POH-SHEN LOH METHOD FOR QUADRATICS

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Abstract. We're gonna approach factoring in a whole new way. Let's say you're just a new algebra 1 student, who only knows *about* factoring, but doesn't actually know how to do it. How would you approach it? How can we find the roots (zeros) of a parabola? Using basic principles, we can build this solution from the ground up, and uncover an even more striking derivation of a beautiful formula.

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1. Some Qualities of Parabolas

So, we know a few things about formulas and their interpretations, if the max exponent that we see (we call this the *degree* of the polynomial) is 2, then it makes a parabola, which looks something like this:

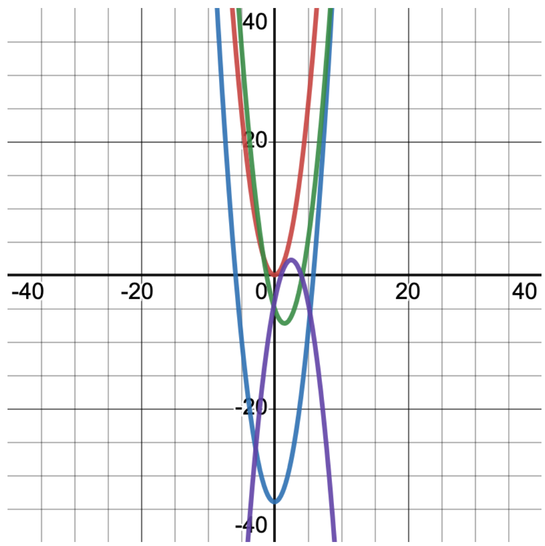

Figure 1. A range of possible parabolas

One of the main qualities that we want to investigate is that the vertex of the parabola is the midpoint between the zeros, or the roots[1]. Of course, the reason why we want these zeros is because, often times we find ourselves needing to solve something like this, note that we're gonna restrict ourselves to cases where $a = 0$, since we can 'force this' by dividing out by a.:

$$x^2 + Bx + C = 0\,.$$

So, if we find the x values that make this equation true, that is: the zeros of the polynomial, then we have found the x values that solve the equation.

Now, let's go back to our property, let's call our roots R and S, arbitrarily and we'll call M the midpoint, i.e., the vertex of the parabola, but since we only care about the x value of the vertex, for now, let's just call it the midpoint.

[1]This can be shown in 2 ways: solve the general form of the quadratic $A(x-h)^2 + k = 0 \implies x = h \pm \sqrt{-\frac{k}{A}}$, so the $\pm$ means that the roots are symmetric about h. The other way is by knowing that a parabola is a conic section of a cone, so because the cone is symmetric, the roots are symmetric.

2. Factoring

We know there's, at most, 2 roots, so we want to split the quadratic into 2 terms. Now, we know that $x^2 = x \cdot x$. So, of course, the root is $x = 0$. So if we add other addends to this equation, how would we split this into two x terms? Well, we would need to subtract (or add) from both terms[2]. Let's just write this down:

$$x^2 + Bx + C = (x - p)(x - r)$$

[3] Notice that r and q better be constants, otherwise there's an x^3 term, which means they cannot possibly be equal. *But*, let's notice something really important here, if $x = p$ or if $x = r$, then this equals 0! What does that mean? It means that p and r *are* the roots, R and S!, so let's rewrite this as:

$$x^2 + Bx + C = (x - R)(x - S)$$

So, we know that we have the roots, but how do we actually *solve* for them? Let's just keep in mind, that we start with the left equation, meaning we *know* B and C. Let's try expanding out the right equation:

$$\begin{aligned}(x - R)(x - S) &= x^2 - Rx - Sx + RS \\ &= x^2 + x(-R - S) + RS \\ \implies x^2 + Bx + C &= x^2 + x(-R - S) + RS\end{aligned}$$

So, we can see, by matching the terms:

$$\implies B = -(R + S), \quad C = RS$$

So, here's where that "find two numbers that are the product of C and the sum of B" (in this case, the negative sum, but that's just because we subtracted the roots, for convenience) thing comes from. But now, we have a problem, this isn't exactly an easy system to solve, especially for an algebra 1 student. This is where we are gonna use or property of the quadratic that we found earlier, that the midpoint is halfway between R and S.

3. Utilizing the Properties of Parabolas

So, there's a distance between R and M, as well as M and S, and that distance is the *same*.

[2]The reason for this is either intuition, and checking, by foiling or the binomial theorem ($(x + y)^n = \binom{n}{k} x^{n-k} y^k$) and plugging in $n = 2$.

[3]Note that you could just as easily add p and r, but if we do it this way, the roots will be p and r, instead of $-p$ and $-r$.

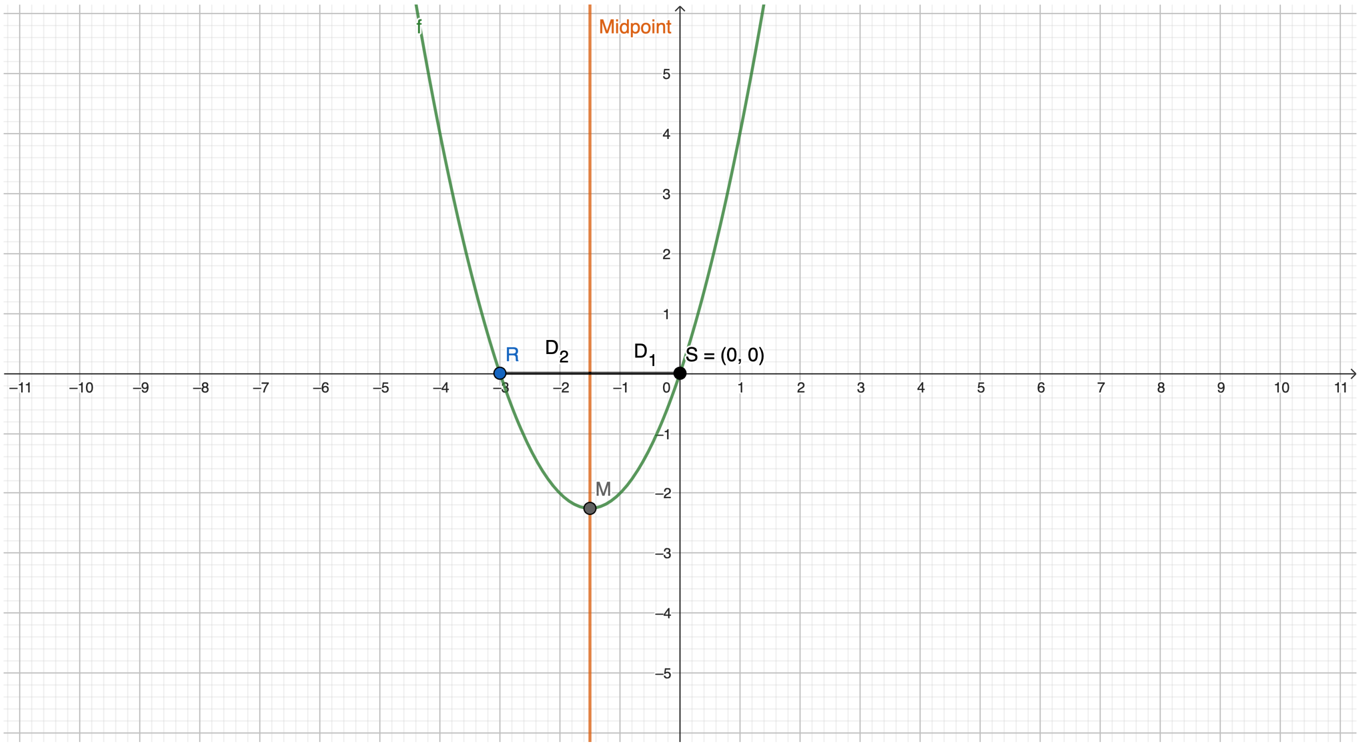

FIGURE 2. Here, we can see that M is halfway between R and S. Note that $D_1 = D_2$

Let's just notate distance as D and, of course, $D \geq 0$. So, now we know:

$$R + D = M \implies R = M - D$$
$$S - D = M \implies S = M + D$$

Well, in our pursuit of a way to solve R and S, we've run into another thing we don't know, M. So, let's think: can we rewrite M as a function of R and S (only using R and S)? Of course, we can, it's directly in the middle of R and S! So, we can calculate it as:

$$M = \frac{R + S}{2}$$

So, we've kinda got ourselves into a circular argument here, that is, until we recall our system: $-(R + S) = B$, which means: $R + S = -B$

and we *know* $-B$. So, we're getting closer, we now have

$$\begin{aligned} M &= -\frac{B}{2} \\ \implies R &= -\frac{B}{2} - D \\ \implies S &= -\frac{B}{2} + D \end{aligned}$$

So, now we got 2 equations and 3 unknowns, do we have another equation to use? Well, if we use $R + S = -B$ again, we just get left with that same equation, again, so let's try the other condition: $RS = C$:

$$\begin{aligned} RS &= \left(-\frac{B}{2} - D\right)\left(-\frac{B}{2} + D\right) = C \\ &= \left(-\frac{B}{2}\right)^2 - D^2 = C \end{aligned}$$

Then, we can solve for D:

$$\implies D = \sqrt{\left(-\frac{B}{2}\right)^2 - C}$$

Notice that we can sub in M

$$D = \sqrt{M^2 - C}$$

So, finally,we have an equation that we can actually solve, since we know B and C, we can find D and if we can find D, we can find R and S.

So the final algorithm is as follows:

(1) Make sure quadratic is of the form $x^2 + Bx + C = 0$, if not you may have to divide perform some algebra, then divide by a.
(2) find the midpoint, $M = -\frac{B}{2}$
(3) find the distance, $D = \sqrt{M^2 - C}$
(4) solve for R and S, $R = M - D$, $S = M + D$; R and S are the roots of the quadratic.

4. A Familiar Formula

Now, as maybe verification that we are correct, or just a corollary, we can prove a very familiar expression.

Let's begin with a general quadratic:

$$ax^2 + bc + c = 0$$

Of course, we need a leading coefficient of 1, so we need to divide out by a:

$$\frac{ax^2+bx+c}{a} = \frac{0}{a}$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

To make this cleaner looking, let $\frac{b}{a} = B$ and $\frac{c}{a} = C$

$$x^2 + Bx + C = 0$$

Let's now solve for M:

$$\begin{aligned} M &= -\frac{B}{2} \\ &= -\frac{\frac{b}{a}}{2} = -\frac{b}{2a} \end{aligned}$$

Now, solve for distance:

$$\begin{aligned} D &= \sqrt{M^2 - C} \\ &= \sqrt{\left(-\frac{b}{2a}\right)^2 - \frac{c}{a}} \\ &= \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \end{aligned}$$

We can simplify, by multiplying $\frac{c}{a}$ by $\frac{4a}{4a}$

$$\begin{aligned} &= \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} \\ &= \sqrt{\frac{b^2 - 4ac}{4a^2}} \end{aligned}$$

Using the properties of exponents, and thereby, properties of square roots:

$$\begin{aligned} &= \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\ &= \frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Now, to find R and S:

$$\begin{aligned} R &= M - D \\ &= -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Likewise for S:

$$\begin{aligned} S &= M + D \\ &= -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

So, we can factor $ax^2 + bx + c = 0$ as:

$$ax^2 + bx + c = \left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = 0$$

The roots of x are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \square$$

Thus, giving us the quadratic equation.

5. Examples

(1) Find roots of $x^2 - 6x + 8$:

$$\begin{aligned} M &= -\frac{-B}{2} \\ &= -\frac{-6}{2} = 3 \end{aligned}$$

Solving for D:

$$\begin{aligned} D &= \sqrt{M^2 - C} \\ &= \sqrt{3^2 - 8} \\ &= \sqrt{9 - 8} = 1 \end{aligned}$$

Solving for R and S:

$$\begin{aligned} R &= M - D \\ &= 3 - 1 = 2 \\ S &= M + D \\ &= 3 + 1 = 4 \end{aligned}$$

Thus, the roots of $x^2 - 6x + 8$ are 2 and 4, so we can factor:

$$x^2 - 6x + 8 = (x-2)(x-4)$$

To be fair, this question was easily done mentally, so let's try a harder one.

(2) Find roots of $x^2 - 2x + \frac{8}{9}$

$$\begin{aligned} M &= -\frac{2}{2} = 1 \\ D &= \sqrt{1^2 - \frac{8}{9}} = \frac{1}{3} \\ R &= 1 - \frac{1}{3} = \frac{2}{3} \\ S &= 1 + \frac{1}{3} = \frac{4}{3} \\ \implies \text{roots} : x &= \frac{2}{3}, \frac{4}{3} \end{aligned}$$

This is a problem which , basically, required the quadratic formula, but here we can factor it, directly.

Of course, this method can also find complex roots, since an algorithm such as this was the reason for the creating the complex numbers.

(3) Find the roots of $x^2 + x + 1$

$$\begin{aligned} M &= -\frac{1}{2} \\ D &= \sqrt{\left(\frac{1}{2}\right)^2 - 1} \\ &= \sqrt{\frac{1}{4} - 1} \\ &= \frac{\sqrt{3}}{2} i \\ R, S &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \end{aligned}$$

Another case to consider is when there are repeated roots, i.e., when the constants are equal, $R = S$. This happens when $D = 0$, there's no distance between the roots. So this happens when $D = \sqrt{M^2 - C} = 0$, that is: $M^2 = C$

(4) Find the roots of $x^2 + 4x + 4$

$$M = -\frac{4}{2} = -2$$

$$D = \sqrt{(-2)^2 - 4} = 0$$

$$R, S = -2$$

Despite being probably slower to compute than both the quadratic formula and traditional factoring, it does give a very intuitive explanation for quadratics. Additionally, it gives a nice algorithm that, when expanded out to more complex polynomials, e.g., cubic and quartic functions, it may be more efficient for a computer to perform these operations, instead of performing one large formula. Below is a (very crude, unpolished) program written in Python, which performs the algorithm.

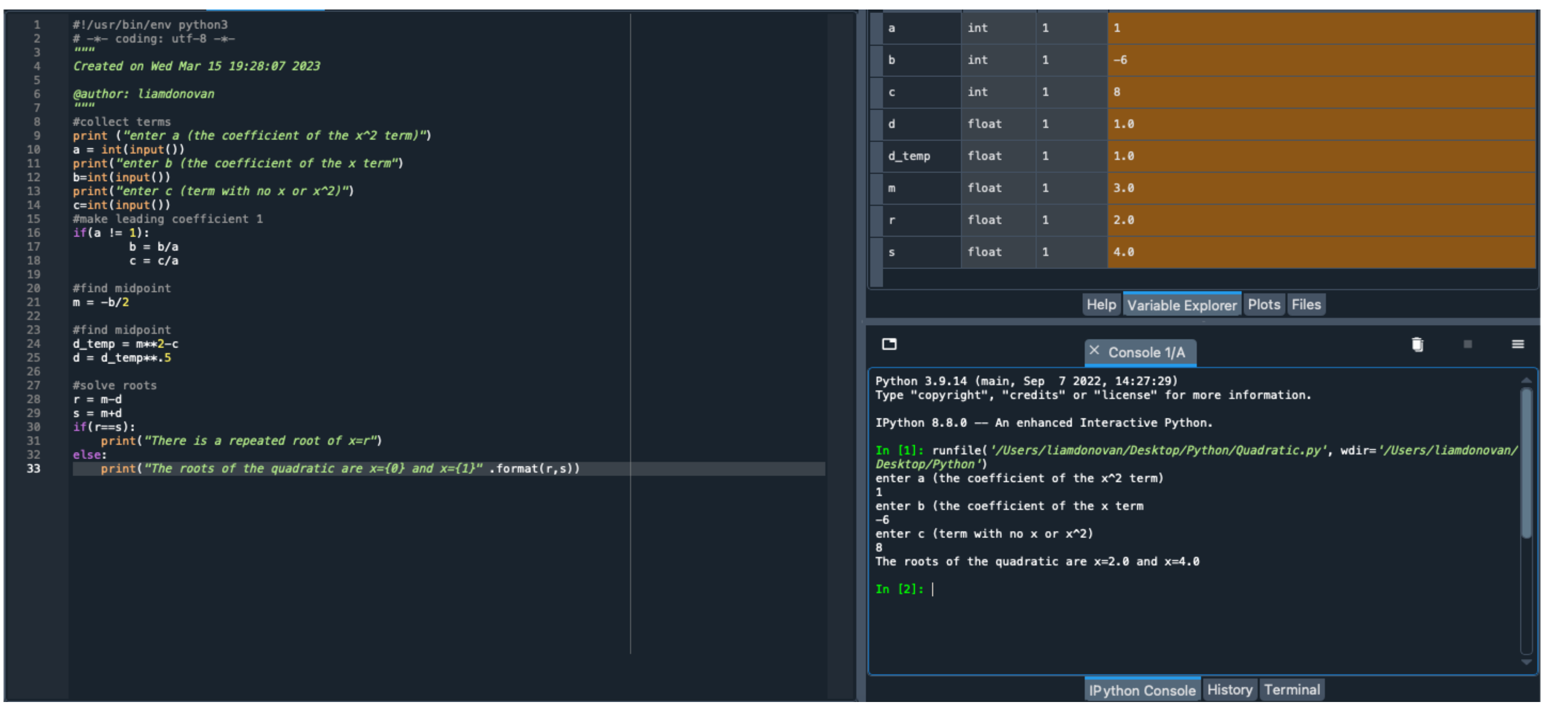

FIGURE 3. Program to compute roots, using the above algorithm