Chapter 3

Demonstration

My system uses this idea of manipulating diagrams "in the mind's eye" to explore and discover geometry theorems. Before describing its internal representations and modules, I will present and discuss several sample interactions with the system. Further implementation details can be found in subsequent chapters.

The system is divided into four main modules: an imperative construction system, a perception-based analyzer, a declarative constraint solver, and a synthesizing learning module. The following examples explore interactions with these modules in increasing complexity.

3.1 Imperative Figure Construction

At its foundation, the system provides a language and engine for performing geometry constructions and building figures.

Example 3.1 presents a simple specification of a figure. Primitives of points, lines, segments, rays, and circles can be combined into polygons and figures and complicated constructions such as the perpendicular bisector of a segment can be abstracted into higher-level construction procedures. The custom special form `let-geo*` emulates the standard `let*` form in Scheme but also annotates the resulting objects with the names and dependencies as specified in this construction.

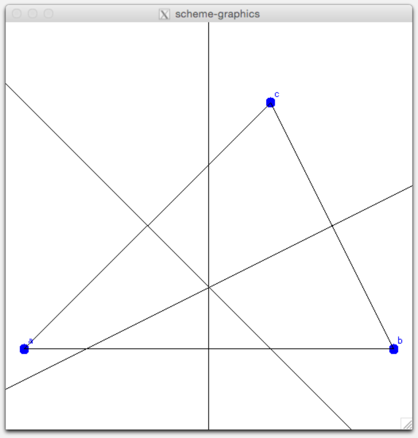
Code Example 3.1: Basic Figure Example

```scheme
(define (triangle-with-perp-bisectors)
  (let-geo* ((a (make-point 0 0))
             (b (make-point 1.5 0))
             (c (make-point 1 1))
             (t (polygon-from-points a b c))
             (pb1 (perpendicular-bisector (make-segment a b)))
             (pb2 (perpendicular-bisector (make-segment b c)))
             (pb3 (perpendicular-bisector (make-segment c a))))
    (figure t pb1 pb2 pb3)))
```

Given such an imperative description of a figure, the system can construct and display an instance of the figure as shown in Example 3.2. The graphics system uses the underlying X window system-based graphics interfaces in MIT Scheme, labels named points (a, b, c), and repositions the coordinate system to display interesting features.

Interaction Example 3.2: Rendering the Basic Figure

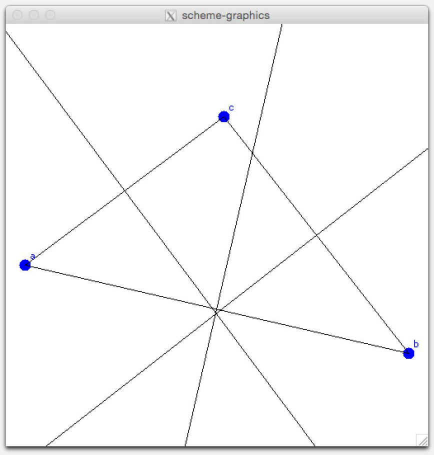
```
=> (show-figure (triangle-with-perp-bisectors))
```

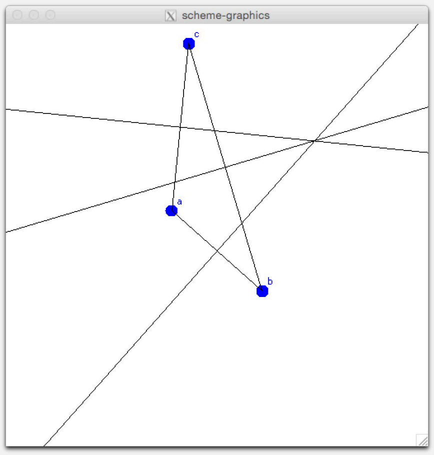


In the first figure, the coordinates of the point were explicitly specified yielding a deterministic instance of the figure. However, as geometry figures often involve arbitrary choices, the construction abstractions support random choices. Figure 3.3 demonstrates the creation of a figure involving an arbitrary triangle. The second formulation `(simple-random-triangle-with-perp-bisectors)` displays a syntax extension provided by `let-geo*` that shortens the common pattern of accessing and naming the components of a random object.

Interaction Example 3.3: Introducing Randomness

```
(define (random-triangle-with-perp-bisectors)
  (let-geo* ((t (random-triangle))
             (a (polygon-point-ref t 0))
             (b (polygon-point-ref t 1))
             (c (polygon-point-ref t 2))
             (pb1 (perpendicular-bisector (make-segment a b)))
             (pb2 (perpendicular-bisector (make-segment b c)))
             (pb3 (perpendicular-bisector (make-segment c a))))
    (figure t pb1 pb2 pb3)))

(define (simple-random-triangle-with-perp-bisectors)
  (let-geo* (((t (a b c)) (random-triangle))
             (pb1 (perpendicular-bisector (make-segment a b)))
             (pb2 (perpendicular-bisector (make-segment b c)))
             (pb3 (perpendicular-bisector (make-segment c a))))
    (figure t pb1 pb2 pb3)))
```

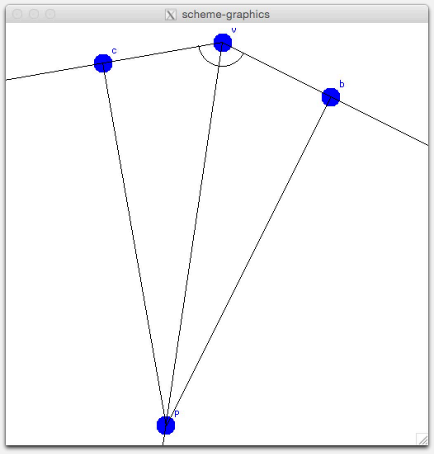


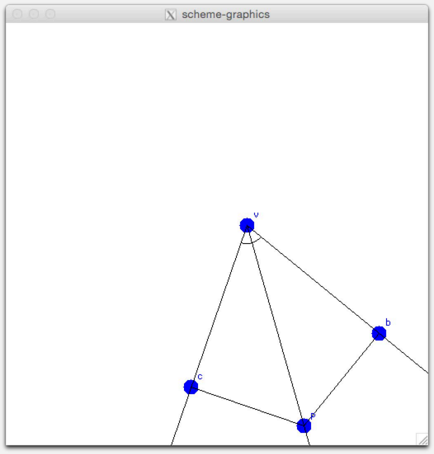


Finally, as examples of more involved constructions, Examples 3.4 and 3.5 demonstrate working with other objects (angles, rays, circles) and construction procedures. Notice that in the angle bisector example the pattern matching syntax extracts the components of an angle (ray, vertex, ray) and segment (endpoints), and that in the Inscribed/Circumscribed example, some intermediary elements are omitted from the final figure list and will not be displayed or analyzed.

Interaction Example 3.4: Angle Bisector Distance

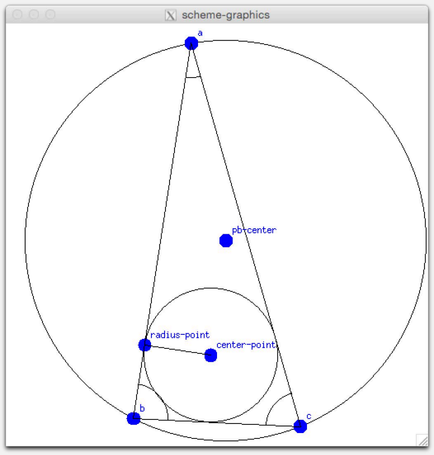
```
(define (angle-bisector-distance)
  (let-geo* (((a (r-1 v r-2)) (random-angle))
             (ab (angle-bisector a))
             (p (random-point-on-ray ab))
             ((s-1 (p b)) (perpendicular-to r-1 p))
             ((s-2 (p c)) (perpendicular-to r-2 p)))
     (figure a r-1 r-2 ab p s-1 s-2)))

=> (show-figure (angle-bisector-distance))
```

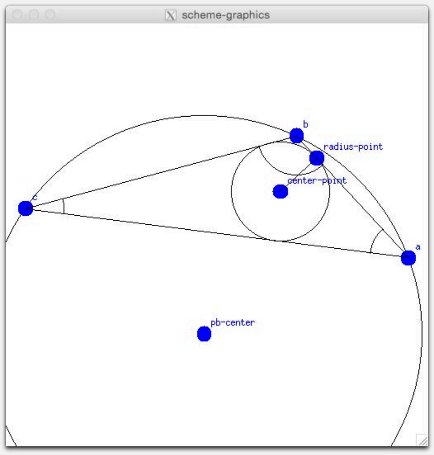




Interaction Example 3.5: Inscribed and Circumscribed Circles

```
(define (inscribed-circumscribed)
  (let-geo* (((t (a b c)) (random-triangle))
             (((a-1 a-2 a-3)) (polygon-angles t))
             (ab1 (angle-bisector a-1))
             (ab2 (angle-bisector a-2))
             ((radius-segment (center-point radius-point))
              (perpendicular-to (make-segment a b)
                                (intersect-linear-elements ab1 ab2)))
             (incircle (circle-from-points
                        center-point
                        radius-point))
             (pb1 (perpendicular-bisector
                   (make-segment a b)))
             (pb2 (perpendicular-bisector
                   (make-segment b c)))
             (pb-center (intersect-lines pb1 pb2))
             (circum-cir (circle-from-points
                          pb-center
                          a)))
    (figure t a-1 a-2 a-3
            pb-center
            radius-segment
            incircle
            circum-cir)))

=> (show-figure (inscribed-circumscribed))
```





The sample images shown alongside these constructions represent images from separate executions of the figure. An additional method for viewing and displaying involves "running an animation" of these constructions in which several instances of the figure are created and displayed, incrementally wiggling each random choice. In generating and wiggling the random values, some effort is taken to avoid degenerate cases or instances where points are too close to one another, as such cases lead to floating-point errors in the numerical analysis.

3.2 Perception and Observation

Given the imperative construction module that enables the specification and construction of geometry figures, the second module focuses on perception and extracting interesting observations from these figures.

Example 3.6 demonstrates the interface for obtaining observations from a figure. An observation is a structure that associates a relationship (concurrent, equal length, parallel) with objects in the figure that satisfy the relationship. Relationships are represented as predicates over typed n-tuples and are checked against all such n-tuples found in the figure under analysis. For example, the perpendicular relationship is checked against all pairs of linear elements in the figure.

The observation objects are complex structures that maintain properties of the underlying relationships and references to the original objects under consideration. However, my custom printer `print-observations` displays them in a more human-readable format.

Interaction Example 3.6: Simple Analysis

```
=> (all-observations (triangle-with-perp-bisectors))

(#[observation 77] #[observation 78] #[observation 79] #[observation 80])

=> (print-observations (all-observations (triangle-with-perp-bisectors)))

((concurrent pb1 pb2 pb3)
 (perpendicular pb1 (segment a b))
 (perpendicular pb2 (segment b c))
 (perpendicular pb3 (segment c a)))
```

The fact that the perpendicular bisector of a segment is equal to that segment isn't very interesting. Thus, as shown in Example 3.7, the analysis module also provides an interface for reporting only the interesting observations. Currently, information about the interesting relationships formed by a perpendicular bisector are specified alongside instructions for how to perform the operation, but a further extension of the learning module could try to infer inductively which properties result from various construction operations.

Interaction Example 3.7: Interesting Analysis

```
=> (print-observations (interesting-observations
                         (triangle-with-perp-bisectors)))

((concurrent pb1 pb2 pb3))
```

For an example with more relationships, Example 3.8 demonstrates the observations and relationships found in a figure with a random parallelogram. These analysis results will be used again later when we demonstrate the system learning definitions for polygons. Note that although the segments, angles, and points were not explicitly listed in the figure, they are extracted from the polygon that is listed. Extensions to the observation model can extract additional points and segments not explicitly listed in the original figure.

Interaction Example 3.8: Parallelogram Analysis

```
(define (parallelogram-figure)
  (let-geo* (((p (a b c d)) (random-parallelogram)))
    (figure p)))

=> (pprint (all-observations (parallelogram-figure)))

((equal-length (segment a b) (segment c d))
 (equal-length (segment b c) (segment d a))
 (equal-angle (angle a) (angle c))
 (equal-angle (angle b) (angle d))
 (supplementary (angle a) (angle b))
 (supplementary (angle a) (angle d))
 (supplementary (angle b) (angle c))
 (supplementary (angle c) (angle d))
 (parallel (segment a b) (segment c d))
 (parallel (segment b c) (segment d a)))
```

3.3 Mechanism-based Declarative Constraint Solver

The first two modules focus on performing imperative constructions to build diagrams and analyze them to obtain interesting symbolic observations and relationships. Alone, these modules could assist a mathematician in building, analyzing, and exploring geometry concepts.

However, an important aspect of automating learning theorems and definitions involves reversing this process and obtaining instances of diagrams by solving provided symbolic constraints and relationships. When we are told to “Imagine a triangle ABC in which AB = BC”, we visualize in our minds eye an instance of such a triangle before continuing with the instructions.

Thus, the third module is a declarative constraint solver. To model the physical concept of building and wiggling components until constraints are satisfied, the system is formulated around solving mechanisms built from bars and joints that must satisfy certain constraints. Such constraint solving is implemented by extending the Propagator Model created by Alexey Radul and Gerald Jay Sussman [**??**] to handle partial information and constraints about geometry positions. Chapter 7 discusses further implementation details.

3.3.1 Bars and Joints

Example 3.9 demonstrates the specification of a very simple mechanism. Mechanisms are created by specifying the bars and joints involved as well as any additional constraints that must be satisfied. This example mechanism is composed of two bars with one joint between them that is constrained to be a right angle.

Code Example 3.9: Very Simple Mechanism

```
(define (simple-mechanism)
  (m:mechanism
   (m:make-named-bar 'a 'b)
   (m:make-named-bar 'b 'c)
   (m:make-named-joint 'a 'b 'c)
   (m:c-right-angle (m:joint 'b))))
```

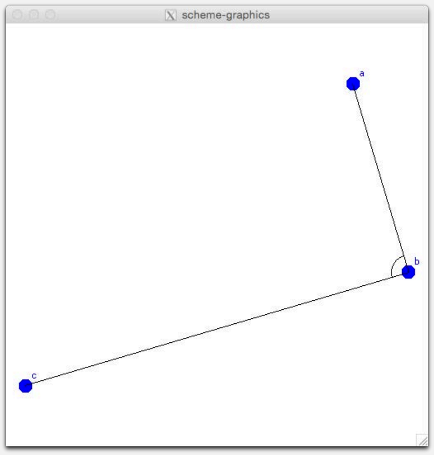
Building a mechanism involves first assembling the bars and joints together so that the named points are identified with one another. Initially, each bar has unknown length and direction, each joint has an unknown angle, and each endpoint has unknown position. Constraints for the bar and joint properties are introduced alongside any explicitly specified constraints.

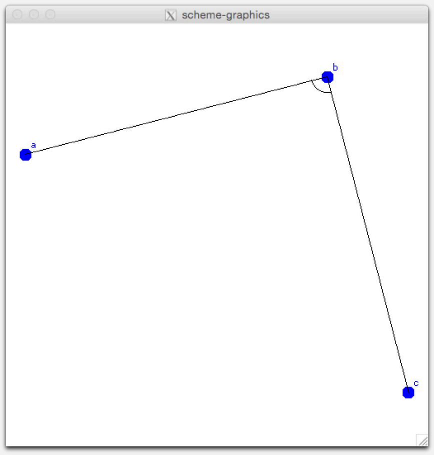
Solving the mechanism involves repeatedly selecting position, lengths, angles, and directions that are not fully specified and selecting values within the domain of that value's current partial information. As values are specified, the wiring of the propagator model propagates further partial information to other values.

The printed statements in Example 3.10 demonstrate that solving the simple mechanism above involves specifying the location of point a, then specifying the length of bar a-b and the direction from a that the bar extends. After those specifications, the joint angle is constrained to be a right angle and the location of point b is known by propagating information about point a and bar a-b's position and length. Thus, the only remaining property to fully specify the figure is the bar length of bar b-c. After building and solving the mechanism, run-mechanism converts it into a figure using the underlying primitives and displays it:

Interaction Example 3.10: Solving the Very Simple Mechanism

```
=> (m:run-mechanism simple-mechanism)

(initializing-point m:bar:a:b-p1 (0 0))
(specifying-bar-length m:bar:a:b .5644024854677596)
(initializing-direction m:bar:a:b-dir (direction 4.999857164003272))
(specifying-bar-length m:bar:b:c 1.1507815910257295)
```





3.3.2 Geometry Examples

These bar and linkage mechanisms can be used to represent the topologies of several geometry figures. Bars correspond to segments and joints correspond to angles. Example 3.11 demonstrates the set of linkages necessary to specify the topology of a triangle. The `m:establish-polygon-topology` procedure simplifies the specification of a closed polygon of joints.

Code Example 3.11: Describing an Arbitrary Triangle

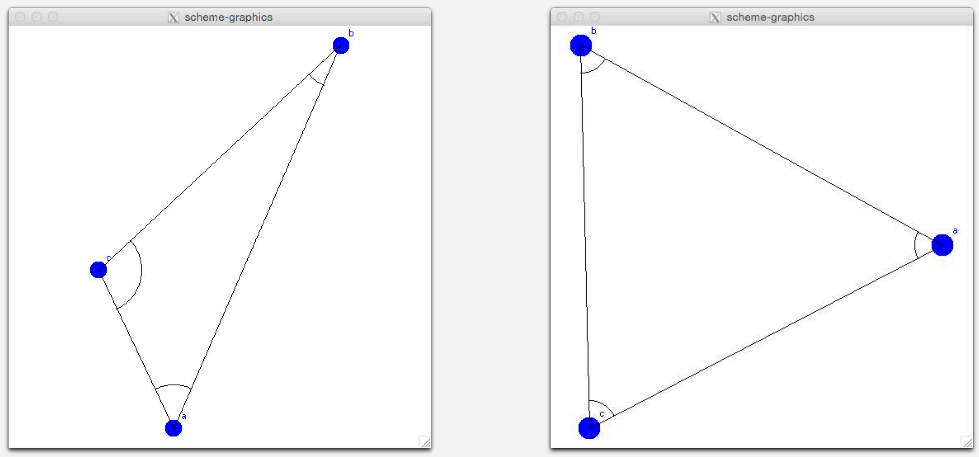
```
(define (arbitrary-triangle)
  (m:mechanism
   (m:make-named-bar 'a 'b)
   (m:make-named-bar 'b 'c)
   (m:make-named-bar 'c 'a)
   (m:make-named-joint 'a 'b 'c)
   (m:make-named-joint 'b 'c 'a)
   (m:make-named-joint 'c 'a 'b)))

(define (simpler-arbitrary-triangle)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)))
```

In example 3.12, Once joints b and c have had their angles specified, propagation sets the angle of joint a to a unique value. The only parameter to specify is the length of one of the bars. The two `initializing-` steps don't affect the resulting shape but determine its position and orientation on the canvas.

In this case, joint angles are specified first. The ordering of what is specified is guided by a heuristic that helps all of the examples shown in this chapter converge to solutions. The heuristic generally prefers specifying the most constrained values first. In some scenarios, specifying values in the wrong order can yield premature contradictions. A planned future extension will attempt to recover from such situations more gracefully by trying other orderings of specifying components.

Interaction Example 3.12: Solving the Triangle

```
=> (m:run-mechanism (arbitrary-triangle))

(specifying-joint-angle m:joint:c:b:a .41203408293499)
(initializing-direction m:joint:c:b:a-dir-1 (direction 3.888926311421853))
(specifying-joint-angle m:joint:a:c:b 1.8745808264593105)
(initializing-point m:bar:c:a-p1 (0 0))
(specifying-bar-length m:bar:c:a .4027149730292784)
```

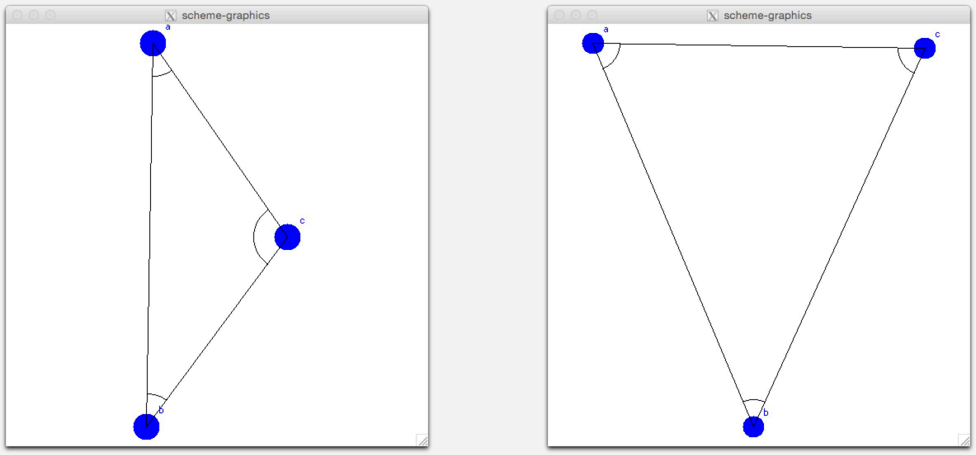


Example 3.13 shows the solving steps involved in solving an isoceles triangle from the fact that its base angles are congruent. Notice that the only two values that must be specified are one joint angle and one bar length. The rest is handled by propagation.

The values used in the propagation involves representing the partial information of where points and angles can be. A specified angle constrains a point to a ray and a specified length constrains a point to be on a circle. As information about a point is merged from several sources, intersecting these rays and circles yields unique solutions for where the points must exist. Although not as dynamic, these representations correspond to physically wiggling and extending the bars until they reach one another.

Interaction Example 3.13: Constraint Solving for Isoceles Triangle

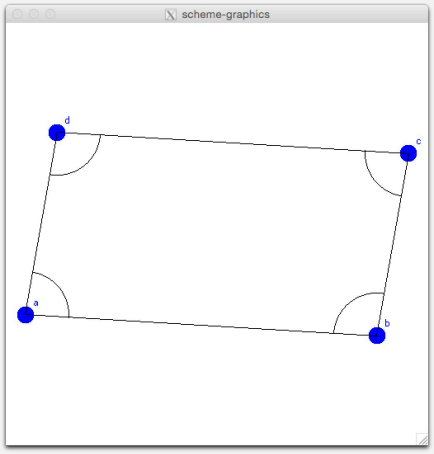
```
(define (isoceles-triangle-by-angles)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c)
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'b))))

=> (m:run-mechanism  isoceles-triangle-by-angles)

(specifying-joint-angle m:joint:c:b:a .6219719886662947)
(initializing-direction m:joint:c:b:a-dir-1 (direction .9330664240883363))
(initializing-point m:bar:b:c-p1 (0 0))
(specifying-bar-length m:bar:b:c .3557699722973674)
```

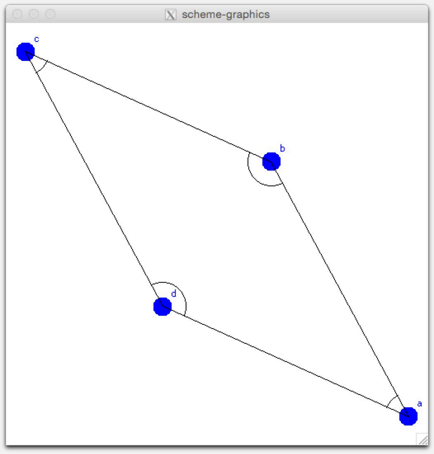


Example 3.14 continues our analysis of properties of the parallelogram. In this case, our constraint solver is able to build figures given the fact that its opposite angles are equal. The fact that these all happen to be parallelograms will be used by the learning module to produce a simpler definition for a parallelogram.

Interaction Example 3.14: Constraint Solving for Parallelogram

```
(define (parallelogram-by-angles)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'c))
   (m:c-angle-equal (m:joint 'b)
                    (m:joint 'd))))

=> (m:run-mechanism parallelogram-by-angles)

(specifying-joint-angle m:joint:c:b:a 1.6835699856637936)
(initializing-angle m:joint:c:b:a-dir-1 (direction 1.3978162819212452))
(initializing-point m:bar:a:b-p1 (0 0))
(specifying-bar-length m:bar:a:b .8152792207652096)
(specifying-bar-length m:bar:b:c .42887899934327023)
```





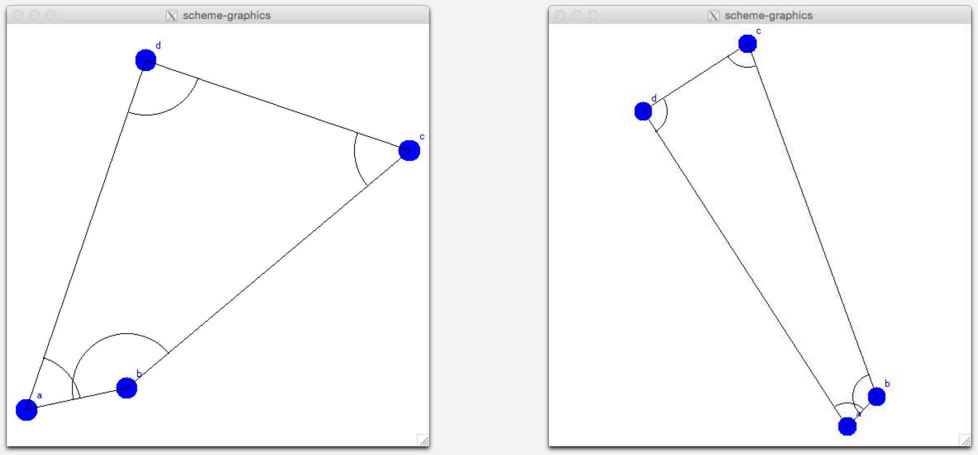
As a more complicated example, Example 3.15 demonstrates the constraint solving from the middle "Is this a rectangle?" question from Chapter 2. Try working this constraint problem by hand. As we see in 3.16, solutions are not all rectangles. Chapter 7 includes a more detailed walkthrough of how this example is solved.

Code Example 3.15: Rectangle Constraints Example

```
(define (is-this-a-rectangle-2)
  (m:mechanism
   (m:establish-polygon-topology 'a 'b 'c 'd)
   (m:c-length-equal (m:bar 'a 'd)
                     (m:bar 'b 'c))
   (m:c-right-angle (m:joint 'd))
   (m:c-angle-equal (m:joint 'a)
                    (m:joint 'c))))
```

Interaction Example 3.16: Solved Constraints

```
=> (m:run-mechanism (is-this-a-rectangle-2))

(specifying-bar-length m:bar:d:a .6742252545577186)
(initializing-direction m:bar:d:a-dir (direction 4.382829365403101))
(initializing-point m:bar:d:a-p1 (0 0))
(specifying-joint-angle m:joint:c:b:a 2.65583669872538)
```



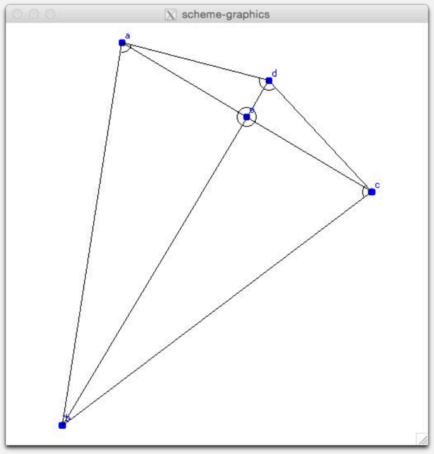
Finally, in addition to solving constraints of the angles and sides for a single polygon, the mechanism system allows for the creation of arbitrary topologies of bars and linkages. In the following examples, we build the topology of a quadrilaterals whose diagonals intersect at a point e and explore the effects of various constraints on these diagonal segments.

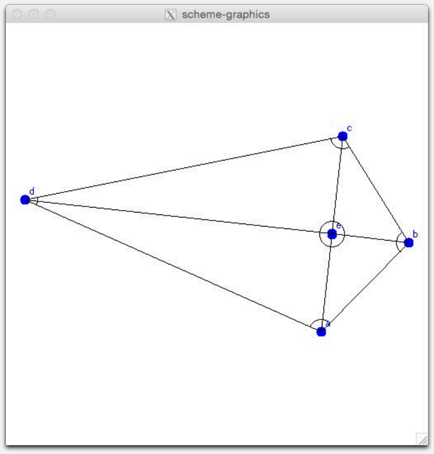
Code Example 3.17: More Involved Topologies for Constraint Solving

```scheme
(define (m:quadrilateral-with-intersecting-diagonals a b c d e)
  (list (m:establish-polygon-topology a b e)
        (m:establish-polygon-topology b c e)
        (m:establish-polygon-topology c d e)
        (m:establish-polygon-topology d a e)
        (m:c-line-order c e a)
        (m:c-line-order b e d)))
```

Interaction Example 3.18: Kites from Diagonal Properties

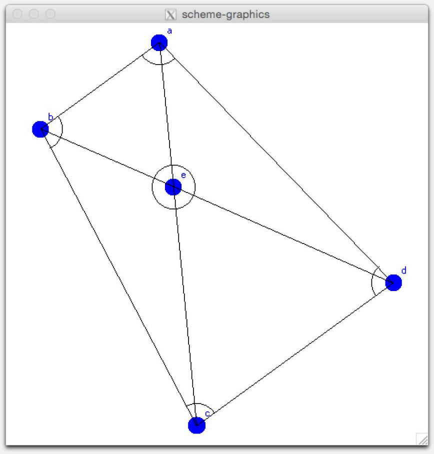
```scheme
(define (kite-from-diagonals)
  (m:mechanism
   (m:quadrilateral-with-intersecting-diagonals 'a 'b 'c 'd 'e)
   (m:c-right-angle (m:joint 'b 'e 'c)) ;; Right Angle in Center
   (m:c-length-equal (m:bar 'c 'e) (m:bar 'a 'e))))

=> (m:run-mechanism kite-from-diagonals)
```

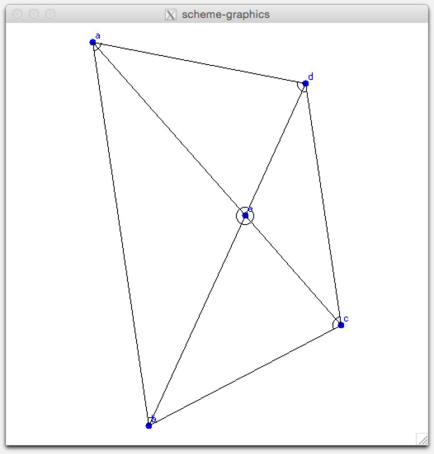




Interaction Example 3.19: Isoceles Trapezoids from Diagonals

```
(define (isoceles-trapezoid-from-diagonals)
  (m:mechanism
   (m:quadrilateral-with-intersecting-diagonals 'a 'b 'c 'd 'e)
   (m:c-length-equal (m:bar 'a 'e) (m:bar 'b 'e))
   (m:c-length-equal (m:bar 'c 'e) (m:bar 'd 'e))))

=> (m:run-mechanism isoceles-trapezoid-from-diagonals)
```

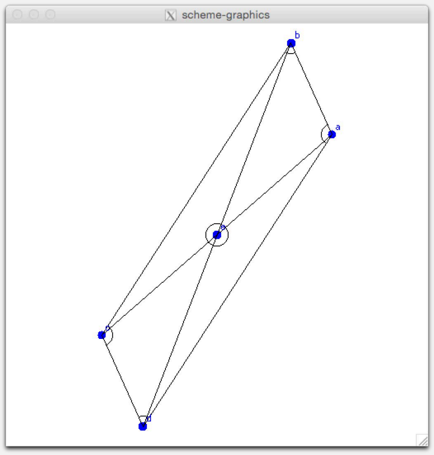


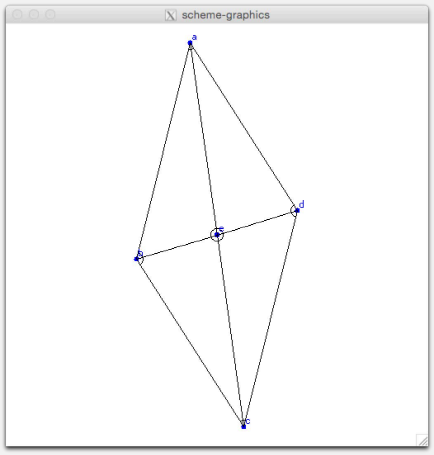


Interaction Example 3.20: Parallelograms from Diagonal Properties

```
(define (parallelogram-from-diagonals)
  (m:mechanism
   (m:quadrilateral-with-intersecting-diagonals 'a 'b 'c 'd 'e)
   (m:c-length-equal (m:bar 'a 'e) (m:bar 'c 'e))
```

```
    (m:c-length-equal (m:bar 'b 'e) (m:bar 'd 'e))))
```





3.4 Learning Module

Finally, given these modules for performing constructions, observing interesting symbolic relationships, and rebuilding figures that satisfy such relationship, a learning module interfaces with these properties to emulate a student that is actively learning geometry.

A user representing the teacher can interact with the system by querying what it knows, teaching it new terms, and asking it to apply its knowledge to new situations.

Example 3.21 shows that the system begins with some knowledge of primitive objects (point, line, ray), and the most basic polygon terms (triangle, quadrilateral). However, upon startup, it knows nothing about higher-level terms such as trapezoids, parallelograms, or isoceles triangles.

Interaction Example 3.21: Querying Terms

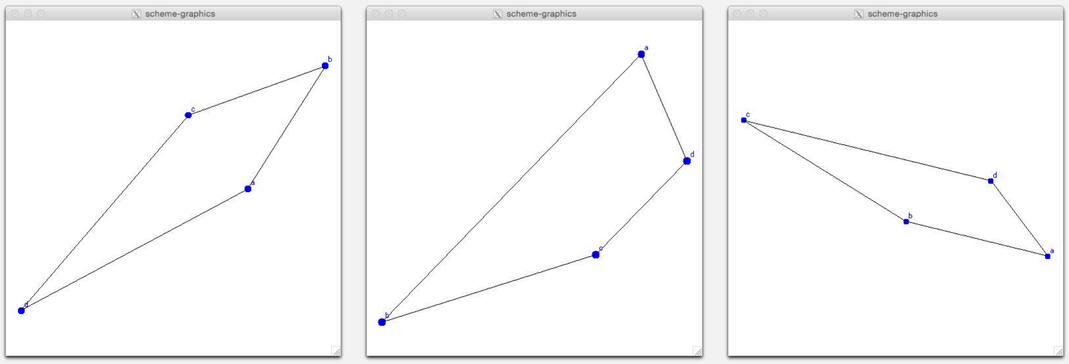
```
=> (what-is 'trapezoid)
unknown

=> (what-is 'line)
primitive-definition

=> (what-is 'triangle)
```

```
(triangle (polygon)
          ((n-sides-3 identity)))
```

A user can create an investigation to help the system learn a new definition by creating a procedure that creates random elements satisfying that definition. Example 3.22 shows the full range of trapezoids created via the `random-trapezoid` procedure.

Interaction Example 3.22: Random Figures

```
=> (show-element (random-trapezoid))
```



The learning module can interface with the perception module to obtain about the given element. In this case (3.23), we see the full dependencies of the elements under consideration instead of simply their names.

Interaction Example 3.23: Analyzing an Element

```
=> (pprint (analyze-element (random-trapezoid)))

((supplementary (polygon-angle 0 <premise>) (polygon-angle 3 <premise>))
 (supplementary (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))
 (parallel (polygon-segment 0 1 <premise>) (polygon-segment 2 3 <premise>)))
```

With these abilities, the system can be taught new definitions by providing a term ('pl for parallelogram) and a generator procedure that produces instances of that element. As shown in example 3.24, after being instructed to learn what a parallelogram is from the `random-parallelogram` procedure, when queried for a definition, we're given the term, then the base definition of this element, then all properties known to be true of such objects.

Interaction Example 3.24: Learning Parallelogram Definition

```
=> (learn-term 'pl random-parallelogram)
done

=> (what-is 'pl)
(pl
 (quadrilateral)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 2 3 <premise>))
  (equal-length (polygon-segment 1 2 <premise>)
                (polygon-segment 3 0 <premise>))
  (equal-angle (polygon-angle 0 <premise>)
               (polygon-angle 2 <premise>))
  (equal-angle (polygon-angle 1 <premise>)
               (polygon-angle 3 <premise>))
  (supplementary (polygon-angle 0 <premise>)
                 (polygon-angle 1 <premise>))
  (supplementary (polygon-angle 0 <premise>)
                 (polygon-angle 3 <premise>))
  (supplementary (polygon-angle 1 <premise>)
                 (polygon-angle 2 <premise>))
  (supplementary (polygon-angle 2 <premise>)
                 (polygon-angle 3 <premise>))
  (parallel (polygon-segment 0 1 <premise>)
            (polygon-segment 2 3 <premise>))
  (parallel (polygon-segment 1 2 <premise>)
            (polygon-segment 3 0 <premise>))))
```

To use such learned knowledge, we can use `is-a?` to test whether other elements are satisfy the definition of the term. As shown in example 3.25, results are correctly returned for any polygon that satisfies the observed properties. In cases where the properties are not satisfied, the system reports the failed conjectures or classifications (e.g. an equaliteral triangle is not a parallelogram: It failed the necessary classification that it must be a quadrilateral because it didn't have 4 sides).

Interaction Example 3.25: Testing Definitions

```
=> (is-a? 'pl (random-parallelogram))
#t

=> (is-a? 'pl (random-rectangle))
#t

=> (is-a? 'pl (polygon-from-points
                (make-point 0 0)
                (make-point 1 0)
                (make-point 2 1)
                (make-point 1 1)))
#t


=> (is-a? 'pl (random-trapezoid))
(failed-conjecture
 (equal-length (polygon-segment 0 1 <premise>)
               (polygon-segment 2 3 <premise>)))

=> (is-a? 'pl (random-equilateral-triangle))
(failed-conjecture (n-sides-4 <premise>))
(failed-classification quadrilateral)

=> (is-a? 'pl (random-segment))
(failed-classification polygon)
(failed-classification quadrilateral)
```

Learning individual definitions is nice, but cool properties arise when definitions build one another. When a new term is learned, the system checks other related terms for overlapping properties to determine where the new definition fits in the lattice of terms. In example 3.26, we see that after learning definitions of kites and rhombuses, the resulting definition of a rhombus is that it a parallelogram and kite that satisfies two additional properties. Later, after learning a rectangle, amazingly, the system shows us that the definition of a square is just a rhombus and rectangle with no additional properties.

Interaction Example 3.26: Building on Definitions

```
=> (learn-term 'kite random-kite)
done

=> (learn-term 'rh random-rhombus)
```

```
done

=> (what-is 'rh)
(rh
 (pl kite)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 3 0 <premise>))
  (equal-length (polygon-segment 1 2 <premise>)
                (polygon-segment 2 3 <premise>))))

=> (learn-term 'rect random-rectangle)
done

=> (learn-term 'square random-square)
done

=> (what-is 'sq)
(sq (rh rectangle) ())
```

Finally, the fun example that integrates all of these systems is learning simpler definitions for these terms. In these examples, `get-simple-definitions` takes a known term, looks up the known observations and properties for that term, and tests using all reasonable subsets of those properties as constraints via the constraint solver. For each subset of properties, if the constraint solver was able to create a diagram satisfying exactly those properties, the resulting diagram is examined as with "is-a" above to see if all the known properties of the original term hold.

If so, the subset of properties is reported as a valid definition of the term, and if the resulting diagram fails some properties, the subset is reported as an invalid (insufficient) set of constraints.

In the example 3.27, we see a trace of finding simple definitions for isoceles triangles and parallelograms. In the first example, the observed properties of an isoceles triangle are that its segments and angles are equal. Via the definitions simplification via constraint solving, we actually discover that the constraints of base angles equal or sides equal are sufficient.

Interaction Example 3.27: Learning Simple Definitions

```
=> (what-is 'isoceles-triangle)

(i-t
 (triangle)
 ((equal-length (polygon-segment 0 1 <premise>)
                (polygon-segment 2 0 <premise>))
  (equal-angle (polygon-angle 1 <premise>) (polygon-angle 2 <premise>))))

=> (get-simple-definitions 'isoceles-triangle)

((invalid-definition ())
 (valid-definition
  ((equal-length (segment a b) (segment c a))))
 (valid-definition
  ((equal-angle (angle b) (angle c))))
 (valid-definition
  ((equal-length (segment a b) (segment c a))
   (equal-angle (angle b) (angle c)))))
```

In the parallelogram example 3.28, some subsets are omitted because the constraint solver wasn't able to solve a diagram given those constraints (to be improved / retried more gracefully in the future). However, the results still show some interesting valid definitions such as the pair of equal opposite angles as explored in Example 3.14 or equal length opposite sides and correctly mark several sets of invalid definitions as not being specific enough.

Interaction Example 3.28: Learning Simple Parallelogram Definitions

```
=> (get-simple-definitions 'pl)

((invalid-definition ())
 (invalid-definition ((equal-length (segment a b) (segment c d))))
 (invalid-definition ((equal-angle (angle a) (angle c))))
 (invalid-definition ((equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-length (segment a b) (segment c d))
   (equal-length (segment b c) (segment d a))))
 (invalid-definition
  ((equal-length (segment b c) (segment d a))
   (equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-angle (angle a) (angle c))
   (equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-length (segment a b) (segment c d))
   (equal-length (segment b c) (segment d a))
   (equal-angle (angle a) (angle c))))
 (valid-definition
  ((equal-length (segment a b) (segment c d))
   (equal-length (segment b c) (segment d a))
   (equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-length (segment a b) (segment c d))
   (equal-angle (angle a) (angle c))
   (equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-length (segment b c) (segment d a))
   (equal-angle (angle a) (angle c))
   (equal-angle (angle b) (angle d))))
 (valid-definition
  ((equal-length (segment a b) (segment c d))
   (equal-length (segment b c) (segment d a))
   (equal-angle (angle a) (angle c))
   (equal-angle (angle b) (angle d)))))
```

This simple definitions implementation is still a work in progress and has room for improvement. For instance, checking all possible subsets is wasteful as any superset of a valid definition is known to be valid and any subset of an invalid definition is known to be invalid. In addition to such checks, in the future I plan to use the knowledge about what properties the insufficient diagram is violating to use as a possible addition to the constraint set. Further extensions could involve generalizing this get-simple-definitions to support other topologies for the initial properties (such as the quadrilaterals being fully specified by their diagonal properties as in Example 3.17)