.999... = 1???

Liam Donovan

March 2023

Abstract

Back when I was in elementary school and my teacher told me that $\frac{1}{3} = .333\cdots$, I was a little skeptical, and I'm sure I'm not alone on this. This idea suggests something very profound, the idea of infinity and the equality of two things. I mean, this statement's truth hunches on the idea that, by multiplying by 3, we get:

$$.999\cdots = 1\,. \tag{1}$$

How can this be true, the left hand side is not some equivalent rewriting of the same number, it is a completely new number, that...goes on forever? How can this be true, and considering that this is such a crucial idea in mathematics, if this is false, all arithmetic comes crumbling down with it. So let's try and verify that these two values are the same, using a few different methods. Then, as a corollary, we can then conclude that

$$.333\cdots = \frac{1}{3} \tag{2}$$

Contents

1 An Infinite Series

1.1 Positional Notation

Before we can even unpack this whole thing, we have to know what we're even looking at when we write a decimal. In risk of getting a little philosophical, what is $.333\cdots$? In fact what does writing a number, in the way we do even mean? Well, normally, we write numbers in a *positional notation*, which means we write them in some order and we multiply the *digit* (the number) in that place by some other value. We call that other value the *base*. So, the next spot to the left is multiplied by that base one more time than the spot to its right. For example, unless you're either a mathematician or a programmer/computer scientist, you probably are only familiar with base 10, the number system that we're used to writing in. The digits are just each possible unique individual number, so we know that every number (at least in base 10[1] is made up up the digits $0, 1, \cdots 9$. So we have 10 digits. So we can express any number in the form of a *sum* of digits and powers of 10 (or whatever base you choose).

Let's think about the number 835, for example: the 8 refers to 800, that is $8 \cdot 100 = 8 \cdot 10^2$. The 3 refers to 30, which is $3 \cdot 10 = 3 \cdot 10^1$. And, of course, $5 = 5 \cdot 10^0$. So, in total, we can just say this is a sum of the products of the base and the digit in spot, increasing right-to-left, i.e.,

$$835 = (8 \cdot 10^2) + (3 \cdot 10^1) + (5 \cdot 10^0)$$

This same principle applies in reverse, as well, it's just that decimals *divide* by the base, i.e., they multiply by the negative base, since we're moving further to the right. So, for example:

$$.835 = 8 \cdot 10^{-1} + 3 \cdot 10^{-2} + 5 \cdot 10^{-3}$$

If we wanted to generalize this out, it just means that a number written as: $\cdots abcd.efg \cdots$ can be written:

$$\cdots abcd.efg \cdots = \cdots a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d \cdot 10^0 + e \cdot 10^{-1} + f \cdot 10^{-2} + g \cdot 10^{-3}$$

So how does this help us solve our problem? We know, using our knowledge of positional notation, that we can rewrite $.999\cdots$ as a sum of powers of 10, multiplied by 9. Like this:

$$.999 \cdots = 9(10^{-1}) + 9(10^{-2}) + 9(10^{-3}) + \cdots \tag{3}$$

Notice that this is, of course, and infinite sum, so we can write it like this:

$$.999 \cdots = \sum_{n=0}^{\infty} 9(10^{-n}) = 9 \sum_{n=0}^{\infty} (10^{-n})$$

[1] Note that the number of digits in a specific base is the base itself, for example base 10, has 10 digits: $\{0, 1, 2, \cdots, 9\}$ and base 2 (also called binary) has 2 digits $\{0, 1\}$

Now, we dig up some old calculus 2 knowledge, the sum of an infinite sum is as follows:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

where a is the 1^{st} element and r is the common ratio, what we multiply by every new element. So, in our case $a = 1$ and $r = \frac{1}{10}$ so, we get:

$$\begin{aligned} 9\sum_{n=1}^{\infty} 10^{-n} &= 9\left(\frac{\frac{1}{10}}{1-\frac{1}{10}}\right) \\ &= 9\left(\frac{\frac{1}{10}}{\frac{9}{10}}\right) \\ &= 9\left(\frac{1}{9}\right) \\ &= 1 \end{aligned}$$

So, we just got:

$$.999\cdots = 1$$

Then by dividing both sides by 3, we get:

$$.333\cdots = \frac{1}{3}$$

2 Cauchy Sequence

So let's think about another way to approach this problem. What's another way to shows that two things are equal? Well, the difference of the two things better be 0, i.e., a and b are equal if $a - b = 0 \iff b - a = 0$.

In our case, since we know that if we only have a certain finite amount of 9s, then they are not equal, since subtracting them doesn't give us 0. For example, $1 - .999 = .001 \neq 0$. So we have to examine what happens if the digits of the 9s approach infinity. This isn't easy to do, except for the, of course, using the infinite sum in part 1. So, we're gonna use something called a *Cauchy Sequence*. A Cauchy Sequence is a sequence that eventually converges to some value and is actually a rigorous definition of the epsilon-delta (ϵ, δ) definition to compute limits. So technically, since we're talking about "places" going to infinity, we're not talking about a sum, but a sequence. Going even further, we're talking about a sequence of the difference of two things, this is called a *metric space*. Let's try and write $.999\cdots$ as a sequence, where the number of digits goes to infinity, we can denote it $(.9)_\infty$. Let's note that it'll be a little 'cleaner' to use fractions.

$$(.9)_\infty = (.9, .99, .999, \cdots) = \left(\frac{9}{10}, \frac{99}{100}, \frac{999}{1000}\cdots\right)$$

So, now let's looks at the Cauchy Sequence that we were looking for[2], $1-(.9)_\infty$:

$$\begin{aligned}1-(.9)_\infty &= \left(1-\frac{9}{10}, 1-\frac{99}{100}, 1-\frac{999}{1000}\cdots\right)\\ &= \left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}\cdots\right)\end{aligned}$$

And what is this? This is just converging to a limit, of decreasing powers of 10. That is:

$$1-(.9)_\infty = \lim_{n\to\infty}\frac{1}{10^n}$$

Using our calculus knowledge, we know this goes to 0, thus

$$\begin{aligned}1-(.9)_\infty &= 0\\ \implies 1 &= (.9)_\infty\\ \implies 1 &= .999\cdots\\ \implies \frac{1}{3} &= .333\cdots\end{aligned}$$

3 Simple Algebra

This one is relatively disputed, since it is, at its heart, a circular argument, i.e., we start with what we want to end with. But let's begin:

$$\text{let } x = .999\cdots$$

by multiplying by 10:

$$10x = 9.999\cdots$$

We can separate that right hand side, into an integer and an infinite decimal:

$$10x == 9 + .999\cdots$$

Notice that now there's an x on the right hand side:

$$\begin{aligned}10x &== 9 + x\\ 9x &= 9\\ x &= 1\\ \implies .999\cdots &= 1\\ \implies .333\cdots &= \frac{1}{3}\end{aligned}$$

Additionally, there's also dispute about arithmetic over the infinite numbers, since we're basically adding and subtract infinite sums. And if the Ramanujan Summation has taught us anything, doing any arithmetic with infinite sums can be a slippery slope. But this is a good way to show to equality without having to resort to calculus.

[2]Technically, we're trying to show that *is* a Cauchy Sequence

4 Final Thoughts

So, this *is* true, after all. This is a nice exercise for students after they finish calculus 2, since they have all the tools they need to compute it. However, the amazing this remains that before the invention of calculus, people still had some knowledge of being true, so they had some concept of infinity, before infinity was even defined.