

Important terms

Triangle inequality

$$|x + y| \leq |x| + |y|$$

Sequences

$$(x_n)_{n \geq 1} \in \mathbb{R}$$

Convergence

(x_n) converges to limit l iff

$$\forall \varepsilon > 0 \in \mathbb{R} \quad \exists N \in \mathbb{N} \text{ st } \forall n > N \in \mathbb{N} \quad |x_n - l| < \varepsilon$$

we write $\lim_{n \rightarrow \infty} (x_n) = l$ or $(x_n) \xrightarrow{n \rightarrow \infty} l$

If no such limit exists, (x_n) diverges.

Properties

$$\exists l \rightarrow \exists !l$$

(x_n) is convergent $\rightarrow (x_n)$ is bounded

Limit laws

If $(x_n) \rightarrow a$ and $(y_n) \rightarrow b$ then

- | | |
|---|--|
| (I)
(II)
(III)
(IV)
(V) | $x_n + y_n \rightarrow a + b$
$x_n - y_n \rightarrow a - b$
$x_n y_n \rightarrow ab$
$\frac{x_n}{y_n} \rightarrow \frac{a}{b} \quad \text{if } b \neq 0$
$ x_n \rightarrow a $ |
|---|--|

Monotone convergence theorem

x_n is bounded above (below) and monotone increasing (decreasing) $\rightarrow x_n$ converges.

Cauchy criterion

(x_n) converges

$$\leftrightarrow$$

$$\forall \varepsilon > 0 \quad \exists N \text{ st } \forall n > N \quad |x_n - x_m| < \varepsilon$$

Cauchy criterion

Let (x_n) be a sequence st $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = l$

Then

1. if $l < 1$ converges
2. if $l > 1$ diverges
3. if $l = 1$?

Squeeze theorem

Let $(x_n) \rightarrow a$ and $(y_n) \rightarrow a$.

If $x_n < a_n < y_n \forall n$, then $a_n \rightarrow a$

Series

A infinite sum (series) s_n of a sequence x_n is defined as a sequence of finite sums.

$$s_n = \sum_{i=0}^n x_n$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=0}^{\infty} (x_n)_i$$

Absolutely convergent $\leftrightarrow \sum_{i=0}^{\infty} |(x_n)_i|$ converges.

x_n is absolutely convergent $\rightarrow x_n$ converges

Thus: Signs don't matter when determining the convergence of a series

Examples

Constant sequence

$$x_n = c \quad c \neq 0$$

$$\rightarrow \lim_{n \rightarrow \infty} s_n = \pm\infty$$

Geometric series

$$x_n = q^n$$

$$s_n = \frac{1 - q^{n+1}}{1 - q}$$

$$= \frac{1}{1 - q} (1 - q^{n+1})$$

if $q < 1$ then s_n converges to $\frac{1}{1-q}$

Harmonic series

$$x_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

$$x_n = \frac{1}{n!}$$

$$\lim_{n \rightarrow \infty} s_n = \sum_{i=1}^{\infty} \frac{1}{n!} = e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)$$

Cheat sheet

Useful limits

$\frac{\sin(x)}{x}$	$\xrightarrow{x \rightarrow 0}$	1
$\frac{\log(1+x)}{x}$	$\xrightarrow{x \rightarrow 0}$	1
$\left(1 - \frac{1}{n}\right)^n$	$\xrightarrow{n \rightarrow \infty}$	e
$\left(1 + \frac{x}{n}\right)^n$	$\xrightarrow{n \rightarrow \infty}$	e^x

$$\exists N \text{ st } \forall n < N$$

$$a_{n+1} - a_n \begin{cases} > 0 \rightarrow \text{monotone increasing} \\ < 0 \rightarrow \text{monotone decreasing} \\ = 0 \rightarrow \text{constant sequence} \end{cases}$$

“Divergence speed tricks”

Principle

The limit of polynomials is determined by the terms with the fastest growth.

Growth hierarchy

$$\forall c \quad c \ll \log n \ll n^c \ll c^n \ll n! \ll n^n$$

For polynomials in the same growth category, the base resp. exponent counts ($c < n^1 < n^2, \dots, n^\infty, 2^n < 3^n, \dots, (\infty)^n$)

Example

For example for taking the limit of a fraction in the form

$$x_n = \frac{a_p n^p + a_{p-1} n^{p-1} \dots a_0 n^0}{b_q n^q + b_{q-1} n^{q-1} \dots b_0 n^0}$$

Only the terms with the highest degrees $a_p n^p$ and $b_q n^q$ determine the limit:

$$(p < q) \rightarrow (x_n \rightarrow 0)$$

$$(p = q) \rightarrow \left(x_n \rightarrow \frac{a_p}{b_q} \right)$$

$$(p > q) \rightarrow (x_n \rightarrow \pm\infty)$$

Check monotony

Just check the difference between a_n and a_{n+1} :