

The Portfolio Choice Channel of Wealth Inequality*

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Abstract

I study how differences in portfolio between households help explain the highly unequal wealth distribution seen in the data. It has been well documented that participation rates are substantially smaller than the ones predicted in standard models of portfolio choice. Also, both participation rates and risky shares are highly increasing in wealth. However, both features are usually absent in workhorse models of wealth accumulation. I introduce portfolio choice and adjustment frictions into an otherwise standard model of households saving behavior. Calibrating it to U.S. household-level data, I show that the model is able provide a better fit of the wealth distribution, explaining roughly half of the gap between top shares predicted by traditional models of wealth accumulation (e.g., [Aiyagari, 1994](#)) and the data. Furthermore, the model is also consistent with well-known facts of households' portfolio choices.

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JEL Classifications: D31 G11, E21 G51,

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1 Introduction

Understanding the right skewed shape of the wealth distribution has been a long standing question in economic debate. This question dates back to [Pareto \(1896\)](#) and has gained increasing interest given recent trends in wealth inequality in the US ([Piketty and Zucman, 2014](#); [Saez and Zucman, 2016](#)). In fact, top 1% of households in the US own 46% of total financial wealth whilst the bottom 50% owns less than 1%.¹

Differences in households wealth accumulation can arise in several ways. For instance, they can be produced by different savings rates or different returns on savings across the wealth distribution. The first channel is commonly asserted in popular debate, however recent evidence from [Fagereng, Holm, Moll and Natvik \(2019\)](#) suggest that net savings are rather flat across the wealth distribution.² The second channel suggests that richer households earn a higher return out of each dollar saved. In that sense, a prominent literature in macroeconomics is starting to emphasize the importance of households portfolio choices in multi-asset environments as a determinant of wealth inequality ([Favilukis, 2013](#); [Gomez, 2018](#); [Kuhn, Schularick and Steins, 2020](#); [Hubmer, Krusell and Smith Jr, 2020](#)).

Moreover, recent empirical evidence suggests that returns on savings are positively correlated with wealth ([Fagereng, Guiso, Malacrino and Pistaferri, 2016](#); [Bach, Calvet and Sodini, 2020](#)). In fact, [Fagereng, Guiso, Malacrino and Pistaferri \(2020\)](#) finds that this result holds even within narrow asset classes. These results suggest that there is a scale-dependent component of wealth accumulation as a whole, as well as a effect on the return on savings within specific assets.³

Among the potential channels, one naturally appealing is the existence of frictions in financial markets that rule-out poor households whilst generating inertia in the upper tail of the distribution. This frictions, broadly defined, may account for the costs in acquiring financial skills, monitoring investments, processing high-frequency financial information, among other. Also, ad hoc assumptions on financial frictions are consistent with findings in the household-finance literature that documents the relatively unresponsive behavior of households out of changes in wealth ([Brunnermeier and Nagel, 2008](#); [Calvet, Campbell and Sodini, 2009](#)).

Intuitively, if households are exposed to persistent changes in wealth, the permanent income hypothesis would predict that they should respond with a large increase in consumption (in terms of the change in wealth). However, empirical evidence by [Fagereng et al. \(2019\)](#) suggests that

¹Source: 2019 U.S. Survey of Consumer Finances.

²They study saving rates using administrative records from Norway. Net savings refer to the change in household's net worth holding asset prices constant. Net saving rate refers to net savings out of total income.

³One way to characterize this feature may be that transactions in financial markets often require households to pay transaction fees, which are decreasing in the scale of the investment.

households marginal propensity to consume out of persistent capital gains is substantially lower than the one predicted in canonical models of wealth accumulation.⁴

Previous work has focused extensively on the role of housing in explaining why households hold onto assets instead of consuming permanent changes to wealth.⁵ This is reasonable as housing is usually the most important asset for a large share of the wealth distribution. Namely, according to the surveys of the Survey of Consumer Finances (SCF) for the period 1998-2019, the average share of housing out of wealth is roughly 50%. However, evidence from [Fagereng et al. \(2019\)](#) suggests that this feature is not exclusively driven by housing and that households hold onto other assets as well.⁶ Further, towards the upper tail of the wealth distribution, housing loses its appeal in explaining wealth inequality as the housing share declines after the median of the wealth distribution. In fact, housing represents the 62% of households assets in the median of the wealth distribution, decreasing monotonically to the 32% on the top 1%.

Another empirical regularity documented in the literature is that both participation rates and shares (out of total wealth) invested in risky assets are substantially smaller than in the workhorse portfolio choice models ([Merton, 1969](#); [Samuelson, 1969](#)). Furthermore, both statistics are highly increasing in wealth, which is in stark contrast with the positive and independent of wealth risky asset share predicted in these models. Therefore, this paper intends to build a bridge between empirical regularities in households finances and theoretical models of wealth accumulation. To do so, I propose a quantitative two-asset heterogeneous agent model with portfolio adjustment frictions and idiosyncratic shocks to both income and wealth. This model builds up from the traditional [Bewley \(1986\)](#); [Huggett \(1993\)](#); [Aiyagari \(1994\)](#) model but with two assets as in [Kaplan, Moll and Violante \(2018\)](#). Also, for the solution method I rely on the upwind finite-difference method proposed by [Achdou, Han, Lasry, Lions and Moll \(2017\)](#) to compute both households policy functions and the stationary distribution of the economy.

Financial frictions are modeled using non-convex adjustment costs as in most of the portfolio choice literature. This assumption relies on empirical evidence from [Vissing-Jorgensen \(2002\)](#) that finds trading frequencies increasing in wealth. In the calibrated model, introducing a relatively small adjustment cost (0.75% percent of adjusters stock) is able to generate a large inaction region in households portfolio decisions. Further, much of the inaction comes from both ends of the distribution. I exploit the intuition of this result by taking the workhorse model in portfolio choice [Merton \(1969\)](#) as starting point and then compute households portfolio share across the wealth

⁴They term this behavior as “saving by holding”.

⁵Among the most common arguments, economist often rely on the double role (as an asset and consumption good) or the indivisibility of housing and the large adjustment costs required to adjust housing stock.

⁶More generally, [Chang, Hong and Karabarbounis \(2018\)](#) documents that life-cycle portfolio choice between homeowners and renters is somewhat similar. Section 2.3 shows that portfolio choice is also similar for both types of households across the wealth distribution.

distribution.

Then, I provide a quantitative exercise in which I compare top shares of the distribution predicted from the baseline model presented below, and the Aiyagari model with portfolio choice *à la* [Merton \(1969\)](#). I show that the model provides a better fit of the wealth distribution while also replicating participation rates seen in the data. Further, I analyze how much of changes in top wealth inequality can be explained by portfolio adjustment frictions by realizing a counterfactual exercise. Finally I propose a more general return process with richer return heterogeneity as an extension for the baseline model.

Therefore, the contribution of this paper is threefold. First, it proposes a quantitative model that is able to match several moments from micro data on households balance sheets. For instance, the model is capable of matching the participation rate in risky assets, as well as the increasing risky share across the wealth distribution. Second, by introducing idiosyncratic wealth shocks and heterogeneity in portfolio composition, the model is able to provide a better fit for top shares of the wealth distribution than traditional models of wealth accumulation without assuming unrealistic portfolio choices from households. Third, the financial frictions introduced in the model are able to narrow roughly half of the gap in top shares, as well as explaining xx% of changes in the top 1% of the wealth distribution.

Related Literature. This paper is situated within three strands of literature. First, it is related to a long-standing literature on the wealth inequality.⁷ This literature dates back to [Pareto \(1896\)](#) with the notion with the notion that upper tail of the wealth distribution follow a power law.⁸ Similarly, random growth theories of the wealth distribution include [Wold and Whittle \(1957\)](#) and [Stiglitz \(1969\)](#). More recently, [Piketty and Zucman \(2014\)](#) present empirical evidence on the evolution of the wealth-to-income ratio on developed economies, while [Saez and Zucman \(2016\)](#) estimates the wealth distribution by using detailed US households balance sheets and tax data.⁹

In the same line, recent empirical work highlights the importance of capital gains for wealth inequality ([Feiveson and Sabelhaus, 2019](#); [Fagereng et al., 2019](#); [Kuhn et al., 2020](#); [Martínez-Toledano, 2020](#)). For instance, [Fagereng et al. \(2019\)](#) finds that saving rates across the wealth distribution crucially depend on whether they include or not capital gains.¹⁰ This is explained because richer households own assets that experience persistent capital gains and, unlike predictions from the permanent income hypothesis, hold onto them (which they term as “saving by holding”).

⁷For a thorough review of the literature, see [Benhabib and Bisin \(2018\)](#)

⁸A random variable S follows a power law if there exist $k, \phi > 0$ such that $\Pr(S > x) = kx^{-\phi}$ for all x .

⁹Also, a prominent literature (e.g., [Gabaix, Lasry, Lions and Moll, 2016](#)) studies quantitative models with the intention of matching micro estimates on both the thick tail inequality and its speed of change.

¹⁰Net savings rate follows from the standard definition of consumption plus savings equal income, whilst gross saving rate include capital gains.

Further, a recent literature has documented empirically the importance of asset prices in wealth inequality (Fagereng et al., 2016, 2020; Bach et al., 2020). Motivated by these findings, an emerging theoretical literature has started to emphasize the importance of portfolio choice, asset prices and heterogeneous returns on saving in wealth inequality (Favilukis, 2013; Gomez, 2018; Cioffi, 2020).

Second, this paper is related to the plethora of research regarding households portfolio choice. This literature was born with seminal contributions of Merton (1969) and Samuelson (1969). Both authors proposed analytic models with closed-form solutions and two sharp predictions: (i) agents should participate at all ages in risky asset markets. (ii) risky share is independent of both age and wealth. However, both predictions are at odds with micro data that consistently (through time and across countries) documents a surprisingly low participation rate among poor households and a steeply increasing risky share across the wealth distribution.¹¹ This discrepancy is most likely explained by unrealistic assumptions in the model as frictionless financial markets and the absence of labor income risk.

Thus, several papers, though mostly in life-cycle settings, have attempted to match micro data estimates by adding more realistic features to the workhorse model.¹² Among the most common features, many incorporate fixed adjustment costs (Duffie and Sun, 1990; Gomes and Michaelides, 2005; Gabaix and Laibson, 2001) and or idiosyncratic labor income risk (see e.g., Chang et al., 2018). Another papers have gone further in the attempt to match life-cycle dynamics of portfolio choice by incorporating per-period participation costs and rare disaster events (Fagereng, Gottlieb and Guiso, 2017).¹³ This paper takes basic features of this literature for the construction of the model.

From an empirical standpoint, evidence from Vissing-Jorgensen (2002) on trading frequencies suggest the existence of fixed adjustment costs. Similarly, she finds that small per period adjustment costs may explain the non-participation of half the nonparticipants.¹⁴ Note that beyond direct monetary fees, participation costs in risky assets may be closely related to the opportunity cost (e.g. in terms of labor or leisure) of monitoring and gathering financial information for investment decisions.¹⁵

Third, this paper intends to contribute to the ongoing strand of literature that incorporates

¹¹See Mankiw and Zeldes (1991); Wachter and Yogo (2010).

¹²See e.g., Bodie, Merton and Samuelson (1992); Heaton and Lucas (1997); Viceira (2001); Haliassos and Michaelides (2003); Cocco, Gomes and Maenhout (2005); Gomes and Michaelides (2003, 2005); Fagereng, Gottlieb and Guiso (2017) and Chang et al. (2018).

¹³They allow for a probability of a negative tail event when investing in stocks to capture the exiting of stock markets after retirement.

¹⁴Namely, a 50 dollar (2000 USD) per period fee.

¹⁵There are several arguments in this line, for example mental accounting, cognitive abilities or other behavioral assumptions (e.g. Christelis et al. (2010)), or even fully rational models where households optimal portfolio choice depends on the cost of acquiring/processing information (e.g. Huang and Liu (2007)).

rich sources of heterogeneity and incomplete markets to macro models. In that sense, this paper is closely related to workhorse heterogeneous agent models (Huggett, 1993; Aiyagari, 1994) but extended to a two asset environment as in Kaplan et al. (2018). Further, this paper takes advantage of the finite-difference method proposed by Achdou et al. (2017) to solve the model and to compute the stationary distribution.

Road map. The rest of the paper is organized as follows. Section 2 presents the main source of data used in this paper, descriptive statistics and some stylized facts. Section 3 introduces the baseline heterogeneous agent model. Section 4 presents the calibration strategy and results of the baseline model. Section 5 introduces a new return process into the baseline model. Section 6 exploits the adjustment frictions to account for changes in top shares of the wealth distribution. Finally, Section 7 concludes.

2 Data and Stylized Facts

2.1 Data Sources and Descriptive Statistics

This section summarizes the main empirical regularities documented in the data. This paper uses the Survey of Consumer Finances (SCF) for the period 1998-2019 as its main data source. This survey is conducted on a triennial basis and offers a rich classification of households asset and liabilities, which allows to thoroughly classify assets according to their risk and liquidity.¹⁶

However, despite the advantages of survey data to explore households balance sheets, given the cross-sectional nature of the SCF, this data is subject to several sources of endogeneity. First, households portfolio allocation are most likely influenced by the business cycle.¹⁷ For example, households may want to liquidate assets to smooth consumption in busts.¹⁸ Second, when pooling data from different surveys to analyze portfolio allocation, the results are subject to cohort effects, as noted by Fagereng et al. (2017). Third, this data structure misses structural changes in variables relevant to households portfolio allocation by treating households from different surveys equally. For instance, Favilukis (2013) argues that the higher participation rates can be explained by a decreasing trend in the participation costs in equity markets. Therefore, this section does not intend to provide causal evidence, but to characterize a set of stylized facts to be covered in the

¹⁶For details on the definition of households assets see Appendix A

¹⁷Though evidence from Brunnermeier and Nagel (2008) and Calvet et al. (2009) suggest that households are rather unresponsive to changes in asset prices

¹⁸Nevertheless, evidence from Kaplan and Violante (2014) suggests that losses from imperfect consumption smoothing are second order relative to gains from higher-return assets, especially in the presence of transaction costs. See also Cochrane (1989); Browning and Crossley (2001).

quantitative model later.

Table 1 presents descriptive statistics for the main assets in households balance sheets.¹⁹ The first panel presents all safe assets broken by specific asset classes. Likewise, the second panel reports specific risky assets. Following Chang et al. (2018), I split several assets (e.g. mutual funds) according to the risk of the underlying asset. For example, equity mutual funds go to risky assets, bonds mutual funds to safe assets, and combinations of both are splitted equally between asset types. The third panel reports total financial wealth according to my baseline definition (See Appendix A). Finally, the fourth panel summarizes net house wealth in the sample.

Table 1: Descriptive Statistics of Households Assets

	Mean	Sd	P10	Median	P90	Part. Rate
<i>Total Safe Assets (S)</i>	112,502	754,336	94	12,838	217,628	0.93
Checking Account	6,423	58,129	0	1,338	11,200	0.84
Savings Accounts	14,809	164,161	0	94	24,617	0.55
Money Market Accounts	5,569	99,903	0	0	1,154	0.12
Cerificates of Deposits	6,874	73,597	0	0	1,817	0.11
Savings bond (safe)	8,001	272,823	0	0	426	0.14
Mutual Funds (safe)	11,185	285,924	0	0	0	0.05
IRA (safe)	17,161	106,690	0	0	22,713	0.19
Pensions (safe)	16,352	91,101	0	0	30,951	0.27
<i>Total Risky Assets (R)</i>	127,304	1,102,643	0	323	205,075	0.51
Brokerage	2,459	112,166	0	0	0	0.02
Stocks	40,557	767,154	0	0	12,315	0.17
Mutual Funds (risky)	26,906	398,752	0	0	3,941	0.11
Savings bond (risky)	1,773	113,179	0	0	0	0.01
IRA (risky)	24,041	143,754	0	0	37,500	0.22
Pensions (risky)	21,536	120,580	0	0	42,646	0.31
<i>Baseline Financial Wealth (S + R)</i>	239,806	1,620,250	105	21,238	437,007	0.94
<i>Net House Wealth</i>	171,922	766,263	0	45,222	367,000	0.68

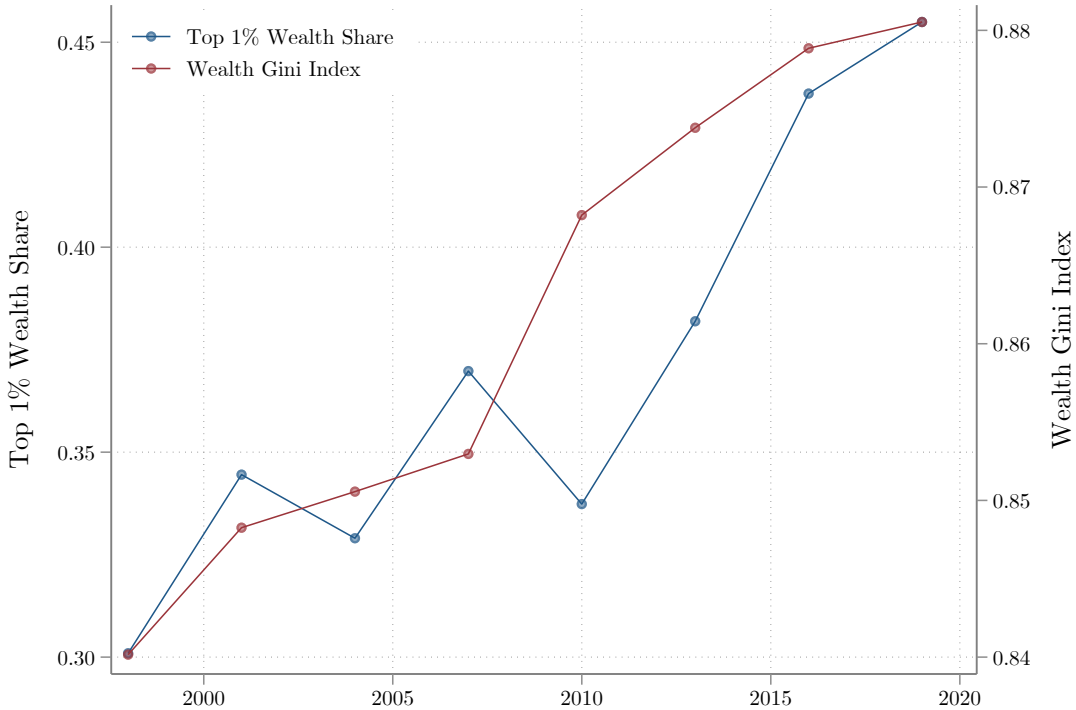
Notes: Data from the SCF for the period 1998-2019. All variables expressed in 2010 USD and deflected using CPI data from the World Bank's World Development Indicators (<https://data.worldbank.org/indicator/FP.CPI.TOTL?locations=US>). Baseline definition of financial wealth follows Chang et al. (2018) (for the specific definition see Appendix A).

¹⁹Also, Table 4 in the Appendix B.1 presents descriptive statistics for demographic variables.

2.2 Stylized Facts on Wealth Inequality and Portfolio Choice

As a first motivating fact, [Gabaix et al. \(2016\)](#) shows that wealth inequality has increased through time. Thus, in Figure 1 I plot the evolution of two common measures of inequality (the top 1% wealth share and the Gini index) for my baseline definition of financial wealth. The data shows a steeply increasing trend over the sample period, with both measures consistently increasing over time. Albeit not explicitly covered in the model, this figure remarks the importance of understanding the drivers of wealth inequality and its dynamic.

Figure 1: Evolution of Financial Wealth Inequality

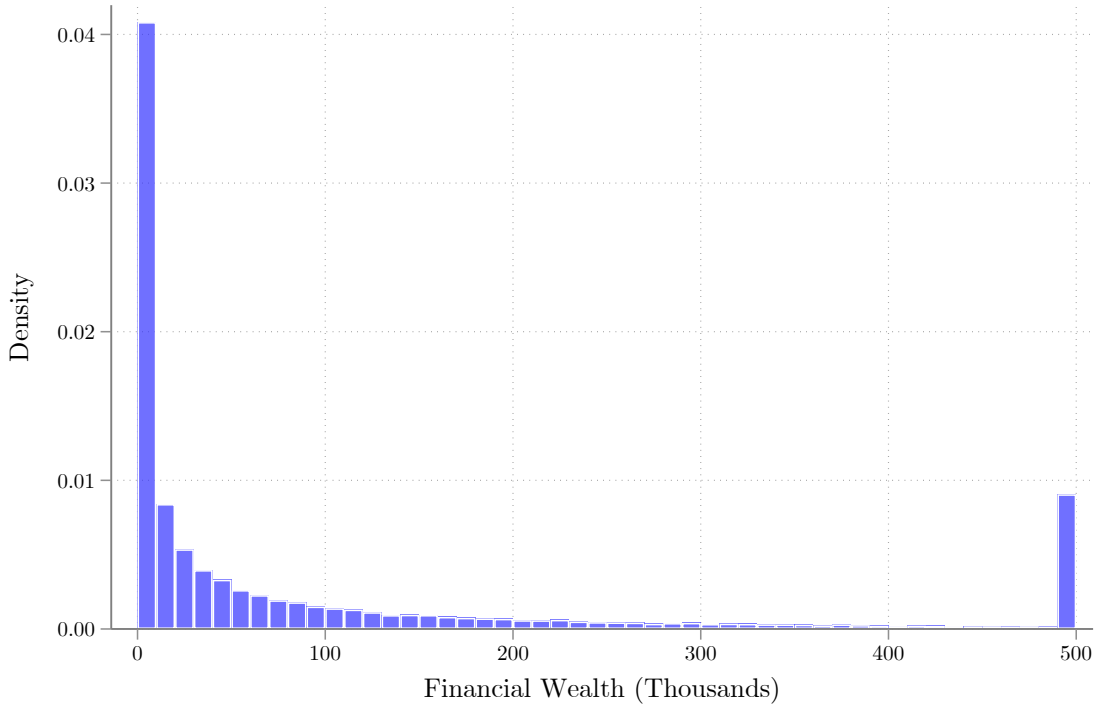


Notes: Data from the SCF for the period 1998-2019 in 2010 USD. Both indicators use Financial Wealth baseline definition (see Appendix A).

Another well-known fact is that the distribution of financial wealth is extremely right skewed. The blue line in Figure 1 supports that fact as the top 1% holds more than 45% of the total financial wealth in 2019. Further, Figure 2 plots the distribution of financial wealth in the SCF in 50 equal-sized bins. Note that over 40% of the households hold less than 10,000 dollars, while roughly a 10% has a financial wealth over 500,000 dollars.

The strikingly uneven distribution of financial wealth may be due different savings rates or different returns to savings. In the spirit of the growth literature, I refer to both types of accumulation as perspiration and inspiration respectively. Namely, richer households may accumulate more wealth because they save at higher rates (perspiration), or because they allocate resources

Figure 2: Financial Wealth Distribution in the SCF



Notes: Data from the SCF for the period 1998-2019 in 2010 thousand USD. Financial Wealth Definition in Appendix A. Weights in this figure were rounded to the nearest integer. Figure 12 in Appendix B.2 presents the distribution broken by asset class.

into higher-return assets (inspiration). Many empirical studies focus on the latter as an important driver for wealth inequality (Piketty and Zucman, 2014; Bach et al., 2020; Fagereng et al., 2020; Kuhn et al., 2020), I take this evidence as an input for my model as there is substantial differences in households balance sheets that have been understudied in the quantitative literature.²⁰

To this end, note from Table 1 that participation rates are substantially higher in safe assets and that the median household is absent in most of the asset markets.²¹ The lack of participation of median-wealth households suggest the existence of heterogeneity in portfolio choices, sensitive to wealth scale. To address this issue, and keeping notation from Table 1, I define risky share as

$$\omega = \frac{R}{R + S} \quad (1)$$

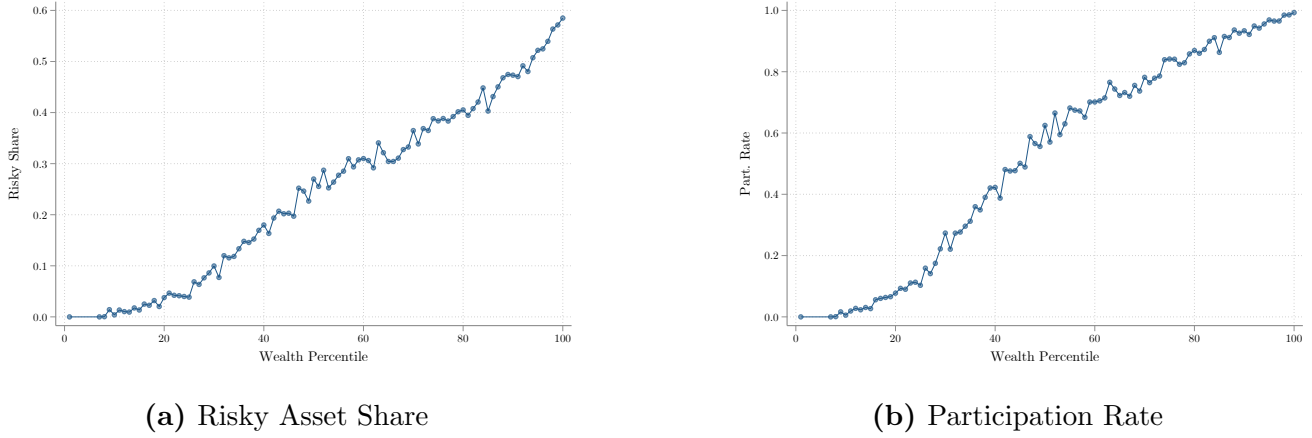
Figure 3 plots households portfolio choices across our baseline financial wealth definition. Panel (A) presents risky shares and reports a steeply increasing share. Panel (B) presents participation

²⁰Both recent improvements in access to reliable micro data in households balance sheets and the development of computational methods to solve macroeconomic models with realistic microfoundations are probably at the core of the increasing interest in this channel.

²¹Also note that housing is the main household asset with a mean stock greater than the mean financial wealth.

rates in risky assets (i.e. $R > 0$), which suggests that this heterogeneity is likely to be produced along the extensive margin.

Figure 3: Portfolio Choices Across the Wealth Distribution



Notes: Risky Share defined in Appendix A. Participation rate in the risky asset is defined as $\mathbf{1}\{R > 0\}$. Each connected point represents the average risky share/participation rate for households in each percentile. Wealth distribution is computed using the baseline definition of financial wealth.

The fact that participation rates are substantially smaller than the ones predicted by theory makes room for theories incorporating different frictions on risky asset participation. For instance, households probably face large costs when monitoring risky asset investments and are exposed to big losses due to delayed adjustment (Duffie and Sun, 1990; Lynch, 1996; Gabaix and Laibson, 2001).²² Similarly, many countries charge operational fees for buying/selling assets, which may be relatively larger for poorer households. This paper builds on this motivation for the quantitative model to be presented in the subsequent section.

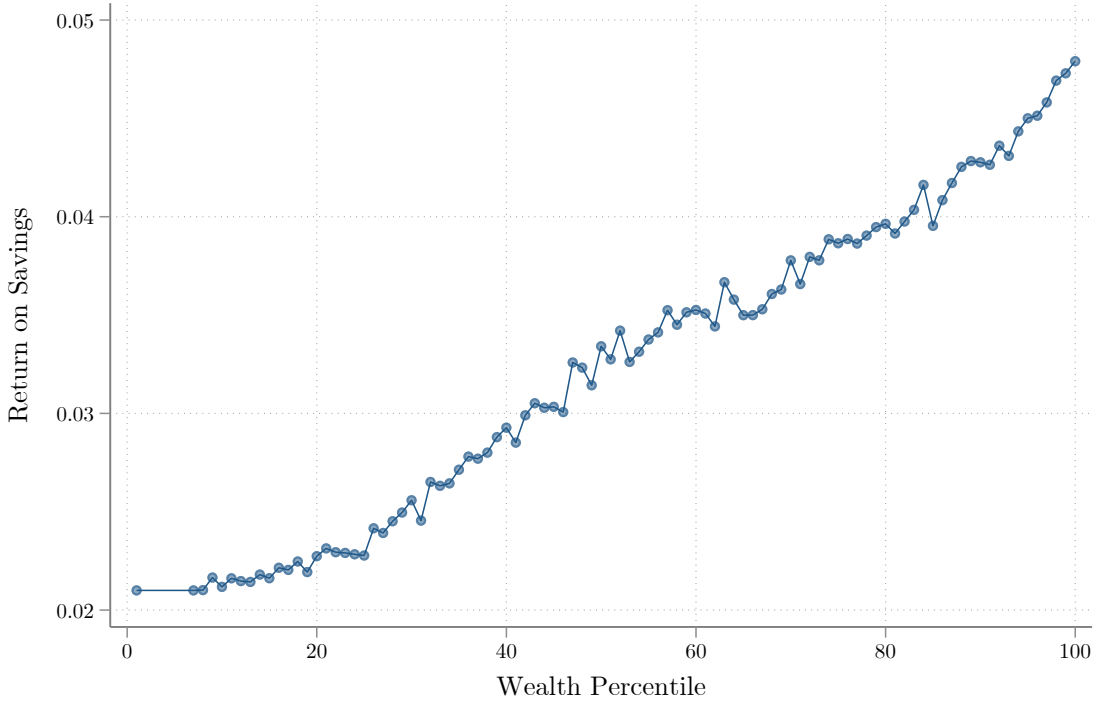
Both participation rates and the risky share are relevant actors in households wealth accumulation if average returns differ across asset classes. A simple back-of-the-envelope example allows to illustrate this point. Consider as a proxy for returns on safe and risky assets the estimates from Xavier (2020) on interest-earning assets and public equity respectively.²³ Then taking this returns as given, compute the average portfolio return for each percentile of the wealth distribution. Figure 4 shows an average return highly increasing in wealth. In fact, the only source of heterogeneity in this calculation is given by heterogeneity in risky shares, and thus the slope may be steeper if returns are increasing with wealth within asset classes, as recent empirical evidence suggest (Fagereng et al., 2020; Xavier, 2020).

Regarding the definition of financial wealth, this paper excludes from its baseline definition to very important assets in households balance sheets: private business and housing. On the

²²For example, spending t hours monitoring investment at an hourly wage w , has an opportunity cost $w \times t$.

²³She estimates an annual return of 2.1% for the former and 6.67% for the later.

Figure 4: Simple Calculation of Return Heterogeneity

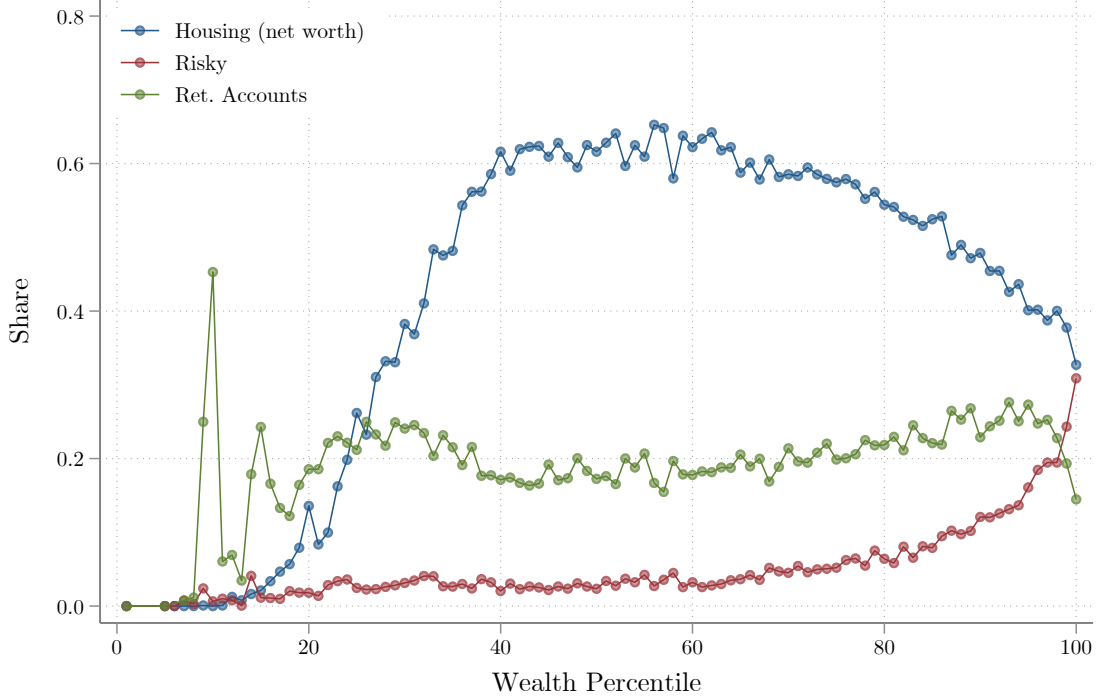


Notes: Risky Share defined in Appendix A. Returns on assets follow annual rates computed by [Xavier \(2020\)](#).

one hand, I abstract from private business, probably the higher return asset, as its nature is somewhat different as most investment assets. Households with private business often participate in management decisions in the company or have private information related to the performance of the business.

On the other hand, dealing with housing in heterogeneous agent models is not straightforward as it constitute both a durable consumption good and an investment asset. [Laibson, Maxted and Moll \(2020\)](#), though on a different setting, build an heterogeneous agent model that explicitly incorporate housing in the utility function, playing the double role described earlier. Despite the fact that this feature allows to build a more realistic environment, results in [Fagereng et al. \(2019\)](#) suggest that "saving by holding" is not a story exclusively driven by housing. Furthermore, using the baseline definition of financial wealth, and incorporating net house wealth, I find that housing loses its appeal to explain trends in wealth inequality at the upper tail of the distribution. Namely, Figure 5 shows that the housing share peaks somewhere around the median of the wealth distribution and then monotonically decreases. On the contrary, other risky assets combined (stocks, risky bonds, mutual funds, annuities, trusts and brokerage) steeply increase at the top of the distribution.

Figure 5: Asset Shares Across the Wealth Distribution



Notes: This figure considers the baseline definition of financial wealth plus housing and retirement account assets for computing both shares and the percentiles of the wealth distribution.

2.3 Sensitivity Analysis

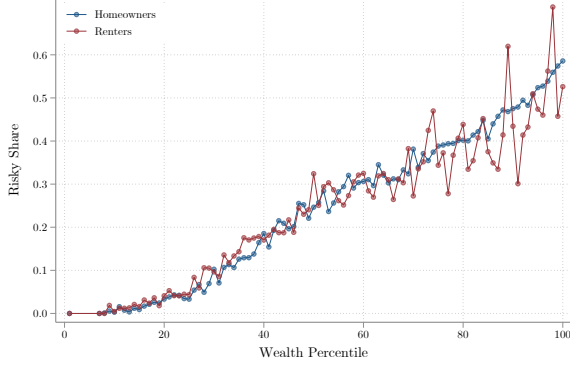
In this subsection I will present risky shares controlling for traditional suspects regarding households savings decisions. First, I will present descriptive evidence on risky shares for groups commonly related to high/low participation in financial markets. Then I will estimate predicted risky shares by using a simple regression model, controlling for classical determinants of portfolio choice (education, income, age).

Figure 6 shows that average risky shares are somewhat similar across the wealth distribution for both homeowners/renters, and college/no college households. The panel on the left shows that despite very few renters at the top of the distribution, their risky share follows a similar trend than the one of homeowners. Similarly, the panel of the right shows that there is no substantial difference in portfolio choices for households with and without college degree. These results are consistent with similar exercises conducted by [Chang et al. \(2018\)](#).

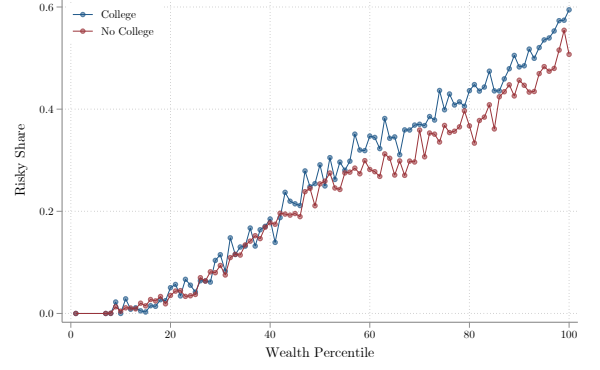
Now for the linear model, I will follow [Fagereng et al. \(2019\)](#) and estimate

$$\omega_{it} = \alpha + \sum_{p=2}^{100} \delta_p D_{it,p} + f(x_{it}) + \mu_t + \varepsilon_{it}, \quad (2)$$

Figure 6: Robustness in Portfolio Decisions



(a) Homeowners versus renters



(b) College versus no college

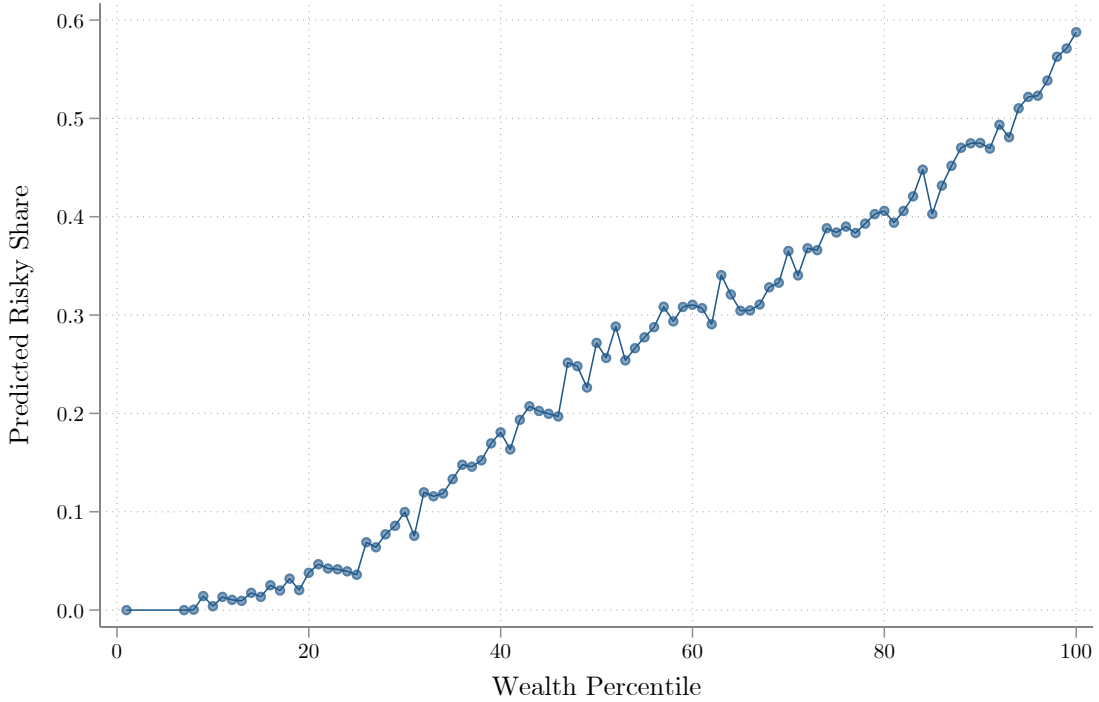
Notes: Homeowners represent households with housing net worth different than 0. College refers to households with a head with a college degree.

where ω_{it} is the risky share of household i , surveyed on the edition t , $D_{it,p}$ a dummy variables for the respective percentile p , μ_t a year dummy and $f(x_{it})$ a vector of control variables that includes, income, homeowner dummy, college dummy, marital status, number of children and third-order polynomial for age.²⁴ After estimating Eq. (2), I compute $\hat{\omega}_{it}$ and then plot the mean predicted risky share by percentile in Figure 7 shows that again, the increasing in wealth risky share is robust to controlling by traditional drivers of portfolio choice.

Hence, there are overall three key stylized facts that will be covered in the next sections. First, the fact that there is surprisingly small participation rates in risky asset. Second, that both shares and participation rates are steeply increasing across the wealth distribution. Third, this papers intends to explain how wealth accumulation is affected by households portfolio choices. In particular, how introducing frictions into an otherwise standard portfolio choice model is able to replicate the highly unequal wealth distribution seen in the data.

²⁴Note that as mentioned, the SCF is not a panel and thus, this is only a cross-sectional model.

Figure 7: Predicted Risky Share



Notes: This figure presents average predicted risky share by percentile. Predicted values of risky share ω are computed from estimating Eq. (2). For coefficients on percentile dummies please refer to Figure 14 in Appendix B.2.

3 An Heterogeneous Agent Model of Portfolio Choice

In this section I describe the quantitative model for the economy. The model features a partial equilibrium model with safe and risky assets. Households face both a consumption-saving decision and a portfolio choice, subject to non-convex adjustment costs. The framework is inspired in traditional entry/exit models from firm dynamics literature (Hopenhayn, 1992).²⁵ It also follows closely the household problem in Kaplan et al. (2018). However, it differs from the later in the way financial frictions are introduced and in the sense that the two assets are not only different in their adjustment cost, but also in the nature of the return process they follow.

Households. Time is continuous and there is no aggregate uncertainty. The economy is populated by a unit mass of infinitely-living households that perceive a utility flow u over consumption $c_t \geq 0$, where u satisfies INADA conditions. Preferences are time-separable and households discount the future at a rate $\rho \geq 0$. Utility is represented by a Constant Relative Risk Aversion

²⁵Other similar applications of stopping time problems can be found in the international trade literature (Melitz, 2003), investment theory (Dixit, 1989), option pricing (Huang and Pang, 2003), among other.

(CRRA) function

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}}{1-\gamma}, & \gamma > 0 \\ \log(c_t), & \gamma = 1. \end{cases} \quad (3)$$

Markets are incomplete in the sense that households are unable to fully insure against shocks. Households can hold two types of assets. (i) a safe asset b with exogenous return rate $r_t^b(b)$, a borrowing limit $\underline{b} > -\infty$ and a wedge between saving and borrowing $\varpi > 0$.²⁶ (ii) a risky asset a with stochastic return r_t^a . The risky asset can only be held in non-negative amounts. Deposits or withdrawals in the risky asset are subject to an adjustment cost $\kappa > 0$.²⁷ This generates an inaction region in households portfolio decisions. The cost can be interpreted directly as transaction fee or indirectly as a monitoring cost, mental accounting, among other.²⁸ Assume for now that the return of the risky asset r_t^a follows a general process of the form

$$dr_t^a = \mu(r_t^a)dt + \sigma(r_t^a)dW_t, \quad (4)$$

where W_t is a standard Brownian motion.²⁹ Labor income follows a two-state Markov process with income streams z_1, z_2 , with $z_2 > z_1$. The income process jumps from state z_1 to z_2 with probability λ_1 , and from z_2 to z_1 with probability λ_2 .^{30,31}

Households differ on their safe wealth b , risky wealth a and the pair idiosyncratic shocks (z, r^a) . At each instant in time t , the state of the economy is the joint distribution of income y and wealth (a, b) .

²⁶That is,

$$r_t^b(b) = \begin{cases} r_t^b, & b > 0 \\ r_t^b + \varpi, & b \in (\underline{b}, 0). \end{cases}$$

Kaplan et al. (2018) refer to this wedge as a “soft borrowing constraint”.

²⁷Note that this model features non-convex adjustment costs whilst Kaplan et al. (2018) uses a kinked, but strictly convex, adjustment cost function.

²⁸See e.g., Gabaix and Laibson (2001); Gomes and Michaelides (2005); Alvarez et al. (2012) among many other. It is also straightforward to allow for a utility cost instead of a monetary cost of adjustment.

²⁹A standard Brownian motion is a stochastic process that satisfies $W(t + \Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}$, $\varepsilon_t \sim \mathcal{N}(0, t)$.

³⁰With this process I implicitly assume that the income process is independent from the return on the risky asset. Even though in life-cycle settings the covariance term may be important to make households shift towards safer assets as they age, in an infinite horizon setting its importance is of second order. Also, the empirical evidence on the correlation between labor income and risky-asset returns is at least mixed. For instance, Cocco et al. (2005) does not find a significant correlation using the U.S. Panel Study of Income Dynamics (PSID), while Fagereng et al. (2017) finds similar results using administrative data for Norway.

³¹Even though it can be easily extended to a more general process (e.g. diffusion), this setting still has economic interpretation: z_1, z_2 may represent unemployed and employed income streams respectively, with λ_1, λ_2 the job creation and job destruction rates. Moreover, despite being very stylized, this process is able to generate key moments from the income process. In particular, for the benchmark calibration presented below, I discretize a Ornstein-Uhlenbeck (OU) process into a two-state Poisson process that replicates the persistence and volatility of the log-income process seen in the data.

Households' asset holdings evolve according to

$$\dot{b}_t = z_t + r_t^b b_t - c_t \quad (5)$$

$$\dot{a}_t = r_t^a a_t, \quad (6)$$

Thus, from Eq. (4) and (6), risky wealth evolves according to³²

$$da_t = \mu(r_t^a)ad_t + \sigma(r_t^a)adW_t \quad (7)$$

Households' value function is given by

$$v(a, b, z) = \max_{\{c_t\}, \tau} \mathbb{E}_0 \int_0^\tau e^{-\rho t} u(c_t) dt + e^{-\rho \tau} \mathbb{E}_0 v^*(a_\tau + b_\tau, z), \quad (8)$$

where expectations are taken over realizations of the labor income and the return on wealth, and

$$v^*(a + b, z) = \max_{a', b'} v(a', b', z) \text{ s.t. } a' + b' = a + b - \kappa. \quad (9)$$

From Eq. (8) note that households choose consumption paths, a stopping time and a re-balanced portfolio upon adjustment in period $t = \tau$. That is, households choose instantaneous consumption flows at every period but keep their portfolio unchanged until idiosyncratic shocks push them out of their inaction range. When that happens, households pay the adjustment cost κ and re-balance their portfolio by solving Eq. (9).

Altogether, households solve Eq. (8) subject to (5)-(6), non-negative constraints over consumption and assets and the exogenous processes for z and r^a . This problem can be expressed recursively in terms of the following Hamilton Jacobi Bellman (HJB) equation

$$\begin{aligned} \rho v(a, b, z) = & \max_c u(c) \\ & \text{Safe Asset : } + \partial_b v(a, b, z)(z + r^b b - c) \\ & \text{Risky Asset : } + \mu(r^a)a\partial_a v(a, b, z) + \frac{\sigma^2(r^a)a^2}{2}\partial_{aa}v(a, b, z) \\ & \text{Labor Income : } + \sum_{z' \in Z} \lambda^{z \rightarrow z'} (v(a, b, z') - v(a, b, z)) \end{aligned} \quad (10)$$

with a constraint that

$$v(a, b, z) \geq v^*(a + b, z) \quad \forall a, b. \quad (11)$$

The intuition behind Eq. (11) is straightforward. Due to the adjustment cost κ , households do not re-balance their portfolio at every period, that is, $v(a, b, z) > v^*(a + b, z)$. Whenever the value of

³²This follows directly from Ito's Lemma.

paying the adjustment cost and re-balancing the portfolio is infinitesimally higher than $v(a, b, z)$, households adjust their portfolio. Hence, $v(a, b, z) = v^*(a + b, z)$ acts as a trigger of the households portfolio decisions.

Further, Eq. (10) is also subject to a state constraint boundary condition given by

$$\partial_b v(a, b, z) \geq u'(z), \quad (12)$$

which is simply a combination of the borrowing constraint in the liquid asset b and the first order condition in Eq. (10). Suppressing dependence of v in (a, b, z) , this problem can be written compactly as³³

$$\min \left\{ \rho v - \max_c \{ u(c) - \mu(r^a)a \partial_a v - \frac{\sigma^2(r^a)a^2}{2} \partial_{aa} v - (z + r^b b - c) \partial_b v - \sum_{z' \in Z} \lambda^{z \rightarrow z'} (v(z') - v(z)) \}, v - \mathcal{M}v \right\} = 0, \quad (13)$$

where $v^* = \mathcal{M}v$ and \mathcal{M} is known as the "intervention operator". This equation is frequently used in mathematics and is called Hamilton Jacobi Bellman Quasi-Variational Inequality (HJBQVI).³⁴

In matrix notation

$$\min \left\{ \rho \mathbf{v} - u(\mathbf{v}) - \mathbf{A}(\mathbf{v}) \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \right\} = 0 \quad (14)$$

Stationary Distribution. The evolution of the joint distribution wealth and income can be described by the Kolmogorov Forward (KF) equation.³⁵ Define $g(a, b, z, t)$ as the density function corresponding to the joint distribution of the states in period t and $s^b(a, b, z)$ as the saving policy function for the safe asset. Then without adjustment, the stationary density satisfies the KF equation

$$0 = -\partial_a (\mu(r^a) a g(a, b, z)) + \frac{1}{2} \partial_{aa} (\sigma^2(r^a) a^2 g(a, b, z)) - \partial_b [s^b(a, b, z) g(a, b, z)] - \lambda^{z \rightarrow z'} g(a, b, z) + \sum_{z' \in Z} \lambda^{z' \rightarrow z} g(a, b, z'), \quad (15)$$

which in matrix notation is simply

$$0 = \mathbf{A}^T g \quad (16)$$

However, Eq. (15) is unable to capture adjustment. To incorporate adjustment we need to reallo-

³³Appendix D.1 presents in detail the derivation of this equation.

³⁴See Bensoussan and Lions (1982, 1984); Barles, Daher and Romano (1995); Bardi and Capuzzo-Dolcetta (2008) and Øksendal (2003)

³⁵Also often referred as the Fokker-Planck equation

cate mass in the adjustment region to their optimal adjustment targets denoted (a^*, b^*) . Luckily, despite the fact that adding this extra terms mathematically is not straightforward, it can be easily dealt with numerically.³⁶

Finally, the stationary distribution $g(a, b, z)$ must satisfy

$$1 = \sum_{z \in Z} \int_0^\infty \int_0^\infty g(a, b, z) da db \quad (17)$$

Hence, the economy can be represented by a system of equations given by (17), the HJB and the KF equations.

4 Baseline Quantitative Analysis

This section presents the main results from solving the model presented in the preceding section. The model is calibrated at an annual frequency using a combination of externally and internally calibrated parameters, targeting specific moments from the data.

4.1 Benchmark Calibration

Labor income. There are four parameters to calibrate for the income process: $\{z_1, z_2, \lambda_1, \lambda_2\}$. These parameters are calibrated as follows. I assume discrete time log-income follows an AR(1) process of the form

$$\log(z_t) = \varphi_z \log(z_{t-1}) + \nu_t \quad (18)$$

Following Guvenen, Kambourov, Kuruscu, Ocampo and Chen (2019) and Aiyagari (1994), I set the autocorrelation $\varphi_z = 0.9$ and the standard deviation of innovations $\sigma_\nu = 0.2$. Then, I recover the drift and the diffusion of the Ornstein-Uhlenbeck process (continuous time analog of the AR(1) process)

$$d \log(z_t) = -\theta_z \log(z_t) + \sigma_z dW_t, \quad (19)$$

where

$$\varphi_z = e^{-\theta_z} \quad (20)$$

$$\sigma_z = \frac{\sigma_\nu^2}{2\theta_z} (1 - e^{-2\theta_z}) \quad (21)$$

³⁶For a theoretical reference see Bertucci (2020). See Appendix C for the numerical approach to deal with this issue.

Average income is normalized to one and income states are $-1, +1$ standard deviations from the mean. That is, $z_1 = 0.79$ and $z_2 = 1.21$. Finally, it can be shown that transition probabilities can be obtained from

$$\lambda^{z \rightarrow z'} = \left[\frac{\theta_z}{2\pi\sigma_z^2(1 - e^{-2\theta_z})} \right] \exp \left[-\frac{\theta_z}{\sigma_z^2} \frac{(\log(z') - \log(z)e^{-\theta_z})^2}{1 - e^{-2\theta_z}} \right], \quad (22)$$

which yields $\lambda_1 = \lambda_2 = 0.25$.

Assets. I use a rather standard calibration for the return on wealth. Namely, the process for the return of the safe and risky assets are calibrated following [Gomes and Michaelides \(2005\)](#). In particular, the annual return of the safe asset r_b is set to 2 percent whereas the drift and the diffusion of the risky asset are constant and set to $\mu = 0.06$ and $\sigma = 0.18$. That is, the average excess of return is 4 percent annually.

The borrowing constraint \underline{b} is set to 1 times the average annual income and the annual wedge is set to $\varpi = 0.06$ as in [Kaplan et al. \(2018\)](#).

Adjustment cost. The non-convexity of the adjustment cost is arguably the most important driver of portfolio choice and wealth inequality in the model. Nevertheless, given the lack of evidence of transaction costs, there is no direct benchmark from the literature. Hence, I internally calibrate this parameter to match the participation rate in the risky asset from the data.

This approach allows to incorporate the extensive margin of asset accumulation, a feature absent in most of the wealth accumulation literature. As it will be shown below, traditional workhorse models that try to replicate the shape of the wealth distribution often require high participation rates, which is in stark contrast with the well-known lack of participation in risky assets from micro data.

Preferences. Preference take standard values from the literature. The coefficient of relative risk aversion parameter γ is set to 2. The discount rate is set to $\rho = 0.053$.

4.2 The role of the adjustment cost κ

This subsection stresses out the importance of financial frictions in households portfolio decisions.³⁷ The intention of introducing frictions in financial markets is to account for different features that drive households portfolio decisions absent in workhorse models in the literature. In that sense, the adjustment cost κ is a reduced form parameter that may represent either direct monetary

³⁷Unless otherwise noted, I will use the baseline calibration presented in the preceding subsection.

Table 2: Baseline Calibration

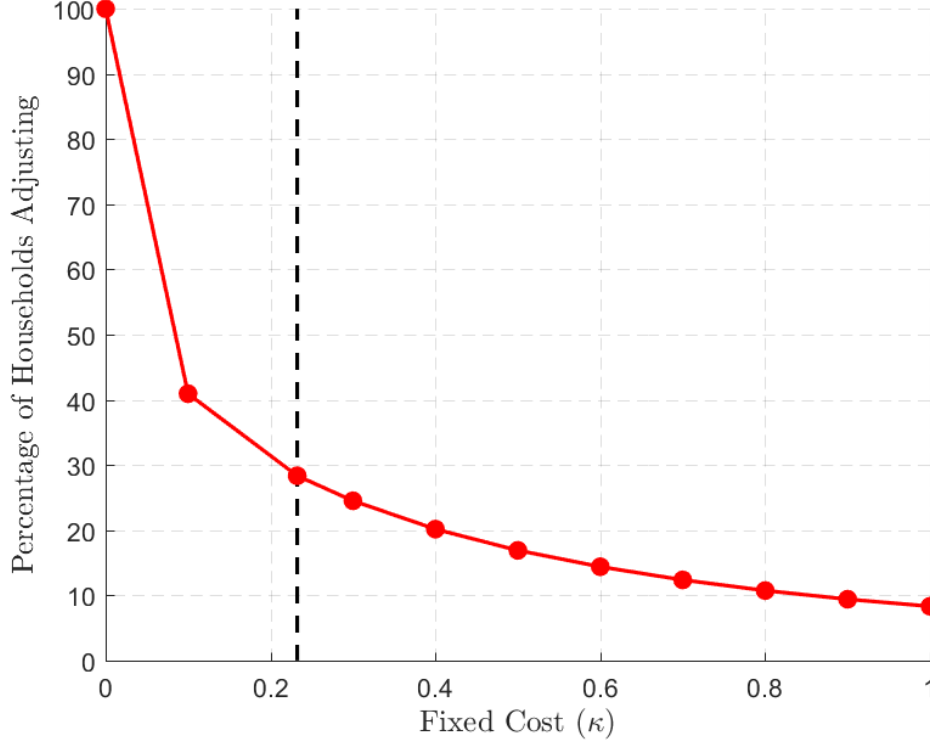
Parameter	Description	Value	Source/Target
<i>Households</i>			
γ	Risk aversion	2	Standard
ρ	Subjective discount rate	0.053	Standard ($\beta = 0.95$)
<i>Assets</i>			
\underline{b}	Borrowing limit	-1	1 times avg. income
ϖ	Interest rate wedge	0.06	Kaplan et al. (2018)
r^b	Safe asset return	0.02	Gomes and Michaelides (2005)
μ	Risky asset drift	0.06	Gomes and Michaelides (2005)
σ	Risky asset volatility	0.18	Gomes and Michaelides (2005)
κ	Adjustment cost	0.23	Participation Rate
<i>Income Process</i>			
z_1, z_2	Income states	0.79, 1.21	$\sigma_z = 0.21, \varphi_z = 0.9, \mathbb{E}(z) = 1$
λ_1, λ_2	Income jumps	0.25, 0.25	Eq. (22)

fees, opportunity costs of planning future investments or more subtle indirect costs like desutility of acquiring financial skills, information processing costs, mental accounting, among many other. Therefore, it is reasonable that the inaction generated in the model is increasing in κ . Figure 8 shows this feature by replicating the share of households adjusting their portfolio allocation as in Kaplan and Violante (2014). Clearly, the figure shows that inaction is increasing in the adjustment cost. In fact, the calibrated model only needs a small adjustment cost to match participation rates seen in the data (roughly 51%). In particular, κ represents an average cost of 0.75% of wealth stock for adjusting households, which is in line with previous studies using transaction costs.³⁸

Furthermore, the model allows to identify from which region of the state-space the adjustment comes from. Namely, Figure 9 presents the adjustment region within the state-space of the economy. Note that there is a relatively large inaction range where poorer households (at the bottom left) and richer households (at the top right) do not adjust their portfolio choice. These results are in line with evidence documenting the importance of the extensive margin in portfolio choice (Chang et al., 2018), as well as evidence in saving by holding at the top of the wealth distribution (Fagereng et al., 2019; Kaplan and Violante, 2014).

³⁸For example, Kaplan and Violante (2014), in a liquid/illiquid two asset model reports a transaction cost of 0.9% of the stock of illiquid assets for adjusting households. Similarly, Alvarez et al. (2012) reports a 1% adjustment cost on durable goods. Also, Individual Retirement Accounts (IRA) are usually subject to costs and penalties for early withdrawals of 10%.

Figure 8: Portfolio Inertia for Different Adjustment Costs



Notes: This figure reports the share of households re-balancing their portfolio (i.e. $v(a, b, z) = v^*(a, b, z)$) out of the total state-space L for different adjustment costs. Vertical line represents baseline calibration for κ .

4.3 The Merton Model as a Useful Benchmark

Note that the model presented in Section 3 is a generalization of the traditional infinite-horizon portfolio allocation model introduced by Merton (1969). In fact, the Merton model can be recovered by assuming $\kappa = 0$ and no labor income ($z_1 = z_2 = 0$). Thus, keeping notation from Section 3, we have that optimal portfolio allocation (hereafter the "Merton rule") yields the following risky share

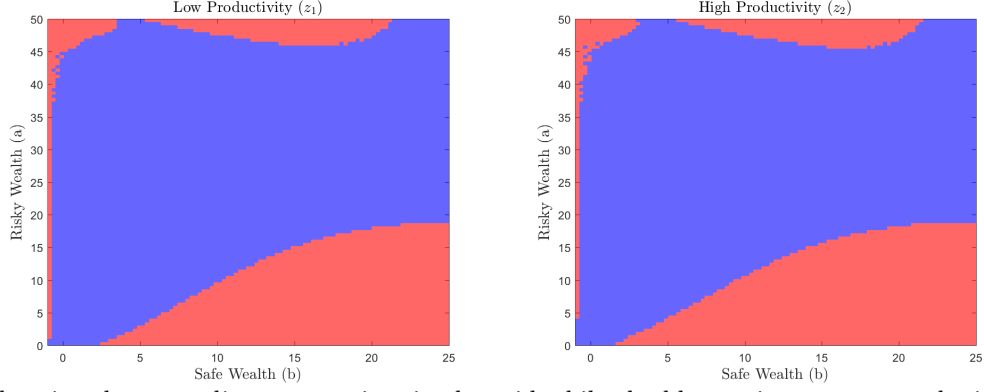
$$\omega^M = \frac{\mu - r^b}{\sigma_r^2 \gamma}, \quad (23)$$

which is time-invariant and independent from wealth.³⁹ The intuition is that as there is no other source of uncertainty aside from the return of the risky asset, and no constraint in the risky asset, households optimally allocate a constant share that depends only on the average excess of return, the volatility of the investment and their personal preferences for risk.

Now if households face labor income uncertainty and frictions in saving and re-balancing their portfolio, risky asset shares are not necessarily optimal. For instance, a poor household will need

³⁹Appendix D.2 solves the model in detail.

Figure 9: Adjustment Region



Notes: The red region denotes adjustment points in the grid while the blue region represents the inaction region. Figure on the left denotes low income (y_1) and the figure in the right the high income (y_2). Figure 15 in Appendix B.2 reports adjustment targets.

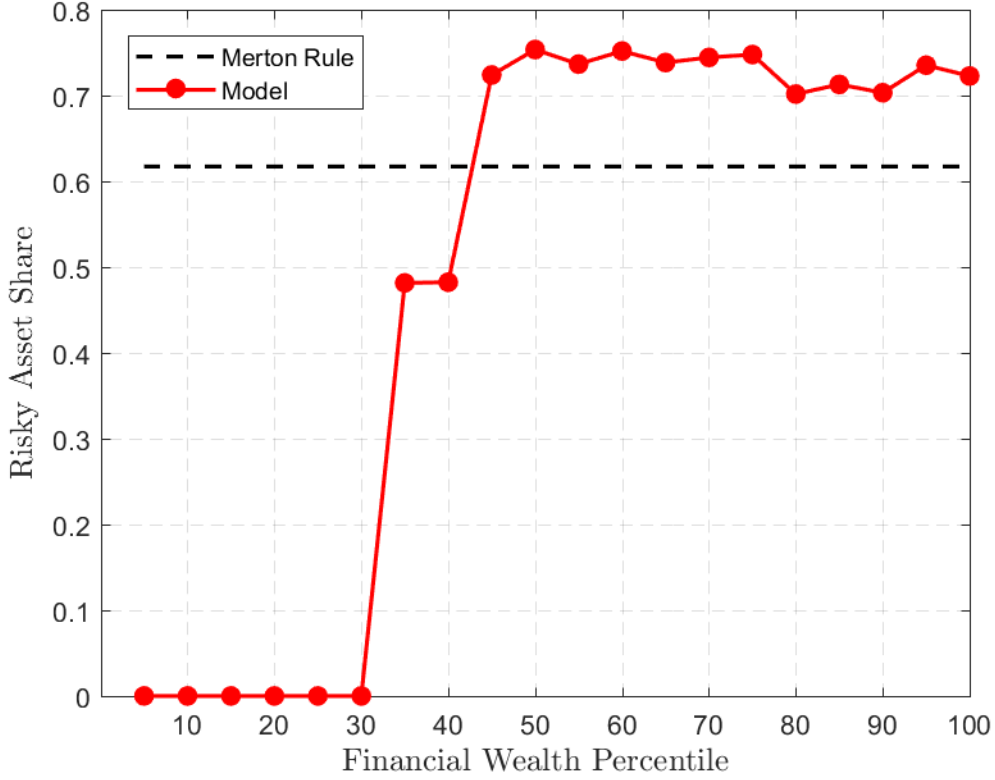
to pay a large (relative to its wealth) cost to enter the risky asset market, and thus, their optimal share should lay below the Merton rule. Similarly, a household with a large initial endowment of the risky asset, may have a share above the Merton rule. However, κ prevents that households to adjust towards their optimal portfolio in a frictionless world.

Consistently, Figure 10 shows that the model replicates that intuition by presenting the average risky share by deciles and comparing it to the Merton rule. In particular, households below percentile 30 do not hold risky assets and thus, they are either hand-to-mouth or restricted to save only in low-return assets. On the contrary, households at the top of the distribution hold a slightly riskier portfolio than the optimal due to frictions in re-balancing their portfolio. Note also that as the adjustment cost is fixed, it is relatively smaller for richer households. Therefore, deviations from Merton rule are substantially higher at the lower end of the distribution. Note also that in the model, over some threshold, say the median of the distribution, risky share is somewhat flat and above the Merton rule. Also, one remark of the model is that given discrete adjustments and the fact that all the risky assets are collapsed into one single asset, the replication of risky shares from the data is imperfect. Extending the model to a N asset environment may help to overcome this issue.

4.4 Stationary Distribution of Wealth

This subsection presents stationary distribution as the solution of the KF equation introduced in Section 3. Since the model cannot be solved analytically, I employ the already mentioned finite-difference upwind scheme method from Achdou et al. (2017) to solve the model (See Appendix C). One of the biggest advantages of this setting is that allows me to easily compute the stationary

Figure 10: Risky Share across Wealth Distribution



distribution of the economy once obtained the transition matrix $\mathbf{A}(\mathbf{v})$. Then with the stationary distribution $g(a, b, z)$, the marginal distribution of both assets in the economy are simply

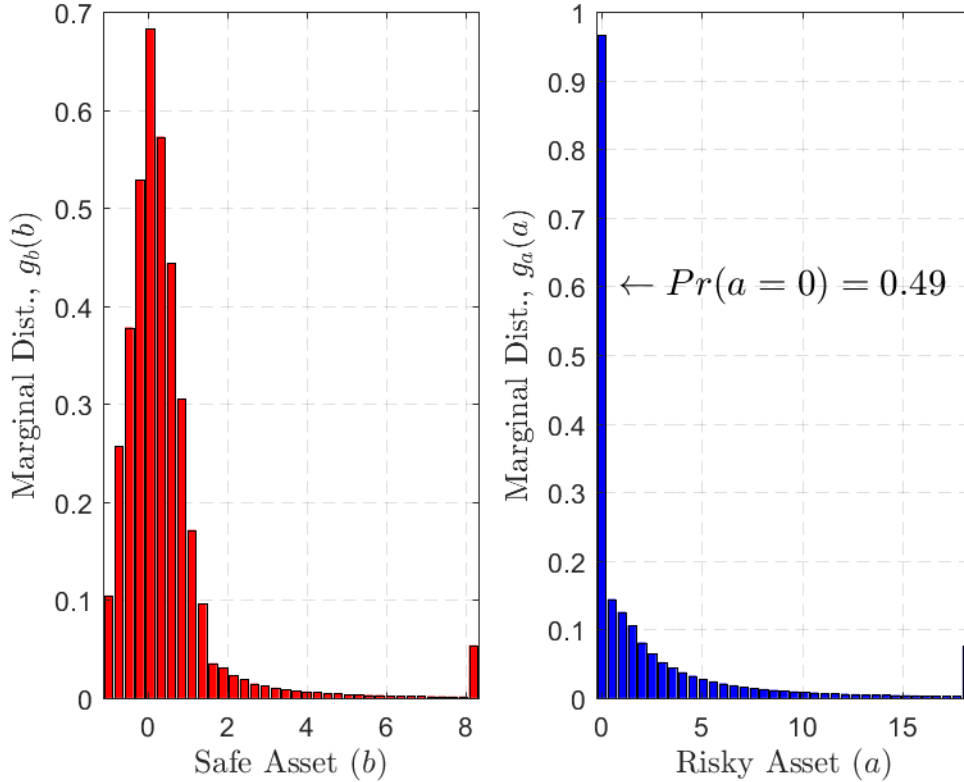
$$g_a(a) = \sum_{z \in Z} \int_{-\infty}^{\infty} g(a, b, z) db \quad (24)$$

$$g_b(b) = \sum_{z \in Z} \int_{-\infty}^{\infty} g(a, b, z) da \quad (25)$$

Figure 11 plots the marginal distribution for both assets. This figure shows a highly unequal distribution of wealth, specially regarding risky asset wealth. Note from the panel on the right that the model replicates the participation rate and broadly the shape of the risky asset distribution (see Figure 12b). In particular, a great mass of individuals with zero or little risky wealth and a large concentration of wealth from a small group of households.

The main channel through which households diverge on their wealth accumulation is by the combination of *ex-post* realizations of idiosyncratic returns to savings (luck) and a scale dependent effect generated by the adjustment cost κ . In the spirit of Gabaix and Laibson (2001), the scale

Figure 11: Marginal Distribution of Assets in Steady State



dependent effect is produced as portfolio choice (and thus returns) are dependent on wealth. Intuitively, low-wealth households are unable to afford the cost to enter risky asset markets. Further, these households are more sensitive to labor income shock as it represents a higher share of their net worth. Conversely, rich households have incentives to keep their risky assets as selling them for larger consumption streams would require paying the adjustment cost. Also, as they are less sensitive to labor income shocks, they prefer to hold large amounts of the high-return asset, despite fluctuations on their labor income.

4.4.1 Two-asset Aiyagari model as a useful benchmark

To show to what extent the model presented in this paper contributes to the literature I take as workhorse model the model of Aiyagari (1994) and include a portfolio allocation problem as in Merton (1969).⁴⁰ That is, heterogeneous households subject to idiosyncratic labor income risk and incomplete markets choose consumption paths and their portfolio without adjustment frictions. Assuming the same processes for labor income and stock returns, this model is simply the particular

⁴⁰Achdou et al. (2017) term this model as “Aiyagari Model with Fat-tailed Wealth Distribution”, for more detail see their online appendix in <https://benjaminmoll.com/papers/>.

case of $\kappa = 0$ in which all differences in accumulation are generated by different realizations of the idiosyncratic shocks.

Despite not targeting explicitly any moment of the wealth distribution, the model provides a reasonable fit of both top shares and the bottom-end of the distribution. Table 3 reports wealth shares obtained with the baseline model and counterfactual shares obtained by assuming $\kappa = 0$ under the calibration presented in Section 4.1. The results show that the model is capable of providing a better fit throughout the entire wealth distribution. On the bottom end the share of the bottom 50% is slightly smaller than in the data, though this can be improved by calibrating the borrowing limit to match this moment.⁴¹ On the other end of the distribution, the top shares predicted in the model (top 10%, 5% and 1%) are smaller, though substantially closer than the two-asset Aiyagari model under the same calibration. In particular, the model is able to narrow the gap between the benchmark model and the data in 54% for the top 10% share, 49% for the top 5% share and 41% for the top 1% share

Another advantage of the model presented in this paper is that it takes into account a well-known fact in the household finance literature absent in most models of wealth accumulation: the surprisingly small participation in risky assets. For instance, the Aiyagari model with portfolio choice and no adjustment frictions is able to match top shares of the wealth distribution under specific calibration, however it assumes a participation rate substantially greater than the one seen in the data.⁴²

Table 3: Wealth Inequality

Measure	Data	Baseline Model	Fat-tail Aiyagari (1994)
Top 1%	37.5	22.2	11.5
Top 5%	64.6	49.6	35.2
Top 10%	77.8	66.1	52.6
Middle 40%	19.5	33.8	38.3
Bottom 50%	0.98	0.10	9.2

Notes: Estimates of Wealth inequality consider baseline Financial Wealth definition and calibration from Section 4.1.

⁴¹Appendix ?? shows that this approach ...

⁴²Achdou et al. (2017) show that under reasonable assumptions on the risky asset process, the stationary wealth distribution follows an asymptotic power law, where $1 - G(a) \sim ma^{-\zeta}$ with tail exponent

$$\zeta = \gamma \left(\frac{2\sigma^2(\rho - r^b)}{(\mu - r^b)^2} - 1 \right),$$

which calibrated to US data ($\zeta \approx 1.5$) provides a good fit for the wealth distribution, though predicts almost a complete participation in risky assets.

5 Wealth Distribution with Richer Return Heterogeneity

This section will introduce a more general process for the risky asset as an extension of the baseline model. The intention is to capture the concentration of shares at the top of the distribution by relying in both economic intuition and empirical data. This will be done by introducing richer heterogeneity in the return process for the risky asset.

The baseline model assumes that the return on savings differ only on the ex-post realization of shocks and the portfolio composition. However, it assumes an *ex-ante* identical process for r^a for all households. Given that the model collapses all risky assets into a one single asset, the baseline process for r^a abstracts from imperfect diversification derived from financial frictions. Think in an economy with multiple risky assets, where each transaction requires paying the adjustment cost κ . Intuitively, the adjustment cost prevents households with little wealth to diversify their portfolio when entering the risky asset markets. Households must pay the adjustment cost for buying a first risky asset, and hence there is no portfolio diversification. When households start accumulating assets, they are able to hold other risky assets and diversify their investment, eventually getting to the mean-variance frontier when wealth is sufficiently high. In other words, it is reasonable to think that the volatility of savings is a function decreasing in wealth, contributing to replicate better the wealth distribution.

Moreover, regarding average excess returns, there is recent empirical evidence that supports the idea of increasing returns in savings even within narrow asset classes (Fagereng et al., 2020; Xavier, 2020). This feature may be easily added by incorporating a scale dependent drift for the return process. Therefore, assume a more general process for the return r_t^a where the evolution of the return is allowed to depend on the scale of the investment. Namely

$$dr_t^a = \mu(a)dt + \sigma(a)dW_t, \quad (26)$$

where

$$\mu(a) = \mu^{min} + \alpha_1 a^{\alpha_2} \quad (27)$$

$$\sigma(a) = \sigma^{max} - \beta_1 a^{\beta_2}, \quad (28)$$

where all parameters are non-negative. It is easy to see that $\frac{\partial \mu(a)}{\partial a} > 0$ and $\frac{\partial \sigma(a)}{\partial a} < 0$. Hence, risky wealth now evolves according to

$$da_t = \mu(a)adt + \sigma(a)adW_t, \quad (29)$$

and the solution method is basically the same as in the baseline model. Now since I incorporated 6

additional parameters, there are seven parameters to calibrate $\{\kappa, \mu^{\min}, \sigma^{\max}, \alpha_1, \alpha_2, \beta_1, \beta_2\}$, I will calibrate them internally to match participation rate (as in the baseline model), and six additional moments from the wealth distribution:

6 The Effect of Adjustment Costs on Top Shares

7 Concluding Remarks

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Appendix A: Data Generation and Definitions

A.1 Survey of Consumer Finances

Households balance sheet data is collected over 8 editions of the Survey of Consumer Finances (SCF) for the period between 1998-2019 (both included). The SCF is a cross-sectional survey of US households conducted on a triennial basis. The survey provides detailed information in households assets, income and demographic characteristics. Roughly 6500 households are randomly selected on each survey, with the intention to represent families from all economic strata. The selection of households is designed by using a multistage area-probability design and tax data. The later source is designed to disproportionately select households relatively wealthy, and usually underrepresented in survey data.

For the construction of the variables, I follow closely the definition of [Chang et al. \(2018\)](#).⁴³ The variables generally remain across surveys but changes to the variables within assets will be mentioned when relevant.

⁴³Also, the replication materials on this paper made easier the processing of surveys in the period 1998-2007.

Checking Accounts, Money Market Accounts: The variables X3506, X3510, X3514, X3518, X3522, X3526 report the amount of money in (up to) six different accounts. The respondent answers in variables X3507, X3511, X3515, X3519, X3523, X3527 whether this account corresponds to a checking account or a money market account.

Savings Account: Savings accounts are defined as the sum of the variables X3804, X3807, X3810, X3813, X3816 for the surveys 1998 and 2001 and X3730, X3736, X3742, X3748, X3754, X3760 from the survey of 2004 onwards.

Certificates of Deposit: The variable X3721 provides the money amount the respondent has in certificates of deposit. Only certificates of deposit that belong to someone related to the household are considered ($X7620 < 4$).

Saving Bonds: Two categories are defined for bonds. Safe bonds consider the sum of variables X3902 (money in US government saving bonds), X3908 (face value of government bonds) and X3910 (money in state and municipal bonds). Risky bonds consider variables X3906 (face value of mortgage backed bonds), X7634 (face value of corporate bonds) and X7633 (face value of foreign bonds).

Life Insurance: Variable X4006 gives the money value of life insurance policies while variable X4010 the amount currently borrowed using these policies. I define life insurance as the maximum between $X4006 - X4010$ and zero.

Miscellaneous Assets: Amount of money owed by friends, relatives or others, money in gold or jewelry and others. Variables X4018, X4022 and X4030 represent money owed to the respondent and X4032 money owed by the respondent. Miscellaneous assets is defined as the difference between the former three and the last variable X4032.

Brokerage Accounts: X3930 gives the dollar value of all the cash or call money accounts at a stock brokerage and X3932 the current balance of margin loans at a stock brokerage. Brokerage account is defined as the former minus the later variable.

Mutual Funds: Variables X3822, X3824, X3826, X3828, X3830 provide the total market value of mutual funds grouped by stocks, tax-free bonds, government-backed bonds, other bonds, and combination funds respectively. Safe mutual funds are defined as $X3824 + X3826 + X3828 + 0.5 \times X3830$ while risky mutual funds consist of $X3822 + 0.5 \times X3830$

Publicly Traded Stocks: The variable X3915 gives the market value of US stocks owned by the respondent, and variable X7641 the market value of stocks of companies outside the US. Stocks is defined as the sum of both variables.

Annuities: The variable X6820 gives the dollar value of annuities for the 1998 and 2001 surveys and X6577 afterwords. Only risky (Stocks, mutual funds or real state) are defined risky annuities, only safe (bonds/interest, CDS/money market) are defined safe annuities, and mixed annuities money is allocated in equal parts on each type (risky/safe).

Trust: Variable X6835 (years 1998-2001) and X6587 (years 2004-2019) provide the money amount of assets in a trust. Allocation between risky and safe trusts follow the same criteria as in annuities.

Individual Retirement Accounts (IRA): For the surveys 1998-2001, variables X3610, X3620 and X3630 report the total money in individual retirement accounts. Variable X3631 reports the detail of the investment. Following [Chang et al. \(2018\)](#), money markets, bonds and universal life policy are considered safe, while stocks, real estate, limited partnership investments and brokerage accounts are considered risky. I allocate 2/3 into the safe account when $X3631 = 4$ (combination of money market, bonds and stocks) and the rest to the risky asset. Similarly, other combinations are divided equally between safe and risky assets.

For the remaining surveys, X6551, X6552, X6553, X6554, X6559, X6560, X6561, X6562, X6567, X6568, X6569 and X6570 report money in individual retirement accounts. Variables X6555, X6563 and X6571 report asset categories. Split accounts accounts are divided equally, interest earning assets are considered safe and stocks, real estate and hedge funds are considered risky.

Pensions: Variables X4226, X4326, X4426, X4826, X4926 and X5026 provide the total amount of money in pension accounts. I subtract possible loans against these accounts by using the variables: X4229, X4329, X4429, X4829, X4929 and X5029. Categories Savings, 401k, defined contribution plan, money purchase plan, tax-deferred annuity, other type of annuity and other are divided equally between safe and risky pensions. Other pension benefits reported in variables X5604, X5612, X5620, X5628, X5636 and X5644 are also equally distributed across assets. In the 2001 survey variables X5604, X5612, X5620, X5628, X5636 and X5644 are allocated using detail on variables X6491, X6492, X6493, X6494, X6495 and X6496. Interest earning assets are considered safe, stocks, real estate and insurance risky, and split or other are equally distributed.

For the surveys 2004-2019, pensions are stored in variables X11232, X11532 (only 2004-2007), X11032, X11132, X11332, X11432. Future pension benefits are reported as in the other surveys (variables X5604, X5612, X5620, X5628, X5636 and X5644), with the exception that 2004-2007 surveys exclude X5636 and X5644. Allocation by asset detail is equal to previous editions of the survey.

Housing: Variables X513 and X526 gives the value of the land and buildings the respondent (partially) owns. Variables X604 and X614 are the value of the site and the mobile home the

respondent owns respectively. Variable X623 is the total value of the home and site if he/she owns both. Variable X716 is the current value of the home/apartment/property the respondent (partially) owns. Variables X1706, X1806 and X1906 (surveys 1998-2016) give the total value of other properties such as real estate investment or vacation houses. Variable X2002 denotes the value of recreational properties and X2012 the remaining value of other properties.

Variables X1409, X1509 and X1619 (until SCF 2010), X1609 (until SCF 2007), X1310, X1328 and X1339 (2013 onwards) provide value of Land contract lending from the respondent. Similarly, variables X1417, X1517 and X1621 (until SCF 2010), X1617 (until SCF 2007), X1318, X1337 and X1324 (2013 onwards) report value of land contract lending to the respondent.

Variables X805, X905 and X1005 give the value of mortgage debt from the respondent, X1108, X1119, X1130, X1136 provide value of lines of credit and X1044 other loans. Finally, housing net worth is defined as the sum of the value of the properties and land contract lending, minus all mortgage debt, lines of credit, land contract borrowing and other loans held by the respondent.

With the previous definitions, I group assets into the following categories:

$$\begin{aligned}
 \textit{Safe Assets} = & \text{Checking Accounts} + \text{Money Market Accounts} + \text{Savings Accounts} \\
 & + \text{Certificates of Deposit} + \text{Safe Saving Bonds} + \text{Life Insurance} + \text{Safe Trusts} \\
 & + \text{Miscellaneous Assets} + \text{Safe Mutual Funds} + \text{Safe Annuities} + \text{Safe IRA} \\
 & + \text{Safe Pensions}
 \end{aligned}$$

$$\begin{aligned}
 \textit{Risky Assets} = & \text{Risky Saving Bonds} + \text{Brokerage Accounts} + \text{Stocks} + \text{Risky Mutual Funds} \\
 & + \text{Risky Annuities} + \text{Risky Trusts} + \text{Risky IRA} + \text{Risky Pensions}
 \end{aligned}$$

And the baseline definition

$$\omega = \frac{\textit{Risky Assets}}{\textit{Risky Assets} + \textit{Safe Assets}}$$

The baseline definition of Financial Wealth used throughout the text is simply

$$FW = \textit{Risky Assets} + \textit{Safe Assets} = R + S \tag{A.1}$$

Finally, alternative definitions may include housing (net of mortgage debt).

Appendix B: Additional Tables and Figures

B.1 Additional Tables

Table 4: Demographic Descriptive Statistics from the SCF

	Mean	Sd	Min	Max
Income	63,975.08	149140.19	0	99320000
Age	50.37	17.32	17	95
Children	1.44	1.78	0	10
1{College Degree}	0.39	0.49	0	1
1{Married}	0.50	0.50	0	1

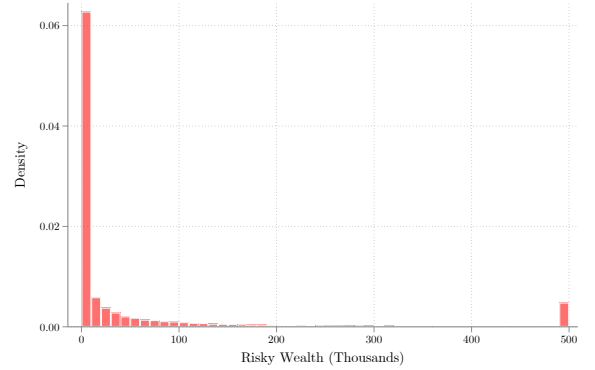
Notes: Data from the SCF for the period 1998-2019 in 2010 US\$. Age and children report the respondent age and number of children. **1{College Degree}** and **1{Married}** report whether the respondent has a college degree and if he/she is currently married.)

B.2 Additional Figures

Figure 12: Financial Wealth Distribution by Asset Class



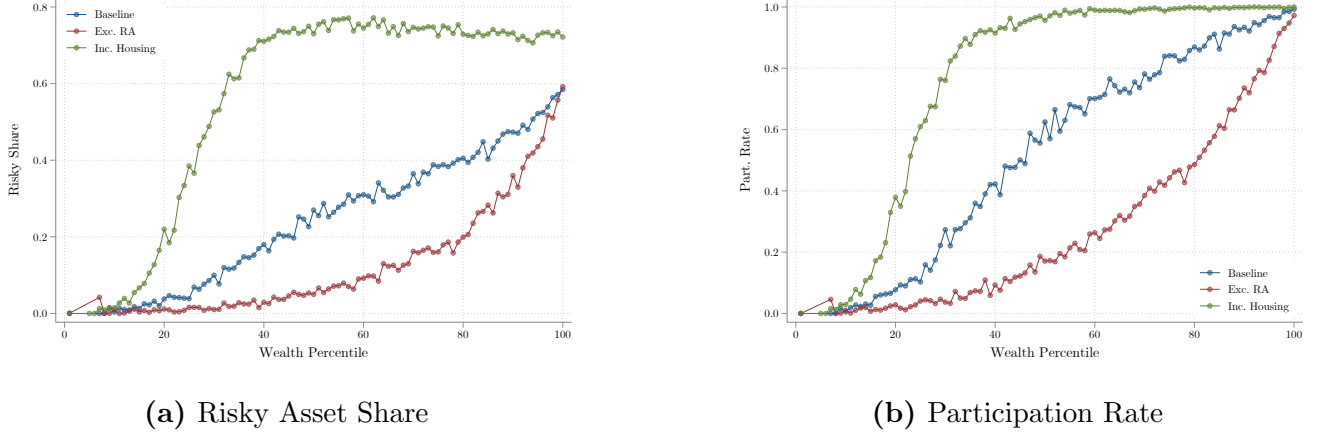
(a) Safe Wealth



(b) Risky Wealth

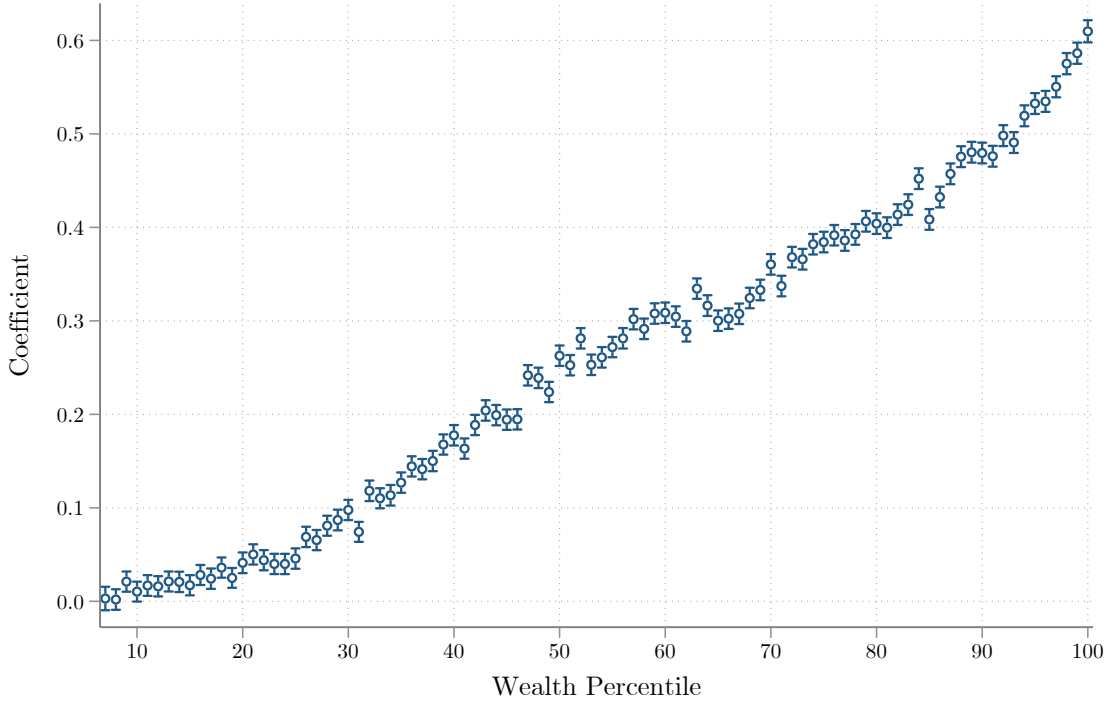
Notes: Data from the SCF for the period 1998-2019 in 2010 thousand USD. Risky/Safe Wealth Definition in Appendix A. Weights in this figure were rounded to the nearest integer.

Figure 13: Portfolio Choice under Different Definitions of Financial Wealth



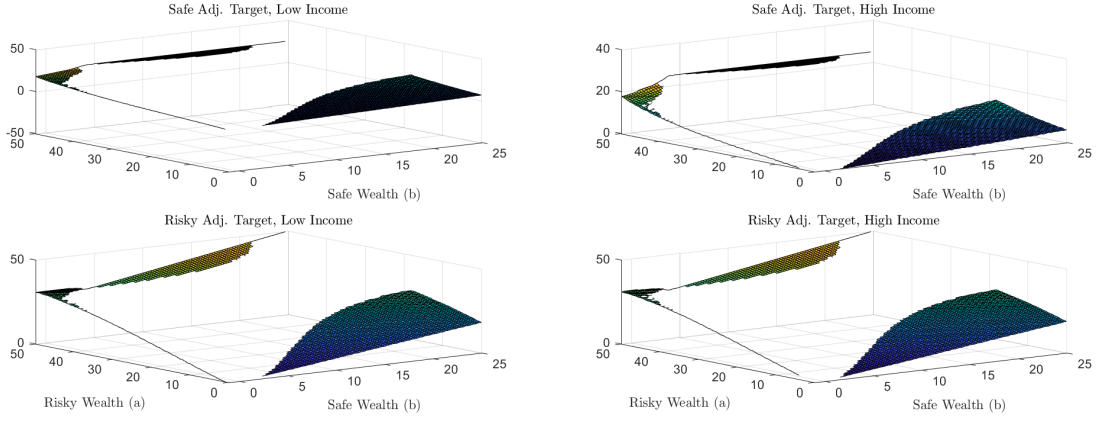
Notes: Risky Share defined in Appendix A. Participation rate in the risky asset is defined as $\mathbb{1}\{R > 0\}$. Each connected point represents the average risky share/participation rate for households in each percentile. Wealth distribution is computed using the baseline definition of financial wealth (blue), and the baseline definition plus retirement accounts (red), housing (green) and both (yellow).

Figure 14: Coefficients on Percentile Dummies for Risky Share Regression



Notes: This figure presents coefficients δ_p from estimating $\omega_{it} = \alpha + \sum_{p=2}^{100} \delta_p D_{it,p} + f(x_{it}) + \mu_t + \varepsilon_{it}$. Circles denote point estimates and lines the respective 95% confidence interval.

Figure 15: Adjustment Targets



Notes: Adjustment targets represent optimal demand for the safe asset b' (first row) and the risky asset a' (second column) for both income profiles.

Figure 16: Risky Asset Marginal Distribution in Steady State

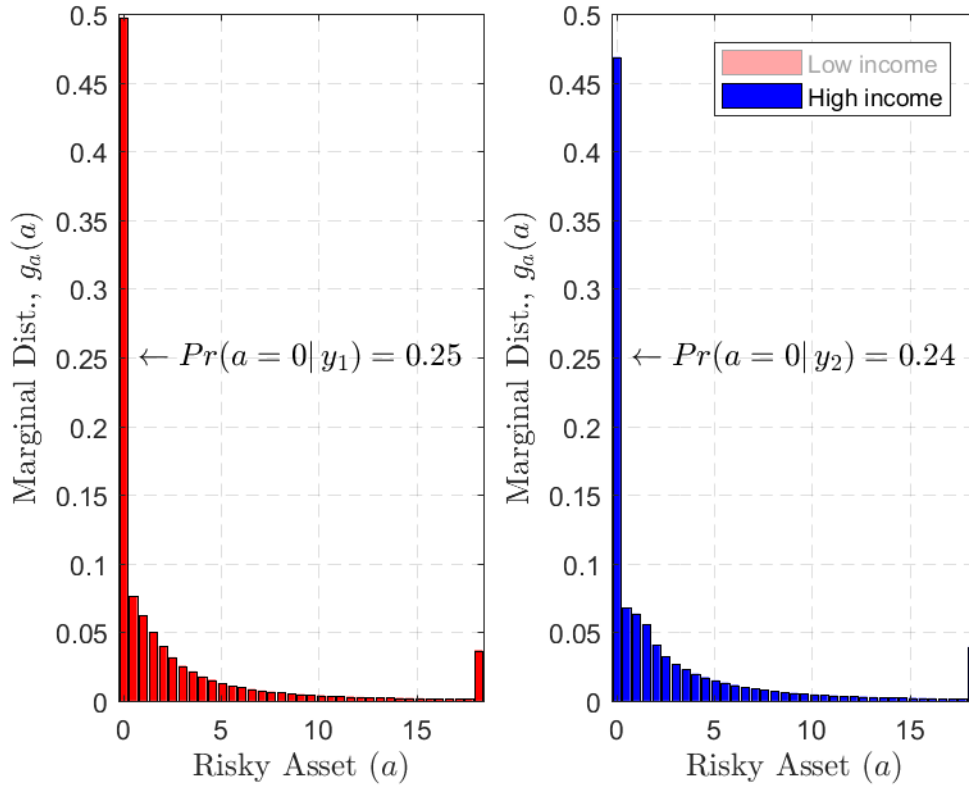
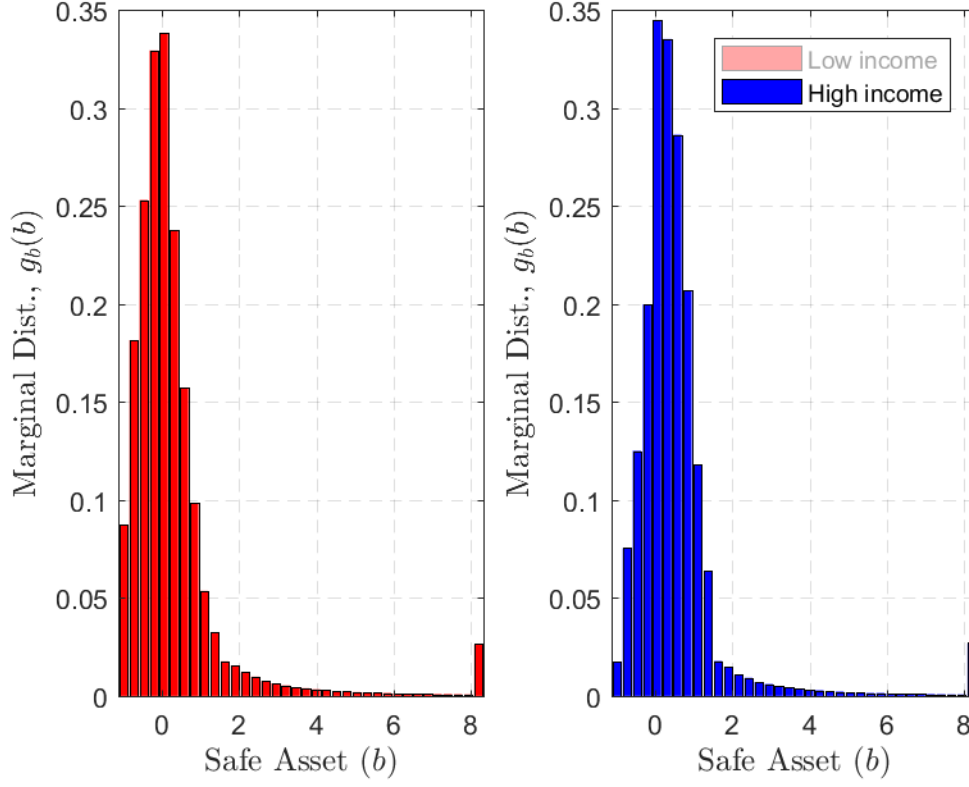


Figure 17: Safe Asset Marginal Distribution in Steady State



Appendix C: Numerical Strategy

C.1 Solving the HJB Equation

Following closely the finite difference method proposed in Achdou et al. (2017) and building upon Benjamin Moll's public access codes⁴⁴, define the equispaced grids $i = 1, \dots, I$, $j = 1, \dots, J$, and $z = 1, \dots, Z$ for the safe asset, risky asset and labor income respectively. Then define $v = v_{i,j,z}$ as the discretized value function. Now define

$$\text{Backward difference: } \partial_{x,B} v = \frac{v_s - v_{s-1}}{\Delta x} \quad (\text{C.1})$$

$$\text{Forward difference: } \partial_{x,F} v = \frac{v_{s+1} - v_s}{\Delta x} \quad (\text{C.2})$$

$$\text{Central difference: } \partial_{xx} v = \frac{v_{s+1} - 2v_s + v_{s-1}}{(\Delta x)^2}, \quad (\text{C.3})$$

⁴⁴<https://benjaminmoll.com/codes/>.

as the forward, backward and central derivative approximations for $x = (a, b)$, $s = (i, j)$, and with Δx as the distance between grid points. The idea is then to use backward approximations whenever the drifts of a and b are negative and forward approximations whenever they are positive. The central approximation is used for $\partial_{aa}v$. Now, the discretized-version of the household problem can be written as follows

$$\min \left\{ \rho \mathbf{v} - u(\mathbf{v}) - \mathbf{A}(\mathbf{v}) \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \right\} = 0, \quad (\text{C.4})$$

where \mathbf{v} and $\mathbf{u}(\mathbf{v})$ are the discrete-time value and utility functions over the stacked state space $L \equiv I \times J \times Z$ and $\mathbf{A}(\mathbf{v})$ is a $L \times L$ squared matrix that captures the evolution of the state variables. The algorithm for the solution is the following:⁴⁵

1. As initial guess \mathbf{v}^0 , use the solution of the no-adjustment case:

$$\rho \mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v}) \mathbf{v} = 0, \quad (\text{C.5})$$

i.e. solve the problem when κ is infinite. Note that as the problem is non-linear in \mathbf{v} , some iteration will be necessary. The guess for this step will be “staying put”

$$v^0 = \frac{(y + r^b b)^{1-\gamma}}{\rho(1-\gamma)} \quad (\text{C.6})$$

Then, from the FOC w.r.t to c , one can compute

$$c = (\partial_b v_0)^{-\frac{1}{\gamma}} \quad (\text{C.7})$$

for both the backward and forward approximations. Then, it is straightforward to compute backward and forward saving policy functions $s^h = s_{i,j,z}^h$, for $h \in \{B, F\}$. Compute consumption and savings using the upwind scheme. That is, choose backward approximation whenever the drift of b is negative and forward when it is positive.

As risky wealth and labor income evolve exogenously, one can compute directly the evolution of these states using the upwind method. Finally, compute the transition matrix $\mathbf{A}(v)$ and solve (C.5). Iterate until convergence.

2. Given \mathbf{v}^n , find \mathbf{v}^{n+1} by solving:

$$\min \left\{ \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} + \rho \mathbf{v}^{n+1} - \mathbf{u}(\mathbf{v}^n) - \mathbf{A}(\mathbf{v}^n) \mathbf{v}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^*(\mathbf{v}^n) \right\} = 0 \quad (\text{C.8})$$

⁴⁵This algorithm is exactly as in https://benjaminmoll.com/wp-content/uploads/2020/06/liquid_illiquid_numerical.pdf but with the exception that now the process for the return of the risky asset is stochastic, and thus risky wealth evolves slightly differently.

This step follows [Huang and Pang \(2003\)](#) and its exercise for American options with transaction costs (see section 2.3 on their paper). Eq. (C.4) can be expressed as

$$\left(\mathbf{v} - \mathbf{v}^*(\mathbf{v})\right)^T \left(\rho \mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v}) \mathbf{v}\right) = 0 \quad (\text{C.9})$$

$$\mathbf{v} \geq \mathbf{v}^*(\mathbf{v}) \quad (\text{C.10})$$

$$\rho \mathbf{v} - \mathbf{u}(\mathbf{v}) - \mathbf{A}(\mathbf{v}) \mathbf{v} \geq 0 \quad (\text{C.11})$$

Now denote $\mathbf{z} = \mathbf{v} - \mathbf{v}^*(v)$ and $\mathbf{B} = \rho \mathbf{I} - \mathbf{A}(\mathbf{v})$. Then Eq. (C.10) is simply $\mathbf{z} \geq 0$ and Eq. (C.11) is

$$\mathbf{B}\mathbf{z} + \mathbf{q} \geq 0, \quad (\text{C.12})$$

where $\mathbf{q} = \mathbf{B}\mathbf{v}^*(\mathbf{v}) - \mathbf{u}(\mathbf{v})$. Hence, the problem can now be written in a standard linear complementarity problem (LCP) form

$$\mathbf{z}^T (\mathbf{B}\mathbf{z} + \mathbf{q}) = 0 \quad (\text{C.13})$$

$$\mathbf{z} \geq 0 \quad (\text{C.14})$$

$$\mathbf{B}\mathbf{z} + \mathbf{q} \geq 0, \quad (\text{C.15})$$

which can be easily dealt with LCP solvers.⁴⁶

3. Iterate until convergence.

C.2 Solving the KF Equation

To ease notation, define the discretized variables as $x \equiv x_{i,j,z}$. Then the discretized-version of Eq. (15), i.e. the evolution of the joint distribution of states without adjustment, can be written as follows

$$0 = -\frac{d}{da}(\mu a g) + \frac{d^2}{da^2} \left(\frac{\sigma^2 a^2}{2} g \right) - \frac{d}{db}(s^b g) - \lambda_k g + \lambda_{-k} g, \quad (\text{C.16})$$

or in matrix notation

$$0 = \mathbf{A}^T \mathbf{g}, \quad (\text{C.17})$$

where \mathbf{A}^T is the transpose of the transition matrix computed in the households problem (i.e. when solving the HJB equation).

In order to deal with adjustment I need to introduce some additional notation. Define (a_k^*, b_k^*) as the optimal adjustment targets, conditional on adjustment. Then denote $\ell = 1, \dots, L$ as the

⁴⁶For solving the model I use the following Newton-based LCP solver in Matlab: <https://www.mathworks.com/matlabcentral/fileexchange/20952-lcp-mcp-solver-newton-based>.

stacked state space and the grid point $k^*(\ell)$ reached from point ℓ upon adjustment. Finally, define the inaction region \mathcal{I} as the set of grid points where $v > v^*(v)$.

To introduce adjustment, I need to incorporate a discretization of the "intervention operator" defined in Section 3. To this end, define the binary matrix \mathbf{M} with elements $M_{\ell,k}$, for $\ell = 1, \dots, L$ and $k = 1, \dots, L$, given by

$$M_{\ell,k} = \begin{cases} 1, & \text{if } \ell \in \mathcal{I} \text{ and } \ell = k \\ 1, & \text{if } \ell \notin \mathcal{I} \text{ and } k^*(\ell) = k \\ 0, & \text{otherwise} \end{cases} \quad (\text{C.18})$$

This matrix moves mass from the adjustment region to their corresponding adjustment targets. Intuitively it is easy to see that when $\ell \in \mathcal{I}$ and $\ell = k$, \mathbf{M} keeps that element on its same position. However, when $\ell \notin \mathcal{I}$, \mathbf{M} moves that mass to the optimal target $k(\ell)$.

To see how to use \mathbf{M} to solve the KF equation, I follow the "operator splitting method" from Achdou et al. (2017) (see their numerical appendix). Namely, defining $\mathbf{g}^n = \mathbf{g}(t^n)$, $n = 1, \dots, N$, I use the following algorithm:

1. Given \mathbf{g}^n , find $\mathbf{g}^{n+\frac{1}{2}}$ from

$$\mathbf{g}^{n+\frac{1}{2}} = \mathbf{M}^T \mathbf{g}^n \quad (\text{C.19})$$

2. Given $\mathbf{g}^{n+\frac{1}{2}}$, find \mathbf{g}^{n+1} from

$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^{n+\frac{1}{2}}}{\Delta t} = (\mathbf{A}\mathbf{M})^T \mathbf{g}^{n+1} \quad (\text{C.20})$$

3. Iterate until convergence.

This algorithm allows to tackle two crucial problems when computing the stationary distribution. (i) how to treat density at grid points within the adjustment region. (ii) how to treat grid points in the inaction region in which idiosyncratic shocks from state variables (summarized in the matrix \mathbf{A}) moves them towards the adjustment region.

Step 1 solves the first problem by moves mass from the adjustment region to their optimal targets in the inaction region. Step 2 solves the second problem by using $\mathbf{A}\mathbf{M}$ as the new transition matrix. Recall that \mathbf{A} is a Poisson transition matrix in which rows denote the starting position of a Poisson process and columns the finishing position. Thus, for each row ℓ , \mathbf{M} takes the entries $A_{\ell,k}$ that end up in the adjustment region, and moves them to columns $k^*(\ell)$ corresponding to their adjustment target. Note that $\mathbf{A}\mathbf{M}$ is still a valid Poisson transition matrix as M only moves

elements from columns and thus, all rows still sum to zero and diagonal elements keep their required properties.⁴⁷

Appendix D: Proofs

D.1 Derivation of the HJBQVI

This subsection derives the HJBQVI presented in Eq. (13). To do so, assume a discrete-time environment, with time periods of length Δ and a discount factor given by $\beta(\Delta) = e^{-\rho\Delta}$. As in Section 3, labor income is given by y_1, y_2 . Transition probability from income y_j to y_{-j} is given by $1 - p_j(\Delta)$, where $p_j(\Delta) = e^{-\lambda_j\Delta}$ is the probability to keep their current income.⁴⁸ Note that the discrete-time analog of the households problem without adjustment can be summarized by the following Bellman equation

$$v_j(a_t, b_t) = \max_c u(c_t)\Delta + \beta(\Delta) \mathbb{E}[v_j(a_{t+\Delta}, b_{t+\Delta})] \quad (\text{D.1})$$

$$s.t. \quad a_{t+\Delta} = r_t^a a_t \Delta + a_t \quad (\text{D.2})$$

$$b_{t+\Delta} = (y_j + r_t^b b_t - c_t)\Delta + b_t, \quad (\text{D.3})$$

for $j = 1, 2$. Given the probability $p_j(\Delta)$ to keep the current income, we have that Eq. (D.1) can be written as follows

$$v_j(a_t, b_t) = \max_c u(c_t)\Delta + \beta(\Delta) \left\{ p_j(\Delta) \mathbb{E}[v_j(a_{t+\Delta}, b_{t+\Delta})] + (1 - p_j(\Delta)) \mathbb{E}[v_{-j}(a_{t+\Delta}, b_{t+\Delta})] \right\} \quad (\text{D.4})$$

For a small enough Δ we have

$$\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta \quad (\text{D.5})$$

$$p_j(\Delta) = e^{-\lambda_j\Delta} \approx 1 - \lambda_j\Delta, \quad (\text{D.6})$$

and thus substituting into (D.4)

$$v_j(a_t, b_t) = \max_c u(c_t)\Delta + (1 - \rho\Delta) \left\{ (1 - \lambda_j\Delta) \mathbb{E}[v_j(a_{t+\Delta}, b_{t+\Delta})] + \lambda_{-j}\Delta \mathbb{E}[v_{-j}(a_{t+\Delta}, b_{t+\Delta})] \right\}, \quad (\text{D.7})$$

⁴⁷Namely, diagonal elements are still non-positive, while the remaining elements are non-negative.

⁴⁸The derivation presented in this subsection can be easily extensible to a different process for labor income.

re-arranging terms

$$v_j(a_t, b_t) = \max_c u(c_t)\Delta + (1 - \rho\Delta) \left\{ \mathbb{E}[v_j(a_{t+\Delta}, b_{t+\Delta})] + \lambda_j \Delta \mathbb{E}[v_{-j}(a_{t+\Delta}, b_{t+\Delta}) - v_j(a_{t+\Delta}, b_{t+\Delta})] \right\} \quad (\text{D.8})$$

Subtracting $(1 - \rho\Delta)v_j(a_t, b_t)$, dividing by Δ and taking $\Delta \rightarrow 0$ we get

$$\rho v_j(a_t, b_t) = \max_c u(c_t) + \frac{\mathbb{E}[dv(a_t, b_t)]}{dt} + \lambda_j (v_{-j}(a_t, b_t) - v_j(a_t, b_t)) \quad (\text{D.9})$$

Finally, we need an expression for the second term on the right-hand side of Eq. (D.9). Given that r^a follows Eq. (4), by Ito's Lemma

$$dv(a_t, b_t) = \left(\partial_b v(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a \partial_a v(a_t, b_t) + \frac{\sigma^2 a^2}{2} \partial_{aa} v(a_t, b_t) \right) dt + \sigma a \partial_a v(a_t, b_t) dW_t, \quad (\text{D.10})$$

taking expectations and noticing that $\mathbb{E}[dW_t] = 0$

$$\frac{\mathbb{E}[dv(a_t, b_t)]}{dt} = \partial_b v(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a \partial_a v(a_t, b_t) + \frac{\sigma^2 a^2}{2} \partial_{aa} v(a_t, b_t) \quad (\text{D.11})$$

Finally, plugging this equation into (D.9) yields Eq. (10). To recover Eq. (13), just note that from the constraint in Eq. (9), $v(a, b) - v^*(a + b) \geq 0$.

D.2 Proof of the Merton Rule

Given no labor income, CRRA utility and no frictions in the financial markets, the household's problem can be written as

$$\begin{aligned} \max_{\{c_t\}, \omega} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \\ s.t. \quad \dot{a} = a_t (\omega(r_t^a - r_t^b) + r_t^b) - c_t \\ dr_t^a = \mu dt + \sigma dW_t, \end{aligned} \quad (\text{D.12})$$

where the HJB equation is given by

$$\rho v(a) = \max_{c, \omega} u(c) + v'(a) [a_t (\omega(\mu - r^b) + r^b) - c] + v''(a) \frac{(\sigma a \omega)^2}{2} \quad (\text{D.13})$$

To find the optimal c^*, ω^* I use a guess-and-verify strategy: guess

$$v_0(a) = \frac{B_0^{-\gamma}}{1-\gamma} a^{1-\gamma}, \quad (\text{D.14})$$

with B_0 a constant to be determined. Then $v'(a) = (B_0 a)^{-\gamma}$ and $v''(a) = -\gamma B_0^{-\gamma} a^{-\gamma-1}$. From the FOC in Eq. (D.13)

$$[c] : c^{-\gamma} = v'(a) \implies c = B_0 a \quad (\text{D.15})$$

$$[\omega] : v'(a)(\mu - r^b)a + v''(a)\sigma^2 a^2 \omega = 0 \implies \omega = \frac{v'(a)(\mu - r^b)}{-v''(a)\sigma^2 a} \quad (\text{D.16})$$

Using Eq. (D.14) in Eq. (D.16) one can obtain Eq. (23). For completeness, replacing Eqs. (D.15)-(D.16) in Eq. (D.13)

$$\rho \frac{B_0^{-\gamma}}{1-\gamma} a^{1-\gamma} = \frac{(B_0 a)^{1-\gamma}}{1-\gamma} + (B_0 a)^{-\gamma} \left[a \left(\frac{(\mu - r^b)^2}{\gamma \sigma^2} + r^b \right) - B_0 a \right] - \gamma B_0^{-\gamma} a^{-\gamma-1} \frac{\sigma^2 a^2 (\mu - r^b)}{2(\gamma \sigma^2)^2} \quad (\text{D.17})$$

Then rearranging terms

$$\frac{\rho}{1-\gamma} = \frac{\gamma B_0}{1-\gamma} + \frac{(\mu - r^b)^2}{2\gamma \sigma^2} + r^b - B_0 \quad (\text{D.18})$$

Lastly, solving for B_0 one can determine the missing parameter

$$B_0 = \frac{1}{2} \left[\rho + \gamma + 1 - (1-\gamma)r^b - \frac{1-\gamma}{\gamma} \frac{(\mu - r^b)^2}{2\sigma^2} \right] \square \quad (\text{D.19})$$