_{\partial B} \oplus \underline{\mathbb{R}}) = w_i(\tau_{\partial B}).$$

Let $\iota : \partial B \to B$ be the inclusion. We get

$$\begin{aligned}(w_1(\partial B)^{r_1} \smile \cdots \smile w_k(\partial B)^{r_k}) \frown [\partial B] &= \iota^*(w_1(B)^{r_1} \smile \cdots \smile w_k(B)^{r_k}) \frown [\partial B] \\ &= (w_1(B)^{r_1} \smile \cdots \smile w_k(B)^{r_n}) \frown \iota_*[\partial B] \\ &= (w_1(B)^{r_1} \smile \cdots \smile w_k(B)^{r_k}) \frown \iota_*(\partial[B]) \\ &= 0,\end{aligned}$$

which concludes the proof. □

Example. — Consider $(0, \cdots, 0, 1)$. Then,

$$w_k(M) \frown [M] \equiv \chi(M) \pmod 2.$$

In fact, AT2Sheet10.3 follows from this.

Example (Stiefel-Whitney classes of projective space)**.** — Recall that

$$H^*(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2) = \mathbb{F}_2[a]/(a^{k+1}).$$

In fact, $a = w_1(\gamma_1) \in H^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ by the naturality of the inclusion and (iii) of the Stiefel-Whitney classes properties.

The relation between the two natural bundles over $\mathbb{R}\mathrm{P}^k$ is the isomorphism

$$\tau_{\mathbb{R}\mathrm{P}^k} \oplus \underline{\mathbb{R}} = \bigoplus_{i=1}^{k+1} \gamma_1.$$

The proof of this takes about two pages in Milnor-Stasheff.

The *total Stiefel Whitney class* is

$$w_{\mathrm{tot}}(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) + \cdots$$

and we may write out $w_{\mathrm{tot}}(\xi \oplus \eta) = w_{\mathrm{tot}}(\xi) \smile w_{\mathrm{tot}}(\eta)$. We can compute

$$w_{\mathrm{tot}}(\tau_{\mathbb{R}\mathrm{P}^k}) = w_{\mathrm{tot}}(\tau_{\mathbb{R}\mathrm{P}^k} \oplus \underline{\mathbb{R}}) = w_{\mathrm{tot}}(\gamma_1^{\oplus(k+1)}) = w_{\mathrm{tot}}(\gamma_1)^{(k+1)} = (1+a)^{k+1}$$

Comparing coefficients we obtain

$$w_i(\tau_{\mathbb{R}\mathrm{P}^k}) = \binom{k+1}{i} a^i \in H^i(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2).$$

The first examples are

$$\begin{aligned}
w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^1) &= 1, \\
w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^2) &= 1 + a + a^2, \\
w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^3) &= 1, \\
w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^4) &= 1 + a + a^4.
\end{aligned}$$

Example. — We discuss $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$. Recall that

$$H^*(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2; \mathbb{F}_2) = \mathbb{F}_2[b, c]/(b^3, c^3)$$

where $b = \mathrm{pr}_1^*(a)$ and $c = \mathrm{pr}_2^*(a)$ for $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$. Moreover, we have $\tau_{\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2} \cong \tau_{\mathbb{R}\mathrm{P}^2} \times \tau_{\mathbb{R}\mathrm{P}^2}$. Therefore we obtain

$$w_{\mathrm{tot}}(\tau_{\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2}) = \mathrm{pr}_1^*(w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^2)) \smile \mathrm{pr}_2^*(w_{\mathrm{tot}}(\mathbb{R}\mathrm{P}^2)) = (1 + b + b^2) \smile (1 + c + c^2).$$

Hence,

$$\begin{aligned}
w_1(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2) &= b + c, \\
w_2(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2) &= b^2 + bc + c^2, \\
w_3(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2) &= b^2 c + bc^2, \\
w_4(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2) &= b^2 c^2.
\end{aligned}$$

Therefore, $w_1^4 = (b+c)^4 = 0$, and so on.

Putting together the last two examples, one can show that

$$[\mathbb{R}P^4] \neq [\mathbb{R}P^2 \times \mathbb{R}P^2] \in \mathbb{N}_4.$$

It suffices to compute some Stiefel-Whitney numbers, and use that these latter are bordism invariants (Proposition III.18).

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