

## Important terms

### Triangle inequality

$$|x + y| \leq |x| + |y|$$

## Sequences

$$(x_n)_{n \geq 1 \in \mathbb{N}} \in \mathbb{R}$$

### Convergence

$(x_n)$  converges to limit  $l$  iff

$$\forall \varepsilon > 0 \in \mathbb{R} \quad \exists N \in \mathbb{N} \text{ st } \forall n > N \in \mathbb{N} \\ |x_n - l| < \varepsilon$$

we write  $\lim_{n \rightarrow \infty} (x_n) = l$  or  $(x_n) \xrightarrow{n \rightarrow \infty} l$

If no such limit exists,  $(x_n)$  diverges.

### Properties

$$\exists l \rightarrow \exists ! l$$

$(x_n)$  is convergent  $\rightarrow (x_n)$  is bounded

### Limit laws

If  $(x_n) \rightarrow a$  and  $(y_n) \rightarrow b$  then

$$\begin{array}{ll} (I) & x_n + y_n \rightarrow a + b \\ (II) & x_n - y_n \rightarrow a - b \\ (III) & x_n y_n \rightarrow ab \\ (IV) & \frac{x_n}{y_n} \rightarrow \frac{a}{b} \quad \text{if } b \neq 0 \\ (V) & |x_n| \rightarrow |a| \end{array}$$

### Monotone convergence theorem

$x_n$  is bounded above (below) and monotone increasing (decreasing)  $\rightarrow x_n$  converges.

### Cauchy criterion

$$\begin{array}{l} (x_n) \text{ converges} \\ \leftrightarrow \\ \forall \varepsilon > 0 \quad \exists N \text{ st } \forall n > N \\ |x_n - x_m| < \varepsilon \end{array}$$

### Cauchy criterion

Let  $(x_n)$  be a sequence st  $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = l$

Then

1. if  $l < 1$  converges
2. if  $l > 1$  diverges
3. if  $l = 1$  ?

### Squeeze theorem

Let  $(x_n) \rightarrow a$  and  $(y_n) \rightarrow a$ .

$$\text{If } x_n < a_n < y_n \forall n, \text{ then } a_n \rightarrow a$$

## Series

A infinite sum (series)  $s_n$  of a sequence  $x_n$  is defined as a sequence of finite sums.

$$s_n = \sum_{i=0}^n x_n \\ \lim_{n \rightarrow \infty} s_n = \sum_{i=0}^{\infty} (x_n)_i$$

Absolutely convergent  $\leftrightarrow \sum_{i=0}^{\infty} |(x_n)_i|$  converges.

$x_n$  is absolutely convergent  $\rightarrow x_n$  converges

Thus: Signs don't matter when determining the convergence of a series

## Examples

### Constant sequence

$$x_n = c \quad c \neq 0 \\ \rightarrow \lim_{n \rightarrow \infty} s_n = \pm \infty$$

### Geometric series

$$x_n = q^n \\ s_n = \frac{1 - q^{n+1}}{1 - q} \\ = \frac{1}{1 - q} (1 - q^{n+1})$$

if  $q < 1$  then  $s_n$  converges to  $\frac{1}{1-q}$

### Harmonic series

$$x_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

$$x_n = \frac{1}{n!}$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=1}^{\infty} \frac{1}{n!} = e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)$$

## Cheat sheet

### Useful limits

$\frac{\sin(x)}{x}$	$\xrightarrow{x \rightarrow 0}$	1
$\frac{\log(1+x)}{x}$	$\xrightarrow{x \rightarrow 0}$	1
$\left(1 - \frac{1}{n}\right)^n$	$\xrightarrow{n \rightarrow \infty}$	$e$
$\left(1 + \frac{x}{n}\right)^n$	$\xrightarrow{n \rightarrow \infty}$	$e^x$

$$\begin{array}{l} \exists N \text{ st } \forall n < N \\ a_{n+1} - a_n \begin{cases} > 0 \rightarrow \text{monotone increasing} \\ < 0 \rightarrow \text{monotone decreasing} \\ = 0 \rightarrow \text{constant sequence} \end{cases} \end{array}$$

### “Divergence speed tricks”

#### Principle

*The limit of polynomials is determined by the terms with the fastest growth.*

### Growth hierarchy

$$\forall c \quad c \ll \log n \ll n^c \ll c^n \ll n! \ll n^n$$

For polynomials in the same growth category, the base resp. exponent counts ( $c < n^1 < n^2, \dots, n^\infty, 2^n < 3^n, \dots, (\infty)^n$ )

#### Example

For example for taking the limit of a fraction in the form

$$x_n = \frac{a_p n^p + a_{p-1} n^{p-1} \dots a_0 n^0}{b_q n^q + b_{q-1} n^{q-1} \dots b_0 n^0}$$

Only the terms with the highest degrees  $a_p n^p$  and  $b_q n^q$  determine the limit:

$$(p < q) \rightarrow (x_n \rightarrow 0)$$

$$(p = q) \rightarrow \left( x_n \rightarrow \frac{a_p}{a_q} \right)$$

$$(p > q) \rightarrow (x_n \rightarrow \pm \infty)$$

### Check monotony

Just check the difference between  $a_n$  and  $a_{n+1}$ :