1.

Exercise

Let $a_n = \frac{\sin\left(\frac{1}{n}\right)}{n}$, starting from $n = 1$ Is a_n monotone? Is it convergent / divergent?

Direct solution

$$\lim_{n\to\infty} a_n = \lim\left(\sin\left(\frac{1}{n}\right)\right)\frac{1}{\lim n}$$

$$= \lim\left(\sin\left(\frac{1}{n}\right)\right)\lim\left(\frac{1}{n}\right)$$

$$= \lim\left(\sin\left(\frac{1}{n}\right)\right)0$$

$$= 0$$

Alternative solution using sqeeze theorem

Since

$$-\frac{1}{n} \leq \frac{\sin\left(\frac{1}{n}\right)}{n} \leq \frac{1}{n}$$

trivially holds, and $\lim \frac{1}{n} = \lim -\frac{1}{n} = 0$, we have $\lim a_n = 0$

2a

Exercise

Consider the sequence $a_n = \frac{n}{4n-1}$. Is it monotone? Is it convergent / divergent?

Intuition

- monotonely decreasing
- converges

Solution

$$a_n = n\left(\frac{1}{4n-1}\right)$$

We can factor out the highest degree (n^1)

$$a_n = \frac{1}{4 - \frac{1}{n}}$$

as $\frac{1}{n} \xrightarrow{n\to\infty} 0$, we have

$$a_n \longrightarrow \frac{1}{4-0} = \frac{1}{4}$$

2b

Exercise

Check if the sequence $a_n = (-1)^n \frac{n^2+\pi}{n}$, starting from $n = 1$

- is monotone
- converges or diverges
- and what its limit is

Solution

Monotonicity

Since $a_n a_{n+1} \leq 0 \forall n \geq 1 \in \mathbb{N}$ due to the $(-1)^n$ term, a_n is alternating, which means it cannot be monotone.

(Con|Div)ergence & Limit

a_n can be split into subsequences

$$\begin{cases} \text{odd } n : -\frac{n^2+\pi}{n} \\ \text{even } n : \frac{n^2+\pi}{n} \end{cases}$$

And because $n^2 \gg n$ and n^2 is in the numerator

$$\frac{n^2 + \pi}{n} \longrightarrow +\infty$$

$$-\frac{n^2 + \pi}{n} \longrightarrow -\infty$$

as $\frac{a}{b} \forall a, ba \gg b \implies \frac{a}{b} \longrightarrow +\infty$.

Upshot

a_n consists of two subsequences $a_n \forall$ even $n \longrightarrow +\infty$ and $a_n \forall$ odd $n \longrightarrow -\infty$. Thus a_n diverges, is unbound and does not have a limit.

3a

$$a_n = \frac{5n^2 - 3n + 2}{3n^2 + 7}$$

$$= \frac{n^2\left(5 - \frac{3}{n} + \frac{2}{n^2}\right)}{n^2\left(3 + \frac{7}{n^2}\right)}$$

$$= \frac{5 - \frac{3}{n} + \frac{2}{n^2}}{3 + \frac{7}{n^2}}$$

We can substitute $a_n = \frac{b}{c}$ which through limit law IV means $a_n \longrightarrow \lim \frac{b}{\lim} c$ and since $\frac{c}{n^k} \longrightarrow 0 \forall c \in \mathbb{R}, k \in \mathbb{N}$ we can neglect all the quotients in this form due to limit law I

$$\lim_{n \to \infty} a_n = \frac{5 - 0 + 0}{3 + 0} = \frac{5}{3}$$

The limit is $\frac{5}{3}$

3b

$$a_n = (-1)^n \frac{\sqrt[4]{n}}{\sqrt[3]{n}}$$

$$a_n = \begin{cases} n \text{ odd} : -\frac{\sqrt[4]{n}}{\sqrt[3]{n}} \\ n \text{ even} : \frac{\sqrt[4]{n}}{\sqrt[3]{n}} \end{cases}$$