Automated Elementary Geometry Theorem Discovery via Inductive Diagram Manipulation

MIT EECS MEng Thesis Proposal

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1 Overview

Understanding elementary geometry is a fundamental reasoning skill, and encompasses a domain both constrained enough to model effectively, yet rich enough to allow for interesting insights. Although elementary geometry knowledge can be conveyed via series of factual definitions, theorems, and proofs, a particularly intriguing aspect of geometry is the ability for students to learn and develop an understanding of core concepts through visual investigation, exploration, and discovery.

These visual reasoning skills reflect many of the cognitive activities used as one interacts with his or her surroundings. Day-to-day decisions regularly rely on visual reasoning processes such as imagining what three dimensional objects look like from other angles, or mentally simulating the effects of one's actions on objects based on a learned understanding of physics and the object's properties. Such skills and inferred rules are developed through repeated observation, followed by the formation and evaluation of conjectures.

Similar to such day-to-day three-dimensional reasoning, visualizing and manipulating 2D geometric diagrams "in the mind's eye" allows one to explore questions such as "what happens if..." or "is it always true that..." to discover new conjectures. Further investigation of examples can increase one's belief in such a conjecture, and an accompanying system of deductive reasoning from basic axioms could prove that an observation is correct.

As an example, a curious student might notice that in a certain drawing of a triangle, the three perpendicular bisectors of the edges are concurrent, and that a circle constructed with center at the point of concurrence intersects all three vertices of the triangle. Given this "interesting observation," the student might explore other triangles to see if this behavior is just coincidence, or conjecture about whether it applies to certain classes of triangles or all triangles in general. After investigating several other examples, the student might have sufficient belief in the conjecture to explore using previously-proven theorems (in this case, correspondences in congruent triangles) to prove the conjecture. My proposed project is a software system that simulates and automates this inductive thought process.

Automating geometric reasoning is not new, and has been an active field in computing and artificial intelligence. Dynamic geometry software, automated proof assistants, deductive databases, and several reformulations into abstract algebra models have been proposed in the last few decades. Although many of these projects have focused on the end goal of obtaining rigorous proofs of geometric theorems, I am particularly interested in exploring and modeling the more creative human-like thought processes of inductively exploring and manipulating diagrams to *discover* new insights about geometry.

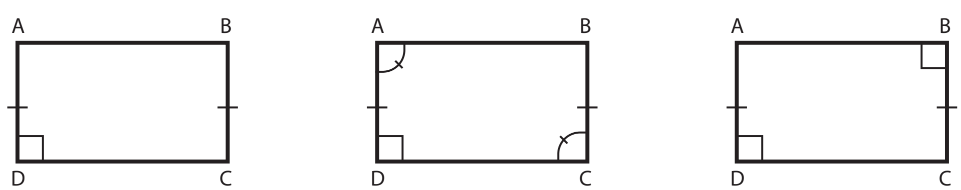
I propose the creation and analysis of an interactive computer system that emulates the curious student described above, and is capable of exploring geometric concepts through inductive investigation. The system will begin with a fairly limited set of factual knowledge regarding basic definitions in geometry and provide means by which a user interacting with the system could "teach" the system additional geometric concepts and theorems by suggesting investigations the system should explore to see if it "notices anything interesting."

To evaluate its recognition of such concepts, the interactive system will provide means for a user to extract the observations and apply its findings to new scenarios. In addition to the automated reasoning and symbolic artificial intelligence aspects of a system that can learn and reason inductively about geometry, the project also has some interesting opportunities to explore educational concepts related to experiential learning, and several extensions to integrate it with existing construction synthesis and proof systems.

2 Manipulating Diagrams "In the Mind's Eye"

Although the field of mathematics has developed a rigorous structure of deductive proofs explaining most findings in geometry, much of human intuition and initial reasoning about geometric ideas come not from applying formal rules, but rather from visually manipulating diagrams "in the mind's eye."

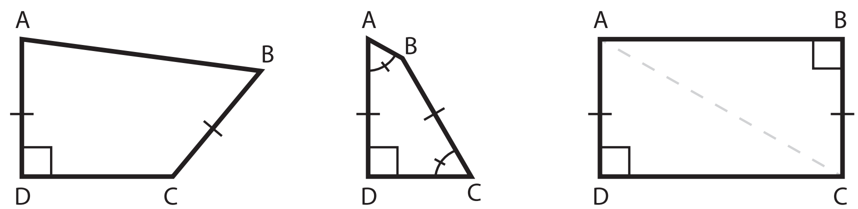
2.1 An Initial Example



Example 1: Of the three diagrams above, determine which have constraints sufficient to restrict the quadrilateral $ABCD$ to always be a rectangle.

An automated deductive solution to this question could attempt to use forward-chaining of known theorems to determine whether there was a logical path that led from the given constraints to the desired result that the quadrilateral shown is a rectangle. However, getting the correct results would require having a rich enough set of inference rules and a valid system of applying them.

A more intuitive visual-reasoning approach usually first explored by humans is to initially verify that the marked constraints hold for the instance of the diagram as drawn and then mentally manipulate or "wiggle" the diagram to see if one can find a nearby counter-example that still satisfies the given constraints, but is not a rectangle. If the viewer is unable to find a counter-example after several attempts, he or she may be sufficiently convinced the conclusion is true, and could commit to exploring a more rigorous deductive proof.



Solution to Example 1: As the reader likely discovered, the first two diagrams can be manipulated to yield instances that are not rectangles, while the third is sufficiently constrained to always represent a rectangle. (This can be proved by adding a diagonal and using the Pythagorean theorem.)

2.2 Diagrams, Figures, and Constraints

This example of manipulation using the "mind's eye" also introduces some terminology helpful in discussing the differences between images as drawn and the spaces of geometric objects they represent. For clarity, a *figure* will refer to an actual configuration of points, lines, and circles drawn on a page. Constraint annotations (congruence or measure) added to a figure create a *diagram*, which represents the entire space of figure *instances* that satisfy the constraints.

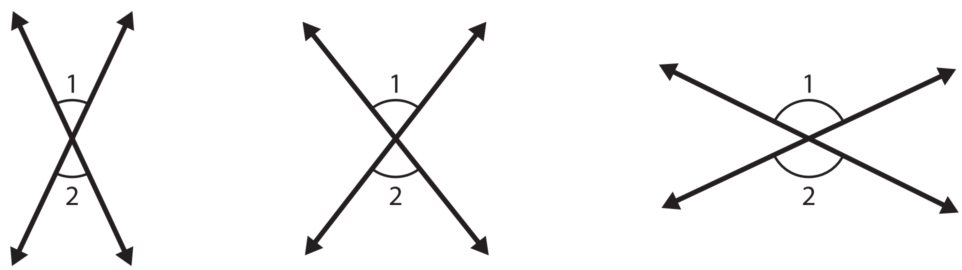
An annotated figure presented on a page is typically an instance of its corresponding diagram. However, it is certainly possible to add annotations to a figure that are not satisfied by that figure, yielding impossible diagrams. In such a case the diagram represents an empty set of satisfying figures.

In the initial example above, the three quadrilaterals figures are drawn as rectangles. It is true that all quadrilateral figures in the space represented by the third diagram are rectangles. However, the space of quadrilaterals represented by the first two diagrams include instances that are not rectangles, as shown above. At this time, we will only start with valid diagrams whose constraints are satisfied in the given figure. However, detecting and explaining inconsistent diagrams could be an interesting extension.

3 Geometry Investigation

These same "mind's eye" reasoning techniques can be used to discover and learn new geometric theorems. Given some "interesting properties" in a particular figure, one can construct other instances of the diagram to examine if the properties appear to hold uniformly, or if they were just coincidences in the initial drawing. Properties that are satisfied repeatedly can be further explored and proved using deductive reasoning. The examples below provide several demonstrations of such inductive investigations.

3.1 Vertical Angles

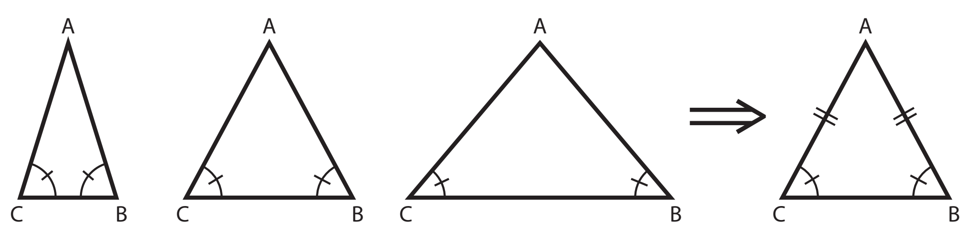


Investigation 1: Construct a pair of vertical angles. Notice anything interesting?

Often one of the first theorems in a geometry course, the fact that vertical angles are equal is one of the simplest examples of applying "mind's eye" visual reasoning. Given the diagram on the left, one could "wiggle" the two lines in his or her mind and imagine how the angles respond. In doing so, one would notice that the lower angle's measure increases and decreases proportionately with that of the top angle. This mental simulation, perhaps accompanied by a few drawn and measured figures, could sufficiently convince the viewer that vertical angles always have equal measure.

Of course, this fact can also be proved deductively by adding up pairs of angles that sum to 180 degrees. However, the inductive manipulations are more reflective of the initial, intuitive process one typically takes when first presented with understanding a diagram.

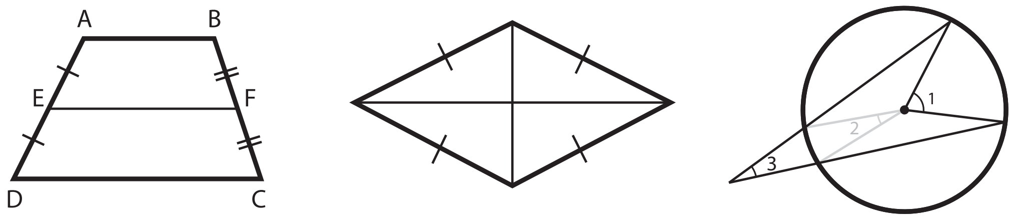
3.2 Elementary Results



Investigation 2: Construct a triangle ABC with $\angle B = \angle C$. Notice anything interesting?

A slightly more involved example includes discovering that if a triangle has two congruent angles, it is isoceles. As above, this fact has a more rigorous proof that involves dropping an altitude from point A and using corresponding parts of congruent triangles to demonstrate the equality of AB and AC. However, the inductive investigation of figures that satisfy the constraints can yield the same conjecture, give students better intuition for what is happening, and help guide the discovery and assembly of known rules to be applied in future situations.

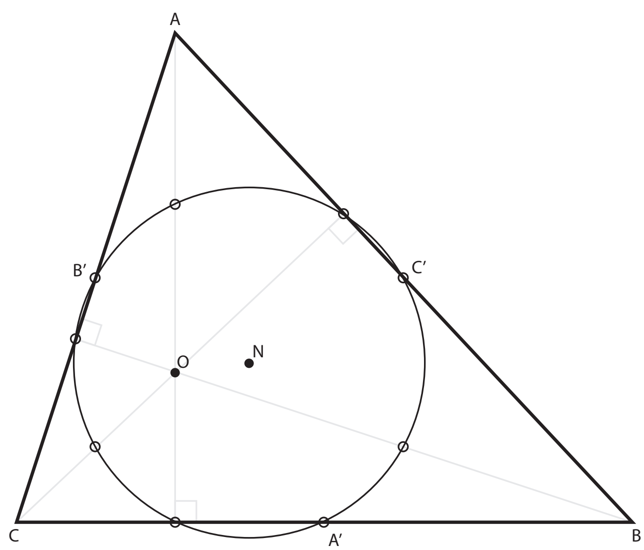
In this and further examples, an important question becomes what properties are considered "interesting" and worth investigating in further instances of the diagram, as discussed in section 4.3. As suggested by the examples in Investigation 3, this can include relations between segment and angle lengths, concurrent lines, collinear points, or parallel and perpendicular lines.



Investigation 3: What is interesting about the relationship between AB, CD, and EF in the trapezoid? What is interesting about the diagonals of a rhombus? What is interesting about $\angle 1$, $\angle 2$, and $\angle 3$?

3.3 Nine Point Circle and Euler Segment

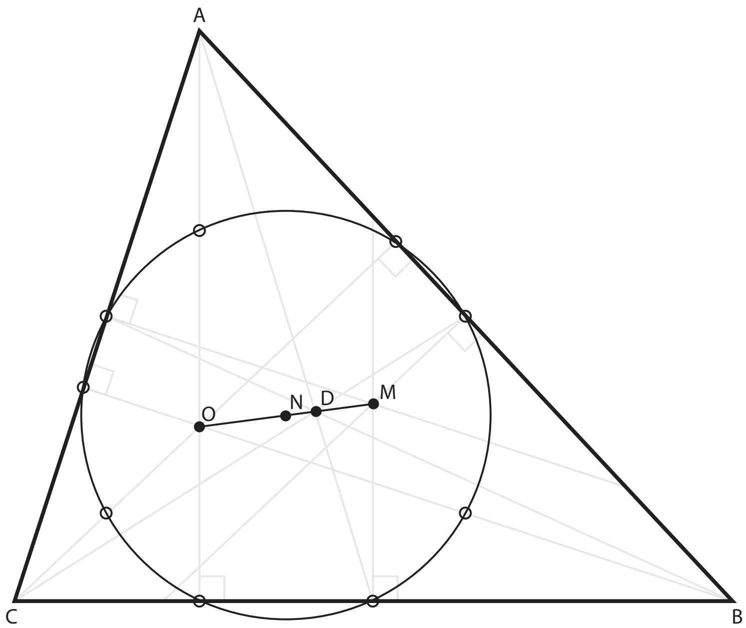
Finally, this technique can be used to explore and discover conjectures well beyond the scope of what one can visualize in his or her head:



Investigation 4a: In triangle ABC, construct the side midpoints A', B', C', and orthocenter O (from altitudes). Then, construct the midpoints of the segments connecting the orthocenter with each triangle vertex. Notice anything interesting?

As a more complicated example, consider the extended investigation of the Nine Point Circle and Euler Segment. As shown in Investigation 4a, the nine points created (feet of the altitudes, midpoints of sides, and midpoints of segments from orthocenter to vertices) are all concentric, lying on a circle with center labeled N.

Upon first constructing this figure, this fact seems almost beyond chance. However, as shown in Investigation 4b (below), further "interesting properties" continue to appear as one constructs the centroid and circumcenter: All four of these special points (O, N, D, and M) are collinear on what is called the *Euler Segment*, and the ratios $ON : ND : DM$ of $3 : 1 : 2$ hold for any triangle.



Investigation 4b: Continue the investigation from 4a by also constructing the centroid D (from medians) and circumcenter M (from perpendicular bisectors). Notice anything interesting?

4 Proposed Software Implementation

My proposed MEng project involves building and analyzing a software implementation that can make such inductive observations and discoveries as described above. Although there are many aspects involved in replicating the human-like reasoning discussed, I will initially focus on a core system that is able to model the inductive exploratory behavior. Further abilities can be added as extensions or applications of the core system as described in section 6.

This core system will provide an interpreter to accept input of construction instructions, an analytic geometry system that can create instances of such constructions, a pattern-finding process to discover "interesting properties", and an interface for reporting findings.

4.1 Interpreting Construction Steps

The first step in such explorations is interpreting an input of the diagram to be explored. To avoid the problems involved with solving constraint systems and the possibility of impossible diagrams, the core system will take as input an explicit list of construction steps that results in an instance of the desired diagram. These instructions can still include arbitrary selections (let P be some point on the line, or let A be some acute angle), but otherwise will be restricted to basic construction operations using a compass and straight edge.

To simplify the input of more complicated diagrams, some of these steps can be abstracted into a library of known construction procedures. For example, although the underlying figures will be limited to very simple objects of points, lines, and angles, the steps of constructing a triangle (three points and three segments) or bisecting a line or angle can be encapsulated into single steps.

4.2 Creating Figures

Given a language for expressing the constructions, the second phase of the system will be to perform such constructions to yield an instance of the diagram. This process will mimic "imagining" manipulations and will result in an analytic representation of the figure with coordinates for each point. Arbitrary choices in the construction ("Let Q be some point not on the line.") will be chosen via an random process, but with an attempt to keep the figures within a reasonable scale to ease human inspection.

4.3 Noticing Interesting Properties

Having constructed a particular figure, the system will need to be able to examine it to find interesting properties. These properties involve facts that appear to be "beyond coincidence". As mentioned in section 3.2, this generally involves relationships between measured values, but can also include "unexpected" configurations of points, lines, and circles. As the system discovers interesting properties, it will reconstruct the diagram using difference choices and observe if the observed properties hold true across many instances of a diagram.

4.4 Reporting Findings

Finally, once the system has discovered some interesting properties that appear repeatedly in instances of a given diagram, it will need a means of reporting its results to the user. Although this could easily be a simple list of all simple relationships, some effort will be taken to avoid repeating observations that obvious in the construction. For example, if a perpendicular bisector of segment AB is requested, the fact that it bisects that segment in every instance is not informative. To do so, the construction process will also maintain a list of facts that can be reasoned from construction assumptions so that these can be omitted in the final reporting.

5 Related Work

The topics of automating geometric proofs and working with diagrams are areas of active research. Several examples of related work can be found in the proceedings of annual conferences such as *Automated Deduction in Geometry* [1] and *Diagrammatic Representation and Inference* [2]. In addition, two papers from the past year combine these concepts with a layer of computer vision interpretation of diagrams. Chen, Song, and Wang present a system that infers what theorems are being illustrated from images of diagrams [3], and a paper by Seo and Hajishirzi describes using textual descriptions of problems to improve recognition of their accompanying figures [4].

Further related work includes descriptions of the educational impacts of dynamic geometry approaches and some software to explore geometric diagrams and proofs. However, such software typically uses alternate approaches to automate such processes, and few focus on inductive reasoning.

5.1 Dynamic Geometry

From an education perspective, there are several texts that emphasize an investigative, conjecture-based approach to teaching such as *Discovering Geometry* by Michael Serra [5], the text I used to learn geometry. Some researchers praise these investigative methods [6] while others question whether it appropriately encourages deductive reasoning skills [7].

5.2 Software

Some of these teaching methods include accompanying software such as Cabri Geometry [8] and the Geometer's Sketchpad [9] designed to enable students to explore constructions interactively. These programs occasionally provide scripting features, but have no proof-related automation.

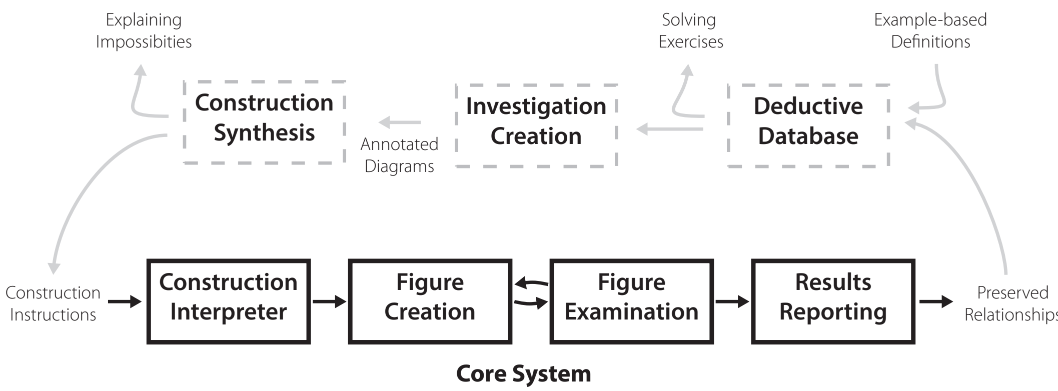
A few more academic analogs of these programs introduce some proof features. For instance, GeoProof [10] integrates diagram construction with verified proofs using a number of symbolic methods carried out by the Coq Proof Assistant, and Geometry Explorer [11] uses a full-angle method of chasing angle relations to check assertions requested by the user. However, none of the software described simulates the exploratory, inductive investigation process used by students first discovering new conjectures.

5.3 Automated Proof and Discovery

Although there are several papers that describe automated discovery or proof in geometry, most of these use alternate, more algebraic methods to prove theorems. These approaches include an area method [12], Wu's Method involving systems of polynomial equations [13], and a system based on Gröbner Bases [14]. Some papers discuss reasoning systems including the construction and application of a deductive database of geometric theorems [15]. However, all of these methods focused either on deductive reasoning or complex algebraic reformulations.

6 Extensions and Applications

There are several approaches that can extend the system and increase its power. These generally involve adding components before and after the core elements to create a more complete geometry reasoning system. Several of these components relate to existing work mentioned above and can use such ideas as a basis for implementation. The diagram demonstrates the connections between the core system and the extensions described below.



System Diagram: Extensions of the core induction system could create a full loop and permit completely independent geometry investigation.

6.1 Deduction Databases and Proof Systems

Making use of the findings of the core components, a deductive automated proof system as discussed above could be used to prove or verify newly discovered conjectures. If proven, these theorems could be stored in a deductive database and used as inference rules in solving exercises. This component would allow the system to demonstrate and apply what it is learning to user-specified tasks and exercises.

6.2 Modeling Base Knowledge

In addition to a deductive database of findings, the system could also include a component in which it could learn new terminology and base definitions. For instance, rather than hard coding what it means for a triangle to be isoceles, the system could allow a user to present example and counterexample figures. Based on patterns discovered in these figures, the system could propose and test definitions to be used in future investigations. This dynamic base knowledge repository could be used throughout the system.

6.3 Construction Synthesis

Preceding the input of the core system, a possible extension is to synthesize construction instructions from an initial diagram. Rather than requiring explicit steps, the system thus could look at an annotated diagram and derive a set of constructions that produces a figure maintaining such constraints. During this process, one would also have to deal with impossible diagrams or complicated constructions. A recent paper by Microsoft Research discusses some of these ideas [16].

6.4 Independent Investigation Creation

Finally, if there is sufficient extension of a deductive library and construction synthesis, an interesting extension would be one in which the system proposes its own diagrams to investigate rather than being prompted from an outside user. This would provide some full circle closure to the discovery process and could even lead to the system creatively devising interesting exercises or exam questions that test what it has acquired.

7 Evaluation of System

The experimentation and evaluation of the system will primarily involve exploring what investigations it is able to pursue and what conjectures it is able to discover. Flexibility in implementation will allow for the modification of base knowledge and characteristics of what is "interesting" to see how these changes affect the system's behavior and whether there is some sort of minimal assumed knowledge that still enables the system to make interesting discoveries.

8 Conclusion

Manipulating geometry diagrams in the "minds eye" reflects the creative human process of discovery, and relates to the experiential aspect of our mind and hand education. Automating this reasoning system can simulate some day to-day interactions and inferences, allowing for an exploration of what can be learned using a more inductive approach. It will be exciting to explore these idea as I continue to pursue my MEng project.

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