Construction of New Completely Regular
$\mathbb{Z}_2\mathbb{Z}_4$-linear Codes from Old [*]

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Abstract. A code $\mathcal{C}$ is said to be $\mathbb{Z}_2\mathbb{Z}_4$-additive if its coordinates can be partitioned into two subsets X and Y, in such a way that the punctured code of $\mathcal{C}$ obtained by removing the coordinates outside X (or, respectively, Y) is a binary linear code (respectively, a quaternary linear code). The binary image of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, through the Gray map, is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, which is not always linear. Given a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code, which is known to be completely regular, some constructions yielding new $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are computed, and the completely regularity of the obtained codes is studied.

Keywords: additive codes, completely regular codes, perfect binary codes, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

1 Introduction

Let $\mathbb{Z}_2^\alpha$ and $\mathbb{Z}_4^\beta$ denote the set of all binary vectors of length α and the set of all quaternary vectors of length β, respectively. Let $GF(q)$ be a Galois field with q elements, q being a power of some prime number, and let $GR(q^m)$ be a Galois ring with cardinality q^m, which comes from an extension of the ring $\mathbb{Z}_q$. The classical Hamming weight $wt(\mathbf{v})$ of a vector $\mathbf{v} \in GF(q)^n$ is the number of coordinates which are different from zero, and the Hamming distance $d(\mathbf{u}, \mathbf{v})$ between two vectors $\mathbf{u}, \mathbf{v} \in GF(q)^n$ denotes the weight of their difference.

Any non-empty subgroup $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*additive code*. Let $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ be the usual *Gray map*, that is, $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, and $\phi(3) = (1,0)$; and let $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, be the *extended Gray map* (Id, ϕ) given by

$$\Phi(u_1, \ldots, u_\alpha | v_1, \ldots, v_\beta) = (u_1, \ldots, u_\alpha | \phi(v_1), \ldots, \phi(v_\beta)).$$

It is worth noting that the extended Gray map is an isometry which transforms Lee distances defined in a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ onto Hamming distances defined in the binary code $C = \Phi(\mathcal{C})$, where the length of C is $n = \alpha + 2\beta$.

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is isomorphic to an abelian structure like $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Thus $\mathcal{C}$ has $|\mathcal{C}| = 2^\gamma 4^\delta$ codewords, where $2^{\gamma+\delta}$ of them are of order two. We call such code $\mathcal{C}$ a $\mathbb{Z}_2\mathbb{Z}_4$-*additive code of type* $(\alpha, \beta; \gamma, \delta)$, and its binary image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*linear code of type* $(\alpha, \beta; \gamma, \delta)$, which may not be linear.

Given a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$, its dual is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, denoted by $\mathcal{C}^\perp$, and defined as the set of vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ which are orthogonal to every codeword in $\mathcal{C}$. We use the following definition (see [3]) of inner product in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2 \left(\sum_{i=1}^\alpha u_i v_i \right) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4, \tag{1}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and computations are made considering zeros and ones in the α binary coordinates as quaternary zeros and ones, respectively.

After the Gray map over $\mathcal{C}^\perp$, the binary code $C_\perp = \Phi(\mathcal{C}^\perp)$, of length $n = \alpha + 2\beta$, is called the $\mathbb{Z}_2\mathbb{Z}_4$-*dual code* of C.

Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of the same length are *monomially equivalent* if one can be obtained from the other by permuting the coordinates and, if necessary, changing the sign of certain quaternary coordinates.

Although a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma', \delta')$ may not have a basis, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma'} \lambda_i \mathbf{u}^{(i)} + \sum_{j=\gamma'+1}^{\gamma'+\delta'} \mu_j \mathbf{v}^{(j)}, \tag{2}$$

where $\lambda_i \in \mathbb{Z}_2$ for $1 \le i \le \gamma'$, $\mu_j \in \mathbb{Z}_4$ for $\gamma' + 1 \le j \le \gamma' + \delta'$ and $\mathbf{u}^{(i)}$, $\mathbf{v}^{(j)}$ are vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of order two and order four, respectively (see [3]).

Vectors $\mathbf{u}^{(i)}$, $\mathbf{v}^{(j)}$ give us a generator matrix $\mathcal{G}$ which can be written as follows:

$$\mathcal{G} = \left(\frac{B_2 | Q_2}{B_1 | Q_1} \right), \tag{3}$$

where B_2 and B_1 are binary matrices of size $(\gamma' \times \alpha)$ and $(\delta' \times \alpha)$, respectively; Q_2 is a $(\gamma' \times \beta)$-matrix whose elements are in $\{0, 2\} \subset \mathbb{Z}_4$ and Q_1 is a quaternary $(\delta' \times \beta)$-matrix with row vectors of order four. Refer to [3] for further details.

Depending on the computations we are doing it might be convenient to take a different representation for the above matrix.

Let $\mathcal{D}$ be a matrix written as follows:

$$\mathcal{D} = \left(\frac{B_2' | Q_2'}{B_1' | Q_1'} \right), \tag{4}$$

where B_2' and B_1' are binary matrices of size $(\psi \times \alpha)$ and $(\theta \times \alpha)$, respectively, in which binary zeros and ones have been represented as quaternary zeros and twos; Q_2' is a $(\psi \times \beta)$-matrix whose elements are in $\{0, 2\} \subset \mathbb{Z}_4$ and Q_1' is a quaternary $(\theta \times \beta)$-matrix with row vectors of order four. We will refer to this

matrix as a *matrix of type* $(\alpha, \beta; \psi, \theta)$. Note that matrix in (4) represents a group homomorphism

$$\mathcal{D} : \mathbb{Z}_2^\psi \times \mathbb{Z}_4^\theta \longrightarrow \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta,$$

where for any $\mathbf{v} = (\lambda_1, \ldots, \lambda_\psi, \mu_{\psi+1}, \ldots, \mu_{\psi+\theta}) \in \mathbb{Z}_2^\psi \times \mathbb{Z}_4^\theta$, vector $\mathcal{D}(\mathbf{v})$ is computed as in (2). From a matrix point of view, this can be done by computing $\mathbf{v}\mathcal{D}$, which yields a vector in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ whose binary zeros and ones are represented, respectively, by quaternary zeros and twos.

Given a code $\mathcal{C}^\perp$ of type $(\alpha, \beta; \gamma, \delta)$, corresponding to the $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code of $\mathcal{C}$, its generator matrix $\mathcal{H}$ is a parity check matrix of $\mathcal{C}$. Thus all codewords in $\mathcal{C}$ are orthogonal to every row vector in $\mathcal{H}$ through the inner product in (1). As for the usual case of codes over a finite field, given a vector $\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we can refer to the syndrome $S_\mathbf{v}$ as the vector in $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ obtained by computing $\mathbf{v}\mathcal{H}^T$, where $\mathcal{H}^T$ is the matrix obtained by writing the rows in $\mathcal{H}$ as column vectors, and it is written as in (4).

Let C be a binary code. The distance of any vector $\mathbf{v} \in \mathbb{Z}_2^n$ to C is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C}\{d(\mathbf{v}, \mathbf{x})\}$, and the covering radius of C is $\rho = \max_{\mathbf{v} \in \mathbb{Z}_2^n}\{d(\mathbf{v}, C)\}$. Let us also define $C(i) = \{\mathbf{v} \in \mathbb{Z}_2^n : d(\mathbf{v}, C) = i\}$, $i = \{1, \ldots, \rho\}$.

We say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are *neighbours* if $d(\mathbf{u}, \mathbf{v}) = 1$. In this paper, the definition of completely regularity given in [9] will be used.

Definition 1. *A code C with covering radius ρ is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number a_l of neighbours in $C(l)$, the same number b_l of neighbours in $C(l+1)$, and the same number c_l of neighbours in $C(l-1)$. Moreover, note that $a_l + b_l + c_l = n$ and $c_0 = b_\rho = 0$. The intersection array of C is then $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$.*

It is well known that any linear completely regular code C implies the existence of a coset distance-regular graph. Binary images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are nonlinear binary codes, but we can also construct the quotient graph which is distance-regular when the initial $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are completely regular.

Given a code C, its *external distance* is the number of nonzero terms in the MacWilliams transform. From any $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ we can construct the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code $\mathcal{C}^\perp$ and, in this case, their respective weight enumerator polynomials are related by the MacWilliams identity [7]. Thus in the specific cases of $\mathbb{Z}_4$ or $\mathbb{Z}_2\mathbb{Z}_4$ additivity, we can also refer to [10] and, hence the external distance of $C = \varPhi(\mathcal{C})$ coincides with the number of nonzero weights occurring in the distance distribution of $C_\perp = \varPhi(\mathcal{C}^\perp)$.

Throughout this paper we will use the concept of *uniformly packed code* (usually called "in the wide sense") defined in [1].

Definition 2 ([1]). *Let C be a binary code of length n and covering radius ρ. We say that C is uniformly packed "in the wide sense" if there exist rational numbers $\tau_0, \ldots, \tau_\rho$ such that, for any $\mathbf{v} \in \mathbb{Z}_2^n$,*

$$\sum_{k=0}^{\rho} \tau_k A_k(\mathbf{v}) = 1,$$

where $A_k(\mathbf{v})$ is the number of codewords in C at distance k from $\mathbf{v}$.

In the following Proposition we summarize a few well known results.

Proposition 1. *Let C be a code (not necessarily linear) with error correcting capability $e \geq 1$, covering radius ρ and external distance s. Then:*

(i) [8] $\rho \leq s$.
(ii) [2] $\rho = s$ *if and only if code C is uniformly packed (in the wide sense).*
(iii) [6] *If C is completely regular, then it is uniformly packed (in the wide sense).*
(iv) [7, 10] *If C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then the external distance s of C coincides with the number of nonzero weights occurring in the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C.*
(v) [7] *If $2e + 1 \geq 2s - 1$, then C is a completely regular code.*

A perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ of length $n = 2^m - 1$ is a binary perfect code, that is, a binary code of minimum distance 3, where all vectors in $\mathbb{Z}_2^n$ are within distance one from a unique codeword.

It is well known [5] that for any $m \geq 2$ and each $\delta \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$ there exists a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of binary length $n = 2^m - 1$, such that its $\mathbb{Z}_2\mathbb{Z}_4$-dual code is of type $(\alpha, \beta; \gamma, \delta)$, where $\alpha = 2^{m-\delta} - 1$, $\beta = 2^{m-1} - 2^{m-\delta-1}$ and $m = \gamma + 2\delta$ (recall that the binary length can be computed as $n = \alpha + 2\beta$). This allows us to write, for a given value of δ, the parity check matrix $\mathcal{H}$ of any $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ corresponding to a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$. This parity check matrix can be expressed in form (3) where the first α columns are all possible nonzero vectors in $\mathbb{Z}_2^{\gamma+\delta}$, and the last β columns are all possible order four vectors in $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, up to scalar multiples, where binary zeros and ones are respectively represented as quaternary zeros and twos (see [4]).

In this paper we construct some completely regular codes by modifying, in various ways, a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C. In Section 2 we extend, puncture and also shorten code C, while in Section 3 we describe a new method to obtain nonlinear completely regular codes in $GF(4)$ by lifting perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes defined over $GF(2)$. The same approach is also taken in Section 4 to obtain uniformly packed codes in $GF(4)$ by lifting extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes over $GF(2)$.

2 Extending, puncturing and shortening perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, with parity check matrix $\mathcal{H}$, such that its binary image $C = \Phi(\mathcal{C})$ is a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code of length $n = 2^m - 1$, for $m \geq 3$, and its $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code $\mathcal{C}^\perp$ is of type $(\alpha, \beta; \gamma, \delta)$.

Let $\hat{\mathcal{C}}$ be the extended code of $\mathcal{C}$, that is, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code whose parity check matrix $\hat{\mathcal{H}}$ is obtained from $\mathcal{H}$ by first adding a zero column at the beginning of the binary part, and then adding the all-twos row vector, where the twos in the first $\alpha + 1$ coordinates are representing binary ones. The binary code $\hat{C} = \Phi(\hat{\mathcal{C}})$ has length $n = 2^m$ and every codeword has even weight.

Let $\dot{\mathcal{C}}$ be the punctured code of $\mathcal{C}$, that is, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code obtained by removing the i-th column of the generator matrix of $\mathcal{C}$ or, equivalently, obtained by deleting the i-th coordinate from each codeword in $\mathcal{C}$. Each time a coordinate is deleted in $\mathcal{C}$, the length of the corresponding binary code $\dot{C} = \Phi(\dot{\mathcal{C}})$ drops by 1 when $1 \leq i \leq \alpha$, or by 2 when $\alpha + 1 \leq i \leq \alpha + \beta$.

Let $\mathcal{C}^*$ be the shortened code of $\mathcal{C}$, that is, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code obtained by taking all codewords in $\mathcal{C}$ having a zero as their i-th component, where $1 \leq i \leq \alpha + \beta$, and subsequently delete the i-th component from these codewords. The procedure of shortening a code obviously decreases its length, thus the binary code $C^* = \Phi(\mathcal{C}^*)$ is of length $n = 2^m - 2$ if $1 \leq i \leq \alpha$, and of length $n = 2^m - 3$ if $\alpha + 1 \leq i \leq \alpha + \beta$.

The following propositions can be proved.

Proposition 2. *External distance.*

(i) *Code $\hat{C}$ has external distance $s = 2$.*
(ii) *Code $\dot{C}$ has external distance $s = 1$.*
(iii) *Code C^* has external distance $s = 2$ when the i-th shortened coordinate is $1 \leq i \leq \alpha$, and $s = 3$ when the i-th shortened coordinate is $\alpha + 1 \leq i \leq \alpha + \beta$.*

Proposition 3. *Covering radius.*

(i) *The covering radius of $\hat{C}$ is $\rho = 2$.*
(ii) *The covering radius of $\dot{C}$ is $\rho = 1$.*
(iii) *The covering radius of C^* is $\rho = 2$.*

Using properties (**ii**) and (**v**) from Proposition 1, the following proposition is straightforward.

Proposition 4. *Uniformly packed codes.*

(i) *Code $\hat{C}$ is uniformly packed.*
(ii) *Code $\dot{C}$ is uniformly packed.*
(iii) *Code C^* is uniformly packed only when the i-th shortened coordinate is $1 \leq i \leq \alpha$.*

Finally, we can state the following proposition which is given without proof.

Proposition 5. *Completely regular codes.*

(i) *Code $\hat{C}$ is a completely regular code with intersection array $(2^m, 2^m - 1; 1, 2^m)$.*

(ii) *Code $\dot{C}$ is a completely regular code with intersection array $(2^m - 2; 2)$ when the i-th punctured coordinate is $1 \leq i \leq \alpha$, and with intersection array $(2^m - 4; 4)$ when $\alpha + 1 \leq i \leq \alpha + \beta$.*

(iii) *Code C^* is completely regular with intersection array $(2^m - 2, 1; 1, 2^m - 2)$, when the i-th shortened coordinate is $1 \leq i \leq \alpha$.*

Remark 1. Notice that in those cases where the extended (shortened) codes are completely regular codes, the intersection array coincides, respectively, with that of the extended (shortened) Hamming binary code of the same length. As for the punctured code, its intersection array coincides with that of the punctured Hamming binary code of the same length when we are puncturing a coordinate within the first α binary coordinates, while it coincides with the intersection array of the 2-punctured Hamming binary code when we are puncturing a quaternary coordinate. In other words, we have constructed new completely regular codes, but with the same intersection array as the corresponding linear codes.

3   Lifting perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Lifting of binary perfect linear codes was previously studied in [12]. In this article we now introduce and study lifted perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

As in Section 2, let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, with parity check matrix $\mathcal{H}$, such that its binary image $C = \Phi(\mathcal{C})$ is a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code of length $n = 2^m - 1$, for $m \geq 3$, and its $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code $\mathcal{C}^\perp$ is of type $(\alpha, \beta; \gamma, \delta)$. We define the code C_r over an extension $GF(2^r)$ of the binary finite field as

$$C_r = C + \xi C + \xi^2 C + \ldots + \xi^{r-1} C,$$

where $\xi^i C$ denotes the result of multiplying every codeword in C by ξ^i, and ξ is a primitive element in $GF(2^r)$. Let $\mathcal{C}_r = \Phi^{-1}(C_r)$ be the corresponding code in $GF(2^r)^\alpha \times GR(4^r)^\beta$. Code $\mathcal{C}_r$ has $\mathcal{H}_r$ as parity check matrix, where $\mathcal{H}_r$ can be seen as an $\alpha \times \beta \times r$ matrix in which each submatrix (h_{ijk}) for a fixed $k \in \{1, \ldots, r\}$ coincides with the parity check matrix $\mathcal{H}$ of $\mathcal{C}$. We will say that code C_r is obtained by lifting the perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C. Note that code C_r may not be linear over $GF(2^r)$.

Any vector $\mathbf{v} \in \mathcal{C}_r$ can be seen as a matrix of size $r \times (\alpha + \beta)$ by representing every coordinate in $GF(2^r)$ from the first α coordinates and every coordinate in $GR(4^r)$ from the last β coordinates, as a column vector in $GF(2)^r$ and in $GR(4)^r$, respectively. This matrix therefore contains r row vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. Moreover, we will represent the binary zeros and ones in the first α coordinates of every such rows as quaternary zeros and twos, respectively. Let this matrix representation of $\mathbf{v}$ be denoted by $[\mathbf{v}]$.

Proposition 6. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code having a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code binary image $C = \Phi(\mathcal{C})$ of length $n = 2^m - 1$, for $m \geq 3$; let $\mathcal{C}_r$ be the code defined over $GF(2^r)^\alpha \times GR(4^r)^\beta$, obtained by lifting $\mathcal{C}$, and let C_r be the corresponding binary image of $\mathcal{C}_r$. For $r = 2$, the covering radius ρ of C_r is 2.*

Proof. Let $\mathbf{v}$ be any vector in $GF(4)^{\alpha} \times GR(16)^{\beta}$ and not in $\mathcal{C}_2$, and let $S_{\mathbf{v}} = \mathbf{v}\mathcal{H}^T \in GF(4)^{\gamma} \times GR(16)^{\delta}$ be its syndrome. We can then write $\mathbf{v} = \mathbf{w} + \mathbf{e}$, where $\mathbf{w} \in \mathcal{C}_2$ and $\mathbf{e}$ is a vector in $GF(4)^{\alpha} \times GR(16)^{\beta}$ of minimum weight such that its syndrome is $S_{\mathbf{e}} = S_{\mathbf{v}}$. Therefore, the distance from $\mathbf{v}$ to $\mathcal{C}_2$ is the weight of vector $\mathbf{e}$, that is $d(\mathbf{v}, \mathcal{C}_2) = wt(\mathbf{e})$.

Let us represent vector $\mathbf{e}$ as a matrix $[\mathbf{e}]$ of two row vectors in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$. Since the covering radius of any perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code is 1, we have that the Lee weight of every row in $[\mathbf{e}]$ is at most 1. Therefore, we know that $wt(\mathbf{e}) \leq 2$, thus $d(\mathbf{v}, \mathcal{C}_2) \leq 2$ and the covering radius of C_2 is $\rho \leq 2$.

However, since C_2 is not a perfect code, we know that ρ cannot be 1. Hence the covering radius ρ of C_2 is 2.

Lemma 1. *Let $\mathbf{v}$ be a vector in $GF(4)^{\alpha} \times GR(16)^{\beta}$, and let $(x, y)^T$ be a column vector in the matrix representation $[\mathbf{v}]$ of $\mathbf{v}$, where $x, y \in \mathbb{Z}_4$. Then,*

$$wt_L(\mathbf{v}) = \sum_{i=1}^{\alpha+\beta} wt_L(([\mathbf{v}]_{1,i}, [\mathbf{v}]_{2,i})^T),$$

where $wt_L((x, y)^T) = \max\{wt_L(x), wt_L(y), wt_L(x - y)\}$.

Recall, when computing the Lee weight defined in Lemma 1, that the first α column vectors in $[\mathbf{v}]$ contain quaternary zeros and twos which are representing binary zeros and ones, respectively. The lemma above let us go further on the Lee weights of codewords in the dual code of $\mathcal{C}_2$.

Proposition 7. *The nonzero Lee weights of the codewords in $\mathcal{C}_2^{\perp}$ are 2^{m-1} and $2^{m-1} + 2^{m-2}$, where $m = \gamma + 2\delta$.*

From the above proposition it is clear that the external distance of C_2 is $s = 2$, which leads us to the following proposition.

Proposition 8. *Code C_2 is uniformly packed.*

We include a technical lemma, which will help us later in Theorem 1.

Lemma 2. *Let $\mathcal{C}_2$ be the code defined over $GF(4)^{\alpha} \times GR(16)^{\beta}$, and let C_2 be the corresponding binary image which we know has covering radius 2. Let μ_i be the number of cosets in $C_2(i)$, for $0 \leq i \leq \rho$. Then,*

$$\mu_1 = 3n; \ \mu_2 = n(n - 1), \ where \ n = \alpha + 2\beta \ .$$

Now, we can prove the main theorem.

Theorem 1. *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, with parity check matrix $\mathcal{H}$, such that its binary image $C = \Phi(\mathcal{C})$ is a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code of length $n = 2^m - 1$, for $m \geq 3$, and its $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code $\mathcal{C}^{\perp}$ is of type $(\alpha, \beta; \gamma, \delta)$. Let $\mathcal{C}_2$ be the code defined over $GF(4)^{\alpha} \times GR(16)^{\beta}$, obtained by lifting $\mathcal{C}$, and let C_2 be the corresponding binary image of $\mathcal{C}_2$. Code C_2 is a completely regular code with intersection array $(3n, 2(n - 1); 1, 6)$.*

Proof. Code C_2 is a projective code, hence $e \geq 1$. From Proposition 6 and Proposition 8 it is easy to see that code C_2 satisfies item (**v**) in Proposition 1 and so, code C_2 is completely regular.

For the computation of the intersection array $(b_0, b_1; c_1, c_2)$, as we know the code is completely regular, we easily have that $c_1 = 1$ and $c_2 = 6$; $|C_2|b_0 = |C_2(1)|c_1$ and $|C_2(1)|b_1 = |C_2(2)|c_2$. From Lemma 2 we know that $|C_2(1)| = 3n|C_2|$ and $|C_2(2)| = n(n-1)|C_2|$. In summary, we have $b_0 = 3n$; $c_1 = 1$; $c_2 = 6$; $b_1 = 6\frac{|C_2(2)|}{|C_2(1)|} = 6\frac{n(n-1)}{3n} = 2(n-1)$.

In general, the lifted code C_r, when $r > 2$, is neither completely regular, nor uniformly packed. We can take a specific example for the case $r = 3$ and show that the corresponding code C_3 has $\rho \neq s$. The same happens, in general, for $r \geq 3$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code whose $\mathbb{Z}_2\mathbb{Z}_4$-additive dual code is of type $(3, 6; 0, 2)$, and having the following parity check matrix:

$$\mathcal{H} = \begin{pmatrix} 2\ 0\ 2 & 0\ 1\ 1\ 1\ 1\ 2 \\ 2\ 2\ 0 & 1\ 0\ 1\ 2\ 3\ 1 \end{pmatrix}. \tag{5}$$

Code $\mathcal{C}$ corresponds to a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ of length $n = 15$. Let $\mathcal{C}_r$ be the code defined over $GF(2^r)^3 \times GR(4^r)^6$, obtained by lifting $\mathcal{C}$, for $r = 3$, and let C_3 be its binary image.

Let $\mathbf{v}$ be any vector in $GF(8)^3 \times GR(64)^6$, and not in C_3. As in the proof of Proposition 6, we have $d(\mathbf{v}, C_3) = wt_L(\mathbf{e})$, where $\mathbf{v} = \mathbf{w} + \mathbf{e}$, $\mathbf{w} \in C_3$, and $\mathbf{e}$ is a vector in $GF(8)^3 \times GR(64)^6$ of minimum weight in the coset of $\mathbf{v}$.

Since the covering radius of any perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code is 1, it follows that the Lee weight of every row in the matrix representation $[\mathbf{e}]$ of vector $\mathbf{e}$ is at most 1. Therefore, the covering radius of C_3 is $\rho \leq 3$.

We proceed to compute the external distance of C_3. Let $\mathbf{w}$ be a codeword in $C_3^\perp$, and let $[\mathbf{w}]$ be its matrix representation.

Let $w_1 = (220|123011)$, $w_2 = (202|011112)$, $w_3 = w_1 + w_2 = (022|130123)$ and $w_4 = 3(w_1 + w_2) = (022|310321)$ be four vectors generated by matrix in (5). Note that the Lee weight of $\mathbf{w}$ is 13 when $[\mathbf{w}]$ has vectors w_1, w_2 and w_3 as rows; it is 15 when $[\mathbf{w}]$ has vectors w_1, w_2 and w_4; it is 8 when $[\mathbf{w}]$ has vector w_1 and two zero rows and, finally, it is 12 when $[\mathbf{w}]$ has vector w_1, w_2 and one zero row. Thus the external distance s of C_3 is $s \geq 4$ and, therefore, C_3 is not a uniformly packed code.

Remark 2. As already mentioned at the beginning of this article, the quotient graph corresponding to the binary image of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code is distance-regular when that $\mathbb{Z}_2\mathbb{Z}_4$-linear code is completely regular.

From the completely regular codes C_2 described in this paper, we obtain distance-regular graphs with classical parameters. The bilinear forms graphs [6, Sec. 9.5] have the same parameters and they were not described, until recently in [12], as coset graphs of completely regular linear codes. The interesting point here is that Theorem 1 is giving now the same description but using nonlinear codes.

4 Lifting extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code having a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code binary image $C = \Phi(\mathcal{C})$ of length $n = 2^m - 1$, for $m \geq 3$, and let $\hat{\mathcal{C}}$ be the extended code of $\mathcal{C}$, studied in Section 2. Let $\hat{\mathcal{C}}_r$, for $r = 2$, be the code defined over $GF(4)^{\alpha+1} \times GR(16)^{\beta}$ obtained by lifting code $\hat{\mathcal{C}}$. Code $\hat{\mathcal{C}}_2$ has $\hat{\mathcal{H}}_2$ as parity check matrix, where $\hat{\mathcal{H}}_2$ is an $(\alpha+1) \times \beta \times 2$ matrix in which each submatrix (h_{ijk}) for a fixed $k \in \{1, 2\}$ coincides with the parity check matrix of $\hat{\mathcal{C}}$, that is $\hat{\mathcal{H}}$. Let $\hat{C}_2$ be the corresponding binary image of $\hat{\mathcal{C}}_2$.

The following proposition can be proved.

Proposition 9. *Code $\hat{C}_2$ has covering radius $\rho = 3$ and it is uniformly packed, but not completely regular.*

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