

# Fun Summary

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# 1 metric spaces

## 1.1 metric spaces

**Definition 1.1.1.** A metric space is a non-empty set  $X$  together with a map

$$d : X \times X \rightarrow \mathbb{R}$$

$$(x, y) \mapsto d(x, y)$$

such that

1.  $d(x, y) = 0$  iff  $x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

*Remark 1.1.2.* ( $d$  admits only positive values)

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$$

**Example 1.1.3.** 1.  $d_2(x, y) = \|x - y\|_2$

$$2. \quad d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$$

**Definition 1.1.4.** (convergence)

A sequence  $(x_n)_{n \in \mathbb{N}}$  in a metric space  $(X, d)$  is said to be convergent to  $x \in X$  if

$$x_n \rightarrow x \text{ in } (X, d)$$

or

$$\lim_{n \rightarrow \infty} x_n = x \text{ in } (x, d)$$

## 1.2 Topology in metric spaces

Let  $(X, d)$  be a metric space.

**Definition 1.2.1.** 1. an open ball is defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

2.  $O \subset X$  is called open if  $\forall y \in O$  there is  $r > 0$  such that  $B_r(y) \subset O$

3.  $A \subset X$  is closed if  $X \setminus A$  is open.

**Theorem 1.2.2.** (*metric spaces are topological spaces*)

Let  $\mathcal{T}$  be the set of open subsets of  $X$ . Then

1.  $\emptyset, X \in \mathcal{T}$

2. if  $U, V \in \mathcal{T}$ , then  $U \cup V \in \mathcal{T}$

3. if  $\{U_i\}_{i \in I} \subset \mathcal{T}$ , then  $\bigcup_{i \in I} U_i \in \mathcal{T}$

**Remark 1.2.3.** 1.  $\emptyset, X$  are closed

2. finite union of closed sets is closed

3. arbitrary intersections of closed sets is closed

**Lemma 1.2.4.**  $A \subset X$  is closed iff  $\forall$  convergent sequences  $(x_n)_{n \in \mathbb{N}} \subset A$  the limit point is in  $A$ .

**Definition 1.2.5.** For  $M \subset X$  we define

$$\overline{M} = \bigcap_{A \supset M, A \text{ closed}}$$

as the closure of  $M$  and

$$M = \bigcup_{O \subset M, O \text{ open}}$$

as the interior of  $M$ .

$\partial M = \overline{M} \setminus M$  is the boundary of  $M$

Attention:

Define the closed ball as  $\overline{B}_r(a) = \{y \in X : d(y, a) \leq r\}$ . Then in general  $\overline{\overline{B}_r(a)} \neq \overline{B}_r(a)$ .

Example: Take  $X \neq \emptyset$  and the trivial metric  $d$ . Then

$$B_1(a) = \{a\} = \overline{B_1(a)}$$

but  $\overline{B}_1(a) = X$ .

### 1.3 separability and completion

Let  $(X, d)$  be a metric space.

**Definition 1.3.1.** 1.  $M \subset X$  is called dense in  $X$  if  $\overline{M} = X$ .

2.  $X$  is called separable if  $X$  has a countable dense subset.

*Remark 1.3.2.*  $M$  is dens in  $X$  iff

$$\forall x \in X \forall \varepsilon > 0 \exists y \in M \text{ s.t. } d(x, y) < \varepsilon$$

**Definition 1.3.3.** 1.  $(x_n)_{n \in \mathbb{N}} \subset X$  is called a Cauchy sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } m, n > N \text{ implies } d(x_n, x_m) < \varepsilon$$

2. A metric space in which all Cauchy sequences converge is called complete.

**Example 1.3.4.** 1.  $(C^0([a, b], \mathbb{R}), d_\infty)$  with  $d_\infty(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|$  is complete.

2.  $(\mathbb{R}^n, d_2)$  with  $d_2(x, y) = \|x - y\|_2$  is complete.

**Lemma 1.3.5.** Let  $(X, d)$  be a complete metric space and  $\emptyset \neq A \subset X$ . Then  $(A, d)$  is complete iff  $A$  is closed.

**Definition 1.3.6.**  $A \subset X$  is called bounded if its diameter

$$\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}$$

is finite.

**Theorem 1.3.7.**  $(X, d)$  is complete iff  $\forall (F_n)_{n \in \mathbb{N}}$  sequences of closed subsets such that  $F_{n+1} \subset F_n$  and  $\text{diam}(F_n) \rightarrow 0$  then

$$\exists! x_0 \in X \text{ s.t. } \bigcap_{n \in \mathbb{N}} F_n = \{x_0\}$$

### 1.4 Continuity

**Definition 1.4.1.** Let  $(X, d_x), (Y, d_y)$  be metric spaces and  $f : X \rightarrow Y$ .  $f$  is continuous in  $x_0$  if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in X \ d_x(x, x_0) < \delta \text{ implies } d_y(f(x), f(x_0)) < \varepsilon$$

With sequences:

$\forall (x_n)_{n \in \mathbb{N}} \subset X$   $x_n \rightarrow x_0$  in  $(X, d_x)$  if it holds  $(f(x_n))_{n \in \mathbb{N}} \subset Y$ ,  $f(x_n) \rightarrow f(x_0)$  in  $(Y, d_y)$

$f$  is continuous if  $f$  is continuous in  $x_0$  for all  $x_0 \in X$ .

In other words  $f$  is continuous if for all  $O \subset Y$  open (closed)  $f^{-1}(O)$  is open (closed) in  $X$ .

Special case:  $f$  is Lipschitz continuous if  $\exists L > 0$  s.t.

$$d_y(f(x), f(y)) \leq L d_x(x, y) \quad \forall x, y \in X$$

$f$  is an isometric if  $\forall x, y \in X$  it holds that  $d_Y(f(y), f(x)) = d_x(x, y)$ .

### 1.5 Compact sets

**Definition 1.5.1.** Let  $(X, d)$  be a metric space and  $A \subset X$ .

1. an open cover of  $A$  is a collection  $\{U_i\}_{i \in I}$  where  $I \neq \emptyset$  is an arbitrary index set of open subsets of  $X$  s.t.  $A \subset \bigcup_{i \in I} U_i$ .
2.  $A$  is compact if every open cover of  $A$  contains a finite subcover i.e. there is  $N \in \mathbb{N}$  and indices  $i_1, \dots, i_N$  such that

$$A \subset U_1 \cup \dots \cup U_N$$

3.  $A$  is sequentially compact if every sequence in  $A$  has a convergence subsequence in  $A$ .
4.  $A$  is called precompact or totally bounded if  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  and  $\exists x_1, \dots, x_N \in X$  such that  $A \subset \bigcup_{i=1}^N B_\varepsilon(x_i)$ .

**Theorem 1.5.2.** Let  $(X, d)$  be a metric space and  $A \subset X$ . The following are equivalent:

1.  $A$  is compact
2.  $A$  is sequentially compact
3.  $(A, d)$  is complete and  $A$  is precompact.

*Remark 1.5.3.* If  $A$  is precompact, then  $\overline{A}$  is precompact. Further, if  $(X, d)$  is complete and  $A \subset X$  then  $A$  is precompact  $\Leftrightarrow \overline{A}$  is compact.

Recall:  $A$  compact  $\Rightarrow$  bounded and closed and  $f : X \rightarrow Y$  continuous with  $A \subset X$  compact, then  $f(A)$  is compact as well. Further, if  $f : A \rightarrow \mathbb{R}$  is continuous and  $A$  is compact, then

$$\exists x_1, x_2 \in A \text{ s.t. } f(x_1) \leq f(x) \leq f(x_2) \forall x \in A$$

Theorem of Heine-Borel:  $A \subset \mathbb{R}^n$  is compact iff  $A$  is closed and bounded.

## 1.6 Theorem of Baire

**Theorem 1.6.1.** Let  $(X, d)$  be a complete metric space and  $\forall n \in \mathbb{N}$  consider  $U_n \subset X$  open and dense. Then

$$\bigcap_{n \in \mathbb{N}} U_n$$

is dense in  $X$ .

*Remark 1.6.2.* 1. Completeness is in general necessary. Consider  $(\mathbb{Q}, d)$  and  $d(x, y) = |x - y|$ . Define a sequence  $x_n$  such that  $\mathbb{Q} = \{x_n \mid n \in \mathbb{N}\}$ . Take  $U_n = \mathbb{Q} \setminus \{x_n\}$  which is open and dense. Then

$$\bigcap_{n \in \mathbb{N}} U_n = \emptyset$$

**Corollary 1.6.3.** Let  $(X, d)$  be a complete metric space. Let  $\forall n \in \mathbb{N}$ ,  $A_n \subset X$  be closed and

$$X = \bigcup_{n \in \mathbb{N}} A_n$$

Then  $\exists N \in \mathbb{N}$  s.t.  $A_N$  has an interior point.

*Remark 1.6.4.* Theorem 1.6.1 is also called Baire category theory.

- In a metric space  $(X, d)$   $A \subset X$  is called nowhere dense if  $\overline{A}$  has no interior points.
- $A$  is called of first category if  $\exists (M_n)_{n \in \mathbb{N}}$  where  $M_n \subset A$  nowhere dense s.t.  $A = \bigcup_{n \in \mathbb{N}} M_n$
- $A$  is called of second category if it is not of first category

Hence the theorem of Baire implies that every complete metric space is of second category.

## 2 Normal spaces and Banach spaces

Let  $X$  be a  $\mathbb{K}$ -vector space where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .



## 2.1 definitions

**Definition 2.1.1.** A map  $\|\cdot\| : X \rightarrow \mathbb{R}$  is called a norm on  $X$  if

1.  $\forall x \in X, \|x\| \geq 0$  and  $\|x\| = 0$  iff  $x = 0$
2.  $\forall \lambda \in \mathbb{K}$  and  $\forall x \in X$  it holds that  $\|\lambda x\| = |\lambda| \cdot \|x\|$
3.  $\forall x, y \in X$  it holds  $\|x + y\| \leq \|x\| + \|y\|$

The pair  $(X, \|\cdot\|)$  is called a normed space.

$p : X \rightarrow \mathbb{R}$  is called a seminorm if  $p(x) \geq 0 \forall x \in X$  and 2. and 3. are also satisfied.

**Example 2.1.2.** 1.  $C^0([0, 1]; \mathbb{R})$  with  $\|f\|_\infty = \max_{x \in [0, 1]} |f(x)|$

2. more general for a compact metric space  $K$ :  $C^0(K, \mathbb{R})$  with  $\|f\|_\infty = \max_{x \in K} |f(x)|$

3.  $C^1([0, 1]; \mathbb{R})$  with  $p(f) = \max_{x \in [0, 1]} |f'(x)|$

4.  $\Omega \subset \mathbb{R}^n$  measurable.  $L^1(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ integrable} \}$  with

$$p : L^1(\Omega) \rightarrow \mathbb{R} : p(f) = \int_{\Omega} |f(x)| dx$$

then  $p$  is a seminorm.

*Remark 2.1.3.* Any normed space is a metric space via

$$d(x, y) = \|x - y\|$$

All concepts from chapter 1 apply.

**Lemma 2.1.4.** Let  $(X, \|\cdot\|)$  be a normed space. Then  $X$  is called separable iff  $\exists A \subset X$  countable such that s.t.  $\overline{\text{span}\{A\}} = X$  where  $\text{span}\{A\} = \{\sum_{i=1}^n \lambda_i x_i\}$  with  $n \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{K}$  and  $x_i \in A$ . Here the closure is defined w.r.t the norm.

**Definition 2.1.5.** A complete normed space is called a Banach space.

## 2.2 Example: $l^p$ -spaces

We consider the vector space  $\mathbb{K}^{\mathbb{N}}$  of sequences in  $\mathbb{K}$ . Let  $x = (x_n)_{n \in \mathbb{N}}$  and  $y = (y_n)_{n \in \mathbb{N}}$ . Define  $x + y = (x_n + y_n)_{n \in \mathbb{N}}$  and  $\lambda x = (\lambda x_n)_{n \in \mathbb{N}}$ .

For  $x \in \mathbb{K}^{\mathbb{N}}$  define

$$\|x\|_{l^p} = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

for  $1 \leq p < \infty$  and

$$\|x\|_{l^\infty} = \sup_{n \in \mathbb{N}} |x_n|$$

else.

Define  $l^p = \{x = (x_n)_{n \in \mathbb{N}} : \|x\|_{l^p} < \infty\}$  for  $1 \leq p \leq \infty$ . We find that  $l^p$  is a subspace of  $\mathbb{K}^\mathbb{N}$  and  $l^p$  is a normed space (for the triangle inequality use the Hölder inequality).

**Theorem 2.2.1.** *For  $1 \leq p \leq \infty$   $l^p$  is a Banach space.*

**Lemma 2.2.2.** *For finite  $p$ ,  $l^p$  is separable while  $l^\infty$  is not.*

### 2.3 Finite dimensional normed spaces

Let  $X$  be a vector space over  $\mathbb{K}$ .  $\exists e_1, \dots, e_n \in X$  s.t.

$$\forall x \in X; \exists \lambda_1, \dots, \lambda_n \in \mathbb{K} : x = \sum_{i=1}^n \lambda_i x_i$$

For  $p \in [1, \infty)$  we define

$$\|x\|_p = \left( \sum_{i=1}^n |\lambda_i|^p \right)^{1/p}$$

and for  $p = \infty$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |\lambda_i|$$

**Definition 2.3.1.** Two norms are equivalent in that

$$\alpha \|\cdot\|_1 \leq \|\cdot\|_2 \leq \beta \|\cdot\|_1$$

**Theorem 2.3.2.** *In a finite dimensional space, all norms are equivalent.*

**Theorem 2.3.3.** *Finite dimensional normed spaces are Banach spaces.*

### 2.4 On the closure of $\overline{B_1(0)}$

**Lemma 2.4.1** (Lemma of Riesz, Lemma of the almost orthogonal element). *Let  $X$  be a normed space.  $U \subset X$  a closed subspace of  $X$  s.t.  $U \neq X$ . Then  $\forall \lambda \in (0, 1) \exists x_\lambda \in X$  s.t.  $\|x_\lambda\| = 1$  and  $\text{dist}(x_\lambda, U) \geq \lambda$ .*

**Theorem 2.4.2.** *In a normed space  $X$ ,  $\overline{B_1(0)}$  is compact iff  $X$  is finite dimensional.*

### 3 A question from approximation theory

#### 3.1 Theorem of Stone-Weierstrass

Let  $X$  be a compact metric space. Then  $(C^0(X), \mathbb{K}), \|\cdot\|_\infty$ , where  $\|f\|_\infty = \max_{x \in X} |f(x)|$  is a Banach space.

Which property of  $A \subset C^0(X, \mathbb{K})$  ensures that  $A$  is dense.

**Definition 3.1.1.**  $A \subset C^0(X, \mathbb{K})$  is called subalgebra, if  $\forall f, g \in A$

1.  $\lambda f + \mu g \in A$  (subspace)
2.  $f \cdot g \in A$

**Example 3.1.2.** •  $\{p : [0, 1] \rightarrow \mathbb{R}\}$  is a subalgebra of  $C^0([0, 1]; \mathbb{R})$ .

- $\{f : [-1, 1] \rightarrow \mathbb{R}; f \text{ continuous and even}\}$  is a subalgebra.

*Remark 3.1.3.* If  $A$  is a subalgebra, then  $\overline{A}$  is also a subalgebra.

**Definition 3.1.4.** Let  $A \subset C^0(X)$  be a subalgebra.

1.  $A$  is called unital if  $1 \in A$
2.  $A$  separates point if  $x, y \in X, x \neq y, \exists f \in A$  s.t.  $f(x) \neq f(y)$ .
3. (if  $\mathbb{K} = \mathbb{C}$ )  $A$  is stable under conjugation if from  $f \in A$  we conclude that also  $\overline{f} \in A$ .

*Remark 3.1.5.* If  $A$  is unital then all constant functions are in  $A$ .

**Lemma 3.1.6.** Consider  $f : [-1, 1] \rightarrow \mathbb{R}$  where  $f(x) = |x|$ . Then  $\exists$  sequence of polynomials  $(p_n)_{n \in \mathbb{N}}$  s.t.

$$p_n \rightarrow f$$

uniformly in  $[-1, 1]$ .

**Lemma 3.1.7.** Let  $A \subset C^0(X, \mathbb{R})$  be a unital subalgebra. Then

1. if  $f \in A$  then  $|f| \in \overline{A}$ .
2. if  $f, g \in A$  then  $\max\{f, g\} \in \overline{A}$  and  $\min\{f, g\} \in \overline{A}$

**Theorem 3.1.8** (Stone-Weierstrass). Let  $A$  be a compact metric space.  $A \subset C^0(X, \mathbb{K})$  is a unital subalgebra that separates points and if  $\mathbb{K} = \mathbb{C}$  is stable under conjugation, then  $A$  is dense in  $C^0(X, \mathbb{K})$  w.r.t  $\|\cdot\|_\infty$ .

### 3.2 Applications

**Theorem 3.2.1** (Theorem of Weierstraß). *Let  $[a, b]$  be a compact interval in  $\mathbb{R}$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $\varepsilon > 0$ . Then  $\exists p : [a, b] \rightarrow \mathbb{R}$  a polynomial s.t.*

$$\|p - f\|_\infty = \sup_{x \in [a, b]} |p(x) - f(x)| < \varepsilon$$

**Definition 3.2.2.** A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is periodic if

$$f(x + t) = f(x)$$

for a  $t \in \mathbb{R}$  and all  $x \in \mathbb{R}$ .

*Remark 3.2.3.* If  $f$  is periodic with period  $t$  then  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$  where  $\tilde{f}(x) = f(t \frac{x}{2\pi})$  is periodic of period  $2\pi$ .

Consider  $C_{2\pi}^0(\mathbb{R}, \mathbb{C})$  the space of continuous  $2\pi$ -periodic functions. We consider the span of  $\{e^{ikx} = \cos(kx) + i \sin(kx), k \in \mathbb{Z}\}$ .

**Definition 3.2.4.** A trigonometric polynomial is a function  $f : \mathbb{R} \rightarrow \mathbb{C}$

$$f(x) = \sum_{k=-N}^N c_k \cdot e^{ikx}$$

with  $c_k \in \mathbb{C}$

**Theorem 3.2.5** (Approximation of periodic functions). *Trigonometric polynomials are dense in  $(C_{2\pi}^0(\mathbb{R}, \mathbb{C}), \|\cdot\|_\infty)$*

#### Application to neural networks

The simplest case of a neural network has  $d$  inputs  $x_1, \dots, x_d$  and one output  $Z$  called a *feed forward* network. Each input influences the output and  $x_i$  might have a weight  $\alpha_i$  associated to it. The output is a function in  $x = (x_1, \dots, x_d)$  and the weights  $\alpha = (\alpha_1, \dots, \alpha_d)$ . For instance, the output is often of the form

$$Z = \sum_{i=1}^d \alpha_i x_i + b$$

where  $b$  is the bias of the network. To make the network slightly stronger, we add a intermediate layer  $y = (y_1, \dots, y_r)$  where each  $x_i$  is connected to each  $y_j$  with the associated weight  $\gamma_{i,j}$ . The  $y$  layer (often called activation) is the connected to the output  $Z$  as above

with weights  $\alpha_j$ . We introduce the relation

$$y_j = \Phi\left(\sum_{i=1}^d \gamma_{j,i} x_i + b\right)$$

for a measurable function  $\Phi$ . Lastly, the output is then given by

$$Z = \sum_{j=1}^r \alpha_j y_j$$

**Definition 3.2.6.** 1.  $A^d = \{a : \mathbb{R}^d \rightarrow \mathbb{R} : a(x) = w^T x + b\}$  where  $w \in \mathbb{R}^d$  and  $b \in \mathbb{R}$ .

2. given  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  measurable  $d \in \mathbb{N}$  define  $\Sigma^d(\Phi) = \{f : \mathbb{R}^d \rightarrow \mathbb{R} : f(x) = \sum_{j=1}^N \alpha_j \Phi(a_j(x)) \text{ with } N \in \mathbb{N}, \alpha_j \in \mathbb{R}, a_j \in A^d\}$  as the set of single hidden layer feed forward networks.

3. A squashing function is a measurable non-decreasing function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  s.t.  $\lim_{x \rightarrow -\infty} \Phi(x) = 0$  and  $\lim_{x \rightarrow \infty} \Phi(x) = 1$ .

**Theorem 3.2.7** (Universal Approximation theorem of Hornik-Stinchcombe-White). *Let  $\Phi$  be a squashing function  $K \subset \mathbb{R}^d$  compact  $f : K \rightarrow \mathbb{R}$  continuous and  $\varepsilon > 0$ . Then  $\exists g \in \Sigma^d(\Phi)$  s.t.*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon$$

## 4 Continuous linear maps

$(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$  are  $K$ -Vector spaces with  $K = \mathbb{R}$  or  $K = \mathbb{C}$ .  $T : X \rightarrow Y$  is called linear if

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

### 4.1 Continuity of linear maps

**Definition 4.1.1.** Let  $T : X \rightarrow Y$  be linear. Then  $T$  is bounded if  $\exists C > 0$  s.t.

$$\|Tx\|_Y \leq C \|x\|_X \quad \forall x \in X$$

or equivalently

$$\sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} \leq C$$

which is also equivalent to

$$\sup_{x \in X, \|x\|_X=1} \|Tx\|_Y \leq C$$

**Theorem 4.1.2.** For  $T : X \rightarrow Y$  linear, the following are equivalent:

1.  $T$  is continuous
2.  $T$  is continuous in 0
3.  $T$  is bounded

**Lemma 4.1.3.** Let  $X$  have infinite dimension. Then  $\exists T : X \rightarrow \mathbb{K}$  linear and not bounded.

**Definition 4.1.4.** Define  $L(X, Y)$  as the set of linear continuous ( $\Leftrightarrow$  bounded) maps from  $X$  to  $Y$ . With the usual addition  $((T_1 + T_2)(x) = T_1(x) + T_2(x))$  and the scalar multiplication  $((\lambda(T))(x) = \lambda T(x))$  this is a vector space.

If  $X = Y$  we write  $L(X)$ . For  $T \in L(X, Y)$

$$\ker T = \{x \in X : Tx = 0\}$$

and

$$\Im(T) = \{y \in Y : \exists x \in X : Tx = y\}$$

## 4.2 Operator norm and dual space

**Theorem 4.2.1.** Let  $X \neq \{0\}$ .

- $L(X, Y)$  with the operator norm  $\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X=1} \|Tx\|_Y$  is a normed space. We have

$$\|Tx\|_Y \leq \|T\| \|x\|_X$$

- If  $Y$  is a Banach space then  $L(X, Y)$  is a Banach space.

**Definition 4.2.2.** For a normed space  $(X, \|\cdot\|_\infty)$  we define the dual space  $X' = L(X, \mathbb{K})$ .

*Remark 4.2.3.*  $X'$  is a Banach space.

## 4.3 Neumann series

**Lemma 4.3.1.** Let  $X, Y, Z$  be three normed spaces. Let  $T \in L(X, Y)$  and  $S \in L(Y, Z)$ . Then  $S \circ T \in L(X, Z)$  and

$$\|S \circ T\| \leq \|S\| \|T\|$$

Let  $T : X \rightarrow Y$  be linear, bounded and bijective. Then  $\exists T^{-1} : Y \rightarrow X$  linear.

**Definition 4.3.2.** Let  $X, Y$  be normed spaces.

1.  $T \in L(X, Y)$  is bijective such that  $T^{-1} \in L(Y, X)$  then  $T$  is called an isomorphism
2.  $X, Y$  are called isomorph if there is  $T : X \rightarrow Y$  isomorphism.
3.  $T \in L(X, Y)$  is called an Isometry if  $\|Tx\| = \|x\|$ .
4.  $X, Y$  are called isometric isomorph if  $\exists T \in L(X, Y)$  an isomorphism that is also an isometry.

*Remark 4.3.3.* The identity  $I_x : X \rightarrow X$  with  $x \mapsto x$  is in  $L(X)$ . Then  $T \in L(X)$  is an isomorphism iff  $\exists S \in L(X)$  s.t.  $S \circ T = I_x$  and  $T \circ S = I_x$

Let  $T \in L(X)$  s.t.  $\|T\| < 1$ . Define  $T^0 = I_x$ ,  $T^n = T \circ T^{n-1}$ . Obviously  $T^n \in L(X)$  for all  $n$ . Now,

$$\left( \sum_{k=0}^n T^k \right)_{n \in \mathbb{N}} \subset L(X)$$

is a Cauchy sequence w.r.t. the operator norm. Hence, if  $X$  is a Banach-Space, so is  $L(X)$  and thus the series converges to a  $S \in L(X)$ . Furthermore

$$\sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1 - \|T\|}$$

Finally, we can also note that  $S = (I_x - T)^{-1}$ .

**Theorem 4.3.4** (Neumann series). *Let  $X$  be a Banach-Space,  $T \in L(X)$  with  $\|T\| < 1$ . The  $I_x - T$  is an isomorphism and*

$$(I_x - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

*is in  $L(X)$ . This is called the Neumann series.*

#### 4.4 The dual space of $l^p$

We only deal with  $1 \leq p < \infty$ .

**Theorem 4.4.1.** *Let  $q \in (1, \infty]$  be s.t.  $\frac{1}{p} + \frac{1}{q} = 1$ . Then the dualspace  $(l^p)'$  is isometric isomorph to  $l^q$ .*

## 5 Theorem of Hahn-Banach

Let  $X$  be a vector space,  $X \neq \{0\}$  over  $\mathbb{K} = \mathbb{R}$ .

### 5.1 Extension Theorem

Given  $U \subset X$  subspace,  $l : U \rightarrow \mathbb{R}$  linear, is there  $L : X \rightarrow \mathbb{R}$  linear such that  $L|_U = l$ ? For this we need Zorn's Lemma:

**Definition 5.1.1.** Let  $M \neq \emptyset$  be a set and  $\leq$  be a partial order on  $M$ , i.e.  $\leq$  satisfies

1. reflexiv:  $x \leq x \ \forall x \in M$
2. antisymmetric:  $x \leq y$  and  $y \leq x \Rightarrow x = y$
3. transitivity  $x \leq y, y \leq z \Rightarrow x \leq z$

Then

- $A \subset M$  is called chain of totally ordered if  $\forall x, y \in A$  either  $x \leq y$  or  $y \leq x$
- $b \in M$  is an upper bound for a chain  $A$  if  $a \leq b$  for all  $a \in A$
- $m \in M$  is called maximal element if

$$\forall x \in M \text{ s.t. } m \leq x \Rightarrow x = m$$

**Lemma 5.1.2** (Zorn). *Let  $M \neq \emptyset$  and  $\leq$  be a partial order on  $M$ . If every chain in  $M$  has an upper bound in  $M$ , then there is a maximal element.*

**Definition 5.1.3.** Let  $X$  be a vector space.  $p : X \rightarrow \mathbb{R}$  is called sublinear if

1.  $p(\lambda x) = \lambda p(x)$  for all  $x \in X, \lambda \geq 0$
2.  $p(x + y) \leq p(x) + p(y)$  for all  $x, y \in X$

**Theorem 5.1.4** (Extension theorem of Hahn-Banach). *Let  $X$  be a vector space over  $\mathbb{R}$ ,  $U \subset X$  a subspace and  $U \neq X$ . Let  $p : X \rightarrow \mathbb{R}$  be a sublinear and  $l : U \rightarrow \mathbb{R}$  be linear s.t.  $l(x) \leq p(x) \ \forall x \in U$ . Then  $\exists L : X \rightarrow \mathbb{R}$  linear s.t.  $L(x) \leq p(x) \ \forall x \in X$  and  $L(x) = l(x) \ \forall x \in U$ .  $L$  is called extension of  $l$ .*

#### Consequences for normed spaces

**Theorem 5.1.5.** *Let  $(X, \|\cdot\|_X)$ ,  $U \subset X$  a subspace of  $X$ , with  $U \neq X$ . Let  $u' \in U' = L(U, \mathbb{R})$ . Then  $\exists x' \in X'$  s.t.  $\|x'\|_{X'} = \|u'\|_{U'}$  such that  $x'(u) = u'(u) \ \forall u \in U$ .*

**Corollary 5.1.6.** *Let  $(X, \|\cdot\|_X)$ ,  $U \subset X$  be a subspace of  $X$  and  $x_0 \in X$  s.t.  $\text{dist}(x_0, U) > 0$ . Then  $\exists x' \in X'$  s.t.  $x'|_U = 0 \ \forall u \in U$  and  $x'(x_0) = \text{dist}(x_0, U)$  with  $\|x'\|_{X'} = 1$ .*

**Corollary 5.1.7.** *Let  $(X, \|\cdot\|_X)$  and  $x_0 \in X$ .*



1. if  $x_0 \neq 0$  then  $\exists F \in X'$  with  $\|F\|_{X'} = 1$  and  $F(x_0) = \|x_0\|_X$ . In particular, for  $x \in X$

$$\|x\|_X = \sup_{F \in X', \|F\|_{X'}=1} |F(x)|$$

2. If  $F(x_0) = 0$  for all  $F \in X'$ , then  $x_0 = 0$ . In particular,  $X'$  separates points of  $X$ .
3.  $U \subset X$  subspace. Then  $U$  is dense in  $X$  iff if for  $x' \in X'$  s.t.  $x'|_U = 0$  it follows  $x' = 0$ .

## 5.2 Separation Theorems

**Definition 5.2.1.** Let  $X$  be a vectorspace over  $\mathbb{R}$ .  $A \subset X$  is called convex, if

$$\forall x, y \in A, \lambda x + (1 - \lambda)y \in A, \forall \lambda \in [0, 1]$$

**Lemma 5.2.2.** Let  $C \subset X$  open and convex with  $O \in C$ . Define  $p_C : X \rightarrow \mathbb{R}$  such that  $p_C(x) = \inf\{\lambda > 0 \text{ s.t. } \frac{x}{\lambda} \in C\}$ . This is called the Minkowski functional. Then  $p_C$  is sublinear and  $C = \{x \in X : p_C(x) < 1\}$ .

**Lemma 5.2.3.** Let  $(X, \|\cdot\|)$  be a normed space and  $A \subset X$  be convex and open,  $A \neq \emptyset$  and  $x_0 \in X \setminus A$ , then  $\exists F \in X'$  s.t.

$$F(x) < F(x_0) \quad \forall x \in A$$

**Definition 5.2.4.** Let  $X \neq \{0\}$  be a  $\mathbb{R}$ -vectorspace.

1.  $H = \{x \in X : f(x) = \alpha\}$  with  $f : X \rightarrow \mathbb{R}$  linear,  $\alpha \in \mathbb{R}$
2.  $A, B \subset X$  are separated by an affine hyperplane  $H$  if  $H = \{f = \alpha\}$  and  $f(a) \leq \alpha \leq f(b) \quad \forall a \in A \quad \forall b \in B$ .
3.  $A, B \subset X$  are strictly separated by an affine Hyperplane  $H$  if  $\exists \varepsilon > 0$  s.t.  $f(a) + \varepsilon \leq \alpha \leq f(b) - \varepsilon$ .

**Theorem 5.2.5** (Separation Theorem of Hahn-Banach). Let  $(X, \|\cdot\|)$ ,  $A, B$  convex,  $A \neq \emptyset$ ,  $B \neq \emptyset$ ,  $A \cap B = \emptyset$  and  $A$  or  $B$  should be open.. Then  $\exists F \in X'$  and  $\delta \in \mathbb{R}$  s.t.

$$F(a) \leq \delta \leq F(b) \quad \forall a \in A, b \in B$$

**Theorem 5.2.6.** Let  $(X, \|\cdot\|)$ ,  $A, B$  convex subsets  $A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$ . Let  $A$  be closed and  $B$  be compact. Then  $\exists F \in X', \exists \varepsilon > 0$  s.t.  $F(a) + \varepsilon \leq F(b) - \varepsilon \quad \forall a \in A, b \in B$ .

## 6 Hilbert Spaces

Let  $X$  be a vector space over  $\mathbb{K}$  where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ .

### 6.1 Inner product space

**Definition 6.1.1.** A map  $\langle \cdot, \cdot \rangle: X \times X \rightarrow \mathbb{K}$  is an inner product on  $X$ , if

1.  $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$
2.  $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
3.  $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4.  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

The pair  $(X, \langle \cdot, \cdot \rangle)$  is called an inner product space also called a pre-Hilbert-space.

An inner product is a symmetric bilinear form if  $\mathbb{K} = \mathbb{R}$  and a sesquilinear form if  $\mathbb{K} = \mathbb{C}$ .

**Theorem 6.1.2** (Cauchy-Schwartz-inequality). *In an inner product space we have*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}$$

**Theorem 6.1.3.** *For an inner product space  $X$  we define  $\| \cdot \| : X \rightarrow [0, \infty)$  by  $\|x\| = \sqrt{\langle x, x \rangle}$ . This is a norm.*

**Definition 6.1.4.** Let  $X$  be an inner product space. Then  $x, y \in X$  are orthogonal if  $\langle x, y \rangle = 0$ . This is written as  $x \perp y$ .

**Corollary 6.1.5.** *If  $x \perp y$ , then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

**Theorem 6.1.6.** *A normed space is an inner product space, iff  $\forall x, y \in X$  the norm satisfies*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

### 6.2 Hilbert spaces

**Definition 6.2.1.** Is an inner product space complete w.r.t. to the induced norm, we call it Hilbert space.

**Theorem 6.2.2** (projection theorem). *Let  $X$  be a Hilbert space,  $A \subset X$  non-empty, convex and closed. Then  $\forall x \in X$*

$$\exists! y \in A \text{ s.t. } \|x - y\| = \text{dist}(x, A)$$

*$y$  is called the best approximation or projection of  $x$  in  $A$ .*

**Theorem 6.2.3** (Characterisation of the best approximation). *Let  $X$  be an inner product space,  $A \subset X$ ,  $A \neq \emptyset$  and convex and  $x \in X$ . Then  $y$  is the best approximation of  $x$  in  $A$  iff*

$$\Re \langle x - y, z - y \rangle \leq 0 \quad \forall z \in A$$

**Definition 6.2.4.** Let  $X$  be an inner product space,  $A \subset X$ , then

$$A^\perp = \{x \in X : x \perp y \quad \forall y \in A\}$$

the orthogonal complement of  $A$ .

*Remark 6.2.5.*  $A^\perp$  is a closed subspace. If  $(x_n)_{n \in \mathbb{N}} \subset A^\perp$ ,  $x_n \rightarrow x$  in  $X$ ,  $\forall n \in \mathbb{N}$  we have  $\langle x_n, y \rangle = 0 \quad \forall y \in A$ . Moreover  $A \subset (A^\perp)^\perp$ .

**Theorem 6.2.6.** *Let  $X$  be a Hilbert space,  $U \subset X$  closed subspace. Then*

$$\forall x \in X \quad \exists! u \in U \text{ s.t. } \|x - u\| = \text{dist}(x, U) = \inf_{z \in U} \|z - u\|$$

*We have  $x - u \in U^\perp$  and  $X = U \oplus U^\perp$ , meaning that  $x = u + v$ ,  $u \in U$ ,  $v \in U^\perp$  uniquely. The  $u$  is called the orthogonal projection of  $x$  in  $U$ .*

**Theorem 6.2.7** (Riesz-Fréchet). *Let  $X \neq \{0\}$  be a Hilbert space.  $\forall F \in X' \quad \exists! y \in X$  s.t.  $F(x) = \langle x, y \rangle$ . Moreover,  $\|F\|_{X'} = \|y\|_X$ . Equivalently*

$$J : X \rightarrow X', \quad (Jy)(x) = \langle x, y \rangle$$

*is a bijective, anti-linear isometry. In particular, if  $X'$  is a Hilbert space, the dual is also a Hilbert space.*

### 6.3 Orthonormal systems

Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space.

**Definition 6.3.1.** Let  $I \neq \emptyset$  be an index set. A family of vectors  $(e_k)_{k \in I} \subset X$  is called an orthonormal system (ONS) if

$$\langle e_i, e_j \rangle = \delta_{i,j}$$

**Theorem 6.3.2** (Schmidt Orthogonalisation theorem). *Let  $\{x_i : i \in I\} \subset X, I \subset \mathbb{N}$  be linearly independent vectors. Then  $\exists$  ONS  $\{e_i : i \in I\}$  s.t.*

$$\text{span}\{x_i : i \in I\} = \text{span}\{e_i : i \in I\}$$

**Lemma 6.3.3** (Bessel's inequality). *Let  $\{e_1, \dots, e_n\}$  be an ONS.  $Y = \text{span}\{e_1, \dots, e_n\}$ . Then  $\forall x \in X$*

$$\inf_{y \in Y} \|x - y\|^2 = \|x - \sum_{i=1}^n \langle x, e_i \rangle e_i\|^2 = \|x\|^2 - \sum_{i=1}^n |\langle x, e_i \rangle|^2 \geq 0$$

**Definition 6.3.4.** If  $I \subset \mathbb{N}$ ,  $(e_n)_{n \in I}$  ONS, then  $\langle x, e_n \rangle$  is called the  $n$ -th Fourier coefficient of  $x$ . W.r.t.  $(e_n)_{n \in I}$ .

**Definition 6.3.5.** An ONS  $(e_n)_{n \in \mathbb{N}}$ ,  $I \subset \mathbb{N}$  is called complete in  $X$  if

$$\overline{\text{span}\{e_n : n \in I\}} = X$$

**Theorem 6.3.6.** *Any separable Hilbert space  $X$  has a complete ONS.*

**Lemma 6.3.7.** *Let  $X$  be a Hilbert space,  $(e_n)_{n \in \mathbb{N}}$  an ONS. Then  $\exists y \in X$  s.t.  $y = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$ .*

**Theorem 6.3.8.** *Let  $X$  be a Hilbert space of infinite dimension,  $(e_n)_{n \in \mathbb{N}}$  an orthonormal system. Then the following are equivalent.*

1.  $(e_n)_{n \in \mathbb{N}}$  is complete
2. if  $x \in X$  s.t.  $\langle x, e_n \rangle = 0 \forall n \in \mathbb{N}$ , then  $x = 0$
3.  $\forall x \in X, x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$  (Fourier series of  $x$ )
4.  $\forall x \in X, \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$ .

**Corollary 6.3.9.** *Any separable infinite-dimensional Hilbert space is isometrically isomorphic to  $\ell^2$ .*

## 7 Spectral theorem for self-adjoint compact operators

We only deal with Hilbert spaces.

### 7.1 Adjoint in Hilbert spaces

Let  $(X, \langle \cdot, \cdot \rangle)$ ,  $(Y, \langle \cdot, \cdot \rangle)$ ,  $T \in L(X, Y)$ . Let  $y \in Y$ . Consider the map

$$X \ni x \mapsto \langle Tx, y \rangle_Y$$

This map is linear and bounded.

$$|\langle Tx, y \rangle_Y| \stackrel{CS}{\leq} \|Tx\|_Y \|y\|_Y \leq \|T\| \|x\|_X \|y\|_Y$$

Thus it is an element of  $X'$ . By the theorem of Riesz-Fréchet

$$\exists! T^*y \in X \text{ s.t. } \langle x, T^*y \rangle_X = \langle Tx, y \rangle_Y \quad \forall x \in X$$

This defines a map  $T^* : Y \rightarrow X$  with  $y \mapsto T^*y$ .

**Definition 7.1.1.**  $T^*$  is the Hilbert space adjoint of  $T$ .

**Lemma 7.1.2.**  $T^* \in L(Y, X)$  and  $\|T^*\| = \|T\|$ .

**Lemma 7.1.3.** Let  $(X, \langle \cdot, \cdot \rangle_X)$ ,  $(Y, \langle \cdot, \cdot \rangle_Y)$ ,  $(Z, \langle \cdot, \cdot \rangle_Z)$  be Hilbert spaces. Let  $T \in L(X, Y)$ ,  $S \in L(Y, Z)$  and  $\lambda \in \mathbb{K}$ . Then

1.  $(S \circ T)^* = T^* S^*$
2.  $(\lambda T)^* = \bar{\lambda} T^*$
3.  $(T^*)^* = T$

**Definition 7.1.4.** Let  $(X, \langle \cdot, \cdot \rangle_X)$  be a Hilbert space and  $T \in L(X)$ .  $T$  is called self-adjoint if  $T^* = T$

**Lemma 7.1.5.** • If  $\mathbb{K} = \mathbb{C}$ ,  $T$  is self-adjoint  $\Leftrightarrow \langle Tx, x \rangle_X \in \mathbb{R} \quad \forall x \in X$

- If  $T$  is self-adjoint, then  $\|T\| = \sup_{x \in X, \|x\|_X=1} |\langle Tx, x \rangle|$

### 7.2 compact operators

Here  $X, Y$  can be only Banach spaces and  $X, Y \neq \{0\}$ .

**Definition 7.2.1.**  $f : X \rightarrow Y$  is compact if  $f$  maps bounded sets in precompact sets.

**Lemma 7.2.2.** Let  $T : X \rightarrow Y$  be linear. Then  $T$  is compact iff  $T(B_1(0))$  is precompact in  $Y$ .

**Notation:**  $K(X, Y) = \{T : X \rightarrow Y \text{ linear and compact}\}$  and  $K(X) = K(X, X)$ .

*Remark 7.2.3.*  $T \in K(X, Y) \Rightarrow T \in L(X, Y)$ .

**Lemma 7.2.4.** 1.  $T \in L(X, Y)$ ,  $S \in L(Y, Z)$ . If  $T$  or  $S$  is compact, then the composition is compact.

2.  $K(X, Y)$  is a closed subspace of  $L(X, Y)$ . In particular  $K(X, Y)$  is a Banach space.

**Definition 7.2.5.** • Let  $H$  be a Hilbert space and  $T \in L(X)$ . Then  $T$  is called self-adjoint if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in X$$

• Let  $X, Y$  be Banach spaces then,  $T \in L(X, Y)$  compact  $\Leftrightarrow T(B_1(0))$  is precompact.

**Lemma 7.2.6.**  $T \in L(X, Y)$  is compact iff  $\forall (x_n)_{n \in \mathbb{N}} \subset X$  bounded  $(T(x_n))_{n \in \mathbb{N}}$  admits a convergent subsequence.

### 7.3 Spectrum

Let  $X$  be a Banach space.

**Definition 7.3.1.** Let  $T \in L(X)$ .

• the resolvent set of  $T$  is

$$\rho(T) = \{\lambda \in \mathbb{K} : (\lambda \cdot Id - T)^{-1} \in L(X)\} \subset \mathbb{K}$$

while  $\sigma(T) = \mathbb{K} \setminus \rho(T)$  is the spectrum of  $T$ .

• the resolvent map  $R : \rho(T) \rightarrow L(X)$  is defined by  $\lambda \mapsto (\lambda Id - T)^{-1}$

• the spectrum of  $T$  is divided into

$$\sigma(T) = \sigma_p(T) \cup \sigma_C(T) \cup \sigma_r(T)$$

where

- $\sigma_P(T) = \{\lambda \in \sigma(T) : \ker(\lambda Id - T) \neq \{0\}\}$  is the point spectrum
- $\sigma_C(T) = \{\lambda \in \sigma(T) \setminus \sigma_P(T) : \text{Im}(\lambda Id - T) \neq X \text{ but } \overline{\text{Im}(\lambda Id - T)} = X\}$
- $\sigma_r(T) = \sigma(T) \setminus (\sigma_P(T) \cup \sigma_C(T))$ .

• the elements of the point spectrum are called eigenvalues and  $x \in X \setminus \{0\} : (I\lambda Id - T)(x) = 0$  is called eigenvector associated to  $\lambda$ .

**Theorem 7.3.2.** For  $T \in L(X)$

1.  $\rho(T)$  is open.
2.  $\sigma(T)$  is compact and

$$\sup_{\lambda \in \sigma(T)} |\lambda| \leq \lim_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} = r(T)$$

is the spectral radius. In particular  $r(T) \leq \|T\|$

#### 7.4 Spectral theorem for self-adjoint compact operators

Let  $X$  be a Hilbert space.

**Lemma 7.4.1.** Let  $T \in K(X)$  self-adjoint. Then  $\|T\|$  or  $-\|T\|$  is an eigenvalue of  $T$ .

**Lemma 7.4.2.** Let  $T \in L(X)$  be self-adjoint. Then all eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal.

**Lemma 7.4.3.** Let  $T \in L(X)$ . If  $M \subset X$  is a closed subspace s.t.  $TM \subset M$ , then  $M^\perp$  is invariant under  $T^*$ .

**Theorem 7.4.4.** Let  $X$  be a Hilbert space,  $T \in K(X)$  self-adjoint. Then  $\exists$  ONS  $(\phi_n)_{n \in I} \subset X$ ,  $I \subset \mathbb{N}$ , and  $\exists (\lambda_n)_{n \in I} \subset \mathbb{R}$  s.t.  $\forall x \in X$

$$Tx = \sum_{n \in I} \lambda_n \langle x, \phi_n \rangle \phi_n$$

In particular  $T\phi_n = \lambda_n \phi_n \forall n \in \mathbb{N}$ . If  $I$  is infinite, then  $\lambda_n \rightarrow 0$ .

**Corollary 7.4.5.** Let  $X$  be a separable Hilbert space with  $\dim X = \infty$  and  $T \in K(X)$  self-adjoint. Then  $\exists$  a complete ONS  $(e_n)_{n \in \mathbb{N}}$  of eigenvectors of  $T$ . In particular  $\forall x \in X$

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

with  $\lambda_n$  being the corresponding eigenvalue to  $e_n$ .

## 8 Reproducing kernel Hilbert spaces

### 8.1 Definitions

Here, we again use  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Further  $X \neq \emptyset$  is simply a set. Also

$$F(X, \mathbb{K}) = \{f : X \rightarrow \mathbb{K} \text{ a map}\}$$

This is a vector space.

**Definition 8.1.1.**  $H \subset F(X, \mathbb{K})$  is a reproducing kernel Hilbert space (RKHS) on  $X$  if

1.  $H$  is a subspace of  $F(X, \mathbb{K})$
2.  $\exists \langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{K}$  inner product, s.t.  $(H, \langle \cdot, \cdot \rangle)$  is a Hilbert space
3.  $\forall x \in X$  the linear map  $E_x : H \rightarrow \mathbb{K}$  with  $E_x(f) = f(x)$  (the evaluation operator) is well-defined and bounded.

Let  $\Omega \subset \mathbb{R}^n$  open,  $H = L^2(\Omega)$  is not a RKHS since evaluation at a point does not make sense for  $f \in L^2(\Omega)$ .

If  $H$  is a RKHS, the evaluation operator  $E_x \in H' \forall x \in X$ . For  $x \in X$ , by Riesz-Fréchet  $\exists! k_x \in H$  s.t.  $E_x(f) = \langle f, k_x \rangle \forall f \in H$ .

**Definition 8.1.2.** The function

$$K : X \times X \rightarrow \mathbb{K}$$

$$(x, y) \mapsto \langle k_y, k_x \rangle$$

is called reproducing kernel of  $H$ .

*Remark 8.1.3.* For  $x, y \in X$  and  $\mathbb{K} = \mathbb{C}$

$$K(x, y) = \langle k_y, k_x \rangle = \overline{\langle k_x, k_y \rangle} = \overline{K(y, x)}$$

while if  $\mathbb{K} = \mathbb{R}$  the kernel is symmetric. Further

$$\|E_x\|^2 = \|k_x\|^2 = \langle k_x, k_x \rangle = K(x, x) \geq 0$$

## 8.2 Theorem of Moore-Aronszajn

**Lemma 8.2.1.** Let  $H$  be a RKHS on  $X$  with kernel  $K$ . Then  $\forall n \in \mathbb{N}$  and  $\forall \{x_1, \dots, x_n\} \subset X$  the matrix

$$(K(x_i, x_j))^n$$

is a positive semidefinite matrix, i.e.

$$\sum_{i,j=1}^n \alpha_j K(x_j, x_i) \overline{\alpha_i} \geq 0 \quad \forall \alpha \in \mathbb{K}^n$$

**Theorem 8.2.2** (Moore-Aronszajn). Let  $X \neq \emptyset$ ,  $K : X \times X \rightarrow \mathbb{K}$  s.t.



1. if  $\mathbb{K} = \mathbb{C}$   $K(x, y) = \overline{K(y, x)}$  and if  $\mathbb{K} = \mathbb{R}$   $K(x, y) = K(y, x)$

2.  $K$  is positive semidefinite

Then there exists a (unique) RKHS on  $K$  with kernel  $K$ . Notation:  $H(K)$ .

### 8.3 An application

Interpolation: Let  $\{x_1, \dots, x_n\} \subset X$  be distinct points.  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  not necessarily distinct. Let  $H$  be a RKHS on  $X$ .

AIM: Find  $f \in H$  s.t. the least square error

$$J(f) = \sum_{i=1}^n |f(x_i) - \lambda_i|^2$$

is minimal at  $f$  and among all minimizers we want the one with minimal norm.

**Theorem 8.3.1.** Let  $H$  be a RKHS on  $X$ .  $\{x_1, \dots, x_n\} \subset X$  distinct points in  $X$ .  $A := (K(x_i, x_j))$  a  $n \times n$ -matrix.  $v = (\lambda_1, \dots, \lambda_n)^T \in \mathbb{K}^n$ . Then  $\exists w \in \mathbb{K}^n$  s.t.  $v - Aw \in \ker(A)$  and

$$H \ni f := \sum_{i=1}^n w_i k_{x_i}$$

satisfies

$$J(f) = \inf_{g \in H} J(g)$$

We have  $k_{x_i} = K(\cdot, x_i)$  and  $f$  is the unique minimizer of minimal norm.