Gae Summary

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Contents

1 Introduction

We will introduce four different models of game theory:

1. Static Model of Complete Information: This model only uses one game round where every player has access to all available information (e.g. coin matching).

2. Dynamic Model of Complete Information: This is essentially the same as the first model but using many different rounds

3. Bayesian Incomplete Information: Similar to the first model but a player may not have complete information about another players strategies (e.g. auctions). That means the game uses hidden information.

4. perfect Bayesian Incomplete Information

2 Strategic games of Complete Information

Definition 2.0.1. A game in strategic form has three components

- the set of (rational) players $P = \{1, 2, ..., n\}$

- a pure strategy space S_i for player i. S_i is the set of all possible decisions of i. All decisions are taken simultaneously and all information is common knowledge.

- a utility function u_i for each player p_i that returns a gain (or loss) for player i for a given strategy vector $s = (s_1, s_2, ..., s_n)$ which is also called a pure-strategy payoff.

Example 2.0.2. *Let $A = [0, 1]$ and $P = \{0, 1\}$. Each player picks a $x_i \in A$ and wants to maximize the measure of $S(x_i) = \{x \in A : |x - x_i| < |x - x_{1-i}|\}$. That means $u_i(x_i) = \lambda(S(x_i))$. The Nash equilibrium in this example is $x_0 = x_1 = \frac{1}{2}$. For three players there is no Nash equilibrium but for any other number there is.*

Definition 2.0.3. Let $i \in P$. We often use the simplified notation s_{-i} as the pure-strategy form of all players except i. That is

$$s_{-i} = (s_1, .., s_{i-1}, s_{i+1}, ..., s_n)$$

Definition 2.0.4. A pure-strategy Nash Equilibrium $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a strategy profile such that $\forall i \in P$ it holds

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ \forall s_i \in S_i$$

Definition 2.0.5. A pure strategy s_i is strictly dominated for player i if there exists

$$\sigma_i' \in \Sigma_i$$

such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-1})$$

for all $s_{-i} \in S_{-i}$. s_i is weakly dominated if there is a σ_i' such that the inequality is weak but strict for at least one s_i.

Definition 2.0.6 (mixed strategy). A mixed strategy σ_i is a probability distribution over pure strategies. The space of mixed strategies of player i will be denoted by Σ_i where $\sigma_i(s_i)$ is the probability that σ_i chooses s_i. The space of mixed strategy profiles is $\Sigma = \times_i \Sigma_i$. The support of a mixed strategy is the set of pure strategies that are assigned a positive probability. The payoff of player i under the profile σ is

$$u_i(s) = \sum_{s \in S} \left(\prod_{j=1}^{|P|} \sigma_j(s_j) \right) u_i(s)$$

Definition 2.0.7 (iterative elimination of strictly dominated strategies). Set $S_i^0 = si$ and $\Sigma_i^0 = \Sigma_i$. Now define S_i^n recursively by

$$S_i^n = \left\{ s_i \in S_i^{n-1} : \text{there is no } \sigma_i \in \Sigma_i^{n-1} \text{ such that } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-1}) \right\}$$

and define

$$\Sigma_i^n = \left\{ s_i \in \Sigma_i : \sigma_i(s_i) > 0 \text{ only if } s_i \in S_i^n \right\}$$

2.1 Monopoly Market: 1-Player Cournot

Assume there is only one player (e.g. a company) that produces one product where production for each item costs $c \in \mathbb{R}$. The company can choose the quantity q to produce. Furthermore there is an $a \in \mathbb{R}$ s.t. for each item sold, the player gains $a - q$. This should model the principle of supply and demand. The question is now, how the profit can be maximised.

We calculate the payoff function w.r.t. to q

$$u(q) = (a - c - q)q = q(a - c) - q^2$$

We optimize via the first derivative

$$0 \overset{!}{=} u'(q) = a - c - 2q$$

Hence u is maximized at $q = \frac{a-c}{2}$.

2.2 Duopoly: 2-Player Cournot

The setting is the same as in the previous section, and the constant c is the same for both companies. Both choose q_i from $\mathbb{N}$. They can sell each item for a prize of $a - q_1 - q_2$. Let (q_1^*, q_2^*) be a strategy profile. We fix q_2^* and compute the optimal value for q_1^* (called the best response).

$$\max_{q_1} u_1(q_1, q_2^*)$$

where

$$u_1(q_1, q_2^*) = q_1(a - q_2^* - c) - q_1^2$$

and again using the first derivative we find

$$q_1^* = \frac{a - q_2^* - c}{2}$$

By symmetry one can find the same value for q_2^* and by a substitution

$$q_1^* = \frac{a - c}{3}$$

By symmetry this must be the unique pure-strategy Nash equilibrium.

3 Dynamic Model of Complete Information

Let $A^0 = (a_1^0, ..., a_n^0)$ be a stage-0 strategy profile. At the beginning of stage 1, know history h^1 which is simply a^0. The possible strategies for player i are now potentially dependent on h^1, which is why we denote the set by $A_i(h^1)$. This is generalised for all stages k.

A pure strategy is thus a plan on how to play in each stage k for a possible history h^k. Let H^k be the set of all possible stage-k histories and let

$$A_i(H^k) = \bigcup_{h^k \in H^k} A_i(h^k)$$

The a pure strategy for player i is a sequence of maps $\{s^k\}_{k=0}^K$ where each s_i^k maps H^k to their set of possible actions.

$$s_i^k(h^k) \in A_i(h^k)$$

The sequence of strategies is thus $a^0 = s^0(\varnothing)$, $a^1 = s^1(a^0)$, $a^2 = s^2(a^0, a^1)$, ...
This is called the path of the strategy profile. The payoff function can be defined via

$$u_i : H^{k+1} \to \mathbb{R}$$

is usually some weighted average of the single-stage payoffs. The weight of stage k is $0 \leq \delta^k \leq 1$ which implies the definition of the payoff u_i

$$u_i = \sum_{t=0}^{T-1} \delta_i^t u_i^t(a^t)$$

where the payoff of stage t is $u_i^t(a^t)$. A pure strategy NE is then a strategy profile s such that no player i can do better with a different strategy, i.e.

$$\forall i \in P, u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i})$$

for all possible s_i.

3.1 Subgame-perfect Nash Equilibrium

If we only look at the game starting from stage k we denote this subgame by $G(h^k)$. Here, all players have common knowledge of the history h^k and the payoff is $u_i(h^k, s^k, ..., s^{T-1})$. Any strategy profile of the entire game induces a strategy profile on the subgame; for player i $s_i|_{h^k}$ is the restriction of s_i to the histories consistent with h^k. In this scenario, a strategy profile s of a multi-stage game is a subgame-perfect equilibrium of for every h^k, the restriction $s|_{h^k}$ to $G(h^k)$ is a Nash equilibrium of $G(h^k)$.

3.2 Stackelberg Duopoly Market Game

Two companies need to choose a quantity $q_i \in [0, \infty)$. In stage 0, company 1 chooses q_1 while company 2 observes and stays quiet. In stage 1, company 2 chooses an answer q_2. As in the Cournot games we assume that both have the same cost s and that there is a constant a such that they can sell for $p(q) = a - q$ where $q = q_1 + q_2$. It is easy to see that

$$q_2 = \frac{a - c - q_1}{2}$$

is the best reaction.
However company 1 knows the best response to their first pick and can therefore maximize

$$u_1(q_1, q_2) = u_1(q_1, \frac{a - c - q_1}{2})$$

and chooses

$$q_1 = \frac{a-c}{2}$$

Thus the pure-strategy Nash Equilibrium of the game in stage 0 is $q_1 = \frac{a-c}{2}$ and in stage 1 with $h^1 = (q_1)$ it is $q_2 = \frac{a-c}{4}$.

3.3 Repeated Finite Prisoner's Dilemma

Since the payoff in each stage equals the sum of this stage and the payoff of the subgame before, we can notice that defecting stays the unique pure-strategy Nash Equilibrium. This is because the first stage is simply the one-round Prisoner's Dilemma. In the second stage, the payoff table changes to the usual table plus the payoff of the previous stage. Since this was zero (which is the Nash Equilibrium) the table doesn't change. This backwards induction can be continued and finally shows that defecting stays the Nash Equilibrium until the last round.

There are however other Nash Equilibria if an infinite number of rounds are played.

3.4 Rubinstein-Stahl Bargaining Game

Two players have to agree on how to share a pie of size 1. In even periods, player 1 proposes a sharing of $(x, 1-x)$ and player 2 accepts or rejects. If player 2 accepts at anytime, the game ends. If player 2 rejects, they can propose their own sharing in every odd period. If player 1 accepts any proposal, the game ends. Notice that periods and stages are not the same. A period consists of two stages. Thus if $(x, 1-x)$ is accepted in period t then the payoffs are $\delta_1^t x$ for player 1 and $\delta_2^t (1-x)$ for player 2.

There is a unique subgame-perfect Nash Equilibrium. Player i always demands a share of $\frac{(1-\delta_j)}{1-\delta_i\delta_j}$. They accept any share greater or equal to $\frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}$ and refuses anything else. Note that $\frac{(1-\delta_j)}{1-\delta_i\delta_j}$ is the highest share for player i that is accepted by player j. They cannot improve their payoff by proposing a smaller share because that will be accepted. Proposing a higher share for themselves would also decrease payoff, since it would be rejected and the payoff for player i in the next round is

$$\delta_i(1 - \frac{1-\delta_i}{1-\delta_i\delta_j}) = \delta_i^2 \frac{1-\delta_i}{1-\delta_i\delta_j} < \frac{1-\delta_i}{1-\delta_i\delta_j}$$

Definition 3.4.1. The continuation payoffs of a strategy profile models the possible payoff a player can achieve if they decide to keep playing (in this case that means refusing the other player's offer).

Using this knowledge, we can prove the uniqueness of the Nash Equilibrium. Define $\underline{v}_i$ and $\overline{v}_i$ to be the lowest and highest continuation payoffs of player i in any perfect

equilibrium of any subgame that begins with player i making an offer. Analogously, define $\underline{w}_j$ and $\overline{w}_j$ to be the highest and lowest values in subgames starting with j. When player 1 makes an offer, player 2 will accept any x such that player 2's share $(1 - x)$ exceeds $\delta_2 \overline{v}_2$. Player 2 cannot expect more than $\overline{v}_2$ in the continuation game following their refusal. Hence

$$\underline{v}_1 \geq 1 - \delta_2 \overline{v}_2$$

By symmetry, player 1 accepts all shares above $\delta 1 \overline{v}_1$ and

$$\underline{v}_2 \geq 1 - \delta_1 \overline{v}_1$$

Since player 2 will never offer player 1 a share greater that $\delta \overline{v}_1$, player 1's continuation payoff when player 2 makes an offer, $\overline{w}_1$ is at most $\delta_1 \overline{v}_1$. Since player 2 can obtain at least $\underline{v}_2$ in the continuation game by rejecting player 1's offer, they will reject any x such that $1 - x < \delta_2 \underline{v}_2$. Player 1's highest equilibrium payoff when making an offer $\overline{v}_1$ satisfies

$$\overline{v}_1 \leq \max\{1 - \delta_2 \underline{v}_2, \delta_1, \overline{w}_1\} \leq \max\{1 - \delta_2 \underline{v}_2, \delta_1^2 \overline{v}_1\}$$

This last maximum is actually equal to $1 - \delta_2 \underline{v}_2$. Combining everything we have shown and doing some number crunching we get $\underline{v}_1 = \overline{v}_1$. Similarly one proves that $\underline{v}_2 = \overline{v}_2$ and $\underline{w}_i = \overline{w}_i$.

3.5 Infinite Cournot Duopoly

Consider the infinitely repeated Cournot-Duopoly game with discount factor δ and marginal cost c for both companies. Assume that both players play the following strategy profile which delivers at each stage the social optimum. "Produce half the monopoly quantity $\frac{q_m}{2} = \frac{a-c}{4}$ in the first period. In the t-th period, produce $\frac{q_m}{2}$ if both players have produced $\frac{q_m}{2}$ in each of the $t - 1$ previous periods. Otherwise produce the Cournot quantity $q_C = \frac{a-c}{3}$."

We now aim to calculate the values for δ where the above strategy is a Nash Equilibrium. If both play $\frac{q_m}{2}$ then both have a payoff of $\frac{(a-c)^2}{8}$. If both play q_C, each player gets $\frac{(a-c)^2}{9}$. If player i is going to produce $\frac{q_m}{2}$ in this stage, then the quantity that maximizes j's profit in this stage is the solution to

$$\max_{q_i} \left(a - q_j - \frac{q_m}{2} - c \right) q_j$$

no matter the stage. This means, that this is valid for any subgame of the infinite multi-stage game. The solution is $q_j = \frac{3(a-c)}{8}$ with associated profit $\frac{9(a-c)^2}{64}$. Thus it is a NE for

both companies to play the given strategy exactly if

$$\frac{1}{1-\delta}(a-c)^2/8 \geq 9(a-c)^2/64 + \frac{\delta}{1-\delta}(a-c)^2/9 \Leftrightarrow \delta \geq \frac{9}{17}$$

3.6 Simple Time Stopping Game

In such a game, every player has the action *stop* which can have positive or negative influence on their payoff. Once a player stops, they can no longer change their strategy. We denote the payoff of the first player to stop by $L(t)$. $F(t)$ is the payoff of the following player if there is one and $B(t)$ is the payoff for both players if they stop simultaneously in a 2-player game. Consider the following example:

Definition 3.6.1 (2-player Cat-Dog-Fight Stopping game)**.** A cat and a dog are fighting for a prize whose current value at any time is $v > 1$. Fighting costs 1 unit per period. If one animal stops fighting at period t the other player wins the prize alone without incurring any cost in that period. The second stopping time is irrelevant. If both stop simultaneously no-one wins the prize. If we consider a per-period discount factor δ the symmetric payoff functions are

$$L(t) = B(t) = -(1 + \delta + ... + \delta^{t-1}) = -\frac{1 - \delta^t}{1 - \delta}$$

for whoever stops first and

$$F(t) = L(t) + \delta^t v$$

We aim to find a subgame-perfect Nash equilibrium. Regardless of the period, we assume both players stop with probability p. For this symmetric mixed strategy to be an equilibrium, it is necessary, that

$$L(t) = p \cdot F(t) + (1 - p)L(t + 1)$$

Equating these gives $p^* = \frac{1}{1+v}$ which ranges from 0 to 1.

Theorem 3.6.2 (One-Stage-Deviation Condition for Finite Games)**.** *Consider a finite multi-stage game with observed actions. Then a strategy profile s is a subgame perfect Nash Equilibrium iff it satisfies the one-stage-deviation condition that no player i can gain by deviating from s in a single stage and conforming to s thereafter.*
More precisely, a profile s is a subgame-perfect NE iff there is no player i and no strategy $\hat{s}_i$ that agrees with s_i except at a single stage t and h^t and such that $\hat{s}_i$ is a better response to s_{-i} than s_i if history h^t is reached.

Definition 3.6.3 (Continuous at Infinity Games). A game is continuous at infinity if for each player i the utility function u_i satisfies

$$\lim_{t \to \infty} \sup_{h, \tilde{h} \, s.t. \, h^t = \tilde{h}^t} \left| u_i(h^t) - u_i(\tilde{h}^t) \right| = 0$$

3.7 Bayesian Games

In this kind of games, a player i has a type θ_i drawn from an objective distribution p. The space of all possible θ_i is called Θ_i. $p(\theta_{-i}|\theta_i)$ is the conditional probability of player i about their opponents' types. The marginal distribution $p_i(\theta_i)$ is strictly positive. The space of pure strategies s_i of player i is denoted by S_i and the mixed strategies σ_i by Σ_i. Finally the payoff function is $u_i(s_1, ..., s_n, \theta_1, ..., \theta_n)$. Strategy spaces, payoff functions, type spaces and prior distributions are all common knowledge and al private information is included in the description of the player's type. The choice of a player is dependent on their type. $\sigma_i(\theta_i)$ is the strategy i chooses when their type is θ_i. Hence, if i knew the strategies of the other players as a function of their type, they could use their beliefs $p(\theta_{-i}|\theta_1)$ to compute the expected utility and find their optimal response.

Definition 3.7.1. A pure-strategy Bayesian Nash Equilibrium in a Bayesian game (i.e. a game of incomplete information) with types θ_i for each player i, prior distribution p and pure-strategy spaces S_i is a NE of the expanded game in which each player i's space of pure strategies is the set of $S_i^{\Theta_i}$ of maps from θ_i to S_i.
Given a strategy profile $s(\cdot)$ and $s_i'(\cdot) \in S_i^{\Theta_i}$ let

$$(S_i'(\theta_i), s_{-i}(\theta_{-i}))$$

denote the value of the strategy profile $(s_i'(\cdot), s_{-i}(\cdot))$ evaluated at $\theta = (\theta_i, \theta_{-i})$. The $s(\cdot)$ is a pure-strategy Bayesian NE if

3.8 perturbed matching coins game

The payoffs of both players is changed in one component.
$u_1(H, H) = 1 + \theta_1$ and $u_2(T, H) = 1 + \theta_2$. The θ_i are from an interval $[-\theta, \theta]$. The symmetric pure strategies $(s_i(\theta_i \geq 0) = H, s_i(\theta_i < 0) = T)$ form a Bayesian Nash Equilibrium. From the point of view of each player, the other chooses either H or T with probability $\frac{1}{2}$. Thus the player should choose H if and only if $\frac{1}{2}(1 + \theta_i) + \frac{1}{2}(-1) \geq 0 \Leftrightarrow \theta_i \geq 0$. When θ converges to 0 the BNE of the incomplete information game converges to the unique NE $(\frac{1}{2}, \frac{1}{2})$ of the complete information game.

3.9 Perturbed battle of the sexes

Again, we change the payoff slightly to be $u_1(M, M) = 2 + \theta_1$ and $u_2(F, F) = 2 + \theta_2$. The θ_i are from an interval $[0, \theta]$. Player 1 plays M if θ_1 exceeds a critical value c and F otherwise. Player 2 plays F if θ_2 exceeds a critical value p and M otherwise. In such a strategy profile, player 1 plays M with probability $\frac{\theta - c}{\theta}$ and player 2 plays F with probability $\frac{\theta - p}{\theta}$. It is easy to see that player 1's expected payoff given player 2's strategy is

$$\frac{p}{\theta}(2 + \theta_1) + (1 - \frac{p}{\theta}) \cdot 0 = \frac{p}{\theta}(2 + \theta_1)$$

and

$$\frac{p}{\theta} \cdot 0 + (1 - \frac{p}{\theta}) \cdot 1 = 1 - \frac{p}{\theta}$$

Therefore, playing M is optimal if and only if $\theta_1 \geq \frac{\theta}{p} - 3$. In the same way we can see that player 2 plays F if and only if $\theta_2 \geq \frac{\theta}{c} - 3$. Thus we get $p = c$ and $p^2 + 3p - \theta = 0$. From this we can get that the probability that player 1 plays M and the probability that player 2 plays F are both equal to

$$1 - \frac{-3 + \sqrt{9 + 4\theta}}{2\theta}$$

which goes to $\frac{2}{3}$ as θ goes to 0.

3.10 Public Good Game

In this game two players have the pure strategies CONTRIBUTE or DON'T (abbreviated by C and D). The payoffs are

$$u_1(C, C) = 1 - c_1, \ u_1(C, D) = 1 - c_1, \ u_1(D, C) = 1, \ u_1(D, D) = 0$$

and

$$u_1(C, C) = 1 - c_2, \ u_2(C, D) = 1, \ u_2(D, C) = 1 - c_2, \ u_2(D, D) = 0$$

A pure strategy is

$$s_i : [\underline{c}, \overline{c}] \to \{C, D\}$$

and player i's payoff is

$$u_i(s_i, s_j) = \max\{s_1, s_2\} - c_i s_i$$

and a BNE is a pair of strategies $(s_1^*(\cdot), s_2^*(\cdot))$ such that for each player i and every possible value of c_i the strategy $s_i^*(c_i)$ maximizes the expected value of u_i. Consider $z_j = \mathbb{P}(s_j^*(c_j) = 1)$ to be the equilibrium probability that player j goes for C. The expected

output for i if they don't contribute then is $z_j \cdot 1$ and $1 - c_j$ if the do. Hence, player i will contribute if their cost is less than $1 - z_j$) which is their gain from the public good times the probability that player j does not contribute. That is, the optimal strategy is $s_i^*(c_i) = 1$ if $c_i < 1 - z_j$ and 0 otherwise. Thus the types of player i who contributes is belong to an interval $[\underline{c}, c_j^*]$ for some c_j^*. Since $z_j = P(c_j^*)$, the equilibrium cutoff level must satisfy $c_i^* = 1 - P(c_j^*)$. Therefore c_1^*, c_2^* must both satisfy the equation $c^* = 1 - P(1 - P(c^*))$. If there is a unique c^* for this equation, then necessarily $c_i^* = c^* = 1 - P(c^*)$. For instance, if P is the uniform distribution on $[0, 1]$, (i.e. $P(c) = \frac{c}{2}$), then c^* is unique and is equal to $\frac{2}{3}$.

3.11 First Prize auction with two types

Let's consider a first-price auction with two types $\underline{\theta}$ (cheap) and $\overline{\theta}$ (generous). Let $\underline{p}, \overline{p}$ be the respective probabilities for a player i. Let's look for an equilibrium where $\underline{\theta}$ bids $\underline{\theta}$ and type $\overline{\theta}$ randomizes according to the continuous distribution $F(s)$ in $[\underline{s}, \overline{s}]$. Clearly $\underline{s} = \underline{\theta}$. If $\underline{s} > \underline{\theta}$ then a player with type $\overline{\theta}$ would be better off bidding just above $\underline{\theta}$ rather than bidding $\underline{s}$ as this would not reduce her probability of winning ans would reduce her payment if they win. For player i with type $\overline{theta}$ to play in an equilibrium with a mixed strategy with support $[\underline{s}, \overline{s}]$, it must hold that

$$(\overline{\theta} - s)[\underline{p} + \overline{p}F(s)] = c \ \forall s \in [\underline{s}, \overline{s}]$$

where c is constant. That is, the expected payoff of the generous type should be constant where for bid s the generous tpe gets $\overline{\theta} - s$ with probability $\underline{p} + \overline{p}F(s)$. as $F(\underline{\theta}) = 0$ the constant c is equal to $(\overline{\theta} - \underline{\theta})\underline{p}$. This $F(\cdot)$ is defined by

$$(\overline{\theta} - s)[\underline{p} + \overline{p}F(s)] = (\overline{p} - \underline{p})$$

Let $G(s) = \underline{p} + \overline{p}F(s)$ denote the cumulative distribution of bids for $s \geq \theta$. Then we get

$$(\overline{\theta} - s)G(s) = (\overline{\theta} - \underline{\theta})\underline{p}$$

Finally $F(\overline{s}) = 1$ implies $(\overline{\theta} - \overline{s}) = (\overline{\theta} - \underline{\theta})\underline{p}$, that is

$$\overline{s} = \overline{p}\overline{\theta} + \underline{p}\underline{\theta}$$