The Portfolio Choice Channel of Wealth Inequality

Lucas Rosso
Universidad de Chile

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Roadmap

- Motivation and stylized facts
- Introduce the model: Solving HA $=$ Solving PDEs
- Main Results: Households behavior $+$ Stationary Distribution
- The *scale* component of wealth accumulation
- Discussion and Extensions

Motivation

Motivation

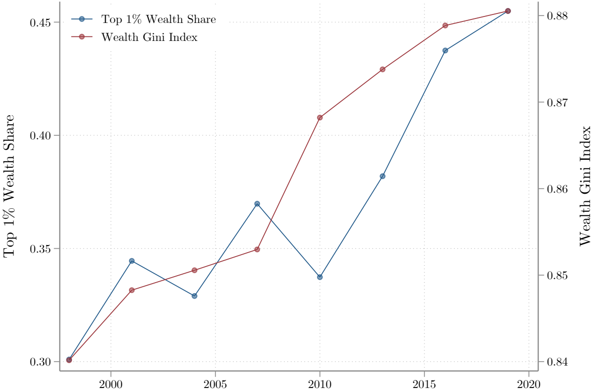
- What is the role of households portfolio choice in wealth inequality?

- Recent evidence suggests that return to savings is highly increasing in wealth Bach et al. (2020); Fagereng et al. (2020)
  - *scale* dependent returns
  - results hold even within narrow asset classes!

- Portfolio choice and *scale* dependence usually abscent in workhorse models of wealth accumulation (e.g. Aiyagari, 1994)

Stylized Facts

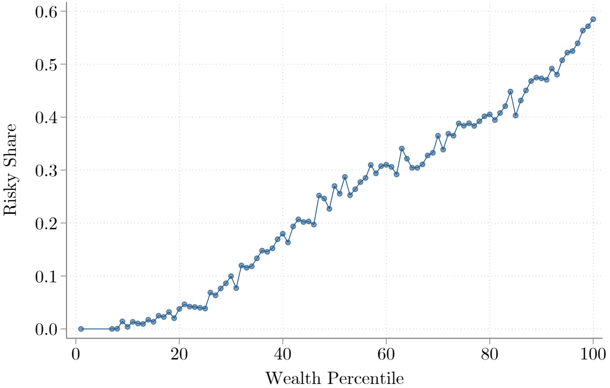
Highly unequal financial wealth distribution and an increasing trend



Notes: Data from the Survey of Consumer Finances (SCF) for the period 1998-2019. Risky assets defined as in Chang et al. (2018) but without including non-actively managed business in Financial Wealth definition. ▸ Detail

Stylized Facts ▸ Robustness

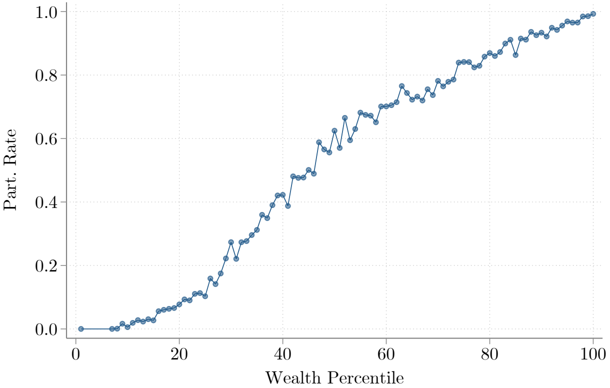
Risky asset share steeply increasing across wealth distribution!



Notes: Data from the Survey of Consumer Finances (SCF) for the period 1998-2019. Risky assets defined as in Chang et al. (2018) but without including non-actively managed business in Financial Wealth definition.

Stylized Facts

Extensive margin matters for portfolio choice



Notes: Data from the Survey of Consumer Finances (SCF) for the period 1998-2019. Risky assets defined as in Chang et al. (2018) but without including non-actively managed business in Financial Wealth definition. Participation rate defined as $\mathbf{1}\{R > 0\}$

Related Literature

Combine two workhorse macro models + financial frictions

- Portfolio choice models Merton (1969); Samuelson (1969)
- Bewley models Bewley (1986); Huggett (1993); Aiyagari (1994)
- Non-convex (fixed) adjustment costs Kaplan and Violante (2014)

Related Work:

1. Empirical evidence of portfolio heterogeneity

   Vissing-Jorgensen (2002); Kuhn et al. (2020); Bach et al. (2020); Fagereng et al. (2020); Martínez-Toledano (2020)

2. Models of wealth inequality with idiosyncratic returns to wealth

   Benhabib et al. (2011, 2015); Gabaix et al. (2016); Gomez (2018); Hubmer et al. (2020); Xavier (2020)

3. Continuous time HA models

   Achdou et al. (2017); Kaplan et al. (2018)

This Paper

- Proposes a model that explicitly incorporates households portfolio decisions.

- Model provides better fit than workhorse models of wealth accumulation
  $\longrightarrow$ and adds more realism to households balance sheets.

- Adjustment cost amplifies wealth inequality by introducing *scale dependence*
  $\longrightarrow$ adjustment cost feeds precautionary channel.

Model

Setup ▸ Advantages

Continuous time, partial-equilibrium heterogeneous agent model with

1. Rich households balance sheets
   - safe and risky assets
   - "hard" and "soft" borrowing constraints
   - fixed adjustment cost in risky asset
   - stochastic returns

2. Uninsurable labor income risk.

Problem consists of solving a system of two PDEs

- Hamilton-Jacobi-Bellman (HJB) equation for individual choices

- Kolmogorov Forward (KF) equation for evolution of distribution

Household Balance Sheets

- Stochastic income follows a two-state Poisson process:

$$z_t \in \{z_L, z_H\}$$

- Safe wealth b_t, risky wealth a_t
- Changes in risky asset holdings entail a fixed adjustment cost κ
$\Longrightarrow$ stopping-time element
- Stochastic return in risky asset:

$$\mathrm{d}r_t^a = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

- Working assumption: Labor income independent from capital income
$\longrightarrow$ second order in infinite-horizon settings (no life cycle)
$\longrightarrow$ consistent with empirical literature Cocco et al. (2005); Fagereng et al. (2017)

Household's Problem

Households are heterogeneous in their wealth (a, b), income z, and the return on savings

$$\upsilon_k(a, b, z) = \max_{\{c_t\},\tau} \mathbb{E}_0 \int_0^\tau e^{-\rho t} u(c_t) + e^{-\rho\tau} \mathbb{E}_0\, \upsilon_k^*(a_\tau + b_\tau, z)$$

$$\begin{aligned}
&\mathrm{d}a_t = \mathrm{d}r_t^a a_t; \\
&\mathrm{d}b_t = (z_t + r_t^b(b_t)b_t - c_t)\mathrm{d}t \\
&z_t \in \{z_L, z_H\} \ \text{Poisson with intensities} \ \lambda_L, \lambda_H \\
&\mathrm{d}r_t^a = \mu\, \mathrm{d}t + \sigma\, \mathrm{d}W_t \\
&a \geq 0; \ b \geq \underline{b},
\end{aligned}$$

where

$$\upsilon_k^*(a + b, z) = \max_{a', b'} \ \upsilon_k(a', b', z) \ \ s.t. \ a' + b' = a + b - \kappa$$

HJB equation

$$
\begin{aligned}
\rho v_k(a,b,z) = \max_c \quad & u(c) + \\
\textit{Safe Asset}: \ & + \partial_b v(a,b,z)(z + r^b b - c) \\
\textit{Risky Asset}: \ & + \mu(r^a) a \partial_a v(a,b,z) + \frac{\sigma^2(r^a)a^2}{2} \partial_{aa} v(a,b,z) \\
\textit{Labor Income}: \ & + \sum_{z' \in Z} \lambda^{z \to z'} \big(v(a,b,z') - v(a,b,z) \big)
\end{aligned}
$$

for $k = L, H$, with a state-constraint boundary condition

$$\partial_b v_k(a, \underline{b}) \geq u'(z_k)$$

and a constraint that

$$v_k(a,b,z) \geq v_k^*(a+b,z) \ \forall \ a, b$$

HJB quasi-variational inequality ▸ Derivation

Suppressing dependence on (a, b, z), the HJBQVI can be written as

$$\min\Big\{\rho v - \max_c\{u(c) - \mu a\, \partial_a v - \frac{\sigma^2 a^2}{2}\partial_{aa} v - (z + r^b b - c)\, \partial_b v \\ - \sum_{z' \in Z} \lambda^{z \to z'} \left(v(z') - v(z)\right), v - \mathcal{M} v\Big\} = 0,$$

where $v_k^* = \mathcal{M} v_k$, and $\mathcal{M}$ is known as the "intervention operator" (See e.g., Azimzadeh et al., 2018)

In matrix notation

$$\min\Big\{\rho \mathbf{v} - u(\mathbf{v}) - \mathbf{A}(\mathbf{v})\, \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v})\Big\} = 0$$

Kolmogorov-Forward Equation

Without adjustment the KF equation is

$$0 = -\partial_a(\mu a g(a,b,z)) + \frac{1}{2}\partial_{aa}(\sigma^2 a^2 g(a,b,z)) - \partial_b[s^b(a,b,z)\, g(a,b,z)] \\ - \lambda^{z\to z'} g(a,b,z) + \lambda^{z'\to z} g(a,b,z'),$$

- Caveat: Mathematical formulation of the KF for impulse control problem is not straightforward!
- However, turns out to be significantly easier to deal once discretized

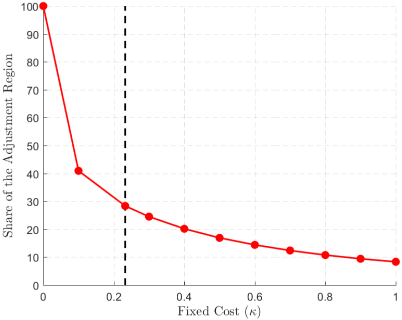
Numerical Solution

Quantitative Analysis

Parametrization

Parameter	Description	Value	Source/Target
Households			
γ	Risk aversion	2	Standard
ρ	Subjective discount rate	0.053	Standard ($\beta = 0.95$)
Assets			
$\underline{b}$	Borrowing limit	-1	1 times avg. income
ϖ	Interest rate wedge	0.06	Kaplan et al. (2018)
r^b	Safe asset return	0.02	Gomes and Michaelides (2005)
μ	Risky asset drift	0.06	Gomes and Michaelides (2005)
σ	Risky asset volatility	0.18	Gomes and Michaelides (2005)
κ	Adjustment cost	0.23	Participation Rate
Income Process			
z_1, z_2	Income states	$0.79, 1.21$	$\sigma_z = 0.21$, $\varphi_z = 0.9$, $\mathbb{E}(z) = 1$
λ_1, λ_2	Income jumps	$0.25, 0.25$	Eq. (1)

The role of κ

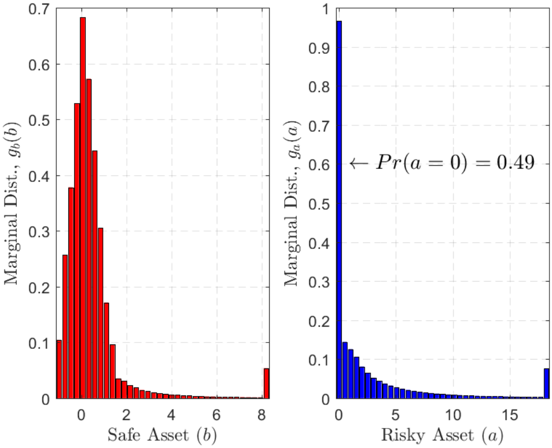


Notes: Connected dots denote the size of the adjustment region out of the total state-space. Vertical line represents the calibrated value for κ

- Small frictions can generate large inaction ranges
- Calibrated κ represents only 0.75% of adjusting households stock.
- Inaction range highly increasing in κ
- Common arguments: brokerage fees, opp cost, processing cost, mental accounting and so on.

Stationary Distribution of Wealth

Model predicts similar distribution of wealth than data



“Fat-tail Aiyagari” as a useful benchmark

Measure	Data	Baseline Model	Fat-tail Aiyagari (1994)
Top 1%	37.5	22.2	11.5
Top 5%	64.6	49.6	35.2
Top 10%	77.8	66.1	52.6
Middle 40%	19.5	33.8	38.3
Bottom 50%	0.98	0.10	9.2

- When $\kappa = 0$, the model reduces to a combination of workhorse models of wealth accumulation (Aiyagari, 1994) $+$ portfolio choice (Merton, 1969) $\longrightarrow$ “Fat-tail Aiyagari”
- Under the same calibration, the introduction of adj. friction (i.e. $\kappa > 0$) substantially improves the fit!

  $\longrightarrow$ adjustment cost narrows the gap in top shares to roughly half
- Still much to go (e.g., no *type* dependence)

The amplifying effect of κ

- Assume wealth inequality increases due to a permanent decrease in labor income risk (Why?)
- How does the adjustment cost affect wealth top shares?
- Turns out that κ amplifies top shares by a factor over 8!
  $\longrightarrow$ *scale* dependence feeds precautionary channel

	Baseline			Fat-tail Aiyagari (1994)		
	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change
Top 1%	22.2	33.9	52.70	11.5	12.2	6.09
Top 5%	49.6	64.1	29.23	35.2	36.6	3.98
Top 10%	66.1	80.1	21.18	52.6	53.6	1.90

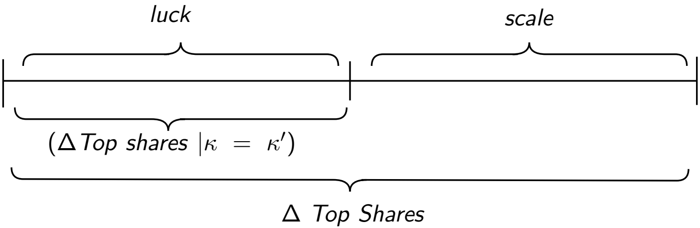
Decomposing top shares into *luck* and *scale*

In the lens of the model, differences in wealth accumulation are generated by

- *luck*: idiosyncratic shocks to income and returns
- *scale*: portfolio re-balancing entails an adjustment cost κ

However, *luck* depends on the participation decision and thus in the *scale component* $\Longrightarrow$ Effects are not additively separable

My approach: re-calibrate κ after a permanent shock (e.g. to the income process) to create counterfactual with equal *scale* component



Decomposing top shares into *luck* and *scale*

	$\sigma_\nu = 0.20$	$\sigma_\nu = 0.18$	% change	% scale	% *luck*
Top 1%	22.2	33.9	52.7	88.0	12.0
Top 5%	49.6	64.1	29.2	88.3	11.7
Top 10%	66.1	80.1	21.2	89.3	10.7

- Roughly 90% of the change in top shares is explained by the *scale* component!
- Results consistent with the amplifying effect discussed earlier

Discussion and Extensions

Richer return heterogeneity and *type* dependence ▸ Example

- Model abstracts from *type* dependence $\longrightarrow$ all differences in wealth accumulation comes from either *luck* or scale dependence
- However, empirical evidence suggests returns are increasing in wealth even within narrow asset classes Fagereng et al. (2020); Xavier (2020)
- Also collapsing all risky assets into one ignores imperfect portfolio diversification

One way to deal with this is assume a more general return process

$$\mathrm{d}r_t^a = \mu(a)\mathrm{d}t + \sigma(a)\mathrm{d}W_t$$

Possible channels: Imperfect portfolio diversification, information frictions, heterogeneous investment opportunities, and so on.

Decreasing Relative Risk Aversion

▸ Results

What if richer households "can afford to take more risk"?

Two opposing forces come to play

- Risk averse households are less willing to hold risky assets
- Risk aversion increases savings which increases wealth and thus participation rates

Two ways to incorporate this:

1. Exogenous preference heterogeneity

$$u^i(c_t) = \frac{c_t^{1-\gamma_i}}{1-\gamma_i}$$

2. Preferences with decreasing RRA (e.g. Stone-Geary utility)

$$u(c_t) = \frac{(c_t - \bar{c})^{1-\gamma}}{1-\gamma},$$

Concluding Remarks

- Portfolio choice matters! $\longrightarrow$ risky share is steeply increasing across wealth distribution.

- Adjustment costs amplify the effect of portfolio choice in inequality by introducing *scale* dependence.

- Portfolio choice + small financial frictions narrow the gap in top wealth shares to $\approx$ half

- Future models should also address *type* to match recent empirical evidence

Thanks!

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Q & A

Safe and Risky asset definitions ◂ Return

I group assets into the following categories:

$$\begin{aligned}
\textit{Safe Assets} =\ & \text{Checking Accounts} + \text{Money Market Accounts} + \text{Savings Accounts} \\
& + \text{Certificates of Deposit} + \text{Safe Saving Bonds} + \text{Life Insurance} \\
& + \text{Safe Trusts} + \text{Miscellaneous Assets} + \text{Safe Mutual Funds} \\
& + \text{Safe Annuities} + \text{Safe IRA} + \text{Safe Pensions}
\end{aligned}$$

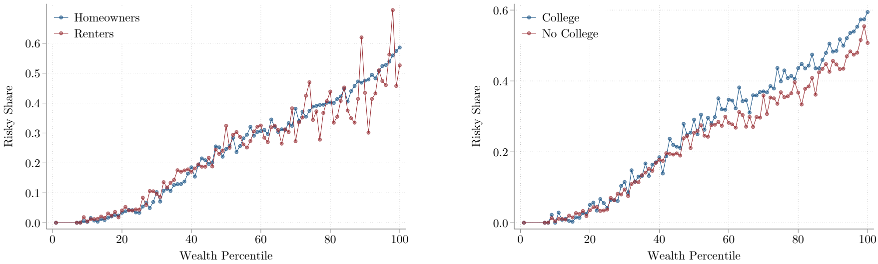
$$\begin{aligned}
\textit{Risky Assets} =\ & \text{Risky Saving Bonds} + \text{Brokerage Accounts} + \text{Stocks} \\
& + \text{Risky Mutual Funds} + \text{Risky Annuities} + \text{Risky Trusts} + \text{Risky IRA} \\
& + \text{Risky Pensions}
\end{aligned}$$

And the baseline definition

$$\omega = \frac{\textit{Risky Assets}}{\textit{Risky Assets} + \textit{Safe Assets}}$$

Robustness in Risky Share

Return



Notes: Homeowners represent households with housing net worth different than 0. College refers to households with a head with a college degree.

Controlling for traditional suspects

Following Fagereng et al. (2019) I estimate a simple model with $\mathbf{x}_{it}$ = age, earnings, education, marital status ...

$$\omega_{it} = \alpha + \sum_{p=2}^{100} \delta_p D_{it,p} + f(\mathbf{x}_{it}) + \mu_t + \varepsilon_{it},$$

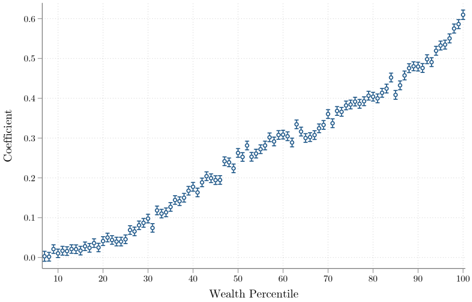
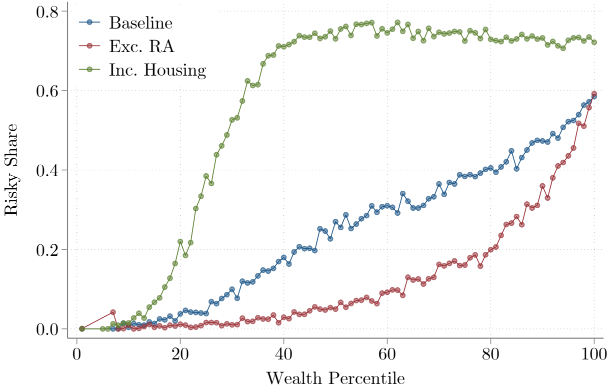


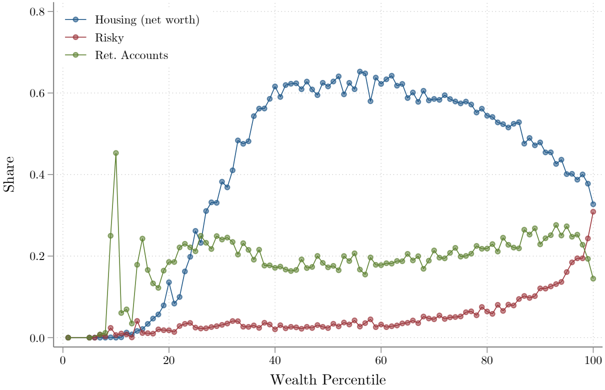
Figure 1: Percentile Dummies δ_p

Alternative Definitions of Financial Wealth



Notes: Wealth distribution is computed using the baseline definition of financial wealth (blue), the baseline definition excluding retirement accounts (red) and the baseline definition including housing net worth (green).

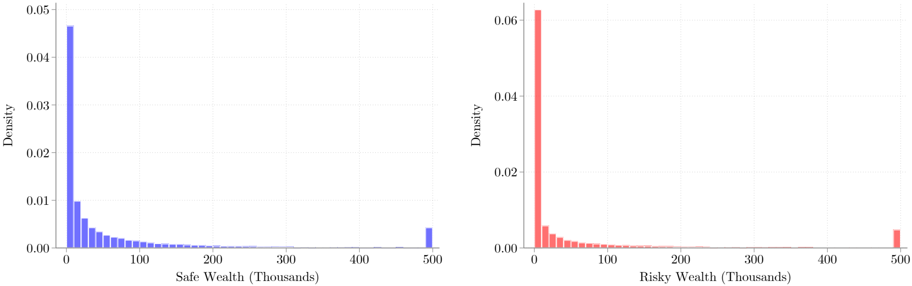
Asset Shares Across Wealth Distribution



Notes: This figure considers the baseline definition of financial wealth plus housing and retirement account assets for computing both shares and the percentiles of the wealth distribution.

Financial Wealth Distribution in the SCF

Risky and Safe Wealth Distribution



Why Continuous time? Ben Moll's take: Back

1. Borrowing Constraint only shows up in boundary conditions
   $\implies$ FOCs always hold with "="

2. FOCs are "static" and can be computed by hand: $c^{-\gamma} = \partial_b v_k$

3. Sparcity: Solving the problem = Inverting giant (but sparse) matrix.

4. Two birds with one stone: diff. operator in KF is the adjoint of opeator in HJB
   $\implies$ after solving HJB, KF comes "for free".

Calibration of the Income Process ◂ Calibration

As in Laibson et al. (2020) I assume an AR(1) process for log-labor income

$$\log(z_t) = \varphi_z \log(z_t) + \nu_t$$

and calibrate $\varphi_z = 0.9$ and $\sigma_\nu = 0.2$ (Guvenen et al., 2019). Then recover the drift and the diffusion of the Ornstein-Uhlenbeck process

$$\mathrm{d}\log(z_t) = -\theta_z \log(z_t) + \sigma_z \mathrm{d}W_t,$$

where

$$\varphi_z = e^{-\theta_z}, \quad \sigma_z = \frac{\sigma_\nu^2}{2\theta_z}(1 - e^{-2\theta_z})$$

Finally, I set z_L, z_H to -1,+1 standard deviations and computer transition probabilities from

$$\lambda^{z \to z'} = \left[\frac{\theta_z}{2\pi\sigma_z^2\,(1 - e^{-2\theta_z})}\right] \exp\left[-\frac{\theta_z}{\sigma_z^2}\frac{(\log(z') - \log(z)e^{-\theta_z})^2}{1 - e^{-2\theta_z}}\right], \quad (1)$$

Derivation of the HJBQVI ◂ Return

Discrete time version of the problem:

$$\begin{aligned} \upsilon_j(a_t, b_t) = & \max_c \ u(c_t)\Delta + \beta(\Delta)\,\mathbb{E}\left[\upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right] \\ s.t. \ \ a_{t+\Delta} = & \ r_t^a a_t \Delta + a_t \\ b_{t+\Delta} = & \ (y_j + r_t^b b_t - c_t)\Delta + b_t, \end{aligned}$$

for $j = L, H$. Given the probability $p_j(\Delta) = e^{-\lambda_j \Delta}$ to keep the current income, we have

$$\begin{aligned} \upsilon_j(a_t, b_t) = \max_c \ u(c_t)\Delta + \beta(\Delta) \Big\{ & p_j(\Delta)\mathbb{E}\left[\upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right] \\ & + (1 - p_j(\Delta))\,\mathbb{E}\left[\upsilon_{-j}(a_{t+\Delta}, b_{t+\Delta})\right] \Big\} \end{aligned}$$

Derivation of the HJBQVI Return

For a small enough Δ we have

$$\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta$$
$$\rho_j(\Delta) = e^{-\lambda_j\Delta} \approx 1 - \lambda_j, \Delta$$

and thus substituting into the equation above

$$\begin{aligned} \upsilon_j(a_t, b_t) = \max_c \; u(c_t)\Delta + (1-\rho\Delta)\Big\{ & (1-\lambda_j\Delta)\mathbb{E}\left[\upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right] \\ & + \lambda_{-j}\Delta\mathbb{E}\left[\upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right]\Big\}, \end{aligned}$$

re-arranging terms

$$\begin{aligned} \upsilon_j(a_t, b_t) = \max_c \; u(c_t)\Delta + (1-\rho\Delta)\Big\{ & \mathbb{E}\left[\upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right] \\ & + \lambda_j\Delta\mathbb{E}\left[\upsilon_{-j}(a_{t+\Delta}, b_{t+\Delta}) - \upsilon_j(a_{t+\Delta}, b_{t+\Delta})\right]\Big\} \end{aligned}$$

Derivation of the HJBQVI

Return

Subtracting $(1-\rho\Delta)\upsilon_j(a_t, b_t)$, dividing by Δ and taking $\Delta \to 0$ we get

$$\rho\upsilon_j(a_t, b_t) = \max_c \; u(c_t) + \frac{\mathbb{E}[\mathrm{d}\upsilon(a_t, b_t)]}{\mathrm{d}t} + \lambda_j\left(\upsilon_{-j}(a_t, b_t) - \upsilon_j(a_t, b_t)\right)\}$$

For the missing term, note that by Ito's Lemma we have

$$\mathrm{d}\upsilon(a_t, b_t) = \Bigg(\partial_b\upsilon(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a\partial_a\upsilon(a_t, b_t) + \frac{\sigma^2 a^2}{2}\partial_{aa}\upsilon(a_t, b_t)\Bigg)\mathrm{d}t + \sigma a\partial_a\upsilon(a_t, b_t)\mathrm{d}W_t,$$

taking expectations and noticing that $\mathbb{E}[\mathrm{d}W_t] = 0$

$$\frac{\mathbb{E}[\mathrm{d}\upsilon(a_t, b_t)]}{\mathrm{d}t} = \partial_b\upsilon(a_t, b_t)(y_t + r_t^b b_t - c_t) + \mu a\partial_a\upsilon(a_t, b_t) + \frac{\sigma^2 a^2}{2}\partial_{aa}\upsilon(a_t, b_t)$$

Numerical Solution

Return

Following Achdou et al. (2017), I use a finite-difference upwind scheme where

$$\text{Backward difference: } \partial_{x,B}\, v = \frac{v_i - v_{i-1}}{\Delta x}$$

$$\text{Forward difference: } \partial_{x,F}\, v = \frac{v_{i+1} - v_i}{\Delta x}$$

$$\text{Central difference: } \partial_{xx}\, v = \frac{v_{i+1} - 2v_i + v_{i-1}}{(\Delta x)^2},$$

for $x \in \{a, b\}$ and where the households problem is discretized as:

$$\min \left\{ \rho \mathbf{v} - u(\mathbf{v}) - A(\mathbf{v})\, \mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \right\} = 0$$

Main idea: Use backward difference when drift is negative and forward difference when positive

Solving the Household's Problem

As mentioned earlier, the discrete-time version of the HJBQVI is given by

$$\min \left\{ \rho \mathbf{v} - u(\mathbf{v}) - A(\mathbf{v})\,\mathbf{v}, \mathbf{v} - \mathbf{v}^*(\mathbf{v}) \right\} = 0$$

- Where A is a $I \times J \times Z$ transition matrix that summarizes the evolution of the state variables.

- Note from the left branch that $u(\cdot)$ depends on υ... Why?

  $\implies$ From FOC: $u'(c) = \partial_b \, \upsilon_k$

Solving the Households' Problem

Algorithm for solution:

1. As initial guess $\mathbf{v}^0$ use the solution to the no-adjustment case:

$$\rho \mathbf{v} - u(\mathbf{v}) - A(\mathbf{v})\,\mathbf{v} = 0$$

2. Given $\mathbf{v}^n$, find $\mathbf{v}^{n+1}$ by solving:

$$\min\left\{\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} + \rho \mathbf{v}^{n+1} - u(\mathbf{v}^n) - A(\mathbf{v}^n)\,\mathbf{v}^{n+1}, \mathbf{v}^{n+1} - \mathbf{v}^*(\mathbf{v}^n)\right\} = 0,$$

3. Iterate until convergence.

Solving the KF Equation

Without adjustment, the solution is given by

$$\mathbf{A}^T g = 0,$$

where $\mathbf{A}^T$ is the transpose of the transition matrix A from the HJB equation.

- Introducing notation: define (a_k^*, b_k^*) as the optimal adjustment targets, $\ell = 1, \ldots, L$ the staked and discretized state-space, $\mathcal{I}$ as the inaction regions and $k^*(\ell)$ reached from the point ℓ upon adjustment

- Define the binary matrix $\mathbf{M}$, with elements $M_{\ell,k}$

$$M_{\ell,k} = \begin{cases} 1, & \text{if } \ell \in \mathcal{I} \text{ and } \ell = k \\ 1, & \text{if } \ell \notin \mathcal{I} \text{ and } k^*(\ell) = k \\ 0, & \text{Otherwise} \end{cases}$$

$\Longrightarrow$ Matrix $\mathbf{M}$ moves points to the adjustment targets.

Solving the KF Equation

This opens two questions:

1. How we treat the density at grid points in the adjustment region?

2. How to treat points in $\mathcal{I}$ but from which the stochastic process for idiosyncratic state variables ends up in the adjustment region?

The following algorithm tackles both problems:

1. Given g^n, find $g^{n+\frac{1}{2}}$ from:

$$g^{n+\frac{1}{2}} = \mathbf{M}^T g^n$$

2. Given $g^{n+\frac{1}{2}}$, find g^{n+1} from:

$$\frac{g^{n+1} - g^{n+\frac{1}{2}}}{\Delta t} = (A\mathbf{M})^T g^{n+1}$$

Imperfect Portfolio Diversification

Return

Assume that the volatility of the risky asset decreases exponentially with risky wealth a at a rate ϑ

$$\sigma(a) = \hat{\sigma} e^{-\vartheta a}$$

I choose the set of parameters Θ that minimizes the weighted deviation between resulting moments $m(\Theta)$ from the model

$$Q(\Theta) = (m - \hat{m}(\Theta))' \mathcal{W} (m - \hat{m}(\Theta))$$

$$\hat{\Theta} = \arg\min_{\Theta} Q(\Theta),$$

Parameter	Value	Target	Model
Fixed adjustment cost (κ)	0.19	51.2[a]	49.7
Exponential decay rate (ϑ)	0.01	77.8[b]	74.7
Average volatility of risky asset ($\bar{\sigma}$)	0.22	0.18[c]	0.21

[a] Risky asset participation rate.

[b] Top 10% wealth share.

[c] Gomes and Michaelides (2005).

Imperfect Portfolio Diversification

Measure	Data	Baseline Model	Imperfect Diversification
Top 1%	37.5	22.2	19.2
Top 5%	64.6	49.6	54.2
Top 10%	77.8	66.1	74.7
Middle 40%	19.5	33.8	26.8
Bottom 50%	0.98	0.10	-0.2

- Better fit for most of the distribution.
- However, predicted top 1% share decreases
  $\Longrightarrow$ Model “needs” volatility to get some households to draw apart!

Decreasing Relative Risk Aversion ◂ Return

I solve both extensions separately by assuming "unemployed" are more risk averse, e.g. $(\gamma_1, \gamma_2) = (1.5, 2.5)$ and calibrate $\bar{c}$ following Achury et al. (2012)

Measure	Data	Baseline Model	Pref. Heterogeneity	Stone-Geary
Top 1%	37.5	22.2	23.0	20.3
Top 5%	64.6	49.6	49.4	47.3
Top 10%	77.8	66.1	65.7	63.8
Middle 40%	19.5	33.8	33.9	34.7
Bottom 50%	0.98	0.10	0.5	1.5

- Results remain overall unchanged $\longrightarrow$ Both forces may offset each other

- Very stylized examples