Analysis I - Summary

lrshsl

Important terms

Triangle inequality

$$\boxed{|x + y| \leq |x| + |y|}$$

Reverse Triangle inequality

$$\boxed{||x| - |y|| \leq |x - y|}$$

Sequences

$$(x_n)_{n\geq 1\in\mathbb{N}} \in \mathbb{R}$$

Convergence

Definition

(x_n) converges to limit l iff

$$\forall \varepsilon > 0 \in \mathbb{R}\ \ \exists N \in \mathbb{N} \text{ st } \forall n > N \in \mathbb{N}$$
$$|x_n - l| < \varepsilon$$

We write $\lim_{n\to\infty}(x_n) = l$ *or* $(x_n) \overset{n\to\infty}{\longrightarrow} l$

If no such limit exists, (x_n) diverges.

Properties

$$\exists l \to \exists! l$$

$$(x_n) \text{ is convergent} \to (x_n) \text{ is bounded}$$

Limit laws

If $(x_n) \longrightarrow a$ and $(y_n) \longrightarrow b$ then

$$\boxed{\begin{array}{lll} (I) & x_n + y_n \longrightarrow a + b & \\ (II) & x_n - y_n \longrightarrow a - b & \\ (III) & x_n y_n \longrightarrow ab & \\ (IV) & \dfrac{x_n}{y_n} \longrightarrow \dfrac{a}{b} & \text{if } b \neq 0 \\ (V) & |x_n| \longrightarrow |a| & \end{array}}$$

Attention

In order to apply these laws the limits of x_n and y_n need to exist and be finite.

Monotonicity

Difference criterion

For a real sequence (a_n) and $d = a_{n+1} - a_n$:

$$\begin{cases} d > 0 : \text{strictly increasing} \\ d < 0 : \text{strictly decreasing} \\ d = 0 : \text{(locally) constant} \end{cases}$$

If this is true $\forall n >$ some N, the sequence is eventually monotone. *One can also use the quotient* $\frac{a_{n+1}}{a_n}$

Monotone convergence theorem

x_n is bounded above (below) and monotone increasing (decreasing) $\to x_n$ converges.

Bolzano-Weierstrass Theorem

*Every **bounded** sequence in $\mathbb{R}$ has a **convergent subsequence**.*

Alternating sequence

Convergence of an alternating sequence

An alternating sequence converges iff its magnitude converges (to zero). Alternating sequences can only converge to zero.

$$\lim_{n\to\infty} a_n = 0 \Longleftrightarrow \lim_{n\to\infty} |a_n| = 0$$

Cauchy criterion

In $\mathbb{R}$:

$$(x_n) \text{ converges}$$
$$\Longleftrightarrow$$
$$\forall \varepsilon > 0 \quad \exists N \text{ st } \forall n, m > N$$
$$|x_n - x_m| < \varepsilon$$

Root limit test

Let (x_n) be a sequence st $\lim_{n\to\infty} \sqrt[n]{|x_n|} = l$

Then

1. if $l < 1$ converges
2. if $l > 1$ diverges
3. if $l = 1$ inconclusive

Squeeze theorem

Let $(x_n) \longrightarrow a$ and $(y_n) \longrightarrow a$.

$$\boxed{\text{If } x_n \leq a_n \leq y_n \forall n, \text{ then } a_n \longrightarrow a}$$

Series

A infinite sum (series) s_n of a sequence x_n is defined as a sequence of finite sums.

$$s_n = \sum_{i=0}^{n} (x_n)_i$$

$$\lim_{n\to\infty} s_n = \sum_{i=0}^{\infty} (x_n)_i$$

Absolutely convergent $\iff \sum_{i=0}^{\infty} |(x_n)_i|$ converges.

x_n is absolutely convergent $\implies x_n$ converges

Thus: *Signs don't matter when determining the absolute convergence of a series*

Series Tests

Nth-Term Test (for Divergence)

If $\lim_{n\to\infty} x_n \neq 0$ or $\nexists$ limit, then $\sum x_n$ **diverges**.

Warning

*If $\lim_{n\to\infty} x_n = 0$, this test is **inconclusive**.*

Comparison Tests

(For positive-term series, $x_n, y_n \geq 0$)

Direct Comparison

Let $0 \leq x_n \leq y_n$ for all large n

- $\sum y_n$ converges $\implies \sum x_n$ converges
- $\sum x_n$ diverges $\implies \sum y_n$ diverges

Limit Comparison

Let $L = \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right)$. If $0 < L < \infty$, then:

$$\sum x_n \quad \text{and} \quad \sum y_n$$

*either **both converge** or **both diverge**.*

Ratio Test

Let $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$

$$\begin{cases} L < 1 : \text{Series converges absolutely} \\ L > 1 : \text{Series diverges} \\ L = 1 : \text{Test is inconclusive} \end{cases}$$

Especially useful for factorials ($n!$) and exponentials (c^n).

Alternating Series Test (Leibniz)

AST

*The series $\sum (-1)^n x_n$ (with $x_n > 0$) converges if **both** are true:*

1. *$x_{n+1} \leq x_n$ for all large n (non-increasing)*
2. *$\lim_{n\to\infty} x_n = 0$*

Examples

Constant sequence

$$x_n = c \quad c \neq 0$$

$$\to \lim_{n\to\infty} s_n = \pm\infty$$

Geometric series

$$x_n = q^n$$

$$s_n = \frac{1 - q^{n+1}}{1-q} = \frac{1}{1-q}(1 - q^{n+1})$$

if $|q| < 1$ then s_n converges to $\frac{1}{1-q}$

Harmonic series

$$x_n = \frac{1}{n}$$

$$\lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

$$x_n = \frac{1}{n!}$$

$$\lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Cheat sheet

Useful limits

$$\boxed{\begin{array}{rcc} \dfrac{\sin(x)}{x} & \xrightarrow{x\to 0} & 1 \\ \dfrac{\log(1+x)}{x} & \xrightarrow{x\to 0} & 1 \\ \left(1+\dfrac{1}{n}\right)^n & \xrightarrow{n\to\infty} & e \\ \left(1+\dfrac{x}{n}\right)^n & \xrightarrow{n\to\infty} & e^x \end{array}}$$

"Divergence speed tricks"

Principle
The limit of polynomials is determined by the terms with the fastest growth.

Growth hierarchy

$$\boxed{\forall c \quad c \ll \log n \ll n^c \ll c^n \ll n! \ll n^n}$$

For polynomials in the same growth category, the base resp. exponent counts ($c < n^1 < n^2, ..., n^\infty$, $2^n < 3^n, ..., (\infty)^n$)

Example

For example for taking the limit of a fraction in the form

$$x_n = \frac{a_p n^p + a_{p-1} n^{p-1} .. a_0 n^0}{b_q n^q + b_{q-1} n^{q-1} .. b_0 n^0}$$

Only the terms with the highest degrees $a_p n^p$ and $b_q n_q$ determine the limit:

$$\begin{array}{lcl} (p < q) & \to & (x_n \longrightarrow 0) \\ (p = q) & \to & \left(x_n \longrightarrow \dfrac{a_p}{a_q}\right) \\ (p > q) & \to & (x_n \longrightarrow \pm\infty) \end{array}$$

Check monotony

Just check the difference between a_n and a_{n+1}:

$$\boxed{\begin{array}{c} \exists N \text{ st } \forall n > N \\ a_{n+1} - a_n \begin{cases} > 0 \to \text{monotone increasing} \\ < 0 \to \text{monotone decreasing} \\ = 0 \to \text{constant sequence} \end{cases} \end{array}}$$

Functions

Domain & Range (Co-Domain)

$$f : D \to R$$

Image

$$\operatorname{Im}(f) := \{f(x) \mid x \in D\}$$

Periodic functions

$$f : D \to R$$

Then:

$$\exists T > 0 \text{ st } \forall x \in D x + T \in D$$
$$\implies f(x) = f(x + T)$$

A periodic function can be defined on some interval and extended periodically:

Let $f(x)$ be a T-periodic function defined for $(0, T]$. $f(x)$ for $x = 3T + 1$ is $f(x - 3T) = f(1)$

Examples

Trigonometric functions

- $\sin(\alpha x), \cos(\alpha x)$ are $\frac{2\pi}{\alpha}$-periodic
- $\tan(\alpha x), \cot(\alpha x)$ are $\frac{\pi}{\alpha}$-periodic

Rational characteristic function

Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = \begin{cases} x \in \mathbb{Q} : 1 \\ \text{otherwise } 0 \end{cases}$

Then f is T-periodic $\forall T \in \mathbb{Q}$

Symmetric functions

A function is odd symmetric *if it is mirrored about the origin, and* even symmetric *if it is mirrored along the y-axis*

From any function $f(x)$ an odd/even symmetric function can be built, for example through:

$$\begin{cases} \text{odd} : \frac{f(x)+f(-x)}{2} \\ \text{even} : \frac{f(x)-f(-x)}{2} \end{cases}$$

The division by 2 is not necessary but convention

$$\boxed{\begin{gathered} f(x) = \text{even version of } f(x) + \\ \text{odd version of } f(x) \end{gathered}}$$

Examples

- cos is even symmetric
- sin is odd symmetric

Hyperbolic functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} : \text{odd}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2} : \text{even}$$