Subspaces

Definition 13.1
We call vectors $v_1, .., v_k \in V$ basis *of* V *if:*

- $\text{span}\{v_1, ..., v_k\} = V$
- $\{v_1, ..., v_k\}$ *are minimal with this property*
  - $\Leftrightarrow \{v_1, ..., v_k\}$ *are linearly independent*
  - *i.e.* $\nexists \lambda_1, ..., \lambda_k \text{ st } \lambda_1 v_1, ... \lambda_k v_k = 0$

Definition 13.2
Any two bases of a vector space have the same number of vectors

Definition 13.3
The dimension *of a vector space* V *is the number of vectors in **any** basis of* V

Definition
a set of vectors are the basis of a vector space iff the matrix with the vectors as columns is invertible $\leftrightarrow$ *its determinant is non-zero.*

Notation for coordinates

Change of basis

$$P_{v \leftarrow w} = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix}$$

- Suppose v and w are two bases of $\mathbb{R}^n$
  - How to find $P_{v \leftarrow w}$ $w_i = a_{1i} v_1 + ...$

$$(v_1, ..., v_n) = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix}$$

$$(w_1, ..., w_n) = \begin{pmatrix} b_{11} & ... & b_{1n} \\ ... & ... & ... \\ b_{n1} & ... & b_{nn} \end{pmatrix}$$

DEF 14.1
Given vector space V *and* W *a function* $f : W \to V$ *is called **linear** if:*

- $f(w + w') = f(w) + f(w'), \quad \forall w, w' \in W$
- $f(\lambda w) = \lambda f(w), \quad \forall \lambda \in \mathbb{R}, \forall w \in \mathbb{W}$

Theorem 14.2
If a matrix a corresponds to a linear function $f : W \to V$ *with respect to bases w and w then*

$$[f(z)]_v = A[z]_w, \quad \forall z \in W$$

Theorem 14.3
$f \cdot g : Z \xrightarrow{g} W \xrightarrow{f} V$

$f \cdot g$ *corresponds to the matrix* $P_{Z \leftarrow W} P_{W \leftarrow V}$

Definition 14.4
$\text{Ker}(f) = \{w \in W \mid f(w) = O_w\} \subset W$
$\text{Im}(f) = \{v \in V \mid \exists w \in W \text{ st } f(w) = v\} \subset V$

$\text{Ker}(f)$ *and* $\text{Im}(f)$ *are subspaces of* W *and* V

Because:

- $O_w \in \text{Ker}(f)$
- $\forall w, w' \in \text{Ker}(f) h_2 w + w' \in \text{Ker}(f)$
- $\forall \lambda \in \mathbb{R}, \forall w \in \text{Ker}(f) h_2 \lambda w \in \text{Ker}(f)$

Analog to that with Im.

Theorem 14.5

- f *is injective* $\Leftrightarrow \text{Ker}(f) = \{o_w\}$
- f *is surjective* $\Leftrightarrow \text{Im}(f) = V$

$$\begin{aligned} f : \mathbb{R}^n \to \mathbb{R}^m, f(x) = Ax \quad & \text{where } A \in \mathbb{R}^{mn} \\ \text{Ker}(f) &= \{x \in \mathbb{R}^n \mid Ax = 0\} \\ &= \text{Ker}(A) = \text{Null}(A) \\ \text{Im}(f) &= \{b \in \mathbb{R}^m \mid Ax = b \text{ has solutions}\} \\ &= \text{Col}(A) = \text{span of columns/column space of } A \\ \text{Row}(A) &= \text{Col}(A^T) = \text{span of rows/row space of } A \end{aligned}$$

Problem

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \rightsquigarrow \mathrm{REF}(A) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find a basis for

$$\mathrm{Ker}(A) = \left\{ \begin{pmatrix} x_1 \\ \dots \\ x_4 \end{pmatrix} \mid A \begin{pmatrix} x_1 \\ \dots \\ x_4 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} -s-t \\ -s \\ s \\ t \end{pmatrix} \right\}_{s,t \in \mathbb{R}}$$

$$= \left\{ s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}_{s,t \in \mathbb{R}}$$

$$\mathrm{Ker}(A) = \mathrm{span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Dimension: $2 = \text{number free columns}$

Problem

Consider vector space spanned by $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \\ 2 \end{pmatrix} \right\}$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \rightsquigarrow \mathrm{REF}(A) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \mathrm{Col}(A^T) = \mathrm{Col}(\mathrm{REF}(A)^T)$$

Pivot rows of $\mathrm{REF}(A)$ are linearly independent $\implies$ pivot rows form a basis

A basis of V is given by $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Problem

$$V = \text{vector space spanned by } \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

Construct the matrix A whose columns are the vectors.

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 2 \end{pmatrix} \rightsquigarrow \mathrm{REF}(A) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \mathrm{Col}(A) \neq \mathrm{Col}(\mathrm{REF}(A))$$

but any linear dependency between the columns of A is reflected by linear dependency between the columns of $\mathrm{REF}(A)$.

The pivots of the $\mathrm{REF}(A)$ show which columns are linearly independent $\Leftrightarrow$ form the basis