Important terms

Triangle inequality

$$|x + y| \leq |x| + |y|$$

Reverse Triangle inequality

$$\|x| - |y\| \leq |x - y|$$

Sequences

$$(x_n)_{n \geq 1 \in \mathbb{N}} \in \mathbb{R}$$

Convergence

(x_n) converges to limit l iff

$$\forall \varepsilon > 0 \in \mathbb{R} \quad \exists N \in \mathbb{N} \text{ st } \forall n > N \in \mathbb{N}$$
$$|x_n - l| < \varepsilon$$

we write $\lim_{n \to \infty} (x_n) = l$ or $(x_n) \overset{n \to \infty}{\longrightarrow} l$

If no such limit exists, (x_n) diverges.

Properties

$$\exists l \to \exists! l$$

$$(x_n) \text{ is convergent} \to (x_n) \text{ is bounded}$$

Limit laws

If $(x_n) \longrightarrow a$ and $(y_n) \longrightarrow b$ then

(I)	$x_n + y_n \longrightarrow a + b$
(II)	$x_n - y_n \longrightarrow a - b$
(III)	$x_n y_n \longrightarrow ab$
(IV)	$\dfrac{x_n}{y_n} \longrightarrow \dfrac{a}{b} \quad \text{if } b \neq 0$
(V)	$|x_n| \longrightarrow |a|$

Attention

In order to apply these laws the limits of x_n and y_n need to exist and be finite.

Monotonicity

Difference criterion

For a real sequence (a_n) and $d = a_{n+1} - a_n$:

$$\begin{cases} d > 0 : \text{strictly increasing} \\ d < 0 : \text{strictly decreasing} \\ d = 0 : \text{(locally) constant} \end{cases}$$

If this is true $\forall n >$ some N, the sequence is eventually monotone. One can also use the quotient $\frac{a_{n+1}}{a_n}$

Monotone convergence theorem

x_n is bounded above (below) and monotone increasing (decreasing) $\to x_n$ converges.

Bolzano-Weierstrass Theorem

*Every **bounded** sequence in $\mathbb{R}$ has a **convergent subsequence**.*

Alternating sequence

Definition

A series is called alternating if it has a form

$$\sum_{i=0}^{\infty} (-1)^i b_i \text{ or } \sum_{i=0}^{\infty} (-1)^{i+1} b_i$$

with $b_i \geq 0 \quad \forall i$

Alternative definition

A series is alternating if $a_k a_{k+1} \leq 0 \quad \forall k \in \mathbb{N}$

Alternating sequence convergence theorem

Let $a_n = \sum_{i=0}^{\infty} a_i$ be alternating st:
1. $\lim_{i \Rightarrow \infty} a_i = 0$
2. $|a_{i+1}| \leq |a_i| \quad \forall i \in \mathbb{N}$

Then a_n converges.

Cauchy criterion

$$
\boxed{
\begin{array}{c}
\text{In } \mathbb{R}: \\[4pt]
(x_n) \text{ converges} \\
\Longleftrightarrow \\
\forall \varepsilon > 0 \quad \exists N \text{ st } \forall n > N \\
|x_n - x_m| < \varepsilon
\end{array}
}
$$

Root limit test

Let (x_n) be a sequence st $\lim_{n\to\infty} \sqrt[n]{|x_n|} = l$

Then

1. if $l < 1$ converges
2. if $l > 1$ diverges
3. if $l = 1$ inconclusive

Squeeze theorem

Let $(x_n) \longrightarrow a$ and $(y_n) \longrightarrow a$.

$$
\boxed{\text{If } x_n \leq a_n \leq y_n \, \forall n, \text{ then } a_n \longrightarrow a}
$$

Series

A infinite sum (series) s_n of a sequence x_n is defined as a sequence of finite sums.

$$
s_n = \sum_{i=0}^{n} (x_n)_i
$$

$$
\lim_{n\to\infty} s_n = \sum_{i=0}^{\infty} (x_n)_i
$$

Absolutely convergent $\Longleftrightarrow \sum_{i=0}^{\infty} |(x_n)_i|$
converges.

x_n is absolutely convergent $\Longrightarrow x_n$ converges

$$
\boxed{\begin{array}{c}\text{Thus: } \textit{Signs don't matter when determining} \\ \textit{the absolute convergence of a series}\end{array}}
$$

Series Tests

Nth-Term Test (for Divergence)

$$
\boxed{\begin{array}{c}\text{If } \lim_{n\to\infty} x_n \neq 0 \text{ or } \nexists \text{ limit, then } \sum x_n \\ \textbf{diverges}.\end{array}}
$$

Warning

If $\lim_{n\to\infty} x_n = 0$, *this test is* ***inconclusive***.

Comparison Tests
(For positive-term series, $x_n, y_n \geq 0$)

Direct Comparison

Let $0 \leq x_n \leq y_n$ *for all large* n
- $\sum y_n$ converges $\Longrightarrow \sum x_n$ converges
- $\sum x_n$ diverges $\Longrightarrow \sum y_n$ diverges

Limit Comparison

Let $L = \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right)$. *If* $0 < L < \infty$,
then:

$$
\sum x_n \quad \text{and} \quad \sum y_n
$$

either ***both converge*** *or* ***both diverge***.

Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right|$

$$
\boxed{
\begin{cases}
L < 1 : \text{Series converges absolutely} \\
L > 1 : \text{Series diverges} \\
L = 1 : \text{Test is inconclusive}
\end{cases}
}
$$

Especially useful for factorials (n!) and exponentials (c^n).

Alternating Series Test (Leibniz)

AST

The series $\sum (-1)^n x_n$ *(with* $x_n > 0$*) converges if* ***both*** *are true:*
1. $x_{n+1} \leq x_n$ *for all large* n *(non-increasing)*
2. $\lim_{n\to\infty} x_n = 0$

Examples

Constant sequence

$$x_n = c \quad c \neq 0$$

$$\rightarrow \lim_{n \to \infty} s_n = \pm\infty$$

Geometric series

$$x_n = q^n$$

$$s_n = \frac{1 - q^{n+1}}{1 - q}$$

$$= \frac{1}{1 - q}(1 - q^{n+1})$$

if $q < 1$ then s_n converges to $\frac{1}{1-q}$

Harmonic series

$$x_n = \frac{1}{n}$$

$$\lim_{n \to \infty} s_n = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

$$x_n = \frac{1}{n!}$$

$$\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n!} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Cheat sheet

Useful limits

$$\frac{\sin(x)}{x} \xrightarrow{x \to 0} 1$$

$$\frac{\log(1 + x)}{x} \xrightarrow{x \to 0} 1$$

$$\left(1 + \frac{1}{n}\right)^n \xrightarrow{n \to \infty} e$$

$$\left(1 + \frac{x}{n}\right)^n \xrightarrow{n \to \infty} e^x$$

$$\exists N \text{ st } \forall n > N$$

$$a_{n+1} - a_n \begin{cases} > 0 \to \text{monotone increasing} \\ < 0 \to \text{monotone decreasing} \\ = 0 \to \text{constant sequence} \end{cases}$$

"Divergence speed tricks"

> **Principle**
> *The limit of polynomials is determined by the terms with the fastest growth.*

Growth hierarchy

$$\forall c \quad c \ll \log n \ll n^c \ll c^n \ll n! \ll n^n$$

For polynomials in the same growth category, the base resp. exponent counts ($c < n^1 < n^2, ..., n^\infty, 2^n < 3^n, ..., (\infty)^n$)

Example
For example for taking the limit of a fraction in the form

$$x_n = \frac{a_p n^p + a_{p-1} n^{p-1}..a_0 n^0}{b_q n^q + b_{q-1} n^{q-1}..b_0 n^0}$$

Only the terms with the highest degrees $a_p n^p$ and $b_q n_q$ determine the limit:

$$(p < q) \quad \to \quad (x_n \longrightarrow 0)$$

$$(p = q) \quad \to \quad \left(x_n \longrightarrow \frac{a_p}{a_q}\right)$$

$$(p > q) \quad \to \quad (x_n \longrightarrow \pm\infty)$$

Check monotony
Just check the difference between a_n and a_{n+1}: