

Fun Summary

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1 metric spaces

1.1 metric spaces

Definition 1.1.1. A metric space is a non-empty set X together with a map

$$d : X \times X \rightarrow \mathbb{R}$$

$$(x, y) \mapsto d(x, y)$$

such that

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$

Remark 1.1.2. (d admits only positive values)

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

Example 1.1.3. 1. $d_2(x, y) = \|x - y\|_2$

$$2. d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$$

Definition 1.1.4. (convergence)

A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is said to be convergent to $x \in X$ if

$$x_n \rightarrow x \text{ in } (X, d)$$

or

$$\lim_{n \rightarrow \infty} x_n = x \text{ in } (x, d)$$

1.2 Topology in metric spaces

Let (X, d) be a metric space.

Definition 1.2.1. 1. an open ball is defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

2. $O \subset X$ is called open if $\forall y \in O$ there is $r > 0$ such that $B_r(y) \subset O$

3. $A \subset X$ is closed if $X \setminus A$ is open.

Theorem 1.2.2. (*metric spaces are topological spaces*)

Let \mathcal{T} be the set of open subsets of X . Then

1. $\emptyset, X \in \mathcal{T}$

2. if $U, V \in \mathcal{T}$, then $U \cup V \in \mathcal{T}$

3. if $\{U_i\}_{i \in I} \subset \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$

Remark 1.2.3. 1. \emptyset, X are closed

2. finite union of closed sets is closed

3. arbitrary intersections of closed sets is closed

Lemma 1.2.4. $A \subset X$ is closed iff \forall convergent sequences $(x_n)_{n \in \mathbb{N}} \subset A$ the limit point is in A .

Definition 1.2.5. For $M \subset X$ we define

$$\overline{M} = \bigcap_{A \supset M, A \text{ closed}}$$

as the closure of M and

$$M = \bigcup_{O \subset M, O \text{ open}}$$

as the interior of M .

$\partial M = \overline{M} \setminus M$ is the boundary of M

Attention:

Define the closed ball as $\overline{B}_r(a) = \{y \in X : d(y, a) \leq r\}$. Then in general $\overline{\overline{B}_r(a)} \neq \overline{B}_r(a)$.

Example: Take $X \neq \emptyset$ and the trivial metric d . Then

$$B_1(a) = \{a\} = \overline{B_1(a)}$$

but $\overline{B}_1(a) = X$.

1.3 separability and completion

Let (X, d) be a metric space.

Definition 1.3.1. 1. $M \subset X$ is called dense in X if $\overline{M} = X$.

2. X is called separable if X has a countable dense subset.

Remark 1.3.2. M is dens in X iff

$$\forall x \in X \forall \varepsilon > 0 \exists y \in M \text{ s.t. } d(x, y) < \varepsilon$$

Definition 1.3.3. 1. $(x_n)_{n \in \mathbb{N}} \subset X$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n > N \text{ implies } d(x_n, x_m) < \varepsilon$$

2. A metric space in which all Cauchy sequences converge is called complete.

Example 1.3.4. 1. $(C^0([a, b], \mathbb{R}), d_\infty)$ with $d_\infty(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|$ is complete.

2. (\mathbb{R}^n, d_2) with $d_2(x, y) = \|x - y\|_2$ is complete.

Lemma 1.3.5. Let (X, d) be a complete metric space and $\emptyset \neq A \subset X$. Then (A, d) is complete iff A is closed.

Definition 1.3.6. $A \subset X$ is called bounded if its diameter

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$$

is finite.

Theorem 1.3.7. (X, d) is complete iff $\forall (F_n)_{n \in \mathbb{N}}$ sequences of closed subsets such that $F_{n+1} \subset F_n$ and $\text{diam}(F_n) \rightarrow 0$ then

$$\exists! x_0 \in X \text{ s.t. } \bigcap_{n \in \mathbb{N}} F_n = \{x_0\}$$

1.4 Continuity

Definition 1.4.1. Let $(X, d_x), (Y, d_y)$ be metric spaces and $f : X \rightarrow Y$. f is continuous in x_0 if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in X \ d_x(x, x_0) < \delta \text{ implies } d_y(f(x), f(x_0)) < \varepsilon$$

With sequences:

$\forall (x_n)_{n \in \mathbb{N}} \subset X \ x_n \rightarrow x_0$ in (X, d_x) if it holds $(f(x_n))_{n \in \mathbb{N}} \subset Y, f(x_n) \rightarrow f(x_0)$ in (Y, d_y)

f is continuous if f is continuous in x_0 for all $x_0 \in X$.

In other words f is continuous if for all $O \subset Y$ open (closed) $f^{-1}(O)$ is open (closed) in X .

Special case: f is Lipschitz continuous if $\exists L > 0$ s.t.

$$d_y(f(x), f(y)) \leq L d_x(x, y) \ \forall x, y \in X$$

f is an isometric if $\forall x, y \in X$ it holds that $d_Y(f(y), f(x)) = d_x(x, y)$.

1.5 Compact sets

Definition 1.5.1. Let (X, d) be a metric space and $A \subset X$.

1. an open cover of A is a collection $\{U_i\}_{i \in I}$ where $I \neq \emptyset$ is an arbitrary index set of open subsets of X s.t. $A \subset \bigcup_{i \in I} U_i$.
2. A is compact if every open cover of A contains a finite subcover i.e. there is $N \in \mathbb{N}$ and indices i_1, \dots, i_N such that

$$A \subset U_1 \cup \dots \cup U_N$$

3. A is sequentially compact if every sequence in A has a convergence subsequence in A .
4. A is called precompact or totally bounded if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ and $\exists x_1, \dots, x_N \in X$ such that $A \subset \bigcup_{i=1}^N B_\varepsilon(x_i)$.

Theorem 1.5.2. Let (X, d) be a metric space and $A \subset X$. The following are equivalent:

1. A is compact
2. A is sequentially compact
3. (A, d) is complete and A is precompact.

Remark 1.5.3. If A is precompact, then \overline{A} is precompact. Further, if (X, d) is complete and $A \subset X$ then A is precompact $\Leftrightarrow \overline{A}$ is compact.

Recall: A compact \Rightarrow bounded and closed and $f : X \rightarrow Y$ continuous with $A \subset X$ compact, then $f(A)$ is compact as well. Further, if $f : A \rightarrow \mathbb{R}$ is continuous and A is compact, then

$$\exists x_1, x_2 \in A \text{ s.t. } f(x_1) \leq f(x) \leq f(x_2) \forall x \in A$$

Theorem of Heine-Borel: $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

1.6 Theorem of Baire

Theorem 1.6.1. Let (X, d) be a complete metric space and $\forall n \in \mathbb{N}$ consider $U_n \subset X$ open and dense. Then

$$\bigcap_{n \in \mathbb{N}} U_n$$

is dense in X .

Remark 1.6.2. 1. Completeness is in general necessary. Consider (\mathbb{Q}, d) and $d(x, y) = |x - y|$. Define a sequence x_n such that $\mathbb{Q} = \{x_n \mid n \in \mathbb{N}\}$. Take $U_n = \mathbb{Q} \setminus \{x_n\}$ which is open and dense. Then

$$\bigcap_{n \in \mathbb{N}} U_n = \emptyset$$

Corollary 1.6.3. Let (X, d) be a complete metric space. Let $\forall n \in \mathbb{N}$, $A_n \subset X$ be closed and

$$X = \bigcup_{n \in \mathbb{N}} A_n$$

Then $\exists N \in \mathbb{N}$ s.t. A_N has an interior point.

Remark 1.6.4. Theorem 1.6.1 is also called Baire category theory.

- In a metric space (X, d) $A \subset X$ is called nowhere dense if \overline{A} has no interior points.
- A is called of first category if $\exists (M_n)_{n \in \mathbb{N}}$ where $M_n \subset A$ nowhere dense s.t. $A = \bigcup_{n \in \mathbb{N}} M_n$
- A is called of second category if it is not of first category

Hence the theorem of Baire implies that every complete metric space is of second category.

2 Normal spaces and Banach spaces

Let X be a \mathbb{K} -vector space where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

2.1 definitions

Definition 2.1.1. A map $\|\cdot\| : X \rightarrow \mathbb{R}$ is called a norm on X if

1. $\forall x \in X, \|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
2. $\forall \lambda \in \mathbb{K}$ and $\forall x \in X$ it holds that $\|\lambda x\| = |\lambda| \cdot \|x\|$
3. $\forall x, y \in X$ it holds $\|x + y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a normed space.

$p : X \rightarrow \mathbb{R}$ is called a seminorm if $p(x) \geq 0 \forall x \in X$ and 2. and 3. are also satisfied.

Example 2.1.2. 1. $C^0([0, 1]; \mathbb{R})$ with $\|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$

2. more general for a compact metric space K : $C^0(K, \mathbb{R})$ with $\|f\|_\infty = \max_{x \in K} |f(x)|$

3. $C^1([0, 1]; \mathbb{R})$ with $p(f) = \max_{x \in [0, 1]} |f'(x)|$

4. $\Omega \subset \mathbb{R}^n$ measurable. $L^1(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ integrable} \}$ with

$$p : L^1(\Omega) \rightarrow \mathbb{R} : p(f) = \int_{\Omega} |f(x)| dx$$

then p is a seminorm.

Remark 2.1.3. Any normed space is a metric space via

$$d(x, y) = \|x - y\|$$

All concepts from chapter 1 apply.

Lemma 2.1.4. Let $(X, \|\cdot\|)$ be a normed space. Then X is called separable iff $\exists A \subset X$ countable such that s.t. $\overline{\text{span}\{A\}} = X$ where $\text{span}\{A\} = \{\sum_{i=1}^n \lambda_i x_i\}$ with $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$ and $x_i \in A$. Here the closure is defined w.r.t the norm.

Definition 2.1.5. A complete normed space is called a Banach space.

2.2 Example: l^p -spaces

We consider the vector space $\mathbb{K}^{\mathbb{N}}$ of sequences in \mathbb{K} . Let $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$. Define $x + y = (x_n + y_n)_{n \in \mathbb{N}}$ and $\lambda x = (\lambda x_n)_{n \in \mathbb{N}}$.

For $x \in \mathbb{K}^{\mathbb{N}}$ define

$$\|x\|_{l^p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$\|x\|_{l^\infty} = \sup_{n \in \mathbb{N}} |x_n|$$

else.

Define $l^p = \{x = (x_n)_{n \in \mathbb{N}} : \|x\|_{l^p} < \infty\}$ for $1 \leq p \leq \infty$. We find that l^p is a subspace of $\mathbb{K}^\mathbb{N}$ and l^p is a normed space (for the triangle inequality use the Hölder inequality).

Theorem 2.2.1. *For $1 \leq p \leq \infty$ l^p is a Banach space.*

Lemma 2.2.2. *For finite p , l^p is separable while l^∞ is not.*

2.3 Finite dimensional normed spaces

Let X be a vector space over \mathbb{K} . $\exists e_1, \dots, e_n \in X$ s.t.

$$\forall x \in X; \exists \lambda_1, \dots, \lambda_n \in \mathbb{K} : x = \sum_{i=1}^n \lambda_i x_i$$

For $p \in [1, \infty)$ we define

$$\|x\|_p = \left(\sum_{i=1}^n |\lambda_i|^p \right)^{1/p}$$

and for $p = \infty$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |\lambda_i|$$

Definition 2.3.1. Two norms are equivalent in that

$$\alpha \|\cdot\|_1 \leq \|\cdot\|_2 \leq \beta \|\cdot\|_1$$

Theorem 2.3.2. *In a finite dimensional space, all norms are equivalent.*

Theorem 2.3.3. *Finite dimensional normed spaces are Banach spaces.*

2.4 On the closure of $\overline{B_1(0)}$

Lemma 2.4.1 (Lemma of Riesz, Lemma of the almost orthogonal element). *Let X be a normed space. $U \subset X$ a closed subspace of X s.t. $U \neq X$. Then $\forall \lambda \in (0, 1) \exists x_\lambda \in X$ s.t. $\|x_\lambda\| = 1$ and $\text{dist}(x_\lambda, U) \geq \lambda$.*

Theorem 2.4.2. *In a normed space X , $\overline{B_1(0)}$ is compact iff X is finite dimensional.*

3 A question from approximation theory

3.1 Theorem of Stone-Weierstrass

Let X be a compact metric space. Then $(C^0(X), \mathbb{K}), \|\cdot\|_\infty$, where $\|f\|_\infty = \max_{x \in X} |f(x)|$ is a Banach space.

Which property of $A \subset C^0(X, \mathbb{K})$ ensures that A is dense.

Definition 3.1.1. $A \subset C^0(X, \mathbb{K})$ is called subalgebra, if $\forall f, g \in A$

1. $\lambda f + \mu g \in A$ (subspace)
2. $f \cdot g \in A$

Example 3.1.2. • $\{p : [0, 1] \rightarrow \mathbb{R}\}$ is a subalgebra of $C^0([0, 1]; \mathbb{R})$.

- $\{f : [-1, 1] \rightarrow \mathbb{R}; f \text{ continuous and even}\}$ is a subalgebra.

Remark 3.1.3. If A is a subalgebra, then \overline{A} is also a subalgebra.

Definition 3.1.4. Let $A \subset C^0(X)$ be a subalgebra.

1. A is called unital if $1 \in A$
2. A separates point if $x, y \in X, x \neq y, \exists f \in A$ s.t. $f(x) \neq f(y)$.
3. (if $\mathbb{K} = \mathbb{C}$) A is stable under conjugation if from $f \in A$ we conclude that also $\overline{f} \in A$.

Remark 3.1.5. If A is unital then all constant functions are in A .

Lemma 3.1.6. Consider $f : [-1, 1] \rightarrow \mathbb{R}$ where $f(x) = |x|$. Then \exists sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ s.t.

$$p_n \rightarrow f$$

uniformly in $[-1, 1]$.

Lemma 3.1.7. Let $A \subset C^0(X, \mathbb{R})$ be a unital subalgebra. Then

1. if $f \in A$ then $|f| \in \overline{A}$.
2. if $f, g \in A$ then $\max\{f, g\} \in \overline{A}$ and $\min\{f, g\} \in \overline{A}$

Theorem 3.1.8 (Stone-Weierstrass). Let A be a compact metric space. $A \subset C^0(X, \mathbb{K})$ is a unital subalgebra that separates points and if $\mathbb{K} = \mathbb{C}$ is stable under conjugation, then A is dense in $C^0(X, \mathbb{K})$ w.r.t $\|\cdot\|_\infty$.

3.2 Applications

Theorem 3.2.1 (Theorem of Weierstraß). *Let $[a, b]$ be a compact interval in \mathbb{R} , $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $\varepsilon > 0$. Then $\exists p : [a, b] \rightarrow \mathbb{R}$ a polynomial s.t.*

$$\|p - f\|_\infty = \sup_{x \in [a, b]} |p(x) - f(x)| < \varepsilon$$

Definition 3.2.2. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is periodic if

$$f(x + t) = f(x)$$

for a $t \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Remark 3.2.3. If f is periodic with period t then $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$ where $\tilde{f}(x) = f(t \frac{x}{2\pi})$ is periodic of period 2π .

Consider $C_{2\pi}^0(\mathbb{R}, \mathbb{C})$ the space of continuous 2π -periodic functions. We consider the span of $\{e^{ikx} = \cos(kx) + i \sin(kx), k \in \mathbb{Z}\}$.

Definition 3.2.4. A trigonometric polynomial is a function $f : \mathbb{R} \rightarrow \mathbb{C}$

$$f(x) = \sum_{k=-N}^N c_k \cdot e^{ikx}$$

with $c_k \in \mathbb{C}$

Theorem 3.2.5 (Approximation of periodic functions). *Trigonometric polynomials are dense in $(C_{2\pi}^0(\mathbb{R}, \mathbb{C}), \|\cdot\|_\infty)$*

Application to neural networks

The simplest case of a neural network has d inputs x_1, \dots, x_d and one output Z called a *feed forward* network. Each input influences the output and x_i might have a weight α_i associated to it. The output is a function in $x = (x_1, \dots, x_d)$ and the weights $\alpha = (\alpha_1, \dots, \alpha_d)$. For instance, the output is often of the form

$$Z = \sum_{i=1}^d \alpha_i x_i + b$$

where b is the bias of the network. To make the network slightly stronger, we add a intermediate layer $y = (y_1, \dots, y_r)$ where each x_i is connected to each y_j with the associated weight $\gamma_{i,j}$. The y layer (often called activation) is then connected to the output Z as above

with weights α_j . We introduce the relation

$$y_j = \Phi\left(\sum_{i=1}^d \gamma_{j,i} x_i + b\right)$$

for a measurable function Φ . Lastly, the output is then given by

$$Z = \sum_{j=1}^r \alpha_j y_j$$

Definition 3.2.6. 1. $A^d = \{a : \mathbb{R}^d \rightarrow \mathbb{R} : a(x) = w^T x + b\}$ where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

2. given $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ measurable $d \in \mathbb{N}$ define $\Sigma^d(\Phi) = \{f : \mathbb{R}^d \rightarrow \mathbb{R} : f(x) = \sum_{j=1}^N \alpha_j \Phi(a_j(x)) \text{ with } N \in \mathbb{N}, \alpha_j \in \mathbb{R}, a_j \in A^d\}$ as the set of single hidden layer feed forward networks.

3. A squashing function is a measurable non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ s.t. $\lim_{x \rightarrow -\infty} \Phi(x) = 0$ and $\lim_{x \rightarrow \infty} \Phi(x) = 1$.

Theorem 3.2.7 (Universal Approximation theorem of Hornik-Stinchcombe-White). *Let Φ be a squashing function $K \subset \mathbb{R}^d$ compact $f : K \rightarrow \mathbb{R}$ continuous and $\varepsilon > 0$. Then $\exists g \in \Sigma^d(\Phi)$ s.t.*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon$$

4 Continuous linear maps

$(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ are K -Vector spaces with $K = \mathbb{R}$ or $K = \mathbb{C}$. $T : X \rightarrow Y$ is called linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

4.1 Continuity of linear maps

Definition 4.1.1. Let $T : X \rightarrow Y$ be linear. Then T is bounded if $\exists C > 0$ s.t.

$$\|Tx\|_Y \leq C \|x\|_X \quad \forall x \in X$$

or equivalently

$$\sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} \leq C$$

which is also equivalent to

$$\sup_{x \in X, \|x\|_X=1} \|Tx\|_Y \leq C$$

Theorem 4.1.2. For $T : X \rightarrow Y$ linear, the following are equivalent:

1. T is continuous
2. T is continuous in 0
3. T is bounded

Lemma 4.1.3. Let X have infinite dimension. Then $\exists T : X \rightarrow \mathbb{K}$ linear and not bounded.

Definition 4.1.4. Define $L(X, Y)$ as the set of linear continuous (\Leftrightarrow bounded) maps from X to Y . With the usual addition $((T_1 + T_2)(x) = T_1(x) + T_2(x))$ and the scalar multiplication $((\lambda(T))(x) = \lambda T(x))$ this is a vector space.

If $X = Y$ we write $L(X)$. For $T \in L(X, Y)$

$$\ker T = \{x \in X : Tx = 0\}$$

and

$$\Im(T) = \{y \in Y : \exists x \in X : Tx = y\}$$

4.2 Operator norm and dual space

Theorem 4.2.1. Let $X \neq \{0\}$.

- $L(X, Y)$ with the operator norm $\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X=1} \|Tx\|_Y$ is a normed space. We have

$$\|Tx\|_Y \leq \|T\| \|x\|_X$$

- If Y is a Banach space then $L(X, Y)$ is a Banach space.

Definition 4.2.2. For a normed space $(X, \|\cdot\|_\infty)$ we define the dual space $X' = L(X, \mathbb{K})$.

Remark 4.2.3. X' is a Banach space.

4.3 Neumann series

Lemma 4.3.1. Let X, Y, Z be three normed spaces. Let $T \in L(X, Y)$ and $S \in L(Y, Z)$. Then $S \circ T \in L(X, Z)$ and

$$\|S \circ T\| \leq \|S\| \|T\|$$

Let $T : X \rightarrow Y$ be linear, bounded and bijective. Then $\exists T^{-1} : Y \rightarrow X$ linear.

Definition 4.3.2. Let X, Y be normed spaces.

1. $T \in L(X, Y)$ is bijective such that $T^{-1} \in L(Y, X)$ then T is called an isomorphism
2. X, Y are called isomorph if there is $T : X \rightarrow Y$ isomorphism.
3. $T \in L(X, Y)$ is called an Isometry if $\|Tx\| = \|x\|$.
4. X, Y are called isometric isomorph if $\exists T \in L(X, Y)$ an isomorphism that is also an isometry.

Remark 4.3.3. The identity $I_x : X \rightarrow X$ with $x \mapsto x$ is in $L(X)$. Then $T \in L(X)$ is an isomorphism iff $\exists S \in L(X)$ s.t. $S \circ T = I_x$ and $T \circ S = I_x$

Let $T \in L(X)$ s.t. $\|T\| < 1$. Define $T^0 = I_x$, $T^n = T \circ T^{n-1}$. Obviously $T^n \in L(X)$ for all n . Now,

$$\left(\sum_{k=0}^n T^k \right)_{n \in \mathbb{N}} \subset L(X)$$

is a Cauchy sequence w.r.t. the operator norm. Hence, if X is a Banach-Space, so is $L(X)$ and thus the series converges to a $S \in L(X)$. Furthermore

$$\sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1 - \|T\|}$$

Finally, we can also note that $S = (I_x - T)^{-1}$.

Theorem 4.3.4 (Neumann series). *Let X be a Banach-Space, $T \in L(X)$ with $\|T\| < 1$. The $I_x - T$ is an isomorphism and*

$$(I_x - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

is in $L(X)$. This is called the Neumann series.

4.4 The dual space of l^p

We only deal with $1 \leq p < \infty$.

Theorem 4.4.1. *Let $q \in (1, \infty]$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then the dualspace $(l^p)'$ is isometric isomorph to l^q .*

5 Theorem of Hahn-Banach

Let X be a vector space, $X \neq \{0\}$ over $\mathbb{K} = \mathbb{R}$.

5.1 Extension Theorem

Given $U \subset X$ subspace, $l : U \rightarrow \mathbb{R}$ linear, is there $L : X \rightarrow \mathbb{R}$ linear such that $L|_U = l$? For this we need Zorn's Lemma:

Definition 5.1.1. Let $M \neq \emptyset$ be a set and \leq be a partial order on M , i.e. \leq satisfies

1. reflexiv: $x \leq x \ \forall x \in M$
2. antisymmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$
3. transitivity $x \leq y, y \leq z \Rightarrow x \leq z$

Then

- $A \subset M$ is called chain of totally ordered if $\forall x, y \in A$ either $x \leq y$ or $y \leq x$
- $b \in M$ is an upper bound for a chain A if $a \leq b$ for all $a \in A$
- $m \in M$ is called maximal element if

$$\forall x \in M \text{ s.t. } m \leq x \Rightarrow x = m$$

Lemma 5.1.2 (Zorn). *Let $M \neq \emptyset$ and \leq be a partial order on M . If every chain in M has an upper bound in M , then there is a maximal element.*

Definition 5.1.3. Let X be a vector space. $p : X \rightarrow \mathbb{R}$ is called sublinear if

1. $p(\lambda x) = \lambda p(x)$ for all $x \in X, \lambda \geq 0$
2. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Theorem 5.1.4 (Extension theorem of Hahn-Banach). *Let X be a vector space over \mathbb{R} , $U \subset X$ a subspace and $U \neq X$. Let $p : X \rightarrow \mathbb{R}$ be a sublinear and $l : U \rightarrow \mathbb{R}$ be linear s.t. $l(x) \leq p(x) \ \forall x \in U$. Then $\exists L : X \rightarrow \mathbb{R}$ linear s.t. $L(x) \leq p(x) \ \forall x \in X$ and $L(x) = l(x) \ \forall x \in U$. L is called extension of l .*

Consequences for normed spaces

Theorem 5.1.5. *Let $(X, \|\cdot\|_X)$, $U \subset X$ a subspace of X , with $U \neq X$. Let $u' \in U' = L(U, \mathbb{R})$. Then $\exists x' \in X'$ s.t. $\|x'\|_{X'} = \|u'\|_{U'}$ such that $x'(u) = u'(u) \ \forall u \in U$.*

Corollary 5.1.6. *Let $(X, \|\cdot\|_X)$, $U \subset X$ be a subspace of X and $x_0 \in X$ s.t. $\text{dist}(x_0, U) > 0$. Then $\exists x' \in X'$ s.t. $x'|_U = 0 \ \forall u \in U$ and $x'(x_0) = \text{dist}(x_0, U)$ with $\|x'\|_{X'} = 1$.*

Corollary 5.1.7. *Let $(X, \|\cdot\|_X)$ and $x_0 \in X$.*

1. if $x_0 \neq 0$ then $\exists F \in X'$ with $\|F\|_{X'} = 1$ and $F(x_0) = \|x_0\|_X$. In particular, for $x \in X$

$$\|x\|_X = \sup_{F \in X', \|F\|_{X'}=1} |F(x)|$$

2. If $F(x_0) = 0$ for all $F \in X'$, then $x_0 = 0$. In particular, X' separates points of X .
3. $U \subset X$ subspace. Then U is dense in X iff if for $x' \in X'$ s.t. $x'|_U = 0$ it follows $x' = 0$.

5.2 Separation Theorems

Definition 5.2.1. Let X be a vectorspace over \mathbb{R} . $A \subset X$ is called convex, if

$$\forall x, y \in A, \lambda x + (1 - \lambda)y \in A, \forall \lambda \in [0, 1]$$

Lemma 5.2.2. Let $C \subset X$ open and convex with $O \in C$. Define $p_C : X \rightarrow \mathbb{R}$ such that $p_C(x) = \inf\{\lambda > 0 \text{ s.t. } \frac{x}{\lambda} \in C\}$. This is called the Minkowski functional. Then p_C is sublinear and $C = \{x \in X : p_C(x) < 1\}$.

Lemma 5.2.3. Let $(X, \|\cdot\|)$ be a normed space and $A \subset X$ be convex and open, $A \neq \emptyset$ and $x_0 \in X \setminus A$, then $\exists F \in X'$ s.t.

$$F(x) < F(x_0) \quad \forall x \in A$$

Definition 5.2.4. Let $X \neq \{0\}$ be a \mathbb{R} -vectorspace.

1. $H = \{x \in X : f(x) = \alpha\}$ with $f : X \rightarrow \mathbb{R}$ linear, $\alpha \in \mathbb{R}$
2. $A, B \subset X$ are separated by an affine hyperplane H if $H = \{f = \alpha\}$ and $f(a) \leq \alpha \leq f(b) \quad \forall a \in A \quad \forall b \in B$.
3. $A, B \subset X$ are strictly separated by an affine Hyperplane H if $\exists \varepsilon > 0$ s.t. $f(a) + \varepsilon \leq \alpha \leq f(b) - \varepsilon$.

Theorem 5.2.5 (Separation Theorem of Hahn-Banach). Let $(X, \|\cdot\|)$, A, B convex, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and A or B should be open.. Then $\exists F \in X'$ and $\delta \in \mathbb{R}$ s.t.

$$F(a) \leq \delta \leq F(b) \quad \forall a \in A, b \in B$$

Theorem 5.2.6. Let $(X, \|\cdot\|)$, A, B convex subsets $A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$. Let A be closed and B be compact. Then $\exists F \in X', \exists \varepsilon > 0$ s.t. $F(a) + \varepsilon \leq F(b) - \varepsilon \quad \forall a \in A, b \in B$.

6 Hilbert Spaces

Let X be a vector space over \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

6.1 Inner product space

Definition 6.1.1. A map $\langle \cdot, \cdot \rangle: X \times X \rightarrow \mathbb{K}$ is an inner product on X , if

1. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$
2. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space also called a pre-Hilbert-space.

An inner product is a symmetric bilinear form if $\mathbb{K} = \mathbb{R}$ and a sesquilinear form if $\mathbb{K} = \mathbb{C}$.

Theorem 6.1.2 (Cauchy-Schwartz-inequality). *In an inner product space we have*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$$

Theorem 6.1.3. *For an inner product space X we define $\| \cdot \| : X \rightarrow [0, \infty)$ by $\|x\| = \sqrt{\langle x, x \rangle}$. This is a norm.*

Definition 6.1.4. Let X be an inner product space. Then $x, y \in X$ are orthogonal if $\langle x, y \rangle = 0$. This is written as $x \perp y$.

Corollary 6.1.5. *If $x \perp y$, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Theorem 6.1.6. *A normed space is an inner product space, iff $\forall x, y \in X$ the norm satisfies*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

6.2 Hilbert spaces

Definition 6.2.1. Is an inner product space complete w.r.t. to the induced norm, we call it Hilbert space.

Theorem 6.2.2 (projection theorem). *Let X be a Hilbert space, $A \subset X$ non-empty, convex and closed. Then $\forall x \in X$*

$$\exists! y \in A \text{ s.t. } \|x - y\| = \text{dist}(x, A)$$

y is called the best approximation or projection of x in A .

Theorem 6.2.3 (Characterisation of the best approximation). *Let X be an inner product space, $A \subset X$, $A \neq \emptyset$ and convex and $x \in X$. Then y is the best approximation of x in A iff*

$$\Re \langle x - y, z - y \rangle \leq 0 \quad \forall z \in A$$

Definition 6.2.4. Let X be an inner product space, $A \subset X$, then

$$A^\perp = \{x \in X : x \perp y \quad \forall y \in A\}$$

the orthogonal complement of A .

Remark 6.2.5. A^\perp is a closed subspace. If $(x_n)_{n \in \mathbb{N}} \subset A^\perp$, $x_n \rightarrow x$ in X , $\forall n \in \mathbb{N}$ we have $\langle x_n, y \rangle = 0 \quad \forall y \in A$. Moreover $A \subset (A^\perp)^\perp$.

Theorem 6.2.6. *Let X be a Hilbert space, $U \subset X$ closed subspace. Then*

$$\forall x \in X \quad \exists! u \in U \text{ s.t. } \|x - u\| = \text{dist}(x, U) = \inf_{z \in U} \|x - z\|$$

We have $x - u \in U^\perp$ and $X = U \oplus U^\perp$, meaning that $x = u + v$, $u \in U$, $v \in U^\perp$ uniquely. The u is called the orthogonal projection of x in U .

Theorem 6.2.7 (Riesz-Fréchet). *Let $X \neq \{0\}$ be a Hilbert space. $\forall F \in X' \quad \exists! y \in X$ s.t. $F(x) = \langle x, y \rangle$. Moreover, $\|F\|_{X'} = \|y\|_X$. Equivalently*

$$J : X \rightarrow X', \quad (Jy)(x) = \langle x, y \rangle$$

is a bijective, anti-linear isometry. In particular, if X' is a Hilbert space, the dual is also a Hilbert space.

6.3 Orthonormal systems

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.

Definition 6.3.1. Let $I \neq \emptyset$ be an index set. A family of vectors $(e_k)_{k \in I} \subset X$ is called an orthonormal system (ONS) if

$$\langle e_i, e_j \rangle = \delta_{i,j}$$

Theorem 6.3.2 (Schmidt Orthogonalisation theorem). *Let $\{x_i : i \in I\} \subset X, I \subset \mathbb{N}$ be linearly independent vectors. Then \exists ONS $\{e_i : i \in I\}$ s.t.*

$$\text{span}\{x_i : i \in I\} = \text{span}\{e_i : i \in I\}$$

Lemma 6.3.3 (Bessel's inequality). *Let $\{e_1, \dots, e_n\}$ be an ONS. $Y = \text{span}\{e_1, \dots, e_n\}$. Then $\forall x \in X$*

$$\inf_{y \in Y} \|x - y\|^2 = \|x - \sum_{i=1}^n \langle x, e_i \rangle e_i\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, e_i \rangle|^2 \geq 0$$

Definition 6.3.4. If $I \subset \mathbb{N}$, $(e_n)_{n \in I}$ ONS, then $\langle x, e_n \rangle$ is called the n -th Fourier coefficient of x . W.r.t. $(e_n)_{n \in I}$.

Definition 6.3.5. An ONS $(e_n)_{n \in \mathbb{N}}$, $I \subset \mathbb{N}$ is called complete in X if

$$\overline{\text{span}\{e_n : n \in I\}} = X$$

Theorem 6.3.6. *Any separable Hilbert space X has a complete ONS.*

Lemma 6.3.7. *Let X be a Hilbert space, $(e_n)_{n \in \mathbb{N}}$ an ONS. Then $\exists y \in X$ s.t. $y = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$.*

Theorem 6.3.8. *Let X be a Hilbert space of infinite dimension, $(e_n)_{n \in \mathbb{N}}$ an orthonormal system. Then the following are equivalent.*

1. $(e_n)_{n \in \mathbb{N}}$ is complete
2. if $x \in X$ s.t. $\langle x, e_n \rangle = 0 \forall n \in \mathbb{N}$, then $x = 0$
3. $\forall x \in X, x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ (Fourier series of x)
4. $\forall x \in X, \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$.

Corollary 6.3.9. *Any separable infinite-dimensional Hilbert space is isometrically isomorphic to ℓ^2 .*

7 Spectral theorem for self-adjoint compact operators

We only deal with Hilbert spaces.

7.1 Adjoint in Hilbert spaces

Let $(X, \langle \cdot, \cdot \rangle)$, $(Y, \langle \cdot, \cdot \rangle)$, $T \in L(X, Y)$. Let $y \in Y$. Consider the map

$$X \ni x \mapsto \langle Tx, y \rangle_Y$$

This map is linear and bounded.

$$|\langle Tx, y \rangle_Y| \stackrel{CS}{\leq} \|Tx\|_Y \|y\|_Y \leq \|T\| \|x\|_X \|y\|_Y$$

Thus it is an element of X' . By the theorem of Riesz-Fréchet

$$\exists! T^*y \in X \text{ s.t. } \langle x, T^*y \rangle_X = \langle Tx, y \rangle_Y \quad \forall x \in X$$

This defines a map $T^* : Y \rightarrow X$ with $y \mapsto T^*y$.

Definition 7.1.1. T^* is the Hilbert space adjoint of T .

Lemma 7.1.2. $T^* \in L(Y, X)$ and $\|T^*\| = \|T\|$.

Lemma 7.1.3. Let $(X, \langle \cdot, \cdot \rangle_X)$, $(Y, \langle \cdot, \cdot \rangle_Y)$, $(Z, \langle \cdot, \cdot \rangle_Z)$ be Hilbert spaces. Let $T \in L(X, Y)$, $S \in L(Y, Z)$ and $\lambda \in \mathbb{K}$. Then

1. $(S \circ T)^* = T^* S^*$
2. $(\lambda T)^* = \bar{\lambda} T^*$
3. $(T^*)^* = T$

Definition 7.1.4. Let $(X, \langle \cdot, \cdot \rangle_X)$ be a Hilbert space and $T \in L(X)$. T is called self-adjoint if $T^* = T$

Lemma 7.1.5. • If $\mathbb{K} = \mathbb{C}$, T is self-adjoint $\Leftrightarrow \langle Tx, x \rangle_X \in \mathbb{R} \quad \forall x \in X$

- If T is self-adjoint, then $\|T\| = \sup_{x \in X, \|x\|_X=1} |\langle Tx, x \rangle|$

7.2 compact operators

Here X, Y can be only Banach spaces and $X, Y \neq \{0\}$.

Definition 7.2.1. $f : X \rightarrow Y$ is compact if f maps bounded sets in precompact sets.

Lemma 7.2.2. Let $T : X \rightarrow Y$ be linear. Then T is compact iff $T(B_1(0))$ is precompact in Y .

Notation: $K(X, Y) = \{T : X \rightarrow Y \text{ linear and compact}\}$ and $K(X) = K(X, X)$.

Remark 7.2.3. $T \in K(X, Y) \Rightarrow T \in L(X, Y)$.

Lemma 7.2.4. 1. $T \in L(X, Y)$, $S \in L(Y, Z)$. If T or S is compact, then the composition is compact.

2. $K(X, Y)$ is a closed subspace of $L(X, Y)$. In particular $K(X, Y)$ is a Banach space.

Definition 7.2.5. • Let H be a Hilbert space and $T \in L(X)$. Then T is called self-adjoint if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in X$$

• Let X, Y be Banach spaces then, $T \in L(X, Y)$ compact $\Leftrightarrow T(B_1(0))$ is precompact.

Lemma 7.2.6. $T \in L(X, Y)$ is compact iff $\forall (x_n)_{n \in \mathbb{N}} \subset X$ bounded $(T(x_n))_{n \in \mathbb{N}}$ admits a convergent subsequence.

7.3 Spectrum

Let X be a Banach space.

Definition 7.3.1. Let $T \in L(X)$.

• the resolvent set of T is

$$\rho(T) = \{\lambda \in \mathbb{K} : (\lambda \cdot Id - T)^{-1} \in L(X)\} \subset \mathbb{K}$$

while $\sigma(T) = \mathbb{K} \setminus \rho(T)$ is the spectrum of T .

- the resolvent map $R : \rho(T) \rightarrow L(X)$ is defined by $\lambda \mapsto (\lambda Id - T)^{-1}$
- the spectrum of T is divided into

$$\sigma(T) = \sigma_p(T) \cup \sigma_C(T) \cup \sigma_r(T)$$

where

- $\sigma_p(T) = \{\lambda \in \sigma(T) : \ker(\lambda Id - T) \neq \{0\}\}$ is the point spectrum
- $\sigma_C(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \text{Im}(\lambda Id - T) \neq X \text{ but } \overline{\text{Im}(\lambda Id - T)} = X\}$
- $\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_C(T))$.

- the elements of the point spectrum are called eigenvalues and $x \in X \setminus \{0\} : (I\lambda Id - T)(x) = 0$ is called eigenvector associated to λ .

Theorem 7.3.2. For $T \in L(X)$

1. $\rho(T)$ is open.
2. $\sigma(T)$ is compact and

$$\sup_{\lambda \in \sigma(T)} |\lambda| \leq \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = r(T)$$

is the spectral radius. In particular $r(T) \leq \|T\|$

7.4 Spectral theorem for self-adjoint compact operators

Let X be a Hilbert space.

Lemma 7.4.1. Let $T \in K(X)$ self-adjoint. Then $\|T\|$ or $-\|T\|$ is an eigenvalue of T .

Lemma 7.4.2. Let $T \in L(X)$ be self-adjoint. Then all eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal.

Lemma 7.4.3. Let $T \in L(X)$. If $M \subset X$ is a closed subspace s.t. $TM \subset M$, then M^\perp is invariant under T^* .

Theorem 7.4.4. Let X be a Hilbert space, $T \in K(X)$ self-adjoint. Then $\exists(\phi_n)_{n \in I} \subset X$, $I \subset \mathbb{N}$, ONS and $\exists(\lambda_n)_{n \in I} \subset \mathbb{R}$ s.t. $\forall x \in X$

$$Tx = \sum_{n \in I} \lambda_n \langle x, \phi_n \rangle \phi_n$$

In particular $T\phi_n = \lambda_n \phi_n \forall n \in \mathbb{N}$. If I is finite, then $\lambda_n \rightarrow 0$.