Lecture 13

Recall

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

- *converges if* $\alpha > 1$
- *diverges if* $\alpha \leq 1$

More general example

$$\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^5 + 2n^3 + 4n}} a_n$$

$$a_n = \frac{n\left(1 + \frac{1}{n}\right)}{\frac{n^5}{2}} \sqrt{1 + \frac{2}{n^2} + \frac{4}{n^4}} = \frac{1}{n^{\frac{3}{2}}} \left(\frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{2}{n^2} + \frac{4}{n^4}}} \right)$$

-> 1 as $n \to \infty$

=> $a_n < \frac{\frac{1}{n^3}}{2} \cdot 2$ for $n >$ same k

More convergence theorems

Alternating series

Definition

A series is called alternating if it has a form $\sum_{i=0}^{\infty} ((-1)^i b_i \vee \sum (-1)^{i+1} b_i$ *with* $b_i \geq 0 \quad \forall i$

Alternatively

A series is alternating if $a_k a_{k+1} \leq 0 \quad \forall k \in \mathbb{N}$

Theorem

Let $a_n = \sum_{i=0}^{\infty} a_i$ *be alternating st:*

1. $\lim_{i \Rightarrow \infty} a_i = 0$
2. $|a_{i+1}| \leq |a_i| \quad \forall i \in \mathbb{N}$

Then a_n *converges.*

Example

$$a_n = (-1)^n \frac{1}{n}$$

$$\sum_{n=0}^{\infty} a_n$$

1. Alternating: $a_n a_{n+1} \leq 0$
2. $\lim_{n \to \infty} a_n = 0$
3. $|a_{n+1}| = \frac{1}{n+1} < \frac{1}{n} = |a_n|$

Therefore $\sum_{i=0}^{\infty} \left((-1)^n \frac{1}{n}\right)$ converges.

Remark
*Alternating sequence criterion guarantees convergence but **not absolute** convergence*

E. g.
$\sum_{i=0}^{\infty}\left((-1)^n \frac{1}{n}\right)$ converges but it's absolute series does not (= harmonic sequence).

Cauchy's criterion

Let (x_n) be a sequence st $\lim_{n\to\infty} \sqrt[n]{|x_n|} = l$

Then

1. if $l < 1$ converges
2. if $l > 1$ diverges
3. if $l = 1$?

Examples

Sequences with $\lim \sqrt{|x_n|} = 1$

- $x_n = (-1)^n$: $\sum\limits_{i=0}^{\infty} x_i$ diverges but (s_n) is bound
- $x_n = n$: $\sum\limits_{i=0}^{\infty} x_i$ diverges but (s_n) is unbound
  - diverges to ∞

$x_n = \frac{1}{n}$, $\sum\limits_{i=0}^{\infty} x_i = +\infty$ -> diverges

$x_n = \frac{1}{n^2}$, $\sum\limits_{i=0}^{\infty} x_n$ converges absolutely

$x_n = \frac{(-1)^n}{n}$, $\sum\limits_{i=0}^{\infty} x_i$ converges but not absolutely

D'Alambert criterion

Let (x_n) be a sequence st $\lim |x_{n+1} \big| x_n|$

Example
D'Alamber:

$$\lim_{k\to\infty} |x_{k+1} \big| x_k| = \lim \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}}$$

$$= \lim \frac{k+1}{k} \frac{2^k}{2^{k+1}} = \lim \frac{k+1}{2k} = \frac{1}{2} < 1$$

=> By D'Alamber $\sum\limits_{k=0}^{\infty} \frac{k}{2^k}$ converges absolutely.