Calculus of Variations

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Contents

1 Introduction

The objects of calculus of variations are so-called functionals, i.e. functions of functions. The main interests are the critical points of these functionals. Since such critical points are often solutions to (elliptic) PDEs. We will therefore look for minima and saddle points.

Example 1.1 (soap bubbles)**.** *Which object encloses a fixed volume and has smallest surface area?*
Formulation:

$$\mathcal{F} : \{\textit{surfaces enclosing volume } v_0\} \to \mathbb{R}_{\geq 0}$$

The minimizer is a sphere.

1.1 mathematical formulation

Let X be a Banach space. Let $\varnothing \neq U \subset X$ and

$$E : U \to \mathbb{R}$$

be a functional.
Now, if E is bounded from below (i.e. $\exists M > 0 : E(u) \geq -M \ \forall u \in U$), is there a $u \in U$ s.t. $E(u) = \inf_{v \in U} E(v)$? Are there other critical points?
There are two ways to approach the first question.

1. if U is open and E is differentiable then look for a solution to

$$E'(u) = 0$$

   (called the classical method in CV). This will lead us to u being a solution to a PDE.

2. take a sequence $(u_n)_{n \in \mathbb{N}} \subset U$ s.t. $E(u_n) \to \inf_{v \in U} E(v)$ (a minimizing sequence).

   (a) Find a topology s.t. $\exists (u_{n_k})_{k \in \mathbb{N}}, u \in U$ with

$$u_{n_k} \to u$$

   in this topology.

   (b) check if $E(u) \leq \liminf_{k \to \infty} E(u_{n_k})$.

2 First variation and convexity

2.1 First variation

We fix X to be a Banach space

Definition 2.1. Let $\varnothing \neq V \subset X$ be open. Let $E : V \to \mathbb{R}$. We say that E is Fréchet differentiable at $u \in V$ if $\exists A \in L(X, \mathbb{R})$ (a linear bounded map) s.t.

$$\lim_{||\varphi||_x \to 0} \frac{E(u+\varphi) - E(u) - A\varphi}{||\varphi||_x} = 0$$

A is called the Fréchet derivative of E at u denoted by $E'(u)$.

Definition 2.2 (First variation)**.** Let $\varnothing \neq V \subset X$, $E : V \to \mathbb{R}$ and $u \in V$. Let $\varphi \in X$ s.t.

$$\exists \delta > 0 \text{ s.t. } u + t\varphi \in V \ \forall t \in (-\delta, \delta)$$

If

$$(-\delta, \delta) \ni t \mapsto E(u + t\varphi) \in \mathbb{R}$$

is differentiable at $t = 0$ we say that E has first in variation u in direction φ and write

$$\delta E(u)(\varphi) = \frac{d}{dt} E(u + t\varphi)|_{t=0}$$

Theorem 2.3 (Fundamental Lemma of CV)**.** *Let $\Omega \subset \mathbb{R}^n$ open, $w \in L^1_{loc}(\Omega)$ (i.e. $\forall K \subset \Omega$ compact, $w \in L^1(K)$). If*

$$\int_\Omega w\varphi \, dx = 0 \ \forall \varphi \in C_0^\infty(\Omega)$$

then $w = 0$ a.e.

Corollary 2.4. *Let $n \in \mathbb{N}$, $u \in L^1_{loc}((a,b))$ s.t.*

$$\int_a^b u \frac{d^n}{dx^n} \varphi \, dx = 0 \ \forall \varphi \in c_0^\infty((a,b))$$

Then $\exists a_1, ..., a_n \in \mathbb{R}$ s.t.

$$u(x) = \sum_{i=0}^{n-1} a_i x^i$$

Definition 2.5. Let $u \in L^1_{loc}(\Omega)$. We say that u is once weakly differentiable if $\forall i \in$

$\{1, 2, ..., n\}$ $\exists v_i \in L^1_{loc}(\Omega)$ s.t.

$$\int_\Omega u \partial_{x_i} \varphi \, dx = - \int_\Omega v_i \varphi \, dx \; \forall \varphi \in C_0^\infty(\Omega)$$

In that case $\partial_i u = v_i$.
Similarly, if $\alpha \in \mathbb{N}_0^n$ we say u is α-weakly differentiable if

$$\exists v_\alpha \in L^1_{loc} \text{ s.t. } \int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int v_\alpha \varphi \, dx$$

Definition 2.6 (Sobolev spaces)**.** For $m \in \mathbb{N}$, $p \in (1, \infty)$, define

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \text{ s.t. } D^\alpha u \in L^p(\Omega) \; \forall \alpha \in \mathbb{N}_0^n, \; |\alpha| \leq m\}$$

This is a vector space and can be equipped with the norm

$$||u||_{W^{m,p}} = \left(\sum_{(\alpha) \leq m} ||D^\alpha u||^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

Theorem 2.7. *$(W^{m,p}(\Omega), ||\cdot||_{W^{m,p}})$ is a Banach space and for $p = 2$ it's a Hilbert space. Finally, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Definition 2.8. $W_0^{m,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)}$, i.w. $f \in W_0^{m,p}(\Omega)$ if

$$\exists (f_n)_{n \in \mathbb{N}} \subset C_0^\infty(\Omega)$$

s.t.

$$||f - f_n||_{W^{m,p}} \to 0, \; n \to \infty$$

Remark 2.9. $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$

Definition 2.10 (weak solution of Poisson equation)**.** We say that u is a weak solution of

$$\begin{cases} -\nabla u & = f, \text{ in } \Omega \\ u & = 0, \text{ on } \partial\Omega \end{cases}$$

if $u \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega (\nabla u \nabla \varphi - f \varphi) \, dx = 0 \; \forall \varphi \in C_0^\infty(\Omega)$$

3 Direct Methods

Lemma 3.1 (Poincaré-inequality). *$\exists C = C(\Omega)$ s.t.*

$$||u||_{L^2} \leq C||\nabla u||_{L^2} \; \forall u \in W_0^{1,2}(\Omega)$$

Lemma 3.2. *Define $||u||_{W_0^{1,2}(\Omega)} = ||\nabla u||_{L^2}$, then $||\cdot||_{W^{1,2}}$ and $||\cdot||_{W_0^{1,2}}$ are equivalent in $W_0^{1,2}(\Omega)$.*

Theorem 3.3. *Every bounded sequence in a Hilbert space admits a weakly convergent subsequence. That means given $(v_k)_{k\in\mathbb{N}} \subset H$ bounded in Hilbert space H $\exists (v_{k_l})_{l\in\mathbb{N}} \exists v \in H$ s.t.*

$$\langle v_{k_l}, w\rangle_H \to \langle v, w\rangle_H \; \forall w \in H$$

Theorem 3.4. *If $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in H, then*

$$||u||_H \leq \liminf_{k\to\infty} ||u_k||_H$$

Remark 3.5. A functional $T : X \to Y$ is compact if $\forall (x_k)_{k\in\mathbb{N}}$ bounded in X, the sequence $(Tx_n)_{n\in\mathbb{N}} \subset Y$ admits a convergent subsequence.

Theorem 3.6. *The inclusion map $W_0^{1,2} \to L^2(\Omega)$ that maps $u \mapsto u$ is compact.*

3.1 Some concepts from functional analysis

Let X be a $\mathbb{R}$-Banach space.

Definition 3.7. We define the dual space

$$X' = L(X, \mathbb{R}) = \{T : X \to \mathbb{R} \text{ linear and continuous}\}$$

Lemma 3.8. *For linear maps, the following are equivalent:*

1. *continuity*
2. *continuity at 0*
3. *boundedness*

Definition 3.9. Similarly, the bidual space

$$X'' = L(X', \mathbb{R})$$

is also a Banach space. There exists a canonical map

$$i_X : X \to X''$$

For $x \in X$ we define the canonical map

$$i_x : X \to X''$$

by

$$i_X(x)(T) = Tx \text{ for } T \in X'$$

i_X is well-defined and $i_X(x)$ is an element of the bidual space. Moreover, the map is injective and an isometry. Spaces X where i_X is also surjective are also called reflexive spaces.

Example 3.10. *All Hilbert spaces, L^p spaces and all Sobolev spaces are reflexive for $1 < p < \infty$.*

Definition 3.11. Let $(x_k)_{k \in \mathbb{N}} \subset X$. x_k converges weakly to $x \in X$ if for all $T \in X'$

$$T(x_k) \to T(x) \text{ for } k \to \infty$$

Theorem 3.12. *Every bounded sequence in a reflexive space admits a weakly convergent subsequence.*

Theorem 3.13. *Let $(X, ||\cdot||)$ be a reflexive Banach space. Let $M \subset X$ and $M \neq \varnothing$ a weakly sequentially closed subset X. Let $E : M \to \mathbb{R}$ s.t.*

1. *$E(y) \to \infty$ if $||y||_X \to \infty$*
2. *E is sequentially weakly lower semi-continuous that is if x_k converges weakly to $x \in X$, than $E(x) \leq \liminf_{k \to \infty} E(x_k)$*

Then, E is bounded from below and $\exists x \in M$ s.t.

$$E(x) = \inf\{E(y) : \; y \in M\}$$

4 A partition problem and functions of bounded variation

Given $\Omega \subset \mathbb{R}^n$ bounded and smooth, is there an $E \subseteq \Omega$ s.t.

$$|E| = |\Omega \setminus E|$$

and $H^{n-1}(\partial E \cap \Omega)$ is minimal?

4.1 Motivation

Definition 4.1. If $E \subset \Omega$ is smooth, then

$$H^{n-1}(\partial E \cap \Omega) = \int_{\partial E \cap \Omega} 1 dS(x) = \int_{\partial E \cap \Omega} \langle v(x), v(x) \rangle dS(x)$$

Let $\varphi \in C : 0^\infty(\Omega; \mathbb{R}^n)$ s.t. $\varphi(x) = \lambda(x) v(x)$ where $\lambda(x) \in [0, 1]$ and $x \in \partial E$. Then

$$H^{n-1}(\partial E \cap \Omega) \geq \int_{\partial E \cap \Omega} \langle \varphi(x), v(x) \rangle dS(x) = \int_{\partial E} \langle \varphi(x), v(x) \rangle dS(x) = \int_{\Omega} \chi_E div(\varphi(x)) dx$$

Since D is smooth, $\exists \psi : \mathbb{R}^n \to \mathbb{R}^n$ with

- $\psi(x) = v(x)$ on $\partial E \cap \Omega$
- $\psi(x) = 0$ "some bit away from Ω"
- $||\psi(x)||_2 \leq 1$ $\forall x \in \mathbb{R}^n$ and $||\psi||_\infty \leq 1$.

Now, let ξ_ε be the bump function on Ω which converges to χ_Ω for $\varepsilon \to 0$. Then $\varphi_\varepsilon = \xi_\varepsilon \psi$ satisfies $\varphi_\varepsilon \in C_0^\infty(\Omega, \mathbb{R}^n)$ and $||\varphi_\varepsilon||_\infty \leq 1$. Now

$$H^{n-1}(\partial E \cap \Omega) \overset{\varepsilon \searrow 0}{\to} \int_{\partial E \cap \Omega} dS(x) = H^{n-1}(\partial E \cap \Omega)$$

Hence

$$H^{n-1}(\partial E \cap \Omega) = \sup\{\int_\Omega \chi_E div\varphi dx \mid \varphi \in C_0^\infty(\Omega, \mathbb{R}^n),\ ||\varphi||_\infty \leq 1\}$$

The idea is to use this expression for non-smooth E.

4.2 Functions of bounded variations

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Definition 4.2. We define for $v \in L^1(\Omega)$

$$\int_\Omega |Dv| = \sup\{\int_\Omega v(x) div\varphi(x) dx \mid \varphi \in C_0^\infty(\Omega; \mathbb{R}^n),\ ||\varphi||_\infty \leq 1\}$$

the total variation of v.
We say that v is of bounded variation, if

$$\int |Dv| < \infty$$

Finally, we define $BV(\Omega) = \{v \in L^1(\Omega) : \int_\Omega |Dv| < \infty\}$

Remark 4.3. It holds that $W^{1,1}(\Omega) \subset BV(\Omega)$. For $v \in W^{1,1}(\Omega)$ it holds that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v| \; dx$$

Finally, above inclusion is even strict: $W^{1,1}(\Omega) \subsetneq BV(\Omega)$

Lemma 4.4. *Let Ω be a bounded domain. Define $||\cdot||_{BV(\Omega)} : BV(\Omega) \to \mathbb{R}$ via*

$$||u||_{BV(\Omega)} = ||u||_{L^1} + \int_\Omega |Du|$$

This is a norm and $(BV(\Omega), ||\cdot||_{BV(\Omega)})$ is a Banach space.

Definition 4.5. Let $E \subset \mathbb{R}^n$ be measurable.

$$P(E,\Omega) = \int_\Omega |D\chi_{E\cap\Omega}|$$

is called the perimeter of E in Ω.

Remark 4.6. If ∂E is smooth, then $P(E,\Omega) = H^{n-1}(\partial E \cap \Omega)$

<u>QUESTION:</u> For $\Omega \subset \mathbb{R}^n$ bounded domain, define

$$M = \{E \subset \Omega \mid \lambda^n(E) = \lambda^n(\Omega \setminus E) = \frac{1}{2}\lambda^n(\Omega)\}$$

Is

$$\inf\{P(E,\Omega) \mid E \in M\}$$

attained?

<u>IDEA:</u> Take $(E_n)_{n\in\mathbb{N}} \subset \Omega$ a minimizing sequence, that is

$$P(E_n,\Omega) = \int_\Omega |D\chi_{E_n\cap\Omega}| \searrow \inf\{P(E,\Omega) \mid E \in M\}$$

4.3 <u>Lower semicontinuity in $BV(\Omega)$</u>

Theorem 4.7. *Let Ω be a bounded domain. Let $(v_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ s.t. $v_k \to v$ in $L^(\Omega)$. Then*

$$\int_\Omega |Dv| \leq \liminf \int_\Omega |Dv_k|$$

In particular, if $\liminf_{k\to\infty} \int_\Omega |Dv_k| < \infty$ then $v \in BV(\Omega)$.

4.4 Lipschitz continuity

Definition 4.8. A function $v : \bar{\Omega} \to \mathbb{R}$ is called Lipschitz continuous if

$$[v]_{0,1} = \sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|} < \infty$$

Using this, one defines $Lip(\Omega) = \{v : \bar{\Omega} \to \mathbb{R} : v \text{ is Lipschitz}\} = C^{0,1}(\Omega)$.
$Lip_{loc}(\Omega) = \{v : \Omega \to \mathbb{R} : v \in Lip(K)\ \forall K \subset \Omega \text{ compact}\}$

Lemma 4.9. *$||\cdot||_{Lip} : Lip(\Omega) \to \mathbb{R}$ defined by*

$$||u||_{Lip} = [u]_{0,1} + ||u||_{C^0}$$

makes $Lip(\Omega)$ a Banach space.

Theorem 4.10. *Let $v \in Lip(\mathbb{R}^n)$. Then v is a.e. classically differentiable. If Ω is a domain, then $v \in Lip_{loc}(\Omega) \Leftrightarrow v \in W^{1,\infty}_{loc}(\Omega)$. If $\partial\Omega$ is C^1, then $v \in Lip(\Omega) \Leftrightarrow v \in W^{1,\infty}(\Omega)$. In this case $[v]_{0,1} = ||\nabla v||_{L^\infty}$.*

Definition 4.11. $Lip(\Omega, g) = \{u \in Lip(\Omega) \mid u \equiv g \text{ on } \partial\Omega\}$
$Lip(\Omega, g, K) = \{u \in Lip(\Omega, g) \mid [u]_{0,1} \leq K\}$

The aim is now to minimize F in $Lip(\Omega, g)$ where

$$F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx$$

Method: Solve minimization in $Lip(\Omega, g, K)$ and then show that the minimizer satisfies $[u]_{0,1} < K$ strictly for a large enough K and conclude that u minimizes F in $Lip(\Omega, g)$.

Lemma 4.12. *Let Ω be a bounded C^1-domain, $g \in Lip(\Omega)$ with $[g]_{0,1} \leq K$. Then $\exists u \in Lip(\Omega, g, K)$ s.t.*

$$F(u) \leq F(v)\ \forall v \in Lip(\Omega, g, K)$$

Lemma 4.13. *Let u^k be a minimizer of F in $Lip(\Omega, g, k)$. If $[u^k]_{0,1} < k$ then u^k minimizes F in $Lip(\Omega, g)$.*

Definition 4.14. Let $v \in Lip(\Omega, K)$. We call v a superminimum (resp. subminimum) for F if $\forall w \in Lip(\Omega, v, K)$ s.t. $w \geq v$ in Ω then $F(v) \leq F(w)$ (resp. $w \leq v$).

Theorem 4.15 (Maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $v \in Lip(\Omega, K)$ be a superminimum, $w \in Lip(\Omega, K)$ be a subminimum s.t. $v \geq w$ on $\partial\Omega$. Then $v \geq w$ in Ω.*

Corollary 4.16. *Let $\Omega \subset \mathbb{R}^n$ open and bounded. Let $v \in Lip(\Omega, K)$ be a superminimum and $w \in Lip(\Omega, K)$ a subminimum. Then*

$$\sup_{\partial\Omega} w - v = \sup_{\Omega} w - v$$

Corollary 4.17. *If $u, \tilde{u} \in Lip(\Omega, g, K)$ are minima of F in $Lip(\Omega, f, K)$ then $u \equiv \tilde{u}$.*

Goal: Prove that u^k satisfies

$$\sup_{x,y\in\Omega, x\neq y} \frac{\left|u^k(x) - u^k(y)\right|}{|x-y|} = [u^k]_{0,1} < K$$

Lemma 4.18. *Let $\Omega \subset \mathbb{R}^n$ bounded and open. Let $u \in Lip(\Omega, g, K)$ be a minimum of F. Then*

$$[u]_{0,1} = \sup_{x\in\Omega, y\in\partial\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|}$$

Lemma 4.19. *Let $a \in \mathbb{R}$, $z \in \mathbb{R}^n$ and $w : \mathbb{R}^n \to \mathbb{R}$ via $w(c) = a + zx$. Then w minimizes F in $Lip(\Omega, w)$.*

5 Obstacle problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with $\partial\Omega \in C^1$. Let $h \in Lip(\Omega)$ s.t. $h|_{\partial\Omega} < 0$ and $h > 0$ somewhere in Ω.
Problem: Is $\inf\{F(u) : u \in Lip(\Omega), u|_{\partial\Omega} = 0, u \geq h\}$ attained? As before $F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}$. That is, does a $u \in M = \{v \in Lip(\Omega) : v|_{\partial\Omega} = 0, v \geq h\}$ s.t.

$$F(u) \leq F(v) \; \forall v \in M$$

exist?
We call h the obstacle.
The difference of this problem to the previous problems is in the first variation. If u is a minimizer in the previous problems, then $\frac{d}{dt}F(u + t\varphi)|_{t=0} = 0 \; \forall \varphi \in C_0^\infty(\Omega)$. Here we have only information form some directions. Since M is convex for all $v \in M$ $tu + (1-t)v$ is again in M for $t \in [0,1]$. Hence, if u is a minimizer, $t \mapsto F(tu + (1-t)v)$ has a minimum at $t = 1$. Hence

$$\frac{d}{dt}F(tu + (1-t)v)|_{t=1} = \int_\Omega \frac{\langle \nabla(tu + (1-t)v), \nabla(u-v)\rangle}{1 + |\nabla(tu + (1-t)v)|^2} \, dx|_{t=1} \leq 0$$

This can be rearranged to

$$\int_\Omega \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \nabla(u-v)\,dx \geq 0\,\forall v \in M$$

This is called a variational inequality.
The strategy to show the existence of a minimizer consists of the following steps

1. add another constraint $[u]_{0,1} \leq k$

2. if minimizer satisfies this with strict inequality then it is a minimizer without the additional constraint

Let $M^k = \{u \in M : [u]_{0,1} \leq k\}$ for $k > [h]_{0,1}$ and note that $\max\{0, h\} \in M^k$. Notice that M^k is also convex.

Lemma 5.1. *$\exists u \in M^k$ s.t. $F(u) \leq F(v)\,\forall v \in M^k$.*

Lemma 5.2. *Let u^k be a minimizer for F in M^k s.t. $[u^k]_{0,1} < k$. Then u^k minimizes F in M.*

Our aim is now to find an a-priori estimate for solutions of variational inequalities of the following type

$$\int_\Omega a(\nabla u)\cdot(\nabla(v-u))\,dx \geq 0\,\forall v \in M \cup M^k$$

where a satisfies the following strong ellipticity condition

$$(a(p)-a(q))\cdot(p-q) > 0\,\forall penis, q \in \mathbb{R}^n, p \neq q$$

Theorem 5.3. *Let $u \in M^k$ be a solution of*

$$\int_\Omega a(\nabla u)\cdot\nabla(v-u)\,dx \geq 0\;\forall v \in M^k$$

where a satisfies the strong ellipticity. Then

$$\sup_{x\neq y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|} \leq \max\left\{[h]_{0,1}, \sup_{x\in\Omega.y\in\partial\Omega}\frac{|u(x)-u(y)|}{|x-y|}\right\}$$

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a convex, bounded domain with $\partial\Omega \in C^1$. Let $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the strong ellipticity and $h \in Lip(\Omega)$, $h|_{\partial\Omega} < 0$ but $h > 0$ somewhere in Ω. Let*

$k > [h]_{0,1}$ and $u \in M^k$ be a solution of

$$\int_\Omega a(\nabla u) \cdot \nabla(v-u)dx \geq 0 \, \forall v \in M^k$$

Then

1. *$u \geq 0$*
2. *u is unique*
3. *$[u]_{0,1} \leq [h]_{0,1} < k$*

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with $\partial\Omega \in C^1$. Let $h \in Lip(\Omega)$, $h|_{\partial\Omega} < 0$ and $h > 0$ somewhere in Ω. Then $\exists! u \in Lip(\Omega)$, $u|_{\partial\Omega} = 0$ and $u \geq h$ in Ω.*

$$F(u) \leq F(v) \forall v \in Lip(\Omega), \; v|_{\partial\Omega)=0}$$

where $F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx$. Moreover, u solves

$$\int_\Omega \frac{\nabla u}{1 + |\nabla u|^2} \nabla(v-u)\, dx \geq 0$$

and u satisfies $[u]_{0,1} \leq [h]_{0,1}$ and $||u||_{L^\infty} \leq [h]_{0,1} diam(\Omega)$.
Moreover, define $I = \{x \in \Omega : u(x) = h(x)\}$ the coincidence set, then

$$\int_\Omega \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \nabla\varphi\, dx = 0 \, \forall \varphi \in Lip(\Omega), \; supp(\varphi) \subset \Omega \setminus I$$

Remark 5.6. Is $I \neq \varnothing$? Is it connected and what is its measure?

6 The Schrödinger equation

We describe an electron in a universe with only one nucleus that attracts the electron. Motivated by the uncertainty principle of Heisenberg, the electron is described by a function $\psi : \mathbb{R}^n \to \mathbb{C}$, $||\psi||_{L^2(\mathbb{R}^n)} = 1$ a wave function.
Idea: $\Omega \subset \mathbb{R}^n$ open, $\int_\Omega |\psi(x)|^2 \, dx$ is the probability that the electron is in Ω.
We consider $n \geq 3$. Define

- Kinetic energy: $\int_{\mathbb{R}^n} |\nabla\psi|^2 \, dx = T(\psi)$
- potential energy: $V(\psi) = \int_{\mathbb{R}^n} v(x) \, |\psi(x)|^2 \, dx$. Typical choice would be $n = 3$ (for obvious reasons), $v(x) = -\frac{z}{|x|}$ with a nucleus of charge z at the origin.

- Total energy: $E(\psi) = T(\psi) + V(\psi)$

Question: Is

$$E_0 = \inf\{E/\psi) : \psi \in W^{1,2}(\mathbb{R}^n),\ ||\psi||_{L^2(\mathbb{R}^n)} = 1\}$$

attained? E_0 is called the ground state energy. If a minimum exists, it is called ground state.

Definition 6.1. Let $f : \mathbb{R}^n \to \mathbb{C}$ then we say that f vanishes at ∞ if $\forall a > 0$

$$\lambda^n(\{x \in \mathbb{R}^n : |f(x)| > a\}) < \infty$$

We define

$$D^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \mid \nabla f \in L^2(\mathbb{R}^n),\ \text{and } f \text{ vanishes at } \infty\}$$

Theorem 6.2. *Let $n \geq 3$. Then $\exists S_n > 0$ s.t. $\forall f \in D^1(\mathbb{R}^n)$*

$$S_n ||f||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq ||\nabla f||_{L^2(\mathbb{R}^n)}$$

Remark 6.3. Let $n \geq 3$, $V \in L^{\frac{n}{2}} + L^\infty(\mathbb{R}^n)$. That means $v = v_1 + v_2$ where $v_1 \in L^{\frac{n}{2}}$, $v_2 \in L^\infty(\mathbb{R}^n)$. Then $\exists \lambda \in \mathbb{R}$ s.t. $\forall \psi \in W^{1,2}(\mathbb{R}^n)$ with $||\psi||_{L^2} = 1$

$$E(\psi) \geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 - \lambda - ||v_2||_{L^2} \geq -\lambda - ||v_2||_{L^2}$$

Using this result, we can conclude that the potential energy is even continuous. That is

$$\int_{\mathbb{R}^n} V(x)\, |\psi_j(x)|^2\, dx \to \int_{\mathbb{R}^n} V(x)\, |\psi_0(x)|^2\, dx$$

for a convergent subsequence of a minimizing sequence ψ_j.

Lemma 6.4. *If $f_k \rightharpoonup f$ in $W^{1,2}(\mathbb{R}^n)$, let $A \subset \mathbb{R}^n$ measurable s.t. $\lambda^n(A) < \infty$. Then*

$$\chi_A f_k \to \chi_A f \ \text{in } L^p(\mathbb{R}^n)$$

for $1 \leq p \leq \frac{2n}{n-2}$.

Lemma 6.5. *Under the same assumptions as in 6.3, and the additional one that v vanishes at ∞, i.e. $\forall a > 0$ it holds that $\lambda^n(\{|V(x)| > a\}) < \infty$, then if $\psi_j \rightharpoonup \psi$ in $W^{1,2}(\mathbb{R}^n)$, $||\psi_j||_{L^2} = 1\ \forall j$ then*

$$\int_{\mathbb{R}^n} V(x)\, |\psi_j(x)|^2\, dx \to \int_{\mathbb{R}^n} V(x)\, |\psi(x)|^2\, dx$$

Theorem 6.6 (existence of the minimizer)**.** *Let* $v \in L^{\frac{n}{2}} + L^\infty$ *and let* v *vanish at* ∞. *If* $E_0 = \inf\{E(\psi) \mid \psi \in W^{1,2}(\mathbb{R}^n), ||\psi||_{L^2(\mathbb{R}^n)}\} < 0$, *then* $\exists \psi_0 \in W^{1,2}(\mathbb{R}^n)$, $||\psi_0|| = 1$ *s.t.* $E(\psi_0) = E_0$. *Moreover,* ψ_0 *is a weak solution of*

$$-\Delta\psi_0 + v\psi_0 = E_0\psi_0$$

i.e. $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \nabla\psi_0\nabla\varphi + v\psi_0\varphi = E_0\int_{\mathbb{R}^n} \psi_0\varphi$$

Remark 6.7.

1. E_0 in the Euler-Lagrange equation can be considered as a Lagrange-multiplier ensuring that the constraint $||\psi||_{L^2} = 1$.

2. We can write
$$E_0 = \inf_{\substack{\psi\in W^{1,2}\\ \psi\neq 0}} \frac{E(\psi)}{||\psi||_{L^2}^2}$$

3. The first condition is important to ensure existence of a minimizer. Indeed, if $v \equiv 0$ there is no ground state ψ_0.
   Let $u \in C_0^\infty(\mathbb{R}^n)$ s.t. $||u||_{L^1} = 1$. Then for $\lambda > 0$ $u_\lambda(x) = \lambda^{\frac{n}{2}}u(\lambda x)$ is a wave function.
$$E(u\lambda) = T(u_\lambda) = \lambda^2 T(u) \overset{\lambda\searrow 0}{\rightarrow} 0$$
   Hence $E_0 = 0$ and $\nexists \psi \in W^{1,2}(\mathbb{R}^n)$ with $||\psi||_{L^2} = 1$ s.t. $E(\psi) = 0$. So a minimizer does not exist.

Remark 6.8. If $E_0 < 0$, one can look at excited states. With the same ideas, if

$$E_1 = \inf\{E(\psi) \mid \psi \in W^{1,2}(\mathbb{R}^n), ||\psi||_{L^2} = 1, \langle\psi, \psi_0\rangle_{L^2} = 0\} < 0$$

there exists $\psi_1 \in W^{1,2}$, $||\psi_1||_{L^2} = 1$, $\langle\psi_1, \psi_0\rangle_{L^2} = 0$ s.t.

$$E(\psi_1) = E_1$$

and one can continue

$$0 > E_k = \inf\{E(\psi) \mid \psi \in W^{1,2}, ||\psi||_{L^2} = 1, \langle\psi, \psi_j\rangle_{L^2} = 0,\ 0 \leq j \leq k-1\}$$

ψ_k satisfies

$$-\Delta\psi_k + v\psi_k = E_k\psi_k,\ E_k = E(\psi_k)$$

and

$$E_k = \inf_{\substack{\psi \in W^{1,2} \\ \psi \perp_{L^2} \psi_j \\ 0 \le j \le k-1}} \frac{E(\psi)}{||\psi||_{L^2}^2}$$

7 Parametric minimal surfaces (the plateau problem)

Let $B = B_1(0) \subset \mathbb{R}^2$ be the unit disk and

$$\gamma : S^1 \to \mathbb{R}^m,\ m \geq 3$$

a C^2 curve that is simple, i.e. γ has no self intersection.
Aim: find a surface of disk type whose boundary is given by γ and with minimal area.

7.1 Some basic concepts from differential geometry

Definition 7.1. A surface $S \subset \mathbb{R}^m$, $m \geq 3$ is a 2-dimensional C^l-submanifold of $\mathbb{R}^m$ where $l \geq 1$, that is $\forall p \in S$ $\exists (U, F, V)$ where $U \subset \mathbb{R}^2$ open. $V \subset \mathbb{R}^m$ open, $p \in V \cap S$ and $F : U \to V \cap S$, $F \in C^2$ is a homeomorphismus and $\forall u \in U$ is $rank(DF(u)) = 2$.

Remark 7.2. The definition implies that locally the graph can be described by the graph of a $u : U \to \mathbb{R}^{m-2}$ and locally the surface can be described as a level set $S \cap V = \{x \in V : g(x) = 0\}$.

Definition 7.3 (tangential space)**.** Let S be a surface in $\mathbb{R}^m$, $p \in S$,

$$T_pS = \{v \in \mathbb{R}^m : \exists c : (-\varepsilon, \varepsilon) \to S \in C^1 \text{ s.t. } c(0) = p, C'(0) = v\}$$

T_pS is the tangential space to S at p with $\dim T_pS = 2$.

If (U, F, V) is a local chart at $p \in S$, then

$$T_pS = span\left\{\frac{\partial F}{\partial u}(u, v), \frac{\partial F}{\partial v}(u, v)\right\}$$

$U \subset \mathbb{R}^2$, $FU \to V \subset \mathbb{R}^m$. Aim: define a scalar product in T_pS. The first fundamental form is

$$g_p : T_pS \times T_pS \to \mathbb{R}$$

with

$$(u, v) \mapsto \langle u, v \rangle_{\mathbb{R}^m}$$

If (U, F, V) is a local chart at p, for $w_1, w_2 \in T_pS$, $w_1 = w_1^1 \frac{\partial F}{\partial u} + w_1^2 \frac{\partial F}{\partial V}$, $w_2 = w_2^1 \frac{\partial F}{\partial u} + w_2^2 \frac{\partial F}{\partial v}$ so that we can represent the first fundamental form with respect to this basis via the matrix $(g_{i,j}(u,v))_{i,j=1}^2$ by

- $g_{1,1}(u,v) = \langle \frac{\partial F}{\partial u}(u,v), \frac{\partial F}{\partial v}(u,v) \rangle$,
- $g_{1,2}(u,v) = g_{2,1}(u,v) = \langle \frac{\partial F}{\partial u}(u,v), \frac{\partial F}{\partial v}(u,v) \rangle$
- $g_{2,2}(u,v) = \langle \frac{\partial F}{\partial v}(u,v), \frac{\partial F}{\partial}(u,v) \rangle$.

We want to define for

$$f : S \to \mathbb{R}$$

continuous $\int_S f(x) dS(x)$
First step: Let (U, F, V) be a local chart and consider f such that $supp(f) \subset V \cap S$.

$$\int_S f(x) dS(x) = \int_u f(F(u,v)) \sqrt{\det(g_{i,j}(u,v))_{i,j=1}^2}\, du\, dv$$

Second step: If S is compact, hen $\exists$ finitely many charts $(U_i, F_i, V_i)_{i=1}$ s.t. $S \subset \bigcup_{i=1}^N (V_i \cap S)$. Then consider $(\psi_i)_{i=1}^N$ a partition of unity associated to this cover.

$$\psi_i \in C_0^\infty(\mathbb{R}^m; [0,1]), supp(\psi_i) \subset V_i,\ \forall i = 1, ..., N \text{ and } 1 = \sum_{i=1}^N \psi_i(x)\ \forall x \in S$$

Then for $fS \to \mathbb{R}$ continuous,

$$\int_S f(x)\, dS(x) = \sum_{i=1}^N \int_s (f\psi_i)(x)\ dS(x)$$

If S is described by only one chart, one can see that

$$area(S) = \int_U \sqrt{\left|\frac{\partial F}{\partial u}\right|^2 \left|\frac{\partial F}{\partial v}\right|^2 - \left(\frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial v}\right)^2}\ d(u,v)$$

Remark 7.4. The area is well-defined for $F \in W^{1,2}(B, \mathbb{R}^m)$.

Definition 7.5.

$$D(F) = \frac{1}{2} \int_B \left|\frac{\partial F}{\partial u}\right|^2 + \left|\frac{\partial F}{\partial v}\right|^2\ d(u,v)$$

is called the Dirichlet energy of F.

7.2 Setup for the problems

Define for $\gamma : \partial B_1(0) \to \mathbb{R}^m$, $m \geq 3$, a closed and simple curve

$$K(\gamma) = \{F \in W^{1,2}(B, \mathbb{R}^m) : F|_{\partial B} \in C^0(\partial B, \mathbb{R}^m) \text{ is a monotone parametrization of } \gamma\}$$

Question 1: $\exists F \in K(\gamma)$ s.t. $A(F) \leq A(\tilde{F})\ \forall \tilde{F} \in K(\gamma)$? This is a difficult question.
Question 2: $\exists F \in K(\gamma)$ s.t.

$$D(F) \leq D(\tilde{F})\ \forall \tilde{F} \in K(\gamma)$$

where $D(F)$ is defined in 7.5.
As we shall see, these two questions are equivalent.

Remark 7.6. $K(\gamma) \neq \varnothing$.

7.3 Euler-Lagrange equation for question 2

Lemma 7.7. *Let $F \in K(\gamma)$ be a minimizer of the Dirichlet energy in $K(\Gamma)$. Then $\Delta F = 0$ on B i.e. F is harmonic and $F \in C^\infty(B)$.*

Lemma 7.8. *If $f \in L^1_{loc}(\Omega)$ such that $\int_\Omega f \Delta\varphi\, dx = 0\ \forall \varphi \in C_0^\infty(\Omega)$. Then $f \in C^\infty(\Omega)$ and $\Delta f = 0$.*

As an abbreviation for the integral making up the Dirichlet-energy $D(F)$ we write $|DF|^2$.

Remark 7.9. Let $X : B_1(0) \to \mathbb{R}^m$ s.t. $\Delta X = 0$. Let $(u_1, u_2) \in B_1(0)$. Then the map

$$z = u_1 + iu_2 \mapsto g(z) = |\partial_{u_1} X|^2 - |\partial_{u_2}|^2 - 2i\partial_{u_1} X \partial_{u_2} X$$

is holomorphic. Indeed by the Cauchy-Riemann equations, g is holomorphic $\iff$ $\partial_{\overline{z}} g = 0$ where

$$\partial_{\overline{z}} g = \frac{1}{2}\left(\partial_{u_1} g + i\partial_{u_2} g\right)$$

It is easily verifiable that $\partial_{\overline{z}} g = 4\partial_z X \cdot \partial_{\overline{z}}\partial_z X = \partial_z X \Delta X$.

Theorem 7.10. *Let $F \in K(\gamma)$ be a critical point of the Dirichlet-energy in $K(\gamma)$. By that we mean that*

1. $\frac{d}{dt} D(F + t\varphi)|_{t=0} = 0\ \forall \varphi \in C_0^\infty(B; \mathbb{R}^m)$
2. $\frac{d}{dt} D(F \circ \varphi_t^{-1}, B_t)|_{t=0} = 0$ *where* $(\varphi_t)_t$ *is a family of diffeomorphisms.* $\varphi_t = Id + t\psi$ *with* $\psi \in C^2(\overline{B_1(0)})$ *and* $\varphi_t : B_1(0) \to B_t = \varphi(B_1(0))$.

Then F solves the following problem:

- $\Delta F = 0$ *in* $B_1(0)$
- $F|_{\partial B_1(0)}$ *is a monotone parametrization of* $\gamma(\partial B_1(0))$
- $|\partial_{u_1} F| = |\partial_{u_2} F|$
- $\langle \partial_{u_1} F, \partial_{u_2} F \rangle = 0$

The last two properties yield that F is a so-called conformal parametrization.

Remark 7.11. If X minimizes the Dirichlet-energy, then X satisfies the second point in the previous theorem.

Lemma 7.12. *If $u \in C^1_{b'}(B;\mathbb{R})$, $r \in (0,1)$*

$$u_r : \partial B_1(0) \to \mathbb{R}$$

$u_r(e^{i\varphi}) = u(re^{i\varphi})$. Then $\exists u_{bd} : \partial B_1(0) \to \mathbb{R}$ s.t.

$$u_r \to u_{bd}$$

in $L^2(\partial B)$.

Definition 7.13. Define $K^*(\gamma) = \{F \in C^1_{b'}(B;\mathbb{R}^m)\}$.

Let $RF \in C^0(\partial B_1(0);\mathbb{R}^m)$ s.t. RF is a weak monotone parametrization of $\Gamma \circ \gamma(\partial B_1(0))$.
IDEA: Take $(\tilde{F}_k)_{k\in\mathbb{N}} \subset K^+(\gamma)$ be a minimizing sequence, $R\tilde{F}_k \in C^0(\partial B_1(0);\mathbb{R}^m)$. We take F_k s.t.

$$-\Delta F_k = 0 \text{ in } B_1(0)$$

$$F_k = R\tilde{F}_k \text{ on } \partial B_1(0)$$

7.4 The Dirichlet problem on the disk

LEt $B = B_1(0) \subset \mathbb{R}^2$ and $\varphi : \partial B_1(0) \to \mathbb{R}$. We study the problem

$$\begin{cases} -\Delta u = 0 & \text{in } B \\ u = \varphi & \text{on } \partial B \end{cases}$$

Note that $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_j^2}$.

Lemma 7.14. *If $u \in C^0(\overline{B})$ is harmonic in B, then*

$$u(x) = \frac{1}{2\pi}(1 - ||x||^2) \int_{\partial B} \frac{u(y)}{|x-y|^2} d_S(y)$$

for $x \in B$.

This suggests to consider

$$E : L^1(\partial B) \to C^\infty(B)$$

where

$$\varphi \mapsto (E_\varphi)(x) = \frac{1}{2\pi}(1 - ||x||^2) \int_{\partial B} \frac{\varphi(y)}{|x-y|^2}\, ds(y)$$

with $x \in B$. We call this the harmonic extension.

Theorem 7.15. *For $\varphi \in C^0(\partial B)$, $\forall x_0 \in \partial B$*

$$\exists \lim_{\substack{x \to x_0 \\ x \in B}} (E_\varphi)(x) = \varphi(x)$$

Hence $E|_{C^0(\partial B)} : C^0(\partial B) \to C^\infty(B) \cap C^0(\overline{B})$.

Theorem 7.16. *Let $v \in C^1_{b'}(B)$ and $v_{bd} \in L^2(\partial B_1(0))$ its restriction to the boundary as before. Let $h = E(v_{bd})$ be the harmonic extension of v_{bd} Then*

$$D(h) \leq D(v)$$

Theorem 7.17 (Maximum principle)**.** *Let $u \in C^0(\overline{B})$ be harmonic in B. Then*

$$\max_{x \in \overline{B}} u(x) = \max_{x \in \partial B} u(x)$$

The same holds for min.

Theorem 7.18 (Poisson formula)**.** *If u is harmonic in B and $K \subset B$ compact, then $\exists d > 0$ s.t.*

$$\sup_K |D_u^\alpha| \leq \left(\frac{2\,|\alpha|}{d}\right)^{|\alpha|} \sup_B u$$

In particular, the uniform limit of harmonic functions is harmonic.

8 Solution of the Plateau Problem

Recall: We want to minimize $D(F) = \frac{1}{2}\int_B |D_xF|^2 + |D_yF|^2\, d(x,y)$ for $F \in K^*(y)$.

Lemma 8.1. *Let $\Phi \in Aut(B)$, $f \in W^{1,2}(B_1(0))$ then $D(f) = D(f \circ \Phi)$.*

Definition 8.2. Let $P_j = \gamma(e^{i(j-1)\frac{2\pi}{3}})$ for $j = 1, 2, 3$.

Lemma 8.3. *Let $f \in C^0(\overline{B}, \mathbb{R}^m) \cap C^1_{b'}(B, \mathbb{R}^m)$, $w_0 \in \overline{B}$ and (r, φ) the polar coordinates centred at w_0. On $B_2(w_0) \cap B$ let*

$$\tilde{f}(r, \varphi) = f(w_0 + re^{i\varphi})$$

Then $\forall \delta \in (0,1)$ $\exists \varrho \in [\delta, \sqrt{\delta}]$ s.t. on

$$C_\varrho = \partial B_\varrho(w_0) \cap B$$

we have

$$\int_{C_\varrho} \left|\partial_\varphi \tilde{f}\right|^2 ds(\varphi) \leq \frac{4\varrho}{|\log \varphi|} D(f)$$

Theorem 8.4. *Let $\Gamma \subset \mathbb{R}^m$ be a closed simple curve,*

$$\Gamma = \gamma(\partial B_1(0))$$

where $\gamma : \partial(B_1(0)) \to \mathbb{R}^m$ s.t.

$K^*(\gamma) = \{F \in C^1_{b'}(B_1(0), \mathbb{R}^m) : RF \in C^0(\partial B_1(0), \mathbb{R}^m)$ *a weak monotone partition of* $\Gamma\} \neq \varnothing$

Then $\exists F \in K^*(\gamma)$ s.t. $D(F) \leq D(\tilde{F})$ $\forall \tilde{F} \in K^*(\gamma)$.

Theorem 8.5. *It holds that $A(X) \leq D(X)$ and equality holds iff X is a conformal parametrization. Further, minimizers of the Dirichlet energy are harmonic and conformal parametrizations.*

Theorem 8.6. *Let $\Gamma = \gamma(\partial B_1(0))$ be a closed simple curve, $\gamma \in C^2$ s.t. $K^*(\gamma) \neq \varnothing$. Then $\exists F \in K^*(\gamma)$ s.t.*

$$D(F) \leq D(\tilde{F})\ \forall \tilde{F} \in K^*(\gamma)$$

Theorem 8.7. *Let $F \in W^{1,2}(B_1(0), \mathbb{R}^m) \cap C^0(\overline{(}B_1(0)), \mathbb{R}^m)$. Then $\forall \varepsilon > 0$ $\exists \tau_\varepsilon : \overline{B_1(0)} \to \overline{B_1(0)}$ homeomorphism s.t. $\tau_\varepsilon \in W^{1,2}$,*

$$F \circ \tau_\varepsilon \in W^{1,2}(B_1(0), \mathbb{R}^m) \cap C^0(\overline{B_1(0)}, \mathbb{R}^m)$$

and

$$D(F \circ \tau_\varepsilon) \leq A(F \circ \tau_\varepsilon) + \varepsilon$$

Corollary 8.8 (Minimizer of the area functional)**.** *Under the assumptions of the previous two theorems*

$$\inf_{F \in K^*(\gamma)} A(F) = \inf_{F \in K^*(\gamma)} D(F)$$

and the infima are attained.

9 A reaction-diffusion equation

Let $p > 1$, $\Omega \subset \mathbb{R}^n$ a C^1-domain, $n \geq 2$ and $\lambda \in \mathbb{R}$. We consider

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1} u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

$u \in W_0^{1,2}(\Omega) \left(= \overline{C_0^\infty(\Omega)}^{||\cdot||_{W^{1,2}}} \right)$ is called a weak solution to this differential equation if

$$\int_\Omega \nabla u \nabla \varphi \, dx = \lambda \int_\Omega u\varphi + \int_\Omega |u|^{p-1} u\varphi \ \forall \varphi \in C_0^\infty(\Omega)$$

Remark 9.1. $u \equiv 0$ is a weak solution. For sufficiently small λ there are other weak solutions.

10 The mountain pass problem

10.1 The Palais-Smale Condition

Let H be a Hilbert space on $\mathbb{R}$. We denote the scalar product by $\langle \cdot, \cdot \rangle$. We use the usual definition of Fréchet-differentiability.

Definition 10.1. Let $E \in C^1(H; \mathbb{R})$. If E is Fréchet-differentiable at u $\forall u \in H$ and the map

$$\begin{cases} E' : H \to L(H; \mathbb{R}) \\ u \mapsto E'(u) \end{cases}$$

is continuous

Definition 10.2. Let $E \in C^1(H; \mathbb{R})$.

1. A sequence $(u_k)_{k \in \mathbb{N}} \subset H$ is called a Palais-Smale sequence for E if

- $\exists \lim_{k\to\infty} E(u_k)$
- $\exists \lim_{k\to\infty} ||\nabla E(u_k)||_H = 0$

2. E satisfies a Palais-Smale condition if every Palais-Smale sequence admits a convergent subsequence (w.r.t. the norm in H).

Definition 10.3. Let $E \in C^1(H;\mathbb{R})$. For $\beta \in \mathbb{R}$, $\delta > 0$, $\varrho > 0$, define

$$K_\beta = \{u \in H \mid E(u) = \beta \wedge \nabla E(u) = 0\}$$

$$E_\beta = \{u \in H \mid E(u) < \beta\}$$

$$N_{\beta,s} = \{u \in H \mid |E(u) - \beta| < \delta \wedge ||\nabla E(u)|| < \delta\}$$

$$U_{\beta,\varrho} = \{u \in H \mid \exists r \in K_\beta : \ ||u - v||_H < \varrho\} = \bigcup_{u \in K_\beta} B_\varrho(u)$$

Lemma 10.4. *Let $E \in C^1(H;\mathbb{R})$ that satisfies the Palais-Smale condition. Then for $\beta \in \mathbb{R}$*

1. *K_β is compact*
2. *$(N_{\beta,\delta})_{\delta>0}$ is a neighbourhood basis of k_β, i.e. Let U open s.t. $U \supset K_\beta$ then $\exists \delta > 0$ s.t. $U \supset N_{\beta,\delta}$*
3. *$(U_{\beta,\varrho})_{\varrho>0}$ is a neighbourhood basis of K_β.*

10.2 A deformation Lemma in H

PROBLEM: In $\mathbb{N}^n$, we could use uniform continuity on compact sets. This no longer holds.
SOLUTION I: Take a deformation $\frac{\partial}{\partial t}\Phi(x,t) \approx -\nabla E(\Phi(x,t))$. This however requires $E \in C^{1,1}$.
SOLUTION II: This works for $e \in C^1$.

Definition 10.5. Let X be a topological space.

1. a covering $(U_i)_{i\in I}$ is a refinement of a cover $(V_j)_{j\in J}$ if $\forall i \in I\ \exists j \in J$ such that $U_i \subset V_j$.
2. a covering of X is called locally finite if $\forall x \in X\ \exists U$ neighbourhood of X such that $U \cap U_i \neq \varnothing$ for finitely many i.
3. X is called para-compact if X is Hausdorff and to any open covering X there exists a locally finite refinement.

Theorem 10.6. *Every metrizable topological space is paracompact.*

Definition 10.7. Let $E \in C^1(H;\mathbb{R})$, $\tilde{H} = \{u \in H \mid \nabla E(u) \neq 0\}$. Then $G : \tilde{H} \to H$ is a locally Lipschitz map is pseudogradient for E if $\forall u \in \tilde{H}$ it holds that

$$||G(u)||_h < 2 \min\{1, ||\nabla E(u)||_H\}$$

$$\langle G(u), \nabla E(u) \rangle > \frac{1}{2} \min\{1, ||\nabla E(u)||_H\} \, ||\nabla E(u)||$$

Theorem 10.8. *Let $E \in C^1(H;\mathbb{R})$, $\tilde{H}$ as above, then E admits a pseudogradient $G : \tilde{H} \to H$*

Theorem 10.9. *Let $E \in C^1(H)$ satisfy the PS condition. Let $\beta \in \mathbb{R}$, $\varepsilon_0 > 0$ and N be a neighbourhood of $K_\beta = \{u \in H \mid E(u) = \beta \wedge \nabla E(u) = 0\}$ (note that $K_\beta = \varnothing, N = \varnothing$ are fine as well).*
The $\exists \varepsilon \in (0, \varepsilon_0)$ and a continuous family of homeomorphisms $\Phi(t, \cdot) : H \to H$, $t \in \mathbb{R}$ such that

1. *$\Phi(t, u) = u$ if $t = 0$ or $\nabla E(u) = 0$ or $|E(u) - \beta| \geq \varepsilon_0$.*

2. *$\forall v \in H$ it holds that $t \mapsto E(\Phi(t, u))$ is non-increasing*

3. *$\Phi(1, E_{\beta+\varepsilon} \setminus N) \subset E_{\beta-\varepsilon}$ and $\Phi(1, E_{\beta+\varepsilon}) \subset E_{\beta-\varepsilon} \cup N$ where $E_\alpha = \{u \in H \mid E(u) < \alpha\}$*

Moreover

$$\Phi(t, \cdot) \circ \Phi(s, \cdot) = \Phi(t + s, \cdot)$$

10.3 Mountain Pass Theorem

Theorem 10.10. *Let H be a Hilbert space, $E \in C^1(H)$ satisfy the PS condition. Moreover, assume*

1. *$E(0) = 0$*

2. *$\exists \varrho > 0, \exists \alpha > 0$ s.t.*
$$||u||_H = \varrho \Rightarrow E(u) \geq \alpha$$

3. *$\exists u_1 \in H$, $||u_1||_H > \varrho$ s.t. $E(u_1) < \alpha$*

Let $P = \{p \in C^0([0,1]; H)$ s.t. $p(0) = 0, p(1) = u_1\}$ and $\beta = \inf_{p \in P} \sup_{u \in p} E(u) \geq \alpha$. Then β is a critical value of E, i.e. $\exists u_{crit}$ critical point of E s.t. $E(u_{crit}) = \beta$.

10.4 An application

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $n \geq 3$. $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function s.t.

$$\exists c > 0, p \in (1, \frac{n+2}{n-2}) : \ |g(x,t)| \leq c(1 + |t|^p)$$

The aim is to show existence of a non-trivial weak solution to

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If $g(x,0) = 0 \ \forall x$, then 0 is a weak solution. Let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$G(x,t) = \int_0^t g(x,s)ds$$

and $e : W_0^{1,2}(\Omega) \to \mathbb{R}$ s.t.

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u)dx$$

Lemma 10.11. *$E \in C^1(W_0^{1,2}(\Omega); \mathbb{R})$ with*

$$DE(u)(\varphi) = \int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega g(x,u)\varphi dx$$

for $u, \varphi \in W_0^{1,2}(\Omega)$.

Lemma 10.12. *If $\exists R_0 > 0$, $\exists q > 2$ s.t. $\forall x \in \Omega$ and $\forall |t| \geq R_0$*

$$0 < qG(x,t) \leq g(x,t)t$$

(i.e. g ahs super-linear growth at ∞) then E satisfies a PS condition.

Theorem 10.13. *If $f_k \to f$ almost everywhere, $f_k \in L^r(\Omega)$, $1 \leq r < \infty$, $|\Omega| < \infty$, then the following are equivalent*

1. *$f_k \to f$ in $L^r(\Omega)$, $f \in L^r(\Omega)$*
2. *uniform equicontinuity of the L^r norm. That is*

$$\sup_k \int_E |f_k|^r \, dx \to 0 \text{ for } |E| \to 0$$

Theorem 10.14. *Additionally, assume that* $\forall \varepsilon > 0 \exists \delta > 0$ *s-t-*

$$\frac{g(x,t)}{t} < \varepsilon \; \forall x \in \Omega \; \forall \, |t| < \delta$$

then the PDE

$$\begin{cases} -\Delta u = g(x,u) \; \text{ in } \Omega \\ u = 0 \; \text{ on } \partial\Omega \end{cases}$$

admits a non-trivial weak solution.