

Consider the optimization problem:

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{y}, \mathbf{z}} & \|\mathbf{W}\mathbf{y}\|_1 + \lambda \|\boldsymbol{\alpha} \cdot \mathbf{z}\|_1 + \frac{\mu}{2} \|\boldsymbol{\Gamma}_0 \mathbf{R}^{-1} \mathbf{x}\|_2^2 + \frac{\mu}{2} \|\boldsymbol{\Gamma}_1 \mathbf{R}^{-1} \mathbf{x}\|_2^2 \\ \text{subject to} & \quad \mathbf{R}^{-1}(\mathbf{z} - \mathbf{x}) = 0 \\ & \quad \mathbf{D}\mathbf{x} + \mathbf{y} = \mathbf{s} \end{aligned} \quad (1)$$

First, let's compute the augmented Lagrangian.

$$\mathcal{L}_\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \gamma) = \|\mathbf{W}\mathbf{y}\|_1 + \lambda \|\boldsymbol{\alpha} \cdot \mathbf{z}\|_1 + \frac{\mu}{2} \|\boldsymbol{\Gamma}_0 \mathbf{R}^{-1} \mathbf{x}\|_2^2 + \frac{\mu}{2} \|\boldsymbol{\Gamma}_1 \mathbf{R}^{-1} \mathbf{x}\|_2^2 + \frac{\rho}{2} \|\mathbf{R}^{-1}(\mathbf{z} - \mathbf{x}) + \frac{\gamma}{\rho}\|_2^2 + \frac{\rho}{2} \|\mathbf{D}\mathbf{x} + \mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho}\|_2^2 \quad (2)$$

$$\frac{\partial}{\partial \mathbf{x}} \mathcal{L}_\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \gamma) = \mu \mathbf{R}^{-1}(\boldsymbol{\Gamma}_0^H \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1^H \boldsymbol{\Gamma}_1) \mathbf{R}^{-1} \mathbf{x} + \rho \mathbf{R}^{-2}(\mathbf{x} - \mathbf{z}) - \gamma + \rho(\mathbf{D}^H \mathbf{D}\mathbf{x} + \mathbf{D}^H(\mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho})) \quad (3)$$

Setting the partial derivative equal to zero:

$$(\rho \mathbf{R}^{-2} + \rho \mathbf{D}^H \mathbf{D} + \mu \mathbf{R}^{-1} \boldsymbol{\Gamma}_0^H \boldsymbol{\Gamma}_0 \mathbf{R}^{-1} + \mu \mathbf{R}^{-1} \boldsymbol{\Gamma}_1^H \boldsymbol{\Gamma}_1 \mathbf{R}^{-1}) \mathbf{x} = \rho \mathbf{R}^{-2} \mathbf{z} + \mathbf{R}^{-1} \gamma - \rho \mathbf{D}^H(\mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho}) \quad (4)$$

$$\mathbf{R}^{-1}(\rho \mathbf{I} + \rho(\mathbf{D}\mathbf{R})^H \mathbf{D}\mathbf{R} + \mu \boldsymbol{\Gamma}_0^H \boldsymbol{\Gamma}_0 + \mu \boldsymbol{\Gamma}_1^H \boldsymbol{\Gamma}_1) \mathbf{R}^{-1} \mathbf{x} = \rho \mathbf{R}^{-2} \mathbf{z} + \mathbf{R}^{-1} \gamma - \rho \mathbf{D}^H(\mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho}) \quad (5)$$

$$(\mathbf{I} + (\mathbf{D}\mathbf{R})^H \mathbf{D}\mathbf{R} + \frac{\mu}{\rho} \boldsymbol{\Gamma}_0^H \boldsymbol{\Gamma}_0 + \frac{\mu}{\rho} \boldsymbol{\Gamma}_1^H \boldsymbol{\Gamma}_1) \mathbf{R}^{-1} \mathbf{x} = \mathbf{R}^{-1} \mathbf{z} + \frac{\gamma}{\rho} - (\mathbf{D}\mathbf{R})^H(\mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho}) \quad (6)$$

$$\mathbf{R}^{-1} \mathbf{x} = (\mathbf{I} + (\mathbf{D}\mathbf{R})^H \mathbf{D}\mathbf{R} + \frac{\mu}{\rho} \boldsymbol{\Gamma}_0^H \boldsymbol{\Gamma}_0 + \frac{\mu}{\rho} \boldsymbol{\Gamma}_1^H \boldsymbol{\Gamma}_1)^{-1} (\mathbf{R}^{-1} \mathbf{z} + \frac{\gamma}{\rho} - (\mathbf{D}\mathbf{R})^H(\mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho})) \quad (7)$$

Looking at the terms of the augmented Lagrangian that depend on \mathbf{y} :

$$\mathcal{L}_\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \gamma) = \|\mathbf{W}\mathbf{y}\|_1 + \frac{\rho}{2} \|\mathbf{D}\mathbf{x} + \mathbf{y} - \mathbf{s} + \frac{\boldsymbol{\eta}}{\rho}\|_2^2 + f_{\rho, \mu}(\mathbf{x}, \mathbf{z}, \gamma) \quad (8)$$

Minimizing the augmented Lagrangian in respect to \mathbf{y}

$$\mathbf{y} = \mathbf{S}_{\frac{\mathbf{W}}{\rho}}(\mathbf{s} - \mathbf{D}\mathbf{x} - \frac{\boldsymbol{\eta}}{\rho}) \quad (9)$$

Looking at the terms of the augmented Lagrangian that depend on \mathbf{z} :

$$\mathcal{L}_\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\eta}, \gamma) = \lambda \|\boldsymbol{\alpha} \cdot \mathbf{z}\|_1 + \frac{\rho}{2} \|\mathbf{R}^{-1}(\mathbf{z} - \mathbf{x}) + \frac{\gamma}{\rho}\|_2^2 + g_{\rho, \mu, \mathbf{W}}(\mathbf{x}, \mathbf{y}, \boldsymbol{\eta}) \quad (10)$$

Minimizing the augmented Lagrangian in respect to \mathbf{z}

$$\mathbf{R}^{-1}\mathbf{z} = \mathbf{S}_{\frac{\lambda\alpha\mathbf{R}}{\rho}}(\mathbf{R}^{-1}\mathbf{x} - \frac{\gamma}{\rho}) \quad (11)$$

The dual variable updates are unchanged from other formulations, so I will omit them in this document.

To summarize:

$$\mathbf{R}^{-1}\mathbf{x} = (\mathbf{I} + (\mathbf{DR})^H \mathbf{DR} + \frac{\mu}{\rho} \mathbf{\Gamma}_0^H \mathbf{\Gamma}_0 + \frac{\mu}{\rho} \mathbf{\Gamma}_1^H \mathbf{\Gamma}_1)^{-1} (\mathbf{R}^{-1}\mathbf{z} + \frac{\gamma}{\rho} - (\mathbf{DR})^H (\mathbf{y} - \mathbf{s} + \frac{\eta}{\rho})) \quad (12)$$

$$\mathbf{y} = \mathbf{S}_{\frac{1}{\rho}}(\mathbf{s} - \mathbf{D}\mathbf{x} - \frac{\eta}{\rho}) \quad (13)$$

$$\mathbf{R}^{-1}\mathbf{z} = \mathbf{S}_{\frac{\lambda\alpha\mathbf{R}}{\rho}}(\mathbf{R}^{-1}\mathbf{x} - \frac{\gamma}{\rho}) \quad (14)$$