Fun Summary

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Contents

1 metric spaces

1.1 metric spaces

Definition 1.1.1. A metric space is a non-empty set X together with a map

$$d : X \times X \to \mathbb{R}$$

$$(x, y) \mapsto d(x, y)$$

such that

1. $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z \in X$

Remark 1.1.2. (d admits only positive values)

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

Example 1.1.3. *1.* $d_2(x, y) = ||x - y||_2$

2. $d(x, y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{else} \end{cases}$

Definition 1.1.4. (convergence)
A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be convergent to $x \in X$ if

$$x_n \to x \text{ in } (X, d)$$

or

$$\lim_{n\to\infty} x_n = x \text{ in } (x, d)$$

1.2 Topology in metric spaces

Let (X, d) be a metric space.

Definition 1.2.1. 1. an open ball is defined by

$$B_r(x) = \{y \in X : \ d(x, y) < r\}$$

2. $O \subset X$ is called open if $\forall y \in O$ there is $r > 0$ such that $B_r(y) \subset O$

3. $A \subset X$ is closed if $X \setminus A$ is open.

Theorem 1.2.2. *(metric spaces are topological spaces)*
Let $\mathcal{T}$ be the set of open subsets of X. Then

1. $\varnothing, X \in \mathcal{T}$

2. if $U, V \in \mathcal{T}$, then $U \cup V \in \mathcal{T}$

3. if $\{U_i\}_{i \in I} \subset \mathcal{T}$, then $\bigcup_{i \in I} \in \mathcal{T}$

Remark 1.2.3. 1. $\varnothing, X$ are closed

2. finite union of closed sets is closed

3. arbitrary intersections of closed sets is closed

Lemma 1.2.4. *$A \subset X$ is closed iff $\forall$ convergent sequences $(x_n)_{n \in \mathbb{N}} \subset A$ the limit point is in A.*

Definition 1.2.5. For $M \subset X$ we define

$$\overline{M} = \bigcap_{A \supset M,\ A \text{ closed}}$$

as the closure of M and

$$M = \bigcup_{O \subset M,\ O \text{ open}}$$

as the interior of M.
$\partial M = \overline{M} \setminus M$ is the boundary of M

<u>Attention:</u>
Define the closed ball as $\overline{B}_r(a) = \{y \in X : \ d(y, a) \leq r\}$. Then in general $\overline{B_r(a)} \neq \overline{B}_r(a)$.
Example: Take $X \neq \varnothing$ and the trivial metric d. Then

$$B_1(a) = \{a\} = \overline{B_1(a)}$$

but $\overline{B}_1(a) = X$.

1.3 separability and completion

Let (X,d) be a metric space.

Definition 1.3.1. 1. $M \subset X$ is called dense in X if $\overline{M} = X$.

2. X is called separable if X has a countable dense subset.

Remark 1.3.2. M is dens in X iff

$$\forall x \in X \ \forall \varepsilon > 0 \ \exists y \in M \text{ s.t. } d(x,y) < \varepsilon$$

Definition 1.3.3. 1. $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } m,n > N \text{ implies } d(x_n, x_m) < \varepsilon$$

2. A metric space in which all Cauchy sequences converge is called complete.

Example 1.3.4. *1.* $(C^0([a,b],\mathbb{R}), d_\infty)$ *with* $d_\infty(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$ *is complete.*

2. $(\mathbb{R}^n, d_2)$ *with* $d_2(x,y) = ||x-y||_2$ *is complete.*

Lemma 1.3.5. *Let* (X,d) *be a complete metric space and* $\varnothing \neq A \subset X$*. Then* (A,d) *is complete iff* A *is closed.*

Definition 1.3.6. $A \subset X$ is called bounded if its diameter

$$diam(A) = \sup\{d(x,y) : \ x,y \in A\}$$

is finite.

Theorem 1.3.7. (X,d) *is complete iff* $\forall (F_n)_{n\in\mathbb{N}}$ *sequences of closed subsets such that* $F_{n+1} \subset F_n$ *and* $diam(F_n) \to 0$ *then*

$$\exists ! x_0 \in X \ s.t. \bigcap_{n\in\mathbb{N} F_n = \{x_0\}}$$

1.4 Continuity

Definition 1.4.1. Let $(X,d_x), (Y,d_y)$ be metric spaces and $f : X \to Y$. f is continuous in x_0 if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x \in X \ d_x(x,x_0) < \delta \text{ implies } d_y(f(x), f(x_0)) < \varepsilon$$

With sequences:

$$\forall (x_n)_{n\in\mathbb{N}} \subset X\ x_n \to x_0 \text{ in } (X, d_x) \text{ if it holds } (f(x_n))_{n\in\mathbb{N}} \subset Y,\ f(x_n) \to f(x_0) \text{ in } (Y, d_y)$$

f is continuous if f is continuous in x_0 for all $x_0 \in X$.
In other words f is continuous if for all $O \subset Y$ open (closed) $f^{-1}(O)$ is open (closed) in X.
Special case: f is Lipschitz continuous if $\exists L > 0$ s.t.

$$d_y(f(x), f(y)) \leq L d_x(x, y)\ \forall x, y \in X$$

f is an isometric if $\forall x, y \in X$ it holds that $d_Y(f(y), f(x)) = d_x(x, y)$.

1.5 Compact sets

Definition 1.5.1. Let (X, d) be a metric space and $A \subset X$.

1. an open cover of A is a collection $\{U_i\}_{i\in I}$ where $I \neq \varnothing$ is an arbitrary index set of open subsets of X s.t. $A \subset \bigcup\limits_{i\in I} U_i$.
2. A is compact if every open cover of A contains a finite subcover i.e. there is $N \in \mathbb{N}$ and indices $i_1, ..., i_N$ such that
$$A \subset U_1 \cup ... \cup U_N$$
3. A is sequentially compact if every sequence in A has a convergence subsequence in A.
4. A is called precompact or totally bounded if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ and $\exists x_1, ..., x_N \in X$ such that $A \subset \bigcup_{i=1}^N B_\varepsilon(x_i)$.

Theorem 1.5.2. *Let (X, d) be a metric scape and $A \subset X$. The following are equivalent:*

1. *A is compact*
2. *A is sequentially compact*
3. *(A, d) is complete and A is precompact.*

Remark 1.5.3. If A is precompact, then $\overline{A}$ is precompact. Further, if (X, d) is complete and $A \subset X$ then A is precompact $\Leftrightarrow$ $\overline{A}$ is compact.

Recall: A compact $\Rightarrow$ bounded and closed and $f : X \to Y$ continuous with $A \subset X$ compact, then $f(A)$ is compact as well. Further, if $f : A \to \mathbb{R}$ is continuous and A is compact, then

$$\exists x_1, x_2 \in A \text{ s.t. } f(x_1) \leq f(x) \leq f(x_2) \; \forall x \in A$$

Theorem of Heine-Borel: $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

1.6 Theorem of Baire

Theorem 1.6.1. *Let (X, d) be a complete metric space and $\forall n \in \mathbb{N}$ consider $U_n \subset X$ open and dense. Then*

$$\bigcap_{n \in \mathbb{N}} U_n$$

is dense in X.

Remark 1.6.2.
1. Completeness is in general necessary. Consider $(\mathbb{Q}, d)$ and $d(x,y) = |x-y|$. Define a sequence x_n such that $\mathbb{Q} = \{x_n \; n \in \mathbb{N}\}$. Take $U_n = \mathbb{Q} \setminus \{x_n\}$ which is open and dense. Then

$$\bigcap_{n \in \mathbb{N}} U_n = \varnothing$$

Corollary 1.6.3. *Let (X, d) be a complete metric space. Let $\forall n \in \mathbb{N}$, $A_n \subset X$ be closed and*

$$X = \bigcup_{n \in \mathbb{N}} A_n$$

Then $\exists N \in \mathbb{N}$ s.t. A_N has an interior point.

Remark 1.6.4. Theorem 1.6.1 is also called Baire category theory.

- In a metric space (X, d) $A \subset X$ is called nowhere dense if $\overline{A}$ has no interior points.
- A is called of first category if $\exists (M_n)_{n \in \mathbb{N}}$ where $M_n \subset A$ nowhere dense s.t. $A = \bigcup_{n \in \mathbb{N}} M_n$
- A is called of second category if it is not of first category

Hence the theorem of Baire implies that every complete metric space is of second category.

2 Normal spaces and Banach spaces

Let X be a $\mathbb{K}$-vector space where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2.1 definitions

Definition 2.1.1. A map $||\cdot|| : X \to \mathbb{R}$ is called a norm on X if

1. $\forall x \in X$, $||x|| \geq 0$ and $||x|| = 0$ iff $x = 0$
2. $\forall \lambda \in \mathbb{K}$ and $\forall x \in X$ it holds that $||\lambda x|| = |\lambda| \cdot ||x||$
3. $\forall x, y \in X$ it holds $||x + y|| \leq ||x|| + ||y||$

The pair $(X, ||\cdot||)$ is called an normed space.

$p : X \to \mathbb{R}$ is called a seminorm if $p(x) \geq 0 \ \forall x \in X$ and 2. and 3. are also satisfied.

Example 2.1.2.
1. *$C^0([0,1];\mathbb{R})$ with $||f||_\infty = \max_{x\in[0,1]} |f(x)|$*
2. *more general for a compact metric space K: $C^0(K,\mathbb{R})$ with $||f||_\infty = \max_{x\in K} |f(x)|$*
3. *$C^1([0,1];\mathbb{R})$ with $p(f) = \max_{x\in[0,1]} |f'(x)|$*
4. *$\Omega \subset \mathbb{R}^n$ measurable. $L^1(\Omega) = \{f : \Omega \to \mathbb{R} : \ f$ integrable $\}$ with*

$$p : L^(\Omega) \to \mathbb{R} : \ p(f) = \int_\Omega |f(x)|\, dx$$

   then p is a seminorm.

Remark 2.1.3. Any normed space is a metric space via

$$d(x,y) = ||x - y||$$

All concepts from chapter 1 apply.

Lemma 2.1.4. *Let $(X, ||\cdot||)$ be a normed space. Then X is called separable iff $\exists A \subset X$ countable such that s.t. $\overline{span\{A\}} = X$ where $span\{A\} = \{\sum_{i=1}^n \lambda_i x_i\}$ with $n \in \mathbb{N}$, $\lambda_i \in K$ and $x_i \in A$. Here the colusre is defined w.r.t the norm.*

Definition 2.1.5. A complete normed space is called a Banach space.

2.2 Example: l^p-spaces

We consider the vector space $\mathbb{K}^\mathbb{N}$ of sequences in in $\mathbb{K}$. Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$.
Define $x + y = (x_n + y_n)_{n\in\mathbb{N}}$ and $\lambda x = (\lambda x_n)_{n\in\mathbb{N}}$.
For $x \in \mathbb{K}^\mathbb{N}$ define

$$||x||_{l^p} = \left(\sum_{n=1}^\infty |x|^p\right)^{1/p}$$

for $1 \leq p < \infty$ and

$$||x||_{l^\infty} = \sup_{n \in \mathbb{N}} |x_n|$$

else.
Define $l^p = \{x = (x_n)_{n\in\mathbb{N}} : ||x||_{l^p} < \infty\}$ for $1 \leq p \leq \infty$. We find that l^p is a subspace of $\mathbb{K}^{\mathbb{N}}$ and l^p is a normed space (for the triangle inequality use the Hölder inequality).

Theorem 2.2.1. *For $1 \leq p \leq \infty$ l^p is a Banach space.*

Lemma 2.2.2. *For finite p, l^p is separable while l^∞ is not.*

2.3 Finite dimensional normed spaces

Let X be a vector space over $\mathbb{K}$. $\exists e_1, ..., e_n \in X$ s.t.

$$\forall x \in X;\ \exists \lambda_1, ..., \lambda_n \in \mathbb{K} :\ x = \sum_{i=1}^{n} \lambda_i x_i$$

For $p \in [1, \infty)$ we define

$$||x||_p = \left(\sum_{i=1}^{n} |\lambda_i|^p \right)^{1/p}$$

and for $p = \infty$

$$||x||_\infty = \max_{1 \leq i \leq n} |\lambda_i|$$

Definition 2.3.1. Two norms are equivalent in that

$$\alpha || \cdot ||_1 \leq || \cdot ||_2 \leq \beta || \cdot ||_1$$

Theorem 2.3.2. *In a finite dimensional space, all norms are equivalent.*

Theorem 2.3.3. *Finite dimensional normed spaces are Banach spaces.*

2.4 On the closure of $\overline{B_1(0)}$

Lemma 2.4.1 (Lemma of Riesz, Lemma of the almost orthogonal element)**.** *Let X be a normed space. $U \subset X$ a closed subspace of X s.t. $U \neq X$. Then $\forall \lambda \in (0, 1) \exists x_\lambda \in X$ s.t. $||x_\lambda|| = 1$ and $dist(x_\lambda, U) \geq \lambda$.*

Theorem 2.4.2. *In a normed space X, $\overline{B_1(0)}$ is compact iff X is finite dimensional.*

3 A question from approximation theory

3.1 Theorem of Stone-Weierstrass

Let X be a compact metric space. Then $(C^0(X), \mathbb{K}), ||\cdot||_\infty$, where $||f||_\infty = \max\limits_{x \in X} |f(x)|$ is a Banach space.
Which property of $A \subset C^0(X, \mathbb{K})$ ensures that A is dense.

Definition 3.1.1. $A \subset C^0(X, \mathbb{K})$ is called subalgebra, if $\forall f, g, \in A$

1. $\lambda f + \mu g \in A$ (subspace)
2. $f \cdot g \in A$

Example 3.1.2.
- $\{p : [0,1] \to \mathbb{R}\}$ *is a subalgebra of* $C^0([0,1]; \mathbb{R})$.
- $\{f : [-1,1] \to \mathbb{R}; f$ *continuous and even*$\}$ *is a subalgebra.*

Remark 3.1.3. If A is a subalgebra, then $\overline{A}$ is also a subalgebra.

Definition 3.1.4. Let $A \subset C^0(X)$ be a subalgebra.

1. A is called unital if $1 \in A$
2. A separates point if $x, y \in X$, $x \neq y$, $\exists f \in A$ s.t. $f(x) \neq f(y)$.
3. (if $\mathbb{K} = \mathbb{C}$) A is stable under conjuguation if from $f \in A$ we conclude that also $\overline{f} \in A$.

Remark 3.1.5. If A is unital then all constant functions are in A.

Lemma 3.1.6. *Consider* $f : [-1,1] \to \mathbb{R}$ *where* $f(x) = |x|$. *Then* $\exists$ *sequence of polynomials* $(p_n)_{n \in \mathbb{N}}$ *s.t.*

$$p_n \to f$$

uniformly in $[-1,1]$.

Lemma 3.1.7. *Let* $A \subset C^0(X, \mathbb{R})$ *be a unital subalgebra. Then*

1. *if* $f \in A$ *then* $|f| \in \overline{A}$.
2. *if* $f, g \in A$ *then* $\max\{f, g\} \in \overline{A}$ *and* $\min\{f, g\} \in \overline{A}$

Theorem 3.1.8 (Stone-Weierstrass)**.** *Let* A *be a compact metric space.* $A \subset C^0(X, \mathbb{K})$ *is a unital subalgebra that separates points and if* $\mathbb{K} = \mathbb{C}$ *is stable under conjugation, then* A *is dense in* $C^0(X, \mathbb{K})$ *w.r.t* $||\cdot||_\infty$.

3.2 Applications

Theorem 3.2.1 (Theorem of Weierstraß)**.** *Let $[a,b]$ be a compact interval in $\mathbb{R}$, $f : [a,b] \to \mathbb{R}$ be a continuous function and $\varepsilon > 0$. Then $\exists p : [a,b] \to \mathbb{R}$ a polynomial s.t.*

$$||p - f||_\infty = \sup_{x\in[a,b]} |p(x) - f(x)| < \varepsilon$$

Definition 3.2.2. A function $f : \mathbb{R} \to \mathbb{C}$ is periodic if

$$f(x + t) = f(x)$$

for a $t \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Remark 3.2.3. If f is periodic with period t then $\tilde{f} : \mathbb{R} \to \mathbb{C}$ where $\tilde{f}(x) = f(t\frac{x}{2\pi})$ is periodic of period 2π.
Consider $C^0_{2\pi}(\mathbb{R},\mathbb{C})$ the space of continuous 2π-periodic functions. We consider the span of $\{e^{ikx} = \cos(kx) + i\sin(kx), k \in \mathbb{Z}\}$.

Definition 3.2.4. A trigonometric polynomial is a function $f : \mathbb{R} \to \mathbb{C}$

$$f(x) = \sum_{k=-N}^{N} c_k \cdot e^{ikx}$$

with $c_k \in \mathbb{C}$

Theorem 3.2.5 (Approximation of periodic functions)**.** *Trigonometric polynomials are dense in $(C^0_{2\pi}(\mathbb{R},\mathbb{C}), ||\cdot||_\infty)$*

Application to neural networks

The simplest case of a neural network has d inputs $x_1, ..., x_d$ and one output Z called a *feed forward* network. Each input influences the output and x_i might have a weight α_i associated to it. The output is a function in $x = (x_1, ..., x_d)$ and the weights $\alpha = (\alpha_1, ..., \alpha_d)$. For instance, the output is often of the form

$$Z = \sum_{i=1}^{d} \alpha_i x_i + b$$

where b is the bias of the network. To make the network slightly stronger, we add a intermediate layer $y = (y_1, ..., y_r)$ where each x_i is connected to each y_j with the associated weight $\gamma_{i,j}$. The y layer (often called activation) is the connected to the output Z as above with weights α_j. We introduce the realtion

$$y_j = \Phi(\sum_{i=1}^{d} \gamma_{j,i} x_i + b)$$

for a measurable function Φ. Lastly, the output is then given by

$$Z = \sum_{j=1}^{r} \alpha_j y_j$$

Definition 3.2.6. 1. $A^d = \{a : \mathbb{R}^d \to \mathbb{R} : \ a(x9 = w^T x + b)\}$ where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

2. given $\Phi : \mathbb{R} \to \mathbb{R}$ measurable $d \in \mathbb{N}$ define $\Sigma^d(\Phi) = \{f : \mathbb{R}^d \to \mathbb{R} : \ f(x) = \sum_{j=1}^{N} \alpha_j \Phi(a_j(x))$ wtih $N \in \mathbb{N}, \alpha_j \in \mathbb{R}, a_j \in A^d\}$ as the set of single hidden layer feed forward networks.

3. A squashing function is a measurable non-decreasing function $\Phi : \mathbb{R} \to \mathbb{R}$ s.t. $\lim_{x \to -\infty} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = 1$.

Theorem 3.2.7 (Universal Approximation theorem of Hornik-Stinchcombe-White)**.** *Let Φ we a squashing function $K \subset \mathbb{R}^d$ compact $f : K \to \mathbb{R}$ continuous and $\varepsilon > 0$. Then $\exists g \in \Sigma^d(\Phi)$ s.t.*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon$$

4 Continuous linear maps

$(X, ||\cdot||_X), (Y, ||\cdot|| : Y)$ are K-Vector spaces with $K = \mathbb{R}$ or $K = \mathbb{C}$. $T : X \to Y$ is called linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

4.1 Continuity of linear maps

Definition 4.1.1. LEt $T : X \to Y$ be linear. Then T is bounded if $\exists C > 0$ s.t.

$$||Tx||_Y \leq C||x||_X \ \forall x \in X$$

or equivalently

$$\sup_{x \in X \setminus \{0\}} \frac{||Tx||_Y}{||x||_X} \leq C$$

which is also equivalent to

$$\sup_{x \in X, ||x||_X = 1} ||Tx||_Y \leq C$$

Theorem 4.1.2. *For $T : X \to Y$ linear, the following are equivalent:*

1. *T is continuous*
2. *T is continuous in 0*
3. *t is bounded*

Lemma 4.1.3. *Let X have infinite dimension. Then $\exists T : X \to \mathbb{K}$ linear and not bounded.*

Definition 4.1.4. Define $L(X, Y)$ as the set of linear continuous ($\Leftrightarrow$ bounded) maps from X to Y. With the usual addition $((T_1 + T_2)(x) = T_1(X) + T_2(x))$ and the scalar multiplication $((\lambda(T)(x)) = \lambda T(x))$ this is a vector space.
If $X = Y$ we write $L(X)$. For $T \in L(X, Y)$

$$\ker T = \{x \in X : Tx = 0\}$$

and

$$\Im(T) = \{y \in Y : \exists x \in X : \; Tx = y\}$$

4.2 Operatornorm and dual space

Theorem 4.2.1. *Let $X \neq \{0\}$.*

- *$L(X, Y)$ with the operatornorm $||T|| = \sup_{x \in X \setminus \{0\}} \frac{||Tx||_Y}{||x||_X} = \sup_{x \in X, ||x||_X = 1} ||Tx||_Y$ is a normed space. We have*
$$||Tx||_Y \leq ||T|| ||x||_X$$
- *If Y is a Banach space then $L(X, Y)$ is a Banach space.*

Definition 4.2.2. For a normed space $(X, ||\cdot||_\infty)$ we define the dual space $X' = L(X, \mathbb{K})$.

Remark 4.2.3. X' is a Banach space.

4.3 Neumann series

Lemma 4.3.1. *Let X, Y, Z be three normed spaces. Let $T \in L(X, Y)$ and $S \in L(Y, Z)$. Then $S \circ T \in L(X, Z)$ and*

$$||S \circ T|| \leq ||S|| ||T||$$

Let $T : X \to Y$ be linear, bounded and bijective. Then $\exists T^-1 : Y \to X$ linear.

Definition 4.3.2. Let X, Y be normed spaces.

1. $T \in L(X,Y)$ is bijective such that $T^-1 \in L(Y,X)$ then T is called an isomorphism
2. X, Y are called isomorph if there is $T : X \to Y$ isomorphism.
3. $T \in L(X,Y)$ is called an Isometry if $||Tx|| = ||x||$.
4. X, Y are called isometric isomorph if $\exists T \in L(X,Y)$ an isomorphism that is also an isometry.

Remark 4.3.3. The identity $I_x : X \to X$ with $x \mapsto x$ is in $L(X)$. Then $T \in L(X)$ is an isomorphism iff $\exists S \in L(X)$ s.t. $S \circ T = I_x$ and $T \circ S = I_x$

Let $T \in L(X)$ s.t $||T|| < 1$. Define $T^0 = I_x$, $T^n = T \circ T^{n-1}$. Obviously $T^n \in L(X)$ for all n. Now,

$$\left(\sum_{k=0}^{n} T^k\right)_{n\in\mathbb{N}} \subset L(X)$$

is a Cauchy sequence w.r.t. the operatornorm. Hence, if X is a Banach-Space, so is $L(X)$ and thus the series converges to a $S \in L(X)$. Furthermore

$$\sum_{k=0}^{\infty} ||T||^k = \frac{1}{1 - ||T||}$$

Finally, we can also note that $S = (I_x - T)^{-1}$.

Theorem 4.3.4 (Neumann series)**.** *Let X be a Banach-Space, $T \in L(X)$ with $||T|| < 1$ The $I_x - T$ is an isomorphism and*

$$(I_x - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

is in $L(X)$. This is called the Neumann series.

4.4 The dual space of l^p

We only deal with $1 \le p < \infty$.

Theorem 4.4.1. *Let $q \in (1, \infty]$ be s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then the dualspace $(l^p)'$ is isometric isomorph to l^q.*

5 Theorem of Hahn-Banach

Let X be a vector space, $X \neq \{0\}$ over $\mathbb{K} = \mathbb{R}$.

5.1 Extension Theorem

Given $U \subset X$ subspace, $l : U \to \mathbb{R}$ linear, is there $L : X \to \mathbb{R}$ linear such that $L|_U = l$? For this we need Zorn's Lemma:

Definition 5.1.1. Let $M \neq \varnothing$ be a set and $\leq$ be a partial order on M, i.e. $\leq$ satisfies

1. reflexiv: $x \leq x \; \forall x \in M$
2. antisymmetric: $x \leq y$ and $y \leq x \Rightarrow x = y$
3. transitivity $x \leq y, y \leq z \Rightarrow x \leq z$

Then

- $A \subset M$ is called chain of totally ordered if $\forall x, y \in A$ either $x \leq y$ or $y \leq x$
- $b \in M$ is an upper bound for a chain A if $a \leq b$ for all $a \in A$
- $m \in M$ is called maximal element if

$$\forall x \in M \text{ s.t. } m \leq x \Rightarrow x = m$$

Lemma 5.1.2 (Zorn)**.** *Let $M \neq \varnothing$ and $\leq$ be a partial order on M. If every chain in M has an upper bound in M, then there is a maximal element.*

Definition 5.1.3. LEt X be a vector space. $p : X \to \mathbb{R}$ is called sublinear if

1. $p(\lambda x) = \lambda p(x)$ for all $x \in X, \lambda \geq 0$
2. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Theorem 5.1.4 (Extension theorem of Hahn-Banach)**.** *Let X be a vecorspace over $\mathbb{R}$, $U \subset X$ a subspace and $U \neq X$. Let $p : X \to \mathbb{R}$ be a subspace $l : U \to \mathbb{R}$ be linear s.t. $l(x) \leq p(x) \; \forall x \in U$. Then $\exists L : X \to \mathbb{R}$ linear s.t. $L(x) \leq p(x) \; \forall x \in X$ and $L(x) = l(x) \; \forall x \in U$. L is called extension of l.*

Consequences for normed spaces

Theorem 5.1.5. *Let $(X, ||\cdot||_X)$, $U \subset X$ a subspace fo X, with $U \neq X$. Let $u' \in U' = L(U, \mathbb{R})$. Then $\exists x' \in X'$ s.t. $||x'||_{X'} = ||u'||_{U'}$ such that $x'(u) = u'(u) \; \forall u \in U$.*

Corollary 5.1.6. *Let $(X, ||\cdot||_X)$, $U \subset X$ be a subspace of X and $x_0 \in X$ s.t. $dist(x_0, U) > 0$. Then $\exists x' \in X'$ s.t. $x'|_U = 0 \; \forall u \in U$ and $x'(x_0) = dist(x_0, U)$ with $||x'||_{X'} = 1$.*

Corollary 5.1.7. *Let $X, ||\cdot||_X)$ and $x_0 \in X$.*

1. *if $x_0 \neq 0$ then $\exists F \in X'$ with $||F||_{X'} = 1$ and $F(x_0) = ||x_0||_X$ In particular, for $x \in X$*
$$||x||_X = \sup_{F \in X', ||F||_{X'}=1} |F(x)|$$
2. *If $F(x_0) = 0$ for all $F \in X'$, then $x_0 = 0$. In particular, X' separates points of X.*
3. *$U \subset X$ subspace. Then U is dense in X iff if for $x' \in X'$ s.t. $x'_{|U} = 0$ it follows $x' = 0$.*

5.2 Separation Theorems

Definition 5.2.1. Let X be a vectorspace over $\mathbb{R}$. $A \subset X$ is called convex, if
$$\forall x, y \in A,\ \lambda x + (1-\lambda)y \in A,\ \forall \lambda \in [0,1]$$

Lemma 5.2.2. *Let $C \subset X$ open and convex with $O \in C$. Define $p_C : X \to \mathbb{R}$ such that $p_C(x) = \inf\{\lambda > 0$ s.t. $\frac{x}{\lambda} \in C\}$. This is called the Minkowski functional.*
Then p_C is sublinear and $C = \{x \in X : p_C(x) < 1\}$.

Lemma 5.2.3. *Let $(X, ||\cdot||)$ be a normed space and $A \subset X$ be convex and open, $A \neq \varnothing$ and $x_0 \in X \setminus A$, then $\exists F \in X'$ s.t.*
$$F(x) < F(x_0)\ \forall x \in A$$

Definition 5.2.4. Let $X \neq \{0\}$ be a $\mathbb{R}$-vectorspace.

1. $H = \{x \in X :\ f(x) = \alpha\}$ with $f : X \to \mathbb{R}$ linear, $\alpha \in \mathbb{R}$
2. $A, B \subset X$ are separated by an affine hyperplane H if $H = \{f = \alpha\}$ and $f(a) \leq \alpha \leq f(b)\ \forall a \in A\ \forall b \in B$.
3. $A, B \subset X$ are strictly separated by an affine Hyperplane H if $\exists \varepsilon > 0$ s.t. $f(a) + \varepsilon \leq \alpha \leq f(b) - \varepsilon$.

Theorem 5.2.5 (Separation Theorem of Hahn-Banach)**.** *Let $(X, ||\cdot||)$, A, B convex, $A \neq \varnothing$, $b \neq \varnothing$, $A \cap B = \varnothing$ and A or B should be open.. Then $\exists F \in X'$ and $\delta \in \mathbb{R}$ s.t.*
$$F(a) \leq \delta \leq F(b)\ \forall a \in A, b \in B$$

Theorem 5.2.6. *Let $(x, ||\cdot||)$, A, B convex subsets $A \neq \varnothing, B \neq \varnothing$, $A \cap B = \varnothing$. Let A be closed and B be compact. Then $\exists F \in X', \exists \varepsilon > 0$ s.t. $F(a) + \varepsilon \leq F(b) - \varepsilon\ \forall a \in A, b \in B$.*

6 Hilbert Spaces

Let X be a vector space over $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

6.1 Inner product space

Definition 6.1.1. A map $\langle \cdot, \cdot \rangle \colon X \times X \to \mathbb{K}$ is an inner product on X, if

1. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$
2. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle x, y \rangle}$
4. $\langle x, y \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

The pair $(X, \langle \cdot, \cdot \rangle)$ is called an inner product space also called a pre-Hilbert-space.
An inner product is a symmetric bilinear form if $\mathbb{K} = \mathbb{R}$ and a sesquilinear form if $\mathbb{K} = \mathbb{R}$.

Theorem 6.1.2 (Cauchy-Schwartz-inequality)**.** *In an inner product space we have*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle}\sqrt{\langle y, y \rangle}$$

Theorem 6.1.3. *For an inner product space X we define $|| \cdot || : X \to [0, \infty)$ by $||x|| = \sqrt{\langle x, x \rangle}$. This is a norm.*

Definition 6.1.4. Let X be an inner product space. Then $x, y \in X$ are orthogonal if $\langle x, y \rangle = 0$. This is written as $x \perp y$.

Corollary 6.1.5. *If $x \perp y$, then*

$$||x + y||^2 = ||x||^2 + ||y||^2$$

Theorem 6.1.6. *A normed space is an inner product space, iff $\forall x, y \in X$ the norm satisfies*

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

6.2 Hilbert spaces

Definition 6.2.1. Is an inner product space complete w.r.t. to the induced norm, we call it Hilbert space.

Theorem 6.2.2 (projection theorem)**.** *Let X be a Hilbert space, $A \subset X$ non-empty, convex and closed. Then* $\forall x \in X$

$$\exists! y \in A \ s.t. \ ||x - y|| = dist(x, A)$$

y is called the best approximation or projection of x in A.

Theorem 6.2.3 (Characterisation of the bes approximation)**.** *Let X be an inner product space, $A \subset X$, $A \neq \varnothing$ and convex and $x \in X$. Then y is the best approximation of x in A iff*

$$\Re\langle x - y, z - y\rangle \leq 0 \ \forall z \in A$$

Definition 6.2.4. Let X be an inner product space, $A \subset X$, then

$$A^{\perp} = \{x \in X : x \perp y \ \forall y \in A\}$$

the orthogonal complement of A.

Remark 6.2.5. $A^{\perp}$ is a closed subspace. If $(x_n)_{n\in\mathbb{N}} \subset A^{\perp}$, $x_n \to x$ in X, $\forall n \in \mathbb{N}$ we have $\langle x_n, y\rangle = 0 \ \forall y \in A$. Moreover $A \subset (A^{\perp})^{\perp}$.

Theorem 6.2.6. *Let X be a Hilbert space, $U \subset X$ closed subspace. Then*

$$\forall x \in X \ \exists! u \in U \ s.t. \ ||x - u|| = dist(x, U) = \inf_{z\in U} ||z - u||$$

We have $x - u \in U^{\perp}$ and $X = U \oplus U^{\perp}$, meaning that $x = u + v$, $u \in U$, $v \in U^{\perp}$ uniquely. The u is called the orthogonal projection of x in U.

Theorem 6.2.7 (Riesz-Fréchet)**.** *Let $X \neq \{0\}$ be a Hilbert space. $\forall F \in X' \ \exists! y \in X$ s.t. $F(x) = \langle x, y\rangle$. Moreover, $||F||_{X'} = ||y||_X$. Equivalently*

$$J : X \to X', \ (Jy)(x) = \langle x, y\rangle$$

is a bijective, anti-linear isometry. In particular, if X' is a Hilbert space, the dual is also a Hilbert space.

6.3 Orthonormal systems

Let $(X\langle\cdot,\cdot\rangle)$ be an inner product space.

Definition 6.3.1. Let $I \neq \varnothing$ be an index set. A family of vectors $(e_k)_{k\in I} \subset X$ is called an orthonormal system (ONS) if

$$\langle e_i, e_j\rangle = \delta_{i,j}$$

Theorem 6.3.2 (Schmidt Orthogonalisation theorem)**.** *Let* $\{x_i : i \in I\} \subset X, I \subset \mathbb{N}$ *be linearly independent vectors. Then* $\exists$ *ONS* $\{e_i : i \in I\}$ *s.t.*

$$span\{x_i : i \in I\} = span\{e_i : i \in I\}$$

Lemma 6.3.3 (Bessel's inequality)**.** *Let* $\{e_1, ..., e_n\}$ *be an ONS.* $Y = span\{e_1, ..., e_n\}$*. Then* $\forall x \in X$

$$\inf_{y \in Y} ||x - y||^2 = ||x - \sum_{i=1}^{n} \langle x_i, e_i \rangle||^2 = ||x||^2 - \sum_{i=1}^{n} |\langle x_i, e_i \rangle|^2 \geq 0$$

Definition 6.3.4. If $I \subset \mathbb{N}$, $(e_n)_{n \in I}$ ONS, then $\langle x, e_n \angle$ is called the n-th Fourier coefficient. of x. W.r.t. $(e_n)_{n \in I}$.

Definition 6.3.5. An ONS $(e_n)_{n \in \mathbb{N}}$, $I \subset \mathbb{N}$ is called complete in X if

$$\overline{span\{e_n : n \in I\}} = X$$

Theorem 6.3.6. *Any separable Hilbert space* X *has a complete ONS.*

Lemma 6.3.7. *Let* X *be a Hilbert space,* $(e_n)_{n \in \mathbb{N}}$ *an ONS. Then* $\exists y \in X$ *s.t.* $y = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$*.*

Theorem 6.3.8. *Let* X *be a Hilbert space of infinite dimension,* $(e_n)_{n \in \mathbb{N}}$ *an orthonormal system. Then the following are equivalent.*

1. $(e_n)_{n \in \mathbb{N}}$ *is complete*
2. *if* $x \in X$ *s.t.* $\langle x, e_n \rangle = 0 \ \forall n \in \mathbb{N}$*, then* $x = 0$
3. $\forall x \in X$*,* $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$ *(Fourier series of* x*)*
4. $\forall x \in X$*,* $||x||^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$*.*

Corollary 6.3.9. *Any separable infinite-dimensional Hilbert space is isometrically isomorphic to* ℓ^2*.*

7 Spectral theorem for self-adjoint compact operators

We only deal with Hilbert spaces.

7.1 Adjoint in Hilbert spaces

Let $(X, \langle\cdot,\cdot\rangle)$, $(Y, \langle\cdot,\cdot\rangle)$, $T \in L(X,Y)$. Let $y \in Y$. Consider the map

$$X \ni x \mapsto \langle Tx, y\rangle_Y$$

This map is linear and bounded.

$$|\langle Tx, y\rangle_Y| \overset{CS}{\leq} ||Tx||_Y ||y||_Y \leq ||T|| ||x||_X ||y||_Y$$

Thus it is an element of X'. By the theorem of Riesz-Fréchet

$$\exists! T^*y \in X \text{ s.t. } \langle x, T^*y\rangle_X = \langle Tx, y\rangle_Y \ \forall x \in X$$

This defines a map $T^* : Y \to X$ with $y \mapsto T^*y$.

Definition 7.1.1. T^* is the Hilbert space adjoint of T.

Lemma 7.1.2. *$T^* \in L(Y, X)$ and $||T^*|| = ||T||$.*

Lemma 7.1.3. *Let $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$, $(Z, \langle\cdot,\cdot\rangle_Z)$ be Hilbert spaces. Let $T \in L(X, y)$, $S \in L(Y, Z)$ and $\lambda \in \mathbb{K}$. Then*

1. $(S \circ T)^* = T^*S^*$
2. $(\lambda T)^* = \overline{\lambda} T^*$
3. $(T^*)^* = T$

Definition 7.1.4. Let $(X, \langle\cdot,\cdot\rangle_X)$ be a Hilbert space and $T \in L(X)$. T is called self-adjoint if $T^* = T$

Lemma 7.1.5.

- *If $\mathbb{K} = \mathbb{C}$, T is self-adjoint $\Leftrightarrow \langle Tx, x\rangle_X \in \mathbb{R} \ \forall x \in X$*
- *If T is self-adjoint, then $||T|| = \sup_{x \in X, ||x||_X = 1} |\langle Tx, x\rangle|$*

7.2 compact operators

Here X, Y can be only Banach spaces and $X, Y \neq \{0\}$.

Definition 7.2.1. $f : X \to Y$ is compact if f maps bounded sets in precompact sets.

Lemma 7.2.2. *Let $T : X \to Y$ be linear. Then T is compact iff $T(B_1(0))$ is precompact in Y.*

Notation: $K(X,Y) = \{T : X \to Y$ linear and compact$\}$ and $K(X) = K(X,X)$.

Remark 7.2.3. $T \in K(X,Y) \Rightarrow T \in L(X,Y)$.

Lemma 7.2.4. *1. $T \in L(X,Y)$, $S \in L(Y,Z)$. If T or S is compact, then the composition is compact.*

2. $K(X,Y)$ is a closed subspace of $L(X,Y)$. In particular $K(X,Y)$ is a Banach space.

Definition 7.2.5. • Let H be a Hilbert space and $T \in L(X)$. Then T is called self-adjoint if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \ \forall x, y \in X$$

- Let X, Y be Banach spaces then, $T \in L(X,Y)$ compact $\Leftrightarrow T(B_1(0))$ is precompact.

Lemma 7.2.6. *$T \in L(X,Y)$ is compact iff $\forall (x_n)_{n\in\mathbb{N}} \subset X$ bounded $(T(x_n))_{n\in\mathbb{N}}$ admits a convergent subsequence.*

7.3 Spectrum

Let X be a Banach space.

Definition 7.3.1. Let $T \in L(X)$.

- the resolvent set of T is

$$\rho(T) = \{\lambda \in \mathbb{K} : (\lambda \cdot Id - T)^{-1} \in L(X)\} \subset \mathbb{K}$$

  while $\sigma(T) = \mathbb{K} \setminus \rho(T)$ is the spectrum of T.

- the resolvent map $R : \rho(T) to L(X)$ is defined by $\lambda \mapsto (\lambda Id - T)^{-1}$

- the spectrum of T is divided into

$$\sigma(T) = \sigma_p(T) \cup \sigma_C(T) \cup \sigma_r(T)$$

  where

  - $\sigma_P(T) = \{\lambda \in \sigma(T) : \ker(\lambda Id - T) \neq \{0\}\}$ is the point spectrum
  - $\sigma_C(T) = \{\lambda \in \sigma(T) \setminus \sigma_P(T) : Im(\lambda Id - T) \neq X$ but $\overline{Im(\lambda Id - T)} = X\}$
  - $\sigma_r(T) = \sigma(T) \setminus (\sigma_P(T) \cup \sigma_C(T))$.

- the elements of the point spectrum are called eigenvalues and $x \in X \setminus \{0\} : (I\lambda Id - T)(x) = 0$ is called eigenvector associated ot λ.

Theorem 7.3.2. *For $T \in L(X)$*

1. *$\rho(T)$ is open.*
2. *$\sigma(T)$ is compact and*

$$\sup_{\lambda \in \sigma(T)} |\lambda| \leq \lim_{m \to \infty} ||T^m||^{\frac{1}{m}} = r(T)$$

is the spectral radius. In particular $r(T) \leq ||T||$

7.4 Spectral theorem for self-adjoint compact operators

Let X be a Hilbert space.

Lemma 7.4.1. *Let $T \in K(X)$ self-adjoint. Then $||T||$ or $-||T||$ is an eigenvalue of T.*

Lemma 7.4.2. *Let $T \in L(X)$ be self-adjoint. Then all eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal.*

Lemma 7.4.3. *Let $T \in L(X)$. If $M \subset X$ is a closed subspace s.t. $TM \subset M$, then $M^\perp$ is invariant under T^*.*

Theorem 7.4.4. *Let X be a Hilbert space, $T \in K(X)$ self-adjoint. Then $\exists$ ONS $(\phi_n)_{n \in I} \subset X,\ I \subset \mathbb{N},$ and $\exists (\lambda_n)_{n \in I} \subset \mathbb{R}$ s.t. $\forall x \in X$*

$$Tx = \sum_{n \in I} \lambda_n \langle x, \phi_n \rangle \phi_n$$

In particular $T\phi_n = \lambda_n \phi_n\ \forall n \in \mathbb{N}$. If I is infinite, then $\lambda_n \to 0$.

Corollary 7.4.5. *Let X be a separable Hilbert space with $\dim X = \infty$ and $T \in K(X)$ self-adjoint. Then $\exists$ a complete ONS $(e_n)_{n \in \mathbb{N}}$ of eigenvectors of T. In particular $\forall x \in X$*

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_N \rangle e_n$$

with λ_n being the corresponding eigenvalue to e_n.

8 Reproducing kernel Hilbert spaces

8.1 Definitions

Here, we again use $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Further $X \neq \varnothing$ is simply a set. Also

$$F(X, \mathbb{K}) = \{f : X \to \mathbb{K} \text{ a map}\}$$

This is a vector space.

Definition 8.1.1. $H \subset F(X, \mathbb{K})$ is a reproducing kernel Hilbert space (RKHS) on X if

1. H is a subspace of $F(X, \mathbb{K})$
2. $\exists \langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ inner product, s.t. $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space
3. $\forall x \in X$ the linear map $E_x : H \to \mathbb{K}$ with $E_x(f) = f(x)$ (the evaluation operator) is well-defined and bounded.

Let $\Omega \subset \mathbb{R}^n$ open, $H = L^2(\Omega)$ is not a RKHS since evaluation at a point does not make sense for $f \in L^2(\Omega)$.
If H is a RKHS, the evaluation operator $E_x \in H' \; \forall x \in X$. For $x \in X$, by Riesz-Fréchet $\exists! k_x \in H$ s.t. $E_x(F) = \langle f, k_x \rangle \; \forall f \in H$.

Definition 8.1.2. The function

$$K : X \times X \to \mathbb{K}$$

$$(x, y) \mapsto \langle k_y, k_x \rangle$$

is called reproducing kernel of H.

Remark 8.1.3. For $x, y \in X$ and $\mathbb{K} = \mathbb{C}$

$$K(x, y) = \langle k_y, k_x \rangle = \overline{\langle k_x, k_y \rangle} = \overline{K(y, x)}$$

while if $\mathbb{K} = \mathbb{R}$ the kernel is symmetric. Further

$$||E_x||^2 = ||k_x||^2 = \langle k_x, k_x \rangle = K(x, x) \geq 0$$

8.2 Theorem of Moore-Aronszajn

Lemma 8.2.1. *Let H be a RKHS on X with kernel K. Then $\forall n \in \mathbb{N}$ and $\forall \{x_1, ..., x_n\} \subset X$ the matrix*

$$(K(x_i, x_j))^n$$

is a positive semidefinite matrix, i.e.

$$\sum_{i,j=1}^{n} \alpha_j K(x_j, x_i) \overline{\alpha_i} \geq 0 \; \forall \alpha \in \mathbb{K}^n$$

Theorem 8.2.2 (Moore-Aronszajn)**.** *Let $X \neq \varnothing$, $K : X \times X \to \mathbb{K}$ s.t.*

1. *if* $\mathbb{K} = \mathbb{C}$ $K(x,y) = \overline{K(y,x)}$ *and if* $\mathbb{K} = \mathbb{R}$ $K(x,y) = K(y,x)$

2. K *is positive semidefinite*

Then there exists a (unique) RKHS on K *with kernel* K. *Notation:* $H(K)$.

8.3 An application

Interpolation: Let $\{x_1, ..., x_n\} \subset X$ be distinct points. $\lambda_1, ..., \lambda_n \in \mathbb{C}$ not necessarily distinct. Let H be a RKHS on X.
AIM: Find $f \in H$ s.t. the least square error

$$J(f) = \sum_{i=1}^{n} |f(x_i) - \lambda_i|^2$$

is minimal at f and among all minimizers we want the one with minimal norm.

Theorem 8.3.1. *Let* H *be a RKHS on* X. $\{x_1, ..., x_n\} \subset X$ *distinct points in* X. $A := (K(x_i, x_j))$ *a* $n \times n$*-matrix.* $v = (\lambda_1, ..., \lambda_n)^T \in \mathbb{K}^n$. *Then* $\exists w \in \mathbb{K}^n$ *s.t.* $v - Aw \in \ker(A)$ *and*

$$H \ni f := \sum_{i=1}^{n} w_i k_{x_i}$$

satisfies

$$J(f) = \inf_{g \in H} J(g)$$

We have $k_{x_i} = K(\cdot, x_i)$ *and* f *is the unique minimizer of minimal norm.*

9 Theorems on continuous linear maps

9.1 uniform boundedness

We need the theorem of Baire a lot in this chapter, so we recall it.

Theorem 9.1.1 (Baire's theorem)**.** *Let* (X, d) *be a complete metric space and* $(U_n)_{n\in\mathbb{N}}$ *s.t.* $U_n \subset X$ *is open and dense* $\forall n \in \mathbb{N}$. *Then*

$$\bigcap_{n\in\mathbb{N}} U_n$$

is dense in X.

Corollary 9.1.2. *Let* (X, d) *be a complete metric space,* $(A_n)_{n\in\mathbb{N}}$ *s.t.* A_n *closed* $\forall n \in \mathbb{N}$ *and* $X = \bigcup_{n\in\mathbb{N}} A_n$. *Then* $\exists N \in \mathbb{N}$ *s.t.* A_N *has an interior point.*

Theorem 9.1.3 (uniform boundedness principle)**.** *Let $X \neq \varnothing$ be a complete metric space, Y a normed space. Let $F \subset C^0(X,Y)$ s.t.*

$$\sup_{f \in F} ||f(x)||_Y < \infty \; \forall x \in X$$

Then $\exists x_0 \in X$ and $\exists r_0 > 0$ s.t.

$$\sup_{x \in \overline{B_{r_0}(x_0)}} \sup_{f \in F} ||f(x)||_Y < \infty$$

Theorem 9.1.4 (Banach-Steinhaus)**.** *Let X Banach space, Y normed space, $\mathcal{T} \subset L(X,Y)$ family such that*

$$\sup_{T \in \mathcal{T}} ||Tx||_Y < \infty \; \forall x \in X$$

Then $\mathcal{T}$ is a bounded set in $L(X,Y)$ i.e.

$$\sup_{T \in \mathcal{T}} ||T||_{L(X,Y)} < \infty$$

Lemma 9.1.5. *Let X be a Banach space, Y a normed space, $(T_n)_{n \in \mathbb{N}} \subset L(X,Y)$ s.t. $\forall x \in X$, $T_n x$ converges in Y. Then $T : X \to Y$ with $x \mapsto \lim_{n \to \infty} T_n x$ is linear and continuous.*

9.2 open map theorem

Definition 9.2.1. Let (X, d_X), (Y, d_Y) be open metric spaces and $f : X \to Y$. Then f is called open if $\forall U \in X$ open $f(U) \subset Y$ is open.

Remark 9.2.2. Let $f : X \to Y$ be bijective. Then f is an open map iff f^{-1} is continuous.

Attention: f continuous and bijective $\not\Longrightarrow$ f^{-1} is continuous. A counterexample is $f : [0,1] \cup (2,3] \to [0,2]$ where

$$f(x) = \begin{cases} x, & x \in [0,1] \\ x-1, & x \in (2,3] \end{cases}$$

Lemma 9.2.3. *Let $T : X \to Y$ be linear, X, Y normed spaces.*

1. *T is open iff $\exists \delta > 0$ s.t. $T(B_1(0)) \supset B_\delta(0)$*
2. *T open $\Rightarrow$ T is surjective*

Theorem 9.2.4 (open map theorem)**.** *If X, Y are Banach spaces, $T \in L(X,Y)$ s.t. T surjective, then T is open.*

Theorem 9.2.5 (theorem of the inverse)**.** *Let X, Y be Banach-spaces, $T \in L(X, Y)$ bijective, then T^{-1} is continuous and in fact $T^{-1} \in L(Y, X)$.*

Corollary 9.2.6. *Let X, Y be Banach. Then any bijective map $T \in L(X, Y)$ is an isomorphism.*

Remark 9.2.7. $T \in L(X)$ where X Banach then

$$\rho(T) = \{\lambda \in \mathbb{K} : (\lambda ID - T)^{-1} \in L(X)\} = \{\lambda \in \mathbb{K} : \lambda Id - T \text{ bijective}\}$$

Theorem 9.2.8. *Let X, Y be Banach. Then $\mathcal{S} = \{T \in L(X, Y) : T$ surjective$\}$ is open in $L(X, Y)$.*

9.3 Closed graph theorem

We work with the graph of an operator. Recall that, given $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$, we can look at the normed space $X \times Y$ equipped with $||(x, y)||_{X\times Y} = ||x||_X + ||y||_Y$.

Definition 9.3.1. Let $T : X \to Y$ linear.

1. $G(T) = \{(x, y) \in X \times Y : y = Tx\}$ is the graph of T
2. T is called a closed linear operator if $G(T)$ is closed.

Remark 9.3.2.
- If $X, Y)$ are Banach spaces, then so is $X \times Y$
- $G(T)$ is a subspace of $X \times Y$ and in particular a Banach space

Lemma 9.3.3. *T is a closed linear operator $\iff$ $\forall (x_n)_{n\in\mathbb{N}} \subset X$ s.t. $x_n \to x$ and $Tx_n \to y$, then necessarily $Tx = y$.*

Theorem 9.3.4 (closed graph theorem)**.** *Let X and Y Banach, $T : X \to Y$ linear. Then T is a linear closed operator iff T is continuous (bounded).*

Remark 9.3.5. If X, Y Banach, $T : X \to Y$ linear, then T is continuous

- iff $\forall (x_n)_{n\in\mathbb{N}} \subset X$ s.t. $x_n \to x$ in X then $Tx_n \to Tx$ in Y
- iff $\forall (x_n)_{n\in\mathbb{N}} \subset X$ s.t. $x_n \to x$ and $Tx_n \to y$.

Definition 9.3.6. Let X, Y be normed spaces and $D \subset X$ a subspace. $T : D \to Y$ linear is called closed if $\forall (x_n)_{n\in\mathbb{N}} \subset D$ s.t. $x_n \to x$ and $Tx_n \to y$ then $x \in D$ and $Tx = y$.

Lemma 9.3.7. *Let X, Y be Banach spaces, $D \subset X$, $T : D \to Y$ linear and clsoed. Define*

$$|| \cdot ||_T : D \to [0, \infty)$$

where

$$||x||_T = ||x||_X + ||Tx||_Y$$

called the graph norm. Then $|| \cdot ||_T$ is a norm, $(D|| \cdot ||_T)$ is a Banach space and

$$T : (D, || \cdot ||_T) \to (Y, || \cdot ||_Y)$$

is continuous.

9.4 Consequences

A central question in mathematics concerns the solvability of equations. Let X, Y be any sets and $f : X \to Y$. Given $y \in Y$ is there an $x \in X$ s.t. $f(x) = y$?.
Here x and Y are normed spaces and $T : X \to Y$ linear. The open map theorem implies that for Banach spaces X and Y, $T : X \to Y$ linear bijective and continuous, then $T^{-1} : Y \to X$ is also continuous. As a consequence, the solution of $Tx = y$ depends continuously on Y. Further $\{T \in L(X, Y) : T \text{ surjective}\}$ is open in $L(X, Y)$, when X and Y are Banach. With the Neumann series, we get

Theorem 9.4.1. *If X, Y Banach,*

$$A = \{T \in L(X, Y) : T \textit{ is an isomorphism}\}$$

is open in $L(X, Y)$. I.e. if $T \in L(X, Y)$ isomorphism $\Rightarrow \exists \rho > 0$ s.t. $\forall S \in L(X, Y)$ s.t. $||S - T|| < \rho$ then S is an isomorphism.

10 L^p-spaces

10.1 Definitions

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

Definition 10.1.1.

$$\mathcal{L}^p(\Omega, \mu) = \{f \in \mathcal{M}(\Omega, \mathbb{R}) : |f|^p \ \mu - \text{integrable}\}$$

for $1 \leq p < \infty$ and

$$\mathcal{L}^\infty = \{f \in \mathcal{M}(\Omega, \mathbb{R}) : \exists N \in \mathcal{A} :\ \mu(N) = 0 :\ \sup_{x \in \Omega \setminus N} |f(x)| < \infty\}$$

We define the functions

$$||f||_p = \left(\int_\Omega |f(x)|^p \, d\mu\right)^{1/p}$$

and

$$||f||_\infty = \text{esssup}|f| = \inf_{N \in \mathcal{A}, \mu(N)=0} \left(\sup_{x \in \Omega \setminus N} |f(x)\right)$$

Lemma 10.1.2. *For $p \in [1, \infty]$, $\mathcal{L}^p(\Omega, \mu)$ are vector spaces. The Hölder and Minkowski inequalities hold. But $||f||_p = 0 \not\Rightarrow f \equiv 0$. In general, only $f = 0$ μ-a.e.*

We define the equivalence relation $\sim$: $f \sim g$ iff $f = g$ μ-a.e.

Definition 10.1.3. For $p \in [1, \infty]$

$$L^p(\Omega, \mu) = \mathcal{L}^p / \sim = \{[f] : f \in \mathcal{L}^p\}$$

Theorem 10.1.4 (Fischer-Riesz)**.** *For $p \in [1, \infty]$, $(L^p(\Omega, \mu), || \cdot ||_p)$ is a Banach space. For $p = 2$, L^2 is a Hilbert space where*

$$\langle f, g\rangle = \int_\Omega f(x)g(x) \, d\mu(x)$$

Remark 10.1.5. If $(f_k)_{k \in \mathbb{N}}$ Cauchy in $(L^p(\Omega, \mu), ||f||_p)$ then $\exists f \in L^p(\Omega, \mu)$ s.t. $f_k \to f$ in $L^p(\Omega, \mu) \not\Rightarrow f_k \to f$ pointwise μ-a.e.
But $\exists$ subsequence $f_{k_j} \to f$ μ-a.e.

10.2 Approximation in L^p

In $\mathbb{R}^n$ with Lebesgue measure: $\Omega \subset \mathbb{R}^n$ measurable, $L^p(\Omega) = L^p(\Omega, \lambda^n)$.

Definition 10.2.1. For $f : \Omega \to \mathbb{R}$ continuous

$$\text{supp}(f) = \overline{\{x \in \Omega :\ f(x) \neq 0\}}$$

is called the support of f.

Definition 10.2.2. Let $C_0^0(\Omega, \mathbb{R}) = \{f : \Omega \to \mathbb{R} :$ f is continuous and *supp*$(f) =$ k compact and $\exists \Omega' \subset \Omega$ open s.t. $k \subset \Omega'\}$ the space of continuous functions with support compactly contained in Ω.

Theorem 10.2.3. *Let $\Omega \subset \mathbb{R}^n$ open, $1 \le p < \infty$. Then $C_0^0(\Omega)$ is dense in $L^p(\Omega)$*

Definition 10.2.4. Similarly we define

$$C_0^k = \{f : \Omega \to \mathbb{R} : \ f \in C^k(\Omega) \text{ and } f \in C_0^0(\Omega; \mathbb{R})\}$$

the space of k-times continuously differentiable functions with compact support in Ω and $C_0^\infty(\Omega) = \bigcap_{k\in\mathbb{N}} C_0^k(\Omega)$ called the set of test functions.

Definition 10.2.5. Define $\phi : \mathbb{R}^n \to \mathbb{R}$ where

$$\phi(x) = \begin{cases} c \cdot \exp(-\frac{1}{1-||x||^2}), & ||x|| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Where $c > 0$ is s.t.

$$\int_{\mathbb{R}^n} \phi(x)\, dx = 1$$

Further, for $\varepsilon > 0$, $\phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right)$$

Then $\int_{\mathbb{R}^n} \phi_\varepsilon(x)\, dx = 1$.

Definition 10.2.6. For $f \in L^1(\Omega)$, $\varepsilon > 0$ and $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f_\varepsilon(x) = \int_\Omega \phi_\varepsilon(x-y) f(y)\, dy$$

called the smoothing of f.

Remark 10.2.7. This is an example of a convolution. For $f, g : \mathbb{R}^n \to \mathbb{R}$ integrable

$$\int_{\mathbb{R}^n} f(x-y)g(y)\, dy = f * g(x) = g * f(x)$$

is the convolution of f and g

Lemma 10.2.8. *Let $\Omega \subset \mathbb{R}^n$ open $f \in L^1(\Omega)$, $\varepsilon > 0$. Then $f\varepsilon \in C^\infty(\mathbb{R}^n)$. If $supp(f) = k \subset \Omega$ compact then for $\varepsilon < dist(k, \partial\Omega)$, $f_\varepsilon \in C_0^\infty(\Omega)$.*

Theorem 10.2.9. *Let $\Omega \subset \mathbb{R}^n$ be open.*

1. *If $f \in C^0(\Omega)$, $K \subset \Omega$ compact, $f_\varepsilon \to f$ uniformly on K.*
2. *If $f \in L^p(\Omega)$, $p \in [1, \infty)$, then $||f_\varepsilon||_p \le ||f||_p$ and $f_\varepsilon \to f$ in $L^p(\Omega)$.*

Corollary 10.2.10. *Let $\Omega \subset \mathbb{R}^n$ be open. Then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$. I.e.*

$$\overline{C_0^\infty(\Omega)}^{||\cdot||_p} = L^p(\Omega)$$

Remark 10.2.11 (Dirac Sequences). $(\phi_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n)$ s.t.

- $\int \phi_k dx = 1$
- $\forall \varepsilon > 0 \lim\limits_{k\to\infty} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \phi_k dx = 0$

allow for a generalization of the above theorem.

Definition 10.2.12. $L^p_{loc}(\Omega) = \{f \in L^0(\Omega) : f \in L^p(K)$ for all compact sets $K \subset \Omega\}$. And further $L^0(\Omega)$ is the space of equivalence classes of a.e. equal measurable functions from $\Omega \to \mathbb{R}$.

Theorem 10.2.13 (Fundamental Lemma in the calculus of variations)**.** *Let $\Omega \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega f(x)\phi(x)dx = 0 \; \forall \phi \in C_0^\infty(\Omega)$$

then $f \equiv 0$ a.e.

10.3 Separability of L^p

Theorem 10.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $p \in [1,\infty)$. Then $L^p(\Omega)$ is separable.*

10.4 Dualspace of $L^p(\Omega)$, $p \in [1,\infty)$

Similar to l^p. Let $q \in (1,\infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{R}^n$ open. Let $g \in L^q(\Omega)$.

$$T_g : L^p(\Omega) \to \mathbb{R}, \; T_g(f) = \int_\Omega f(x)g(x)dx$$

Then, by Hölder,

$$T_g \in L^p(\Omega)', \; ||T_g||_{L^p(\Omega)} \leq ||g||_q$$

Theorem 10.4.1. *Let $\Omega \subset \mathbb{R}^n$ open, $p \in [1,\infty)$ and $q \in (1,\infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then $J : L^q(\Omega) \to L^p(\Omega)$ with $g \mapsto T_g$ is an isometric isomorphism.*

Theorem 10.4.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and $\nu : \mathcal{A} \to \mathbb{R}$ a bounded signed measure, i.e.*

- $\nu(\varnothing) = 0$
- *ν is σ-additive*

- *the total variation*

$$||\nu||_{var} = \sup\{\sum_{k=1}^{n} |\nu(E_i)| : n \in \mathbb{N} \textit{ and } E_1, ..., E_n \in \mathcal{A} \textit{ are pairwise disjoint sets}\}$$

  is finite

Then the following are equivalent:

1. $\exists! f \in L^1(\Omega, \mu)$ *s.t.* $\nu(A) = \int_A f \, d\nu$
2. ν *is absolutely continuous w.r.t.* μ, *i.e.*

$$\forall A \in \mathcal{A} : \mu(A) = 0 \Rightarrow \nu(A) = 0$$

Remark 10.4.3. In 1, one often uses the notation $f = \frac{d\nu}{d\mu}$ and calls this function Radon-Nikodym derivative of ν w.r.t. μ.

11 Reflexive Spaces and Weak Convergence

11.1 Reflexive Spaces

Let $X \neq \{0\}$ be a normed space and X' be its dual.

Definition 11.1.1. $X'' = (X')' = L(X', \mathbb{K})$ is the bi-dualspace of X.

There is a natural map between X and X''. This is $i_X : X \to X''$, defined by

$$x \mapsto i_X(x) \in X'', \text{ i.e. } i_X(x) : X' \to \mathbb{K}$$

That is $i_X(x)(f) = f(x) \, \forall f \in X'$.
i_X is linear and bounded.

Definition 11.1.2. $i_X : X \to X''$ as above is called canonical evaluation map.