The Brilliantly Clever Pi and Gamma Functions

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Abstract

In mathematics, we often like to generalize our results, it's sort of in the spirit of the subject. Mathematics is rigorous, and hates leaving 'holes' in its content.

For example: the definition $x^n = \underbrace{x \cdot x \cdot x \ldots}_{n}$ breaks down, when n is not a whole, positive number. $x^{\frac{1}{2}} =$???; how can I multiply by something $\frac{1}{2}$ times?

To extend out the exponents, we use a property, that we can see from the numbers that we *can* find to define all the other exponents; namely: $x^{n+m} = x^n \cdot x^m$.

Now, we're gonna talk about the factorial, $x! = x(x-1)\ldots 1$. We, again, run into a problem that this definition breaks, when x is not a positive, whole number (we call those *natural numbers*, denoted $\mathbb{N}$). So, being the mathematicians that we are, we're gonna try and extend out the factorial! We call this an *analytical continuation*.

So, we know that $x!$ better still give us what we expect, when x is a natural number (or 0). However, this isn't nearly as easy as it is with exponents, we really only have one property of factorials, that could help us, that $x! = x(x-1)!$, but that doesn't really help us extend this out. So we're gonna have to build something completely new. The result is not only very satisfying, but is also widely used, in statistics, number theory, and combinatorics (which is almost all abstract mathematics).

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1 Initial Ideas and Motivations

So, we know that we want this function to take in a value, x and output $x!$ if x is a natural number, then, hopefully, this will helps us compute the non-natural numbers.

We want this effect of 'continued multiplication'. This should remind us of exponents. t^x but, we know that the value of x should decrease as we go. Doesn't that sound like the power rule, when we take a derivative? E.g., $\frac{\mathrm{d}}{\mathrm{d}t}t^x = xt^{x-1}$.

Here's another problem: if x is not a natural number, then this will never reach 1, since we're just subtracting 1 from it over and over. So, we need almost a 'force stop' or a 'break' for the operation. One that also forces the iteration to include 1. What's something that iterates, that also forces certain values onto a function? I think of *integration by parts*, it can iterate many times, and we can make it a define integral to force the function have to take certain values. Even better, integration by parts includes a part that that is *differentiated*, so we seem to be onto something here.
Recall: Integration by parts (for definite integals):

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = f(t)g(t)\Big|_a^b - \int_a^b g(t)f'(t)\,\mathrm{d}t$$

So far, here's what we've got:

- We should have an definite integral
- The thing inside the integral should include t^x, which will be f(t), since we're gonna differentiate it
- Output of integral should be $x!$ if x is a natural number (denoted $x \in \mathbb{N}$)

2 Euler's Factorial Function (Π-Function)

So, we still need to figure out a few things, what's $g(t)$ and what are the bounds, a, b? Let's first just try our the integration and see what happens:

$$x! = \int_a^b t^x g'(t)\,\mathrm{d}t$$

by Integration by parts, let $g(t) = g(t), f(t) = t^x \implies f'(x) = xt^{x-1}$

$$= t^x g(t)\Big|_a^b - \int_a^b xt^{x-1}g(t)\,\mathrm{d}t$$

since x is a constant, in t, we can factor it out of the integral:

$$= t^x g(t)\Big|_a^b - x\int_a^b t^{x-1}g(t)\,\mathrm{d}t$$

again, by integration by parts, let $g(t) = g(t), f(t) = xt^{x-1} \implies f'(t) = x(x-1)t^{x-2}$

$$= t^x g(t)\Big|_a^b - x\left(xt^{x-1}g(t)\Big|_a^b - (x-1)\int_a^b t^{x-2}g(t)\,\mathrm{d}t\right)$$
$$\ldots$$

So, we can see that our derivative is giving us these $x(x-1)(x-2)\ldots$ terms, like we want it to, but these $f(t)g(t)$ terms are giving us some trouble. We want these to be 0. Since we're gonna keep differentiating $f(t)$, and integrating $g(t)$we want all of these to be 0, we can write the condition like this:

$$f^{(n)}(t)g^{(-n)}(t)\Big|_a^b = 0 \quad \forall n \leq x \tag{1}$$

note that if we differentiate the polynomial n times, we get $n!t$, so if we differentiate more than that, it's just 0, so $n \leq x$. Also note that $g^{(-n)}$ refers to repeated integration.

Additionally, we want a $g(t)$ that won't stop the integration early.

Eventually, we can see that the exponent will be 0, meaning that we'll get $\int_a^b g^{(-x)}(t)\,\mathrm{d}t$, which will stop the integration, since we won't continue integration by parts at that point. Therefore, we need to pick a $g(t)$ that won't stop the integration. Thus, we don't want it to 'go away'. Can we think of any functions that make us integrate multiple times? Polynomials work, but then we just have one big polynomial, when we multiply, terminating integration by parts. $\sin t, \cos t$ will result in repeated integration, but they actually resolve each other, in that once we integrate twice, we will see a repeated integral, which resolves integration by parts, making out 'max' iterations 2.

Another choice is e^t. This is good, since if $g'(t) = e^t\,\mathrm{d}t \implies g(t) = e^t$ meaning that this will never terminate, until the final integral, when $f(t) \to 1$.

Let's keep in mind the we still need to satisfy (1), let's plug in $g^{(-n)}(x) = e^t$:

$$f^{(n)}(t)e^t\Big|_a^b = 0$$

To make this a bit easier to see, since the behavior of polynomials will be the same, for our purposes, consider, without loss of generality, that we're referring to the 0^{th} derivative of $f(t)$, i.e., $f(t)$

$$t^x e^t\Big|_a^b = 0$$

[1] Of course, at $t = 0$, this will be 0, since we have a polynomial. When does e^t go to 0? That's when $t \to -\infty$. So our bounds should be $a = -\infty, b = 0$, right? Well, yeah that would work, for now, but let me show you the problem we would run into: Let's look at what we've got so far:

$$\begin{aligned}
x! &= \int_a^b t^x e^t\,\mathrm{d}t && (1)\\
&= -x\left(-\int_{-\infty}^0 t^{x-1}e^t\,\mathrm{d}t\right) && (2)\\
&= -x(-(x-1))\left(-\int_{-\infty}^0 t^{n-2}e^t\,\mathrm{d}t\right) && (3)\\
&\dots && (4)
\end{aligned}$$

Can you see a problem? The fact that these negative signs are there means that if we were looking for the factorial of an odd number (where the negatives don't cancel), we would get $-(x!)$. While we could take the absolute value of the function, there's a bit of a 'nicer' way to do this. Can we think of some way to get rid of these negative signs? Well, we just defined $g'(x) = e^t\,\mathrm{d}t$, couldn't we just have defined it as: $g'(x) = e^{-t}\,\mathrm{d}t$? Then $g(t) = -e^{-t}$. This would cancel the negative sign.

Of course, since we just changed $g(t)$, we need to change our bounds. Since $f(t)$ defined our 0 bound, it doesn't change, but when does $g(t) = e^{-t}$ go to 0? Well, we said $e^{-\infty} \to 0$, so in this case, we need $t \to \infty$. So, we set $b = \infty$. So, we think we have a function that can represent, and hopefully, extend out the factorial:

$$x! = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$$

2.1 Proofs

We need to show that for all natural numbers, as well as 0, this holds. So let's try:
Before we jump into the proof, we need to confirm that all the $f(t)g(t)$ terms will be 0:

Lemma 2.1. *Suppose that $a_n(t)$ is a monomial, with degree n, $a_n(t) = a_n t^n$, where a_n is a constant, $a_n \in \mathbb{R} \quad \forall n \in \mathbb{N}$:*

$$a_n(t)\cdot -e^{-t}\Big|_0^\infty = 0 \qquad \forall a_n \in \mathbb{R}, \forall n \in \mathbb{N}$$

Proof.

$$\begin{aligned}
a_n(t)\cdot -e^{-t}\Big|_0^\infty &= -\frac{a_n(t)}{e^t}\Big|_0^\infty\\
&= -\left(\lim_{t\to\infty}\left(\frac{a_n(t)}{e^t}\right) - \left(\frac{a_n(0)}{e^0}\right)\right)
\end{aligned}$$

[1] Notice that this holds for all derivatives of t^x

Of course, $\frac{a_n(0)}{e^t} = \frac{0}{1} = 0$

$$\begin{aligned} &= -\lim_{t\to\infty}\left(\frac{a_n(t)}{e^t}\right) \\ &= -\left(\frac{a_n t^\infty}{e^\infty}\right) \\ &= \frac{\infty}{\infty} \end{aligned}$$

thus, we can use L'Hopital's Rule. Note that after n derivatives:

$$a_n^{(n)}(t) = n!a_n$$

which is a constant, thus:

$$\begin{aligned} a_n^{(n+1)}(t) &= 0 \\ \implies -\lim_{t\to\infty}\frac{a^{(n+1)}(t)}{e^t} &= -\lim_{t\to\infty}\frac{0}{e^t} \\ &= 0 \end{aligned}$$

Thus,

$$a_n(t)\cdot -e^{-t}\Big|_0^\infty = 0 \qquad \forall a_n \in \mathbb{R}, \forall n \in \mathbb{N}$$

$\square$

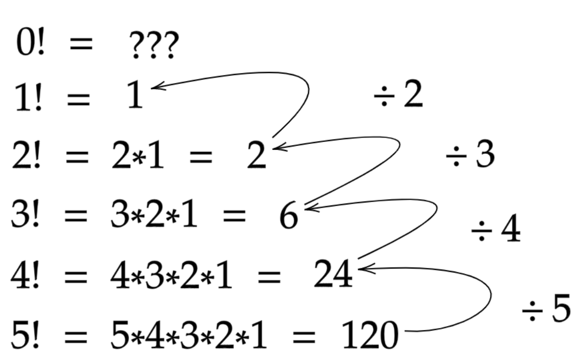
Of course, if we are trying to prove that 0! holds under this integral, we need to know what 0! is.

Definition.

$$0! = 1$$

Well, we need to know *why* $0! = 1$. It is a definition, since it doesn't fit under the definition of the factorial, but why is it defined this way?
One way we can define 0!, is to try and spot a pattern.:

Figure 1: Pattern for factorials



From this week can see that this definition is, indeed, natural.
Another way this is naturally defined, is that factorials were first used to describe how many ways you could arrange a certain number of objects, i.e., the number of ways to arrange n objects is $n!$.
In our case, if we have no objects, there certainly is only 1 way to arrange this set, thus $0! = 1$

Now we begin our actual goal of proving that this is a valid representation of the factorial.

Claim.

$$\int_0^\infty t^x e^{-t}\,\mathrm{d}t = x! \quad x \in \mathbb{N} \cup \{0\}$$

Proof. First, we verify the special case, $0! = 1$:

$$\begin{aligned}
0! &= \int_0^\infty t^0 e^{-t}\,\mathrm{d}t \\
&= \int_0^\infty e^{-t}\,\mathrm{d}t \\
&= -e^{-t}\big|_0^\infty \\
&= (-e^(-\infty) - (-e^0) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

Now, we consider the case where x is a natural number:

Let $x \in \mathbb{N}$ By integration by parts, let $f(t) = t^x \implies f'(t) = xt^{x-1}$, $g'(t) = e^{-t}dt \implies g(t) - e^{-t}$

$$\begin{aligned}
\int_0^\infty t^x e^{-t}\,\mathrm{d}t &= \left(t^x \cdot -e^{-t}\right)\big|_0^\infty - \int_0^\infty -e^{-t}(xt^{x-1})\,\mathrm{d}t \\
&= \left(t^x - e^{-t}\right)\big|_0^\infty + x\int_0^\infty e^{-t}(t^{x-1})\,\mathrm{d}t
\end{aligned}$$

computing $\int_0^\infty e^{-t}(t^{x-1})\,\mathrm{d}t$:

let $f(t) = t^{x-1} \implies f'(t) = (x-1)t^{x-2}$, $g'(t) = e^{-t}dt \implies g(t) - e^{-t}$

$$= \left((x-1)t^{x-1} \cdot -e^{-t}\right)\big|_0^\infty + (x-1)\int_0^\infty e^{-t}(t^{x-2})\,\mathrm{d}t$$

$$\implies \int_0^\infty t^x e^{-t}\,\mathrm{d}t = \left(t^x - e^{-t}\right)\big|_0^\infty + x\left[\left((x-1)t^{x-1} \cdot -e^{-t}\right)\big|_0^\infty + (x-1)\int_0^\infty e^{-t}(t^{x-2})\,\mathrm{d}t\right]$$

We can cancel the $f(t)g(t)$ terms, by the above lemma, which leaves us with:

$$\int_0^\infty t^x e^{-t}\,\mathrm{d}t = x\left[(x-1)\int_0^\infty e^{-t}(t^{x-2})\,\mathrm{d}t\right]$$

notice that, after the 1^{st} iteration, the factor in front of the integral is x, then after the 2^{nd} iteration, the factor is $x-1$. We can keep going and verify that the factor in front of the 3^{rd} factor is $x-2$ and so on. So the factor in front of the n^{th} iteration (as long as $n \leq x$) would be $x-(n-1)$, which means: $f(t) = t^0$. So, after x iterations, the factor would be $x-(x-1) = 1$. So, we would see:

$$\int_0^\infty t^x e^{-t}\,\mathrm{d}t = x(x-1)(x-2)\ldots 1\int_0^\infty t^0 e^{-t}\,\mathrm{d}t$$

Notice this last integral is just $0!$, which we showed was equal to 1. So we get:

$$\int_0^\infty t^x e^{-t}\,\mathrm{d}t = x!$$

For some natural number, x $\square$

This function is called *Euler's Factorial Function* or *The Pi Function* (Π-*function)*, since we denote it with an uppercase pi.

Definition (Euler's Factorial Function)**.**

$$\Pi(x) = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$$

if x is a natural number, $\Pi(x) = x!$

2.2 A More Rigorous Proof

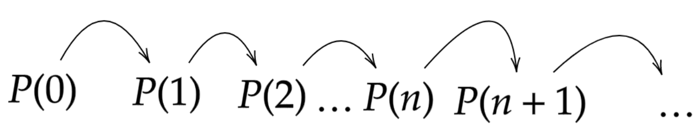
To be completely honest, the previous proof was quite 'hand-wavy' and not rigorous, so we're gonna use another method, that is very natural for natural numbers, call *Proof by Induction*.

2.2.1 Proof by Induction

Here's the idea: if we are trying to prove that some iterative process holds over the natural numbers, since we are iterating, the previous step will appear in the next step. For example: for a statement $P(n)$, where $n \in \mathbb{N}$ is true, then if we can use $P(n)$ to prove $P(n+1)$, then it holds for all natural numbers. So, if we were to assume that $P(n)$ is true, then prove that $P(n+1)$ is true, using $P(n)$, then all the n except for the first one should all be true. Basically, we just showed that, all the statements that have an element are true, since the one before it is true. Which means, that all we have to do is prove the first case, $P(0)$[2] then we get this 'domnino effect' where all the other $P(n)$ are true. So, if we can prove that:

$$P(0) \equiv T, \qquad P(n) \equiv T \implies P(n+1) \equiv T$$

Which means



We call proving $P(0)$ the *base case* and proving $P(n) \equiv T \implies P(n+1) \equiv T$ the *inductive step* or the *inductive hypothesis*.

[2]It's worth mentioning that P(0) doesn't *have* to be 0 in the function, it's just the first case we want to prove.

So, now we do this to prove our specific case:

Proof. We want to prove that $x! = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$ for all natural numbers. So, our base case is: $P(x = 0)$:

$$0! = 1 = \int_0^\infty t^0 e^{-t}\,\mathrm{d}t$$

We already proved this, so we can move on knowing $P(x = 0) \equiv T$. Our inductive step would then be: $P(x) \equiv T \implies P(x+1) \equiv T$:
Assuming that $x! = \int_0^\infty t^x e^{-t}\,\mathrm{d}t$, we need to prove that $(x+1)! = \int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t$

$$(x+1) \stackrel{?}{=} \int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t$$

by integration by parts, let $f(t) = x^{k+1} \implies f'(x) = (k+1)x^k, \qquad g'(x) = e^{-t} \implies g(t) = -e^{-t}$

$$\int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t = t^{x+1}(-e^{-t})\big|_0^\infty - \int_0^\infty (x+1) - e^{-t} t^x\,\mathrm{d}t$$

by lemma (2.1) we know that the $f(t)g(t)$ term equals 0. Thus,

$$\int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t = (x+1)\int_0^\infty x^t e^{-t}\,\mathrm{d}t$$

We can see that the remaining integral is just $\int_0^\infty t^x e^{-t}\,\mathrm{d}t = x!$:

$$\int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t = (x+1)x!$$

We can show the property that we stated in the abstract here:

Lemma 2.2. *Suppose that x is a natural number:*

$$(x+1)! = (x+1)x!$$

Proof.

$$x! := x(x-1)(x-2)\dots 1$$
$$(x+1)! = (x+1)\underbrace{x(x-1)(x-2)\dots 1}_{x!}$$
$$\implies (x+1)! = (x+1)x!$$

□

Thus, by applying this lemma, we find

$$\int_0^\infty t^{x+1} e^{-t}\,\mathrm{d}t = (x+1)x! = (x+1)!$$

So, $\forall x \in \mathbb{N}\ \int_0^\infty t^x e^{-t}\,\mathrm{d}t = x!$ □

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3 The Gamma Function

First, a bit of historical background: In 1730, Euler wrote a letter to Goldbach (yes, *that* Goldbach), where he explained that he derived this function:

$$x! = \int_0^\infty (-\ln s)^x \, \mathrm{d}s$$

If we were to substitute:

$$t = -\ln s \implies -t = \ln s \implies e^{-t} = s \implies -e^{-t}\,\mathrm{d}t = \mathrm{d}s, \quad x \in [0,\infty) \implies s \in (-\infty, 0]$$

$$\begin{aligned} \implies x! &= \int_{-\infty}^{0} \left(-\ln\left(e^{-t}\right)^x - e^{-t}\,\mathrm{d}t\right. \\ &= -\int_{-\infty}^{0} t^x (e^{-t})\,\mathrm{d}t \\ &= \int_0^\infty t^x e^{-t}\,\mathrm{d}t = \Pi(x) \end{aligned}$$

So this function is equivalent to the one we found earlier, Euler definitely knew this. In fact the formalization of the '!' notation for the factorial didn't occur until 1808. Also in 1808, Legendre, a French mathematician, wrote the function with the Greek letter gamma, instead of pi; $\Gamma(x)$. We aren't sure why he did this, but it became much more popular. In addition, Legendre preferred the form:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t$$

Notice that this is just the Pi function shifted down by one, i.e.,

$$\Pi(x-1) = \Gamma(x) = (x-1)!$$

if x is a natural number.

We're not sure *why* Legendre did this, but here's one theory:

Claim. $\Pi(-1)$ is undefined.

Proof.

$$\Pi(-1) = (-1)! = \int_0^\infty t^{-1} e^{-t}\, \mathrm{d}t$$

by integration by parts: $f(t) = t^{-1} \implies f'(t) = -1t^{-2}, \qquad g'(t) = e^{-t}\,\mathrm{d}t \implies g(t) = -e^{-t}$

$$\int_0^\infty t^{-1} e^{-t}\, \mathrm{d}t = \frac{1}{e^t t^2}\bigg|_0^\infty - \int_0^\infty -e^{-t}(-t^{-2})\, \mathrm{d}t$$

Just looking at the first term:

$$\frac{1}{e^t t^2}\bigg|_0^\infty = \lim_{t\to\infty} \frac{1}{e^t t^2} - \frac{1}{e^0 0^2}$$

$\frac{1}{e^0 0^2}$ is not defined so the integral is undefined.

$\square$

Also notice that, for any negative integer, this also diverges, since we continue to differentiate t^{-2}, so we get something of the form:

$$\frac{c}{e^t t^{c_0}}\bigg|_0^\infty$$

where c, c_0 are constants. Which is undefined.

Anyways, since $\Pi(-1)$ isn't defined, it is theorized that Legendre wanted the first discontinuity to occur at 0, rather than -1 , i.e.,

$$x \in \mathbb{Z}^- \implies \Pi(x)\, DNE$$
$$x \leq 0 \implies \Gamma(x)\, DNE$$

Anyways, we will only refer to the gamma function now, but just remember it's the same function as the Pi function.

4 Extending out the Factorials

As an example, let's examine (what I think to be) the most fascinating result from the gamma function:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Of course, if we attempted integration by parts, like we did with -1, it would seem that $\Gamma(x)$ is undefined for all numbers less than 1, however some of these, we actually *can* compute.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{1/2-1} e^{-t}\, \mathrm{d}t = \int_0^\infty t^{-1/2} e^{-t}$$

Let's notice, if we were to differentiate $t^{1/2}$ we would get $\frac{1}{2}t^{-1/2}$, so we may get something to cancel, if we try a u-sub.

let $u = t^1/2 \implies \mathrm{d}u = \frac{1}{2}t^{-1/2}\,\mathrm{d}t \implies t = u^2, t^{-1/2} = 1/u \quad \mathrm{d}t = 2u\,\mathrm{d}u$ and $t = u^2 \implies$ if $t \in [0,\infty)$ then $u \in [0,\infty)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{u} e^{-u^2} 2u\, \mathrm{d}u$$
$$= 2\int_0^\infty e^{-u^2}\, \mathrm{d}u$$

This integral is closely related to the *Gaussian Integral.*

Theorem (Gaussian Integral).

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

The derivation of the Gaussian Integral, as well as its various properties can be found here. Since the *Gaussian function* ($f(u) = e^{-u^2}$) is even ($f(-u) = f(u)$)(maybe its most important quality):

$$2\int_{-\infty}^{\infty} e^{-u^2}\, \mathrm{d}u = \int_{0}^{\infty} e^{-u^2}\, \mathrm{d}u$$ [3]

So this means we can rewrite the integral as:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-u^2}\, \mathrm{d}u\right)\\ &= \sqrt{\pi}\end{aligned}$$