RSA Cryptosystem and Prerequisites

Liam Donovan

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Overview

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Background Information: Modular Arithmetic

For most of what we do in cryptography, we will use modular arithmetic, that is: instead of the mathematics of the quotient in a division of two integers, we will look at the remainder. For this we will need a few results.

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$$10 = 2(5) + 0$$

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In this case, we cannot write *a* as a group of *b*, without a remainder, so a/b will not be an integer.
It should be clear that if we make as many groups of *b* as we possible can, that *q* and *r* will be unique.

Modular Congruence

If two integers, a, c have the same remainder, when divided by b, we say they are congruent mod b.[Sma03] We denote this by

$$a \equiv b \pmod{c}$$

Another definition of this relies on the fact that if we subtract the division algorithm of a and b, then the r terms will cancel, meaning that $b \mid (a - b)$.

A Familiar Example

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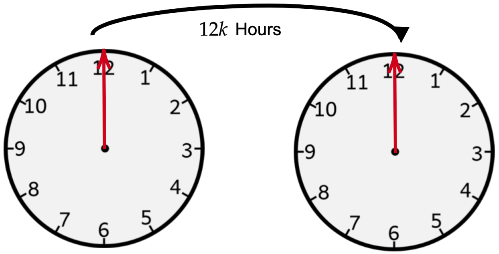

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Likewise, consider after $3 + 12k$ hours, we would land at 3 and after $4 + 12k$ we would land at 4. Since it takes 12 hours to go back around to 3 after the first pass.

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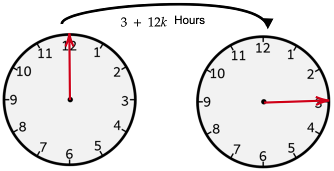

Figure: After $3 + 12k$ hours, the clock is at 3.

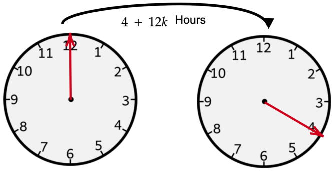

Figure: After $4k + 12k$ hours, the clock is at 4.

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x exists if and only if $gcd(a, c) = 1$. Where gcd refers to the greatest common factor, the largest integer which divides both a and c. In this case, we say a, c are *coprime*. That is: they share no common factors, except 1, which is a factor of all integers.
In practice, a computer performs the extended Euclidean algorithm, which can compute x.

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- Further strengthens the encryption If the attacker does not know which group we are working with everything will seem random.

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  - So, we use large primes to create a large semiprime to encrypt our data. We can hide the prime factors, while publicizing the semiprime.

Euler's Totient Function

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- A property of Euler's totient function is that is is multiplicative
  - This means that if a, b are coprime, $\varphi(ab) = \varphi(a)\varphi(b)$.
  - So, if we have two primes, p, q, $\varphi(pq) = \varphi(p)\varphi(q) = (p-1)(q-1)$
  - I can use the semiprime pq to create another value that the attacker will not know, $\varphi(pq)$.

Setting It Up

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3. Choose an integer e, which is less than $\varphi(n)$, but greater than 2. e for "encryption".
4. Find the modular multiplicative inverse of $e \pmod{\varphi(n)}$. Let it be d, for "decryption", $ed \equiv 1 \pmod{\varphi(n)}$

The Process

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5. Then, Alice recovers m, since $m^{ed} \equiv m \pmod{n}$

Proof That RSA Works

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To prove this we need a quick theorem.

Euler's Theorem

Theorem

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Claim 5.1

Let $gcd(m, n) = 1 = gcd(e, \varphi(n))$ and $ed \equiv 1 \pmod{\varphi(n)}$

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$$\begin{aligned} \varphi(n) &\mid ed - 1 \\ \implies \exists z \in \mathbb{Z} \quad z\varphi(n) &= ed - 1 \\ 1 + z\varphi(n) &= ed \end{aligned}$$

Now we can plug this into the exponent:

$$\begin{aligned} m^{1+z\varphi(n)} &\equiv m \pmod{n} \\ m \cdot m^{z\varphi(n)} &\equiv m \pmod{n} \\ m \cdot (1)^z &\equiv m \pmod{n} \\ m &\equiv m \pmod{n} \end{aligned}$$

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