Automated Elementary Geometry Theorem Discovery via Inductive Diagram Manipulation

by

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Abstract

In this thesis, I created and analyzed an interactive computer system capable of exploring geometry concepts through inductive investigation. My system begins with a limited set of knowledge about basic geometry and enables a user interacting with the system to "teach" the system additional geometry concepts and theorems by suggesting investigations the system should explore to see if it "notices anything interesting." The system uses random sampling and physical simulations to emulate the more human-like processes of manipulating diagrams "in the mind's eye." It then uses symbolic pattern matching and a propagator-based truth maintenance system to appropriately generalize findings and propose newly discovered theorems. These theorems can be rigorously proved using external proof assistants, but also be used by the system to assist in its explorations of new, higher-level concepts. Through a series of simple investigations similar to an introductory course in geometry, the system has been able to propose and learn a few dozen standard geometry theorems, and through more self-directed explorations, it has discovered several interesting properties and theorems not typically covered in standard mathematics courses.

Thesis Supervisor: Gerald J. Sussman
Title: Panasonic Professor of Electrical Engineering

Acknowledgments

This is the acknowledgements section. You should replace this with your own acknowledgements.

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Chapter 1

Introduction

In this thesis, I develop and analyze an interactive computer system that emulates a student learning geometry concepts through inductive investigation. Although geometry knowledge can be conveyed via a series of factual definitions, theorems, and proofs, my system focuses on a more investigative approach in which an external teacher guides the student to "discover" new definitions and theorems via explorations and self-directed inquiry.

My system emulates such a student by beginning with a fairly limited knowledge set regarding basic definitions in geometry and providing a means by which a user interacting with the system can "teach" additional geometric concepts and theorems by suggesting investigations the system should explore to see if it "notices anything interesting."

To enable such learning, my project includes the combination of four intertwined modules: an imperative geometry construction interpreter to build constructions, a declarative geometry constraint solver to solve and test specifications, an observation-based perception module to notice interesting properties, and a learning module to analyze information from the other modules and integrate it into new definition and theorem discoveries.

To evaluate its recognition of such concepts, my system provides means for a user to extract the observations and apply its findings to new scenarios. Through a series of simple investigations similar to an introductory course in geometry, the system has been able to propose and learn a few dozen standard geometry theorems. Furthermore, through more self-directed explorations, it has discovered several interesting properties and theorems not typically covered in standard mathematics courses.

1.1 Document Structure

Chapter 2 further discusses motivation of the system and presents some examples of diagram manipulation, emphasizing the technique of visualizing diagrams "in the mind's eye."

Chapter 9 discusses some related work to automated geometry theorem discovery and proof, as well as a comparison with existing dynamic geometry systems.

Chapter 4 further introduces the system modules and discusses how they work together in the discovery of new definitions and theorems.

Chapter 6 describes the implementation and function of the imperative construction module that enables the system to carry out constructions.

Chapters 8 describes the implementation and function of the propagator-based declarative geometry constraint solver that builds instances of diagrams satisfying declarative constraints.

Chapter 7 describes the implementation and function of the perception module focused on observing interesting properties in diagrams. A key question involves determining "what is interesting".

Chapter 5 describes the analyzer module which integrates results from the other systems to create new discoveries. Main features include filtering out obvious or known results to focus on the most interesting discoveries, the persistence and storage of definitions and theorems, and an interface to apply these findings to new situations.

Chapter 10 discusses several of the definitions and theorems results the overall system has been able to discover and learn.

Chapter 11 evaluates the strengths and weaknesses of the system. Future work and possible extensions are discussed.

Chapter 2

Motivation and Examples

Understanding elementary geometry is a fundamental reasoning skill, and encompasses a domain both constrained enough to model effectively, yet rich enough to allow for interesting insights. Although elementary geometry knowledge can be conveyed via series of factual definitions, theorems, and proofs, a particularly intriguing aspect of geometry is the ability for students to learn and develop an understanding of core concepts through visual investigation, exploration, and discovery.

These visual reasoning skills reflect many of the cognitive activities used as one interacts with his or her surroundings. Day-to-day decisions regularly rely on visual reasoning processes such as imagining what three dimensional objects look like from other angles, or mentally simulating the effects of one's actions on objects based on a learned understanding of physics and the object's properties. Such skills and inferred rules are developed through repeated observation, followed by the formation and evaluation of conjectures.

Similar to such day-to-day three-dimensional reasoning, visualizing and manipulating 2D geometric diagrams "in the mind's eye" allows one to explore questions such as "what happens if…" or "is it always true that…" to discover new conjectures. Further investigation of examples can increase one's belief in such a conjecture, and an accompanying system of deductive reasoning from basic axioms could prove that an observation is correct.

As an example, a curious student might notice that in a certain drawing of a triangle, the three perpendicular bisectors of the edges are concurrent, and that a circle constructed with center at the point of concurrence intersects all three vertices of the triangle. Given this “interesting observation”, the student might explore other triangles to see if this behavior is just coincidence, or conjecture about whether it applies to certain classes of triangles or all triangles in general. After investigating several other examples, the student might have sufficient belief in the conjecture to explore using previously-proven theorems (in this case, correspondences in congruent triangles) to prove the conjecture. My proposed project is a software system that simulates and automates this inductive thought process.

Automating geometric reasoning is not new, and has been an active field in computing and artificial intelligence. Dynamic geometry software, automated proof assistants, deductive databases, and several reformulations into abstract algebra models have been proposed in the last few decades. Although many of these projects have focused on the end goal of obtaining rigorous proofs of geometric theorems, I am particularly interested in exploring and modeling the more creative human-like thought processes of inductively exploring and manipulating diagrams to *discover* new insights about geometry.

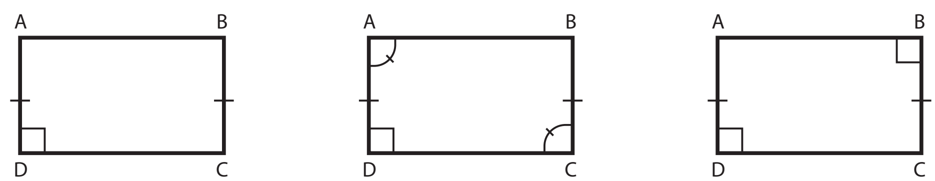
The interactive computer system presented in this thesis emulates the curious student described above, and is capable of exploring geometric concepts through inductive investigation. The system begins with a fairly limited set of factual knowledge regarding basic definitions in geometry and provides means by which a user interacting with the system can “teach” the system additional geometric concepts and theorems by suggesting investigations the system should explore to see if it “notices anything interesting.”

To evaluate its recognition of such concepts, the interactive system provide means for a user to extract the observations and apply such findings to new scenarios. In addition to the automated reasoning and symbolic artificial intelligence aspects of a system that can learn and reason inductively about geometry, the project also has some interesting opportunities to explore educational concepts related to experiential learning, and several extensions to integrate it with existing construction synthesis and proof systems.

2.1 Manipulating Diagrams "In the Mind's Eye"

Although the field of mathematics has developed a rigorous structure of deductive proofs explaining most findings in geometry, much of human intuition and initial reasoning about geometric ideas come not from applying formal rules, but rather from visually manipulating diagrams "in the mind's eye." Consider the following example:

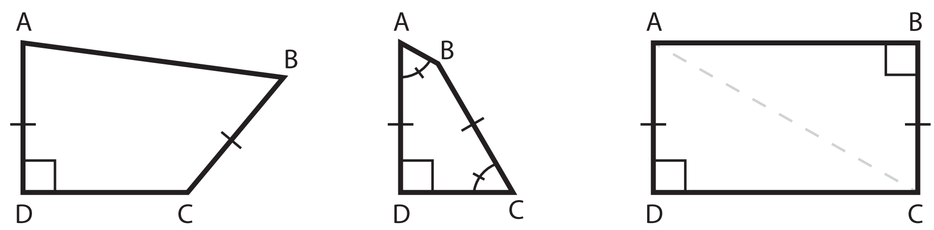
2.1.1 An Initial Example



Example 1: Of the three diagrams above, determine which have constraints sufficient to restrict the quadrilateral $ABCD$ to always be a rectangle.

An automated deductive solution to this question could attempt to use forward-chaining of known theorems to determine whether there was a logical path that led from the given constraints to the desired result that the quadrilateral shown is a rectangle. However, getting the correct results would require having a rich enough set of inference rules and a valid logic system for applying them.

A more intuitive visual-reasoning approach usually first explored by humans is to initially verify that the marked constraints hold for the instance of the diagram as drawn and then mentally manipulate or "wiggle" the diagram to see if one can find a nearby counter-example that still satisfies the given constraints, but is not a rectangle. If the viewer is unable to find a counter-example after several attempts, he or she may be sufficiently convinced the conclusion is true, and could commit to exploring a more rigorous deductive proof.



Solution to Example 1: As the reader likely discovered, the first two diagrams can be manipulated to yield instances that are not rectangles, while the third is sufficiently constrained to always represent a rectangle. (This can be proven by adding a diagonal and using the Pythagorean theorem.)

2.1.2 Diagrams, Figures, and Constraints

This example of manipulation using the "mind's eye" also introduces some terminology helpful in discussing the differences between images as drawn and the spaces of geometric objects they represent. For clarity, a *figure* will refer to an actual configuration of points, lines, and circles drawn on a page. Constraint annotations (congruence or measure) added to a figure create a *diagram*, which represents the entire space of figure *instances* that satisfy the constraints.

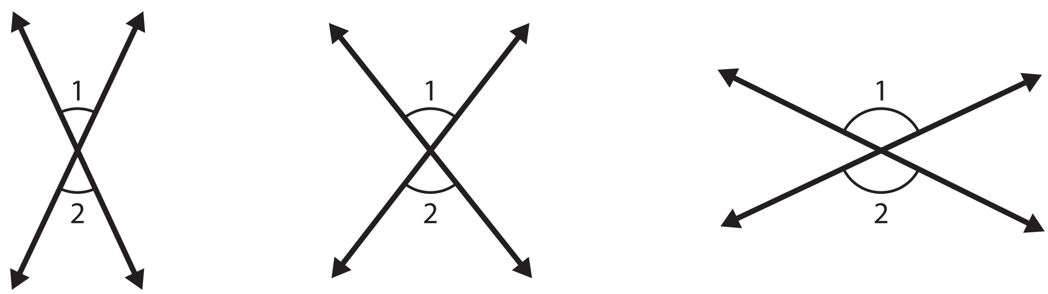
An annotated figure presented on a page is typically an instance of its corresponding diagram. However, it is certainly possible to add annotations to a figure that are not satisfied by that figure, yielding impossible diagrams. In such a case the diagram represents an empty set of satisfying figures.

In the initial example above, the three quadrilaterals figures are drawn as rectangles. It is true that all quadrilateral figures in the space represented by the third diagram are rectangles. However, the space of quadrilaterals represented by the first two diagrams include instances that are not rectangles, as shown above. At this time, the system only accepts diagrams whose constraints are satisfied in a given figure. However, detecting and explaining impossible diagrams, purely from their set of constraints could be an interesting extension.

2.2 Geometry Investigation

These same "mind's eye" reasoning techniques can be used to discover and learn new geometric theorems. Given some "interesting properties" in a particular figure, one can construct other instances of the diagram to examine if the properties appear to hold uniformly, or if they were just coincidences in the initial drawing. Properties that are satisfied repeatedly can be further explored and proved using deductive reasoning. The examples below provide several demonstrations of such inductive investigations.

2.2.1 Vertical Angles

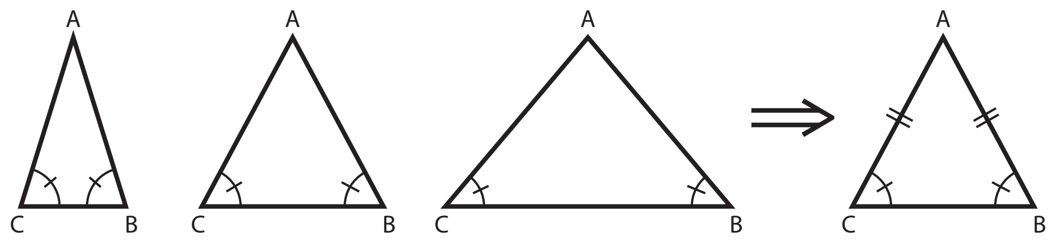


Investigation 1: Construct a pair of vertical angles. Notice anything interesting?

Often one of the first theorems in a geometry course, the fact that vertical angles are equal is one of the simplest examples of applying "mind's eye" visual reasoning. Given the diagram on the left, one could "wiggle" the two lines in his or her mind and imagine how the angles respond. In doing so, one would notice that the lower angle's measure increases and decreases proportionately with that of the top angle. This mental simulation, perhaps accompanied by a few drawn and measured figures, could sufficiently convince the viewer that vertical angles always have equal measure.

Of course, this fact can also be proved deductively by adding up pairs of angles that sum to 180 degrees, or by using a symmetry arguments. However, the inductive manipulations are more reflective of the initial, intuitive process one typically takes when first presented with understanding a problem.

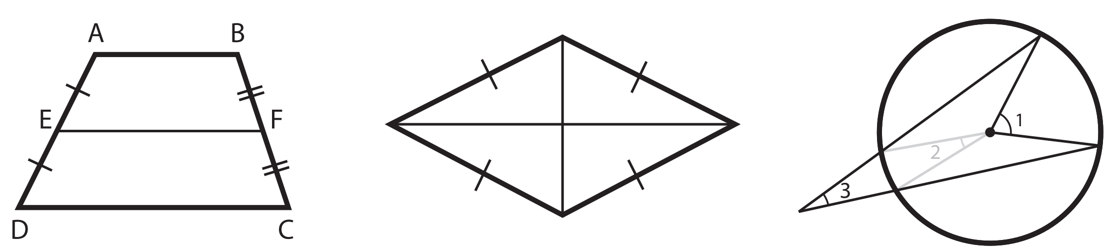
2.2.2 Elementary Results



Investigation 2: Construct a triangle ABC with $\angle B = \angle C$. Notice anything interesting?

A slightly more involved example includes discovering that if a triangle has two congruent angles, it is isoceles. As above, this fact has a more rigorous proof that involves dropping an altitude from point A and using corresponding parts of congruent triangles to demonstrate the equality of AB and AC. However, the inductive investigation of figures that satisfy the constraints can yield the same conjecture, give students better intuition for what is happening, and help guide the discovery and assembly of known rules to be applied in future situations.

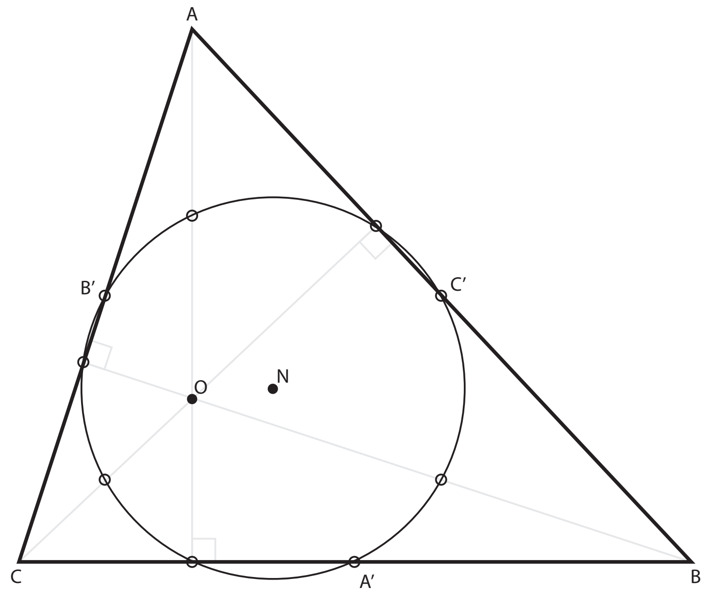
In this and further examples, an important question becomes what properties are considered "interesting" and worth investigating in further instances of the diagram, as discussed in section 4.3.3. As suggested by the examples in Investigation 3, this can include relations between segment and angle lengths, concurrent lines, collinear points, or parallel and perpendicular lines.



Investigation 3: What is interesting about the relationship between AB, CD, and EF in the trapezoid? What is interesting about the diagonals of a rhombus? What is interesting about $\angle 1$, $\angle 2$, and $\angle 3$?

2.2.3 Nine Point Circle and Euler Segment

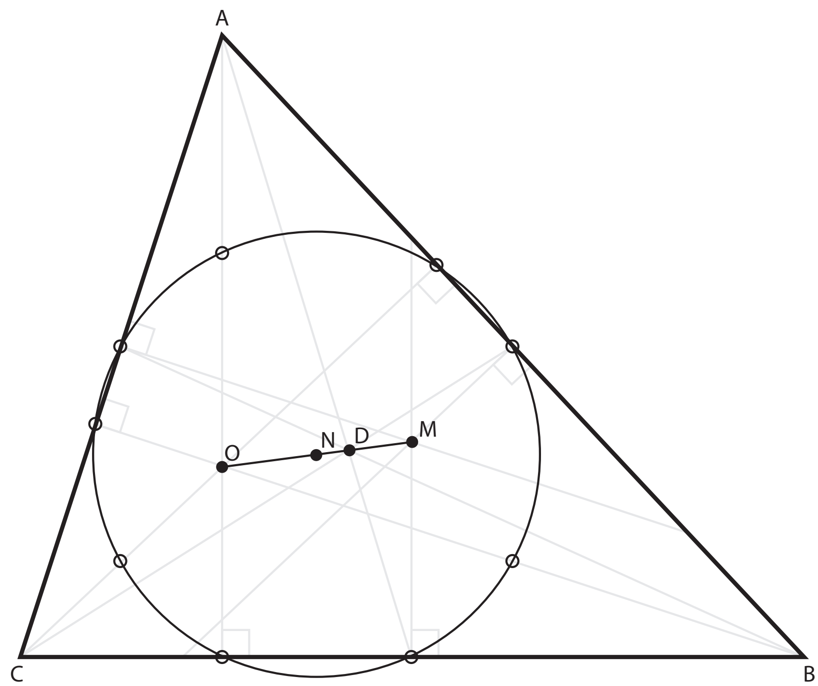
Finally, this technique can be used to explore and discover conjectures well beyond the scope of what one can visualize in his or her head:



Investigation 4a: In triangle ABC, construct the side midpoints A', B', C', and orthocenter O (from altitudes). Then, construct the midpoints of the segments connecting the orthocenter with each triangle vertex. Notice anything interesting?

As a more complicated example, consider the extended investigation of the Nine Point Circle and Euler Segment. As shown in Investigation 4a, the nine points created (feet of the altitudes, midpoints of sides, and midpoints of segments from orthocenter to vertices) are all concentric, lying on a circle with center labeled N.

Upon first constructing this figure, this fact seems almost beyond chance. However, as shown in Investigation 4b (below), further "interesting properties" continue to appear as one constructs the centroid and circumcenter: All four of these special points (O, N, D, and M) are collinear on what is called the *Euler Segment*, and the ratios $ON : ND : DM$ of $3 : 1 : 2$ hold for any triangle.



Investigation 4b: Continue the investigation from 4a by also constructing the centroid D (from medians) and circumcenter M (from perpendicular bisectors). Notice anything interesting?

(Maybe I'll try to add in some more concluding remarks about this "mind's eye" concept.)

Chapter 3

Demonstration

My system uses this idea of manipulating diagrams "in the mind's eye" to explore and discover geometry theorems. Before discussing some of the internal representations and modules, I will briefly describe the goals of the system to provide direction and context to understand the components.

Listing 3.1: Getting labels

```
(define (arbitrary-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)))
```

Listing 3.2: Constraint Solving for Isoceles Triangle

```
(define (isoceles-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-length-equal (m:bar 'a 'b)
                     (m:bar 'b 'c))))
```

Listing 3.3: Constraint Solving for Isoceles Triangle

```
(define (parallelogram-by-angles)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'c))
```

```
  (m:c-angle-equal (m:joint 'b)
                   (m:joint 'd))))
```

Chapter 4

System Overview

My system uses this idea of manipulating diagrams "in the mind's eye" to explore and discover geometry theorems. Before discussing some of the internal representations and modules, I will briefly describe the goals of the system to provide direction and context to understand the components.

4.1 Goals

The end goal of the system is for it to be to notice and learn interesting concepts in Geometry from inductive explorations.

Because these ideas are derived from inductive observation, we will typically refer to them as conjectures. Once the conjectures are reported, they can easily be integrated into existing automated proof systems if a deductive proof is desired.

The conjectures explored in this system can be grouped into three areas: definitions, properties, and theorems.

Properties Properties include all the facts derived from a single premise. "Opposite angles in a rhombus are equal" or "The midpoint of a segment divides it into two equal-length segments".

Definitions Definitions classify and differentiate an object from other objects. For instance"What is a rhombus?" yields the definition that it is a quadrilateral (classification) with four equal sides (differentiation). For definitions, the system will attempt to simplify definition properties to more minimal sets, provide alternative formations, and use pre-existing definitions when possible: "A Square is a rhombus and a rectangle"

Theorems Theorems are very similar to properties but involve several premises. For instance, theorems about triangles may involve the construction of angle bisectors, incenters or circumcenters, or the interaction among several polygons in the same diagram.

Finally, given a repository of these conjectures about geometry, the system will be able to apply its findings in future investigations by examining elements to display its knowledge of definitions, and focusing future investigations by omitting results implied by prior theorems.

4.2 Diagram Representations

The system and modules are built around three core representations. As discussed in the motivation section, we use the term "diagram" to represent the abstract geometric object represented by these means:

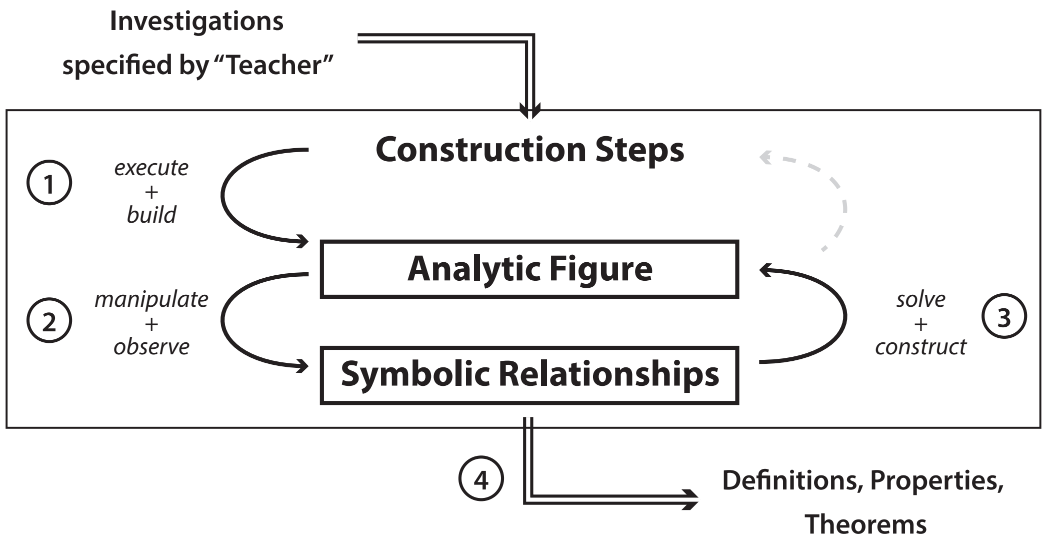
Construction Steps The main initial representation of most diagrams is a series of construction steps. These generally make up the input investigation from an external user trying to teach the system a concept. In some investigations, the actual construction steps are opaque to the system (as in a teacher that provides a process to "magically" produce rhombuses), but often, the construction steps use processes known by the system so that the resulting figures can include dependency information about how the figure was built.

Analytic Figure The second representation is an analytic figure for a particular instance of a diagram. This representation can be drawn and includes coordinates for all points in the diagram. This representation is used by the perception module to observe interesting relationships.

Symbolic Relationships Finally, the third representation is a collection of symbolic relationships or constraints on elements of the diagram. These are initially formed from the results of the perception module, but may also be introduced as known properties for certain premises and construction steps. These symbolic relationships can be further tested and simplified to discover which sets of constraints subsume one another.

While construction steps are primarily used as input and to generate examples, as the system investigates a figure, the analytic figure and symbolic relationship models get increasingly intertwined. The "mind's eye" perception aspects of observing relationships in the analytic figure lead to new symbolic relationships and a propagator-like approach of wigging solutions to the symbolic constraints yields new analytic figures.

As relationships are verified and simplified, results are output and stored in the student's repository of geometry knowledge. This process is depicted in the figure below and components are described in the following chapters.



System Overview: Given construction steps for an investigation an external teacher wishes the student perform, the system first (1) uses its imperative construction module to execute these construction steps and build an analytic instance of the diagram. Then, (2) it will manipulate the diagram by "wiggling" random choices and use the perception module to observe interesting relationships. Given these relationships, it will (3) use the declarative propagator-based constraint solver to reconstruct a diagram satisfying a subset of the constraints to determine which are essential in the original diagram. Finally (4), a learning module will monitor the overall process, omit already-known results, and assemble a repository of known definitions, properties, and theorems.

4.2.1 Modules

These four modules include an imperative geometry construction interpreter used to build diagrams, a declarative geometry constraint solver to solve and test specifications, an observation-based perception module to notice interesting properties, and a learning module to analyze information from the other modules and integrate it into new definition and theorem discoveries.

4.3 Sample Interaction

This core system provides an interpreter to accept input of construction instructions, an analytic geometry system that can create instances of such constructions, a pattern-finding process to discover "interesting properties", and an interface for reporting findings.

4.3.1 Interpreting Construction Steps

The first step in such explorations is interpreting an input of the diagram to be explored. To avoid the problems involved with solving constraint systems and the possibility of impossible diagrams, the core system takes as input explicit construction steps that results in an instance of the desired diagram. These instructions can still include arbitrary selections (let P be some point on the line, or let A be some acute angle), but otherwise are restricted to basic construction operations using a compass and straight edge.

To simplify the input of more complicated diagrams, some of these steps can be abstracted into a library of known construction procedures. For example, although the underlying figures are be limited to very simple objects of points, lines, and angles, the steps of constructing a triangle (three points and three segments) or bisecting a line or angle can be encapsulated into single steps.

4.3.2 Creating Figures

Given a language for expressing the constructions, the second phase of the system is to perform such constructions to yield an instance of the diagram. This process mimics "imagining" manipulations and results in an analytic representation of the figure with coordinates for each point. Arbitrary choices in the construction ("Let Q be some point not on the line.") are chosen via an random process, but with an attempt to keep the figures within a reasonable scale to ease human inspection.

4.3.3 Noticing Interesting Properties

Having constructed a particular figure, the system examines it to find interesting properties. These properties involve facts that appear to be "beyond coincidence". This generally involves relationships between measured values, but can also include "unexpected" configurations of points, lines, and circles. As the system discovers interesting properties, it will reconstruct the diagram using different choices and observe if the observed properties hold true across many instances of a diagram.

4.3.4 Simplifying Definitions and Known Facts

4.3.5 Reporting Findings

Finally, once the system has discovered some interesting properties that appear repeatedly in instances of a given diagram, it reports its results to the user via the learning module. Although this includes a simple list of all simple relationships, effort is taken to avoid repeating observations that obvious in the construction. For example, if a perpendicular bisector of segment AB is requested, the fact that it bisects that segment in every instance is not informative. To do so, the construction process interacts with properties known in the learning module to maintain a list of facts that can be reasoned from construction assumptions so that these can be omitted in the final reporting.

4.4 Example Interaction

[For now see walkthrough in the "results" chapter. Will add a good, simple example here]

Chapter 5

Learning Module

5.1 Overview

The Learning and learning module is one the core elements integrating information from the other components of the system. It maintains

5.2 Interactions with Learning Module

The learning module provides the primary interface by which users interact with the system. As such, it provides means by which users can both query the system to discover and use what it has known, as well as to teach the system information by suggesting investigations it should undertake.

5.2.1 Querying

A simple way of interacting with the learning module is to ask it for what it knows about various geometry concepts or terms. For definitions, the results provide the classification (that a rhombus is a parallelogram), and a set of minimal properties that differentiates that object from its classification. Further querying can present all known properties of the named object as well as theorems involving that term.

```
(what-is 'rhombus)
```

5.2.2 Learning Definitions

To learn a new definition, the system must be given the name of the term being learned as well as a procedure that will generate arbitrary instances of that definition. To converge to the correct definition, that random procedure should present a wide diversity of instances (i.e. the random-parallelogram procedure should produce all sorts of parallelograms, not just rectangles). However, reconciling mixed information about what constitutes a term could be an interesting extension.

```
(learn-term ’parallelogram random-parallelogram)
```

5.2.3 Applying Learned Properties

To apply the learned conjectures, the learning system can use its repository of geometry knowledge to tell you what a given figure is or to point out old vs. new properties seen in a specific figure:

```
(analyze-figure figure)
(examine (random-rhombus))
```

5.2.4 Performing Investigations

Investigations are similar to analyzing various figures above except that they have the intent of the analysis results being placed in the geometry knowledge repository. This separation also allows for dependence information about where properties were derived from.

5.3 Representing Discoveries

Discoveries are represented within a lattice of premises (discoveries about quadrilaterals $<$ discoveries about rhombuses $<$ discoveries about squares, but are separate from discoveries about circles or segments).

5.3.1 Placement of discoveries

Given this lattice structure, an interesting question when exploring new properties

5.3.2 Ordering of discoveries

An issue with this system is that often discoveries can be in slightly different formats. As such, for each relationship, we establish a consistent ordering of elements and use pattern matching to

For example, assertions about equality of segments $|AB| = |CD|$ are independent of the ordering of points within the elements.

5.3.3 Pattern Matching against existing conjectures

Based on dependencies, we replace the lowest-level random dependencies with arbitrary pattern elements `(?  s1 ,segment)` for instance. Then, when new conjectures are being considered, we attempt to pattern match based on existing elements to see if there is a redundant observation.

Chapter 6

Imperative Construction System

6.1 Overview

The first module is an imperative system for performing geometry constructions. This is the typical input method for generating coordinate-backed instances of figures and thus declares.

We will first discuss the basic underlying structures that comprise figures, then describe the higher-order language used to specify construction steps.

6.2 Basic Structures

6.2.1 Points

Points form the basis of most elements. Throughout the system, points are labeled and used to identify other elements.

6.3 Linear Elements

The linear elements of Segments, Lines, and Rays are built upon points. Initially the internal representation of lines were that of two points, but to simplify manipulations,

To better specify angles (see below), all linear elements, including segments and lines are directioned. Thus, a line pointing. Predicates exist that compare lines for equality ignoring

6.3.1 Angles

Initially angles were represented as three points, now vertex + two directions. CCW orientation. Methods exist to determine them from various pairs of linear elements, uses directionality of linear elements to determine which "quadrant" of the angle is desired.

Given a figure, methods exist to extract angles from the diagrams in analysis rather than specifying each angle of interest while creating the diagram.

6.3.2 Math Support

Some "core" math structures to help these calculations: Direction represents a direction in [0, 2pi], fixes principal value [0, 2pi], and support various operations for direction intervals (basic intersection, adding, shifting, etc). Currently all represented by single theta value, could generalize via generics to dx, dy, or theta depending on computation source.

6.4 Higher-level structures

In addition to the basic geometry structures, the system uses several grouping structures to combine and abstract the basic figure elements into higher-level figures elements.

For closure of combinators, all these higher level objects are also "Diagram objects".

6.4.1 Polygons

Polygons are represented as groups of points.

6.4.2 Figures

Figures are currently groups of elements. In the creation of figures we extract additional information and build a graph out of adjacent components for use in the analysis stages.

6.5 Construction Operations

6.5.1 Traditional constructions

Midpoint, perpendicular line, bisectors

6.5.2 Intersections

Generic intersections, mathematically based at line/line or line/circle at the core. Other intersections also add the check that the resulting point(s) are on the elements.

6.5.3 Measurement-based operations

A "Ruler + Protractor" is generally not permitted in traditional construction problems. However, sometimes its nice to be able to use measurements to more quickly compute a result (e.g. angle bisector by halving angle) vs. going through the whole ray/circle based construction process.

6.5.4 Transformations

Currently, rotate about a point or translate by a vector. Also interfaces for by *random* point or vector.

6.6 Randomness

6.6.1 Random Choices

At the basis of all random

6.6.2 Remembering choices

6.6.3 Backtracking

Currently, the system does not backtrack based on random choices. However, there are plans to perform checks on randomly-generated elements that are too close to one another and to retry the random choice to avoid degenerate choices.

6.6.4 Avoiding almost-degenerate points

As discussed above, randomly making choices in

6.6.5 Animating choices

I animate over a small range within the specified random range. Top-level infrastructure determinies frames, sleeping, etc. Constructions can request to animate functions of one arg [0, 1]. As the figure and animation is run, each call to randomize gets a call to random whenever their value is non-false.

6.7 Dependencies

6.7.1 Implementation

Eq-properties, etc.

6.7.2 Naming

Sometimes derived if unknown, figure out how name metadata relates to the dependencies.

6.7.3 Forcing higher-level random dependenceis

"Inverts" the dependency tree that would otherwise usually go down to points. set-dependency! as random-square. When given an element by the teacher, generally we don't know how the construction was performed.

6.7.4 Dependency-less diagrams

In some cases, the dependency structure of a figure can be wiped.

6.8 Construction Language

Constructions and instruction-based investigations are specified by scheme procedures that return the desired figures.

6.8.1 Macros

I created a let-geo* special form that is similar to Scheme's (let ...) form, but sets the element names as specified so they can be more easily referred to later.

6.8.2 Multiple Assignment

In let-geo*, I also permit some constructions to optionally map to multiple assignments of names, such as the case in which you create a triangle and simulatneously want to store and name the triangle's vertex points.

6.9 Graphics

The system integrates with Scheme's graphics system for the X Window System to display the figures for the users. The graphical viewer can include labels and highlight specific elements, as well as display animations representing the "wiggling" of the diagram.

Chapter 7

Perception Module

7.1 Overview

Given a module that executes construction steps to build analytic figures, we need a way of "seeing" these figures in our mind's eye. Thus, the perception module is primarily concerned with the task of examining the figure and observing interesting properties in figure.

7.1.1 Extracting segments and angles

The observation module also builds and traverses a graph-representation of the object of connectedness and adjacencies to extract more segments and angles, or include intersections of elements in its investigation.

Auxillary Segments

In some circumstances, the system can insert and consider segments between all pairs of points. Although this can sometimes produce interesting results, it can often lead to too many elements being considered. This option is off by default but can be enabled in a self-exploration mode.

7.1.2 What is Interesting?

Concurrent points, collinear points, equal angles, supplementary/complementary angles, parallel, perpendicular elements, concentric points, (future:) ratios between measurements, etc.

7.1.3 Removing Obvious Properties

This module makes use of available dependency information to eliminate some obvious properties. At this phase, the eliminations arise only from basic geometry knowledge "hard-coded" into the system, and not upon any specific prior-learned formula.

Trivial relations

Points being on lines, segments, circles directly dependent on that point.

Branch Relations

Other examples include "branch" relations. [REF: Chen, Song, etc.]. ABCD on a line with AB = CD also means that AC = BD, for instance.

7.2 Representations

A "Relationship" object represents a type of relationship, a "Observation" object refers to a specific observation seen in a figure.

Chapter 8

Declarative Geometry Constraint Solver

8.1 Overview

The final module is a declarative geometric constraint solver. Given a user-specified topology of the diagram and various constraints, this system is able to solve those constraints and instantiate a diagram that satisfies them if possible.

This system is implemented using propagators, involved the creation of new partial information about point regions and direction intervals, and focuses on a

Future efforts involve a backtrack-search mechanism if constraints fail, and a system of initializing the diagram with content from an existing figure, kicking out and wiggling arbitrary premises, and seeing how the resulting diagram properties respond.

8.2 Mechanical Analogies

The geometry constraint solver - physical manipulation, simulation, and “wiggling”.

8.2.1 Bar and Joint Linkages

Bars have endpoints, directions and length. Joints have a vertex point and two directions. Currently, most joints are directioned and have max value of 180 degrees.

8.2.2 Mechanism

The Mechanism in our declarative system is analogous to Figure, grouping elements. Also computes various caching and lookup tables to more easily access elements.

8.3 Partial Information

8.3.1 Regions

Propagating partial information across bars and joints yields a new region system: Regions include point sets of one or more possible points, an entire ray, or an entire arc. These rays and arcs are from an anchored bar with only one of direction or length specified, for instance.

8.3.2 Direction Intervals

Ranges of intervals. Full circle + invalid intervals. Adding and subtracting intervals of direction and thetas gets complicated at times.

Challenges with intersection, multiple segments. Eventually just return nothing is okay.

8.4 Propagator Constraints

System uses propagators to solve these mechanism constraints.

8.4.1 Basic Linkage Constraints

Direction, dx, dy, length, thetas. "Bars" + "Joints"

8.4.2 Higher Order Constraints

Angle sum of polygon, or scan through polygon and ensure that the angles don't not match. Example is equilateral triangle, for instance... Could also observe always "60 degrees" as an interesting fact and put that in as a constraint. They're alebgraically quite similar, but my propagators currently don't perform symbolic algebra.

8.5 Solving: Specification Ordering

Given a wired diagram, process is repeatedly specifying values for elements

8.5.1 Anchored vs. Specified vs. Initialized

8.6 Backtracking

If it can't build a figure with a given set of specifications, it will first try some neighboring values, then backtrack and try a new value for the previous element. After a number of failed attempts, it will abort and claim that at this time, it is unable to build a diagram satisfying the constraints.

(This doesn't mean that it is impossible: Add analysis/info about what it can/-can't solve)

8.7 Interfacing with existing diagrams

Converts between figures and symbolic relationships.

8.8 Specification Interface

Establish Polygon Topology

Nice techniques for establishing polygon topology.

Chapter 9

Related Work

[Need to update with a few more references I've found, and some relating to the mechanical simulation aspects]

The topics of automating geometric proofs and working with diagrams are areas of active research. Several examples of related work can be found in the proceedings of annual conferences such as *Automated Deduction in Geometry* [?] and *Diagrammatic Representation and Inference* [?]. In addition, two papers from the past year combine these concepts with a layer of computer vision interpretation of diagrams. Chen, Song, and Wang present a system that infers what theorems are being illustrated from images of diagrams [?], and a paper by Seo and Hajishirzi describes using textual descriptions of problems to improve recognition of their accompanying figures [?].

Further related work includes descriptions of the educational impacts of dynamic geometry approaches and some software to explore geometric diagrams and proofs. However, such software typically uses alternate approaches to automate such processes, and few focus on inductive reasoning.

9.1 Dynamic Geometry

From an education perspective, there are several texts that emphasize an investigative, conjecture-based approach to teaching such as *Discovering Geometry* by Michael Serra [?], the text I used to learn geometry. Some researchers praise these investigative methods [?] while others question whether it appropriately encourages deductive reasoning skills [?].

9.2 Software

Some of these teaching methods include accompanying software such as Cabri Geometry [?] and the Geometer's Sketchpad [?] designed to enable students to explore constructions interactively. These programs occasionally provide scripting features, but have no proof-related automation.

A few more academic analogs of these programs introduce some proof features. For instance, GeoProof [?] integrates diagram construction with verified proofs using a number of symbolic methods carried out by the Coq Proof Assistant, and Geometry Explorer [?] uses a full-angle method of chasing angle relations to check assertions requested by the user. However, none of the software described simulates the exploratory, inductive investigation process used by students first discovering new conjectures.

9.3 Automated Proof and Discovery

Although there are several papers that describe automated discovery or proof in geometry, most of these use alternate, more algebraic methods to prove theorems. These approaches include an area method [?], Wu's Method involving systems of polynomial equations [?], and a system based on Gröbner Bases [?]. Some papers discuss reasoning systems including the construction and application of a deductive database of geometric theorems [?]. However, all of these methods focused either on deductive reasoning or complex algebraic reformulations.

Chapter 10

Results

10.1 Overview

Isoceles triangle angles vs. theorems, bisectors of kite, lots of cool collinear / concurrent points in Triangles, for instance.

Will add more diagrams, explanations, better examples, etc.

```
;;; Sample:

;;;;;;;;;;;;;;;;;;;;;;;;;; Looking up terms ;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Starts with limited knowledge

(what-is 'square)
=>  unknown

(what-is 'rhombus)
=>  unknown

;;; Knows primitive objects
```

```
(what-is 'line)
=>  primitive-definition



(what-is 'point)
=>  primitive-definition



(what-is 'polygon)
=>  primitive-definition



;;; And some built-in non-primitives

(what-is 'triangle)
=>  (triangle (polygon) ((n-sides-3 <premise>)))

(what-is 'quadrilateral)
=> (quadrilateral (polygon) ((n-sides-4 <premise>)))



;;;;;;;;;;;; Can idenitfy whether elements satisfy these ;;;;;;;;;;;;;

(is-a? 'polygon (random-square))
;Value: #t

(is-a? 'quadrilateral (random-square))
;Value: #t

(is-a? 'triangle (random-square))
```

```
=> (failed-observation (n-sides-3 <premise>))
;Value: #f

(is-a? 'segment (random-square))
;Value: #f

(is-a? 'line (random-line))
;Value: #t

;;;;;;;;;;;;;;;;; Can learn and explain new terms ;;;;;;;;;;;;;;;;;;

(what-is 'isoc-t)
=>  unknown

(learn-term 'isoc-t random-isoceles-triangle)
--- Results ---
(48
 (discovered
  (equal-length (polygon-segment 0 1 (random-isoceles-triangle 1))
                (polygon-segment 2 0 (random-isoceles-triangle 1)))
  (equal-angle (polygon-angle 1 (random-isoceles-triangle 1))
               (polygon-angle 2 (random-isoceles-triangle 1)))))

((equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 0 <premise>))
 (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>)))
;Value: done



(what-is 'isoc-t)
```

```
(isoc-t
 (triangle)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 2 0 <premise>))
  (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))))
;Unspecified return value

(is-a? ’isoc-t (random-isoceles-triangle))
;Value: #t

(is-a? ’isoc-t (random-equilateral-triangle))
;Value: #t

(is-a? ’isoc-t (random-triangle))
(failed-observation
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 0 <premise>)))

;Value: #f

(learn-term ’equi-t random-equilateral-triangle)
--- Results ---
(32
 (discovered
  (equal-length (polygon-segment 0 1 (random-equilateral-triangle 1))
                (polygon-segment 1 2 (random-equilateral-triangle 1)))
  (equal-length (polygon-segment 0 1 (random-equilateral-triangle 1))
                (polygon-segment 2 0 (random-equilateral-triangle 1)))
  (equal-length (polygon-segment 1 2 (random-equilateral-triangle 1))
                (polygon-segment 2 0 (random-equilateral-triangle 1)))
```

```
  (equal-angle (polygon-angle 0 (random-equilateral-triangle 1))
               (polygon-angle 1 (random-equilateral-triangle 1)))
  (equal-angle (polygon-angle 0 (random-equilateral-triangle 1))
               (polygon-angle 2 (random-equilateral-triangle 1)))
  (equal-angle (polygon-angle 1 (random-equilateral-triangle 1))
               (polygon-angle 2 (random-equilateral-triangle 1)))))

((equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 1 2 <premise>))
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 0 <premise>))
 (equal-length (polygon-segment 1 2 <premise>)
               (polygon-segment 2 0 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
 (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>)))

;Value: done

(what-is 'equi-t)
(equi-t
 (isoc-t)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 1 2 <premise>))
  (equal-length (polygon-segment 1 2 <premise>)
                (polygon-segment 2 0 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))))
;Unspecified return value
```

```
(is-a? 'equi-t (random-isoceles-triangle))
(failed-observation
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 1 2 <premise>)))
;Value: #f



(is-a? 'equi-t (random-equilateral-triangle))
;Value: #t



;;;;;;;;;;;;;;; Let's learn some basic quadrilaterals ;;;;;;;;;;;;;;;;

(learn-term 'pl random-parallelogram)
;Value: done

(what-is 'pl)
(pl
 (quadrilateral)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 2 3 <premise>))
  (equal-length (polygon-segment 1 2 <premise>)
                (polygon-segment 3 0 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
  (equal-angle (polygon-angle 1 <premise>) (polygon-angle 3 <premise>))
  (supplementary (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
  (supplementary (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
  (supplementary (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
  (supplementary (polygon-angle 2 <premise>) (polygon-angle 3 <premise>))
```

```
  (parallel (polygon-segment 0 1 <premise>) (polygon-segment 2 3 <premise>))
  (parallel (polygon-segment 1 2 <premise>) (polygon-segment 3 0 <premise>))))

;Unspecified return value

(learn-term 'kite random-kite)
--- Results ---
((equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 1 2 <premise>))
 (equal-length (polygon-segment 2 3 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>)))
;Value: done


(what-is 'kite)
(kite
 (quadrilateral)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 1 2 <premise>))
  (equal-length (polygon-segment 2 3 <premise>)
                (polygon-segment 3 0 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))))
;Unspecified return value


(learn-term 'rh random-rhombus)
--- Results ---
((equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 1 2 <premise>))
```

```
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 3 <premise>))
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-length (polygon-segment 1 2 <premise>)
               (polygon-segment 2 3 <premise>))
 (equal-length (polygon-segment 1 2 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-length (polygon-segment 2 3 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
 (equal-angle (polygon-angle 1 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
 (supplementary (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
 (supplementary (polygon-angle 2 <premise>) (polygon-angle 3 <premise>))
 (parallel (polygon-segment 0 1 <premise>) (polygon-segment 2 3 <premise>))
 (parallel (polygon-segment 1 2 <premise>) (polygon-segment 3 0 <premise>)))

;Value: done

(what-is 'rh)
(rh
 (pl kite)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 3 0 <premise>))
  (equal-length (polygon-segment 1 2 <premise>)
                (polygon-segment 2 3 <premise>))))

;Unspecified return value
```

```
(learn-term 'rectangle random-rectangle)
;Value: done

(what-is 'rectangle)
(rectangle
 (pl)
 ((concentric (polygon-point 0 <premise>)
              (polygon-point 1 <premise>)
              (polygon-point 2 <premise>)
              (polygon-point 3 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
  (equal-angle (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
  (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
  (equal-angle (polygon-angle 2 <premise>) (polygon-angle 3 <premise>))
  (supplementary (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
  (supplementary (polygon-angle 1 <premise>) (polygon-angle 3 <premise>))
  (perpendicular (polygon-segment 0 1 <premise>)
                 (polygon-segment 1 2 <premise>))
  (perpendicular (polygon-segment 0 1 <premise>)
                 (polygon-segment 3 0 <premise>))
  (perpendicular (polygon-segment 1 2 <premise>)
                 (polygon-segment 2 3 <premise>))
  (perpendicular (polygon-segment 2 3 <premise>)
                 (polygon-segment 3 0 <premise>))))

;Unspecified return value

(learn-term 'sq random-square)
```

```
--- Results ---
((concentric (polygon-point 0 <premise>)
             (polygon-point 1 <premise>)
             (polygon-point 2 <premise>)
             (polygon-point 3 <premise>))
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 1 2 <premise>))
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 3 <premise>))
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-length (polygon-segment 1 2 <premise>)
               (polygon-segment 2 3 <premise>))
 (equal-length (polygon-segment 1 2 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-length (polygon-segment 2 3 <premise>)
               (polygon-segment 3 0 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
 (equal-angle (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
 (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
 (equal-angle (polygon-angle 1 <premise>) (polygon-angle 3 <premise>))
 (equal-angle (polygon-angle 2 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 0 <premise>) (polygon-angle 1 <premise>))
 (supplementary (polygon-angle 0 <premise>) (polygon-angle 2 <premise>))
 (supplementary (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
 (supplementary (polygon-angle 1 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 2 <premise>) (polygon-angle 3 <premise>))
 (parallel (polygon-segment 0 1 <premise>) (polygon-segment 2 3 <premise>))
```

```
 (parallel (polygon-segment 1 2 <premise>) (polygon-segment 3 0 <premise>))
 (perpendicular (polygon-segment 0 1 <premise>)
                (polygon-segment 1 2 <premise>))
 (perpendicular (polygon-segment 0 1 <premise>)
                (polygon-segment 3 0 <premise>))
 (perpendicular (polygon-segment 1 2 <premise>)
                (polygon-segment 2 3 <premise>))
 (perpendicular (polygon-segment 2 3 <premise>)
                (polygon-segment 3 0 <premise>)))

;Value: done



;;; For example: Notice that when it reports what a square is, it says
its a rectangle and a rhombus!

(what-is ’sq)
(sq (rectangle rh) ())

;Unspecified return value
```

Chapter 11

Conclusion

11.1 Overview

To be concluded ...

11.2 Extensions

Possible extensions include integrating with existing automated proof systems (Coq, etc.)

Also: learning construction procedures from the declarative constraint solver's solution.

Appendix A

Code Listings

Listings

Listing A.1: load.scm

```scheme
;;; load.scm -- Load the system

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Utilities ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (reset)
  (ignore-errors (lambda () (close)))
  (ge (make-top-level-environment))
  (load "load"))

(define (load-module subdirectory)
  (let ((cur-pwd (pwd)))
    (cd subdirectory)
    (load "load")
    (cd cur-pwd)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Load Modules ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(for-each (lambda (m) (load-module m))
          '("lib"
            "core"
            "figure"
            "graphics"
            "manipulate"
            "perception"
            "learning"
            "content"))
(load "main")

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Initialize ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(set! *random-state* (fasload "a-random-state"))
(initialize-scheduler)
(initialize-student)

'done-loading
```

Listing A.2: main.scm

```
(define (i-t-figure)
  (let-geo* (((t (a b c)) (random-isoceles-triangle)))
    (figure t)))


(define (midpoint-figure)
  (let-geo* (((s (a b)) (random-segment))
             (m (segment-midpoint s)))
    (figure s m)))

(define (random-rhombus-figure)
  (let-geo* (((r (a b c d)) (random-rhombus)))
    (figure r)))

;;; Other Examples:

(define (debug-figure)
  (let-geo* (((r (a b c d)) (random-parallelogram))
             (m1 (midpoint a b))
             (m2 (midpoint c d)))
    (figure r m1 m2 (make-segment m1 m2))))

(define (demo-figure)
  (let-geo* (((t (a b c)) (random-isoceles-triangle))
             (d (midpoint a b))
             (e (midpoint a c))
             (f (midpoint b c))

             (l1 (perpendicular (line-from-points a b) d))
             (l2 (perpendicular (line-from-points a c) e))
             (l3 (perpendicular (line-from-points b c) f))

             (i1 (intersect-lines l1 l2))
             (i2 (intersect-lines l1 l3))

             (cir (circle-from-points i1 a)))

    (figure
     (make-segment a b)
     (make-segment b c)
     (make-segment a c)
     a b c l1 l2 l3 cir
     i1 i2)))

(define (circle-line-intersect-test)
  (let-geo* ((cir (random-circle))
             ((rad (a b)) (random-circle-radius cir))
             (p (random-point-on-segment rad))
             (l (random-line-through-point p))
             (cd (intersect-circle-line cir l))
             (c (car cd))
             (d (cadr cd)))
    (figure cir rad p l c d)))

(define (circle-test)
  (let-geo* ((a (random-point))
             (b (random-point))
             (d (distance a b))
             (r (rand-range
                 (* d 0.5)
                 (* d 1)))
             (c1 (make-circle a r))
             (c2 (make-circle b r))
             (cd (intersect-circles c1 c2))
             (c (car cd))
             (d (cadr cd)))
```

```
    (figure (polygon-from-points a c b d))))

(define (line-test)
  (let-geo* ((a (random-point))
             (b (random-point))
             (c (random-point))
             (d (random-point))
             (l1 (line-from-points a b))
             (l2 (line-from-points c d))
             (e (intersect-lines l1 l2))
             (f (random-point-on-line l1))
             (cir (circle-from-points e f)))
    (figure a b c d l1 l2 e f cir)))

(define (angle-test)
  (let-geo* (((t (a b c)) (random-triangle))
             (a-1 (smallest-angle (angle-from-points a b c)))
             (a-2 (smallest-angle (angle-from-points b c a)))
             (a-3 (smallest-angle (angle-from-points c a b)))
             (l1 (angle-bisector a-1))
             (l2 (angle-bisector a-2))
             (l3 (angle-bisector a-3))
             (center-point
              (intersect-lines (ray->line l1)
                               (ray->line l2)))
             (radius-line
              (perpendicular (line-from-points b c)
                             center-point))
             (radius-point
              (intersect-lines radius-line
                               (line-from-points b c)))
             (cir (circle-from-points
                   center-point
                   radius-point))
             (pb1 (perpendicular-bisector
                   (make-segment a b)))
             (pb2 (perpendicular-bisector
                   (make-segment b c)))
             (pb-center (intersect-lines pb1 pb2))
             (circum-cir (circle-from-points
                          pb-center
                          a)))
    (figure t cir a-1 a-2 a-3
            pb-center
            circum-cir
            center-point)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;; Run commands

(define current-figure demo-figure)

(define c
  (if (environment-bound? (the-environment) 'c)
      c
      (canvas)))

(define (close)
  (ignore-errors (lambda () (graphics-close (canvas-g c)))))

(define *num-inner-loop* 5)
(define *num-outer-loop* 5)


(define (run-figure current-figure-proc)
  (let ((analysis-data (make-analysis-collector)))
    (run-animation
     (lambda ()
```

```scheme
      (let ((current-figure (current-figure-proc)))
        (draw-figure current-figure c)
        (let ((analysis-results (analyze-figure current-figure)))
          (save-results (print analysis-results) analysis-data))
        )))
    (display "--- Results ---\n")
    (analyze-figure current-figure)
    (print-analysis-results analysis-data)))

(define interesting-figures
  (list
   debug-figure
   parallel-lines-converse
   perpendicular-bisector-equidistant
   perpendicular-bisector-converse
   demo-figure
   linear-pair
   vertical-angles
   corresponding-angles
   cyclic-quadrilateral))

(define (r)
  (for-each (lambda (figure)
              (run-figure figure))
            interesting-figures)
  'done)

;(r)
```

Listing A.3: learning/load.scm

```
;;; load.scm -- Load learning module
(for-each (lambda (f) (load f))
          '("definitions"
            "student"
            "core-knowledge"
            "simplifier"))
```

Listing A.4: learning/core-knowledge.scm

```scheme
;;; core-knowledge.scm -- Core knowledge of a student

;;; Commentary:

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Adding to student ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (provide-core-knowledge student)
  (for-each (lambda (def)
              (add-definition! student def))
            primitive-definitions)
  (for-each (lambda (def)
              (add-definition! student def))
            built-in-definitions))

;;;;;;;;;;;;;;;;;;;;;;;;; Primitive definitions ;;;;;;;;;;;;;;;;;;;;;;;;;

(define primitive-definitions
  (list
   (make-primitive-definition 'point point? random-point)
   (make-primitive-definition 'line line? random-line)
   (make-primitive-definition 'segment segment? random-segment)
   (make-primitive-definition 'polygon polygon? random-polygon)
   (make-primitive-definition 'circle circle? random-circle)
   (make-primitive-definition 'angle angle? random-angle)))

;;;;;;;;;;;;;;;;;;;;;;;; Built-in Definitions ;;;;;;;;;;;;;;;;;;;;;;;;

(define (polygon-n-sides-observation n)
  (make-observation
   '()
   (make-polygon-n-sides-relationship n)
   (list (with-source car '<premise>))))

(define built-in-definitions
  (list
   ;; Triangle
   (make-restrictions-definition
    'triangle '(polygon)
    (list (polygon-n-sides-observation 3))
    random-triangle)
   ;; Quadrilateral
   (make-restrictions-definition
    'quadrilateral '(polygon)
    (list (polygon-n-sides-observation 4))
    random-quadrilateral)

   ;; Isoceles Triangle!
   #|
   (make-restrictions-definition
    'isoceles-triangle 'triangle
    (list (lambda (t)
            (let* ((a (polygon-point-ref t 0))
                   (b (polygon-point-ref t 1))
                   (c (polygon-point-ref t 2)))
              (segment-equal-length? (make-segment a b)
                                     (make-segment a c)))))

   random-isoceles-triangle))
|#
   ))
```

Listing A.5: learning/definitions.scm

```scheme
;;; definitions.scm --- representation and interaction with definitions

;;; Commentary:

;; Ideas:
;; - primitive definitions

;; Future:
;; - relationship-based definitions

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;; Basic Structure ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <definition>
  (%make-definition name classifications observations predicate generator)
  definition?
  (name definition-name)
  (classifications definition-classifications)
  (observations definition-observations)
  (predicate definition-predicate set-definition-predicate!)
  (generator definition-generator))

(define (make-primitive-definition name predicate generator)
  (%make-definition name '() '()  predicate generator))

(define (primitive-definition? def)
  (and (definition? def)
       (null? (definition-classifications def))))

;;;;;;;;;;;;;;;;;;;;; Higher-order Definitions ;;;;;;;;;;;;;;;;;;;;;

(define (make-restrictions-definition
         name classifications observations generator)
  (%make-definition name classifications observations #f generator))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Formatting ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (print-definition def)
  (list (definition-name def)
        (definition-classifications def)
        (map print (definition-observations def))))

(defhandler print print-definition
  definition?)

(define (print-primitive-definition def)
  'primitive-definition)

(defhandler print print-primitive-definition
  primitive-definition?)
```

Listing A.6: learning/simplifier.scm

```scheme
;;; simplifier.scm --- simplifies definitions

;;; Commentary:

;; Ideas:
;; - interfaces to manipulator

;; Future:
;; - Support more complex topologies.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;; Main Interface ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (simplify-definition
         n-sides
         relationships)
  #f)

(define (relationships->constraints relationships)
  '())

(define (relationship->constraint rel)
  '())

(define (establish-polygon-topology-for-n-gon n-sides)
  (cond ((= n-sides 3)
         (m:establish-polygon-topology 'a 'b 'c))
        ((= n-sides 4)
         (m:establish-polygon-topology 'a 'b 'c 'd))))

(define (relationships->figure n-sides relationships)
  (initialize-scheduler)
  (let ((m (apply
            m:mechanism
            (cons (establish-polygon-topology-for-n-gon n-sides)
                  (relationships->constraints relationships)))))
    (m:build-mechanism m)
    (m:solve-mechanism m)
    (let ((f (m:mechanism->figure m)))
      f)))
```

Listing A.7: learning/student.scm

```scheme
;;; student.scm -- base model of a student's knowlege

;;; Commentary:

;; Ideas:
;; - Definitions, constructions, theorems
;; - "What is"

;; Future:
;; - Simplifiers of redudant / uninsteresting info
;; - Propose own investigations?

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Student Structure ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <student>
  (%make-student definitions)
  student?
  (definitions student-definitions))

(define (make-student)
  (%make-student (make-key-weak-eq-hash-table)))


;;;;;;;;;;;;;;;;;;;;;;; Building Predicates ;;;;;;;;;;;;;;;;;;;;;;;;

(define (build-predicate-for-definition s def)
  (let ((classifications (definition-classifications def))
        (observations (definition-observations def)))
    (let ((classification-predicate
           (lambda (obj)
             (every
              (lambda (classification)
                (or ((definition-predicate (student-lookup s classification))
                     obj)
                    (begin (if *explain*
                               (pprint `(failed-classification
                                         ,classification)))
                           #f)))
              classifications))))
      (lambda args
        (and (apply classification-predicate args)
             (every (lambda (o) (satisfies-observation o args))
                    observations))))))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Definitions ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (add-definition! s def)
  (if (not (definition-predicate def))
      (set-definition-predicate!
       def
       (build-predicate-for-definition s def)))
  (hash-table/put! (student-definitions s)
                   (definition-name def)
                   def))

(define (lookup-definition s name)
  (hash-table/get (student-definitions s)
                  name
                  #f))

;;;;;;;;;;;;;;;;;;;;;;;;; Current Student ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define *current-student* #f)

```

```scheme
(define (student-lookup s term)
  (lookup-definition s term))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Query ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (lookup term)
  (let ((result (student-lookup *current-student* term)))
    (if (not result)
        'unknown
        result)))

(define (what-is term)
  (pprint (lookup term)))

(define *explain* #f)

(define (is-a? term obj)
  (show-element obj)
  (let ((def (lookup term)))
    (if (eq? def 'unknown)
        `(,term unknown)
        (fluid-let ((*explain* #t))
          ((definition-predicate def) obj)))))

(define (internal-is-a? term obj)
  (let ((def (lookup term)))
    (if (eq? def 'unknown)
        `(,term unknown)
        ((definition-predicate def) obj))))

(define (show-me term)
  (let ((def (lookup term)))
    (if (eq? def 'unknown)
        `(,term unknown)
        (show-element ((definition-generator def))))))

(define (examine object)
  (show-element object)
  (let ((base-terms (filter (lambda (term)
                              (internal-is-a? term object))
                            (hash-table/key-list
                             (student-definitions *current-student*)))))
    base-terms))

;;;;;;;;;;;;;;;;;;;;;;; Simplifying base terms ;;;;;;;;;;;;;;;;;;;;;;;

(define (simplify-base-terms terms)
  (let ((parent-terms (append-map
                       (lambda (t) (definition-classifications (lookup t)))
                           terms)))
    (filter (lambda (t) (not (memq t parent-terms)))
            terms)))

;;;;;;;;;;;;;;;;;;;;;;;; Graphics Interfaces ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (show-element element)
  (if (polygon? element)
      (name-polygon element))
  (draw-figure (figure element) c))

;;;;;;;;;;;;;;;;;;;;;;;; Initializing Student ;;;;;;;;;;;;;;;;;;;;;;;;

(define (initialize-student)
  (let ((s (make-student)))
    (provide-core-knowledge s)
    (set! *current-student* s)))
```

```
(define (learn-term term object-generator)
  (let ((v (lookup term)))
    (if (not (eq? v 'unknown))
        (pprint `(already-known ,term))
        (let ((example (name-polygon (object-generator))))
          (let* ((base-terms (examine example))
                 (simple-base-terms (simplify-base-terms base-terms))
                 (base-definitions (map lookup base-terms))
                 (base-observations (flatten (map definition-observations
                                                  base-definitions)))
                 (fig (figure (with-dependency '<premise> example)))
                 (observations (analyze-figure fig))
                 (simplified-observations
                  (simplify-observations observations base-observations)))
            (run-figure (lambda () (figure (object-generator))))
            (pprint observations)
            (let ((new-def
                   (make-restrictions-definition
                    term
                    simple-base-terms
                    simplified-observations
                    object-generator)))
              (add-definition! *current-student* new-def)
              'done))))))
```

Listing A.8: learning/walkthrough.scm

```
;;; Sample:

;;;;;;;;;;;;;;;;;;;;;;;;;; Looking up terms ;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Starts with limited knowledge

(what-is 'square)

(what-is 'rhombus)

;;; Knows primitive objects

(what-is 'line)

(what-is 'point)

(what-is 'polygon)

;;; And some built-in non-primitives

(what-is 'triangle)

(what-is 'quadrilateral)

;;;;;;;;;;;; Can idenitfy whether elements satisfy these ;;;;;;;;;;;;

(is-a? 'polygon (random-square))

(is-a? 'quadrilateral (random-square))

(is-a? 'triangle (random-square))

(is-a? 'segment (random-square))

(is-a? 'line (random-line))

;;;;;;;;;;;;;;;;;; Can learn and explain new terms ;;;;;;;;;;;;;;;;;;

(what-is 'isoc-t)

(learn-term 'isoc-t random-isoceles-triangle)

(what-is 'isoc-t)

(is-a? 'isoc-t (random-isoceles-triangle))

(is-a? 'isoc-t (random-equilateral-triangle))

(is-a? 'isoc-t (random-triangle))

(learn-term 'equi-t random-equilateral-triangle)

(what-is 'equi-t)

(is-a? 'equi-t (random-isoceles-triangle))

(is-a? 'equi-t (random-equilateral-triangle))

;;;;;;;;;;;;;;; Let's learn some basic quadrilaterals ;;;;;;;;;;;;;;;

(learn-term 'pl random-parallelogram)

(what-is 'pl)

(learn-term 'kite random-kite)

```

```
(what-is 'kite)

(learn-term 'rh random-rhombus)

(what-is 'rh)

(learn-term 'rectangle random-rectangle)

(what-is 'rectangle)

(learn-term 'sq random-square)

(what-is 'sq)
```

Listing A.9: figure/load.scm

```scheme
;;; load.scm -- Load figure
(for-each (lambda (f) (load f))
          '("core"
            "line"
            "direction"
            "direction-interval"
            "vec"
            "measurements"
            "angle"
            "bounds"
            "circle"
            "point"
            "constructions"
            "intersections"
            "figure"
            "math-utils"
            "polygon"
            "metadata"
            "dependencies"
            "randomness"
            "transforms"))
```

Listing A.10: figure/core.scm

```scheme
;;; core.scm --- Core definitions used throughout the figure elements

;;; Commentary:

;; Ideas:
;; - Some gemeric handlers used in figure elements

;; Future:
;; - figure-element?, e.g.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Element Component ;;;;;;;;;;;;;;;;;;;;;;;;;

(define element-component
  (make-generic-operation
   2 'element-component
   (lambda (el i)
     (error "No component procedure for element" el))))

(define (component-procedure-from-getters . getters)
  (let ((num-getters (length getters)))
    (lambda (el i)
      (if (not (<= 0 i (- num-getters 1)))
          (error "Index out of range for component procedure: " i))
      ((list-ref getters i)
       el))))

(define (declare-element-component-handler handler type)
  (defhandler element-component handler type number?))

(declare-element-component-handler list-ref list?)

#|
Example Usage:

(declare-element-component-handler
 (component-procedure-from-getters car cdr)
 pair?)

(declare-element-component-handler vector-ref vector?)

(element-component '(3 . 4 ) 1)
;Value: 4

(element-component #(1 2 3) 2)
;Value: 3
|#
```

Listing A.11: figure/line.scm

```
;;; line.scm --- Line

;;; Commentary:

;; Ideas:
;; - Linear Elements: Segments, Lines, Rays
;; - All have direction
;; - Conversions to directions, extending.
;; - Lines are point + direction, but hard to access point
;; - Means to override dependencies for random segments

;; Future:
;; - Simplify direction requirements
;; - Improve some predicates, more tests
;; - Fill out more dependency information

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Segments ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type &lt;segment&gt;
  (%segment p1 p2)
  segment?
  (p1 segment-endpoint-1)
  (p2 segment-endpoint-2))

(define (set-segment-dependency! segment dependency)
  (set-dependency! segment dependency)
  (set-dependency!
   (segment-endpoint-1 segment)
   `(segment-endpoint-1 segment))
  (set-dependency!
   (segment-endpoint-2 segment)
   `(segment-endpoint-2 segment)))

;;; Alternate, helper constructors

(define (make-segment p1 p2)
  (let ((seg (%segment p1 p2)))
    (set-element-name!
     seg
     (symbol '*seg*: (element-name p1) '- (element-name p2)))
    (with-dependency
     `(segment ,p1 ,p2)
     seg)))

(define (make-auxiliary-segment p1 p2)
  (with-dependency
   `(aux-segment ,p1 ,p2)
   (make-segment p1 p2)))

(declare-element-component-handler
 (component-procedure-from-getters segment-endpoint-1
                                   segment-endpoint-2)
 segment?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Lines ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <line>
  (%make-line point dir)
  line?
  (point line-point)
  (dir line-direction)) ;; Point on the line

(define make-line %make-line)

```

```scheme
(define (line-from-points p1 p2)
  (make-line p1 (direction-from-points p1 p2)))

(define (line-from-point-direction p dir)
  (make-line p dir))

;;; TODO, use for equality tests?
(define (line-offset line)
  (let ((direction (direction-from-points p1 p2))
        (x1 (point-x p1))
        (y1 (point-y p1))
        (x2 (point-x p2))
        (y2 (point-y p2)))
    (let ((offset (/ (- (* x2 y1)
                        (* y2 x1))
                     (distance p1 p2))))
      (%make-line direction offset))))

;;; TODO: Figure out dependencies for these
(define (two-points-on-line line)
  (let ((point-1 (line-point line)))
   (let ((point-2 (add-to-point
                   point-1
                   (unit-vec-from-direction (line-direction line)))))
     (list point-1 point-2))))

(define (line-p1 line)
  (car (two-points-on-line line)))

(define (line-p2 line)
  (cadr (two-points-on-line line)))


;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Rays ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <ray>
  (make-ray initial-point direction)
  ray?
  (initial-point ray-endpoint)
  (direction ray-direction))

(define (ray-from-point-direction p dir)
  (make-ray p dir))

(define (ray-from-points endpoint p1)
  (make-ray endpoint (direction-from-points endpoint p1)))

(define (shorten-ray-from-point r p)
  (if (not (on-ray? p r))
      (error "Can only shorten rays from points on the ray"))
  (ray-from-point-direction p (ray-direction r)))

;;;;;;;;;;;;;;;;;;;;;; Constructors from angles ;;;;;;;;;;;;;;;;;;;;;;

(define (ray-from-arm-1 a)
  (let ((v (angle-vertex a))
        (dir (angle-arm-1 a)))
    (make-ray v dir)))

(define (ray-from-arm-2 a)
  (ray-from-arm-1 (reverse-angle a)))

(define (line-from-arm-1 a)
  (ray->line (ray-from-arm-1 a)))

(define (line-from-arm-2 a)
  (ray->line (ray-from-arm-2 a)))
```

```scheme
;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Transforms ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define flip (make-generic-operation 1 'flip))

(define (flip-line line)
  (make-line
   (line-point line)
   (reverse-direction (line-direction line))))
(defhandler flip flip-line line?)

(define (flip-segment s)
  (make-segment (segment-endpoint-2 s) (segment-endpoint-1 s)))
(defhandler flip flip-segment segment?)

(define (reverse-ray r)
  (make-ray (ray-endpoint r)
            (reverse-direction (ray-direction r))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Operations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (segment-length seg)
  (distance (segment-endpoint-1 seg)
            (segment-endpoint-2 seg)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (linear-element? x)
  (or (line? x)
      (segment? x)
      (ray? x)))

(define (parallel? a b)
  (direction-parallel? (->direction a)
                       (->direction b)))

(define (perpendicular? a b)
  (direction-perpendicular? (->direction a)
                            (->direction b)))

(define (segment-equal? s1 s2)
  (and
   (point-equal? (segment-endpoint-1 s1)
                 (segment-endpoint-1 s2))
   (point-equal? (segment-endpoint-2 s1)
                 (segment-endpoint-2 s2))))

(define (segment-equal-ignore-direction? s1 s2)
  (or (segment-equal? s1 s2)
      (segment-equal? s1 (flip-segment s2))))

(define (segment-equal-length? seg-1 seg-2)
  (close-enuf? (segment-length seg-1)
               (segment-length seg-2)))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Conversions ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Ray shares point p1
(define (segment->ray segment)
  (make-ray (segment-endpoint-1 segment)
            (direction-from-points
             (segment-endpoint-1 segment)
             (segment-endpoint-2 segment))))

(define (ray->line ray)
  (make-line (ray-endpoint ray)
             (ray-direction ray)))

(define (segment->line segment)
```

```
  (ray->line (segment->ray segment)))

(define (line->direction l)
  (line-direction l))

(define (ray->direction r)
  (ray-direction r))

(define (segment->direction s)
  (direction-from-points
   (segment-endpoint-1 s)
   (segment-endpoint-2 s)))

(define (segment->vec s)
  (sub-points
   (segment-endpoint-2 s)
   (segment-endpoint-1 s)))

(define ->direction (make-generic-operation 1 '->direction))
(defhandler ->direction line->direction line?)
(defhandler ->direction ray->direction ray?)
(defhandler ->direction segment->direction segment?)

(define ->line (make-generic-operation 1 '->line))
(defhandler ->line identity line?)
(defhandler ->line segment->line segment?)
(defhandler ->line ray->line ray?)
```

Listing A.12: figure/direction.scm

```
;;; direction.scm --- Low-level direction structure

;;; Commentary:

;; A Direction is equivalent to a unit vector pointing in some direction.

;; Ideas:
;; - Ensures range [0, 2pi]

;; Future:
;; - Could generalize to dx, dy or theta

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Direction Structure ;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <direction>
  (%direction theta)
  direction?
  (theta direction-theta))

(define (make-direction theta)
  (%direction (fix-angle-0-2pi theta)))

(define (print-direction dir)
  `(direction ,(direction-theta dir)))
(defhandler print print-direction direction?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Arithemtic ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (add-to-direction dir radians)
  (make-direction (+ (direction-theta dir)
                     radians)))
;;; D2 - D1
(define (subtract-directions d2 d1)
  (if (direction-equal? d1 d2)
      0
      (fix-angle-0-2pi (- (direction-theta d2)
                          (direction-theta d1)))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Operations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; CCW
(define (rotate-direction-90 dir)
  (add-to-direction dir (/ pi 2)))

(define (reverse-direction dir)
  (add-to-direction dir pi))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (direction-equal? d1 d2)
  (or (close-enuf? (direction-theta d1)
                   (direction-theta d2))
      (close-enuf? (direction-theta (reverse-direction d1))
                   (direction-theta (reverse-direction d2)))))

(define (direction-opposite? d1 d2)
  (close-enuf? (direction-theta d1)
               (direction-theta (reverse-direction d2))))

(define (direction-perpendicular? d1 d2)
  (let ((difference (subtract-directions d1 d2)))
    (or (close-enuf? difference (/ pi 2))
        (close-enuf? difference (* 3 (/ pi 2))))))

```

```
(define (direction-parallel? d1 d2)
  (or (direction-equal? d1 d2)
      (direction-opposite? d1 d2)))
```

Listing A.13: figure/direction.scm

```
;;; direction.scm --- Low-level direction structure

;;; Commentary:

;; A Direction is equivalent to a unit vector pointing in some direction.

;; Ideas:
;; - Ensures range [0, 2pi]

;; Future:
;; - Could generalize to dx, dy or theta

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Direction Structure ;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <direction>
  (%direction theta)
  direction?
  (theta direction-theta))

(define (make-direction theta)
  (%direction (fix-angle-0-2pi theta)))

(define (print-direction dir)
  `(direction ,(direction-theta dir)))
(defhandler print print-direction direction?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Arithemtic ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (add-to-direction dir radians)
  (make-direction (+ (direction-theta dir)
                     radians)))
;;; D2 - D1
(define (subtract-directions d2 d1)
  (if (direction-equal? d1 d2)
      0
      (fix-angle-0-2pi (- (direction-theta d2)
                          (direction-theta d1)))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Operations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; CCW
(define (rotate-direction-90 dir)
  (add-to-direction dir (/ pi 2)))

(define (reverse-direction dir)
  (add-to-direction dir pi))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (direction-equal? d1 d2)
  (or (close-enuf? (direction-theta d1)
                   (direction-theta d2))
      (close-enuf? (direction-theta (reverse-direction d1))
                   (direction-theta (reverse-direction d2)))))

(define (direction-opposite? d1 d2)
  (close-enuf? (direction-theta d1)
               (direction-theta (reverse-direction d2))))

(define (direction-perpendicular? d1 d2)
  (let ((difference (subtract-directions d1 d2)))
    (or (close-enuf? difference (/ pi 2))
        (close-enuf? difference (* 3 (/ pi 2))))))

```

```
(define (direction-parallel? d1 d2)
  (or (direction-equal? d1 d2)
      (direction-opposite? d1 d2)))
```

Listing A.14: figure/vec.scm

```scheme
;;; vec.scm --- Low-level vector structures

;;; Commentary:

;; Ideas:
;; - Simplifies lots of computation, cartesian coordiates
;; - Currently 2D, could extend

;; Future:
;; - Could generalize to allow for polar vs. cartesian vectors

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Vector Structure ;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <vec>
  (make-vec dx dy)
  vec?
  (dx vec-x)
  (dy vec-y))

;;; Transformations of Vectors
(define (vec-magnitude v)
  (let ((dx (vec-x v))
        (dy (vec-y v)))
    (sqrt (+ (square dx) (square dy)))))

;;;;;;;;;;;;;;;;;;;;; Alternate Constructors ;;;;;;;;;;;;;;;;;;;;;

(define (unit-vec-from-direction direction)
  (let ((theta (direction-theta direction)))
   (make-vec (cos theta) (sin theta))))

(define (vec-from-direction-distance direction distance)
  (scale-vec (unit-vec-from-direction direction) distance))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Conversions ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (vec->direction v)
  (let ((dx (vec-x v))
        (dy (vec-y v)))
    (make-direction (atan dy dx))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Operations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Returns new vecs

(define (rotate-vec v radians)
  (let ((dx (vec-x v))
        (dy (vec-y v))
        (c (cos radians))
        (s (sin radians)))
    (make-vec (+ (* c dx) (- (* s dy)))
              (+ (* s dx) (* c dy)))))

(define (scale-vec v c)
  (let ((dx (vec-x v))
        (dy (vec-y v)))
    (make-vec (* c dx) (* c dy))))

(define (scale-vec-to-dist v dist)
  (scale-vec (unit-vec v) dist))

(define (reverse-vec v)
  (make-vec (- (vec-x v))
            (- (vec-y v))))
```

```
(define (rotate-vec-90 v)
  (let ((dx (vec-x v))
        (dy (vec-y v)))
    (make-vec (- dy) dx)))

(define (unit-vec v)
  (scale-vec v (/ (vec-magnitude v))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (vec-equal? v1 v2)
  (and (close-enuf? (vec-x v1)  (vec-x v2))
       (close-enuf? (vec-y v1)  (vec-y v2))))

(define (vec-direction-equal? v1 v2)
  (direction-equal?
   (vec->direction v1)
   (vec->direction v2)))

(define (vec-perpendicular? v1 v2)
  (close-enuf?
   (* (vec-x v1) (vec-x v2))
   (* (vec-y v1) (vec-y (reverse-vec v2)))))
```

Listing A.15: figure/measurements.scm

```scheme
;;; measurements.scm

;;; Commentary:

;; Ideas:
;; - Measurements primarily for analysis
;; - Occasionally used for easily duplicating angles or segments

;; Future:
;; - Arc Measure

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Distance ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (distance p1 p2)
  (sqrt (+ (square (- (point-x p1)
                      (point-x p2)))
           (square (- (point-y p1)
                      (point-y p2))))))

;;; Sign of distance is positive if the point is to the left of
;;; the line direction and negative if to the right.
(define (signed-distance-to-line point line)
  (let ((p1 (line-p1 line))
        (p2 (line-p2 line)))
    (let ((x0 (point-x point))
          (y0 (point-y point))
          (x1 (point-x p1))
          (y1 (point-y p1))
          (x2 (point-x p2))
          (y2 (point-y p2)))
      (/ (+ (- (* x0 (- y2 y1)))
            (* y0 (- x2 x1))
            (- (* x2 y1))
            (* y2 x1))
         (* 1.0
            (sqrt (+ (square (- y2 y1))
                     (square (- x2 x1)))))))))

(define (distance-to-line point line)
  (abs (signed-distance-to-line point line)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Angles ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (angle-measure a)
  (let* ((d1 (angle-arm-1 a))
         (d2 (angle-arm-2 a)))
    (subtract-directions d1 d2)))

;;;;;;;;;;;;;;;;;;;;;;;;; Measured Elements ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (measured-point-on-ray r dist)
  (let* ((p1 (ray-p1 r))
         (p2 (ray-p2 r))
         (v (sub-points p1 p2))
         (scaled-v (scale-vec-to-dist v dist)))
    (add-to-point p1 scaled-v)))

(define (measured-angle-ccw p1 vertex radians)
  (let* ((v1 (sub-points p1 vertex))
         (v-rotated (rotate-vec v (- radians))))
    (angle v1 vertex v-rotated)))

(define (measured-angle-cw p1 vertex radians)
  (reverse-angle (measured-angle-ccw p1 vertex (- radians))))
```

Listing A.16: figure/angle.scm

```scheme
;;; angle.scm --- Angles

;;; Commentary:

;; Ideas:
;; - Initially three points, now vertex + two directions
;; - Counter-clockwise orientation
;; - Uniquely determining from elements forces directions
;; - naming of "arms" vs. "directions"

;; Future Ideas:
;; - Automatically discover angles from diagrams (e.g. from a pile of
;;     points and segments)
;; - Angle intersections

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Angles ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; dir1 and dir2 are directions of the angle arms
;;; The angle sweeps from dir2 *counter clockwise* to dir1
(define-record-type <angle>
  (make-angle dir1 vertex dir2)
  angle?
  (dir1 angle-arm-1)
  (vertex angle-vertex)
  (dir2 angle-arm-2))

(declare-element-component-handler
 (component-procedure-from-getters
  ray-from-arm-1
  angle-vertex
  ray-from-arm-2)
 angle?)

;;;;;;;;;;;;;;;;;;;;; Transformations on Angles ;;;;;;;;;;;;;;;;;;;;;;

(define (reverse-angle a)
  (let ((d1 (angle-arm-1 a))
        (v (angle-vertex a))
        (d2 (angle-arm-2 a)))
    (make-angle d2 v d1)))

(define (smallest-angle a)
  (if (> (angle-measure a) pi)
      (reverse-angle a)
      a))

;;;;;;;;;;;;;;;;;;;;;;; Alternate Constructors ;;;;;;;;;;;;;;;;;;;;;;;

(define (angle-from-points p1 vertex p2)
  (let ((arm1 (direction-from-points vertex p1))
        (arm2 (direction-from-points vertex p2)))
    (make-angle arm1 vertex arm2)))

(define (smallest-angle-from-points p1 vertex p2)
  (smallest-angle (angle-from-points p1 vertex p2)))

;;;;;;;;;;;;;;;;;;;; Angle from pairs of elements ;;;;;;;;;;;;;;;;;;;;

(define angle-from (make-generic-operation 2 'angle-from))

(define (angle-from-lines l1 l2)
  (let ((d1 (line->direction l1))
        (d2 (line->direction l2))
        (p (intersect-lines l1 l2)))
```

```
    (make-angle d1 p d2)))
(defhandler angle-from angle-from-lines line? line?)

(define (angle-from-line-ray l r)
  (let ((vertex (ray-endpoint r)))
    (assert (on-line? vertex l)
            "Angle-from-line-ray: Vertex of ray not on line")
    (let ((d1 (line->direction l))
          (d2 (ray->direction r)))
      (make-angle d1 vertex d2))))
(defhandler angle-from angle-from-line-ray line? ray?)

(define (angle-from-ray-line r l)
  (reverse-angle (angle-from-line-ray l r)))
(defhandler angle-from angle-from-ray-line ray? line?)

(define (angle-from-segment-segment s1 s2)
  (define (angle-from-segment-internal s1 s2)
    (let ((vertex (segment-endpoint-1 s1)))
      (let ((d1 (segment->direction s1))
            (d2 (segment->direction s2)))
        (make-angle d1 vertex d2))))
  (cond ((point-equal? (segment-endpoint-1 s1)
                       (segment-endpoint-1 s2))
         (angle-from-segment-internal s1 s2))
        ((point-equal? (segment-endpoint-2 s1)
                       (segment-endpoint-1 s2))
         (angle-from-segment-internal (flip s1) s2))
        ((point-equal? (segment-endpoint-1 s1)
                       (segment-endpoint-2 s2))
         (angle-from-segment-internal s1 (flip s2)))
        ((point-equal? (segment-endpoint-2 s1)
                       (segment-endpoint-2 s2))
         (angle-from-segment-internal  (flip s1) (flip s2)))
        (else (error "Angle-from-segment-segment must share vertex"))))
(defhandler angle-from angle-from-segment-segment segment? segment?)

(define (smallest-angle-from a b)
  (smallest-angle (angle-from a b)))

;;;;;;;;;;;;;;;;;;;;;;; Predicates on Angles ;;;;;;;;;;;;;;;;;;;;;;;

(define (angle-measure-equal? a1 a2)
  (close-enuf? (angle-measure a1)
               (angle-measure a2)))

(define (supplementary-angles? a1 a2)
  (close-enuf? (+ (angle-measure a1)
                  (angle-measure a2))
               pi))

(define (complementary-angles? a1 a2)
  (close-enuf? (+ (angle-measure a1)
                  (angle-measure a2))
               (/ pi 2.0)))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Definitions ;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; TODO? Consider learning or putiting elsewhere
(define (linear-pair? a1 a2)
  (define (linear-pair-internal? a1 a2)
    (and (point-equal? (angle-vertex a1)
                       (angle-vertex a2))
         (direction-equal? (angle-arm-2 a1)
                           (angle-arm-1 a2))
         (direction-opposite? (angle-arm-1 a1)
                              (angle-arm-2 a2))))
  (or (linear-pair-internal? a1 a2)
```

```
      (linear-pair-internal? a2 a1)))

(define (vertical-angles? a1 a2)
  (and (point-equal? (angle-vertex a1)
                     (angle-vertex a2))
       (direction-opposite? (angle-arm-1 a1)
                            (angle-arm-1 a2))
       (direction-opposite? (angle-arm-2 a1)
                            (angle-arm-2 a2))))
```

Listing A.17: figure/bounds.scm

```scheme
;;; bounds.scm --- Graphics Bounds

;;; Commentary:

;; Ideas:
;; - Logic to extend segments to graphics bounds so they can be drawn.

;; Future:
;; - Separate logical bounds of figures from graphics bounds
;; - Combine logic for line and ray (one vs. two directions)
;; - Should these be a part of "figure" vs. "graphics"
;; - Remapping of entire figures to different canvas dimensions

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;; Bounds Constants ;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Max bounds of the graphics window

(define *g-min-x* -1)
(define *g-max-x*  1)
(define *g-min-y* -1)
(define *g-max-y*  1)

;;;;;;;;;;;;;;; Conversion to segments for Graphics ;;;;;;;;;;;;;;;;

(define (extend-to-max-segment p1 p2)
  (let ((x1 (point-x p1))
        (y1 (point-y p1))
        (x2 (point-x p2))
        (y2 (point-y p2)))
    (let ((dx (- x2 x1))
          (dy (- y2 y1)))
      (cond
       ((= 0 dx) (make-segment
                  (make-point x1 *g-min-y*)
                  (make-point x1 *g-max-y*)))
       ((= 0 dy) (make-segment
                  (make-point *g-min-x* y1)
                  (make-point *g-min-y* y1)))
       (else
        (let ((t-xmin (/ (- *g-min-x* x1) dx))
              (t-xmax (/ (- *g-max-x* x1) dx))
              (t-ymin (/ (- *g-min-y* y1) dy))
              (t-ymax (/ (- *g-max-y* y1) dy)))
          (let* ((sorted (sort (list t-xmin t-xmax t-ymin t-ymax) <))
                 (min-t (cadr sorted))
                 (max-t (caddr sorted))
                 (min-x (+ x1 (* min-t dx)))
                 (min-y (+ y1 (* min-t dy)))
                 (max-x (+ x1 (* max-t dx)))
                 (max-y (+ y1 (* max-t dy))))
            (make-segment (make-point min-x min-y)
                          (make-point max-x max-y)))))))))

(define (ray-extend-to-max-segment p1 p2)
  (let ((x1 (point-x p1))
        (y1 (point-y p1))
        (x2 (point-x p2))
        (y2 (point-y p2)))
    (let ((dx (- x2 x1))
          (dy (- y2 y1)))
      (cond
       ((= 0 dx) (make-segment
                  (make-point x1 *g-min-y*)
                  (make-point x1 *g-max-y*)))
```

```
      ((= 0 dy) (make-segment
                 (make-point *g-min-x* y1)
                 (make-point *g-min-y* y1)))
      (else
       (let ((t-xmin (/ (- *g-min-x* x1) dx))
             (t-xmax (/ (- *g-max-x* x1) dx))
             (t-ymin (/ (- *g-min-y* y1) dy))
             (t-ymax (/ (- *g-max-y* y1) dy)))
         (let* ((sorted (sort (list t-xmin t-xmax t-ymin t-ymax) <))
                (min-t (cadr sorted))
                (max-t (caddr sorted))
                (min-x (+ x1 (* min-t dx)))
                (min-y (+ y1 (* min-t dy)))
                (max-x (+ x1 (* max-t dx)))
                (max-y (+ y1 (* max-t dy))))
           (make-segment p1
                         (make-point max-x max-y)))))))))
```

Listing A.18: figure/circle.scm

```scheme
;;; circle.scm --- Circles

;;; Commentary:

;; Ideas:
;; - Currently rather limited support for circles

;; Future:
;; - Arcs, tangents, etc.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Circle structure ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <circle>
  (make-circle center radius)
  circle?
  (center circle-center)
  (radius circle-radius))

;;;;;;;;;;;;;;;;;;;;; Alternate Constructions ;;;;;;;;;;;;;;;;;;;;;;;

(define (circle-from-points center radius-point)
  (make-circle center
          (distance center radius-point)))

;;;;;;;;;;;;;;;;;;;;;;;;; Points on circle ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (point-on-circle-in-direction cir dir)
  (let ((center (circle-center cir))
        (radius (circle-radius cir)))
    (add-to-point
     center
     (vec-from-direction-distance
      dir radius))))
```

Listing A.19: figure/point.scm

```scheme
;;; point.scm --- Point

;;; Commentary:

;; Ideas:
;; - Points are the basis for most elements

;; Future:
;; - Transform to different canvases
;; - Have points know what elements they are on.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Point Structure ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <point>
  (make-point x y)
  point?
  (x point-x)
  (y point-y))

(define (print-point p)
  `(point ,(point-x p) ,(point-y p)))

(defhandler print
  print-point point?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (point-equal? p1 p2)
  (and (close-enuf? (point-x p1)
                    (point-x p2))
       (close-enuf? (point-y p1)
                    (point-y p2))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Operations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; P2 - P1
(define (sub-points p2 p1)
  (let ((x1 (point-x p1))
        (x2 (point-x p2))
        (y2 (point-y p2))
        (y1 (point-y p1)))
    (make-vec (- x2 x1)
              (- y2 y1))))

;;; Direction from p1 to p2
(define (direction-from-points p1 p2)
  (vec->direction (sub-points p2 p1)))

(define (add-to-point p vec)
  (let ((x (point-x p))
        (y (point-y p))
        (dx (vec-x vec))
        (dy (vec-y vec)))
    (make-point (+ x dx)
                (+ y dy))))
```

Listing A.20: figure/constructions.scm

```
;;; constructions.scm --- Constructions

;;; Commentary:

;; Ideas:
;; - Various logical constructions that can be peformed on elements
;; - Some higher-level constructions...

;; Future:
;; - More constructions?
;; - Separation between compass/straightedge and compound?
;; - Experiment with higher-level vs. learned constructions

;;; Code:

;;;;;;;;;;;;;;;;;;;;;; Segment Constructions ;;;;;;;;;;;;;;;;;;;;;;;

(define (midpoint p1 p2)
  (let ((newpoint
         (make-point (avg (point-x p1)
                          (point-x p2))
                     (avg (point-y p1)
                          (point-y p2)))))
    (with-dependency
     `(midpoint ,(element-dependency p1) ,(element-dependency p2))
     (with-source (lambda (premise)
                    (midpoint
                     ((element-source p1) premise)
                     ((element-source p1) premise)))
                  newpoint))))

(define (segment-midpoint s)
  (let ((p1 (segment-endpoint-1 s))
        (p2 (segment-endpoint-2 s)))
    (with-dependency
     `(segment-midpoint ,s)
     (with-source (lambda (premise)
                    (segment-midpoint
                     ((element-source s) premise)))
                  (midpoint p1 p2)))))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; TODO: Where to put these?
(define (on-segment? p seg)
  (let ((seg-start (segment-endpoint-1 seg))
        (seg-end (segment-endpoint-2 seg)))
    (let ((seg-length (distance seg-start seg-end))
          (p-length (distance seg-start p))
          (dir-1 (direction-from-points seg-start p))
          (dir-2 (direction-from-points seg-start seg-end)))
      (or (point-equal? seg-start p)
          (and (direction-equal? dir-1 dir-2)
               (or
                (point-equal? seg-end p)
                (< p-length seg-length)))))))

(define (on-line? p l)
  (let ((line-pt (line-point l))
        (line-dir (line-direction l)))
    (or (point-equal? p line-pt)
        (let ((dir-to-p (direction-from-points p line-pt)))
          (or (direction-equal? line-dir dir-to-p)
              (direction-equal? line-dir (reverse-direction dir-to-p)))))))

(define (on-ray? p r)
```

```scheme
  (let ((ray-endpt (ray-endpoint r))
        (ray-dir (ray-direction r)))
    (or (point-equal? ray-endpt p)
        (let ((dir-to-p (direction-from-points ray-endpt p)))
          (direction-equal? dir-to-p ray-dir)))))

;;;;;;;;;;;;;;;;;;;;;; Construction of lines ;;;;;;;;;;;;;;;;;;;;;;;

(define (perpendicular linear-element point)
  (let* ((direction (->direction linear-element))
         (rotated-direction (rotate-direction-90 direction)))
    (make-line point rotated-direction)))

;;; endpoint-1 is point, endpoint-2 is on linear-element
(define (perpendicular-to linear-element point)
  (let ((pl (perpendicular linear-element point)))
    (let ((i (intersect-linear-elements pl (->line linear-element))))
      (make-segment point i))))

(define (perpendicular-line-to linear-element point)
  (let ((pl (perpendicular linear-element point)))
    pl))

(define (perpendicular-bisector segment)
  (let ((midpt (segment-midpoint segment)))
    (perpendicular (segment->line segment)
                   midpt)))

(define (angle-bisector a)
  (let* ((d1 (angle-arm-1 a))
         (d2 (angle-arm-2 a))
         (vertex (angle-vertex a))
         (radians (angle-measure a))
         (half-angle (/ radians 2))
         (new-direction (add-to-direction d2 half-angle)))
    (make-ray vertex new-direction)))

(define (polygon-angle-bisector polygon vertex-angle)
  (angle-bisector (polygon-angle polygon vertex-angle)))

;;;;;;;;;;;;;;;;;;;; Higher-order constructions ;;;;;;;;;;;;;;;;;;;;

(define (circumcenter t)
  (let ((p1 (polygon-point-ref t 0))
        (p2 (polygon-point-ref t 1))
        (p3 (polygon-point-ref t 2)))
    (let ((l1 (perpendicular-bisector (make-segment p1 p2)))
          (l2 (perpendicular-bisector (make-segment p1 p3))))
      (intersect-linear-elements l1 l2))))

;;;;;;;;;;;;;;;;;;;; Concurrent Linear Elements ;;;;;;;;;;;;;;;;;;;;

(define (concurrent? l1 l2 l3)
  (let ((i-point (intersect-linear-elements l1 l2)))
    (and i-point
         (on-element? i-point l3))))

(define (concentric? p1 p2 p3 p4)
  (and (not (point-equal? p1 p2))
       (not (point-equal? p1 p3))
       (not (point-equal? p1 p4))
       (not (point-equal? p2 p3))
       (not (point-equal? p2 p4))
       (not (point-equal? p3 p4))
       (let ((pb-1 (perpendicular-bisector
                    (make-segment p1 p2)))
             (pb-2 (perpendicular-bisector
                    (make-segment p2 p3)))
```

```
             (pb-3 (perpendicular-bisector
                    (make-segment p3 p4))))
         (concurrent? pb-1 pb-2 pb-3))))

(define (concentric-with-center? center p1 p2 p3)
  (let ((d1 (distance center p1))
        (d2 (distance center p2))
        (d3 (distance center p3)))
    (and (close-enuf? d1 d2)
         (close-enuf? d1 d3))))
```

Listing A.21: figure/intersections.scm

```scheme
;;; intersections.scm --- Intersections

;;; Commentary:

;; Ideas:
;; - Unified intersections
;; - Separation of core computations

;; Future:
;; - Amb-like selection of multiple intersections, or list?
;; - Deal with elements that are exactly the same

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Computations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; http://en.wikipedia.org/wiki/Line%E2%80%93line_intersection
;;; line 1 through p1, p2 with line 2 through p3, p4
(define (intersect-lines-by-points p1 p2 p3 p4)
  (let ((x1 (point-x p1))
        (y1 (point-y p1))
        (x2 (point-x p2))
        (y2 (point-y p2))
        (x3 (point-x p3))
        (y3 (point-y p3))
        (x4 (point-x p4))
        (y4 (point-y p4)))
    (let* ((denom
            (det (det x1 1 x2 1)
                 (det y1 1 y2 1)
                 (det x3 1 x4 1)
                 (det y3 1 y4 1)))
           (num-x
            (det (det x1 y1 x2 y2)
                 (det x1  1 x2  1)
                 (det x3 y3 x4 y4)
                 (det x3  1 x4  1)))
           (num-y
            (det (det x1 y1 x2 y2)
                 (det y1  1 y2  1)
                 (det x3 y3 x4 y4)
                 (det y3  1 y4  1))))
      (if (= denom 0)
          '()
          (let
              ((px (/ num-x denom))
               (py (/ num-y denom)))
            (list (make-point px py)))))))

;;; http://mathforum.org/library/drmath/view/51836.html
(define (intersect-circles-by-centers-radii c1 r1 c2 r2)
  (let* ((a (point-x c1))
         (b (point-y c1))
         (c (point-x c2))
         (d (point-y c2))
         (e (- c a))
         (f (- d b))
         (p (sqrt (+ (square e)
                     (square f))))
         (k (/ (- (+ (square p) (square r1))
                  (square r2))
               (* 2 p))))
    (if (> k r1)
        (error "Circle's don't intersect")
        (let* ((t (sqrt (- (square r1)
                           (square k))))
```

```
               (x1 (+ a (/ (* e k) p)))
               (y1 (+ b (/ (* f k) p)))
               (dx (/ (* f t) p))
               (dy (- (/ (* e t) p))))
          (list (make-point (+ x1 dx)
                            (+ y1 dy))
                (make-point (- x1 dx)
                            (- y1 dy)))))))

;;; Intersect circle centered at c with radius r and line through
;;; points p1, p2
;;; http://mathworld.wolfram.com/Circle-LineIntersection.html
(define (intersect-circle-line-by-points c r p1 p2)
  (let ((offset (sub-points (make-point 0 0) c)))
    (let ((p1-shifted (add-to-point p1 offset))
          (p2-shifted (add-to-point p2 offset)))
      (let ((x1 (point-x p1-shifted))
            (y1 (point-y p1-shifted))
            (x2 (point-x p2-shifted))
            (y2 (point-y p2-shifted)))
        (let* ((dx (- x2 x1))
               (dy (- y2 y1))
               (dr (sqrt (+ (square dx) (square dy))))
               (d (det x1 x2 y1 y2))
               (disc (- (* (square r) (square dr)) (square d))))
          (if (< disc 0)
              (list)
              (let ((x-a (* d dy))
                    (x-b (* (sgn dy) dx (sqrt disc)))
                    (y-a (- (* d dx)))
                    (y-b (* (abs dy) (sqrt disc))))
                (let ((ip1 (make-point
                            (/ (+ x-a x-b) (square dr))
                            (/ (+ y-a y-b) (square dr))))
                      (ip2 (make-point
                            (/ (- x-a x-b) (square dr))
                            (/ (- y-a y-b) (square dr)))))
                  (if (close-enuf? 0 disc) ;; Tangent
                      (list (add-to-point ip1 (reverse-vec offset)))
                      (list (add-to-point ip1 (reverse-vec offset))
                            (add-to-point ip2 (reverse-vec offset)))))))))))

;;;;;;;;;;;;;;;;;;;;;;;; Basic Intersections ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (intersect-lines-to-list line1 line2)
  (let ((p1 (line-p1 line1))
        (p2 (line-p2 line1))
        (p3 (line-p1 line2))
        (p4 (line-p2 line2)))
    (intersect-lines-by-points p1 p2 p3 p4)))

(define (intersect-lines line1 line2)
  (let ((i-list (intersect-lines-to-list line1 line2)))
    (if (null? i-list)
        (error "Lines don't intersect")
        (car i-list))))

(define (intersect-circles cir1 cir2)
  (let ((c1 (circle-center cir1))
        (c2 (circle-center cir2))
        (r1 (circle-radius cir1))
        (r2 (circle-radius cir2)))
    (intersect-circles-by-centers-radii c1 r1 c2 r2)))

(define (intersect-circle-line cir line)
  (let ((center (circle-center cir))
        (radius (circle-radius cir))
        (p1 (line-p1 line))
```

```
        (p2 (line-p2 line)))
    (intersect-circle-line-by-points center radius p1 p2)))

(define standard-intersect
  (make-generic-operation 2 'standard-intersect))

(defhandler standard-intersect
  intersect-lines-to-list line? line?)

(defhandler standard-intersect
  intersect-circles circle? circle?)

(defhandler standard-intersect
  intersect-circle-line circle? line?)

(defhandler standard-intersect
  (flip-args intersect-circle-line) line? circle?)

;;;;;;;;;;;;;;;;;;;;;;; Generic intersection ;;;;;;;;;;;;;;;;;;;;;;;

(define (intersect-linear-elements el-1 el-2)
  (let ((i-list (standard-intersect (->line el-1)
                                    (->line el-2))))
    (if (null? i-list)
        #f
        (let ((i (car i-list)))
          (if (or (not (on-element? i el-1))
                  (not (on-element? i el-2)))
              #f
              i)))))

;;;;;;;;;;;;;;;;;;;;;;;;;;; On Elements ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define on-element? (make-generic-operation 2 'on-element?))

(defhandler on-element? on-segment? point? segment?)
(defhandler on-element? on-line? point? line?)
(defhandler on-element? on-ray? point? ray?)
```

Listing A.22: figure/figure.scm

```scheme
;;; figure.scm --- Figure

;;; Commentary:

;; Ideas:
;; - Gathers elements that are part of a figure
;; - Helpers to extract relevant elements

;; Future:
;; - Convert to record type like other structures
;; - Extract points automatically?

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;; Figure Structure ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (figure . elements)
  (cons 'figure elements))
(define (figure-elements figure)
  (cdr figure))

(define (all-figure-elements figure)
  (append (figure-elements figure)
          (figure-points figure)
          (figure-linear-elements figure)))

(define (figure? x)
  (and (pair? x)
       (eq? (car x 'figure))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Getters ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (figure-filter predicate figure)
  (filter predicate (figure-elements figure)))

(define (figure-points figure)
  (dedupe-by point-equal?
   (append (figure-filter point? figure)
           (append-map (lambda (polygon) (polygon-points polygon))
                       (figure-filter polygon? figure))
           (append-map (lambda (s)
                         (list (segment-endpoint-1 s)
                               (segment-endpoint-2 s)))
                       (figure-filter segment? figure)))))

(define (figure-angles figure)
  (append (figure-filter angle? figure)
          (append-map (lambda (polygon) (polygon-angles polygon))
                      (figure-filter polygon? figure))))

(define (figure-segments figure)
  (append (figure-filter segment? figure)
          (append-map (lambda (polygon) (polygon-segments polygon))
                      (figure-filter polygon? figure))))

(define (figure-linear-elements figure)
  (append (figure-filter linear-element? figure)
          (append-map (lambda (polygon) (polygon-segments polygon))
                      (figure-filter polygon? figure))))
```

Listing A.23: figure/math-utils.scm

```scheme
;;; math-utils.scm --- Math Helpers

;;; Commentary:

;; Ideas:
;; - All angles are [0, 2pi]
;; - Other helpers

;; Future:
;; - Add more as needed, integrate with scmutils-basic

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Angles ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define pi (* 4 (atan 1)))

(define (fix-angle-0-2pi a)
  (float-mod a (* 2 pi)))

(define (rad->deg rad)
  (* (/ rad (* 2 pi)) 360))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Modular ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (float-mod num mod)
  (- num
     (* (floor (/ num mod))
        mod)))

;;;;;;;;;;;;;;;;;;;;;;;;; Basic Operators ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (avg a b)
  (/ (+ a b) 2))

(define (sgn x)
  (if (< x 0) -1 1))

;;;;;;;;;;;;;;;;;;;;;;;;;; Linear Alegbra ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (det a11 a12 a21 a22)
  (- (* a11 a22) (* a12 a21)))

;;;;;;;;;;;;;;;;;;;;; Extensions of Max/Min ;;;;;;;;;;;;;;;;;;;;;;;

(define (min-positive . args)
  (min (filter (lambda (x) (>= x 0)) args)))

(define (max-negative . args)
  (min (filter (lambda (x) (<= x 0)) args)))
```

Listing A.24: figure/polygon.scm

```scheme
;;; polygon.scm --- Polygons

;;; Commentary:

;; Ideas:
;; - Points and (derived) segments define polygon

;; Future
;; - Figure out dependencies better
;; - Other operations, angles? diagonals? etc.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Polygon Structure ;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Data structure for a polygon, implemented as a list of
;;; points in counter-clockwise order.
;;; Drawing a polygon will draw all of its points and segments.
(define-record-type <polygon>
  (%polygon n-points points)
  polygon?
  (n-points polygon-n-points)
  (points %polygon-points))

(define (polygon-from-points . points)
  (let ((n-points (length points)))
    (%polygon n-points points)))

(define ((ngon-predicate n) obj)
  (and (polygon? obj)
       (= n (polygon-n-points obj))))

;;;;;;;;;;;;;;;;;;;;;;;;;; Polygon Points ;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Internal reference for polygon points
(define (polygon-point-ref polygon i)
  (if (not (<= 0 i (- (polygon-n-points polygon) 1)))
      (error "polygon point index not in range"))
  (list-ref (%polygon-points polygon) i))

(define (polygon-points polygon)
  (map (lambda (i) (polygon-point polygon i))
       (iota (polygon-n-points polygon))))

;;; External polygon points including dependencies
(define (polygon-point polygon i)
  ;;: TODO: Handle situations where polygon isn't terminal dependency
  (with-dependency ;;-if-unknown
   `(polygon-point ,i ,(element-dependency polygon))
   (with-source
    (lambda (p) (polygon-point (car p) i))
    (polygon-point-ref polygon i))))

(declare-element-component-handler
 polygon-point
 polygon?)

(define (polygon-index-from-point polygon point)
  (index-of
   point
   (%polygon-points polygon)
   point-equal?))

(define (name-polygon polygon)
  (for-each (lambda (i)
              (set-element-name! (polygon-point-ref polygon i)
```

```
                                 (nth-letter-symbol (+ i 1))))
            (iota (polygon-n-points polygon)))
  polygon)

;;;;;;;;;;;;;;;;;;;;;;;;; Polygon Segments ;;;;;;;;;;;;;;;;;;;;;;;;;

;;; i and j are indices of adjacent points
(define (polygon-segment polygon i j)
  (let ((n-points (polygon-n-points polygon)))
   (cond
    ((not (or (= i (modulo (+ j 1) n-points))
              (= j (modulo (+ i 1) n-points))))
     (error "polygon-segment must be called with adjacent indices"))
    ((or (>= i n-points)
         (>= j n-points))
     (error "polygon-segment point index out of range"))
    (else
     (let* ((p1 (polygon-point-ref polygon i))
            (p2 (polygon-point-ref polygon j))
            (segment (make-segment p1 p2)))
       ;;: TODO: Handle situations where polygon isn't terminal dependency
       (with-dependency
        ‘(polygon-segment ,i ,j ,polygon)
        (with-source
         (lambda (p) (polygon-segment (car p) i j))
         segment)))))))

(define (polygon-segments polygon)
  (let ((n-points (polygon-n-points polygon)))
    (map (lambda (i)
           (polygon-segment polygon i (modulo (+ i 1) n-points)))
         (iota n-points))))

;;;;;;;;;;;;;;;;;;;;;;;;;; Polygon Angles ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define polygon-angle
  (make-generic-operation 2 'polygon-angle))

(define (polygon-angle-by-index polygon i)
  (let ((n-points (polygon-n-points polygon)))
    (cond
     ((not (<= 0 i (- n-points 1)))
      (error "polygon-angle point index out of range"))
     (else
      (let* ((v (polygon-point-ref polygon i))
             (a1p (polygon-point-ref polygon
                                     (modulo (- i 1)
                                             n-points)))
             (a2p (polygon-point-ref polygon
                                     (modulo (+ i 1)
                                             n-points)))
             (angle (angle-from-points a1p v a2p)))
        (with-dependency
         ‘(polygon-angle ,i ,polygon)
         (with-source
          (lambda (p) (polygon-angle-by-index (car p) i))
          angle)))))))

(defhandler polygon-angle
  polygon-angle-by-index
  polygon? number?)

(define (polygon-angle-by-point polygon p)
  (let ((i (polygon-index-from-point polygon p)))
    (if (not i)
        (error "Point not in polygon" (list p polygon)))
    (polygon-angle-by-index polygon i)))
```

```
(defhandler polygon-angle
  polygon-angle-by-point
  polygon? point?)

(define (polygon-angles polygon)
  (map (lambda (i) (polygon-angle-by-index polygon i))
       (iota (polygon-n-points polygon))))
```

Listing A.25: figure/metadata.scm

```scheme
;;; metadata.scm - Element metadata

;;; Commentary:

;; Ideas:
;; - Currently, names
;; - Dependencies grew here, but are now separate

;; Future:
;; - Point/Linear/Circle adjacency - walk like graph

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Names ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (set-element-name! element name)
  (eq-put! element 'name name)
  element)

(define (element-name element)
  (or (eq-get element 'name)
      '*unnamed*))
```

Listing A.26: figure/dependencies.scm

```scheme
;;; dependencies.scm --- Dependencies of figure elements

;;; Commentary:

;; Ideas:
;; - Use eq-properties to set dependencies of elements
;; - Some random elements are gien external/random dependencies
;; - For some figures, override dependencies of intermediate elements

;; Future:
;; - Expand to full dependencies
;; - Start "learning" and generalizing

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Sources ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (set-source! element source)
  (eq-put! element 'source source))

(define (with-source source element)
  (set-source! element source)
  element)

(define (element-source element)
  (or (eq-get element 'source)
      '*unknown-source*))

;;;;;;;;;;;;;;;;;;;;;;; Setitng Dependencies ;;;;;;;;;;;;;;;;;;;;;;;

(define (set-dependency! element dependency)
  (eq-put! element 'dependency dependency))

(define (with-dependency dependency element)
  (set-dependency! element dependency)
  element)


(define (with-dependency-if-unknown dependency element)
  (if (dependency-unknown? element)
      (with-dependency dependency element)
      element))
;;;;;;;;;;;;;;;;;;;;;;; Unknown Dependencies ;;;;;;;;;;;;;;;;;;;;;;;

(define *unknown-dependency* (list '*unknown-dependency*))
(define (unknown-dependency? x)
  (eq? x *unknown-dependency*))

(define (dependency-unknown? element)
  (unknown-dependency? (element-dependency element)))

(define dependency-known? (notp dependency-unknown?))
;;;;;;;;;;;;;;;;;;;;;; Accessing Dependencies ;;;;;;;;;;;;;;;;;;;;;;

(define (element-dependency element)
  (or (eq-get element 'dependency)
      *unknown-dependency*))

;;;;;;;;;;;;;;;;;;;;;;; Random Dependencies ;;;;;;;;;;;;;;;;;;;;;;;;
(define (make-random-dependency tag)
  (%make-random-dependency tag 0))

(define-record-type <random-dependency>
  (%make-random-dependency tag num)
  random-dependency?
  (tag random-dependency-tag)
```

```
  (num %random-dependency-num set-random-dependency-num!))

(define (random-dependency-num rd)
  (let ((v (%random-dependency-num rd)))
    (if (= v 0)
        0
        v)))

(define (print-random-dependency rd)
  (list (random-dependency-tag rd)
        (random-dependency-num rd)))
(defhandler print print-random-dependency random-dependency?)

(define (number-figure-random-dependencies! figure)
  (define *random-dependency-num* 1)
  (map (lambda (el)
         (let ((dep (element-dependency el)))
           (cond ((random-dependency? dep)
                  (set-random-dependency-num!
                   dep
                   *random-dependency-num*)
                  (set! *random-dependency-num*
                        (+ *random-dependency-num* 1))))))
       (figure-elements figure))
  'done)

(define element-dependencies->list
  (make-generic-operation
   1 'element-dependencies->list
   (lambda (x) x)))

(define (element-dependency->list el)
  (element-dependencies->list
   (element-dependency el)))

(defhandler element-dependencies->list
  element-dependency->list
  dependency-known?)

(defhandler element-dependencies->list
  print-random-dependency
  random-dependency?)

(defhandler element-dependencies->list
  (lambda (l)
    (map element-dependencies->list l))
  list?)
```

Listing A.27: figure/randomness.scm

```scheme
;;; randomness.scm --- Random creation of elements

;;; Commentary:

;; Ideas:
;; - Random points, segments, etc. essential to system
;; - Separated out animation / persistence across frames

;; Future:
;; - Better random support
;; - Maybe separating out "definitions" (random square, etc.)

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Base: Random Scalars ;;;;;;;;;;;;;;;;;;;;;;;

(define (internal-rand-range min-v max-v)
  (if (close-enuf? min-v max-v)
      (error "range is too close for rand-range"
             (list min-v max-v))
      (let ((interval-size (max *machine-epsilon* (- max-v min-v))))
        (persist-value (+ min-v (random (* 1.0 interval-size)))))))

(define (safe-internal-rand-range min-v max-v)
  (let ((interval-size (max 0 (- max-v min-v))))
    (internal-rand-range
     (+ min-v (* 0.1 interval-size))
     (+ min-v (* 0.9 interval-size)))))

;;;;;;;;;;;;;;;;;;;;;;;;;; Animated Ranges ;;;;;;;;;;;;;;;;;;;;;;;;;

(define *wiggle-ratio* 0.15)

;;; Will return floats even if passed integers
;;; TODO: Rename to animated?
(define (rand-range min max)
  (let* ((range-size (- max min))
         (wiggle-amount (* range-size *wiggle-ratio*))
         (v (internal-rand-range min (- max wiggle-amount))))
    (animate-range v (+ v wiggle-amount))))

;;; Random Values - distances, angles

(define (rand-theta)
  (rand-range 0 (* 2 pi)))

(define (rand-angle-measure)
  (rand-range (* pi 0.05) (* .95 pi)))

(define (rand-obtuse-angle-measure)
  (rand-range (* pi 0.55) (* .95 pi)))

(define (random-direction)
  (let ((theta (rand-theta)))
    (make-direction theta)))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Random Points ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define *point-wiggle-radius* 0.05)
(define (random-point)
  (let ((x (internal-rand-range -0.8 0.8))
        (y (internal-rand-range -0.8 0.8)))
    (random-point-around (make-point x y))))

(define (random-point-around p)
  (let ((x (point-x p))
```

```
        (y (point-y p)))
    (let ((theta (internal-rand-range 0 (* 2 pi)))
          (d-theta (animate-range 0 (* 2 pi))))
      (let ((dir (make-direction (+ theta d-theta))))
        (with-dependency
         (make-random-dependency 'random-point)
         (add-to-point
          (make-point x y)
          (vec-from-direction-distance dir *point-wiggle-radius*)))))))

;;; TODO: Maybe separate out reflection about line?
(define (random-point-left-of-line line)
  (let* ((p (random-point))
         (d (signed-distance-to-line p line))
         (v (rotate-vec-90
             (unit-vec-from-direction
              (line-direction line)))))
    (if (> d 0)
        p
        (add-to-point p (scale-vec v (* 2 (- d)))))))

(define (random-point-between-rays r1 r2)
  (let ((offset-vec (sub-points (ray-endpoint r2)
                            (ray-endpoint r1))))
    (let ((d1 (ray-direction r1))
          (d2 (ray-direction r2)))
      (let ((dir-difference (subtract-directions d2 d1)))
        (let ((new-dir (add-to-direction
                        d1
                        (internal-rand-range 0.05 dir-difference))))
          (random-point-around
           (add-to-point
            (add-to-point (ray-endpoint r1)
                          (vec-from-direction-distance
                           new-dir
                           (internal-rand-range 0.05 0.9)))
            (scale-vec offset-vec
                       (internal-rand-range 0.05 0.9)))))))))

(define (random-point-on-segment seg)
  (let* ((p1 (segment-endpoint-1 seg))
         (p2 (segment-endpoint-2 seg))
         (t (rand-range 0.0 1.0))
         (v (sub-points p2 p1)))
    (add-to-point p1 (scale-vec v t))))

;;; TODO: Fix this for new construction
(define (random-point-on-line l)
  (let* ((p1 (line-p1 l))
         (p2 (line-p2 l))
         (seg (extend-to-max-segment p1 p2))
         (sp1 (segment-endpoint-1 seg))
         (sp2 (segment-endpoint-2 seg))
         (t (rand-range 0.0 1.0))
         (v (sub-points sp2 sp1)))
    (add-to-point sp1 (scale-vec v t))))

(define (random-point-on-ray r)
  (let* ((p1 (ray-endpoint r))
         (dir (ray-direction r))
         (p2 (add-to-point p1 (unit-vec-from-direction dir)))
         (seg (ray-extend-to-max-segment p1 p2))
         (sp1 (segment-endpoint-1 seg))
         (sp2 (segment-endpoint-2 seg))
         (t (rand-range 0.05 1.0))
         (v (sub-points sp2 sp1)))
    (add-to-point sp1 (scale-vec v t))))
```

```
(define (random-point-on-circle c)
  (let ((dir (random-direction)))
    (point-on-circle-in-direction c dir)))

(define (n-random-points-on-circle-ccw c n)
  (let* ((thetas
          (sort
           (make-initialized-list n (lambda (i) (rand-theta)))
           <)))
    (map (lambda (theta)
           (point-on-circle-in-direction c
                                         (make-direction theta)))
         thetas)))

;;;;;;;;;;;;;;;;;;;;;; Random Linear Elements ;;;;;;;;;;;;;;;;;;;;;;

(define (random-line)
  (let ((p (random-point)))
    (with-dependency
     (make-random-dependency 'random-line)
     (random-line-through-point p))))

(define (random-segment)
  (let ((p1 (random-point))
        (p2 (random-point)))
    (let ((seg (make-segment p1 p2)))
      (set-segment-dependency!
       seg
       (make-random-dependency 'random-segment))
      seg)))

(define (random-ray)
  (let ((p (random-point)))
    (random-ray-from-point p)))

(define (random-line-through-point p)
  (let ((v (random-direction)))
    (line-from-point-direction p v)))

(define (random-ray-from-point p)
  (let ((v (random-direction)))
    (ray-from-point-direction p v)))

(define (random-horizontal-line)
  (let ((p (random-point))
        (v (make-vec 1 0)))
    (line-from-point-vec p v)))

(define (random-vertical-line)
  (let ((p (random-point))
        (v (make-vec 0 1)))
    (line-from-point-vec p v)))

;;;;;;;;;;;;;;;;;;;;;; Random Circle Elements ;;;;;;;;;;;;;;;;;;;;;;

(define (random-circle-radius circle)
  (let ((center (circle-center circle))
        (radius (circle-radius circle))
        (angle (random-direction)))
    (let ((radius-vec
           (scale-vec (unit-vec-from-direction
                       (random-direction))
                      radius)))
      (let ((radius-point (add-to-point center radius-vec)))
        (make-segment center radius-point)))))

(define (random-circle)
  (let ((pr1 (random-point))
```

```
        (pr2 (random-point)))
    (circle-from-points (midpoint pr1 pr2) pr1)))

(define (random-angle)
  (let* ((v (random-point))
         (d1 (random-direction))
         (d2 (add-to-direction
              d1
              (rand-angle-measure))))
    (smallest-angle (make-angle d1 v d2))))

;;;;;;;;;;;;;;;;;;;;;;;;; Random Polygons ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (random-n-gon n)
  (if (< n 3)
      (error "n must be > 3"))
  (let* ((p1 (random-point))
         (p2 (random-point)))
    (let ((ray2 (reverse-ray (ray-from-points p1 p2))))
      (let lp ((n-remaining (- n 2))
               (points (list p2 p1)))
        (if (= n-remaining 0)
            (apply polygon-from-points (reverse points))
            (lp (- n-remaining 1)
                (cons (random-point-between-rays
                       (reverse-ray (ray-from-points (car points)
                                                     (cadr points)))
                       ray2)
                      points)))))))

(define (random-polygon)
  (random-n-gon (+ 3 (random 5))))

;;;;;;;;;;;;;;;;;;;;;;;;; Random Triangles ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (random-triangle)
  (let* ((p1 (random-point))
         (p2 (random-point))
         (p3 (random-point-left-of-line (line-from-points p1 p2))))
    (with-dependency
     (make-random-dependency 'random-triangle)
     (polygon-from-points p1 p2 p3))))

(define (random-equilateral-triangle)
  (let* ((s1 (random-segment))
         (s2 (rotate-about (segment-endpoint-1 s1)
                           (/ pi 3)
                           s1)))
    (with-dependency
     (make-random-dependency 'random-equilateral-triangle)
     (polygon-from-points
      (segment-endpoint-1 s1)
      (segment-endpoint-2 s1)
      (segment-endpoint-2 s2)))))

(define (random-isoceles-triangle)
  (let* ((s1 (random-segment))
         (base-angle (rand-angle-measure))
         (s2 (rotate-about (segment-endpoint-1 s1)
                           base-angle
                           s1)))
    (with-dependency
     (make-random-dependency 'random-isoceles-triangle)
     (polygon-from-points
      (segment-endpoint-1 s1)
      (segment-endpoint-2 s1)
      (segment-endpoint-2 s2)))))
```

```
;;;;;;;;;;;;;;;;;;;;;;; Random Quadrilaterals ;;;;;;;;;;;;;;;;;;;;;;;;

(define (random-quadrilateral)
  (with-dependency
   (make-random-dependency 'random-quadrilateral)
   (random-n-gon 4)))

(define (random-square)
  (let* ((s1 (random-segment))
         (p1 (segment-endpoint-1 s1))
         (p2 (segment-endpoint-2 s1))
         (p3 (rotate-about p2
                           (- (/ pi 2))
                           p1))
         (p4 (rotate-about p1
                           (/ pi 2)
                           p2)))
    (with-dependency
     (make-random-dependency 'random-square)
     (polygon-from-points p1 p2 p3 p4))))

(define (random-rectangle)
  (let* ((r1 (random-ray))
         (p1 (ray-endpoint r1))
         (r2 (rotate-about (ray-endpoint r1)
                           (/ pi 2)
                           r1))
         (p2 (random-point-on-ray r1))
         (p4 (random-point-on-ray r2))
         (p3 (add-to-point
              p2
              (sub-points p4 p1))))
    (with-dependency
     (make-random-dependency 'random-rectangle)
     (polygon-from-points
      p1 p2 p3 p4))))

(define (random-parallelogram)
  (let* ((r1 (random-ray))
         (p1 (ray-endpoint r1))
         (r2 (rotate-about (ray-endpoint r1)
                           (rand-angle-measure)
                           r1))
         (p2 (random-point-on-ray r1))
         (p4 (random-point-on-ray r2))
         (p3 (add-to-point
              p2
              (sub-points p4 p1))))
    (with-dependency
     (make-random-dependency 'random-parallelogram)
     (polygon-from-points p1 p2 p3 p4))))

(define (random-kite)
  (let* ((r1 (random-ray))
         (p1 (ray-endpoint r1))
         (r2 (rotate-about (ray-endpoint r1)
                           (rand-obtuse-angle-measure)
                           r1))
         (p2 (random-point-on-ray r1))
         (p4 (random-point-on-ray r2))
         (p3 (reflect-about-line
              (line-from-points p2 p4)
              p1)))
    (with-dependency
     (make-random-dependency 'random-parallelogram)
     (polygon-from-points p1 p2 p3 p4))))

(define (random-rhombus)
```

```
(let* ((s1 (random-segment))
       (p1 (segment-endpoint-1 s1))
       (p2 (segment-endpoint-2 s1))
       (p4 (rotate-about p1
                         (rand-angle-measure)
                         p2))
       (p3 (add-to-point
            p2
            (sub-points p4 p1))))
  (with-dependency
   (make-random-dependency 'random-rhombus)
   (polygon-from-points p1 p2 p3 p4))))
```

Listing A.28: figure/transforms.scm

```scheme
;;; transforms.scm --- Transforms on Elements

;;; Commentary:

;; Ideas:
;; - Generic transforms - rotation and translation
;; - None mutate points, just return new copies.

;; Future:
;; - Translation or rotation to match something
;; - Consider mutations?
;; - Reflections?

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Rotations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (rotate-point-about rot-origin radians point)
  (let ((v (sub-points point rot-origin)))
    (let ((rotated-v (rotate-vec v radians)))
      (add-to-point rot-origin rotated-v))))

(define (rotate-segment-about rot-origin radians seg)
  (define (rotate-point p) (rotate-point-about rot-origin radians p))
  (make-segment (rotate-point (segment-endpoint-1 seg))
                (rotate-point (segment-endpoint-2 seg))))

(define (rotate-ray-about rot-origin radians r)
  (define (rotate-point p) (rotate-point-about rot-origin radians p))
  (make-ray (rotate-point-about rot-origin radians (ray-endpoint r))
            (add-to-direction (ray-direction r) radians)))

(define (rotate-line-about rot-origin radians l)
  (make-line (rotate-point-about rot-origin radians (line-point l))
             (add-to-direction (line-direction l) radians)))

(define rotate-about (make-generic-operation 3 'rotate-about))
(defhandler rotate-about rotate-point-about point? number? point?)
(defhandler rotate-about rotate-ray-about point? number? ray?)
(defhandler rotate-about rotate-segment-about point? number? segment?)
(defhandler rotate-about rotate-line-about point? number? line?)

(define (rotate-randomly-about p elt)
  (let ((radians (rand-angle-measure)))
    (rotate-about p radians elt)))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Translations ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (translate-point-by vec point)
  (add-to-point point vec))

(define (translate-segment-by vec segment)
  (define (translate-point p) (translate-point-by vec p))
  (make-segment (translate-point (segment-endpoint-1 seg))
                (translate-point (segment-endpoint-2 seg))))

(define (translate-ray-by vec r)
  (make-ray (translate-point-by vec (ray-endpoint r))
            (ray-direction r)))

(define (translate-line-by vec l)
  (make-line (translate-point-by vec (line-point l))
             (line-direction l)))

(define (translate-angle-by vec a)
  (define (translate-point p) (translate-point-by vec p))
```

```scheme
  (make-angle (angle-arm-1 a)
              (translate-point (angle-vertex a))
              (angle-arm-2 a)))

(define translate-by (make-generic-operation 2 'rotate-about))
(defhandler translate-by translate-point-by vec? point?)
(defhandler translate-by translate-ray-by vec? ray?)
(defhandler translate-by translate-segment-by vec? segment?)
(defhandler translate-by translate-line-by vec? line?)
(defhandler translate-by translate-angle-by vec? angle?)

;;; Reflections

(define (reflect-about-line line p)
  (if (on-line? p line)
      p
      (let ((s (perpendicular-to line p)))
        (let ((v (segment->vec s)))
          (add-to-point
           p
           (scale-vec v 2))))))

;;;;;;;;;;;;;;;;;;;;;;;;; Random Translation ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (translate-randomly-along-line l elt)
  (let* ((vec (unit-vec-from-direction (line->direction l)))
         (scaled-vec (scale-vec vec (rand-range 0.5 1.5))))
    (translate-by vec elt)))

(define (translate-randomly elt)
  (let ((vec (rand-translation-vec-for elt)))
    (translate-by vec elt)))

(define (rand-translation-vec-for-point p1)
  (let ((p2 (random-point)))
    (sub-points p2 p1)))

(define (rand-translation-vec-for-segment seg)
  (rand-translation-vec-for-point (segment-endpoint-1 seg)))

(define (rand-translation-vec-for-ray r )
  (rand-translation-vec-for-point (ray-endpoint r)))

(define (rand-translation-vec-for-line l)
  (rand-translation-vec-for-point (line-point l)))

(define rand-translation-vec-for
  (make-generic-operation 1 'rand-translation-vec-for))
(defhandler rand-translation-vec-for
  rand-translation-vec-for-point point?)
(defhandler rand-translation-vec-for
  rand-translation-vec-for-segment segment?)
(defhandler rand-translation-vec-for
  rand-translation-vec-for-ray ray?)
(defhandler rand-translation-vec-for
  rand-translation-vec-for-line line?)
```

Listing A.29: perception/load.scm

```
;;; load.scm -- Load perception
(for-each (lambda (f) (load f))
          '("relationship"
            "observation"
            "analyzer"))
```

Listing A.30: perception/observation.scm

```scheme
;;; observation.scm -- observed relationships

;;; Commentary:

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Observation ;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <observation>
  (make-observation premises relationship args)
  observation?
  (premises observation-premises)
  (relationship observation-relationship)
  (args observation-args))

(define (observation-equal? obs1 obs2)
  (equal? (print-observation obs1)
          (print-observation obs2)))

(define (print-observation obs)
  (cons
   (print (observation-relationship obs))
   (map element-dependencies->list (observation-args obs))))

(defhandler print print-observation observation?)


;;;;;;;;;;;;;;;;;;;;;;;; Checking observation ;;;;;;;;;;;;;;;;;;;;;;;;

(define (satisfies-observation obs new-premise)
  (let ((new-args
         (map (lambda (arg)
                ((element-source arg) new-premise))
              (observation-args obs)))
        (rel (observation-relationship obs)))
    (or (relationship-holds rel new-args)
        (begin (if *explain*
                   (pprint `(failed-observation ,obs)))
               #f))))

;;;;;;;;;;;;;;;;;;;;; Simplifying observations ;;;;;;;;;;;;;;;;;;;;;

(define (simplify-observations observations base-observations)
  (define memp (member-procedure observation-equal?))
  (filter
   (lambda (o) (not (memp o base-observations)))
   observations))
```

Listing A.31: perception/analyzer.scm

```scheme
;;; analyzer.scm --- Tools for analyzing Diagram

;;; Commentary

;; Ideas:
;; - Analyze figrue to dermine properties "beyond coincidence"
;; - Use dependency structure to eliminate some obvious examples.

;; Future:
;; - Add More "interesting properties"
;; - Create storage for learned properties.
;; - Output format, add names
;; - Separate "discovered" from old properties.

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;; Main Interface ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (analyze-figure figure)
  (analyze figure))

;;; Given a figure, report what's interesting
(define (analyze figure)
  (number-figure-random-dependencies! figure)
  (let* ((points (figure-points figure))
         (angles (figure-angles figure))
         (implied-segments '() ; (point-pairs->segments (all-pairs points))
                           )
         (linear-elements (append
                           (figure-linear-elements figure)
                           implied-segments))
         (segments (append (figure-segments figure)
                           implied-segments)))
    (append
     (extract-relationships points
                            (list concurrent-points-relationship
                                  concentric-relationship
                                  concentric-with-center-relationship))
     (extract-relationships segments
                             (list equal-length-relationship))
     (extract-relationships angles
                             (list equal-angle-relationship
                                   supplementary-angles-relationship
                                   complementary-angles-relationship))
     (extract-relationships linear-elements
                             (list parallel-relationship
                                   perpendicular-relationship
                                   )))))

(define (extract-relationships elements relationships)
  (append-map (lambda (r)
                (extract-relationship elements r))
              relationships))

(define (extract-relationship elements relationship)
  (map (lambda (tuple)
         (make-observation '() relationship tuple))
       (report-n-wise
        (relationship-arity relationship)
        (relationship-predicate relationship)
        elements)))

;;;;;;;;;;;;;;;;;;;;;; Cross products, pairs ;;;;;;;;;;;;;;;;;;;;;;;;

;;; General proceudres for generating pairs
(define (all-pairs elements)
```

```scheme
  (all-n-tuples 2 elements))

(define (all-n-tuples n elements)
  (cond ((zero? n) '(()))
        ((< (length elements) n) '())
        (else
         (let lp ((elements-1 elements))
           (if (null? elements-1)
               '()
               (let ((element-1 (car elements-1))
                     (n-minus-1-tuples
                      (all-n-tuples (- n 1) (cdr elements-1))))
                 (append
                  (map
                   (lambda (rest-tuple)
                     (cons element-1 rest-tuple))
                   n-minus-1-tuples)
                  (lp (cdr elements-1)))))))))

;;;;;;;;;;;;;;;;;;;;;;;;; Obvious Segments ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (segment-for-endpoint p1)
  (let ((dep (element-dependency p1)))
    (and dep
         (or (and (eq? (car dep) 'segment-endpoint-1)
                  (cadr dep))
             (and (eq? (car dep) 'segment-endpoint-2)
                  (cadr dep))))))

(define (derived-from-same-segment? p1 p2)
  (and
   (segment-for-endpoint p1)
   (segment-for-endpoint p2)
   (eq? (segment-for-endpoint p1)
        (segment-for-endpoint p2))))

(define (polygon-for-point p1)
  (let ((dep (element-dependency p1)))
    (and dep
         (and (eq? (car dep) 'polygon-point)
              (cons (caddr dep)
                    (cadr dep))))))

(define (adjacent-in-same-polygon? p1 p2)
  (let ((poly1 (polygon-for-point p1))
        (poly2 (polygon-for-point p2)))
    (and poly1 poly2
         (eq? (car poly1) (car poly2))
         (or (= (abs (- (cdr poly1)
                        (cdr poly2)))
                1)
             (and (= (cdr poly1) 0)
                  (= (cdr poly2) 3))
             (and (= (cdr poly1) 3)
                  (= (cdr poly2) 0))))))

(define (point-pairs->segments ppairs)
  (filter (lambda (segment) segment)
          (map (lambda (point-pair)
                 (let ((p1 (car point-pair))
                       (p2 (cadr point-pair)))
                   (and (not (point-equal? p1 p2))
                        (not (derived-from-same-segment? p1 p2))
                        (not (adjacent-in-same-polygon? p1 p2))
                        (make-auxiliary-segment
                         (car point-pair)
                         (cadr point-pair))))) ; TODO: Name segment
               ppairs)))
```

```
;;;;;;;;;;;;;;;;;;;;;;;;; Dealing with pairs ;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Check for pairwise equality
(define ((nary-predicate n predicate) tuple)
  (apply predicate tuple))

;;; Merges "connected-components" of pairs
(define (merge-pair-groups elements pairs)
  (let ((i 0)
        (group-ids (make-key-weak-eq-hash-table))
        (group-elements (make-key-weak-eq-hash-table))) ; Map from pair
    (for-each (lambda (pair)
                (let ((first (car pair))
                      (second (cadr pair)))
                  (let ((group-id-1 (hash-table/get group-ids first i))
                        (group-id-2 (hash-table/get group-ids second i)))
                    (cond ((and (= group-id-1 i)
                                (= group-id-2 i))
                           ;; Both new, new groups:
                           (hash-table/put! group-ids first group-id-1)
                           (hash-table/put! group-ids second group-id-1))
                          ((= group-id-1 i)
                           (hash-table/put! group-ids first group-id-2))
                          ((= group-id-2 i)
                           (hash-table/put! group-ids second group-id-1)))
                    (set! i (+ i 1)))))
              pairs)
    (for-each (lambda (elt)
                (hash-table/append group-elements
                                   (hash-table/get group-ids elt 'invalid)
                                   elt))
              elements)
    (hash-table/remove! group-elements 'invalid)
    (hash-table/datum-list group-elements)))

(define (report-n-wise n predicate elements)
  (let ((tuples (all-n-tuples n elements)))
    (filter (nary-predicate n predicate) tuples)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Results: ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (make-analysis-collector)
  (make-equal-hash-table))

(define (save-results results data-table)
  (hash-table/put! data-table results
                   (+ 1 (hash-table/get data-table results 0))))

(define (print-analysis-results data-table)
  (hash-table/for-each
   data-table
   (lambda (k v)
     (pprint (list v (cons 'discovered k))))))
```

Listing A.32: graphics/load.scm

```scheme
;;; load.scm -- Load graphics
(for-each (lambda (f) (load f))
          '("appearance"
            "graphics"))
```

Listing A.33: graphics/appearance.scm

```scheme
(define (with-color color element)
  (eq-put! element 'color color)
  element)

(define default-element-color
  (make-generic-operation 1
                          'default-element-color
                          (lambda (e) "black")))

(defhandler default-element-color (lambda (e) "blue") point?)
(defhandler default-element-color (lambda (e) "black") segment?)

(define (element-color element)
  (or (eq-get element 'color)
      (default-element-color element)))
```

Listing A.34: graphics/graphics.scm

```
(define (draw-figure figure canvas)
  (clear-canvas canvas)
  (for-each
   (lambda (element)
     (canvas-set-color canvas (element-color element))
     ((draw-element element) canvas))
   (all-figure-elements figure))
  (for-each
   (lambda (element)
     (canvas-set-color canvas (element-color element))
     ((draw-label element) canvas))
   (all-figure-elements figure))
  (graphics-flush (canvas-g canvas))
  )

(define draw-element
  (make-generic-operation 1 'draw-element
                          (lambda (e) (lambda (c) 'done))))

(define draw-label
  (make-generic-operation 1 'draw-label (lambda (e) (lambda (c)'done))))

(define (add-to-draw-element! predicate handler)
  (defhandler draw-element
    (lambda (element)
      (lambda (canvas)
        (handler canvas element)))
    predicate))

(define (add-to-draw-label! predicate handler)
  (defhandler draw-label
    (lambda (element)
      (lambda (canvas)
        (handler canvas element)))
    predicate))


(define *point-radius* 0.02)
(define (draw-point canvas point)
  (canvas-fill-circle canvas
               (point-x point)
               (point-y point)
               *point-radius*))
(define (draw-point-label canvas point)
  (canvas-draw-text canvas
                    (+ (point-x point) *point-radius*)
                    (+ (point-y point) *point-radius*)
                    (symbol->string (element-name point))))

(define (draw-segment canvas segment)
  (let ((p1 (segment-endpoint-1 segment))
        (p2 (segment-endpoint-2 segment)))
    (canvas-draw-line canvas
                      (point-x p1)
                      (point-y p1)
                      (point-x p2)
                      (point-y p2))))
(define (draw-segment-label canvas segment)
  (let ((v (vec-from-direction-distance (rotate-direction-90
                                         (segment->direction segment))
                                        (* 2 *point-radius*)))
        (m (segment-midpoint segment)))
    (let ((label-point (add-to-point m v)))
      (canvas-draw-text canvas
                        (point-x label-point)
                        (point-y label-point)
```

```
                        (symbol->string (element-name segment))))))

(define (draw-line canvas line)
  (let ((p1 (line-p1 line)))
   (let ((p2 (add-to-point
              p1
              (unit-vec-from-direction (line-direction line)))))
     (draw-segment canvas (extend-to-max-segment p1 p2)))))

(define (draw-ray canvas ray)
  (let ((p1 (ray-endpoint ray)))
    (let ((p2 (add-to-point
               p1
               (unit-vec-from-direction (ray-direction ray)))))
      (draw-segment canvas (ray-extend-to-max-segment p1 p2)))))

(define (draw-circle canvas c)
  (let ((center (circle-center c))
        (radius (circle-radius c)))
    (canvas-draw-circle canvas
                        (point-x center)
                        (point-y center)
                        radius)))

(define *angle-mark-radius* 0.1)
(define (draw-angle canvas a)
  (let* ((vertex (angle-vertex a))
         (d1 (angle-arm-1 a))
         (d2 (angle-arm-2 a))
         (angle-start (direction-theta d2))
         (angle-end (direction-theta d1)))
    (canvas-draw-arc canvas
                     (point-x vertex)
                     (point-y vertex)
                     *angle-mark-radius*
                     angle-start
                     angle-end)))

;;; Add to generic operations

(add-to-draw-element! point? draw-point)
(add-to-draw-element! segment? draw-segment)
(add-to-draw-element! circle? draw-circle)
(add-to-draw-element! angle? draw-angle)
(add-to-draw-element! line? draw-line)
(add-to-draw-element! ray? draw-ray)

(add-to-draw-label! point? draw-point-label)

;;; Canvas for x-graphics

(define (x-graphics) (make-graphics-device 'x))

(define (canvas)
  (let ((g (x-graphics)))
    (graphics-enable-buffering g)
    (list 'canvas g)))

(define (canvas-g canvas)
  (cadr canvas))

(define (canvas? x)
  (and (pair? x)
       (eq? (car x 'canvas))))

(define (clear-canvas canvas)
  (graphics-clear (canvas-g canvas)))
```

```scheme
(define (canvas-draw-circle canvas x y radius)
  (graphics-operation (canvas-g canvas)
                      'draw-circle
                      x y radius))

(define (canvas-draw-text canvas x y text)
  (graphics-draw-text (canvas-g canvas) x y text))

(define (canvas-draw-arc canvas x y radius
                         angle-start angle-end)
  (let ((angle-sweep
         (fix-angle-0-2pi (- angle-end
                             angle-start))))
    (graphics-operation (canvas-g canvas)
                        'draw-arc
                        x y radius radius
                        (rad->deg angle-start)
                        (rad->deg angle-sweep)
                        #f)))

(define (canvas-fill-circle canvas x y radius)
  (graphics-operation (canvas-g canvas)
                      'fill-circle
                      x y radius))

(define (canvas-draw-line canvas x1 y1 x2 y2)
  (graphics-draw-line (canvas-g canvas)
                      x1 y1
                      x2 y2))

(define (canvas-set-color canvas color)
  (graphics-operation (canvas-g canvas) 'set-foreground-color color)
  )
```

Listing A.35: manipulate/load.scm

```scheme
;;; load.scm -- Load manipulate
(for-each (lambda (f) (load f))
          '("linkages"
            "region"
            "constraints"
            "topology"
            "mechanism"
            "main"))
```

Listing A.36: manipulate/linkages.scm

```
;;; linkages.scm ---  Bar/Joint propagators between directions and coordinates

;;; Commentary:

;; Ideas:
;; - Join "Identify" bars and joints to build mechanism
;;   versions of diagrams
;; - Use propagator system to deal with partial information
;; - Used Regions for partial info about points,
;; - Direction Intervals for partial info about joint directions.

;; Future:
;; - Other Linkages?
;; - Draw partially assembled linkages

;;; Example:

#|
 (let* ((s1 (m:make-bar))
        (s2 (m:make-bar))
        (j (m:make-joint)))
   (m:instantiate (m:joint-theta j) (/ pi 2) 'theta)
   (c:id (m:bar-length s1)
         (m:bar-length s2))
   (m:instantiate-point (m:bar-p2 s1) 4 0 'bar-2-endpoint)
   (m:instantiate-point (m:bar-p1 s1) 2 -2 'bar-2-endpoint)
   (m:identify-out-of-arm-1 j s1)
   (m:identify-out-of-arm-2 j s2)
   (run)
   (m:examine-point (m:bar-p2 s2)))
|#

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;; TMS Interfaces ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:instantiate cell value premise)
  (add-content cell
               (make-tms (contingent value (list premise)))))

(define (m:examine-cell cell)
  (let ((v (content cell)))
    (cond ((nothing? v) v)
          ((tms? v)
           (contingent-info (tms-query v)))
          (else v))))

(defhandler print
  (lambda (cell) (print (m:examine-cell cell)))
  cell?)

(define (m:contradictory? cell)
  (contradictory? (m:examine-cell cell)))

;;;;;;;;;;;;;;;;;;;;;;;; Reversing directions ;;;;;;;;;;;;;;;;;;;;;;;;

(define m:reverse-direction
  (make-generic-operation 1 'm:reverse-direction))
(defhandler m:reverse-direction
  reverse-direction direction?)
(defhandler m:reverse-direction
  reverse-direction-interval direction-interval?)

(propagatify m:reverse-direction)

(define (ce:reverse-direction input-cell)
```

```scheme
  (let-cells (output-cell)
    (name! output-cell (symbol 'reverse- (name input-cell)))
    (p:m:reverse-direction input-cell output-cell)
    (p:m:reverse-direction output-cell input-cell)
    output-cell))

;;;;;;;;;;;;;;;;;;;;;;; Adding to directions ;;;;;;;;;;;;;;;;;;;;;;;

(define (m:add-interval-to-direction d i)
  (if (empty-interval? i)
      (error "Cannot add empty interval to direction"))
  (make-direction-interval-from-start-dir-and-size
   (add-to-direction d (interval-low i))
   (- (interval-high i)
      (interval-low i))))

(define (m:add-interval-to-standard-direction-interval di i)
  (if (empty-interval? i)
      (error "Cannot add empty interval to direction"))
  (let ((di-size (direction-interval-size di))
        (i-size (- (interval-high i)
                   (interval-low i)))
        (di-start (direction-interval-start di)))
    (make-direction-interval-from-start-dir-and-size
     (add-to-direction di-start (interval-low i))
     (+ di-size i-size))))

(define (m:add-interval-to-full-circle-direction-interval fcdi i)
  (if (empty-interval? i)
      (error "Cannot add empty interval to direction"))
  fcdi)

(define (m:add-interval-to-invalid-direction-interval fcdi i)
  (if (empty-interval? i)
      (error "Cannot add empty interval to direction"))
  (error "Cannot add to invalid direction in"))

(define m:add-to-direction
  (make-generic-operation 2 'm:add-to-direction))

(defhandler m:add-to-direction
  m:add-interval-to-direction direction? interval?)

(defhandler m:add-to-direction
  add-to-direction direction? number?)

(defhandler m:add-to-direction
  m:add-interval-to-standard-direction-interval
  standard-direction-interval? interval?)

(defhandler m:add-to-direction
  m:add-interval-to-full-circle-direction-interval
  full-circle-direction-interval? interval?)

(defhandler m:add-to-direction
  m:add-interval-to-invalid-direction-interval
  invalid-direction-interval? interval?)

(defhandler m:add-to-direction
  shift-direction-interval direction-interval? number?)

(propagatify m:add-to-direction)

;;;;;;;;;;;;;;;;;;;;;; Subtracting directions ;;;;;;;;;;;;;;;;;;;;;;

(defhandler generic-negate
  (lambda (i) (mul-interval i -1)) %interval?)
```

```
(define (m:standard-direction-interval-minus-direction di d)
  (if (within-direction-interval? d di)
      (make-interval
       0
       (subtract-directions (direction-interval-end di) d))
      (make-interval
       (subtract-directions (direction-interval-start di) d)
       (subtract-directions (direction-interval-end di) d))))

(define (m:full-circle-direction-interval-minus-direction di d)
  (make-interval
   0 (* 2 pi)))

(define (m:direction-minus-standard-direction-interval d di)
  (if (within-direction-interval? d di)
      (make-interval
       0
       (subtract-directions d (direction-interval-start di)))
      (make-interval
       (subtract-directions d (direction-interval-end di))
       (subtract-directions d (direction-interval-start di)))))

(define (m:direction-minus-full-circle-direction-interval d di)
  (make-interval
   0 (* 2 pi)))

(define m:subtract-directions
  (make-generic-operation 2 'm:subtract-directions))

(defhandler m:subtract-directions
  subtract-directions direction? direction?)

;;; TODO: Support Intervals for thetas?
(defhandler m:subtract-directions
  (lambda (di1 di2)
    nothing)
  direction-interval? direction-interval?)

(defhandler m:subtract-directions
  m:standard-direction-interval-minus-direction
  standard-direction-interval? direction?)

(defhandler m:subtract-directions
  m:full-circle-direction-interval-minus-direction
  full-circle-direction-interval? direction?)

(defhandler m:subtract-directions
  m:direction-minus-standard-direction-interval
  direction? standard-direction-interval?)

(defhandler m:subtract-directions
  m:direction-minus-full-circle-direction-interval
  direction? full-circle-direction-interval?)

(propagatify m:subtract-directions)

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Vec ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
(define-record-type <m:vec>
  (%m:make-vec dx dy length direction)
  m:vec?
  (dx m:vec-dx)
  (dy m:vec-dy)
  (length m:vec-length)
  (direction m:vec-direction))


;;; Allocate and wire up the cells in a vec
(define (m:make-vec vec-id)
```

```
  (let-cells (dx dy length direction)
    (name! dx (symbol vec-id '-dx))
    (name! dy (symbol vec-id '-dy))
    (name! length (symbol vec-id '-len))
    (name! direction (symbol vec-id '-dir))

    (p:make-direction
     (e:atan2 dy dx) direction)
    (p:sqrt (e:+ (e:square dx)
                 (e:square dy))
            length)
    (p:* length (e:direction-cos direction) dx)
    (p:* length (e:direction-sin direction) dy)
    (%m:make-vec dx dy length direction)))

(define (m:print-vec v)
  `(m:vec (,(print (m:vec-dx v))
           ,(print (m:vec-dy v)))
          ,(print (m:vec-length v))
          ,(print (m:vec-direction v))))

(defhandler print m:print-vec m:vec?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Point ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
(define-record-type <m:point>
  (%m:make-point x y region)
  m:point?
  (x m:point-x)
  (y m:point-y)
  (region m:point-region))

;;; Allocate cells for a point
(define (m:make-point id)
  (let-cells (x y region)
    (name! x (symbol id '-x))
    (name! y (symbol id '-y))
    (name! region (symbol id '-region))
    (p:m:x-y->region x y region)
    (p:m:region->x region x)
    (p:m:region->y region y)
    (%m:make-point x y region)))

(define (m:x-y->region x y)
  (m:make-singular-point-set (make-point x y)))

(propagatify m:x-y->region)

(define (m:region->x region)
  (if (m:singular-point-set? region)
      (point-x (m:singular-point-set-point region))
      nothing))

(define (m:region->y region)
  (if (m:singular-point-set? region)
      (point-y (m:singular-point-set-point region))
      nothing))

(propagatify m:region->x)
(propagatify m:region->y)

(define (m:instantiate-point p x y premise)
  (m:instantiate (m:point-x p)
                 x premise)
  (m:instantiate (m:point-y p)
                 y premise)
  (m:instantiate (m:point-region p)
                 (m:make-singular-point-set (make-point x y))
                 premise))
```

```
(define (m:examine-point p)
  (list 'm:point
        (m:examine-cell (m:point-x p))
        (m:examine-cell (m:point-y p))))

(define (m:print-point p)
  `(m:point ,(print (m:point-x p))
            ,(print (m:point-y p))
            ,(print (m:point-region p))))

(defhandler print m:print-point m:point?)

;;; Set p1 and p2 to be equal
(define (m:identify-points p1 p2)
  (for-each (lambda (getter)
              (c:id (getter p1)
                    (getter p2)))
            (list m:point-x m:point-y m:point-region)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Bar ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:bar>
  (%m:make-bar p1 p2 vec)
  m:bar?
  (p1 m:bar-p1)
  (p2 m:bar-p2)
  (vec m:bar-vec))

(define (m:bar-direction bar)
  (m:vec-direction (m:bar-vec bar)))

(define (m:bar-length bar)
  (m:vec-length (m:bar-vec bar)))

(define (m:print-bar b)
  `(m:bar
    ,(print (m:bar-name b))
    ,(print (m:bar-p1 b))
    ,(print (m:bar-p2 b))
    ,(print (m:bar-vec b))))

(defhandler print m:print-bar m:bar?)

;;; Allocate cells and wire up a bar
(define (m:make-bar bar-id)
  (let ((bar-key (m:make-bar-name-key bar-id)))
    (let ((p1 (m:make-point (symbol bar-key '-p1)))
          (p2 (m:make-point (symbol bar-key '-p2))))
      (name! p1 (symbol bar-key '-p1))
      (name! p2 (symbol bar-key '-p2))
      (let ((v (m:make-vec bar-key)))
        (c:+ (m:point-x p1)
             (m:vec-dx v)
             (m:point-x p2))
        (c:+ (m:point-y p1)
             (m:vec-dy v)
             (m:point-y p2))
        (let ((bar (%m:make-bar p1 p2 v)))
          (m:p1->p2-bar-propagator p1 p2 bar)
          (m:p2->p1-bar-propagator p2 p1 bar)
          bar)))))

;;; TODO: Combine p1->p2 / p2->p1
(define (m:x-y-direction->region px py direction)
  (if (direction? direction)
      (let ((vertex (make-point px py)))
        (m:make-ray vertex direction))
```

```
      nothing))

(propagatify m:x-y-direction->region)

(define (m:x-y-length-di->region px py length dir-interval)
  (if (direction-interval? dir-interval)
      (let ((vertex (make-point px py)))
        (m:make-arc vertex length dir-interval))
      nothing))

(propagatify m:x-y-length-di->region)

(define (m:p1->p2-bar-propagator p1 p2 bar)
  (let ((p1x (m:point-x p1))
        (p1y (m:point-y p1))
        (p2r (m:point-region p2))
        (length (m:bar-length bar))
        (dir (m:bar-direction bar)))
    (p:m:x-y-direction->region p1x p1y dir p2r)
    (p:m:x-y-length-di->region p1x p1y length dir p2r)))

(define (m:p2->p1-bar-propagator p2 p1 bar)
  (let ((p2x (m:point-x p2))
        (p2y (m:point-y p2))
        (p1r (m:point-region p1))
        (length (m:bar-length bar))
        (dir (m:bar-direction bar)))
    (p:m:x-y-direction->region p2x p2y (ce:reverse-direction dir) p1r)
    (p:m:x-y-length-di->region p2x p2y length (ce:reverse-direction dir) p1r)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Joint  ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;;; Direction-2 is counter-clockwise from direction-1 by theta
(define-record-type <m:joint>
  (%m:make-joint vertex dir-1 dir-2 theta)
  m:joint?
  (vertex m:joint-vertex)
  (dir-1 m:joint-dir-1)
  (dir-2 m:joint-dir-2)
  (theta m:joint-theta))

(define *max-joint-swing* pi)

(define (m:make-joint joint-id)
  (let ((joint-key (m:make-joint-name-key joint-id)))
   (let ((vertex (m:make-point (symbol joint-key '-vertex))))
     (let-cells (dir-1 dir-2 theta)
       (name! dir-1 (symbol joint-key '-dir-1))
       (name! dir-2 (symbol joint-key '-dir-2))
       (name! theta (symbol joint-key '-theta))
       (name! vertex (symbol joint-key '-vertex))
       (p:m:add-to-direction
        dir-1 theta dir-2)
       (p:m:add-to-direction
        dir-2 (e:negate theta) dir-1)
       (p:m:subtract-directions
        dir-2 dir-1
        theta)
       (m:instantiate theta (make-interval 0 *max-joint-swing*) 'theta)
       (%m:make-joint vertex dir-1 dir-2 theta)))))

(define (m:print-joint j)
  `(m:joint
    ,(print (m:joint-name j))
    ,(print (m:joint-dir-1 j))
    ,(print (m:joint-vertex j))
    ,(print (m:joint-dir-2 j))
    ,(print (m:joint-theta j))))
```

```
(defhandler print m:print-joint m:joint?)

;;; TOOD: Abstract?
(define (m:identify-out-of-arm-1 joint bar)
  (m:set-endpoint-1 bar joint)
  (m:set-joint-arm-1 joint bar)
  (m:identify-points (m:joint-vertex joint)
                     (m:bar-p1 bar))
  (c:id (m:joint-dir-1 joint)
        (m:bar-direction bar)))

(define (m:identify-out-of-arm-2 joint bar)
  (m:set-endpoint-1 bar joint)
  (m:set-joint-arm-2 joint bar)
  (m:identify-points (m:joint-vertex joint)
                     (m:bar-p1 bar))
  (c:id (m:joint-dir-2 joint)
        (m:bar-direction bar)))

(define (m:identify-into-arm-1 joint bar)
  (m:set-endpoint-2 bar joint)
  (m:set-joint-arm-1 joint bar)
  (m:identify-points (m:joint-vertex joint)
                     (m:bar-p2 bar))
  (c:id (ce:reverse-direction (m:joint-dir-1 joint))
        (m:bar-direction bar)))

(define (m:identify-into-arm-2 joint bar)
  (m:set-endpoint-2 bar joint)
  (m:set-joint-arm-2 joint bar)
  (m:identify-points (m:joint-vertex joint)
                     (m:bar-p2 bar))
  (c:id (ce:reverse-direction (m:joint-dir-2 joint))
        (m:bar-direction bar)))

;;;;;;;;;;;;;;;;;;;;;;;; Storing Adjacencies ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:set-endpoint-1 bar joint)
  (eq-append! bar 'm:bar-endpoints-1 joint))

(define (m:bar-endpoints-1 bar)
  (or (eq-get bar 'm:bar-endpoints-1)
      '()))

(define (m:set-endpoint-2 bar joint)
  (eq-append! bar 'm:bar-endpoints-2 joint))

(define (m:bar-endpoints-2 bar)
  (or (eq-get bar 'm:bar-endpoints-2)
      '()))

(define (m:set-joint-arm-1 joint bar)
  (eq-put! joint 'm:joint-arm-1 bar))

(define (m:joint-arm-1 joint)
  (eq-get joint 'm:joint-arm-1))

(define (m:set-joint-arm-2 joint bar)
  (eq-put! joint 'm:joint-arm-2 bar))

(define (m:joint-arm-2 joint)
  (eq-get joint 'm:joint-arm-2))

;;;;;;;;;;;;;;;;;;;;;;;;;;; Named Linkages  ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:make-bar-name-key bar-id)
  (symbol 'm:bar:
          (m:bar-id-p1-name bar-id) ':
```

```
         (m:bar-id-p2-name bar-id)))

(define (m:make-joint-name-key joint-id)
  (symbol 'm:joint:
          (m:joint-id-dir-1-name joint-id) ':
          (m:joint-id-vertex-name joint-id) ':
          (m:joint-id-dir-2-name joint-id)))

(define (m:name-element! element name)
  (eq-put! element 'm:name name))

(define (m:element-name element)
  (or (eq-get element 'm:name)
      '*unnamed*))

(define (m:make-named-bar p1-name p2-name)
  (let ((bar (m:make-bar (m:bar p1-name p2-name))))
    (m:name-element! (m:bar-p1 bar) p1-name)
    (m:name-element! (m:bar-p2 bar) p2-name)
    bar))

(define (m:bar-name bar)
  (m:bar
   (m:element-name (m:bar-p1 bar))
   (m:element-name (m:bar-p2 bar))))

(define (m:bars-name-equivalent? bar-1 bar-2)
  (or (m:bar-id-equal?
       (m:bar-name bar-1)
       (m:bar-name bar-2))
      (m:bar-id-equal?
       (m:bar-name bar-1)
       (m:reverse-bar-id (m:bar-name bar-2)))))

(define (m:bar-p1-name bar)
  (m:element-name (m:bar-p1 bar)))

(define (m:bar-p2-name bar)
  (m:element-name (m:bar-p2 bar)))

(define (m:make-named-joint arm-1-name vertex-name arm-2-name)
  (let ((joint-id (m:joint arm-1-name
                           vertex-name
                           arm-2-name)))
   (let ((joint (m:make-joint joint-id)))
     (m:name-element! (m:joint-dir-1 joint) arm-1-name)
     (m:name-element! (m:joint-vertex joint) vertex-name)
     (m:name-element! (m:joint-dir-2 joint) arm-2-name)
     joint)))

(define (m:joint-name joint)
  (m:joint
   (m:joint-dir-1-name joint)
   (m:joint-vertex-name joint)
   (m:joint-dir-2-name joint)))

(define (m:joint-vertex-name joint)
  (m:element-name (m:joint-vertex joint)))

(define (m:joint-dir-1-name joint)
  (m:element-name (m:joint-dir-1 joint)))

(define (m:joint-dir-2-name joint)
  (m:element-name (m:joint-dir-2 joint)))

;;;;;;;;;;;;;;;;; Symbolic Bar / Joint Identifiers ;;;;;;;;;;;;;;;;;

;;; Maybe Move?
```

```scheme

(define-record-type <m:bar-id>
  (%m:make-bar-id p1-name p2-name)
  m:bar-id?
  (p1-name m:bar-id-p1-name)
  (p2-name m:bar-id-p2-name))

(define (m:bar-id-equal? bar-id-1 bar-id-2)
  (and (eq? (m:bar-id-p1-name bar-id-1)
            (m:bar-id-p1-name bar-id-2))
       (eq? (m:bar-id-p2-name bar-id-1)
            (m:bar-id-p2-name bar-id-2))))

(define (m:bar p1-name p2-name)
  (%m:make-bar-id p1-name p2-name))

(defhandler print m:make-bar-name-key m:bar-id?)

(define (m:reverse-bar-id bar-id)
  (%m:make-bar-id (m:bar-id-p2-name bar-id)
                  (m:bar-id-p1-name bar-id)))

;;; Joints:

(define-record-type <m:joint-vertex-id>
  (%m:make-joint-verex-id vertex-name)
  m:joint-vertex-id?
  (vertex-name m:joint-vertex-id-name))

(define-record-type <m:joint-id>
  (%m:make-joint-id dir-1-name vertex-name dir-2-name)
  m:joint-id?
  (dir-1-name m:joint-id-dir-1-name)
  (vertex-name m:joint-id-vertex-name)
  (dir-2-name m:joint-id-dir-2-name))

(defhandler print m:make-joint-name-key m:joint-id?)

(define (m:joint arg1 . rest)
  (cond ((null? rest)
         (%m:make-joint-verex-id arg1))
        ((= 2 (length rest))
         (%m:make-joint-id arg1 (car rest) (cadr rest)))
        (else
         (error "m:joint was called with the wrong number of arguments."))))

;;;;;;;;;;;;; Tables and Accessors for named linkages ;;;;;;;;;;;;;;
(define (m:make-bars-by-name-table bars)
  (let ((table (make-key-weak-eqv-hash-table)))
    (for-each (lambda (bar)
                (let ((key (m:make-bar-name-key (m:bar-name bar))))
                  (if (hash-table/get table key #f)
                      (error "Bar key already in bar name table" key))
                  (hash-table/put! table key bar)))
              bars)
    table))

;;; Unordered
(define (m:find-bar-by-id table bar-id)
  (or (hash-table/get table
                      (m:make-bar-name-key bar-id)
                      #f)
      (hash-table/get table
                      (m:make-bar-name-key (m:reverse-bar-id bar-id))
                      #f)))

;;; Joints:

```

```scheme
(define (m:make-joints-by-vertex-name-table joints)
  (let ((table (make-key-weak-eq-hash-table)))
    (for-each
     (lambda (joint)
       (let ((key (m:joint-vertex-name joint)))
         (hash-table/put!
          table key
          (cons
           joint (hash-table/get table
                                 key
                                 '())))))
     joints)
    table))

(define (m:find-joint-by-vertex-name table vertex-name)
  (let ((joints (hash-table/get table
                                vertex-name
                                #f)))
    (cond ((null? joints) #f)
          ((= (length joints) 1)
           (car joints))
          (else (error "Vertex name not unique among joints"
                       (map m:joint-name joints))))))

(define (m:make-joints-by-name-table joints)
  (let ((table (make-key-weak-eq-hash-table)))
    (for-each (lambda (joint)
                (hash-table/put! table
                                 (m:make-joint-name-key (m:joint-name joint))
                                 joint))
              joints)
    table))

;;; dir-2 is CCW from dir-1
(define (m:find-joint-by-id table joint-id)
  (hash-table/get
   table
   (m:make-joint-name-key joint-id)
   #f))

;;;;;;;;;;;;;;;;;;;;;; Operations using Names ;;;;;;;;;;;;;;;;;;;;;;

(define (m:identify-joint-bar-by-name joint bar)
  (let ((vertex-name (m:joint-vertex-name joint))
        (dir-1-name (m:joint-dir-1-name joint))
        (dir-2-name (m:joint-dir-2-name joint))
        (bar-p1-name (m:bar-p1-name bar))
        (bar-p2-name (m:bar-p2-name bar)))
    (cond ((eq? vertex-name bar-p1-name)
           (cond ((eq? dir-1-name bar-p2-name)
                  (m:identify-out-of-arm-1 joint bar))
                 ((eq? dir-2-name bar-p2-name)
                  (m:identify-out-of-arm-2 joint bar))
                 (else (error "Bar can't be identified with joint - no arm"
                              bar-p2-name))))
          ((eq? vertex-name bar-p2-name)
           (cond ((eq? dir-1-name bar-p1-name)
                  (m:identify-into-arm-1 joint bar))
                 ((eq? dir-2-name bar-p1-name)
                  (m:identify-into-arm-2 joint bar))
                 (else (error "Bar can't be identified with joint - no arm"
                              bar-p1-name))))
          (else (error "Bar can't be identified with joint - no vertex"
                       vertex-name)))))

;;;;;;;;;;;;;;;;;;;;;;;; Degrees of Freedom  ;;;;;;;;;;;;;;;;;;;;;;;

(define (m:specified? cell #!optional predicate)
```

```
  (let ((v (m:examine-cell cell)))
    (and
     (not (nothing? v))
     (or (default-object? predicate)
         (predicate v)))))

(define (m:bar-length-specified? bar)
  (m:specified? (m:bar-length bar)) number?)

(define (m:bar-direction-specified? bar)
  (m:specified? (m:bar-direction bar)) direction?)

(define (m:joint-theta-specified? joint)
  (m:specified? (m:joint-theta joint)) number?)

;;;;;;;;;;;;;;;;;;;;;;;;; Point Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:point-specified? p)
  (and (m:specified? (m:point-x p) number?)
       (m:specified? (m:point-y p) number?)))

(define (m:point-contradictory? p)
  (or (m:contradictory? (m:point-x p))
      (m:contradictory? (m:point-y p))
      (m:contradictory? (m:point-region p))))

;;;;;;;;;;;;;;;;;;;;;;;;;; Bar Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:bar-p1-specified? bar)
  (m:point-specified? (m:bar-p1 bar)))

(define (m:bar-p2-specified? bar)
  (m:point-specified? (m:bar-p2 bar)))

(define (m:bar-p1-contradictory? bar)
  (m:point-contradictory? (m:bar-p1 bar)))

(define (m:bar-p2-contradictory? bar)
  (m:point-contradictory? (m:bar-p2 bar)))

(define (m:bar-anchored? bar)
  (or (m:bar-p1-specified? bar)
      (m:bar-p2-specified? bar)))

(define (m:bar-directioned? bar)
  (and (m:bar-anchored? bar)
       (m:specified? (m:bar-direction bar) direction?)))

(define (m:bar-direction-contradictory? bar)
  (or (m:contradictory? (m:bar-direction bar))
      (m:contradictory? (m:vec-dx (m:bar-vec bar)))
      (m:contradictory? (m:vec-dy (m:bar-vec bar)))))

(define (m:bar-length-specified? bar)
  (and (m:specified? (m:bar-length bar) number?)))

(define (m:bar-direction-specified? bar)
  (and (m:specified? (m:bar-direction bar) number?)))

(define (m:bar-length-contradictory? bar)
  (m:contradictory? (m:bar-length bar)))

(define (m:bar-length-dir-specified? bar)
  (and (m:bar-length-specified? bar)
       (m:bar-direction-specified? bar)))

(define (m:bar-fully-specified? bar)
  (and (m:bar-p1-specified? bar)
```

```scheme
      (m:bar-p2-specified? bar)))

(define (m:bar-contradictory? bar)
  (or (m:bar-p1-contradictory? bar)
      (m:bar-p2-contradictory? bar)
      (m:bar-direction-contradictory? bar)
      (m:bar-length-contradictory? bar)))

;;;;;;;;;;;;;;;;;;;;;;;;; Joint Predicates ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:joint-dir-1-specified? joint)
  (m:specified? (m:joint-dir-1 joint) direction?))

(define (m:joint-dir-1-contradictory? joint)
  (m:contradictory? (m:joint-dir-1 joint)))

(define (m:joint-dir-2-specified? joint)
  (m:specified? (m:joint-dir-2 joint) direction?))

(define (m:joint-dir-2-contradictory? joint)
  (m:contradictory? (m:joint-dir-2 joint)))

(define (m:joint-theta-contradictory? joint)
  (m:contradictory? (m:joint-theta joint)))

(define (m:joint-anchored? joint)
  (or (m:joint-dir-1-specified? joint)
      (m:joint-dir-2-specified? joint)))

(define (m:joint-anchored-and-arm-lengths-specified? joint)
  (and (m:joint-anchored? joint)
       (m:bar-length-specified? (m:joint-arm-1 joint))
       (m:bar-length-specified? (m:joint-arm-2 joint))))

(define (m:joint-specified? joint)
  (m:specified? (m:joint-theta joint) number?))

(define (m:joint-dirs-specified? joint)
  (and
   (m:joint-dir-1-specified? joint)
   (m:joint-dir-2-specified? joint)))

(define (m:joint-fully-specified? joint)
  (and
   (m:point-specified? (m:joint-vertex joint))
   (m:joint-dir-1-specified? joint)
   (m:joint-dir-2-specified? joint)))

(define (m:joint-contradictory? joint)
  (or
   (m:point-contradictory? (m:joint-vertex joint))
   (m:joint-dir-1-contradictory? joint)
   (m:joint-dir-2-contradictory? joint)
   (m:joint-theta-contradictory? joint)))

;;;;;;;;;;;;;;;;;;;;;;;; Specifying Values ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:joint-theta-if-specified joint)
  (let ((theta-v (m:examine-cell
                  (m:joint-theta joint))))
    (if (number? theta-v) theta-v
        0)))

(define (m:bar-max-inner-angle-sum bar)
  (let ((e1 (m:bar-endpoints-1 bar))
        (e2 (m:bar-endpoints-2 bar)))
    (if (or (null? e1)
            (null? e2))
```

```
      0
      (+ (apply max (map m:joint-theta-if-specified e1))
         (apply max (map m:joint-theta-if-specified e2))))))

(define (m:joint-bar-sums joint)
  (let ((b1 (m:joint-arm-1 joint))
        (b2 (m:joint-arm-2 joint)))
    (and (m:bar-length-specified? b1)
         (m:bar-length-specified? b2)
         (+ (m:examine-cell (m:bar-length b1))
            (m:examine-cell (m:bar-length b2))))))

(define (m:random-theta-for-joint joint)
  (let ((theta-range (m:examine-cell (m:joint-theta joint))))
    (if (interval? theta-range)
        (begin
          (safe-internal-rand-range
           (interval-low theta-range)
           (interval-high theta-range)))
        (error "Attempting to specify theta for joint"))))

(define (m:random-bar-length)
  (internal-rand-range 0.1 0.9))

(define (m:initialize-bar bar)
  (if (not (m:bar-anchored? bar))
      (m:instantiate-point (m:bar-p1 bar) 0 0 'initialize))
  (let ((random-dir (random-direction)))
    (m:instantiate (m:bar-direction bar)
                   random-dir 'initialize)
    (pp `(initializing-bar ,(print (m:bar-name bar))
                           ,(print random-dir)))))

(define (m:initialize-joint joint)
  (m:instantiate-point (m:joint-vertex joint) 0 0 'initialize)
  (pp `(initializing-joint ,(print (m:joint-name joint)))))

;;;;;;;;;; Assembling named joints into diagrams ;;;;;;;

(define (m:assemble-linkages bars joints)
  (let ((bar-table (m:make-bars-by-name-table bars)))
    (for-each
     (lambda (joint)
       (let ((vertex-name (m:joint-vertex-name joint))
             (dir-1-name (m:joint-dir-1-name joint))
             (dir-2-name (m:joint-dir-2-name joint)))
         (for-each
          (lambda (dir-name)
            (let ((bar (m:find-bar-by-id
                        bar-table
                        (m:bar vertex-name
                               dir-name))))
              (if (eq? bar #f)
                  (error "Could not find bar for" vertex-name dir-name))
              (m:identify-joint-bar-by-name joint bar)))
          (list dir-1-name dir-2-name))))
     joints)))

#|
 ;; Simple example of "solving for the third point"
 (begin
   (initialize-scheduler)
   (let ((b1 (m:make-named-bar 'a 'c))
         (b2 (m:make-named-bar 'b 'c))
         (b3 (m:make-named-bar 'a 'b))
         (j1 (m:make-named-joint 'b 'a 'c))
         (j2 (m:make-named-joint 'c 'b 'a))
         (j3 (m:make-named-joint 'a 'c 'b)))
```

```
    (m:assemble-linkages
     (list b1 b2 b3)
     (list j2 j3 j1))

    (m:initialize-joint j1)
    (c:id (m:bar-length b1) (m:bar-length b2))

    (m:instantiate (m:bar-length b3) 6 'b3-len)
    (m:instantiate (m:bar-length b1) 5 'b1-len)
    (run)
    (m:examine-point (m:bar-p2 b1))))
 ;Value: (m:point 3 4)

|#

;;;;;;;;;;;;;;;;;;; Converstion to Figure Elements ;;;;;;;;;;;;;;;;;;;

;;; TODO: Extract dependencies from TMS? or set names

(define (m:point->figure-point m-point)
  (if (not (m:point-specified? m-point))
      (let ((r (m:examine-cell (m:point-region m-point))))
        (m:region->figure-elements r))
      (let ((p (make-point (m:examine-cell (m:point-x m-point))
                           (m:examine-cell (m:point-y m-point)))))
        (set-element-name! p (m:element-name m-point))
        p)))

(define (m:bar->figure-segment m-bar)
  (if (not (m:bar-fully-specified? m-bar))
      #f
      (let ((p1 (m:point->figure-point (m:bar-p1 m-bar)))
            (p2 (m:point->figure-point (m:bar-p2 m-bar))))
        (and (point? p1)
             (point? p2)
             (make-segment p1 p2)))))

(define (m:joint->figure-angle m-joint)
  (if (not (m:joint-fully-specified? m-joint))
      #f
      (make-angle (m:examine-cell (m:joint-dir-2 m-joint))
                  (m:point->figure-point (m:joint-vertex m-joint))
                  (m:examine-cell (m:joint-dir-1 m-joint)))))
```

Listing A.37: manipulate/region.scm

```
;;; regions.scm --- Region Information

;;; Commentary:

;; Ideas:
;; - Points, Lines, Circles, Intersections
;; - For now, semicircle (joints only go to 180deg to avoid
;;     multiple solns.)

;; Future:
;; - Differentiate regions with 2 deg. of freedom
;; - Improve contradiction objects

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Point Sets ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:point-set>
  (%m:make-point-set points)
  m:point-set?
  (points m:point-set-points))

(define (m:make-point-set points)
  (%m:make-point-set points))

(define (m:make-singular-point-set point)
  (m:make-point-set (list point)))

(define (m:in-point-set? p point-set)
  (pair? ((member-procedure point-equal?) p (m:point-set-points point-set))))

(define (m:singular-point-set? x)
  (and (m:point-set? x)
       (= 1 (length (m:point-set-points x)))))

(define (m:singular-point-set-point ps)
  (if (not (m:singular-point-set? ps))
      (error "Not a singular point set"))
  (car (m:point-set-points ps)))

(define (m:point-sets-equivalent? ps1 ps2)
  (define delp (delete-member-procedure list-deletor point-equal?))
  (define memp (member-procedure point-equal?))
  (let lp ((points-1 (m:point-set-points ps1))
           (points-2 (m:point-set-points ps2)))
    (if (null? points-1)
        (null? points-2)
        (let ((p1 (car points-1)))
          (if (memp p1 points-2)
              (lp (cdr points-1)
                  (delp p1 points-2))
              #f)))))

(define (m:print-point-set ps)
  (cons 'm:point-set
        (map (lambda (p) (list 'point (point-x p) (point-y p)))
             (m:point-set-points ps))))

(defhandler print
  m:print-point-set m:point-set?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Rays ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:ray>
  (%m:make-ray endpoint direction)
  m:ray?
```

```
  (endpoint m:ray-endpoint)
  (direction m:ray-direction))

(define m:make-ray %m:make-ray)

(define (m:ray->figure-ray m-ray)
  (with-color "red"
              (make-ray (m:ray-endpoint m-ray)
                        (m:ray-direction m-ray))))

(define (m:on-ray? p ray)
  (let ((endpoint (m:ray-endpoint ray)))
    (or (point-equal? p endpoint)
        (let ((dir (direction-from-points endpoint p)))
          (direction-equal? dir (m:ray-direction ray))))))

(define (m:p2-on-ray ray)
  (add-to-point (m:ray-endpoint ray)
                (unit-vec-from-direction (m:ray-direction ray))))

(define (m:rays-equivalent? ray1 ray2)
  (and (point-equal? (m:ray-endpoint ray1)
                     (m:ray-endpoint ray2))
       (direction-equal? (m:ray-direction ray1)
                         (m:ray-direction ray2))))

(define (m:print-ray ray)
  (let ((endpoint (m:ray-endpoint ray)))
    `(m:ray (,(point-x endpoint)
             ,(point-y endpoint))
            ,(direction-theta (m:ray-direction ray)))))

(defhandler print
  m:print-ray m:ray?)

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Arcs ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:arc>
  (m:make-arc center-point radius dir-interval)
  m:arc?
  (center-point m:arc-center)
  (radius m:arc-radius)
  (dir-interval m:arc-dir-interval))

;;; Start direction + ccw pi radian
(define (m:make-semi-circle center radius start-direction)
  (m:make-arc center radius
              (make-direction-interval start-direction
                                       (reverse-direction start-direction))))

(define (m:on-arc? p arc)
  (let ((center-point (m:arc-center arc))
        (radius (m:arc-radius arc)))
    (let ((distance (distance p center-point))
          (dir (direction-from-points center-point p)))
      (and (close-enuf? distance radius)
           (within-direction-interval?
            dir
            (m:arc-dir-interval arc))))))

(define (m:arcs-equivalent? arc1 arc2)
  (and (point-equal? (m:arc-center arc1)
                     (m:arc-center arc2))
       (close-enuf? (m:arc-radius arc1)
                    (m:arc-radius arc2))
       (direction-interval-equal?
        (m:arc-dir-interval arc1)
        (m:arc-dir-interval arc2))))
```

```scheme

(define (m:print-arc arc)
  (let ((center-point (m:arc-center arc))
        (dir-interval (m:arc-dir-interval arc)))
    `(m:arc (,(point-x center-point)
             ,(point-y center-point))
            ,(m:arc-radius arc)
            (,(direction-theta (direction-interval-start dir-interval))
             ,(direction-theta (direction-interval-end dir-interval))))))

(defhandler print
  m:print-arc
  m:arc?)

;;;;;;;;;;;;;;;;;;;;;;; Contradiction Objects ;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:region-contradiction>
  (m:make-region-contradiction error-regions)
  m:region-contradiction?
  (error-regions m:contradiction-error-regions))

;;; TODO: Maybe differeniate by error values
(define (m:region-contradictions-equivalent? rc1 rc2) #t)

(define (m:region-contradiction->figure-elements rc)
  (map m:region->figure-elements (m:contradiction-error-regions rc)))

;;;;;;;;;;;;;;;;;;;;;;; Specific Intersections ;;;;;;;;;;;;;;;;;;;;;;

(define (m:intersect-rays ray1 ray2)
  (let ((endpoint-1 (m:ray-endpoint ray1))
        (endpoint-2 (m:ray-endpoint ray2))
        (dir-1 (m:ray-direction ray1))
        (dir-2 (m:ray-direction ray2)))
    (if (direction-equal? dir-1 dir-2)
        (cond ((m:on-ray? endpoint-1 ray2) ray1)
              ((m:on-ray? endpoint-2 ray1) ray2)
              ;; TODO: Determine error value
              (else (m:make-region-contradiction (list ray1 ray2))))
        (let ((ray1-p2 (m:p2-on-ray ray1))
              (ray2-p2 (m:p2-on-ray ray2)))
          (let ((intersections
                 (intersect-lines-by-points endpoint-1 ray1-p2
                                            endpoint-2 ray2-p2)))
            (if (not (= 1 (length intersections)))
                (m:make-region-contradiction (list ray1 ray2))
                (let ((intersection (car intersections)))
                 (if (and (m:on-ray? intersection ray1)
                          (m:on-ray? intersection ray2))
                     (m:make-point-set (list intersection))
                     ;; TODO: Determine error value
                     (m:make-region-contradiction (list ray1 ray2)))))))))

(define (m:intersect-arcs arc1 arc2)
  (let ((c1 (m:arc-center arc1))
        (c2 (m:arc-center arc2))
        (r1 (m:arc-radius arc1))
        (r2 (m:arc-radius arc2)))
    (if (point-equal? c1 c2)
        (if (close-enuf? r1 r2)
            (m:make-arc c1 r1
                        (intersect-direction-intervals
                         (m:arc-dir-interval arc1)
                         (m:arc-dir-interval arc2)))
            (m:make-region-contradiction (list arc1 arc2)))
        (let ((intersections
               (intersect-circles-by-centers-radii
                c1 r1 c2 r2)))
```

```scheme
          (let ((points
                 (filter (lambda (p)
                           (and (m:on-arc? p arc1)
                                (m:on-arc? p arc2)))
                         intersections)))
            (if (> (length points) 0)
                (m:make-point-set points)
                ;; TODO: Determine error value
                (m:make-region-contradiction (list arc1 arc2))))))))

(define (m:intersect-ray-arc ray arc)
  (let ((center (m:arc-center arc))
        (radius (m:arc-radius arc))
        (endpoint (m:ray-endpoint ray))
        (ray-p2 (m:p2-on-ray ray)))
    (let ((intersections
           (intersect-circle-line-by-points
            center radius endpoint ray-p2)))
      (let ((points
             (filter (lambda (p)
                       (and (m:on-ray? p ray)
                            (m:on-arc? p arc)))
                     intersections)))
        (if (> (length points) 0)
            (m:make-point-set points)
            ;; TODO: Determine error value
            (m:make-region-contradiction (list ray arc)))))))

(define (m:intersect-arc-ray arc ray)
  (m:intersect-ray-arc ray arc))

;;;;;;;;;;;;;;;;;;; Intersecting with Point Sets ;;;;;;;;;;;;;;;;;;;

(define m:in-region? (make-generic-operation 2 'm:in-region?))

(defhandler m:in-region? m:in-point-set? point? m:point-set?)
(defhandler m:in-region? m:on-ray? point? m:ray?)
(defhandler m:in-region? m:on-arc? point? m:arc?)
(defhandler m:in-region? (lambda (p r) #f) point? m:region-contradiction?)

(define (m:intersect-point-set-with-region ps1 region)
  (let ((results
         (let lp ((points-1 (m:point-set-points ps1))
                  (point-intersections '()))
           (if (null? points-1)
               point-intersections
               (let ((p1 (car points-1)))
                 (if (m:in-region? p1 region)
                     (lp (cdr points-1)
                         (cons p1 point-intersections))
                     (lp (cdr points-1)
                         point-intersections)))))))
    (if (> (length results) 0)
        (m:make-point-set results)
        ;;; TODO: Determine error value
        (m:make-region-contradiction (list ps1 region)))))

(define (m:intersect-region-with-point-set region ps)
  (m:intersect-point-set-with-region ps region))

;;;;;;;;;;;;;;;; Generic Intersect Regions "Merge" ;;;;;;;;;;;;;;;;

(define m:intersect-regions (make-generic-operation 2 'm:intersect-regions))

;;; Same Type
(defhandler m:intersect-regions
  m:intersect-rays m:ray? m:ray?)
(defhandler m:intersect-regions
```

```
  m:intersect-arcs m:arc? m:arc?)

;;; Arc + Ray
(defhandler m:intersect-regions
  m:intersect-ray-arc m:ray? m:arc?)
(defhandler m:intersect-regions
  m:intersect-arc-ray m:arc? m:ray?)

;;; Point Sets
(defhandler m:intersect-regions
  m:intersect-region-with-point-set any? m:point-set?)
(defhandler m:intersect-regions
  m:intersect-point-set-with-region m:point-set? any?)

;;; Contradictions
(defhandler m:intersect-regions (lambda (a b) a) m:region-contradiction? any?)
(defhandler m:intersect-regions (lambda (a b) b) any? m:region-contradiction?)

;;;;;;;;;;;;;;;;;;;;;;; Generic Equivalency ;;;;;;;;;;;;;;;;;;;;;;;;;

(define m:region-equivalent?
  (make-generic-operation 2 'm:region-equivalent? (lambda (a b) #f)))

(defhandler m:region-equivalent?
  m:point-sets-equivalent? m:point-set? m:point-set?)

(defhandler m:region-equivalent?
  m:rays-equivalent? m:ray? m:ray?)

(defhandler m:region-equivalent?
  m:arcs-equivalent? m:arc? m:arc?)

(defhandler m:region-equivalent?
  m:region-contradictions-equivalent?
  m:region-contradiction?
  m:region-contradiction?)

;;;;;;;;;;;;;;;;;; Interface to Propagator System ;;;;;;;;;;;;;;;;;;

(define (m:region? x)
  (or (m:point-set? x)
      (m:ray? x)
      (m:arc? x)
      (m:region-contradiction? x)))


(defhandler equivalent? m:region-equivalent? m:region? m:region?)

(defhandler merge m:intersect-regions m:region? m:region?)

(defhandler contradictory? m:region-contradiction? m:region?)

#|
 Simple Examples
 (pp (let-cells (c)
    (add-content c (m:make-arc (make-point 1 0) (sqrt 2)
                               (make-direction-interval
                                (make-direction (/ pi 8))
                                (make-direction (* 7 (/ pi 8))))))

    (add-content c (m:make-ray (make-point -3 1) (make-direction 0)))
    (add-content c (m:make-ray (make-point 1 2)
                 (make-direction (* 7 (/ pi 4)))))
    (content c)))

 (let ((a (make-point 0 0))
       (b (make-point 1 0))
       (c (make-point 0 1))
```

```
      (d (make-point 1 1)))
    (let-cells (cell)
      (add-content cell
                   (make-tms
                    (contingent (m:make-point-set (list a b c))
                                '(a))))
      (add-content cell
                   (make-tms
                    (contingent (m:make-point-set (list a d))
                                '(a))))
      (pp (tms-query (content cell)))))
|#
;;;;;;;;;;;;;;;;;;;;;;;;; To Figure elements ;;;;;;;;;;;;;;;;;;;;;;;;;

(define m:region->figure-elements
  (make-generic-operation 1 'm:region->figure-elements (lambda (r) #f )))

(defhandler m:region->figure-elements
  m:ray->figure-ray
  m:ray?)

(defhandler m:region->figure-elements
  m:region-contradiction->figure-elements
  m:region-contradiction?)
```

Listing A.38: manipulate/constraints.scm

```scheme
;;; constraints.scm --- Constraints for mechanisms

;;; Commentary:

;; Ideas:
;; - Abstraction for specifying constraints
;; - Length, angle equality
;; - Perpendicular / Parellel

;; Future:
;; - Constraints for other linkages?

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Constraint Structure ;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:constraint>
  (m:make-constraint type args constraint-procedure)
  m:constraint?
  (type m:constraint-type)
  (args m:constraint-args)
  (constraint-procedure m:constraint-procedure))

;;;;;;;;;;;;;;;;;;;;;;;;; Constraint Types ;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:c-length-equal bar-id-1 bar-id-2)
  (m:make-constraint
   'm:c-length-equal
   (list bar-id-1 bar-id-2)
   (lambda (m)
     (let ((bar-1 (m:lookup m bar-id-1))
           (bar-2 (m:lookup m bar-id-2)))
       (c:id
        (m:bar-length bar-1)
        (m:bar-length bar-2))))))

(define (m:c-angle-equal joint-id-1 joint-id-2)
  (m:make-constraint
   'm:c-angle-equal
   (list joint-id-1 joint-id-2)
   (lambda (m)
     (let ((joint-1 (m:lookup m joint-id-1))
           (joint-2 (m:lookup m joint-id-2)))
       (c:id (m:joint-theta joint-1)
             (m:joint-theta joint-2))))))

(define (m:c-right-angle joint-id)
  (m:make-constraint
   'm:right-angle
   (list joint-id)
   (lambda (m)
     (let ((joint (m:lookup m joint-id)))
       (c:id
        (m:joint-theta joint)
        (/ pi 2))))))

;;; p2 between p1 p3 in a line
(define (m:c-line-order p1-id p2-id p3-id)
  (list
   (m:make-named-bar p1-id p2-id)
   (m:make-named-bar p2-id p3-id)
   (m:make-named-joint p1-id p2-id p3-id)
   (m:c-full-angle (m:joint p1-id p2-id p3-id))))

(define (m:c-full-angle joint-id)
  (m:make-constraint
```

```
   'm:full-angle
   (list joint-id)
   (lambda (m)
     (let ((joint (m:lookup m joint-id)))
       (c:id
        (m:joint-theta joint)
        pi)))))

(define (m:equal-joints-in-sum equal-joint-ids
                               all-joint-ids
                               total-sum)
  (m:make-constraint
   'm:equal-joints-in-sum
   all-joint-ids
   (lambda (m)
     (let ((all-joints (m:multi-lookup m all-joint-ids))
           (equal-joints (m:multi-lookup m equal-joint-ids)))
       (let ((other-joints
              (set-difference all-joints equal-joints eq?)))
         (c:id (m:joint-theta (car equal-joints))
               (ce:/
                (ce:- total-sum
                      (ce:multi+ (map m:joint-theta other-joints)))
                (length equal-joints))))))))

;;;;;;;;;;;; Applying and Marking Constrained Elements ;;;;;;;;;;;;;

(define (m:constrained? element)
  (not (null? (m:element-constraints element))))

(define (m:element-constraints element)
  (or (eq-get element 'm:constraints)
      '()))

(define (m:set-element-constraints! element constraints)
  (eq-put! element 'm:constraints constraints))

(define (m:mark-constraint element constraint)
  (m:set-element-constraints!
   element
   (cons constraint
         (m:element-constraints element))))

(define (m:apply-constraint m constraint)
  (for-each (lambda (element-id)
              (m:mark-constraint
               (m:lookup m element-id)
               constraint))
            (m:constraint-args constraint))
  ((m:constraint-procedure constraint) m))

;;;;;;;;;;;;;;;;;;;;;;;;; Propagator Utils ;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (ce:multi+ cells)
  (cond ((null? cells) 0)
        ((null? (cdr cells)) (car cells))
        (else
         (ce:+ (car cells)
               (ce:multi+ (cdr cells))))))
```

Listing A.39: manipulate/topology.scm

```scheme
;;; topology.scm --- Helpers for establishing topology for mechanism

;;; Commentary:

;; Ideas:
;; - Simplify listing out all bar and joint orderings
;; - Start with basic polygons, etc.

;; Future:
;; - Figure out making multi-in/out joints: (all pairs?)

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;; Establish-topology ;;;;;;;;;;;;;;;;;;;;;;;;;

;;; CCW point names
(define (m:establish-polygon-topology . point-names)
  (if (< (length point-names) 3)
      (error "Min polygon size: 3"))
  (let ((extended-point-names
         (append point-names
                 (list (car point-names) (cadr point-names)))))
    (let ((bars
           (map (lambda (p1-name p2-name)
                  (m:make-named-bar p1-name p2-name))
                point-names
                (cdr extended-point-names)))
          (joints
           (map (lambda (p1-name vertex-name p2-name)
                  (m:make-named-joint p1-name vertex-name p2-name))
                (cddr extended-point-names)
                (cdr extended-point-names)
                point-names)))
      (append bars joints))))
```

Listing A.40: manipulate/mechanism.scm

```
;;; mechanism.scm --- Group of Bars / Joints

;;; Commentary:

;; Ideas:
;; - Grouping of bars and joints
;; - Integrate with establishing toplogy

;; Future:
;; - Also specify constraints with it
;; - Convert to Diagram

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;; Mechanism Structure ;;;;;;;;;;;;;;;;;;;;;;;;

(define-record-type <m:mechanism>
    (%m:make-mechanism bars joints constraints
                       bar-table joint-table joint-by-vertex-table)
    m:mechanism?
    (bars m:mechanism-bars)
    (joints m:mechanism-joints)
    (constraints m:mechanism-constraints)
    (bar-table m:mechanism-bar-table)
    (joint-table m:mechanism-joint-table)
    (joint-by-vertex-table m:mechanism-joint-by-vertex-table))

(define (m:make-mechanism bars joints constraints)
  (let ((bar-table (m:make-bars-by-name-table bars))
        (joint-table (m:make-joints-by-name-table joints))
        (joint-by-vertex-table (m:make-joints-by-vertex-name-table joints)))
    (%m:make-mechanism bars joints constraints
                       bar-table joint-table joint-by-vertex-table)))

(define (m:mechanism . args)
  (let ((elements (flatten args)))
    (let ((bars (m:dedupe-bars (filter m:bar? elements)))
          (joints (filter m:joint? elements))
          (constraints (filter m:constraint? elements)))
      (m:make-mechanism bars joints constraints))))

(define (m:print-mechanism m)
  `((bars ,(map print (m:mechanism-bars m)))
    (joints ,(map print (m:mechanism-joints m)))
    (constraints ,(map print (m:mechanism-constraints m)))))

(defhandler print m:print-mechanism m:mechanism?)

;;;;;;;;;;;;;;;;;;;;;;;;;; Deduplication ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:dedupe-bars bars)
  (dedupe (member-procedure m:bars-name-equivalent?) bars))


;;;;;;;;;;;;;;;;;;;;;;;;;;;; Accessors ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:mechanism-joint-by-vertex-name m vertex-name)
  (m:find-joint-by-vertex-name
   (m:mechanism-joint-by-vertex-table m)
   vertex-name))

(define (m:mechanism-joint-by-names m dir-1-name vertex-name dir-2-name)
  (m:find-joint-by-names
   (m:mechanism-joint-table m)
   dir-1-name vertex-name dir-2-name))

```

```
(define (m:multi-lookup m ids)
  (map (lambda (id) (m:lookup m id)) ids))

(define (m:lookup m id)
  (cond ((m:bar-id? id) (m:find-bar-by-id
                         (m:mechanism-bar-table m)
                         id))
        ((m:joint-id? id) (m:find-joint-by-id
                           (m:mechanism-joint-table m)
                           id))
        ((m:joint-vertex-id? id) (m:find-joint-by-vertex-name
                                  (m:mechanism-joint-by-vertex-table m)
                                  (m:joint-vertex-id-name id)))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Specified ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:mechanism-fully-specified? mechanism)
  (and (every m:bar-fully-specified? (m:mechanism-bars mechanism))
       (every m:joint-fully-specified? (m:mechanism-joints mechanism))))

(define (m:mechanism-contradictory? mechanism)
  (or (any m:bar-contradictory? (m:mechanism-bars mechanism))
      (any m:joint-contradictory? (m:mechanism-joints mechanism))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Specify ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Should these be in Linkages?

(define *any-dir-specified* #f)
(define *any-point-specified* #f)

(define (any-one l)
  (let ((i (random (length l))))
    (list-ref l i)))

(define (m:pick-bar bars)
  (car (sort-by-key bars (negatep m:bar-max-inner-angle-sum))))

(define m:pick-joint-1 any-one)

(define (m:pick-joint joints)
  (car
   (append
    (sort-by-key
     (filter m:joint-bar-sums joints)
     m:joint-bar-sums)
    (filter (notp m:joint-bar-sums) joints))))

(define (m:specify-angle-if-first-time cell)
  (if (not *any-dir-specified*)
      (let ((dir (random-direction)))
        (set! *any-dir-specified* #t)
        (pp `(initializing-angle ,(name cell) ,(print dir)))
        (m:instantiate cell dir 'first-time-angle))))

(define (m:specify-point-if-first-time point)
  (if (not *any-point-specified*)
      (begin
        (set! *any-point-specified* #t)
        (pp `(initializing-point ,(name point) (0 0)))
        (m:instantiate-point point 0 0 'first-time-point))))

(define (m:specify-bar bar)
  (let ((v (m:random-bar-length)))
    (pp `(specifying-bar ,(print (m:bar-name bar)) ,v))
    (m:instantiate (m:bar-length bar) v 'specify-bar)
    (m:specify-angle-if-first-time (m:bar-direction bar))
    (m:specify-point-if-first-time (m:bar-p1 bar))))
```

```
(define (m:specify-joint joint)
  (let ((v (m:random-theta-for-joint joint)))
    (pp `(specifying-joint ,(print (m:joint-name joint)) ,v))
    (m:instantiate (m:joint-theta joint) v 'specify-joint)
    (m:specify-angle-if-first-time (m:joint-dir-1 joint))))

(define (m:initialize-joint-vertex joint)
  (m:specify-point-if-first-time (m:joint-vertex joint)))

(define (m:initialize-joint-direction joint)
  (m:specify-angle-if-first-time (m:joint-dir-1 joint)))

(define (m:initialize-bar-p1 bar)
  (m:specify-point-if-first-time (m:bar-p1 bar)))

(define (m:specify-joint-if m predicate)
  (let ((joints (filter (andp predicate (notp m:joint-specified?))
                        (m:mechanism-joints m))))
    (and (not (null? joints))
         (m:specify-joint (m:pick-joint joints)))))

(define (m:initialize-joint-if m predicate)
  (let ((joints (filter (andp predicate (notp m:joint-specified?))
                        (m:mechanism-joints m))))
    (and (not (null? joints))
         (let ((j (m:pick-joint joints)))
           (m:initialize-joint-direction j)))))

(define (m:specify-bar-if m predicate)
  (let ((bars (filter (andp predicate (notp m:bar-length-specified?))
                      (m:mechanism-bars m))))
    (and (not (null? bars))
         (m:specify-bar (m:pick-bar bars)))))

(define (m:initialize-bar-if m predicate)
  (let ((bars (filter (andp predicate (notp m:bar-length-specified?))
                      (m:mechanism-bars m))))
    (and (not (null? bars))
         (m:initialize-bar-p1 (m:pick-bar bars)))))

(define (m:specify-something m)
  (or
   (m:specify-bar-if m m:constrained?)
   (m:specify-joint-if m m:constrained?)
   (m:specify-joint-if m m:joint-anchored-and-arm-lengths-specified?)
   (m:specify-joint-if m m:joint-anchored?)
   (m:specify-bar-if m m:bar-directioned?)
   (m:specify-bar-if m m:bar-anchored?)
   (m:initialize-joint-if m m:joint-dirs-specified?)
   (m:initialize-bar-if m m:bar-length-dir-specified?)
   (m:initialize-bar-if m m:bar-direction-specified?)
   (m:initialize-bar-if m m:bar-length-specified?)
   (m:initialize-joint-if m m:joint-anchored?)
   (m:initialize-joint-if m true-proc)
   (m:initialize-bar-if m true-proc)))

;;;;;;;;;;;;;;;;;;;;;;;; Applying constraints ;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:apply-mechanism-constraints m)
  (for-each (lambda (c)
              (m:apply-constraint m c))
            (m:mechanism-constraints m)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Build ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define (m:identify-vertices m)
  (for-each (lambda (joints)
```

```
              (let ((first-vertex (m:joint-vertex (car joints))))
                (for-each (lambda (joint)
                            (m:identify-points first-vertex
                                               (m:joint-vertex joint)))
                          (cdr joints))))
            (hash-table/datum-list (m:mechanism-joint-by-vertex-table m))))

(define (m:build-mechanism m)
  (m:identify-vertices m)
  (m:assemble-linkages (m:mechanism-bars m)
                       (m:mechanism-joints m))
  (m:apply-mechanism-constraints m))

(define (m:initialize-solve)
  (set! *any-dir-specified* #f)
  (set! *any-point-specified* #f))

(define *m* #f)
(define (m:solve-mechanism m)
  (set! *m* m)
  (m:initialize-solve)
  (let lp ()
    (run)
    (cond ((m:mechanism-contradictory? m)
           (m:draw-mechanism m c)
           (error "Contradictory mechanism built"))
          ((not (m:mechanism-fully-specified? m))
           (if (m:specify-something m)
               (lp)
               (error "Couldn't find anything to specify.")))
          (else 'mechanism-built))))

#|
 (begin
   (initialize-scheduler)
   (m:build-mechanism
    (m:mechanism
     (m:establish-polygon-topology 'a 'b 'c))))
|#

;;;;;;;;;;;;;;;;;;;;;;; Conversion to Figure ;;;;;;;;;;;;;;;;;;;;;;;

(define (m:mechanism->figure m)
  (let ((points
         (map (lambda (joint)
                (m:point->figure-point (m:joint-vertex joint)))
              (m:mechanism-joints m)))
        (segments (map m:bar->figure-segment (m:mechanism-bars m)))
        (angles (map m:joint->figure-angle (m:mechanism-joints m))))
    (apply figure (flatten (filter (lambda (x) (or x))
                           (append points segments angles))))))

(define (m:draw-mechanism m c)
  (draw-figure (m:mechanism->figure m) c))

#|
(let lp ()
  (initialize-scheduler)
  (let ((m (m:mechanism
            (m:establish-polygon-topology 'a 'b 'c 'd))))
    (pp (m:joint-anchored? (car (m:mechanism-joints m))))
    (m:build-mechanism m)
    (m:solve-mechanism m)
    (let ((f (m:mechanism->figure m)))
      (draw-figure f c)
      (pp (analyze-figure f)))))
|#
```

Listing A.41: manipulate/main.scm

```
;;; main.scm --- Main definitions and code for running the
;;; manipulation / mechanism-based code

;;; Examples

(define (arbitrary-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)))

(define (arbitrary-right-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-right-angle (m:joint 'a))))

(define (arbitrary-right-triangle-2)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-right-angle (m:joint 'c))))

(define (quadrilateral-with-diagonals a b c d)
  (list
   (m:establish-polygon-topology a b c d)
   (m:establish-polygon-topology a b c)
   (m:establish-polygon-topology b c d)
   (m:establish-polygon-topology c d a)
   (m:establish-polygon-topology d a c)))

(define (quadrilateral-with-diagonals-intersection a b c d e)
  (list
   (quadrilateral-with-diagonals a b c d)
   (m:establish-polygon-topology a b e)
   (m:establish-polygon-topology b c e)
   (m:establish-polygon-topology c d e)
   (m:establish-polygon-topology d a e)
   (m:c-line-order c e a)
   (m:c-line-order b e d)))

(define (quad-diagonals)
  (m:mechanism
   ;; Setup abcd with e in the middle:
   (quadrilateral-with-diagonals-intersection 'a 'b 'c 'd 'e)

   ;; Right Angle in Center:
   (m:c-right-angle (m:joint 'b 'e 'c))

   ;; Diagonals Equal
   ;;(m:c-length-equal (m:bar 'c 'a) (m:bar 'b 'd))
   (m:c-length-equal (m:bar 'c 'e) (m:bar 'a 'e))
   ;;(m:c-length-equal (m:bar 'b 'e) (m:bar 'd 'e))

   ;; Make it a square:
   ;;(m:c-length-equal (m:bar 'c 'e) (m:bar 'b 'e))
   ))

;;; Works:
(define (isoceles-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-length-equal (m:bar 'a 'b)
                     (m:bar 'b 'c))))

(define (isoceles-triangle-by-angles)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'b))
```

```
  (m:equal-joints-in-sum
   (list (m:joint 'a) (m:joint 'b))
   (list (m:joint 'a) (m:joint 'b) (m:joint 'c))
   pi)))

;;; Often works:
(define (arbitrary-quadrilateral)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)))

;;; Always works:
(define (parallelogram-by-sides)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-length-equal (m:bar 'a 'b)
                     (m:bar 'c 'd))
   (m:c-length-equal (m:bar 'b 'c)
                     (m:bar 'd 'a))))

(define (kite-by-sides)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-length-equal (m:bar 'a 'b)
                     (m:bar 'b 'c))
   (m:c-length-equal (m:bar 'c 'd)
                     (m:bar 'd 'a))))

(define (rhombus-by-sides)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-length-equal (m:bar 'a 'b)
                     (m:bar 'b 'c))
   (m:c-length-equal (m:bar 'b 'c)
                     (m:bar 'c 'd))
   (m:c-length-equal (m:bar 'c 'd)
                     (m:bar 'a 'd))))

;;; Never works:
(define (parallelogram-by-angles)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'c))
   (m:c-angle-equal (m:joint 'b)
                    (m:joint 'd))

   (m:equal-joints-in-sum
    (list (m:joint 'a) (m:joint 'c))
    (list (m:joint 'a) (m:joint 'b) (m:joint 'c) (m:joint 'd))
    (* 2 pi))
   (m:equal-joints-in-sum
    (list (m:joint 'b) (m:joint 'd))
    (list (m:joint 'a) (m:joint 'b) (m:joint 'c) (m:joint 'd))
    (* 2 pi))))

(define *m*)
(define (m:run-mechanism mechanism-proc)
  (initialize-scheduler)
  (let ((m (mechanism-proc)))
    (set! *m* m)
    (m:build-mechanism m)
    (m:solve-mechanism m)
    (let ((f (m:mechanism->figure m)))
      (draw-figure f c)
      ;;(pp (analyze-figure f))
      )))

#|
```

```
(let lp ()
  (initialize-scheduler)
  (pp 'start)
  (m:run-mechanism
   (lambda ()
     (m:mechanism
      ;;(m:establish-polygon-topology 'a 'b 'c)
      (m:make-named-bar 'a 'b)
      (m:make-named-bar 'b 'c)
      (m:make-named-bar 'c 'a)
      (m:make-named-joint 'c 'b 'a)
      (m:make-named-joint 'a 'c 'b)
      (m:make-named-joint 'b 'a 'c)

      (m:make-named-bar 'a 'd)
      (m:make-named-bar 'b 'd)
      (m:make-named-joint 'd 'a 'b)
      (m:make-named-joint 'a 'b 'd)
      (m:make-named-joint 'b 'd 'a)

      (m:make-named-bar 'c 'd)
      (m:make-named-joint 'a 'd 'c)
      (m:make-named-joint 'c 'a 'd)
      (m:make-named-joint 'd 'c 'a))))
  (lp))

(let lp ()
  (initialize-scheduler)
  (let ((m (m:mechanism
            (m:establish-polygon-topology 'a 'b 'c 'd))))
    (m:build-mechanism m)
    (m:solve-mechanism m)
    (let ((f (m:mechanism->figure m)))
      (draw-figure f c)
      (pp (analyze-figure f)))))
|#
```

Listing A.42: content/load.scm

```scheme
;;; load.scm -- Load learning module
(for-each (lambda (f) (load f))
          '("investigations"))
```

Listing A.43: content/investigations.scm

```scheme

;;; [1] Linear Pair Conjecture
;;; Givens: Angles a-1 and a-2 form a linear pair
;;; Goal: m(a-1) + m(a-2) = 180 degrees
(define (linear-pair)
  (let-geo* ((a (random-point))
             (l1 (random-line-through-point a))
             (r (random-ray-from-point a))
             (a-1 (smallest-angle-from l1 r))
             (a-2 (smallest-angle-from r (flip l1))))
    (figure a l1 r a-1 a-2)))

;;; [2] Vertical Angles Conjecture
;;; Givens: Angles a-1 and a-2 are vertical angles
;;; Goal: m(a-1) = m(a-2)
(define (vertical-angles)
  (let-geo* ((l1 (random-line))
             (c (random-point-on-line l1))
             (l2 (rotate-randomly-about c l1))
             (a-1 (smallest-angle-from l1 l2))
             (a-2 (smallest-angle-from (flip l1) (flip l2))))
    (figure l1 c l2 a-1 a-2)))

;;; [3a] Corresponding Angles Conjecture
;;; Givens: - Lines l1 and l2 are parallel
;;;         - Line l3 is a transversal
;;;         - a-1 and a-2 are resulting corresponding angles
;;; Goal: m(a-1) = m(a-2)
(define (corresponding-angles)
  (let-geo* ((l1 (random-line))
             (l2 (translate-randomly l1))
             (a (random-point-on-line l1))
             (b (random-point-on-line l2))
             (l3 (line-from-points a b))
             (a-1 (smallest-angle-from l3 l2))
             (a-2 (smallest-angle-from l3 l1)))
            (figure l1 l2 a b l3 a-1 a-2)))
;;; TODO: Translate randomly *multiple*
;;; TODO: Multiple return values

;;; [3b, 3c] Interior / alternate interior: ordering of angles and

;;; [4] Converse of Parallel lines
;;; Givens: - m(a-1) = m(a-2)
;;;         - a-1, a-2, are either CA, AIA, AEA, etc. of Lines l1, l2
;;; Goal: lines l1 and l2 are parallel
(define (parallel-lines-converse)
  (let-geo* ((a-1 (random-angle))
             (l3 (line-from-arm-1 a-1))
             (a-2 (translate-randomly-along-line l3 a-1))
             (l1 (line-from-arm-2 a-1))
             (l2 (line-from-arm-2 a-2)))
    (figure a-1 a-2 l1 l2 l3)))

;;; [5] Perpendicular bisector conjecture
;;; Givens: - p is a point on perpendicular bisector of segment (a, b)
;;; Goal: p is equidistant from a and b
(define (perpendicular-bisector-equidistant)
  (let-geo* (((s (a b)) (random-segment))
             (l1 (perpendicular-bisector s))
             (p (random-point-on-line l1)))
            (figure s l1 p)))
;;; TODO: Analyze equal segments not actually there...

;;; [6] Converse of perpendicular bisector conjecture
;;; Given: - a and b are equidistant from point p
```

```
;;; Goal: p is on the perpendicular bisector of a, b
(define (perpendicular-bisector-converse)
  (let-geo* ((p (random-point))
             (a (random-point))
             (b (rotate-randomly-about p a))
             (s (make-segment a b))
             (pb (perpendicular-bisector s)))
            (figure p a b s pb)))
;;; TODO: aux-segment

;;; [7] Shortest distance conjecture
;;; Givens: arbitrary point p, point a on line l
;;; Goal: Discover that shortest distance to line is along perpendicular
(define (shortest-distance)
  (let-geo* ((p (random-point))
             (l (random-line))
             (a (random-point-on-line l)))
    (figure p l a (make-auxiliary-segment p a))))
;;; TODO: Tricky, figure out how to minimize value, specify "minimize" property?

;;; [8] Angle bisector conjecture
;;; Given: angle a-1 of rays r-1, r-2, point a on angle-bisector l1
;;; Goal: Distnace from a to r-1 = distance a to r-2

(define (angle-bisector-distance)
  (let-geo* (((a (r-1 v r-2)) (random-angle))
             (ab (angle-bisector a))
             (p (random-point-on-ray ab))
             ((s-1 (p b)) (perpendicular-to r-1 p))
             ((s-2 (p c)) (perpendicular-to r-2 p)))
     (figure a r-1 r-2 ab p s-1 s-2)))
;;; Interesting, dependent on "shortest distance" from prior conjecture

;;; [9] Angle bisector concurrency
;;; Given: Triangle abc with angle-bisectors l1, l2, l3
;;; Goal: l1, l2, l3 are concurrent
(define (angle-bisector-concurrency)
  (let-geo* (((t1 (a b c)) (random-triangle))
             (((a-1 a-2 a-3)) (polygon-angles t1))
             (l1 (polygon-angle-bisector t1 a))
             (l2 (polygon-angle-bisector t1 b))
             (l3 (polygon-angle-bisector t1 c)))
    (figure t1 l1 l2 l3)))
;;; TODO: Concurrency of lines
;;; TODO: Draw markings for angle bisector

;;; [10] Perpendicular Bisector Concurrency
;;; Given: Triangle ABC with sides s1, s2, s3, perpendicular bisectors
;;; l1, l2, l3
;;; Goal: l1, l2, l3 are concurrent
(define (perpendicular-bisector-concurrency)
  (let-geo* (((t (a b c)) (random-triangle))
             (l1 (perpendicular-bisector (make-segment a b)))
             (l2 (perpendicular-bisector (make-segment b c)))
             (l3 (perpendicular-bisector (make-segment c a))))
    (figure t l1 l2 l3)))

;;; [11] Altitude Concurrency
;;; Given: Triangle ABC with altituds alt-1, alt2, alt-3
;;; Goal: alt-1, alt-2, alt-3 are concurrent
(define (altitude-concurrency)
  (let-geo* (((t (a b c)) (random-triangle))
             (alt-1 (perpendicular-line-to (make-segment b c) a))
             (alt-2 (perpendicular-line-to (make-segment a c) b))
             (alt-3 (perpendicular-line-to (make-segment a b) c)))
            (figure t alt-1 alt-2 alt-3)))
;;; TODO: Resist redundant concurrencies
;;; TODO: See if it can provide/learn a name for this point?
```

```
;;;  [12] Circumcenter Conjecture
(define (circumcenter-figure)
  (let-geo* (((t (a b c)) (random-triangle))
             (c-center (circumcenter t)))
    (figure t c-center (circle-from-points c-center a))))
;;; TODO: Circumcenter macro?

;;; [13] Incenter Conjecture
;;; [14] Median Concurrency Conjecture
;;; [15] Centroid Ratio Conjecture
;;; [16] Center of Gravity Conjecture
;;; [Exp.1] Euler Line Conjecture
;;; [Exp.2] Euler Segment Conjecture
;;; [17] Triangle Sum Conjecture
;;; [18] Isoceles Triangle Conjecture
;;; [19] Converse of Isoceles Triangle Conjecture
;;; [20] Triangle Inequality Conjecture
;;; [21] Side-Angle Inequaity Conjecture
;;; [22] Triangle Exterior Angle Conjecture
;;; [23] SSS Congruence Conjecture
;;; [24] SAS Congruence Conjecture
;;; [24b] SSA - Congruencey?
;;; TODO: Provide some property to consider truth
;;; [25] ASA Congruence Conjecture
;;; [26] SAA Congruence Conjecture
;;; [26b] AAA - Congruency?
;;; [27] Vertex Angle Bisector Conjecture
;;; [28] Equilateral/Eqiangular Triangle Conjecture
;;; [29] Quadrilateral Sum Conjecture
;;; [30] Pentagon Sum Conjecture
;;; [31] Polygon Sum Conjecture
;;; [32] Exterior Angle Sum Conjecture
;;; [33] Equiangular Polygon Conjecture
;;; [34] Kite Angles Conjecture
;;; [35] Kite Diagonals Conjecture
;;; [36] Kite Diagonal Biesctor Conjecture
;;; [37] Kite Angle Bisector Conjecture
;;; [38] Trapezoid Consecutive Angles Conjecture
;;; [39] Isoceles Trapezoid Conjecture
;;; [40] Isoceles Trapezoid Diagonals Conjecture
;;; [41] Three Midsegments Conjecture
;;; [42] Triangle Midsegment Conjecture
;;; [43] Trapezoid Midsegment Conjecture
;;; [44] Parallelogram Opposite Angles Conjecture

(define (parallelogram-opposite-angles)
  (let-geo*
      (((p (a b c d)) (random-parallelogram)))
    (figure p)))
#|
;;; [45] Parallelogram Consecutive Angles Conjecture
;;; [46] Parallelogram Opposite Sides Conjecture
;;; [47] Parallelogram Diagonals Conjecture
;;; [48] Double-Edged Straitedge Conjecture
;;; [49] Thombus Diagonals Conjecture
;;; [50] Rhombus Angles Conjecture
;;; [51] Rectangle Diagonals Conjecture
;;; [52] Square Diagonals Conjecture
;;; [53] Tangent Conjecture
;;; [54] Tangent Segment Conjecture
;;; [55] Chord Central Angles Conjecture
;;; [56] Chord Arcs Conjecture
;;; [57] Perpendicular to a Chord Conjecture
;;; [58] Chord Distance to Center Conjecture
;;; [59] Perpendicular Bisector of a Chord Conjecture
;;; [60] Inscribed Angle Conjecture
;;; [61] Inscribed Angles Intercepting Arcs Conjecture
```

```
;;; [62] Angles Inscribed in a Semicircle Conjecture
;;; [63] Cyclic Quadrilateral Conjecture
|#
(define (cyclic-quadrilateral)
  (let-geo*
   ((cir (random-circle))
    (((a b c d)) (n-random-points-on-circle-ccw cir 4))
    (q (polygon-from-points a b c d)))
   (figure q)))
#|
;;; [64] Parallel Lines Intercepted Arcs Conjecture
;;; [65] Circumference Conjecture
;;; [66] Arc Length Conjecture
;;; [Exp.3] Intersecting Seacants Conjecture
;;; [Exp.4] Intersecting Chords Conjecture
;;; [Exp.5] Tangent-Secant Conjecture
;;; [Exp.6] Intersecting Tangents Conjecture
;;; [Exp.7] Tangent-Chord Conjecture
;;; [67] Reflection Line Conjecture
;;; [68] Coordinate Transforms Conjecture
;;; [69] Minimal Path Conjecture
;;; [70] Reflections Across Parallel Lines Conjecture
;;; [71] Reflections Across Intersecting Lines Conjecture
;;; [72] Tessellating Triangles Conjecture
;;; [73] Tesselating Quadrilateral Conjecture
;;; [74] Rectangle Area Conjecture
;;; [75] Parallelogram Area Conjecture
;;; [76] Triangle Area Conjecture
;;; [77] Trapezoid Area Conjecture
;;; [78] Kite Area Conjecture
;;; [79] Regular Polygon Area Conjecture
;;; [80] Circle Area Conjecture
;;; [81] Pythagorean Theorem
;;; [82] Converse of Pythagorean Theorem
;;; [83] Isoceles Right Triangle Conjecture
;;; [84] 30-60-90 Triangle Conjecture
;;; [85] Distance Formula
;;; [86] Prism-Cylinder Volume Conjecture
;;; [87] Pyramid-Cone Volume Conjecture
;;; [Exp.8] Platonic Solids
;;; [88] Sphere Volume Conjecture
;;; [89] Sphere Surface Area Conjecture
;;; [91] AA Similarity Conjecture
;;; [92] SSS Similarity Conjecture
;;; [93] SAS Similarity Conjecture
;;; [94] Proportional Parts Conjecture
;;; [95] Angle Bisector / Opposite Side Conjecture
;;; [96] Proportional Area Conjecture
;;; [97] Proportional Volumes Conjecture
;;; [98] Parallel/Porportionality Conjecture
;;; [99] Extended Parallel/Proportionality Conjecture
;;; [100] SAS Triangle Area Conjecture
;;; [101] Las of Sines
;;; [102] Law of Cosines
;;; [Exp.9] Special Constructions
|#
```

Listing A.44: core/load.scm

```
;;; load.scm -- Load core
(for-each (lambda (f) (load f))
          '("utils"
            "macros"
            "print"
            "animation"))
```

Listing A.45: core/animation.scm

```scheme
;;; animation.scm --- Animating and persisting values in figure constructions

;;; Commentary:

;; Ideas:
;; - Animate a range
;; - persist randomly chosen values across frames

;; Future:
;; - Backtracking, etc.
;; - Save continuations?

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;; Configurations ;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define *animation-steps* 15)

;; ~30 Frames per second:
(define *animation-sleep* 30)

;;;;;;;;;;;;;;;;;;;;;;;;; Internal Constants ;;;;;;;;;;;;;;;;;;;;;;;;;
(define *is-animating?* #f)
(define *animation-value* 0)
(define *next-animation-index* 0)
(define *animating-index* 0)

(define (run-animation f-with-animations)
  (fluid-let ((*is-animating?* #t)
              (*persistent-values-table* (make-key-weak-eq-hash-table)))
   (let lp ((animate-index 0))
     (fluid-let
         ((*animating-index* animate-index))
       (let run-frame ((frame 0))
         (fluid-let ((*next-animation-index* 0)
                     (*next-value-index* 0)
                     (*animation-value*
                      (/ frame (* 1.0 *animation-steps*))))
           (f-with-animations)
           (sleep-current-thread *animation-sleep*)
           (if (< frame *animation-steps*)
               (run-frame (+ frame 1))
               (if (< *animating-index* (- *next-animation-index* 1))
                   (lp (+ animate-index 1))))))))))

;;;;;;;;;;;;;;;;;;;;;;;; Animating Functions ;;;;;;;;;;;;;;;;;;;;;;;;;

;;; f should be a function of one float argument in [0, 1]
(define (animate f)
  (let ((my-index *next-animation-index*))
    (set! *next-animation-index* (+ *next-animation-index* 1))
    (f (cond ((< *animating-index* my-index) 0)
             ((= *animating-index* my-index) *animation-value*)
             ((> *animating-index* my-index) 1)))))

(define (animate-range min max)
  (animate (lambda (v)
             (+ min
                (* v (- max min))))))

;;;;;;;;;;;;;;;;;;;;;;;;;;;; Persistence ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define *persistent-values-table* #f)
(define *next-value-index* 0)

(define (persist-value v)
```

```
(if (not *is-animating?*)
    v
    (let* ((my-index *next-value-index*)
           (table-value (hash-table/get
                         *persistent-values-table*
                         my-index
                         #f)))
      (set! *next-value-index* (+ *next-value-index* 1))
      (or table-value
          (begin
            (hash-table/put! *persistent-values-table*
                             my-index
                             v)
            v)))))
```

Listing A.46: core/macros.scm

```
;;; macros.scm --- Macros for let-geo* to assign names and variables
;;; to elements

;;; Commentary:

;; Ideas:
;; - Basic naming
;; - Multiple assignment

;; Future:
;; - Warn about more errors
;; - More efficient multiple-assignment for lists

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;; Expanding Assignment ;;;;;;;;;;;;;;;;;;;;;;;

(define *multiple-assignment-symbol* '*multiple-assignment-result*)

(define (expand-multiple-assignment lhs rhs)
  (expand-compound-assignment
   (list *multiple-assignment-symbol* lhs)
   rhs))

(define (make-component-assignments key-name component-names)
  (map (lambda (name i)
         (list name `(element-component ,key-name ,i)))
       component-names
       (iota (length component-names))))

(define (expand-compound-assignment lhs rhs)
  (if (not (= 2 (length lhs)))
      (error "Malformed compound assignment LHS (needs 2 elements): " lhs))
  (let ((key-name (car lhs))
        (component-names (cadr lhs)))
    (if (not (list? component-names))
        (error "Component names must be a list:" component-names))
    (let ((main-assignment (list key-name rhs))
          (component-assignments (make-component-assignments
                                  key-name
                                  component-names)))
      (cons main-assignment
            component-assignments))))

(define (expand-assignment assignment)
  (if (not (= 2 (length assignment)))
      (error "Assignment in letgeo* must be of length 2, found:" assignment))
  (let ((lhs (car assignment))
        (rhs (cadr assignment)))
    (if (list? lhs)
        (if (= (length lhs) 1)
            (expand-multiple-assignment (car lhs) rhs)
            (expand-compound-assignment lhs rhs))
        (list assignment))))

(define (expand-assignments assignments)
  (append-map expand-assignment assignments))

;;;;;;;;;;;;;;;;;;;;;; Extract Variable Names ;;;;;;;;;;;;;;;;;;;;;;

(define (variables-from-assignment assignment)
  (flatten (list (car assignment))))

(define (variables-from-assignments assignments)
  (append-map variables-from-assignment assignments))

```

```
(define (set-name-expressions symbols)
  (map (lambda (s)
         `(set-element-name! ,s (quote ,s)))
       symbols))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; let-geo* ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

;;; Syntax for setting names for geometry objects declared via let-geo
(define-syntax let-geo*
  (sc-macro-transformer
   (lambda (exp env)
     (let ((assignments (cadr exp))
           (body (caddr exp)))
       (let ((new-assignments (expand-assignments assignments))
             (variable-names (variables-from-assignments assignments)))
         (let ((result`(let*
                           ,new-assignments
                         ,@(set-name-expressions variable-names)
                         ,body)))
           result))))))
```

Listing A.47: core/print.scm

```scheme

;;; print.scm --- Print things nicely

;;; Commentary:
;;; - Default printing is not very nice for many of our record structure

;;; Code:

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;; Print ;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;

(define print
  (make-generic-operation 1 'print (lambda (x) x)))

(defhandler print
  (lambda (p) (cons (print (car p))
                    (print (cdr p))))
  pair?)

(defhandler print
  (lambda (l) (map print l))
  list?)

(define (pprint x)
  (pp (print x))
  (display "\n"))
```

Listing A.48: core/utils.scm

```
;;; close-enuf? floating point comparison from scmutils
;;; Origin: Gerald Jay Sussman

(define *machine-epsilon*
  (let loop ((e 1.0))
    (if (= 1.0 (+ e 1.0))
        (* 2 e)
        (loop (/ e 2)))))

(define *sqrt-machine-epsilon*
  (sqrt *machine-epsilon*))

#|
 (define (close-enuf? h1 h2 tolerance)
   (<= (magnitude (- h1 h2))
       (* .5 (max tolerance *machine-epsilon*)
          (+ (magnitude h1) (magnitude h2) 2.0))))
|#

(define (close-enuf? h1 h2 #!optional tolerance scale)
  (if (default-object? tolerance)
      (set! tolerance (* 10 *machine-epsilon*)))
  (if (default-object? scale)
      (set! scale 1.0))
  (<= (magnitude (- h1 h2))
      (* tolerance
         (+ (* 0.5
               (+ (magnitude h1) (magnitude h2)))
            scale))))

(define (assert boolean error-message)
  (if (not boolean) (error error-message)))

(define (flatten list)
  (cond ((null? list) '())
        ((list? (car list))
         (append (flatten (car list))
                 (flatten (cdr list))))
        (else (cons (car list) (flatten (cdr list))))))

(define ((notp predicate) x)
  (not (predicate x)))

(define ((andp p1 p2) x)
  (and (p1 x)
       (p2 x)))

(define (true-proc . args) #t)
(define (false-proc . args) #f)

(define (identity x) x)

;;; ps1 \ ps2
(define (set-difference set1 set2 member-predicate)
  (define delp (delete-member-procedure list-deletor member-predicate))
  (let lp ((set1 set1)
           (set2 set2))
    (if (null? set2)
        set1
        (let ((e (car set2)))
          (lp (delp e set1)
              (cdr set2))))))

(define (eq-append! element key val)
  (eq-put! element key
           (cons val
```

```scheme
                 (or (eq-get element key) '())))))

(define (sort-by-key l key)
  (sort l (lambda (v1 v2)
            (< (key v1)
               (key  v2)))))

(define (sort-by-key-2 l key)
  (let ((v (sort-by-key-2 l key)))
    (pprint (map (lambda (x) (cons (name x) (key x))) v))
    v))

(define ((negatep f) x)
  (- (f x)))

(define ((flip-args f) x y)
  (f y x))

(define (index-of el list equality-predicate)
  (let lp ((i 0)
           (l list))
    (cond ((null? l) #f)
          ((equality-predicate (car l) el)
           i)
          (else (lp (+ i 1) (cdr l))))))

;;; (nth-letter-symbol 1) => 'a , 2 => 'b, etc.
(define (nth-letter-symbol i)
  (symbol (make-char (+ 96 i) 0)))

(define (hash-table/append table key element)
  (hash-table/put! table
                   key
                   (cons element
                         (hash-table/get table key '()))))

(define (dedupe-by equality-predicate elements)
  (dedupe (member-procedure equality-predicate) elements))

(define (dedupe member-predicate elements)
  (cond ((null? elements) '())
        (else
         (let ((b1 (car elements)))
           (if (member-predicate b1 (cdr elements))
               (dedupe member-predicate (cdr elements))
               (cons b1 (dedupe member-predicate (cdr elements))))))))
```

Bibliography