

# Calculus of Variations

December 10, 2024

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# 1 Introduction

The objects of calculus of variations are so-called functionals, i.e. functions of functions. The main interests are the critical points of these functionals. Since such critical points are often solutions to (elliptic) PDEs. We will therefore look for minima and saddle points.

**Example 1.1** (soap bubbles). *Which object encloses a fixed volume and has smallest surface area?*

*Formulation:*

$$\mathcal{F} : \{\text{surfaces enclosing volume } v_0\} \rightarrow \mathbb{R}_{\geq 0}$$

*The minimizer is a sphere.*

## 1.1 mathematical formulation

Let  $X$  be a Banach space. Let  $\emptyset \neq U \subset X$  and

$$E : U \rightarrow \mathbb{R}$$

be a functional.

Now, if  $E$  is bounded from below (i.e.  $\exists M > 0 : E(u) \geq -M \forall u \in U$ ), is there a  $u \in U$  s.t.  $E(u) = \inf_{v \in U} E(v)$ ? Are there other critical points?

There are two ways to approach the first question.

1. if  $U$  is open and  $E$  is differentiable then look for a solution to

$$E'(u) = 0$$

(called the classical method in CV). This will lead us to  $u$  being a solution to a PDE.

2. take a sequence  $(u_n)_{n \in \mathbb{N}} \subset U$  s.t.  $E(u_n) \rightarrow \inf_{v \in U} E(v)$  (a minimizing sequence).

- (a) Find a topology s.t.  $\exists (u_{n_k})_{k \in \mathbb{N}}, u \in U$  with

$$u_{n_k} \rightarrow u$$

in this topology.

- (b) check if  $E(u) \leq \liminf_{k \rightarrow \infty} E(u_{n_k})$ .

## 2 First variation and convexity

### 2.1 First variation

We fix  $X$  to be a Banach space

**Definition 2.1.** Let  $\emptyset \neq V \subset X$  be open. Let  $E : V \rightarrow \mathbb{R}$ . We say that  $E$  is Fréchet differentiable at  $u \in V$  if  $\exists A \in L(X, \mathbb{R})$  (a linear bounded map) s.t.

$$\lim_{\|\varphi\|_x \rightarrow 0} \frac{E(u + \varphi) - E(u) - A\varphi}{\|\varphi\|_x} = 0$$

$A$  is called the Fréchet derivative of  $E$  at  $u$  denoted by  $E'(u)$ .

**Definition 2.2** (First variation). Let  $\emptyset \neq V \subset X$ ,  $E : V \rightarrow \mathbb{R}$  and  $u \in V$ . Let  $\varphi \in X$  s.t.

$$\exists \delta > 0 \text{ s.t. } u + t\varphi \in V \forall t \in (-\delta, \delta)$$

If

$$(-\delta, \delta) \ni t \mapsto E(u + t\varphi) \in \mathbb{R}$$

is differentiable at  $t = 0$  we say that  $E$  has first in variation  $u$  in direction  $\varphi$  and write

$$\delta E(u)(\varphi) = \frac{d}{dt} E(u + t\varphi)|_{t=0}$$

**Theorem 2.3** (Fundamental Lemma of CV). Let  $\Omega \subset \mathbb{R}^n$  open,  $w \in L^1_{loc}(\Omega)$  (i.e.  $\forall K \subset \Omega$  compact,  $w \in L^1(K)$ ). If

$$\int_{\Omega} w\varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

then  $w = 0$  a.e.

**Corollary 2.4.** Let  $n \in \mathbb{N}$ ,  $u \in L^1_{loc}((a, b))$  s.t.

$$\int_a^b u \frac{d^n}{dx^n} \varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty((a, b))$$

Then  $\exists a_1, \dots, a_n \in \mathbb{R}$  s.t.

$$u(x) = \sum_{i=0}^{n-1} a_i x^i$$

**Definition 2.5.** Let  $u \in L^1_{loc}(\Omega)$ . We say that  $u$  is once weakly differentiable if  $\forall i \in$

$\{1, 2, \dots, n\} \exists v_i \in L^1_{loc}(\Omega)$  s.t.

$$\int_{\Omega} u \partial_{x_i} \varphi \, dx = - \int_{\Omega} v_i \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

In that case  $\partial_i u = v_i$ .

Similarly, if  $\alpha \in \mathbb{N}_0^n$  we say  $u$  is  $\alpha$ -weakly differentiable if

$$\exists v_\alpha \in L^1_{loc} \text{ s.t. } \int_{\Omega} u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} v_\alpha \varphi \, dx$$

**Definition 2.6** (Sobolev spaces). For  $m \in \mathbb{N}$ ,  $p \in (1, \infty)$ , define

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \text{ s.t. } D^\alpha u \in L^p(\Omega) \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq m\}$$

This is a vector space and can be equipped with the norm

$$\|u\|_{W^{m,p}} = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

**Theorem 2.7.**  $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}})$  is a Banach space and for  $p = 2$  it's a Hilbert space. Finally,  $C^\infty(\Omega) \cap W^{m,p}(\Omega)$  is dense in  $W^{m,p}(\Omega)$ .

**Definition 2.8.**  $W_0^{m,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)}$ , i.w.  $f \in W_0^{m,p}(\Omega)$  if

$$\exists (f_n)_{n \in \mathbb{N}} \subset C_0^\infty(\Omega)$$

s.t.

$$\|f - f_n\|_{W^{m,p}} \rightarrow 0, \quad n \rightarrow \infty$$

*Remark 2.9.*  $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$

**Definition 2.10** (weak solution of Poisson equation). We say that  $u$  is a weak solution of

$$\begin{cases} -\Delta u &= f, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{cases}$$

if  $u \in W_0^{1,2}(\Omega)$  and

$$\int_{\Omega} (\nabla u \nabla \varphi - f \varphi) \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

### 3 Direct Methods

**Lemma 3.1** (Poincaré-inequality).  $\exists C = C(\Omega)$  s.t.

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} \quad \forall u \in W_0^{1,2}(\Omega)$$

**Lemma 3.2.** Define  $\|u\|_{W_0^{1,2}(\Omega)} = \|\nabla u\|_{L^2}$ , then  $\|\cdot\|_{W^{1,2}}$  and  $\|\cdot\|_{W_0^{1,2}}$  are equivalent in  $W_0^{1,2}(\Omega)$ .

**Theorem 3.3.** Every bounded sequence in a Hilbert space admits a weakly convergent subsequence. That means given  $(v_k)_{k \in \mathbb{N}} \subset H$  bounded in Hilbert space  $H$   $\exists (v_{k_l})_{l \in \mathbb{N}} \exists v \in H$  s.t.

$$\langle v_{k_l}, w \rangle_H \rightarrow \langle v, w \rangle_H \quad \forall w \in H$$

**Theorem 3.4.** If  $(u_k)_{k \in \mathbb{N}}$  converges weakly to  $u$  in  $H$ , then

$$\|u\|_H \leq \liminf_{k \rightarrow \infty} \|u_k\|_H$$

*Remark 3.5.* A functional  $T : X \rightarrow Y$  is compact if  $\forall (x_k)_{k \in \mathbb{N}}$  bounded in  $X$ , the sequence  $(Tx_n)_{n \in \mathbb{N}} \subset Y$  admits a convergent subsequence.

**Theorem 3.6.** The inclusion map  $W_0^{1,2} \rightarrow L^2(\Omega)$  that maps  $u \mapsto u$  is compact.

#### 3.1 Some concepts from functional analysis

Let  $X$  be a  $\mathbb{R}$ -Banach space.

**Definition 3.7.** We define the dual space

$$X' = L(X, \mathbb{R}) = \{T : X \rightarrow \mathbb{R} \text{ linear and continuous}\}$$

**Lemma 3.8.** For linear maps, the following are equivalent:

1. continuity
2. continuity at 0
3. boundedness

**Definition 3.9.** Similarly, the bidual space

$$X'' = L(X', \mathbb{R})$$

is also a Banach space. There exists a canonical map

$$i_X : X \rightarrow X''$$

For  $x \in X$  we define the canonical map

$$i_x : X \rightarrow X''$$

by

$$i_X(x)(T) = Tx \text{ for } T \in X'$$

$i_X$  is well-defined and  $i_X(x)$  is an element of the bidual space. Moreover, the map is injective and an isometry. Spaces  $X$  where  $i_X$  is also surjective are also called reflexive spaces.

**Example 3.10.** All Hilbert spaces,  $L^p$  spaces and all Sobolev spaces are reflexive for  $1 < p < \infty$ .

**Definition 3.11.** Let  $(x_k)_{k \in \mathbb{N}} \subset X$ .  $x_k$  converges weakly to  $x \in X$  if for all  $T \in X'$

$$T(x_k) \rightarrow T(x) \text{ for } k \rightarrow \infty$$

**Theorem 3.12.** Every bounded sequence in a reflexive space admits a weakly convergent subsequence.

**Theorem 3.13.** Let  $(X, \|\cdot\|)$  be a reflexive Banach space. Let  $M \subset X$  and  $M \neq \emptyset$  a weakly sequentially closed subset  $X$ . Let  $E : M \rightarrow \mathbb{R}$  s.t.

1.  $E(y) \rightarrow \infty$  if  $\|y\|_X \rightarrow \infty$
2.  $E$  is sequentially weakly lower semi-continuous that is if  $x_k$  converges weakly to  $x \in X$ , then  $E(x) \leq \liminf_{k \rightarrow \infty} E(x_k)$

Then,  $E$  is bounded from below and  $\exists x \in M$  s.t.

$$E(x) = \inf\{E(y) : y \in M\}$$

## 4 A partition problem and functions of bounded variation

Given  $\Omega \subset \mathbb{R}^n$  bounded and smooth, is there an  $E \subseteq \Omega$  s.t.

$$|E| = |\Omega \setminus E|$$

and  $H^{n-1}(\partial E \cap \Omega)$  is minimal?

#### 4.1 Motivation

**Definition 4.1.** If  $E \subset \Omega$  is smooth, then

$$H^{n-1}(\partial E \cap \Omega) = \int_{\partial E \cap \Omega} 1 dS(x) = \int_{\partial E \cap \Omega} \langle v(x), v(x) \rangle dS(x)$$

Let  $\varphi \in C : 0^\infty(\Omega; \mathbb{R}^n)$  s.t.  $\varphi(x) = \lambda(x)v(x)$  where  $\lambda(x) \in [0, 1]$  and  $x \in \partial E$ . Then

$$H^{n-1}(\partial E \cap \Omega) \geq \int_{\partial E \cap \Omega} \langle \varphi(x), v(x) \rangle dS(x) = \int_{\partial E} \langle \varphi(x), v(x) \rangle dS(x) = \int_{\Omega} \chi_E \operatorname{div}(\varphi(x)) dx$$

Since  $D$  is smooth,  $\exists \psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with

- $\psi(x) = v(x)$  on  $\partial E \cap \Omega$
- $\psi(x) = 0$  “some bit away from  $\Omega$ ”
- $\|\psi(x)\|_2 \leq 1 \ \forall x \in \mathbb{R}^n$  and  $\|\psi\|_\infty \leq 1$ .

Now, let  $\xi_\varepsilon$  be the bump function on  $\Omega$  which converges to  $\chi_\Omega$  for  $\varepsilon \rightarrow 0$ . Then  $\varphi_\varepsilon = \xi_\varepsilon \psi$  satisfies  $\varphi_\varepsilon \in C_0^\infty(\Omega, \mathbb{R}^n)$  and  $\|\varphi_\varepsilon\|_\infty \leq 1$ . Now

$$H^{n-1}(\partial E \cap \Omega) \xrightarrow{\varepsilon \rightarrow 0} \int_{\partial E \cap \Omega} dS(x) = H^{n-1}(\partial E \cap \Omega)$$

Hence

$$H^{n-1}(\partial E \cap \Omega) = \sup \left\{ \int_{\Omega} \chi_E \operatorname{div} \varphi dx \mid \varphi \in C_0^\infty(\Omega, \mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\}$$

The idea is to use this expression for non-smooth  $E$ .

#### 4.2 Functions of bounded variations

Let  $\Omega \subset \mathbb{R}^n$  be open and bounded.

**Definition 4.2.** We define for  $v \in L^1(\Omega)$

$$\int_{\Omega} |Dv| = \sup \left\{ \int_{\Omega} v(x) \operatorname{div} \varphi(x) dx \mid \varphi \in C_0^\infty(\Omega; \mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\}$$

the total variation of  $v$ .

We say that  $v$  is of bounded variation, if

$$\int |Dv| < \infty$$



Finally, we define  $BV(\Omega) = \{v \in L^1(\Omega) : \int_{\Omega} |Dv| < \infty\}$

*Remark 4.3.* It holds that  $W^{1,1}(\Omega) \subset BV(\Omega)$ . For  $v \in W^{1,1}(\Omega)$  it holds that

$$\int_{\Omega} |Dv| = \int_{\Omega} |\nabla v| \, dx$$

Finally, above inclusion is even strict:  $W^{1,1}(\Omega) \subsetneq BV(\Omega)$

**Lemma 4.4.** *Let  $\Omega$  be a bounded domain. Define  $\|\cdot\|_{BV(\Omega)} : BV(\Omega) \rightarrow \mathbb{R}$  via*

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1} + \int_{\Omega} |Du|$$

*This is a norm and  $(BV(\Omega), \|\cdot\|_{BV(\Omega)})$  is a Banach space.*

**Definition 4.5.** Let  $E \subset \mathbb{R}^n$  be measurable.

$$P(E, \Omega) = \int_{\Omega} |D\chi_{E \cap \Omega}|$$

is called the perimeter of  $E$  in  $\Omega$ .

*Remark 4.6.* If  $\partial E$  is smooth, then  $P(E, \Omega) = H^{n-1}(\partial E \cap \Omega)$

QUESTION: For  $\Omega \subset \mathbb{R}^n$  bounded domain, define

$$M = \{E \subset \Omega \mid \lambda^n(E) = \lambda^n(\Omega \setminus E) = \frac{1}{2} \lambda^n(\Omega)\}$$

Is

$$\inf\{P(E, \Omega) \mid E \in M\}$$

attained?

IDEA: Take  $(E_n)_{n \in \mathbb{N}} \subset \Omega$  a minimizing sequence, that is

$$P(E_n, \Omega) = \int_{\Omega} |D\chi_{E_n \cap \Omega}| \searrow \inf\{P(E, \Omega) \mid E \in M\}$$

#### 4.3 Lower semicontinuity in $BV(\Omega)$

**Theorem 4.7.** *Let  $\Omega$  be a bounded domain. Let  $(v_k)_{k \in \mathbb{N}} \subset BV(\Omega)$  s.t.  $v_k \rightarrow v$  in  $L^1(\Omega)$ .*

*Then*

$$\int_{\Omega} |Dv| \leq \liminf \int_{\Omega} |Dv_k|$$

*In particular, if  $\liminf_{k \rightarrow \infty} \int_{\Omega} |Dv_k| < \infty$  then  $v \in BV(\Omega)$ .*

#### 4.4 Lipschitz continuity

**Definition 4.8.** A function  $v : \bar{\Omega} \rightarrow \mathbb{R}$  is called Lipschitz continuous if

$$[v]_{0,1} = \sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|} < \infty$$

Using this, one defines  $Lip(\Omega) = \{v : \bar{\Omega} \rightarrow \mathbb{R} : v \text{ is Lipschitz}\} = C^{0,1}(\Omega)$ .

$Lip_{loc}(\Omega) = \{v : \Omega \rightarrow \mathbb{R} : v \in Lip(K) \ \forall K \subset \Omega \text{ compact}\}$

**Lemma 4.9.**  $\|\cdot\|_{Lip} : Lip(\Omega) \rightarrow \mathbb{R}$  defined by

$$\|u\|_{Lip} = [u]_{0,1} + \|u\|_{C^0}$$

makes  $Lip(\Omega)$  a Banach space.

**Theorem 4.10.** Let  $v \in Lip(\mathbb{R}^n)$ . Then  $v$  is a.e. classically differentiable. If  $\Omega$  is a domain, then  $v \in Lip_{loc}(\Omega) \Leftrightarrow v \in W_{loc}^{1,\infty}(\Omega)$ . If  $\partial\Omega$  is  $C^1$ , then  $v \in Lip(\Omega) \Leftrightarrow v \in W^{1,\infty}(\Omega)$ . In this case  $[v]_{0,1} = \|\nabla v\|_{L^\infty}$ .

**Definition 4.11.**  $Lip(\Omega, g) = \{u \in Lip(\Omega) \mid u \equiv g \text{ on } \partial\Omega\}$

$Lip(\Omega, g, K) = \{u \in Lip(\Omega, g) \mid [u]_{0,1} \leq K\}$

The aim is now to minimize  $F$  in  $Lip(\Omega, g)$  where

$$F(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx$$

Method: Solve minimization in  $Lip(\Omega, g, K)$  and then show that the minimizer satisfies  $[u]_{0,1} < K$  strictly for a large enough  $K$  and conclude that  $u$  minimizes  $F$  in  $Lip(\Omega, g)$ .

**Lemma 4.12.** Let  $\Omega$  be a bounded  $C^1$ -domain,  $g \in Lip(\Omega)$  with  $[g]_{0,1} \leq K$ . Then  $\exists u \in Lip(\Omega, g, K)$  s.t.

$$F(u) \leq F(v) \ \forall v \in Lip(\Omega, g, K)$$

**Lemma 4.13.** Let  $u^k$  be a minimizer of  $F$  in  $Lip(\Omega, g, k)$ . If  $[u^k]_{0,1} < k$  then  $u^k$  minimizes  $F$  in  $Lip(\Omega, g)$ .

**Definition 4.14.** Let  $v \in Lip(\Omega, K)$ . We call  $v$  a superminimum (resp. subminimum) for  $F$  if  $\forall w \in Lip(\Omega, v, K)$  s.t.  $w \geq v$  in  $\Omega$  then  $F(v) \leq F(w)$  (resp.  $w \leq v$ ).

**Theorem 4.15** (Maximum principle). Let  $\Omega \subset \mathbb{R}^n$  be open and bounded. Let  $v \in Lip(\Omega, K)$  be a superminimum,  $w \in Lip(\Omega, K)$  be a subminimum s.t.  $v \geq w$  on  $\partial\Omega$ . Then  $v \geq w$  in  $\Omega$ .

**Corollary 4.16.** *Let  $\Omega \subset \mathbb{R}^n$  open and bounded. Let  $v \in Lip(\Omega, K)$  be a superminimum and  $w \in Lip(\Omega, K)$  a subminimum. Then*

$$\sup_{\partial\Omega} w - v = \sup_{\Omega} w - v$$

**Corollary 4.17.** *If  $u, \tilde{u} \in Lip(\Omega, g, K)$  are minima of  $F$  in  $Lip(\Omega, f, K)$  then  $u \equiv \tilde{u}$ .*

Goal: Prove that  $u^k$  satisfies

$$\sup_{x, y \in \Omega, x \neq y} \frac{|u^k(x) - u^k(y)|}{|x - y|} = [u^k]_{0,1} < K$$

**Lemma 4.18.** *Let  $\Omega \subset \mathbb{R}^n$  bounded and open. Let  $u \in Lip(\Omega, g, K)$  be a minimum of  $F$ . Then*

$$[u]_{0,1} = \sup_{x \in \Omega, y \in \partial\Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}$$

**Lemma 4.19.** *Let  $a \in \mathbb{R}$ ,  $z \in \mathbb{R}^n$  and  $w : \mathbb{R}^n \rightarrow \mathbb{R}$  via  $w(c) = a + zx$ . Then  $w$  minimizes  $F$  in  $Lip(\Omega, w)$ .*

## 5 Obstacle problems

Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex domain with  $\partial\Omega \in C^1$ . Let  $h \in Lip(\Omega)$  s.t.  $h|_{\partial\Omega} < 0$  and  $h > 0$  somewhere in  $\Omega$ .

Problem: Is  $\inf\{F(u) : u \in Lip(\Omega), u|_{\partial\Omega} = 0, u \geq h\}$  attained? As before  $F(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2}$ . That is, does a  $u \in M = \{v \in Lip(\Omega) : v|_{\partial\Omega} = 0, v \geq h\}$  s.t.

$$F(u) \leq F(v) \quad \forall v \in M$$

exist?

We call  $h$  the obstacle.

The difference of this problem to the previous problems is in the first variation. If  $u$  is a minimizer in the previous problems, then  $\frac{d}{dt}F(u + t\varphi)|_{t=0} = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$ . Here we have only information from some directions. Since  $M$  is convex for all  $v \in M$   $tu + (1 - t)v$  is again in  $M$  for  $t \in [0, 1]$ . Hence, if  $u$  is a minimizer,  $t \mapsto F(tu + (1 - t)v)$  has a minimum at  $t = 1$ . Hence

$$\frac{d}{dt}F(tu + (1 - t)v)|_{t=1} = \int_{\Omega} \frac{\langle \nabla(tu + (1 - t)v), \nabla(u - v) \rangle}{1 + |\nabla(tu + (1 - t)v)|^2} dx|_{t=1} \leq 0$$

This can be rearranged to

$$\int_{\Omega} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \nabla(u - v) dx \geq 0 \forall v \in M$$

This is called a variational inequality.

The strategy to show the existence of a minimizer consists of the following steps

1. add another constraint  $[u]_{0,1} \leq k$
2. if minimizer satisfies this with strict inequality then it is a minimizer without the additional constraint

Let  $M^k = \{u \in M : [u]_{0,1} \leq k\}$  for  $k > [h]_{0,1}$  and note that  $\max\{0, h\} \in M^k$ . Notice that  $M^k$  is also convex.

**Lemma 5.1.**  $\exists u \in M^k$  s.t.  $F(u) \leq F(v) \forall v \in M^k$ .

**Lemma 5.2.** Let  $u^k$  be a minimizer for  $F$  in  $M^k$  s.t.  $[u^k]_{0,1} < k$ . Then  $u^k$  minimizes  $F$  in  $M$ .

Our aim is now to find an a-priori estimate for solutions of variational inequalities of the following type

$$\int_{\Omega} a(\nabla u) \cdot (\nabla(v - u)) dx \geq 0 \forall v \in M \cup M^k$$

where  $a$  satisfies the following strong ellipticity condition

$$(a(p) - a(q)) \cdot (p - q) > 0 \forall p, q \in \mathbb{R}^n, p \neq q$$

**Theorem 5.3.** Let  $u \in M^k$  be a solution of

$$\int_{\Omega} a(\nabla u) \cdot \nabla(v - u) dx \geq 0 \forall v \in M^k$$

where  $a$  satisfies the strong ellipticity. Then

$$\sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|} \leq \max \left\{ [h]_{0,1}, \sup_{x \in \Omega, y \in \partial\Omega} \frac{|u(x) - u(y)|}{|x - y|} \right\}$$

**Theorem 5.4.** Let  $\Omega \subset \mathbb{R}^n$  be a convex, bounded domain with  $\partial\Omega \in C^1$ . Let  $a : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy the strong ellipticity and  $h \in Lip(\Omega)$ ,  $h|_{\partial\Omega} < 0$  but  $h > 0$  somewhere in  $\Omega$ . Let

$k > [h]_{0,1}$  and  $u \in M^k$  be a solution of

$$\int_{\Omega} a(\nabla u) \cdot \nabla(v - u) dx \geq 0 \quad \forall v \in M^k$$

Then

1.  $u \geq 0$
2.  $u$  is unique
3.  $[u]_{0,1} \leq [h]_{0,1} < k$

**Theorem 5.5.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex domain with  $\partial\Omega \in C^1$ . Let  $h \in Lip(\Omega)$ ,  $h|_{\partial\Omega} < 0$  and  $h > 0$  somewhere in  $\Omega$ . Then  $\exists! u \in Lip(\Omega)$ ,  $u|_{\partial\Omega} = 0$  and  $u \geq h$  in  $\Omega$ .

$$F(u) \leq F(v) \quad \forall v \in Lip(\Omega), \quad v|_{\partial\Omega} = 0$$

where  $F(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx$ . Moreover,  $u$  solves

$$\int_{\Omega} \frac{\nabla u}{1 + |\nabla u|^2} \nabla(v - u) dx \geq 0$$

and  $u$  satisfies  $[u]_{0,1} \leq [h]_{0,1}$  and  $\|u\|_{L^\infty} \leq [h]_{0,1} \text{diam}(\Omega)$ .

Moreover, define  $I = \{x \in \Omega : u(x) = h(x)\}$  the coincidence set, then

$$\int_{\Omega} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \nabla \varphi dx = 0 \quad \forall \varphi \in Lip(\Omega), \quad \text{supp}(\varphi) \subset \Omega \setminus I$$

*Remark 5.6.* Is  $I \neq \emptyset$ ? Is it connected and what is its measure?

## 6 The Schrödinger equation

We describe an electron in a universe with only one nucleus that attracts the electron. Motivated by the uncertainty principle of Heisenberg, the electron is described by a function  $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$  a wave function.

Idea:  $\Omega \subset \mathbb{R}^n$  open,  $\int_{\Omega} |\psi(x)|^2 dx$  is the probability that the electron is in  $\Omega$ .

We consider  $n \geq 3$ . Define

- Kinetic energy:  $\int_{\mathbb{R}^n} |\nabla \psi|^2 dx = T(\psi)$
- potential energy:  $V(\psi) = \int_{\mathbb{R}^n} v(x) |\psi(x)|^2 dx$ . Typical choice would be  $n = 3$  (for obvious reasons),  $v(x) = -\frac{z}{|x|}$  with a nucleus of charge  $z$  at the origin.

- Total energy:  $E(\psi) = T(\psi) + V(\psi)$

Question: Is

$$E_0 = \inf\{E(\psi) : \psi \in W^{1,2}(\mathbb{R}^n), \|\psi\|_{L^2(\mathbb{R}^n)} = 1\}$$

attained?  $E_0$  is called the ground state energy. If a minimum exists, it is called ground state.

**Definition 6.1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  then we say that  $f$  vanishes at  $\infty$  if  $\forall a > 0$

$$\lambda^n(\{x \in \mathbb{R}^n : |f(x)| > a\}) < \infty$$

We define

$$D^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} \mid \nabla f \in L^2(\mathbb{R}^n), \text{ and } f \text{ vanishes at } \infty\}$$

**Theorem 6.2.** Let  $n \geq 3$ . Then  $\exists S_n > 0$  s.t.  $\forall f \in D^1(\mathbb{R}^n)$

$$S_n \|f\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^2(\mathbb{R}^n)}$$

*Remark 6.3.* Let  $n \geq 3$ ,  $V \in L^{\frac{n}{2}} + L^\infty(\mathbb{R}^n)$ . That means  $v = v_1 + v_2$  where  $v_1 \in L^{\frac{n}{2}}$ ,  $v_2 \in L^\infty(\mathbb{R}^n)$ . Then  $\exists \lambda \in \mathbb{R}$  s.t.  $\forall \psi \in W^{1,2}(\mathbb{R}^n)$  with  $\|\psi\|_{L^2} = 1$

$$E(\psi) \geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 - \lambda - \|v_2\|_{L^2} \geq -\lambda - \|v_2\|_{L^2}$$

Using this result, we can conclude that the potential energy is even continuous. That is

$$\int_{\mathbb{R}^n} V(x) |\psi_j(x)|^2 dx \rightarrow \int_{\mathbb{R}^n} V(x) |\psi_0(x)|^2 dx$$

for a convergent subsequence of a minimizing sequence  $\psi_j$ .

**Lemma 6.4.** If  $f_k \rightharpoonup f$  in  $W^{1,2}(\mathbb{R}^n)$ , let  $A \subset \mathbb{R}^n$  measurable s.t.  $\lambda^n(A) < \infty$ . Then

$$\chi_A f_k \rightarrow \chi_A f \text{ in } L^p(\mathbb{R}^n)$$

for  $1 \leq p \leq \frac{2n}{n-2}$ .

**Lemma 6.5.** Under the same assumptions as in 6.3, and the additional one that  $v$  vanishes at  $\infty$ , i.e.  $\forall a > 0$  it holds that  $\lambda^n(\{|V(x)| > a\}) < \infty$ , then if  $\psi_j \rightharpoonup \psi$  in  $W^{1,2}(\mathbb{R}^n)$ ,  $\|\psi_j\|_{L^2} = 1 \forall j$  then

$$\int_{\mathbb{R}^n} V(x) |\psi_j(x)|^2 dx \rightarrow \int_{\mathbb{R}^n} V(x) |\psi(x)|^2 dx$$

**Theorem 6.6** (existence of the minimizer). *Let  $v \in L^{\frac{n}{2}} + L^\infty$  and let  $v$  vanish at  $\infty$ . If  $E_0 = \inf\{E(\psi) \mid \psi \in W^{1,2}(\mathbb{R}^n), \|\psi\|_{L^2(\mathbb{R}^n)} < \infty\} < 0$ , then  $\exists \psi_0 \in W^{1,2}(\mathbb{R}^n)$ ,  $\|\psi_0\|_{L^2} = 1$  s.t.  $E(\psi_0) = E_0$ . Moreover,  $\psi_0$  is a weak solution of*

$$-\Delta\psi_0 + v\psi_0 = E_0\psi_0$$

i.e.  $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \nabla\psi_0 \nabla\varphi + v\psi_0\varphi = E_0 \int_{\mathbb{R}^n} \psi_0\varphi$$

*Remark 6.7.* 1.  $E_0$  in the Euler-Lagrange equation can be considered as a Lagrange-multiplier ensuring that the constraint  $\|\psi\|_{L^2} = 1$ .

2. We can write

$$E_0 = \inf_{\substack{\psi \in W^{1,2} \\ \psi \neq 0}} \frac{E(\psi)}{\|\psi\|_{L^2}^2}$$

3. The first condition is important to ensure existence of a minimizer. Indeed, if  $v \equiv 0$  there is no ground state  $\psi_0$ .

Let  $u \in C_0^\infty(\mathbb{R}^n)$  s.t.  $\|u\|_{L^1} = 1$ . Then for  $\lambda > 0$   $u_\lambda(x) = \lambda^{\frac{n}{2}}u(\lambda x)$  is a wave function.

$$E(u\lambda) = T(u_\lambda) = \lambda^2 T(u) \xrightarrow{\lambda \searrow 0} 0$$

Hence  $E_0 = 0$  and  $\nexists \psi \in W^{1,2}(\mathbb{R}^n)$  with  $\|\psi\|_{L^2} = 1$  s.t.  $E(\psi) = 0$ . So a minimizer does not exist.

*Remark 6.8.* If  $E_0 < 0$ , one can look at excited states. With the same ideas, if

$$E_1 = \inf\{E(\psi) \mid \psi \in W^{1,2}(\mathbb{R}^n), \|\psi\|_{L^2} = 1, \langle \psi, \psi_0 \rangle_{L^2} = 0\} < 0$$

there exists  $\psi_1 \in W^{1,2}$ ,  $\|\psi_1\|_{L^2} = 1$ ,  $\langle \psi_1, \psi_0 \rangle_{L^2} = 0$  s.t.

$$E(\psi_1) = E_1$$

and one can continue

$$0 > E_k = \inf\{E(\psi) \mid \psi \in W^{1,2}, \|\psi\|_{L^2} = 1, \langle \psi, \psi_j \rangle_{L^2} = 0, 0 \leq j \leq k-1\}$$

$\psi_k$  satisfies

$$-\Delta\psi_k + v\psi_k = E_k\psi_k, \quad E_k = E(\psi_k)$$

and

$$E_k = \inf_{\substack{\psi \in W^{1,2} \\ \psi \perp_{L^2} \psi_j \\ 0 \leq j \leq k-1}} \frac{E(\psi)}{\|\psi\|_{L^2}^2}$$

## 7 Parametric minimal surfaces (the plateau problem)

Let  $B = B_1(0) \subset \mathbb{R}^2$  be the unit disk and

$$\gamma : S^1 \rightarrow \mathbb{R}^m, \quad m \geq 3$$

a  $C^2$  curve that is simple, i.e.  $\gamma$  has no self intersection.

Aim: find a surface of disk type whose boundary is given by  $\gamma$  and with minimal area.

### 7.1 Some basic concepts from differential geometry

**Definition 7.1.** A surface  $S \subset \mathbb{R}^m$ ,  $m \geq 3$  is a 2-dimensional  $C^l$ -submanifold of  $\mathbb{R}^m$  where  $l \geq 1$ , that is  $\forall p \in S \exists (U, F, V)$  where  $U \subset \mathbb{R}^2$  open.  $V \subset \mathbb{R}^m$  open,  $p \in V \cap S$  and  $F : U \rightarrow V \cap S$ ,  $F \in C^2$  is a homeomorphism and  $\forall u \in U$  is  $\text{rank}(DF(u)) = 2$ .

*Remark 7.2.* The definition implies that locally the graph can be described by the graph of a  $u : U \rightarrow \mathbb{R}^{m-2}$  and locally the surface can be described as a level set  $S \cap V = \{x \in V : g(x) = 0\}$ .

**Definition 7.3** (tangential space). Let  $S$  be a surface in  $\mathbb{R}^m$ ,  $p \in S$ ,

$$T_p S = \{v \in \mathbb{R}^m : \exists c : (-\varepsilon, \varepsilon) \rightarrow S \in C^1 \text{ s.t. } c(0) = p, c'(0) = v\}$$

$T_p S$  is the tangential space to  $S$  at  $p$  with  $\dim T_p S = 2$ .

If  $(U, F, V)$  is a local chart at  $p \in S$ , then

$$T_p S = \text{span} \left\{ \frac{\partial F}{\partial u}(u, v), \frac{\partial F}{\partial v}(u, v) \right\}$$

$U \subset \mathbb{R}^2$ ,  $F : U \rightarrow V \subset \mathbb{R}^m$ . Aim: define a scalar product in  $T_p S$ . The first fundamental form is

$$g_p : T_p S \times T_p S \rightarrow \mathbb{R}$$

with

$$(u, v) \mapsto \langle u, v \rangle_{\mathbb{R}^m}$$



If  $(U, F, V)$  is a local chart at  $p$ , for  $w_1, w_2 \in T_p S$ ,  $w_1 = w_1^1 \frac{\partial F}{\partial u} + w_1^2 \frac{\partial F}{\partial v}$ ,  $w_2 = w_2^1 \frac{\partial F}{\partial u} + w_2^2 \frac{\partial F}{\partial v}$  so that we can represent the first fundamental form with respect to this basis via the matrix  $(g_{i,j}(u, v))_{i,j=1}^2$  by

- $g_{1,1}(u, v) = \langle \frac{\partial F}{\partial u}(u, v), \frac{\partial F}{\partial u}(u, v) \rangle$ ,
- $g_{1,2}(u, v) = g_{2,1}(u, v) = \langle \frac{\partial F}{\partial u}(u, v), \frac{\partial F}{\partial v}(u, v) \rangle$
- $g_{2,2}(u, v) = \langle \frac{\partial F}{\partial v}(u, v), \frac{\partial F}{\partial v}(u, v) \rangle$ .

We want to define for

$$f : S \rightarrow \mathbb{R}$$

continuous  $\int_S f(x) dS(x)$

First step: Let  $(U, F, V)$  be a local chart and consider  $f$  such that  $\text{supp}(f) \subset V \cap S$ .

$$\int_S f(x) dS(x) = \int_U f(F(u, v)) \sqrt{\det(g_{i,j}(u, v))_{i,j=1}^2} du dv$$

Second step: If  $S$  is compact, then  $\exists$  finitely many charts  $(U_i, F_i, V_i)_{i=1}^N$  s.t.  $S \subset \bigcup_{i=1}^N (V_i \cap S)$ . Then consider  $(\psi_i)_{i=1}^N$  a partition of unity associated to this cover.

$$\psi_i \in C_0^\infty(\mathbb{R}^m; [0, 1]), \text{supp}(\psi_i) \subset V_i, \forall i = 1, \dots, N \text{ and } 1 = \sum_{i=1}^N \psi_i(x) \forall x \in S$$

Then for  $f : S \rightarrow \mathbb{R}$  continuous,

$$\int_S f(x) dS(x) = \sum_{i=1}^N \int_S (f \psi_i)(x) dS(x)$$

If  $S$  is described by only one chart, one can see that

$$\text{area}(S) = \int_U \sqrt{\left| \frac{\partial F}{\partial u} \right|^2 \left| \frac{\partial F}{\partial v} \right|^2 - \left( \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v} \right)^2} d(u, v)$$

*Remark 7.4.* The area is well-defined for  $F \in W^{1,2}(B, \mathbb{R}^m)$ .

**Definition 7.5.**

$$D(F) = \frac{1}{2} \int_B \left| \frac{\partial F}{\partial u} \right|^2 + \left| \frac{\partial F}{\partial v} \right|^2 d(u, v)$$

is called the Dirichlet energy of  $F$ .

## 7.2 Setup for the problems

Define for  $\gamma : \partial B_1(0) \rightarrow \mathbb{R}^m$ ,  $m \geq 3$ , a closed and simple curve

$$K(\gamma) = \{F \in W^{1,2}(B, \mathbb{R}^m) : F|_{\partial B} \in C^0(\partial B, \mathbb{R}^m) \text{ is a monotone parametrization of } \gamma\}$$

Question 1:  $\exists F \in K(\gamma)$  s.t.  $A(F) \leq A(\tilde{F}) \forall \tilde{F} \in K(\gamma)$ ? This is a difficult question.

Question 2:  $\exists F \in K(\gamma)$  s.t.

$$D(F) \leq D(\tilde{F}) \forall \tilde{F} \in K(\gamma)$$

where  $D(F)$  is defined in 7.5.

As we shall see, these two questions are equivalent.

*Remark 7.6.*  $K(\gamma) \neq \emptyset$ .

## 7.3 Euler-Lagrange equation for question 2

**Lemma 7.7.** *Let  $F \in K(\gamma)$  be a minimizer of the Dirichlet energy in  $K(\Gamma)$ . Then  $\Delta F = 0$  on  $B$  i.e.  $F$  is harmonic and  $F \in C^\infty(B)$ .*

**Lemma 7.8.** *If  $f \in L^1_{loc}(\Omega)$  such that  $\int_\Omega f \Delta \varphi dx = 0 \forall \varphi \in C_0^\infty(\Omega)$ . Then  $f \in C^\infty(\Omega)$  and  $\Delta f = 0$ .*

As an abbreviation for the integral making up the Dirichlet-energy  $D(F)$  we write  $|DF|^2$ .

*Remark 7.9.* Let  $X : B_1(0) \rightarrow \mathbb{R}^m$  s.t.  $\Delta X = 0$ . Let  $(u_1, u_2) \in B_1(0)$ . Then the map

$$z = u_1 + iu_2 \mapsto g(z) = |\partial_{u_1} X|^2 - |\partial_{u_2} X|^2 - 2i\partial_{u_1} X \partial_{u_2} X$$

is holomorphic. Indeed by the Cauchy-Riemann equations,  $g$  is holomorphic  $\iff \partial_{\bar{z}} g = 0$  where

$$\partial_{\bar{z}} g = \frac{1}{2} (\partial_{u_1} g + i\partial_{u_2} g)$$

It is easily verifiable that  $\partial_{\bar{z}} g = 4\partial_z X \cdot \partial_{\bar{z}} \partial_z X = \partial_z X \Delta X$ .

**Theorem 7.10.** *Let  $F \in K(\gamma)$  be a critical point of the Dirichlet-energy in  $K(\gamma)$ . By that we mean that*

1.  $\frac{d}{dt} D(F + t\varphi)|_{t=0} = 0 \forall \varphi \in C_0^\infty(B; \mathbb{R}^m)$
2.  $\frac{d}{dt} D(F \circ \varphi_t^{-1}, B_t)|_{t=0} = 0$  where  $(\varphi_t)_t$  is a family of diffeomorphisms.  $\varphi_t = Id + t\psi$  with  $\psi \in C^2(\overline{B_1(0)})$  and  $\varphi_t : B_1(0) \rightarrow B_t = \varphi(B_1(0))$ .

Then  $F$  solves the following problem:

- $\Delta F = 0$  in  $B_1(0)$
- $F|_{\partial B_1(0)}$  is a monotone parametrization of  $\gamma(\partial B_1(0))$
- $|\partial_{u_1} F| = |\partial_{u_2} F|$
- $\langle \partial_{u_1} F, \partial_{u_2} F \rangle = 0$

The last two properties yield that  $F$  is a so-called conformal parametrization.