Fun Summary

November 16, 2023

Contents

1 metric spaces

1.1 metric spaces

Definition 1.1.1. A metric space is a non-empty set X together with a map

$$d : X \times X \to \mathbb{R}$$

$$(x, y) \mapsto d(x, y)$$

such that

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$

Remark 1.1.2. (d admits only positive values)

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

Example 1.1.3. *1.* $d_2(x, y) = ||x - y||_2$

2. $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$

Definition 1.1.4. (convergence)
A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be convergent to $x \in X$ if

$$x_n \to x \text{ in } (X, d)$$

or

$$\lim_{n\to\infty} x_n = x \text{ in } (x, d)$$

1.2 Topology in metric spaces

Let (X, d) be a metric space.

Definition 1.2.1. 1. an open ball is defined by

$$B_r(x) = \{y \in X :\ d(x, y) < r\}$$

2. $O \subset X$ is called open if $\forall y \in O$ there is $r > 0$ such that $B_r(y) \subset O$

3. $A \subset X$ is closed if $X \setminus A$ is open.

Theorem 1.2.2. *(metric spaces are topological spaces)*
Let $\mathcal{T}$ be the set of open subsets of X. Then

1. $\varnothing, X \in \mathcal{T}$

2. if $U, V \in \mathcal{T}$, *then* $U \cup V \in \mathcal{T}$

3. if $\{U_i\}_{i \in I} \subset \mathcal{T}$, *then* $\bigcup_{i \in I} \in \mathcal{T}$

Remark 1.2.3. 1. $\varnothing, X$ are closed

2. finite union of closed sets is closed

3. arbitrary intersections of closed sets is closed

Lemma 1.2.4. *$A \subset X$ is closed iff $\forall$ convergent sequences $(x_n)_{n \in \mathbb{N}} \subset A$ the limit point is in A.*

Definition 1.2.5. For $M \subset X$ we define

$$\overline{M} = \bigcap_{A \supset M,\ A \text{ closed}}$$

as the closure of M and

$$M = \bigcup_{O \subset M,\ O \text{ open}}$$

as the interior of M.
$\partial M = \overline{M} \setminus M$ is the boundary of M

Attention:
Define the closed ball as $\overline{B}_r(a) = \{y \in X : \ d(y,a) \leq r\}$. Then in general $\overline{B_r(a)} \neq \overline{B}_r(a)$.
Example: Take $X \neq \varnothing$ and the trivial metric d. Then

$$B_1(a) = \{a\} = \overline{B_1(a)}$$

but $\overline{B}_1(a) = X$.

1.3 separability and completion

Let (X, d) be a metric space.

Definition 1.3.1. 1. $M \subset X$ is called dense in X if $\overline{M} = X$.

2. X is called separable if X has a countable dense subset.

Remark 1.3.2. M is dens in X iff

$$\forall x \in X \; \forall \varepsilon > 0 \; \exists y \in M \text{ s.t. } d(x, y) < \varepsilon$$

Definition 1.3.3. 1. $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ s.t. } m, n > N \text{ implies } d(x_n, x_m) < \varepsilon$$

2. A metric space in which all Cauchy sequences converge is called complete.

Example 1.3.4. *1.* $(C^0([a, b], \mathbb{R}), d_\infty)$ *with* $d_\infty(f, g) = \max\limits_{x\in[a,b]} |f(x) - g(x)|$ *is complete.*

2. $(\mathbb{R}^n, d_2)$ *with* $d_2(x, y) = ||x - y||_2$ *is complete.*

Lemma 1.3.5. *Let* (X, d) *be a complete metric space and* $\varnothing \neq A \subset X$*. Then* (A, d) *is complete iff* A *is closed.*

Definition 1.3.6. $A \subset X$ is called bounded if its diameter

$$diam(A) = \sup\{d(x, y) : \; x, y \in A\}$$

is finite.

Theorem 1.3.7. (X, d) *is complete iff* $\forall (F_n)_{n\in\mathbb{N}}$ *sequences of closed subsets such that* $F_{n+1} \subset F_n$ *and* $diam(F_n) \to 0$ *then*

$$\exists! x_0 \in X \; s.t. \bigcap_{n\in\mathbb{N} F_n = \{x_0\}}$$

1.4 Continuity

Definition 1.4.1. Let $(X, d_x), (Y, d_y)$ be metric spaces and $f : X \to Y$. f is continuous in x_0 if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall x \in X \; d_x(x, x_0) < \delta \text{ implies } d_y(f(x), f(x_0)) < \varepsilon$$

With sequences:

$$\forall (x_n)_{n\in\mathbb{N}} \subset X \; x_n \to x_0 \text{ in } (X, d_x) \text{ if it holds } (f(x_n))_{n\in\mathbb{N}} \subset Y,\; f(x_n) \to f(x_0) \text{ in } (Y, d_y)$$

f is continuous if f is continuous in x_0 for all $x_0 \in X$.
In other words f is continuous if for all $O \subset Y$ open (closed) $f^{-1}(O)$ is open (closed) in X.
Special case: f is Lipschitz continuous if $\exists L > 0$ s.t.

$$d_y(f(x), f(y)) \leq L d_x(x, y) \; \forall x, y \in X$$

f is an isometric if $\forall x, y \in X$ it holds that $d_Y(f(y), f(x)) = d_x(x, y)$.

1.5 Compact sets

Definition 1.5.1. Let (X, d) be a metric space and $A \subset X$.

1. an open cover of A is a collection $\{U_i\}_{i\in I}$ where $I \neq \varnothing$ is an arbitrary index set of open subsets of X s.t. $A \subset \bigcup\limits_{i\in I} U_i$.
2. A is compact if every open cover of A contains a finite subcover i.e. there is $N \in \mathbb{N}$ and indices $i_1, ..., i_N$ such that

$$A \subset U_1 \cup ... \cup U_N$$

3. A is sequentially compact if every sequence in A has a convergence subsequence in A.
4. A is called precompact or totally bounded if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ and $\exists x_1, ..., x_N \in X$ such that $A \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i)$.

Theorem 1.5.2. *Let (X, d) be a metric scape and $A \subset X$. The following are equivalent:*

1. *A is compact*
2. *A is sequentially compact*
3. *(A, d) is complete and A is precompact.*

Remark 1.5.3. If A is precompact, then $\overline{A}$ is precompact. Further, if (X, d) is complete and $A \subset X$ then A is precompact $\Leftrightarrow$ $\overline{A}$ is compact.

Recall: A compact $\Rightarrow$ bounded and closed and $f : X \to Y$ continuous with $A \subset X$ compact, then $f(A)$ is compact as well. Further, if $f : A \to \mathbb{R}$ is continuous and A is compact, then

$$\exists x_1, x_2 \in A \text{ s.t. } f(x_1) \le f(x) \le f(x_2)\ \forall x \in A$$

Theorem of Heine-Borel: $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

1.6 Theorem of Baire

Theorem 1.6.1. *Let (X, d) be a complete metric space and $\forall n \in \mathbb{N}$ consider $U_n \subset X$ open and dense. Then*

$$\bigcap_{n\in\mathbb{N}} U_n$$

is dense in X.

Remark 1.6.2.

1. Completeness is in general necessary. Consider $(\mathbb{Q}, d)$ and $d(x, y) = |x - y|$. Define a sequence x_n such that $\mathbb{Q} = \{x_n\ n \in \mathbb{N}\}$. Take $U_n = \mathbb{Q} \setminus \{x_n\}$ which is open and dense. Then

$$\bigcap_{n\in\mathbb{N}} U_n = \varnothing$$

Corollary 1.6.3. *Let (X, d) be a complete metric space. Let $\forall n \in \mathbb{N}$, $A_n \subset X$ be closed and*

$$X = \bigcup_{n\in\mathbb{N}} A_n$$

Then $\exists N \in \mathbb{N}$ s.t. A_N has an interior point.

Remark 1.6.4. Theorem 1.6.1 is also called Baire category theory.

- In a metric space (X, d) $A \subset X$ is called nowhere dense if $\overline{A}$ has no interior points.
- A is called of first category if $\exists (M_n)_{n\in\mathbb{N}}$ where $M_n \subset A$ nowhere dense s.t. $A = \bigcup_{n\in\mathbb{N}} M_n$
- A is called of second category if it is not of first category

Hence the theorem of Baire implies that every complete metric space is of second category.

2 Normal spaces and Banach spaces

Let X be a $\mathbb{K}$-vector space where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2.1 definitions

Definition 2.1.1. A map $||\cdot|| : X \to \mathbb{R}$ is called a norm on X if

1. $\forall x \in X$, $||x|| \geq 0$ and $||x|| = 0$ iff $x = 0$
2. $\forall \lambda \in \mathbb{K}$ and $\forall x \in X$ it holds that $||\lambda x|| = |\lambda| \cdot ||x||$
3. $\forall x, y \in X$ it holds $||x + y|| \leq ||x|| + ||y||$

The pair $(X, ||\cdot||)$ is called an normed space.

$p : X \to \mathbb{R}$ is called a seminorm if $p(x) \geq 0$ $\forall x \in X$ and 2. and 3. are also satisfied.

Example 2.1.2. *1.* $C^0([0,1];\mathbb{R})$ *with* $||f||_\infty = \max_{x\in[0,1]} |f(x)|$

2. more general for a compact metric space K*:* $C^0(K, \mathbb{R})$ *with* $||f||_\infty = \max_{x\in K} |f(x)|$

3. $C^1([0,1];\mathbb{R})$ *with* $p(f) = \max_{x\in[0,1]} |f'(x)|$

4. $\Omega \subset \mathbb{R}^n$ *measurable.* $L^1(\Omega) = \{f : \Omega \to \mathbb{R} :\ f$ *integrable* $\}$ *with*

$$p : L^(\Omega) \to \mathbb{R} :\ p(f) = \int_\Omega |f(x)|\, dx$$

then p *is a seminorm.*

Remark 2.1.3. Any normed space is a metric space via

$$d(x, y) = ||x - y||$$

All concepts from chapter 1 apply.

Lemma 2.1.4. *Let* $(X, ||\cdot||)$ *be a normed space. Then* X *is called separable iff* $\exists A \subset X$ *countable such that s.t.* $\overline{span\{A\}} = X$ *where* $span\{A\} = \{\sum_{i=1}^n \lambda_i x_i\}$ *with* $n \in \mathbb{N}$, $\lambda_i \in K$ *and* $x_i \in A$. *Here the colusre is defined w.r.t the norm.*

Definition 2.1.5. A complete normed space is called a Banach space.

2.2 Example: l^p-spaces

We consider the vector space $\mathbb{K}^\mathbb{N}$ of sequences in in $\mathbb{K}$. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$.
Define $x + y = (x_n + y_n)_{n\in\mathbb{N}}$ and $\lambda x = (\lambda x_n)_{n\in\mathbb{N}}$.
For $x \in \mathbb{K}^\mathbb{N}$ define

$$||x||_{l^p} = \left(\sum_{n=1}^\infty |x|^p\right)^{1/p}$$

for $1 \leq p < \infty$ and

$$||x||_{l^\infty} = \sup_{n\in\mathbb{N}} |x_n|$$

else.

Define $l^p = \{x = (x_n)_{n\in\mathbb{N}} : ||x||_{l^p} < \infty\}$ for $1 \leq p \leq \infty$. We find that l^p is a subspace of $\mathbb{K}^{\mathbb{N}}$ and l^p is a normed space (for the triangle inequality use the Hölder inequality).

Theorem 2.2.1. *For $1 \leq p \leq \infty$ l^p is a Banach space.*

Lemma 2.2.2. *For finite p, l^p is separable while l^∞ is not.*

2.3 Finite dimensional normed spaces

Let X be a vector space over $\mathbb{K}$. $\exists e_1, ..., e_n \in X$ s.t.

$$\forall x \in X;\ \exists \lambda_1, ..., \lambda_n \in \mathbb{K}:\ x = \sum_{i=1}^{n} \lambda_i x_i$$

For $p \in [1, \infty)$ we define

$$||x||_p = \left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{1/p}$$

and for $p = \infty$

$$||x||_\infty = \max_{1\leq i\leq n} |\lambda_i|$$

Definition 2.3.1. Two norms are equivalent in that

$$\alpha||\cdot||_1 \leq ||\cdot||_2 \leq \beta||\cdot||_1$$

Theorem 2.3.2. *In a finite dimensional space, all norms are equivalent.*

Theorem 2.3.3. *Finite dimensional normed spaces are Banach spaces.*

2.4 On the closure of $\overline{B_1(0)}$

Lemma 2.4.1 (Lemma of Riesz, Lemma of the almost orthogonal element)**.** *Let X be a normed space. $U \subset X$ a closed subspace of X s.t. $U \neq X$. Then $\forall \lambda \in (0, 1) \exists x_\lambda \in X$ s.t. $||x_\lambda|| = 1$ and $dist(x_\lambda, U) \geq \lambda$.*

Theorem 2.4.2. *In a normed space X, $\overline{B_1(0)}$ is compact iff X is finite dimensional.*

3 A question from approximation theory

3.1 Theorem of Stone-Weierstrass

Let X be a compact metric space. Then $(C^0(X), \mathbb{K}), ||\cdot||_\infty$, where $||f||_\infty = \max\limits_{x\in X} |f(x)|$ is a Banach space.
Which property of $A \subset C^0(X, \mathbb{K})$ ensures that A is dense.

Definition 3.1.1. $A \subset C^0(X, \mathbb{K})$ is called subalgebra, if $\forall f, g, \in A$

1. $\lambda f + \mu g \in A$ (subspace)

2. $f \cdot g \in A$

Example 3.1.2.

- $\{p : [0, 1] \to \mathbb{R}\}$ *is a subalgebra of* $C^0([0, 1]; \mathbb{R})$.

- $\{f : [-1, 1] \to \mathbb{R}; f$ *continuous and even*$\}$ *is a subalgebra.*

Remark 3.1.3. If A is a subalgebra, then $\overline{A}$ is also a subalgebra.

Definition 3.1.4. Let $A \subset C^0(X)$ be a subalgebra.

1. A is called unital if $1 \in A$

2. A separates point if $x, y \in X$, $x \neq y$, $\exists f \in A$ s.t. $f(x) \neq f(y)$.

3. (if $\mathbb{K} = \mathbb{C}$) A is stable under conjuguation if from $f \in A$ we conclude that also $\overline{f} \in A$.

Remark 3.1.5. If A is unital then all constant functions are in A.

Lemma 3.1.6. *Consider* $f : [-1, 1] \to \mathbb{R}$ *where* $f(x) = |x|$. *Then* $\exists$ *sequence of polynomials* $(p_n)_{n\in\mathbb{N}}$ *s.t.*

$$p_n \to f$$

uniformly in $[-1, 1]$.

Lemma 3.1.7. *Let* $A \subset C^0(X, \mathbb{R})$ *be a unital subalgebra. Then*

1. *if* $f \in A$ *then* $|f| \in \overline{A}$.

2. *if* $f, g \in A$ *then* $\max\{f, g\} \in \overline{A}$ *and* $\min\{f, g\} \in \overline{A}$

Theorem 3.1.8 (Stone-Weierstrass)**.** *Let* A *be a compact metric space.* $A \subset C^0(X, \mathbb{K})$ *is a unital subalgebra that separates points and if* $\mathbb{K} = \mathbb{C}$ *is stable under conjugation, then* A *is dense in* $C^0(X, \mathbb{K})$ *w.r.t* $||\cdot||_\infty$.

Theorem 3.2.1 (Theorem of Weierstraß). *Let $[a, b]$ be a compact interval in $\mathbb{R}$, $f : [a, b] \to \mathbb{R}$ be a continuous function and $\varepsilon > 0$. Then $\exists p : [a, b] \to \mathbb{R}$ a polynomial s.t.*

$$||p - f||_\infty = \sup_{x \in [a,b]} |p(x) - f(x)| < \varepsilon$$

Definition 3.2.2. A function $f : \mathbb{R} \to \mathbb{C}$ is periodic if

$$f(x + t) = f(x)$$

for a $t \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Remark 3.2.3. If f is periodic with period t then $\tilde{f} : \mathbb{R} \to \mathbb{C}$ where $\tilde{f}(x) = f(t\frac{x}{2\pi})$ is periodic of period 2π.
Consider $C^0_{2\pi}(\mathbb{R}, \mathbb{C})$ the space of continuous 2π-periodic functions. We consider the span of $\{e^{ikx} = \cos(kx) + i \sin(kx), k \in \mathbb{Z}\}$.

Definition 3.2.4. A trigonometric polynomial is a function $f : \mathbb{R} \to \mathbb{C}$

$$f(x) = \sum_{k=-N}^{N} c_k \cdot e^{ikx}$$

with $c_k \in \mathbb{C}$

Theorem 3.2.5 (Approximation of periodic functions). *Trigonometric polynomials are dense in $(C^0_{2\pi}(\mathbb{R}, \mathbb{C}), || \cdot ||_\infty)$*

Application to neural networks

The simplest case of a neural network has d inputs $x_1, ..., x_d$ and one output Z called a *feed forward* network. Each input influences the output and x_i might have a weight α_i associated to it. The output is a function in $x = (x_1, ..., x_d)$ and the weights $\alpha = (\alpha_1, ..., \alpha_d)$. For instance, the output is often of the form

$$Z = \sum_{i=1}^{d} \alpha_i x_i + b$$

where b is the bias of the network. To make the network slightly stronger, we add a intermediate layer $y = (y_1, ..., y_r)$ where each x_i is connected to each y_j with the associated weight $\gamma_{i,j}$. The y layer (often called activation) is the connected to the output Z as above with weights α_j. We introduce the realtion

$$y_j = \Phi(\sum_{i=1}^{d} \gamma_{j,i} x_i + b)$$

for a measurable function Φ. Lastly, the output is then given by

$$Z = \sum_{j=1}^{r} \alpha_j y_j$$

Definition 3.2.6. 1. $A^d = \{a : \mathbb{R}^d \to \mathbb{R} : \ a(x9 = w^T x + b)\}$ where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

2. given $\Phi : \mathbb{R} \to \mathbb{R}$ measurable $d \in \mathbb{N}$ define $\Sigma^d(\Phi) = \{f : \mathbb{R}^d \to \mathbb{R} : \ f(x) = \sum_{j=1}^{N} \alpha_j \Phi(a_j(x))$ wtih $N \in \mathbb{N}, \alpha_j \in \mathbb{R}, a_j \in A^d\}$ as the set of single hidden layer feed forward networks.

3. A squashing function is a measurable non-decreasing function $\Phi : \mathbb{R} \to \mathbb{R}$ s.t. $\lim_{x \to -\infty} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = 1$.

Theorem 3.2.7 (Universal Approximation theorem of Hornik-Stinchcombe-White)**.** *Let Φ we a squashing function $K \subset \mathbb{R}^d$ compact $f : K \to \mathbb{R}$ continuous and $\varepsilon > 0$. Then $\exists g \in \Sigma^d(\Phi)$ s.t.*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon$$

4 Continuous linear maps

$(X, ||\cdot||_X), (Y, ||\cdot|| : Y)$ are K-Vector spaces with $K = \mathbb{R}$ or $K = \mathbb{C}$. $T : X \to Y$ is called linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

4.1 Continuity of linear maps

Definition 4.1.1. LEt $T : X \to Y$ be linear. Then T is bounded if $\exists C > 0$ s.t.

$$||Tx||_Y \leq C ||x||_X \ \forall x \in X$$

or equivalently

$$\sup_{x \in X \setminus \{0\}} \frac{||Tx||_Y}{||x||_X} \leq C$$

which is also equivalent to

$$\sup_{x \in X, ||x||_X = 1} ||Tx||_Y \leq C$$

Theorem 4.1.2. *For $T : X \to Y$ linear, the following are equivalent:*

1. *T is continuous*
2. *T is continuous in 0*
3. *t is bounded*

Lemma 4.1.3. *Let X have infinite dimension. Then $\exists T : X \to \mathbb{K}$ linear and not bounded.*

Definition 4.1.4. Define $L(X,Y)$ as the set of linear continuous ($\Leftrightarrow$ bounded) maps from X to Y. With the usual addition ($(T_1 + T_2)(x) = T_1(X) + T_2(x)$) and the scalar multiplication ($(\lambda(T)(x)) = \lambda T(x)$) this is a vector space.
If $X = Y$ we write $L(X)$. For $T \in L(X,Y)$

$$\ker T = \{x \in X : Tx = 0\}$$

and

$$\Im(T) = \{y \in Y : \exists x \in X : \; Tx = y\}$$

4.2 Operatornorm and dual space

Theorem 4.2.1. *Let $X \neq \{0\}$.*

- *$L(X,Y)$ with the operatornorm $||T|| = \sup_{x \in X \setminus \{0\}} \frac{||Tx||_Y}{||x||_X} = \sup_{x \in X, ||x||_X = 1} ||Tx||_Y$ is a normed space. We have*
$$||Tx||_Y \leq ||T|| ||x||_X$$
- *If Y is a Banach space then $L(X,Y)$ is a Banach space.*

Definition 4.2.2. For a normed space $(X, ||\cdot||_\infty)$ we define the dual space $X' = L(X, \mathbb{K})$.

Remark 4.2.3. X' is a Banach space.

4.3 Neumann series

Lemma 4.3.1. *Let X, Y, Z be three normed spaces. Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. Then $S \circ T \in L(X,Z)$ and*

$$||S \circ T|| \leq ||S|| ||T||$$

Let $T : X \to Y$ be linear, bounded and bijective. Then $\exists T^-1 : Y \to X$ linear.

Definition 4.3.2. Let X, Y be normed spaces.

1. $T \in L(X,Y)$ is bijective such that $T^-1 \in L(Y,X)$ then T is called an isomorphism
2. X, Y are called isomorph if there is $T : X \to Y$ isomorphism.
3. $T \in L(X,Y)$ is called an Isometry if $||Tx|| = ||x||$.
4. X, Y are called isometric isomorph if $\exists T \in L(X,Y)$ an isomorphism that is also an isometry.

Remark 4.3.3. The identity $I_x : X \to X$ with $x \mapsto x$ is in $L(X)$. Then $T \in L(X)$ is an isomorphism iff $\exists S \in L(X)$ s.t. $S \circ T = I_x$ and $T \circ S = I_x$

Let $T \in L(X)$ s.t $||T|| < 1$. Define $T^0 = I_x$, $T^n = T \circ T^{n-1}$. Obviously $T^n \in L(X)$ for all n. Now,

$$\left(\sum_{k=0}^{n} T^k\right)_{n\in\mathbb{N}} \subset L(X)$$

is a Cauchy sequence w.r.t. the operatornorm. Hence, if X is a Banach-Space, so is $L(X)$ and thus the series converges to a $S \in L(X)$. Furthermore

$$\sum_{k=0}^{\infty} ||T||^k = \frac{1}{1 - ||T||}$$

Finally, we can also note that $S = (I_x - T)^{-1}$.

Theorem 4.3.4 (Neumann series). *Let X be a Banach-Space, $T \in L(X)$ with $||T|| < 1$ The $I_x - T$ is an isomorphism and*

$$(I_x - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

is in $L(X)$. This is called the Neumann series.

4.4 The dual space of l^p

We only deal with $1 \le p < \infty$.

Theorem 4.4.1. *Let $q \in (1, \infty]$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then the dualspace $(l^p)'$ is isometric isomorph to l^q.*

5 Theorem of Hahn-Banach

Let X be a vector space, $X \neq \{0\}$ over $\mathbb{K} = \mathbb{R}$.

5.1 Extension Theorem

Given $U \subset X$ subspace, $l : U \to \mathbb{R}$ linear, is there $L : X \to \mathbb{R}$ linear such that $L|_U = l$? For this we need Zorn's Lemma:

Definition 5.1.1. Let $M \neq \varnothing$ be a set and $\leq$ be a partial order on M, i.e. $\leq$ satisfies

1. reflexiv: $x \leq x\ \forall x \in M$
2. antisymmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$
3. transitivity $x \leq y, y \leq z \Rightarrow x \leq z$

Then

- $A \subset M$ is called chain of totally ordered if $\forall x, y \in A$ either $x \leq y$ or $y \leq x$
- $b \in M$ is an upper bound for a chain A if $a \leq b$ for all $a \in A$
- $m \in M$ is called maximal element if

$$\forall x \in M \text{ s.t. } m \leq x \Rightarrow x = m$$

Lemma 5.1.2 (Zorn)**.** *Let $M \neq \varnothing$ and $\leq$ be a partial order on M. If every chain in M has an upper bound in M, then there is a maximal element.*

Definition 5.1.3. LEt X be a vector space. $p : X \to \mathbb{R}$ is called sublinear if

1. $p(\lambda x) = \lambda p(x)$ for all $x \in X, \lambda \geq 0$
2. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Theorem 5.1.4 (Extension theorem of Hahn-Banach)**.** *Let X be a vecorspace over $\mathbb{R}$, $U \subset X$ a subspace and $U \neq X$. Let $p : X \to \mathbb{R}$ be a subspace $l : U \to \mathbb{R}$ be linear s.t. $l(x) \leq p(x)\ \forall x \in U$. Then $\exists L : X \to \mathbb{R}$ linear s.t. $L(x) \leq p(x)\ \forall x \in X$ and $L(x) = l(x)\ \forall x \in U$. L is called extension of l.*

Consequences for normed spaces

Theorem 5.1.5. *Let $(X, ||\cdot||_X)$, $U \subset X$ a subspace fo X, with $U \neq X$. Let $u' \in U' = L(U, \mathbb{R})$. Then $\exists x' \in X'$ s.t. $||x'||_{X'} = ||u'||_{U'}$ such that $x'(u) = u'(u)\ \forall u \in U$.*

Corollary 5.1.6. *Let $(X, ||\cdot||_X)$, $U \subset X$ be a subspace of X and $x_0 \in X$ s.t. $dist(x_0, U) > 0$. Then $\exists x' \in X'$ s.t. $x'|_U = 0\ \forall u \in U$ and $x'(x_0) = dist(x_0, U)$ with $||x'||_{X'} = 1$.*

Corollary 5.1.7. *Let $X, ||\cdot||_X)$ and $x_0 \in X$.*

1. *if $x_0 \neq 0$ then $\exists F \in X'$ with $||F||_{X'} = 1$ and $F(x_0) = ||x_0||_X$ In particular, for $x \in X$*

$$||x||_X = \sup_{F \in X', ||F||_{X'}=1} |F(x)|$$

2. *If $F(x_0) = 0$ for all $F \in X'$, then $x_0 = 0$. In particular, X' separates points of X.*

3. *$U \subset X$ subspace. Then U is dense in X iff if for $x' \in X'$ s.t. $x'_{|U} = 0$ it follows $x' = 0$.*

5.2 Separation Theorems

Definition 5.2.1. Let X be a vectorspace over $\mathbb{R}$. $A \subset X$ is called convex, if

$$\forall x, y \in A,\ \lambda x + (1-\lambda)y \in A,\ \forall \lambda \in [0,1]$$

Lemma 5.2.2. *Let $C \subset X$ open and convex with $O \in C$. Define $p_C : X \to \mathbb{R}$ such that $p_C(x) = \inf\{\lambda > 0$ s.t. $\frac{x}{\lambda} \in C\}$. This is called the Minkowski functional.*
Then p_C is sublinear and $C = \{x \in X : p_C(x) < 1\}$.

Lemma 5.2.3. *Let $(X, ||\cdot||)$ be a normed space and $A \subset X$ be convex and open, $A \neq \varnothing$ and $x_0 \in X \setminus A$, then $\exists F \in X'$ s.t.*

$$F(x) < F(x_0)\ \forall x \in A$$

Definition 5.2.4. Let $X \neq \{0\}$ be a $\mathbb{R}$-vectorspace.

1. $H = \{x \in X :\ f(x) = \alpha\}$ with $f : X \to \mathbb{R}$ linear, $\alpha \in \mathbb{R}$

2. $A, B \subset X$ are separated by an affine hyperplane H if $H = \{f = \alpha\}$ and $f(a) \leq \alpha \leq f(b)\ \forall a \in A\ \forall b \in B$.

3. $A, B \subset X$ are strictly separated by an affine Hyperplane H if $\exists \varepsilon > 0$ s.t. $f(a) + \varepsilon \leq \alpha \leq f(b) - \varepsilon$.

Theorem 5.2.5 (Separation Theorem of Hahn-Banach)**.** *Let $(X, ||\cdot||)$, A, B convex, $A \neq \varnothing$, $b \neq \varnothing$, $A \cap B = \varnothing$ and A or B should be open.. Then $\exists F \in X'$ and $\delta \in \mathbb{R}$ s.t.*

$$F(a) \leq \delta \leq F(b)\ \forall a \in A, b \in B$$

Theorem 5.2.6. *Let $(x, ||\cdot||)$, A, B convex subsets $A \neq \varnothing, B \neq \varnothing$, $A \cap B = \varnothing$. Let A be closed and B be compact. Then $\exists F \in X', \exists \varepsilon > 0$ s.t. $F(a) + \varepsilon \leq F(b) - \varepsilon\ \forall a \in A, b \in B$.*