Loose Ends from Tutorial 2

Various stuff

This week wasn't as smooth sailing as I would have expected, but I guess there were some slightly sneaky pitfalls... I'll comment the exercises one by one.

Problem 4 (Natural transformations and homotopies). Everybody did this one correctly. I just want to point out something since multiple people seemed to be bothered by it: it is true that simplicial homotopies do not necessarily have inverses, but it still makes sense to talk about homotopy equivalence, by considering the equivalence relation generated by the non-symmetric relation "there is a simplicial homotopy equivalence between". This is nothing strange, weak homotopy equivalence of topological spaces (and quasi-isomorphism of chain complexes, and really weak equivalence in any category with weak equivalences) is defined this way.

Problem 5 (Corepresenting the homotopy coherent nerve). Actually, nothing to say about this one, it was just an annoying verification.

Problem 6 (Right/left fibrations are conservative). This was the sneaky one! The first part, showing that right and left fibrations are conservative, wasn't an issue, I'll just note the following obvious (but fundamental) fact, which might be useful to state explicitly once (and then it should become second nature):

Fact. — *For f a morphism in a quasicategory $\mathcal{C}$, the following are equivalent:*

- *f is an equivalence,*
- *f admits an (homotopy) inverse f',*
- *f admits an (homotopy) preinverse e and an (homotopy) postinverse g,*
- *f admits an (homotopy) preinverse e and e admits an (homotopy) preinverse d,*
- *f admits an (homotopy) postinverse g and g admits an (homotopy) postinverse h,*

If these equivalent conditions apply, then $f \simeq d \simeq h$ and $f' \simeq e \simeq g$, and all of them are equivalences (or in other words we can work with equivalences in quasicategories the same way we work with homotopy equivalences).

The second point instead (essentially you had to show that right and left fibrations are isofibrations), nobody got right! The reason is that you cannot include $\Lambda^1_0 = \{0\}$ as the endpoint of Δ^1 and then lift against a left fibration as if that was a legit horn inclusion! So, how to do it? It goes like this: since $g : X \to FY$ is an equivalence, we can consider a postinverse $f : FY \to X$, and lift it (correctly!) to some map $\tilde{f} : Y \to \tilde{X}$, which is also an equivalence (by the fact above plus left fibrations being conservative). Now, considering a preinverse $h : \tilde{X} \to Y$ of $\tilde{f}$, we have that g and $F(h)$ are homotopic (again by the fact above, of course): therefore we can consider one more lifting problem (as an exercise figure out which one) to get some map $\tilde{g}$ with the desired property that $F(\tilde{g}) = g$.

The solution that everybody wrote down would have worked of course if g was a morphism $FX \to Y$, and in fact that would have sufficed to prove that left fibrations are isofibrations: one can easily show, using an argument similar to the one in the last paragraph, that for inner fibrations (which isofibrations are by definition) the lifting of morphisms $X \to FY$ or $FX \to Y$ are equivalent properties (and perhaps this would have made for a less confusing exercise...).

(Bonus question if you feel like it: by the fact above inverses of equivalences are unique up to homotopy (although certainly not unique), but are they unique up to contractible choice? What does that even mean?)

Problem 7 (Weakly saturated hull of boundary inclusions). I also don't think anybody got full marks on this one. First: the template provided in the notes was meant to help showing that a monomorphism is in the weakly saturated hull of the boundary inclusions, but one also has to prove that the weakly saturated hull of boundary inclusions is made just of monomorphisms! It's not difficult, sure, it follows from injections of sets being closed under the operations that make up a weakly saturated hull (it's also not formal: this is not true in any category), but I think it deserved at the very least to be mentioned (most people completely ignored this).

Now, once one has proved that monomorphisms are weakly saturated, the core of the exercise was filling in the details in the template provided in the notes, which is meant to show that any monomorphism is in the weakly saturated hull of boundary inclusions. First, one should reduce the problem to inclusions of simplicial sets using that retractions are in the weakly saturated hull; this is probably not strictly necessary, but at least a nice touch. Then one proves that a simplicial set is the colimit of his skeleta (nobody had problems with this), and afterwards that you get the inclusions between successive skeleta as some cobase change of coproducts of boundary inclusion: this most people did ok, but few had a readable solution which actually checked all the details. To make it readable, I would have checked the relevant pushout square levelwise (as opposed to checking the universal property, which was also a viable proof, though), using the simple fact that a commutative square of sets

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

is a pushout if and only if f induces a bijection $C \smallsetminus A \to D \smallsetminus B$.

You can find a written-up solution of the second step (or something that closely resembles it, at least) on my very old Algebraic Topology I notes (Proposition V.6), and for the third step you can essentially just repeat the arguments in the first and second step with the skeleta relative A (if I find a minute I might write down the solution in full), but instead I would suggest to read the proof on Kerodon which is as well explained as it gets. Also, one final subtlety that Lurie takes care of: a priori we don't have infinite coproducts in the weakly saturated hull, you need to write the map

$$j : \coprod_{\sigma \in X_k^{\mathrm{nd}}} \partial\Delta^k \hookrightarrow \coprod_{\sigma \in X_k^{\mathrm{nd}}} \Delta^k$$

as a transfinite composition!