Probability and Statistics for Engineers Homework Five TMATH 390

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Contents

Problem 1

A college library has five copies of a certain text on reserve. Two copies (1 and 2) are first editions, and the other three (3, 4, and 5) are second editions. A student examines these books in random order, stopping only when a second edition has been selected. One outcome is 4, and another is 215.

(a) List all outcomes in the sample space S.
(b) Let A denote the event that exactly one book must be examined. What outcomes are in A?
(c) Let B be the event that book 4 is the one selected. What outcomes are in B?
(d) Let C be the event that book 2 is not examined. What outcomes are in C?

Answer to a The sample space of an experiment are all of the possible outcomes of a chance experiment. The sample space for this chance experiment is:

$$S = [3, 4, 5, 13, 14, 15, 23, 24, 25, 123, 124, 125, 213, 214, 215]$$

Answer to b The event that exactly one book must be examined means that the student randomly picked up a second edition book first:

$$A = [3, 4, 5]$$

Answer to c The event that book 4 is selected doesn't necessarily mean that it is the first book selected, but it could also be the second or third after one or both of the first two first edition books were selected:

$$B = [4, 14, 25, 124, 214]$$

Answer to d The event that book 2 is not examined means that we could have any simple event that does not contain 2:

$$C = [3, 4, 5, 13, 14, 15]$$

Problem 2

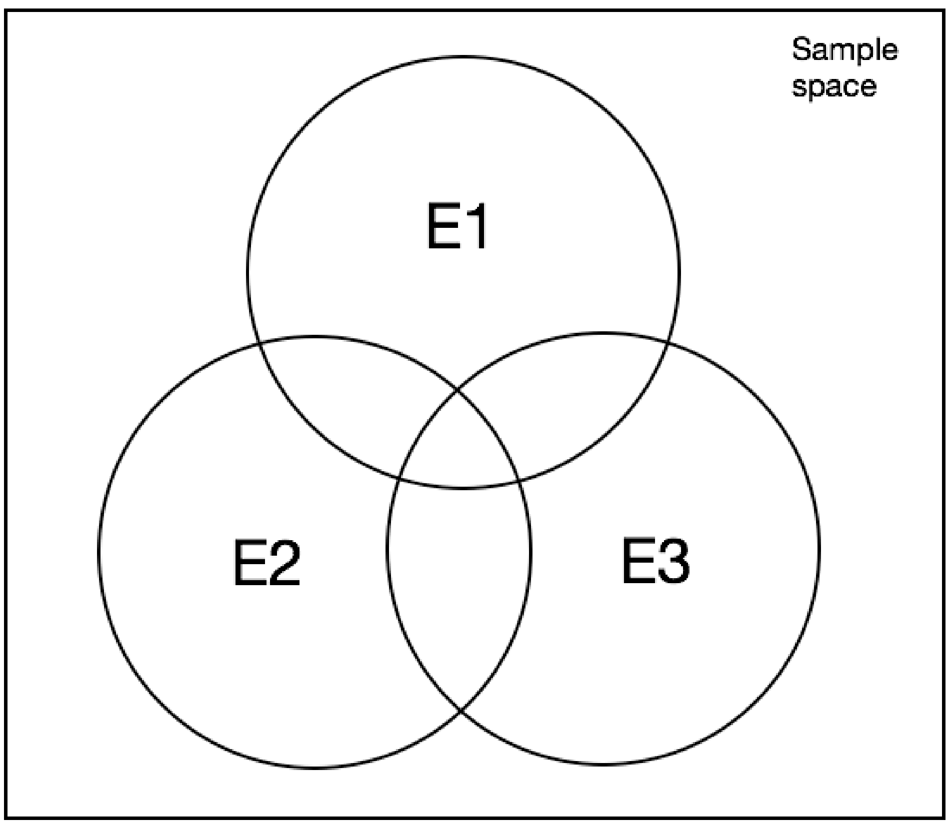
An engineering firm is constructing power plants at three different sites.
Define the events E_1, E_2, and E_3 as follows:

- E_1 = the plant at site 1 is completed by the contract date.
- E_2 = the plant at site 2 is completed by the contract date.
- E_3 = the plant at site 3 is completed by the contract date.

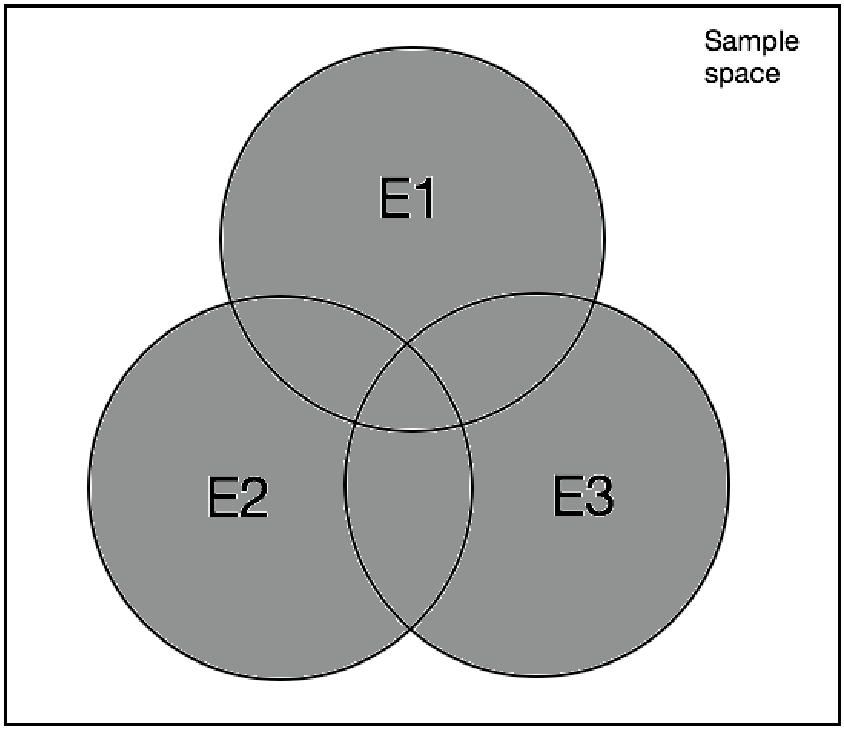
Draw a Venn diagram that depicts these three events as intersecting circles. Shade the region on the Venn diagram corresponding to each of the following events (redraw the Venn diagram for each question):

(a) At least one plant is completed by the contract date.
(b) All plants are completed by the contract date.
(c) None of the plants are completed by the contract date.
(d) Only the plant at site 1 is completed by the contract date.
(e) Exactly one of the three plants is completed by the contract date.
(f) Either the plant at site 1 or site 2 or both of the two plants are completed by the contract date.

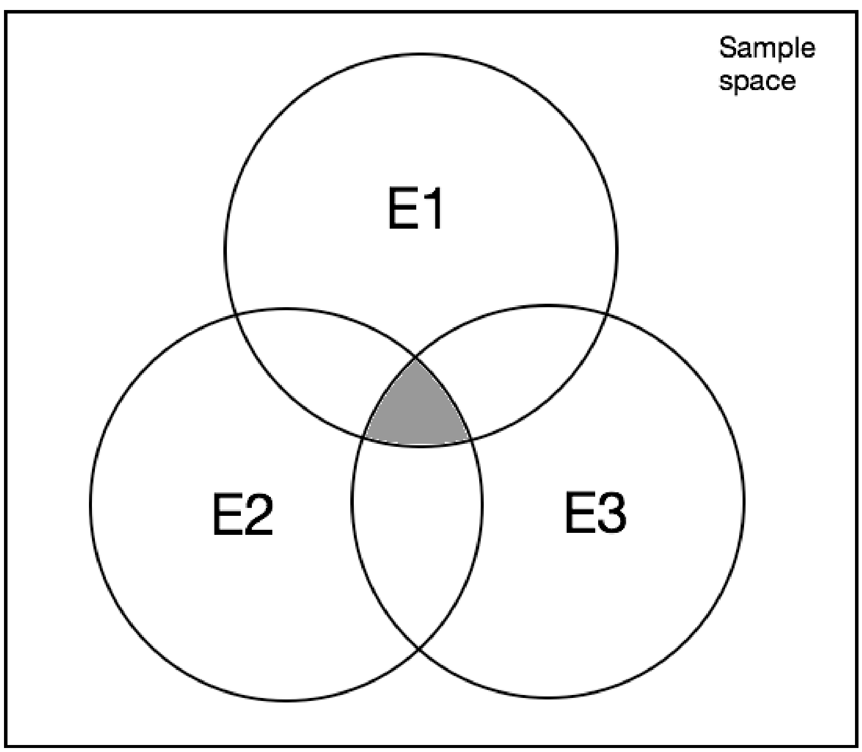
Original Venn diagram:



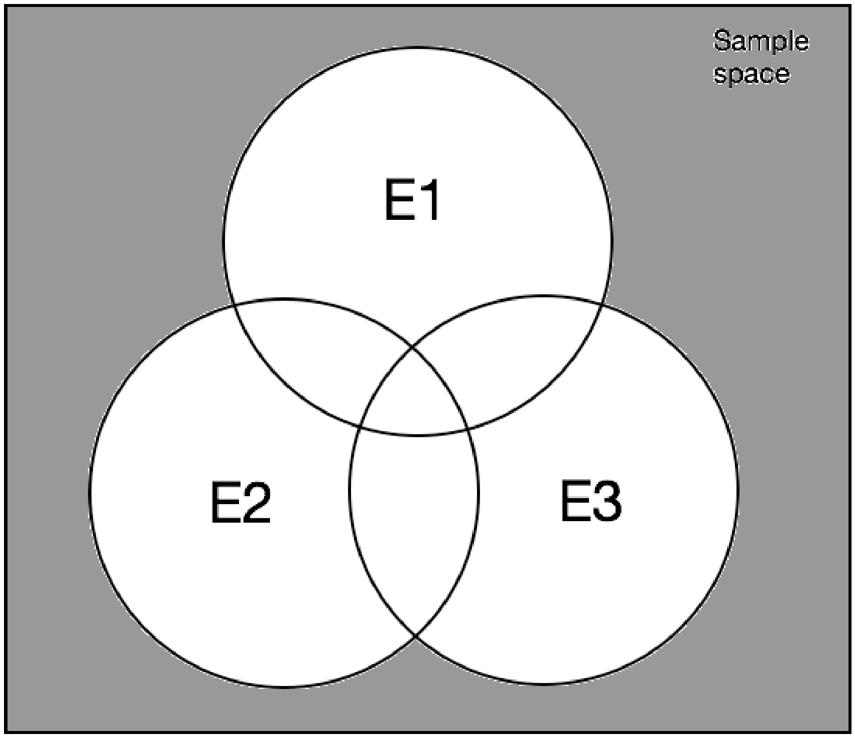
Answer to part a



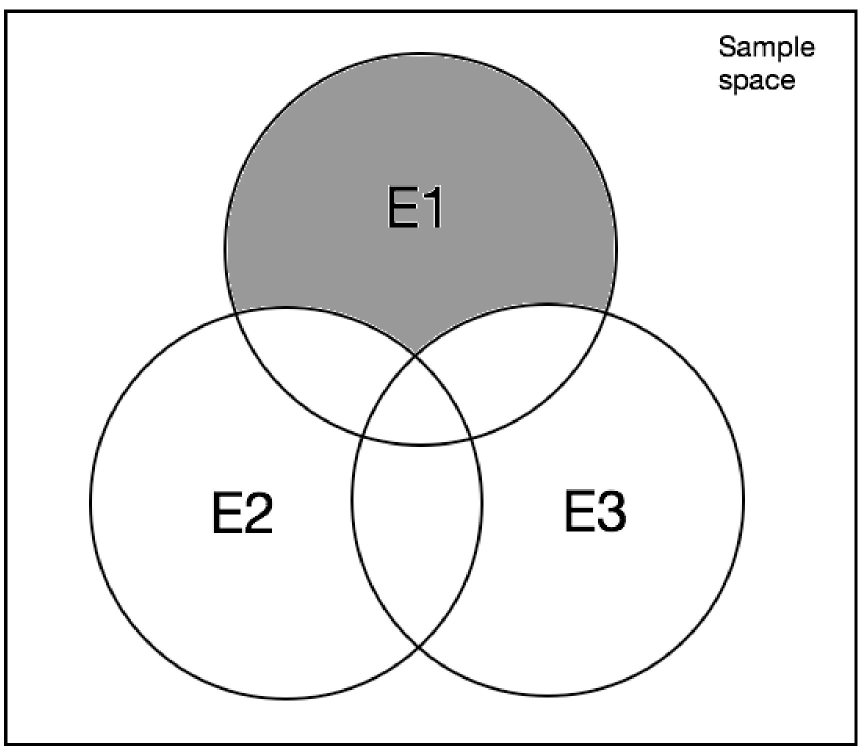
Answer to part b



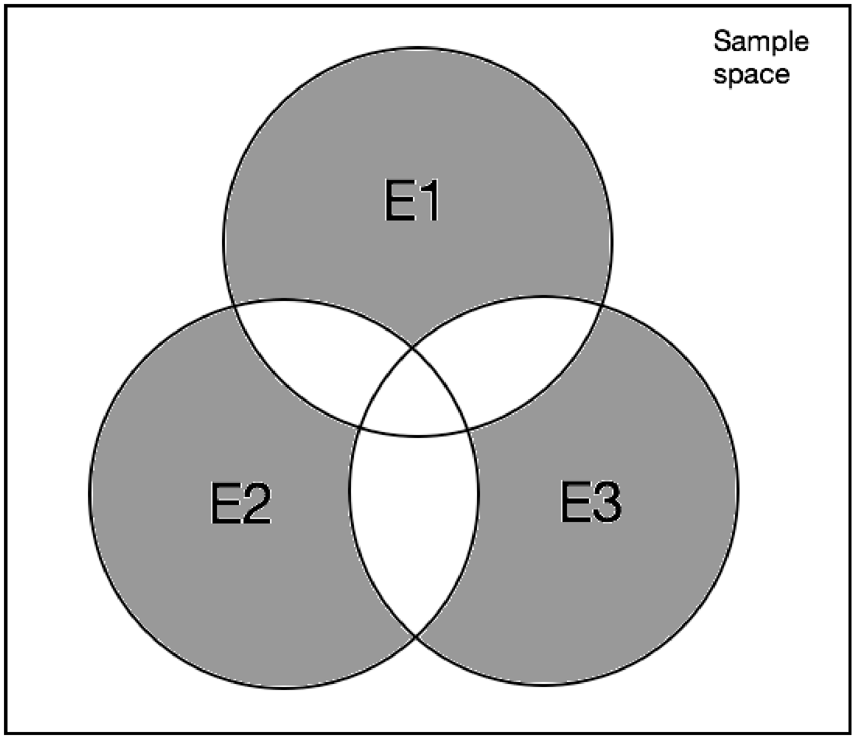
Answer to part c



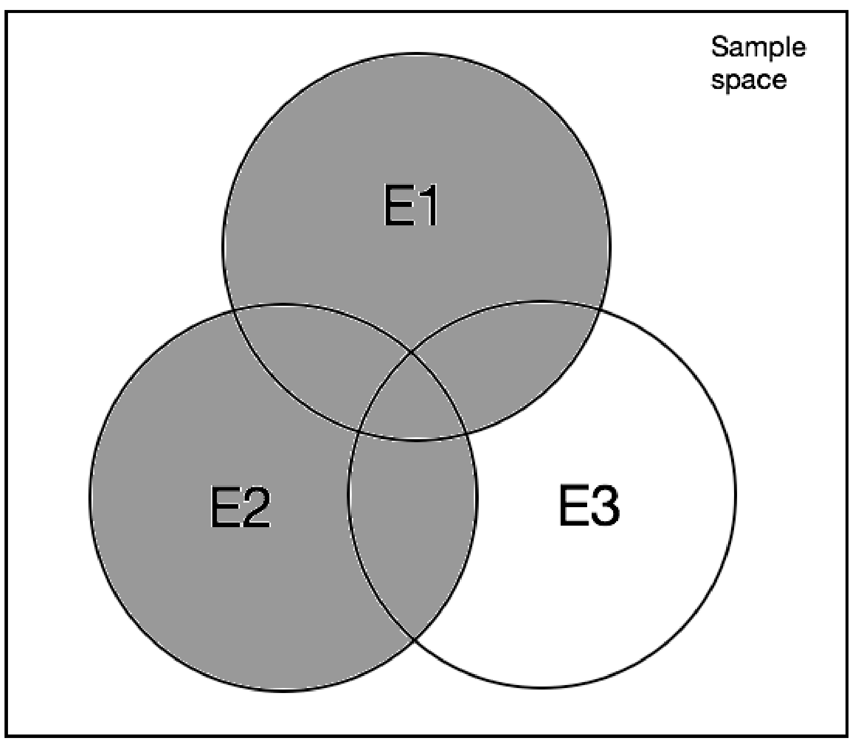
Answer to part d



Answer to part e



Answer to part f



Problem 3

Let $A_i = i$th student got a perfect score on midterm 1, for $i = 1, ..., 18$.

(a) Interpret $\left(\bigcap_{i=1}^{18} A_i\right)'$ in English.

(b) Interpret $\bigcup_{i=1}^{18} A_i'$ in English. Is it equivalent to part a?

Answer to a $\left(\bigcap_{i=1}^{18} A_i\right)'$ is the same as saying the complement of A_1, and A_2, and ... and A_18. Basically, students 1 through 18 did not get a perfect score on the first midterm, but any other combination of these students getting a perfect score on the first midterm is possible (i.e. A_1 and A_2).

Answer to b $\bigcup_{i=1}^{18} A_i'$ is the same as saying A_1', or A_2', or ... or A_{18}'. Basically, any combination of students 1 through 18 can get a perfect score on their first midterm, however it is not possible for all 18 students to get a perfect score.

Part a and part b are the same.

Problem 4

Using everything we have learned about events and probability, prove that

$$P(A|B) + P(A'|B) = 1$$

Do not assume that events A and B are independent (i.e. $P(A \cap B) \neq P(A) \cdot P(B)$). Explain each step of your calculation.

Answer to problem 4 $P(A|B)$ = the conditional probability of A occurring given that event B has already occurred. We know that these probabilities must be between 0 and 1, so $0 \leq P(A|B) \leq 1$ and $0 \leq P(A'|B) \leq 1$. Also, from the definition, we know that event B has already occurred, so we're really just adding together the probabilities of wether A will occur or not. $P(A|B)$ is the probability that A will occur and $P(A'|B)$ is the probability that A will not occur. Since there are only two possibilities, then the summation of those two possibilities should equal 1, or a 100% chance. Meaning that there is a 100% chance that A will either occur or not occur given that B has already occurred.

Problem 5

Five companies (A, B, C, D, and E) that make electrical relays compete each year to be the sole supplier of relays to a major automobile manufacturer. The auto company's records show that the probabilities of choosing a company to be the sole supplier are

Supplier chose:	A	B	C	D	E
Probability:	0.30	0.20	0.10	0.25	0.15

(a) Suppose that supplier E goes out of business this year, leaving the remaining 4 companies to compete with one another. What are the new probabilities of companies A, B, C, and D being chosen ad the sole supplier this year?
(b) Suppose the auto company narrows the choice of suppliers to companies A and C. What is the probability that company A is chosen this year?

Answer to a If company E were to go out of business then to determine the probabilities of the other companies being chosen, we have to use the equation:

$$P(\text{Other Company}|E')$$

$$= \frac{P(\text{Other Company})}{P(E')}$$

This gives us a new table:

A	B	C	D
0.3529	0.2352	0.1176	0.2941

Answer to b If we narrowed down our decision to companies A and C, then we would use the equations:

$$P(A|B' + D' + E') = \frac{P(A)}{P(B' + D' + E')}$$

$$= \frac{0.3}{1 - (0.2 + 0.25 + 0.15)}$$
$$= 0.75$$

and

$$P(C|B' + D' + E') = \frac{P(C)}{P(B' + D' + E')}$$
$$= \frac{0.1}{1 - (0.2 + 0.25 + 0.15)}$$
$$= 0.25$$

Therefore there is a 75% chance of choosing company A and a 25% chance of choosing company C.

Problem 6

Number 5.16 in your book on page 214:
In Exercise 6, suppose that there is a probability of 0.01 that a digit is incorrectly sent over a communication channel (i.e., that a digit sent as a 1 is received as a 0, or vice versa). Consider a message that consists of exactly 60% 1s.

(a) What is the proportion of 1s received at the end of the channel?
(b) If a 1 is received, what is the probability that a 1 was sent? Hint: Use the tree diagram from Exercise 6.

Answer to a Since our original message consists of 60% 1s and since those 1s have a 99% chance of staying a 1 (1 - 0.01 from the probability that they will change) and since we also have 40% 0s in the original message and they have a 1% chance of turning into 1s then we can use this calculation to determine the proportion of 1s in the final message:

$$(0.6 \times 0.99) + (0.4 \times 0.01) = 0.598$$

So 59.8% of the received message will be comprised of 1s.

Answer to b If a 1 is received then there is a 99% chance that a 1 was sent since there is a 1% chance of an error. We also know that the message being sent consists of 60% 1s. Therefore we use the formula:

$$P(S|R) = \frac{P(S \text{ and } R)}{P(R)}$$

$$= \frac{0.99(0.6)}{0.598}$$
$$= 0.993311$$

So we have a 99.33% chance that the sent bit will be a 1 if we receive a 1.

Problem 7

Number 5.19 in your book on page 214:
In forensic science, the probability that any two people match with respect to a given characteristic (hair color, blood type, etc.) is called the *probability of a match*. Suppose that the frequencies of blood phenotypes in the population are as follows:

A	B	AB	O
0.42	0.10	0.04	0.44

(a) What is the probability that two randomly chosen people both have blood type A?
(b) Repeat the calculation in part (a) for the other three blood types.
(c) Find the probability that two randomly chosen people have matching blood types. Note: A person can have only one phenotype.
(d) The probability that two people do not match for a given characteristic is called *discriminating power*. What is the discriminating power for the comparison of two people's blood types in part (c)?

Answer to a Since we have a 42% chance of randomly picking a person with blood type A, then randomly picking two people with this blood type would be $0.42 \times 0.42 = 0.1764$. We have a 17.6% chance of randomly choosing two people with blood type A.

Answer to b Repeating the same calculation in part A, we get:
$0.10 \times 0.10 = 0.01$
$0.04 \times 0.04 = 0.0016$
$0.44 \times 0.44 = 0.1936$
So we have a 1% chance of choosing two people with blood type B, a 0.2% chance of choosing two people with blood type AB, and a 19.4% chance of choosing two people with blood type O.

Answer to c To find this probability, all we have to do is sum together the probabilities found in part a and b:

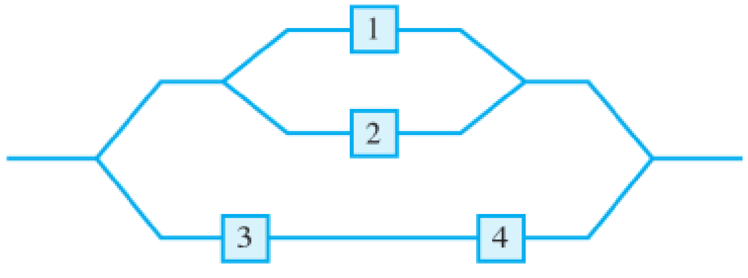
$$0.01 + 0.0016 + 0.1936 + 0.1764 = 0.3816$$

We have a 38.2% chance of choosing two people that have the same blood type.

Answer to d The probability that two people do not match blood types is basically the opposite of the previous question so $1 - 0.3816 = 0.6184$. We have a 61.8% chance of choosing two people who don't have a matching blood type.

Problem 8

Number 5.21 in your book on page 214:
Consider a system of components connected as shown in the following figure:



Components 1 and 2 are connected in parallel, so that their subsystem functions correctly if either component 1 or 2 functions. Components 3 and 4 are connected in series, so their subsystem works only if both components work correctly.

If all components work independently of one another and P(a given component works) = 0.9, calculate the probability that the entire system works correctly.

Answer to problem 8 To solve this problem we need to find the probability $P((1 \text{ or } 2) \text{ or } (3 \text{ and } 4))$ with $P(1) = P(2) = P(3) = P(4) = 0.9$.

$$P(1 \text{ or } 2) = 1 - P(1')P(2') = 1 - 0.01 = 0.99$$

$$P(3 \text{ and } 4) = P(3)P(4) = (0.9)(0.9) = 0.81$$

Now we can treat the top and bottom parts of the system as single events A and B respectively where $P(A) = 0.99$ and $P(B) = 0.81$.

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.99 + 0.81 - 0.8019 \\ &= 0.9981 \end{aligned}$$

So there is a 99.81% chance that the entire system works correctly.

Problem 9

Let p be the sample proportion, written as $p = \frac{n_f}{n}$, where n = sample size, and n_f = the number of females in the sample. Show that $E[p] = \pi$ and $V[p] = \frac{\pi(1-\pi)}{n}$, where π = proportion of females in population. Hint: Use what you know about Binomial distributions.

Answer to problem 9 Recall that for the binomial distribution:

$$E[X] = n\pi$$

$$V[X] = n\pi(1-\pi) = E[X^2] - (E[X])^2$$

so

$$E[p] = E[\frac{n_f}{n}]$$

Since n is a constant

$$= \frac{1}{n}E[n_f] = \frac{n\pi}{n} = \pi$$

and

$$V[p] = V[\frac{n_f}{n}]$$

Since n is a constant

$$= \frac{1}{n^2}V[n_f] = \frac{n\pi(1-\pi)}{n^2} = \frac{\pi(1-\pi)}{n}$$

Problem 10

A survey of the members of a large professional engineering society is conducted to determine their views on proposed changes to an ASTM measurement standard. Suppose that 80% of the entire membership favor the proposed changes. Hint: Use the result from the previous problem.

(a) Calculate the mean and standard deviation of the sampling distribution of the proportion of engineers in samples of size 25 who favor the proposed changes.
(b) Calculate the mean and standard deviation of the sampling distribution of the proportion of engineers in samples of size 100 who favor the proposed changes.

Answer to a Using the idea that $E[p] = \pi$ and $V[p] = \frac{\pi(1-\pi)}{n}$:

$$E[p] = \pi = 0.8$$

$$\sigma = \sqrt{V[p]} = \sqrt{\frac{0.8(1-0.8)}{25}}$$

$$\sigma = 0.08$$

Answer to b Using the idea that $E[p] = \pi$ and $V[p] = \frac{\pi(1-\pi)}{n}$:

$$E[p] = \pi = 0.8$$

$$\sigma = \sqrt{V[p]} = \sqrt{\frac{0.8(1-0.8)}{100}}$$

$$\sigma = 0.04$$

Problem 11

The lifetime of a certain battery is normally distributed with a mean value of 8 hours and a standard deviation of 1 hour. There are four such batteries in a package.

(a) What is the probability that the average lifetime of the four batteries exceeds 9 hours?
(b) If T denotes the average lifetime of the four batteries in a randomly selected package, find the numerical value of T_0 for which $P(T \geq T_0) = 0.95$.

Answer to a In order to solve this, we need to find $P(x > 9)$. In this case, since we know that the lifetime of the battery is normally distributed, then we can use the standard normal table to help us find our answer. Now we use $P(z > \frac{9-\mu_{\bar{x}}}{\sigma_{\bar{x}}})$ where $\mu_{\bar{x}} = 8$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{2}$. From this, $P(z > 2) = 1 - P(z < 2)$. Using the table, we get 0.0228 for the proportion of time that the lifetime exceeds 9 hours. We could also say that we have a 2.28% chance that the average lifetime of the four batteries will exceed 9 hours.

Answer to b For this problem, we know that $P(T \geq T_0) = 0.95$. From this, we know that $P(T \geq T_0) = 1 - P(T < T_0)$ so $P(T < T_0) = 0.05$. Now we just do a reverse table look up and we get the following:

$$-1.645 = \frac{T_0 - 8}{0.5}$$

$$T_0 = 7.1775$$

Problem 12

The number of flaws x on an electroplated automobile grill is known to have the following probability mass functions: $p(0) = 0.6$, $p(1) = 0.2$, $p(2) = 0.1$, $p(3) = 0.1$.

(a) Calculate the mean and standard deviation of x.
(b) What are the mean and standard deviation of the sampling distribution of the average number of flaws per grill in a random sample of 50 grills?
(c) For a random sample of 50 grills, calculate the approximate probability that the average number of flaws per grill exceeds 0.8.

Answer to a For the mean:

$$\begin{aligned}\mu &= \sum xp(x) \\ &= 0(0.6) + 1(0.2) + 2(0.1) + 3(0.1) \\ &= 0.7\end{aligned}$$

For the standard deviation

$$\begin{aligned}\sigma &= \sqrt{\sum(x-\mu)^2 p(x)} \\ &= \sqrt{((0-0.7)^2 0.6) + ((1-0.7)^2 0.2) + ((2-0.7)^2 0.1) + ((3-0.7)^2 0.1)} \\ &= \sqrt{0.294 + 0.018 + 0.169 + 0.529} \\ &= \sqrt{1.01} \\ &= 1.01\end{aligned}$$

Answer to b For the mean:

$$\mu_{\bar{x}} = \mu = 0.7$$

For the standard deviation:

$$\begin{aligned}\sigma_{\bar{x}} &= \frac{\sigma}{\sqrt{n}} \\ &= \frac{1.01}{\sqrt{50}} \\ &= 0.1421\end{aligned}$$

Answer to c We need to find $P(x > 0.8)$ which is the same as saying $P(z > \frac{0.8-\mu}{\sigma}$ since we are looking at a normal distribution. From this we get $P(z > 0.70359)$. Now we just have to look at the standard normal table. Since we want something greater than a number, then we can just do $1 - P(z < 0.704)$ to get the same thing. From looking at the table, we get

$$P(z > 0.704) = 1 - 0.7580$$
$$P(z > 0.704) = 0.242$$