Loose Ends from Tutorial 0

Simplicial groups are Kan complexes

In the tutorial I had to rush through the proof of this result, so as promised here's a write-up. The proof I have written here is adapted from [May], and it proceeds by describing an algorithm to construct the horn fillings. In the end it really is the same proof I already gave at the tutorial, just in a slightly different vest (I guess I changed my mind and now I think this more explicit version is overall nicer than the slightly slicker one I presented, if you liked the other version more just ask at the next tutorial and you shall receive a better explanation).

Proposition. — *Let X be a simplicial group. Then the underlying simplicial set of X is a Kan complex.*

Proof. We want to show that given any arbitrary n-horn in X, there exists a lift to the n-simplex. Let $\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_n$ be the faces of an arbitrary n-horn in X; we want to exhibit them as faces of a single n-simplex θ. First, we will construct a simplex η with $d_t\eta = \theta_t$ for all $t < k$.

If $k = 0$, there is nothing to do. Otherwise, we will inductively construct n-simplices $\eta_0, \dots, \eta \coloneqq \eta_{k-1}$ such that $d_t\eta_m = \theta_t$ for all $t \leq m$. For $m = 0$ we can set $\eta_0 = s_0\theta_0$. Then assuming we have already constructed η_{m-1}, we define $\upsilon_m \coloneqq s_m(\theta_m \cdot d_m\eta_{m-1}^{-1})$ and $\eta_m \coloneqq \upsilon_m \cdot \eta_{m-1}$. If now $t < m$, (using that the structure maps are group morphisms!)

$$\begin{aligned}
d_t\upsilon_m &= d_t s_m(\theta_m \cdot d_m\eta_{m-1}^{-1}) \\
&= s_{m-1}(d_t\theta_m \cdot d_t d_m\eta_{m-1}^{-1}) \\
&= s_{m-1}(d_{m-1}\theta_t \cdot d_{m-1}d_t\eta_{m-1}^{-1}) \\
&= s_{m-1}d_{m-1}(\theta_t \cdot \theta_t^{-1}) = 1
\end{aligned}$$

while $d_m\upsilon_m = \theta_m \cdot d_m\eta_{m-1}^{-1}$. Thus, $d_t\eta_m = d_t\eta_{m-1} = \theta_t$ and

$$d_m\eta_m = \theta_m \cdot d_m\eta_{m-1}^{-1} \cdot d_m\eta_{m-1} = \theta_m$$

as desired.

To conclude, we will inductively construct $\tau_n, \dots, \tau_k$ (note that we're moving downward this time!) such that $d_t\tau_m = \tau_t$ for all $t < k$ as well as all $t > m$; then $\theta \coloneqq \tau_k$ will be the simplex filling the given horn. For $m = n$ the simplex η constructed above already does the job. If now τ_{m+1} has already been constructed, then we set $\zeta_m = s_m(\tau_{m+1} \cdot d_{m+1}\tau_{m+1}^{-1})$, $\tau_m = \zeta_m \cdot \tau_{m+1}$. A similar computation to the one above then shows that τ_m has the desired properties. □

A different proof, more geometric and using a bit more technology (not sure it's really worth it, one way or another you have to get your hands dirty anyway) is given on page 14 of this document by Joyal and Tierney, and yet another proof which seems worth reading is given in Higher Algebra (Proposition 1.3.2.11): at a first glance I don't get it, but it's probably sleek.

References

[May] J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, 1967.