1 Standard Form

I need to compute gradients, exploiting my Cholesky factorization for efficiency.

$$\boldsymbol{y} = (\rho \mathbf{I} + \boldsymbol{D}^H \boldsymbol{D})^{-1} \boldsymbol{x} \tag{1}$$

As an abreviation:

$$\boldsymbol{A} = \rho \mathbf{I} + \boldsymbol{D}^H \boldsymbol{D} \tag{2}$$

$$\boldsymbol{y} = \boldsymbol{A}^{-1} \boldsymbol{x} \tag{3}$$

I need the derivative of the output $\boldsymbol{y}$ in respect to dictionary weights $\boldsymbol{D}$. Using the matrix cookbook, I get the following:

$$\frac{\partial \boldsymbol{y}}{\partial D_{i,j}} = -\boldsymbol{A}^{-1} \frac{\partial \boldsymbol{A}}{\partial D_{i,j}} \boldsymbol{A}^{-1} \boldsymbol{x} \tag{4}$$

The derivation is boring and tedious, but I computed the derivatives of $\boldsymbol{A}$ and verified it by comparing to an empirical derivative in MATLAB.

$$\frac{\partial \boldsymbol{A}}{\partial \operatorname{real}(D_{i,j})} = \boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T \tag{5}$$

$$\frac{\partial \boldsymbol{A}}{\partial \operatorname{imag}(D_{i,j})} = -1j * \boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + 1j * \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T \tag{6}$$

It is necessary to compute separate partial derivatives for the real and imaginary components because approach direction in the complex plane matters for this function.

Putting it all together, I have the following:

$$\frac{\partial \boldsymbol{A}^{-1} \boldsymbol{x}}{\partial \operatorname{real}(D_{i,j})} = -\boldsymbol{A}^{-1} (\boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T) \boldsymbol{A}^{-1} \boldsymbol{x} \tag{7}$$

$$\frac{\partial \boldsymbol{A}^{-1} \boldsymbol{x}}{\partial \operatorname{imag}(D_{i,j})} = -\boldsymbol{A}^{-1} (-1j * \boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + 1j * \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T) \boldsymbol{A}^{-1} \boldsymbol{x} \tag{8}$$

To use within the automatic differentiation framework in TensorFlow with loss L, I need to be able to compute $\frac{\partial L}{\partial \operatorname{real}(D_{i,j})}$ and $\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})}$ as a function of $\nabla_{\boldsymbol{y}} L$, where $\nabla_{\boldsymbol{y}} L$ is a column vector with $(\nabla_{\boldsymbol{y}} L)[i] = \frac{\partial L}{\partial \operatorname{real}(\boldsymbol{y}_i)} + j \frac{\partial L}{\partial \operatorname{imag}(\boldsymbol{y}_i)}$.

Given these notations, for generic composite function $f(g(\boldsymbol{x}))$ with real output, the following equations can be derived using chain rule:

$$\frac{\partial f}{\partial \operatorname{real}(\boldsymbol{x})} = \operatorname{real}((\nabla_{g(\boldsymbol{x})} f)^H \frac{\partial g}{\partial \operatorname{real}(\boldsymbol{x})}) \tag{9}$$

$$\frac{\partial f}{\partial \operatorname{imag}(\boldsymbol{x})} = \operatorname{real}((\nabla_{g(\boldsymbol{x})} f)^H \frac{\partial g}{\partial \operatorname{imag}(\boldsymbol{x})}) \tag{10}$$

Applying this general rule to the actual problem at hand:

$$\frac{\partial L}{\partial \operatorname{real}(D_{i,j})} = \operatorname{real}((\nabla_{\boldsymbol{y}} L)^H \frac{\partial \boldsymbol{y}}{\partial \operatorname{real}(D_{i,j})}) \tag{11}$$

$$\frac{\partial L}{\partial \operatorname{real}(D_{i,j})} = -\operatorname{real}((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}(\boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T) \boldsymbol{A}^{-1} \boldsymbol{x}) \tag{12}$$

$$\frac{\partial L}{\partial \operatorname{real}(D_{i,j})} = -\operatorname{real}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{D} \boldsymbol{A}^{-1} \boldsymbol{x})) - \operatorname{real}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{D}^H \boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{A}^{-1} \boldsymbol{x})) \tag{13}$$

Returning for the partial derivative in respect to the imaginary component I have:

$$\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})} = \operatorname{real}((\nabla_{\boldsymbol{y}} L)^H \frac{\partial \boldsymbol{y}}{\partial \operatorname{imag}(D_{i,j})}) \tag{14}$$

$$\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})} = -\operatorname{real}((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}(-1j * \boldsymbol{e}_j \boldsymbol{e}_i^T \boldsymbol{D} + 1j * \boldsymbol{D}^H \boldsymbol{e}_i \boldsymbol{e}_j^T) \boldsymbol{A}^{-1} \boldsymbol{x}) \tag{15}$$

$$\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})} = -\operatorname{real}(-1j*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{D} \boldsymbol{A}^{-1} \boldsymbol{x})) - \operatorname{real}(1j*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{D}^H \boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{A}^{-1} \boldsymbol{x})) \tag{16}$$

I would like to combine this with the partial derivative in respect to the real component.

$$\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})} = -\operatorname{real}(-1j*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{D} \boldsymbol{A}^{-1} \boldsymbol{x})) - \operatorname{real}(-1j*(-(\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{D}^H \boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{A}^{-1} \boldsymbol{x})) \tag{17}$$

Using the fact that $\operatorname{real}(-1j * \boldsymbol{x}) = \operatorname{imag}(\boldsymbol{x})$:

$$\frac{\partial L}{\partial \operatorname{imag}(D_{i,j})} = -\operatorname{imag}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{D} \boldsymbol{A}^{-1} \boldsymbol{x})) - \operatorname{imag}(-((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{D}^H \boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{A}^{-1} \boldsymbol{x})) \tag{18}$$

In a slight abuse of notation, I define $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \operatorname{real}(x)} + j \frac{\partial f}{\partial \operatorname{imag}(x)}$
So,

$$\frac{\partial L}{\partial D_{i,j}} = -((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{D} \boldsymbol{A}^{-1} \boldsymbol{x}) - ((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1} \boldsymbol{D}^H \boldsymbol{e}_i)^* (\boldsymbol{e}_j^T \boldsymbol{A}^{-1} \boldsymbol{x})^* \tag{19}$$

Since we're looking at scalars, conjugates can be replaced with a Hermitian transpose. ($\boldsymbol{A}^{-1}$ is Hermitian, so it's Hermitian transpose is itself.)

$$\frac{\partial L}{\partial D_{i,j}} = -(\boldsymbol{e}_i^T \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{x})((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{e}_j) - (\boldsymbol{e}_i^T \boldsymbol{D}\boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)(\boldsymbol{x}^H \boldsymbol{A}^{-1}\boldsymbol{e}_j) \tag{20}$$

Note in each term is the product of the ith element of a column vector and the jth element of a row vector. Therefore, I can compile the entire gradient in respect to matrix $\boldsymbol{D}$ by summing the outer products:

$$\nabla_{\boldsymbol{D}} L = -(\boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{x})(\boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)^H - (\boldsymbol{D}\boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)(\boldsymbol{A}^{-1}\boldsymbol{x})^H \tag{21}$$

2 Woodbury Form

I need to compute gradients, exploiting my Cholesky factorization for efficiency.

$$\boldsymbol{y} = (\rho\mathbf{I} + \boldsymbol{D}\boldsymbol{D}^H)^{-1}\boldsymbol{x} \tag{22}$$

As an abreviation:

$$\boldsymbol{A} = \rho\mathbf{I} + \boldsymbol{D}\boldsymbol{D}^H \tag{23}$$

$$\boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{x} \tag{24}$$

I need the derivative of the output $\boldsymbol{y}$ in respect to dictionary weights $\boldsymbol{D}$. Using the matrix cookbook, I get the following:

$$\frac{\partial \boldsymbol{y}}{\partial D_{i,j}} = -\boldsymbol{A}^{-1}\frac{\partial \boldsymbol{A}}{\partial D_{i,j}}\boldsymbol{A}^{-1}\boldsymbol{x} \tag{25}$$

The derivation is boring and tedious, but I computed the derivative of $\boldsymbol{A}$ and verified it using empirical tests in MATLAB.

$$\frac{\partial \boldsymbol{A}}{\partial\,\mathrm{real}(D_{i,j})} = \boldsymbol{e}_i\boldsymbol{e}_j^T\boldsymbol{D}^H + \boldsymbol{D}\boldsymbol{e}_j\boldsymbol{e}_i^T \tag{26}$$

$$\frac{\partial \boldsymbol{A}}{\partial\,\mathrm{imag}(D_{i,j})} = 1j * \boldsymbol{e}_i\boldsymbol{e}_j^T\boldsymbol{D}^H - 1j * \boldsymbol{D}\boldsymbol{e}_j\boldsymbol{e}_i^T \tag{27}$$

Putting it all together, I have the following:

$$\frac{\partial \boldsymbol{A}^{-1}\boldsymbol{x}}{\partial\,\mathrm{real}(D_{i,j})} = -\boldsymbol{A}^{-1}(\boldsymbol{e}_i\boldsymbol{e}_j^T\boldsymbol{D}^H + \boldsymbol{D}\boldsymbol{e}_j\boldsymbol{e}_i^T)\boldsymbol{A}^{-1}\boldsymbol{x} \tag{28}$$

$$\frac{\partial \boldsymbol{A}^{-1}\boldsymbol{x}}{\partial\,\mathrm{imag}(D_{i,j})} = -\boldsymbol{A}^{-1}(1j * \boldsymbol{e}_i\boldsymbol{e}_j^T\boldsymbol{D}^H - 1j * \boldsymbol{D}\boldsymbol{e}_j\boldsymbol{e}_i^T)\boldsymbol{A}^{-1}\boldsymbol{x} \tag{29}$$

To use within the automatic differentiation framework in TensorFlow with loss L, I need to be able to compute $\frac{\partial L}{\partial\,\mathrm{real}(D_{i,j})}$ and $\frac{\partial L}{\partial\,\mathrm{imag}(D_{i,j})}$ as a function of $\nabla_{\boldsymbol{y}} L$, where $\nabla_{\boldsymbol{y}} L$ is a column vector with $(\nabla_{\boldsymbol{y}} L)[i] = \frac{\partial L}{\partial\,\mathrm{real}(\boldsymbol{y}_i)} + j\frac{\partial L}{\partial\,\mathrm{imag}(\boldsymbol{y}_i)}$.

Given these notations, for generic composite function $f(g(\boldsymbol{x}))$ with real output, the following equations can be derived using chain rule:

$$\frac{\partial f}{\partial\,\mathrm{real}(\boldsymbol{x})} = \mathrm{real}((\nabla_{g(\boldsymbol{x})} f)^H \frac{\partial g}{\partial\,\mathrm{real}(\boldsymbol{x})}) \tag{30}$$

$$\frac{\partial f}{\partial\,\mathrm{imag}(\boldsymbol{x})} = \mathrm{real}((\nabla_{g(\boldsymbol{x})} f)^H \frac{\partial g}{\partial\,\mathrm{imag}(\boldsymbol{x})}) \tag{31}$$

Applying this general rule to the actual problem at hand:

$$\frac{\partial L}{\partial\,\mathrm{real}(D_{i,j})} = \mathrm{real}((\nabla_{\boldsymbol{y}} L)^H \frac{\partial \boldsymbol{y}}{\partial\,\mathrm{real}(D_{i,j})}) \tag{32}$$

$$\frac{\partial L}{\partial\,\mathrm{real}(D_{i,j})} = -\,\mathrm{real}((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}(\boldsymbol{e}_i \boldsymbol{e}_j^T \boldsymbol{D}^H + \boldsymbol{D}\boldsymbol{e}_j \boldsymbol{e}_i^T)\boldsymbol{A}^{-1}\boldsymbol{x}) \tag{33}$$

$$\frac{\partial L}{\partial\,\mathrm{real}(D_{i,j})} = -\,\mathrm{real}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{D}^H \boldsymbol{A}^{-1}\boldsymbol{x})) - \mathrm{real}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\boldsymbol{x})) \tag{34}$$

Returning for the partial derivative in respect to the imaginary component I have:

$$\frac{\partial L}{\partial\,\mathrm{imag}(D_{i,j})} = \mathrm{real}(\nabla_{\boldsymbol{y}} L)^H \frac{\partial \boldsymbol{y}}{\partial\,\mathrm{imag}(D_{i,j})}) \tag{35}$$

$$\frac{\partial L}{\partial\,\mathrm{imag}(D_{i,j})} = -\,\mathrm{real}((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}(1j * \boldsymbol{e}_i \boldsymbol{e}_j^T \boldsymbol{D}^H - 1j * \boldsymbol{D}\boldsymbol{e}_j \boldsymbol{e}_i^T)\boldsymbol{A}^{-1}\boldsymbol{x}) \tag{36}$$

$$\frac{\partial L}{\partial\,\mathrm{imag}(D_{i,j})} = -\,\mathrm{real}(1j*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{D}^H \boldsymbol{A}^{-1}\boldsymbol{x})) - \mathrm{real}(-1j*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\boldsymbol{x})) \tag{37}$$

I would like to combine this expression with the partial derivative in respect to the real component.

$$\frac{\partial L}{\partial\,\mathrm{imag}(D_{i,j})} = -\,\mathrm{imag}(-1*((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{e}_i)(\boldsymbol{e}_j^T \boldsymbol{D}^H \boldsymbol{A}^{-1}\boldsymbol{x})) - \mathrm{imag}(((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\boldsymbol{x})) \tag{38}$$

$$\frac{\partial L}{\partial D_{i,j}} = -((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{e}_i)^*(\boldsymbol{e}_j^T \boldsymbol{D}^H \boldsymbol{A}^{-1}\boldsymbol{x})^* - (\frac{\partial L}{\partial \boldsymbol{y}} \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j)(\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\boldsymbol{x}) \tag{39}$$

$$\frac{\partial L}{\partial D_{i,j}} = -(\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)(\boldsymbol{x}^H \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j) - (\boldsymbol{e}_i^T \boldsymbol{A}^{-1}\boldsymbol{x})((\nabla_{\boldsymbol{y}} L)^H \boldsymbol{A}^{-1}\boldsymbol{D}\boldsymbol{e}_j) \tag{40}$$

$$\nabla_{\boldsymbol{D}} L = -(\boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)(\boldsymbol{D}^H \boldsymbol{A}^{-1}\boldsymbol{x})^H - (\boldsymbol{A}^{-1}\boldsymbol{x})(\boldsymbol{D}^H \boldsymbol{A}^{-1}\nabla_{\boldsymbol{y}} L)^H \tag{41}$$