Consider the optimization problem:

$$
\begin{aligned}
&\min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1 \\
&\text{subject to } \boldsymbol{R}^{-1}\boldsymbol{z}-\boldsymbol{R}^{-1}\boldsymbol{x}=0 \\
&\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}=0
\end{aligned} \tag{1}
$$

This optimization problem has the Lagrangian function:

$$\mathrm{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\eta})=\frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1+\boldsymbol{\gamma}^H\boldsymbol{R}^{-1}(\boldsymbol{z}-\boldsymbol{x})+\boldsymbol{\eta}^H(\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}) \tag{2}$$

Now, we can augment the Lagrangian.

$$\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=\frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1+\boldsymbol{\gamma}^H\boldsymbol{R}^{-1}(\boldsymbol{z}-\boldsymbol{x})+\boldsymbol{\eta}^H(\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x})+\frac{\rho}{2}\|\boldsymbol{R}^{-1}(\boldsymbol{z}-\boldsymbol{x})\|_2^2+\frac{\rho}{2}\|\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}\|_2^2 \tag{3}$$

$$\nabla_{\boldsymbol{x}}\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=-\boldsymbol{R}^{-1}\boldsymbol{\gamma}-\boldsymbol{D}^H\boldsymbol{\eta}+\rho\boldsymbol{R}^{-2}\boldsymbol{x}-\rho\boldsymbol{R}^{-2}\boldsymbol{z}+\rho\boldsymbol{D}^H\boldsymbol{D}\boldsymbol{x}-\rho\boldsymbol{D}^H\boldsymbol{y} \tag{4}$$

For $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma}$ such that $\nabla_{\boldsymbol{x}}\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=0$:

$$(\boldsymbol{R}^{-2}+\boldsymbol{D}^H\boldsymbol{D})\boldsymbol{x}=\boldsymbol{R}^{-2}\boldsymbol{z}+\frac{\boldsymbol{R}^{-1}\boldsymbol{\gamma}}{\rho}+\boldsymbol{D}^H\boldsymbol{y}+\frac{\boldsymbol{D}^H\boldsymbol{\eta}}{\rho} \tag{5}$$

$$\boldsymbol{R}^{-1}(\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))\boldsymbol{R}^{-1}\boldsymbol{x}=\boldsymbol{R}^{-2}\boldsymbol{z}+\frac{\boldsymbol{R}^{-1}\boldsymbol{\gamma}}{\rho}+\boldsymbol{D}^H\boldsymbol{y}+\frac{\boldsymbol{D}^H\boldsymbol{\eta}}{\rho} \tag{6}$$

$$(\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))\boldsymbol{R}^{-1}\boldsymbol{x}=\boldsymbol{R}^{-1}\boldsymbol{z}+\frac{\boldsymbol{\gamma}}{\rho}+(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{y}+\frac{(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{\eta}}{\rho} \tag{7}$$

$$\boldsymbol{R}^{-1}\boldsymbol{x}=(\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))^{-1}(\boldsymbol{R}^{-1}\boldsymbol{z}+\frac{\boldsymbol{\gamma}}{\rho}+(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{y}+\frac{(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{\eta}}{\rho}) \tag{8}$$

$$\min_{\boldsymbol{x}}\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=\boldsymbol{R}(\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))^{-1}(\boldsymbol{R}^{-1}\boldsymbol{z}+\frac{\boldsymbol{\gamma}}{\rho}+(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{y}+\frac{(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{\eta}}{\rho}) \tag{9}$$

$$\nabla_{\boldsymbol{y}}\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=\boldsymbol{W}^H\boldsymbol{W}\boldsymbol{y}-\boldsymbol{W}^H\boldsymbol{s}+\boldsymbol{\eta}+\rho\boldsymbol{y}-\rho\boldsymbol{D}\boldsymbol{x} \tag{10}$$

$$\min_{\boldsymbol{y}}\mathrm{L}_\rho(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma})=(\rho\mathbf{I}+\boldsymbol{W}^T\boldsymbol{W})^{-1}(\boldsymbol{W}\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho})) \tag{11}$$

$$\min_{\boldsymbol{y}} \mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma}) = \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho})) & \text{within signal domain} \\ \boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho} & \text{outside signal domain} \end{cases} \tag{12}$$

$$\min_{\boldsymbol{z}} \mathrm{L}_{\rho}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\boldsymbol{\gamma}) = \mathrm{S}_{\frac{\lambda \boldsymbol{R}^2}{\rho}}(\boldsymbol{x}-\frac{\boldsymbol{R}\boldsymbol{\gamma}}{\rho}) \tag{13}$$

From this, we can get the update equations:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{R}(\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))^{-1}(\boldsymbol{R}^{-1}\boldsymbol{z}^{(k)}+\frac{\boldsymbol{\gamma}^{(k)}}{\rho}+(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{y}^{(k)}+\frac{(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{\eta}^{(k)}}{\rho}) \tag{14}$$

$$\boldsymbol{y}^{(k+1)} = \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho})) & \text{within signal domain} \\ \boldsymbol{D}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho} & \text{outside signal domain} \end{cases} \tag{15}$$

$$\boldsymbol{z}^{(k+1)} = \mathrm{S}_{\frac{\lambda \boldsymbol{R}^2}{\rho}}(\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{R}\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{16}$$

$$\boldsymbol{\gamma}^{(k+1)} = \boldsymbol{\gamma}^{(k)}+\rho\boldsymbol{R}^{-1}(\boldsymbol{z}^{(k+1)}-\boldsymbol{x}^{(k+1)}) \tag{17}$$

$$\boldsymbol{\eta}^{(k+1)} = \boldsymbol{\eta}^{(k)}+\rho(\boldsymbol{y}^{(k+1)}-\boldsymbol{D}\boldsymbol{x}^{(k+1)}) \tag{18}$$

However, these equations will be simpler using slightly different updates.

$$\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)} = (\mathbf{I}+(\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}))^{-1}(\boldsymbol{R}^{-1}\boldsymbol{z}^{(k)}+\frac{\boldsymbol{\gamma}^{(k)}}{\rho}+(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{y}^{(k)}+\frac{(\boldsymbol{D}\boldsymbol{R})^H\boldsymbol{\eta}^{(k)}}{\rho}) \tag{19}$$

$$\boldsymbol{y}^{(k+1)} = \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho})) & \text{within signal domain} \\ \boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho} & \text{outside signal domain} \end{cases} \tag{20}$$

$$\boldsymbol{R}^{-1}\boldsymbol{z}^{(k+1)} = \mathrm{S}_{\frac{\lambda \boldsymbol{R}}{\rho}}(\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{21}$$

$$\boldsymbol{\gamma}^{(k+1)} = \boldsymbol{\gamma}^{(k)}+\rho(\boldsymbol{R}^{-1}\boldsymbol{z}^{(k+1)}-\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)}) \tag{22}$$

$$\boldsymbol{\eta}^{(k+1)} = \boldsymbol{\eta}^{(k)}+\rho(\boldsymbol{y}^{(k+1)}-\boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{x}^{(k+1)}) \tag{23}$$

In the above equations, the dictionary never appears unscaled (without $\boldsymbol{R}$), and the coefficients are always scaled by $\boldsymbol{R}^{-1}$. Unfortunately, $\boldsymbol{R}$ still must be calculated for the $\boldsymbol{z}$-update.

For reference while coding, I would like to adjust into SPORCO package notation. I can add an S subscript to prevent confusion.

$$\boldsymbol{x}_S^{(k)} = \boldsymbol{R}^{-1}\boldsymbol{x}^{(k)} \tag{24}$$

$$\boldsymbol{y}_S^{(k)} = \begin{bmatrix} \boldsymbol{y}^{(k)} \\ \boldsymbol{R}^{-1}\boldsymbol{z}^{(k)} \end{bmatrix} \tag{25}$$

$$\boldsymbol{u}_S^{(k)} = \begin{bmatrix} \boldsymbol{\eta}^{(k)} \\ \boldsymbol{\gamma}^{(k)} \end{bmatrix} \tag{26}$$

$$\boldsymbol{A}_S = \begin{bmatrix} -\boldsymbol{D} \\ -\mathbf{I} \end{bmatrix} \tag{27}$$

$$\boldsymbol{B}_S = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_0 & \boldsymbol{P}_1 \end{bmatrix} \tag{28}$$

$$\boldsymbol{C}_S = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{29}$$

$$\boldsymbol{Q}_S = \mathbf{I} + (\boldsymbol{D}\boldsymbol{R})^H(\boldsymbol{D}\boldsymbol{R}) \tag{30}$$

$$\boldsymbol{D}_S = \boldsymbol{D}\boldsymbol{R} \tag{31}$$

$$\boldsymbol{x}_S^{(k+1)} = \boldsymbol{Q}_S^{-1}(\boldsymbol{P}_1\boldsymbol{y}_S^{(k)} + \frac{\boldsymbol{P}_1\boldsymbol{u}_S^{(k)}}{\rho} + \boldsymbol{D}_S^H\boldsymbol{P}_0\boldsymbol{y}_S^{(k)} + \frac{\boldsymbol{D}_S^H\boldsymbol{P}_0\boldsymbol{u}_S^{(k)}}{\rho}) \tag{32}$$

$$\boldsymbol{y}_S^{(k+1)} = \begin{bmatrix} \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s} + \rho(\boldsymbol{D}_S\boldsymbol{x}_S^{(k+1)} - \frac{\boldsymbol{P}_0\boldsymbol{u}_S^{(k)}}{\rho})) & \text{within signal domain} \\ \boldsymbol{D}_S\boldsymbol{x}_S^{(k+1)} - \frac{\boldsymbol{P}_0\boldsymbol{u}_S^{(k)}}{\rho} & \text{outside signal domain} \end{cases} \\ \mathrm{S}_{\frac{\lambda\boldsymbol{R}}{\rho}}(\boldsymbol{x}_S^{(k+1)} - \frac{\boldsymbol{P}_1\boldsymbol{u}_S^{(k)}}{\rho}) \end{bmatrix}$$

(33)

$$\boldsymbol{u}_S^{(k+1)} = \boldsymbol{u}_S^{(k)} + \rho(\boldsymbol{A}_S\boldsymbol{x}_S^{(k+1)} + \boldsymbol{B}_S\boldsymbol{y}_S^{(k+1)}) \tag{34}$$