TMATH 390
Probability and Statics for Engineers
NOTES

Ben Foster*
benf94@uw.edu

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Abstract

This document contains the notes from the course TMATH 390 and does not necessarily contain all the information provided by the instructor.

*Institute of Technolgy, University of Washington Tacoma

Contents

1 Data and Distributions

The discipline of statistics teaches us how to make intelligent judgments and informed decisions in the presence of uncertainty and variation.

1.1 Populations, Samples, and Processes

An investigation will typically focus on a well-defined collection of objects constituting a **population** of interest. In one study, the population might consist of all gelatin capsules of a particular type produced during a specified period. When desired information is available for all objects in the population, we have what is called a **census**. A subset of the population – a **sample**– is usually selected in some prescribed manner to get feedback on the population.

1.2 Visual Displays for Univariate Data

Stem-and-Leaf Displays These can be an effective way to organize numerical data without expending much effort. It is based on separating each observation into two parts: a **stem**, consisting of one or more leading digits, and a **leaf**, consisting of the remaining or trailing digit(s). Example:

0	4
1	1345678889
2	1223456666777889999
3	01122333445556666777777888899999
4	11122222334444556666667778888899 9
5	00111222233455666667777888899
6	01111244455666778

Table 1: Percentage binge drinkers at each of 140 colleges

1.2.1 Dotplots

A dotplot is an attractive summary of numerical data when the data set is reasonably small or there are relatively few distinct data values. Each observation is presented by a dot above the corresponding location on a horizontal measurement scale. When a value occurs more than once, there is a dot for each occurrence, and these dots are stacked vertically. Example: Here is data on state-by-state appropriations for higher education as a percentage of state and local tax revenue for fiscal year 2009-2010:

14.0	3.1	8.6	9.6	7.4	4.0	4.5	6.5	6.1	8.8
8.2	8.6	6.4	6.7	8.0	8.5	9.4	9.5	4.6	6.8
3.9	6.9	6.3	11.9	5.8	5.8	9.9	5.9	2.7	4.2
14.9	4.0	12.1	8.0	5.2	9.2	6.8	4.3	3.9	9.6
8.0	8.6	8.6	8.7	3.1	5.8	6.2	8.7	6.8	8.9

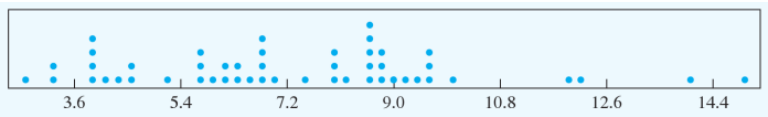


Figure 1: example caption

1.2.2 Histograms

Some numerical data is obtained by counting to determine the value of a variable, whereas other data is obtained by taking measurements. The prescription for drawing a histogram is different for these two cases.

A variable is **Discrete** if its set of possible values either is finite or else can be listed in an infinite sequence. A variable is **continuous** if its possible values consist of an entire interval on the number line.
Consider data consisting of observations on a discrete variable x. The **Frequency** of any particular x value is the number of times that value occurs in the data set. The **Relative Frequency** of a value is the fraction or proportion of time the value occurs.

$$\text{relative frequency of a value} = \frac{\text{number of times the value occurs}}{\text{number of observations in the data set}}$$

Here is an example of a histogram:

Board Size	Frequency	Relative Frequency	Board Size	Frequency	Relative Frequency
4	3	0.0147	19	0	0.0000
5	12	0.0588	20	0	0.0000
6	13	0.0637	21	1	0.0049
7	25	0.1225	22	0	0.0000
8	24	0.1176	23	0	0.0000
9	42	0.2059	24	1	0.0049
10	23	0.1127	25	0	0.0000
11	19	0.0931	26	0	0.0000
12	16	0.0784	27	0	0.0000
13	11	0.0539	28	0	0.0000
14	5	0.0245	29	0	0.0000
15	4	0.0196	30	0	0.0000
16	1	0.0049	31	0	0.0000
17	3	0.0147	32	1	0.0049
18	0	0.0000		204	0.9997

Table 2: Table of board members for hospitals

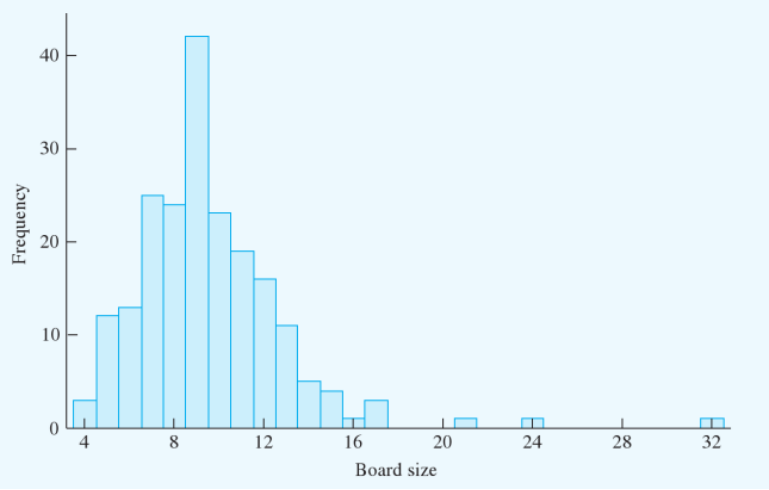


Figure 2: Example of a Histogram

Constructing a Histogram for Continuous Data: Equal Class Widths Determine the frequency and relative frequency for each class. Mark the class boundaries on a horizontal measurement axis. Above each class interval, draw a rectangle whose height is the corresponding relative frequency (or frequency).

Constructing a Histogram for Continuous Data: Unequal Class Widths After determining the frequencies and relative frequencies, calculate the height of each rectangle using the formula:

$$\text{rectangle height} = \frac{\text{relative frequency of the class}}{\text{class width}}$$

The resulting rectangle heights are usually called *densities*, and the vertical scale is the **density scale**. This prescription will also work when the class widths are equal.

1.2.3 Histogram Shapes

Histograms come in a variety of shapes. A **unimodal** histogram is one that rises to a single peak and then declines. A **bimodal** histogram has two different peaks. Bimodality occurs when the data set consists of observations on two quite different kinds of individuals or objects. A histogram with more than two peaks is said to be **multimodal**. Of course, the number of peaks may well depend on the choice of class intervals, particularly with a small number of observations.

A histogram is **symmetric** if the left half is a mirror image of the right half. A unimodal histogram is **positively skewed** if the right or upper tail is stretched out compared with the left or lower tail, and **negatively skewed** if the longer tail extends to the left.

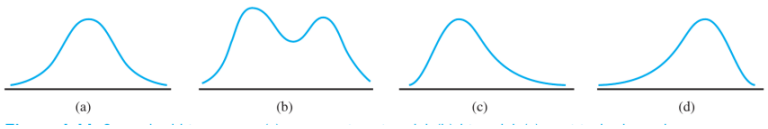


Figure 3: Smoothed Histograms: (a) symmetric unimodal; (b) bimodal; (c) positively skewed; (d) negatively skewed

1.2.4 Categorical Data

A histogram for categorical data is often called a **bar chart**. In come cases, there will be a natural ordering of classes (for example, freshman, sophomore, junior, senior), whereas in other cases, the order will be arbitrary (Honda, Yamaha, Ford, etc.). A **Pareto diagram** is a bar chart resulting from a quality control study in which category data represents a different type of product nonconformity or production problem.

1.3 Describing Distributions

In Section 1.2, we saw that a histogram could be used to describe how values of a variable x are distributed in a data set. In practice, a histogram is virtually always constructed from sample data.

1.3.1 Continuous Distributions

1.4 The Normal Distribution

1.5 Other Continuous Distributions

1.6 Several Useful Discrete Distributions

2 Numerical Summary Measures

2.1 Measures of Center

2.2 Measures of Variability

2.3 More Detailed Summary Quantities

2.4 Quantile Plots

Constructing a Quantile plot can take a little more work than constructing a regular distribution. For example, when making a Normal Quantile Plot, you would use the following definition for a sample quantile: Let $x_{(1)}$ denote the smallest sample observation, $x_{(2)}$ the second smallest observation,..., and $x_{(n)}$ the largest sample observation. For $i = 1, ..., n, x_{(i)}$ is the $[(i - 0.5/n]$th sample quantile.
Therefore, to make the Normal Quantile Plot, you would use the coordinates:

$$\left(\left(\frac{0.5}{n}\right) \text{th quantile}, x_{(1)}\right), ..., \left(\left(\frac{i-0.5}{n}\right) \text{th quantile}, x_{(n)}\right)$$

The plot of this, if a true normal distribution, should fall close to a 45° angle or a line with a slope of 1 passing through the point (0,0).

If you get a normal distribution that isn't standard, then you would use this:

$$\text{quantile for normal } (\mu, \sigma)\text{distribution} = \mu + (\text{corresponding z quantile})\sigma$$

3 Bivariate and Multivariate Data and Distributions

Bivariate and Multivariate Data and Distributions:

A multivariate data set consists of observations made simultaneously on two or more variables. One important special case is that of bivariate data, in which observations on only two variables, x and y are available. We will also discuss the correlation coefficient which is a measure of how strongly two variables are related.

3.1 Scatterplots

Scatterplots are the best way to graphically describe Bivariate data sets. In R, typing in the command `plot(A,B)` will print out the scatter plot with A being x and B being Y where as the command `plot(A ~ B)` treats A as y and B as x. Here is an example of a scatterplot with histograms related to both variables attached to it:

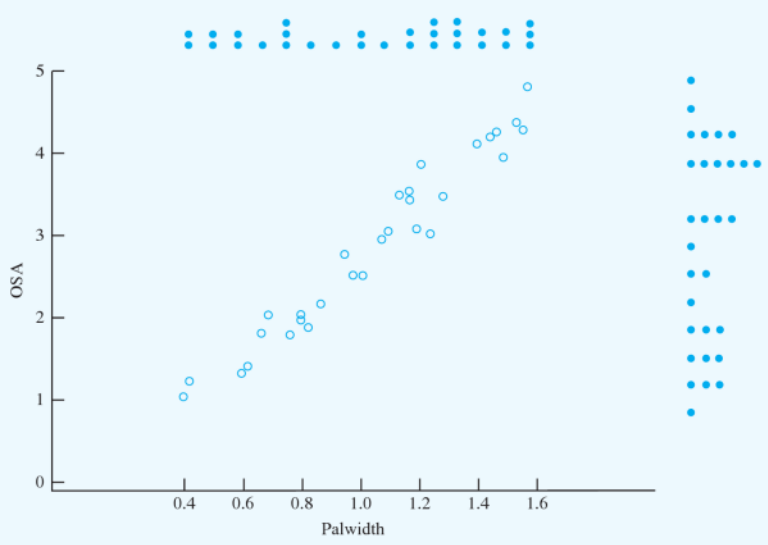


Figure 4: Example of a scatterplot

3.2 Correlation

In order to make precise statements and draw reliable conclusions from the data, we must go beyond pictures and find **Correlation Coefficient** which is a quantitative assessment of the strength of relationship between x and y values in a set of pairs.

A *positive* relationship is one where both x and y tend to increase together. A *negative* relationship is one where y tends to decrease as x increases. A strong positive or negative relationship can be linear or curved in appearance so long as x and y tend to be clustered together in particular pattern.

Pearson's sample correlation This sample correlation is r given by the following equation:

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum(y_i - \bar{y})^2}}$$
$$= \frac{S_{xy}}{\sqrt{S_{xx}}\sqrt{S_{yy}}}$$

Computing formulas for the three summation quantities are:

$$S_{xx} = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}$$

$$S_{yy} = \sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}$$
$$S_{xy} = \sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n}$$

Properties of r

- The value of r does not depend on the unit of measurement for either variable, meaning the correlation coefficient measures the inherent strength of relationship between two numerical variables.
- The value of r does not depend on which of the two variables is labeled x.
- The value of r is between -1 and +1. A value near the upper limit is indicative of a substantial positive relationship, whereas an r close to the lower limit suggests a prominent negative relationship.
- r = 1 only when all the points in a scatterplot of the data lie exactly on a straight line that slopes upward. Similarly, r = -1 only when all the points lie exactly on a downward sloping line.
- The value of r is a measure of the extent to which x and y are **linearly** related. The extent to which the points in the scatterplot fall close to a straight line.

The Population Correlation Coefficient Pearson's r measures how strongly the x and y values in a *sample* of pairs are related to one another. There is an analogous measure of how strongly x and y are related in the entire population of pairs from which the sample was obtained. This is called the **population correlation coefficient** and is denoted by ρ. ρ satisfies properties paralleling those of r:

- ρ is a number between -1 and +1 that does not depend on the unit of measurement for either x or y, or on which variable is labeled x and which is labeled y.
- $\rho = +1$ or -1 if and only if all pairs in the population lie exactly on a straight line.

Correlation and Causation Be sure to remember that just because a value of r is close to 1 does not mean that relatively large values of one variable *cause* relatively large values of the other variable.

3.3 Fitting a Line to Bivariate Data

Given two numerical values of x and y, the general objective of *regression analysis* i s to use the information about x to draw some type of conclusion concerning y. The different roles played by the two variables are reflected in standard terminology: y is called the **dependent** or **response variable**, and x is referred to as the **independent, predictor, or explanatory variable**.
A scatterplot of y vs x frequently exhibits a linear pattern. In such cases, it is natural to summarize the relationship between the variables by finding a line that is as close as possible to the points in the plot.

Fitting a Straight Line Often when making a scatterplot, you can place a line that generally summarizes the scatterplot and then each point will have a deviation from that line. In order to find the best fitting line, we need to find the **Least Squares Line**. To find that we have to minimize the following sum:

$$\sum [y_i - (a + bx_i)]^2 = [y_1 - (a + bx_1)]^2 + ... + [y_n - (a + bx_n)]^2$$

To find this equation, let $g(\tilde{a}, \tilde{b}) = \sum [y_i - (\tilde{a} + \tilde{b}x_i)]^2$. Then the intercept a and the slope b of the least squares line are the values of $\tilde{a}$ and $\tilde{b}$ that minimize $g(\tilde{a}, \tilde{b})$. These minimizing values are obtained by taking the partial derivative of the g function first with respect to $\tilde{a}$ and then with respect to $\tilde{b}$, and equating these two partial derivatives to zero.

The slope b of the least squares line is given by

$$b = \frac{\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)/n}{\sum x_i^2 - \left(\sum x_i\right)^2/n} = \frac{S_{xy}}{S_{xx}}$$

The vertical intercept a of the least squares line is given by

$$a = \bar{y} - b\bar{x}$$

The equation of the least squares line is often written as $\hat{y} = a + bx$, where the hat above the y emphasizes that $\hat{y}$ is a prediction of y that results from the substitution of any particular x value into the equation.
The least squares line should not be used to make a prediction for an x value much beyond the range of the data. The danger of extrapolation is that the fitted relationship may not be valid for such x values.

Regression The term comes from the relationship between the least squares line and the sample correlation coefficient. A little Algebraic manipulation yields:

$$b = r\left(\frac{s_y}{s_x}\right)\hat{y} = \bar{y} + r\left(\frac{s_y}{s_x}\right)(x - \bar{x}$$

When $-1 < r < 1$, for *any* x value, the corresponding predicted value $\hat{y}$ will be closer in terms of standard deviations to $\bar{y}$ than is x to $\bar{x}$; that is, $\hat{y}$ is pulled toward (regressed toward) the mean y value. This **regression effect** was first noticed by Sir Francis Galton in the late 1800s when he studied the relation between father's height and son's height; The predicted height of a son was always closer to the mean height than was his father's height.

Assessing the Fit of the Least Squares Line How much of the observed variation in y can be attributed to the approximate linear relationship and the fact that x is varying? A quantitative assessment is based on the vertical deviations from the least squares line.
Variation in y can be effectively be explained by an approximate straight-line relationship when the points in the scatterplot fall close to the least squares line – that is, when the residuals are small in magnitude. A natural measure of variation about the least squared line is the sum of the squared residuals.
Residual sum of squares, denoted by **SSResid**, is given by:

$$\text{SSResid} = \sum(y_i - \hat{y})^2 = (y_1 - \hat{y}_1)^2 + ... + (y_n - \hat{y}_n)^2$$

(Alternatively called *error sum of squares* and denoted by SSE).
Total sum of squares, denoted by **SSTo**, is defined as

$$\text{SSTo} = \sum(y_i - \bar{y})^2$$

Alternatie notation for SSTo is S_{yy}, and a computing formula is

$$\sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}$$

A computing formula for residual sum of squares makes it unnecessary to calculate the residuals:

$$\text{SSResid} = \text{SSTo} - bS_{xy}$$

because b and S_{xy} have the same sign, bS_{xy} is a positive quantity unless $b = 0$, so the computing formula shows that SSResid = SSTo if $b = 0$ and SSResid ¡ SSTo otherwise.
SSResid is the amount of variation in y that cannot be attributed to the linear relationship between x and y.

The **coefficient of determination**, denoted by r^2, is given by

$$r^2 = 1 - \frac{\text{SSResid}}{\text{SSTo}}$$

It is the proportion of variation in the observed y values that can be explained by a linear relationship between x and y in the sample.

Standard Deviation About the Least Squares Line The **standard deviation about the least squares line** is given by

$$s_e = \sqrt{\frac{\text{SSResid}}{n-2}}$$

Roughly speaking, s_e is the typical amount by which an observation deviates from the least squares line.

Plotting the Residuals (Optional A desirable plot exhibits no particular pattern, such as curvature or much greater spread in one part of the plot than in another part. Looking at a residual plot after fitting a line amounts to examining y after removing any linear dependence on x. This can sometimes more clearly show the existence of a nonlinear relationship.

A point that is far off typically means that there is some unusual behavior such a recording error, non-standard experimental condition or atypical experimental subject.

3.4 Nonlinear Relationships

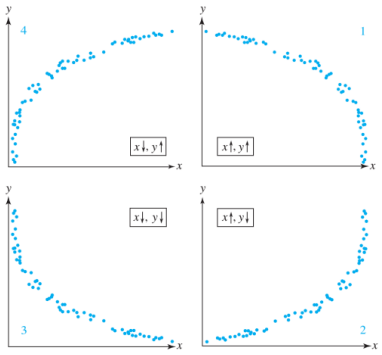
A scatterplot of bivariate data frequently shows curvature rather than a linear pattern. In this section, we discuss several different ways to fit a curve to such data.

Power Transformations Suppose that the general pattern is curved and monotonic. In this case, it is often possible to find a **power transformation** for x and y so that there is a linear pattern in a scatterplot of the transformed data. By a power transformation, we mean the use of exponents p and q such that the transformed values are $x\prime = x^p$ and/ or $y\prime = y^q$; the relevant scatterplot is of the $(x\prime, y\prime)$ pairs.

Transformed value $= (originalvalue)^{POWER}$

Power Transformation ladder:

Power	Transformed value	Name
3	$(Originalvalue)^3$	Cube
2	$(Originalvalue)^2$	Square
1	Original value	No transformation
$\frac{1}{2}$	$\sqrt{Originalvalue}$	Square root
$\frac{1}{3}$	$\sqrt[3]{Originalvalue}$	Cube root
0	Log(original value)	Logarithm
-1	$\frac{1}{originalvalue}$	Reciprocal



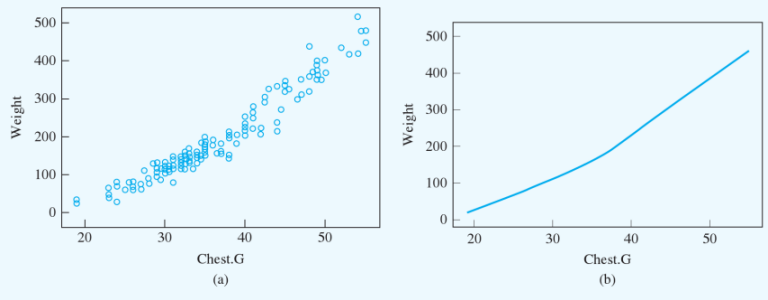
Fitting a Polynomial Function Sometimes the general patter of curvature in a scatterplot is not monotonic. In such instances, it is reasonable to fit a quadratic function $a + b_1x + b_2x^2$, whose graph is a parabola, to the data. If the quadratic coefficient b_2 is positive, the parabola turns upward, whereas it turns downward if b_2 is negative. Just as in fitting a straight line, the principle of least squares can be employed to find the best-fit quadratic. The **least squares coefficients** $a, b_1,$ **and** b_2 are the values of $\tilde{a}, \tilde{b_1},$ and$\tilde{b_2}$ that minimize

$$g(\tilde{a}, \tilde{b_1}, \tilde{b_2}) = \sum_i [y_i - (\tilde{a} + \tilde{b_1}x_i + \tilde{b_2}x_i^2)]^2$$

Which is the sum of squared vertical deviations from the points in the scatterplot to the parabola determined by the quadratic with coefficients $\tilde{a}, \tilde{b_1}, \tilde{b_2}$. Taking the partial derivative of the g function first with respect to $\tilde{a}$, then with respect to $\tilde{b_1}$, and finally with respect to $\tilde{b_2}$, and equating these three expressions to zero gives three equations in three unknowns. These *normal equations* are again linear in the unknowns, but because there are three rather than just two, there is no explicit elementary expression for their solution. Instead, matrix algebra must be used to solve the system numerically for each different data set.

The methodology employed to fit a quadratic is easily extended to fit a higher-order polynomial. For example, using the principle of least squares to fit a cubic equation gives a system of normal equations consisting of four equations in four unknowns. The arithmetic is best left to a statistical computer package. In practice, a cubic equation is rarely fit to data, and iti s virtually never appropriate to fit anything of higher order than this.

Smoothing a Scatterplot Sometimes the pattern in a scatterplot is too complex for a line or curve of a particular type (e.g., exponential or parabolic) to give a good fit. Statisticians have recently developed some more flexible methods that permit a wide variety of patterns to be modeled using the same fitting procedure. One such method is LOWESS, short for *locally weighted scatterplot smoother*. Let $(x*, y*)$ denote a particular one of the n (x, y) pairs in the sample. The $\hat{y}$ value corresponding to $(x*, y*)$ is obtained by fitting a straight line using only a specified percentage of the data (e.g, 25%) whose x values are closest to $x*$. Furthermore, rather than use "ordinary" least squares, which gives equal weight to all points, those with x values closer to $x*$ are more heavily weighted than those whose x values are farther away. The height of the resulting line above $x*$ is the fitted value $\hat{y}*$. This process is repeated for each of the n points, so n different lines are fit (you surely wouldn't want to do all this by hand). Finally, the fitted points are connected to produce a LOWESS curve.



3.5 using More Than One Predictor

In many situations, predictions of y values can be improved and more observed y variation can be explained by utilizing information in two or more explanatory variables. Notation is a bit more complex than in the case of a single predictor. Let

$$k = \text{number of explanatory variables or predictors} \quad (1)$$

$$n = \text{sample size} \quad (2)$$

3.6 Joint Distributions

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