This is clever and all, but $\boldsymbol{BA}$ has the same eigenvalues as $\boldsymbol{AB}$, and if $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{BA}$, then $\boldsymbol{Ay}$ is an eigenvector of $\boldsymbol{AB}$. Much simpler methods exist to solve this problem. Still interested in the complicated way? Read on.

$$\boldsymbol{u}\boldsymbol{v}^H + \boldsymbol{v}\boldsymbol{u}^H = \lambda_1 \boldsymbol{x}\boldsymbol{x}^H + \lambda_2 \boldsymbol{y}\boldsymbol{y}^H \tag{1}$$

Task: Given $\boldsymbol{u}$ and $\boldsymbol{v}$, find λ_1, λ_2, $\boldsymbol{x}$, and $\boldsymbol{y}$
Let $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^H + \boldsymbol{v}\boldsymbol{u}^H$

$$\boldsymbol{Au} = (\boldsymbol{v}^H\boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u}^H\boldsymbol{u})\boldsymbol{v} \tag{2}$$

$$\boldsymbol{Av} = (\boldsymbol{v}^H\boldsymbol{v})\boldsymbol{u} + (\boldsymbol{u}^H\boldsymbol{v})\boldsymbol{v} \tag{3}$$

$$\boldsymbol{A}^2\boldsymbol{u} = (\boldsymbol{v}^H\boldsymbol{u})((\boldsymbol{v}^H\boldsymbol{u})\boldsymbol{u} + (\boldsymbol{u}^H\boldsymbol{u})\boldsymbol{v}) + (\boldsymbol{u}^H\boldsymbol{u})((\boldsymbol{v}^H\boldsymbol{v})\boldsymbol{u} + (\boldsymbol{u}^H\boldsymbol{v})\boldsymbol{v}) \tag{4}$$

$$\boldsymbol{A}^2\boldsymbol{u} = ((\boldsymbol{v}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{u}) + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}))\boldsymbol{u} + ((\boldsymbol{v}^H\boldsymbol{u})(\boldsymbol{u}^H\boldsymbol{u}) + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{u}^H\boldsymbol{v}))\boldsymbol{v} \tag{5}$$

$$\boldsymbol{A}^2\boldsymbol{u} = ((\boldsymbol{v}^H\boldsymbol{u})^2 + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}))\boldsymbol{u} + (2\operatorname{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}(\boldsymbol{u}^H\boldsymbol{u}))\boldsymbol{v} \tag{6}$$

$$\boldsymbol{A}^2\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{Au}\end{bmatrix}\begin{bmatrix}-c \\ -b\end{bmatrix} \tag{7}$$

Need to solve for b and c.

$$\begin{bmatrix}\boldsymbol{u} & \boldsymbol{Au}\end{bmatrix} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}\begin{bmatrix}1 & \boldsymbol{v}^H\boldsymbol{u} \\ 0 & \boldsymbol{u}^H\boldsymbol{u}\end{bmatrix} \tag{8}$$

$$\begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{Au}\end{bmatrix}\frac{1}{\boldsymbol{u}^H\boldsymbol{u}}\begin{bmatrix}\boldsymbol{u}^H\boldsymbol{u} & -\boldsymbol{v}^H\boldsymbol{u} \\ 0 & 1\end{bmatrix} \tag{9}$$

$$\boldsymbol{A}^2\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{v}\end{bmatrix}\begin{bmatrix}(\boldsymbol{v}^H\boldsymbol{u})^2 + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}) \\ 2\operatorname{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}(\boldsymbol{u}^H\boldsymbol{u})\end{bmatrix} \tag{10}$$

$$\boldsymbol{A}^2\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{Au}\end{bmatrix}\frac{1}{\boldsymbol{u}^H\boldsymbol{u}}\begin{bmatrix}\boldsymbol{u}^H\boldsymbol{u} & -\boldsymbol{v}^H\boldsymbol{u} \\ 0 & 1\end{bmatrix}\begin{bmatrix}(\boldsymbol{v}^H\boldsymbol{u})^2 + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}) \\ 2\operatorname{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}(\boldsymbol{u}^H\boldsymbol{u})\end{bmatrix} \tag{11}$$

$$\boldsymbol{A}^2\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u} & \boldsymbol{Au}\end{bmatrix}\begin{bmatrix}(\boldsymbol{v}^H\boldsymbol{u})^2 + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}) - 2\operatorname{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}\boldsymbol{v}^H\boldsymbol{u} \\ 2\operatorname{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}\end{bmatrix} \tag{12}$$

$$\lambda^2 + b\lambda + c = 0 \tag{13}$$

$$b = -2\,\mathrm{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\} \tag{14}$$

$$c = 2\,\mathrm{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}\boldsymbol{v}^H\boldsymbol{u} - (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}) - (\boldsymbol{v}^H\boldsymbol{u})^2 \tag{15}$$

$$\lambda = \mathrm{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\} \pm \frac{1}{2}\sqrt{\mathrm{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}^2 + 4((\boldsymbol{v}^H\boldsymbol{u})^2 + (\boldsymbol{u}^H\boldsymbol{u})(\boldsymbol{v}^H\boldsymbol{v}) - 2\,\mathrm{Re}\left\{\boldsymbol{v}^H\boldsymbol{u}\right\}\boldsymbol{v}^H\boldsymbol{u})} \tag{16}$$

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