Consider the optimization problem:

$$\begin{aligned} &\min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1 \\ &\text{subject to } \boldsymbol{z}-\boldsymbol{x}=0 \\ &\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}=0 \end{aligned} \tag{1}$$

This optimization problem has the Lagrangian function:

$$\mathrm{L}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\eta})\frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1+\boldsymbol{\gamma}^H(\boldsymbol{z}-\boldsymbol{x})+\boldsymbol{\eta}^H(\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}) \tag{2}$$

Now, we can augment the Lagrangian. I will use the modification from the normalization document in my notes for one of the augmentation terms.

$$\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=\frac{1}{2}\|\boldsymbol{s}-\boldsymbol{W}\boldsymbol{y}\|_2^2+\lambda\|\boldsymbol{z}\|_1+\boldsymbol{\gamma}^H(\boldsymbol{z}-\boldsymbol{x})+\boldsymbol{\eta}^H(\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x})+\frac{\rho}{2}\|\boldsymbol{\Lambda}^{\frac{1}{2}}(\boldsymbol{z}-\boldsymbol{x})\|_2^2+\frac{\rho}{2}\|\boldsymbol{y}-\boldsymbol{D}\boldsymbol{x}\|_2^2 \tag{3}$$

$$\nabla_{\boldsymbol{x}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=-\boldsymbol{\gamma}-\boldsymbol{D}^H\boldsymbol{\eta}+\rho\boldsymbol{\Lambda}\boldsymbol{x}-\rho\boldsymbol{\Lambda}\boldsymbol{z}+\rho\boldsymbol{D}^H\boldsymbol{D}\boldsymbol{x}-\rho\boldsymbol{D}^H\boldsymbol{y} \tag{4}$$

$$\min_{\boldsymbol{x}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=(\rho\boldsymbol{\Lambda}+\rho\boldsymbol{D}^H\boldsymbol{D})^{-1}(\rho\boldsymbol{D}^H(\boldsymbol{y}+\frac{\boldsymbol{\eta}}{\rho})+\rho\boldsymbol{\Lambda}\boldsymbol{z}+\boldsymbol{\gamma}) \tag{5}$$

$$\min_{\boldsymbol{x}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=(\boldsymbol{\Lambda}+\boldsymbol{D}^H\boldsymbol{D})^{-1}(\boldsymbol{D}^H(\boldsymbol{y}+\frac{\boldsymbol{\eta}}{\rho})+\boldsymbol{\Lambda}\boldsymbol{z}+\frac{\boldsymbol{\gamma}}{\rho}) \tag{6}$$

$$\min_{\boldsymbol{y}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=(\rho\mathbf{I}+\boldsymbol{W}^T\boldsymbol{W})^{-1}(\boldsymbol{W}\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho})) \tag{7}$$

$$\min_{\boldsymbol{y}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=\begin{cases}\frac{1}{1+\rho}(\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho})) & \text{within signal domain}\\ \boldsymbol{D}\boldsymbol{x}-\frac{\boldsymbol{\eta}}{\rho} & \text{outside signal domain}\end{cases} \tag{8}$$

$$\min_{\boldsymbol{z}}\mathrm{L}_{\rho,\boldsymbol{\Lambda}}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{\eta},\gamma)=\mathrm{S}_{\frac{\lambda\boldsymbol{\Lambda}^{-1}}{\rho}}(\boldsymbol{x}-\frac{\boldsymbol{\Lambda}^{-1}\boldsymbol{\gamma}}{\rho}) \tag{9}$$

From this, we can get the update equations:

$$\boldsymbol{x}^{(k+1)}=(\boldsymbol{\Lambda}+\boldsymbol{D}^H\boldsymbol{D})^{-1}(\boldsymbol{D}^H(\boldsymbol{y}^{(k)}-\boldsymbol{s}+\frac{\boldsymbol{\eta}^{(k)}}{\rho})+\boldsymbol{\Lambda}\boldsymbol{z}^{(k)}+\frac{\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{10}$$

$$\boldsymbol{y}^{(k+1)}=\begin{cases}\frac{1}{1+\rho}(\boldsymbol{s}+\rho(\boldsymbol{D}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho})) & \text{within signal domain}\\ \boldsymbol{D}\boldsymbol{x}^{(k+1)}-\frac{\boldsymbol{\eta}^{(k)}}{\rho} & \text{outside signal domain}\end{cases} \tag{11}$$

$$\boldsymbol{z}^{(k+1)} = \mathrm{S}_{\frac{\lambda\boldsymbol{\Lambda}^{-1}}{\rho}}(\boldsymbol{x}^{(k+1)} - \frac{\boldsymbol{\Lambda}^{-1}\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{12}$$

$$\boldsymbol{\gamma}^{(k+1)} = \boldsymbol{\gamma}^{(k)} + \rho\boldsymbol{\Lambda}(\boldsymbol{z}^{(k+1)} - \boldsymbol{x}^{(k+1)}) \tag{13}$$

$$\boldsymbol{\eta}^{(k+1)} = \boldsymbol{\eta}^{(k)} + \rho(\boldsymbol{y}^{(k+1)} - \boldsymbol{D}\boldsymbol{x}^{(k+1)}) \tag{14}$$

Now, it will be helpful to add a version of scaled ADMM.
Recall that

$$(\boldsymbol{\Lambda} + \boldsymbol{D}^H\boldsymbol{D})^{-1} = \boldsymbol{\Lambda}^{-\frac{1}{2}}(\mathbf{I} + \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{D}^H\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^{-1}\boldsymbol{\Lambda}^{-\frac{1}{2}} \tag{15}$$

So,

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{\Lambda}^{-\frac{1}{2}}(\mathbf{I}+\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{D}^H\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^{-1}\boldsymbol{\Lambda}^{-\frac{1}{2}}(\boldsymbol{D}^H(\boldsymbol{y}^{(k)} - \boldsymbol{s} + \frac{\boldsymbol{\eta}^{(k)}}{\rho}) + \boldsymbol{\Lambda}\boldsymbol{z}^{(k)} + \frac{\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{16}$$

$$\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} = (\mathbf{I}+(\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^H\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^{-1}((\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^H(\boldsymbol{y}^{(k)} - \boldsymbol{s} + \frac{\boldsymbol{\eta}^{(k)}}{\rho}) + \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{z}^{(k)} + \frac{\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{17}$$

$$\boldsymbol{y}^{(k+1)} = \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s} + \rho(\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} - \frac{\boldsymbol{\eta}^{(k)}}{\rho})) & \text{within signal domain} \\ \boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} - \frac{\boldsymbol{\eta}^{(k)}}{\rho} & \text{outside signal domain} \end{cases} \tag{18}$$

$$\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{z}^{(k+1)} = \mathrm{S}_{\frac{\lambda\boldsymbol{\Lambda}^{-\frac{1}{2}}}{\rho}}(\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} - \frac{\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\gamma}^{(k)}}{\rho}) \tag{19}$$

$$\frac{\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\gamma}^{(k+1)}}{\rho} = \frac{\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\gamma}^{(k)}}{\rho} + \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{z}^{(k+1)} - \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} \tag{20}$$

$$\frac{\boldsymbol{\eta}^{(k+1)}}{\rho} = \frac{\boldsymbol{\eta}^{(k)}}{\rho} + \boldsymbol{y}^{(k+1)} - \boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x}^{(k+1)} \tag{21}$$

So, now we can simplify this with some substitutions:

$$\boldsymbol{A} = \mathbf{I} + (\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}})^H\boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}} \tag{22}$$

$$\boldsymbol{D} = \boldsymbol{D}\boldsymbol{\Lambda}^{-\frac{1}{2}} \tag{23}$$

$$\boldsymbol{x} = \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{x} \tag{24}$$

$$\boldsymbol{z} = \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{z} \tag{25}$$

$$\boldsymbol{\gamma} = \frac{\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\gamma}}{\rho} \tag{26}$$

$$\boldsymbol{\eta} = \frac{\boldsymbol{\eta}}{\rho} \tag{27}$$

With these substitutions, we arrive at the following:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{A}^{-1}(\boldsymbol{D}^H(\boldsymbol{y}^{(k)} - \boldsymbol{s} + \boldsymbol{\eta}^{(k)}) + \boldsymbol{z}^{(k)} + \boldsymbol{\gamma}^{(k)}) \tag{28}$$

$$\boldsymbol{y}^{(k+1)} = \begin{cases} \frac{1}{1+\rho}(\boldsymbol{s} + \rho(\boldsymbol{D}\boldsymbol{x}^{(k+1)} - \boldsymbol{\eta}^{(k)})) & \text{within signal domain} \\ \boldsymbol{D}\boldsymbol{x}^{(k+1)} - \boldsymbol{\eta}^{(k)} & \text{outside signal domain} \end{cases} \tag{29}$$

$$\boldsymbol{z}^{(k+1)} = \mathrm{S}_{\frac{\lambda\boldsymbol{\Lambda}^{-\frac{1}{2}}}{\rho}}(\boldsymbol{x}^{(k+1)} - \boldsymbol{\gamma}^{(k)}) \tag{30}$$

$$\boldsymbol{\gamma}^{(k+1)} = \boldsymbol{\gamma}^{(k)} + \boldsymbol{z}^{(k+1)} - \boldsymbol{x}^{(k+1)} \tag{31}$$

$$\boldsymbol{\eta}^{(k+1)} = \boldsymbol{\eta}^{(k)} + \boldsymbol{y}^{(k+1)} - \boldsymbol{D}\boldsymbol{x}^{(k+1)} \tag{32}$$