An Intuitive Approach to Deriving the Power and Taylor Series

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Abstract

The power series and the Taylor series generally breezed over sections of a first year university calculus course. Often times, the formulas are stated but never fully explained. However, more effort to explain these objects should be taken, considering that they give way Euler's formula, which gives way to almost all of complex numbered functions. From this, we find several applications in electronics, number theory, and combinatorics.

In this article, we're gonna look at how to derive the power and Taylor series, in a way that is natural. We can show that these equations are not overwhelmingly difficult to understand and that anybody, with some series knowledge and some critical thinking, can come up with these formulas, too!

Contents

1 What's a Power Series?

We're (hopefully) all familiar with polynomials. Here's a classic example:

$$x^2 + x + 1 = 0.$$

Now, if we're observant, notice that we could totally write this as a series, right?

$$\sum_{n=0}^{2} x^n = 1 + x + x^2$$

So, we can write polynomials, with degree d (the highest power is d) as a series:

$$\sum_{n=0}^{d} x^n$$

You can probably already see a few problems with this. Of course, if we had an example where the we'd need to be a little more careful, if we had a series where the coefficients weren't one. For example:

$$3x^2 + 2x + 5$$

Since the coefficients are all different, we need a new way to express this. Since we're really only concerned with the powers of x, let's basically just write the coefficients as a separate term, then state them separately. Since all the coefficients are different, we have to express that they change as the series progresses. That is, there needs to be an n "attached" to the constant. We can just subscript this as c_n (c for 'constant'), which shows that as n changes, the constant changes, also.

$$\sum_{n=0}^{2} c_n x^n = 5 + 2x + 3x^2$$

where $c_0 = 5, c_1 = 2, c_2 = 3$.

I suppose if all the c_n we equal, you could just factor it out of the series, all together.

This also fixed a problem where if every power of x wasn't present, then we would have trouble expressing it as a series. For example:

$$x^2 + 4$$

This is tough, since series are continuous, they don't have gaps. But, luckily, we can just say that the missing place has a constant of zero.

$$\sum_{n=0}^{n=2} c_n x^n = 4 + x^2$$

where $c_0 = 4, c_1 = 0, c_2 = 1$.

So, we can write any polynomial, of degree d, as:

$$\sum_{n=0}^{d} c_n x^n$$

Now, most of what students learn in this calculus class is series and how (and if) we can solve them, if they are infinitely big. So, we're gonna do the same think here. Let's let n go to infinity.

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \dots$$

Note that now we need to worry about the constants not falling into a pattern, since we have an infinite number of them, we can't list them all. Luckily, we usually care more about *whether* the series converges or not, and if the series converges without the constants, then it will converge with them, too, since as the terms go to 0, the product of the constant, $c_n(x-a)^n$ will also go to 0.

Now, we have another case: how about when our function is a shifted version of another one? For example: $(x-2)^2 + (x-2)$ is just $x^2 + x$ shifted the right by 2; that is, wherever there is an x in $x^2 + x$, there is an $(x-2)^2$ in $(x-2)^2 + (x-2)$.

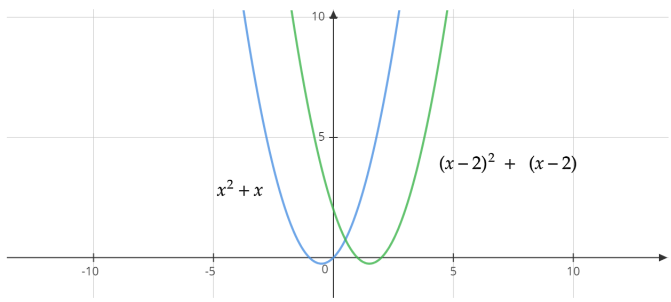

Figure 1: All figures made using Mathcha, by the author of this article.

$$\sum_{n=0}^{2} c_n (x-2) = (x-2) + (x-2)^2$$

where $c_0 = 0, c_{1,2} = 1$.

Of course, we could just factor out $(x-2)^2 + (x-2)$, but when we go to infinitely many terms, we cannot, so it's easier to just put into our formula now. Also, it will make our lives a bit easier, since if we know that the base function converges, the shifted one should also converge, since we're not changing the terms, just where they appear in the sum. Since this idea that we are just plugging in something different in for x, by adding or subtracting a constant, we can just put another constant, a in the expression.

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Here, a is the rightward shift of the function (as we showed earlier).

Students who have already taken this course will recognize this as the power series formula. Basically, it's just an infinitely long polynomial. Now, we can formalize this:

Definition.

A power series is a series that can be written in the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

2 When Does it Converge?

So, a power series is an infinitely long polynomial, with its power becoming infinitely large. The idea that the exponents are getting infinitely big means it's probably not gonna converge most of the time. If we've learned anything while tackling these series problems: it's that if we're adding infinitely many things, those things better be getting smaller. That is: for $\sum_{n=0}^{\infty} t_n$, as $n \to \infty$, $t \to 0$. Otherwise, this is gonna diverge to ∞ or $-\infty$.

So, of course, we can't really do anything about the powers, so we need to find values of $x - a$ that converge. So, we want to find the largest value of $x - a$ where the series doesn't diverge, so we want to find $x - a < R$. We denote this R since we call it the **radius of convergence**. Of course, if we also need to think about when the series diverges toward $-\infty$, too. This means that if $-(x - a)$ get too large, then we also don't converge. So, to cover both of our cases, we just take the absolute value of $x - a$.

$$|(x - a)| < R \tag{1}$$

Like most of the other tests we use, we aren't sure what happens when $|x - a| = R$. We just have to check these cases.

Of course, we are looking at values of x where the series converges (since a is fixed), so let's try and rearrange this as an inequality on x.

$$\begin{aligned} |x - a| &< R \\ -R > x - a &> R \\ a - R > x &> R + a \end{aligned}$$

We call this region the **interval of convergence**. Notice that the of the interval is $2R$, so the radius of convergence is how far the center, a, is from the edge of the interval of convergence, which is why its called the *radius* of convergence.

We find the radius and interval of convergence by using the convergence tests we've been using.

Example.

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$$

Usually, the easiest test is gonna be the ratio or the root test. Let's do the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)}{4^{n+1}}(x+3)^{n+1} \cdot \frac{4^n}{(-1)^n n(x+3)^n} \right|$$

The ratio test tells us that if $L < 1$, then this series converges. Remember that our goal is to find the x values for which the series converges, so we need to find the x values which makes $L < 1$. Now, we can continue, by cancelling a bunch of terms.

$$L = \lim_{n\to\infty} \left| \frac{-1(n+1)(x+3)}{4n} \right|$$

Now, since we have a limit of n, we can take $|x+3|$ and -1 out of the limit.

$$L = |-(x+3)| \lim_{n\to\infty} \left| \frac{n+1}{4n} \right|$$

We can use L'Hopitals rule on that limit, to get $\frac{1}{4}$. Also, the absolute value of a negative value is equal to the absolute value of its positive value.

$$L = |(x+3)| \frac{1}{4}$$

Like we said before, we need $L < 1$, so we can plug that in.

$$\begin{aligned} |(x+3)| \frac{1}{4} &< 1 \\ |x+3| &< 4 \end{aligned}$$

So now, we have the number which bounds the convergence of $x - a$. Isn't that the radius of convergence? So we know that $R = 4$. Now all we need to do is complete the inequality to find the interval of convergence.

$$\begin{aligned} -4 < x+3 &< 4 \\ -7 < x &< 1 \end{aligned}$$

So if $x \in (-7, 1)$, then it converges. Now, we have to check the end points, since the ratio test doesn't account for these cases.
$x = -7$:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-7+3)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n \end{aligned}$$

We can cancel 4^n here, if we separate out $(-4)^n$ into $(-1)^n 4^n$.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n n}{4^n} 4^n$$

Cancelling the 4^n and combining the $(-1)^n$ terms:

$$= \sum_{n=0}^{\infty} (-1)^{2n} n$$

Notice that this is -1 to some even power, like $(-1)^0, (-1)^2, (-1)^4$, etc. and -1 to an even power is 1. Alternatively, we could write it as $(-1)^2(1)^n = 1$

$$= \sum_{n=0}^{\infty} n$$

Of course, this is the sum of all the positive whole numbers, which diverges, since the limit of the terms doesn't go to 0 (divergence test). So, $x = -7$ diverges, which means it is not in the interval of convergence.

Now, let's check $x = 1$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n} 4^n \tag{2}$$

$$= \sum_{n=0}^{\infty} (-1)^n n \tag{3}$$

Again, this diverges by the divergence test. So, $x = 1$ also diverges, so we are left with just the region of $-7 < x < 1$ as the interval of convergence.

3 Relation to the Geometric Series

Now, can we think of any series that has increasing powers? One common example is the **geometric series**. This is the sum of a bunch of terms with increasing exponents:

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 \dots \tag{4}$$

You may remember that we call r the common ratio; the thing we multiply each term by to get to the next term, the **common ratio**. We could easily just replace r with x and this would be just like our definition of our power series, with the no shift ($a = 0$ in our original equation) and all of the constants are equal, so $c_n = c, \ \forall n$. Since all of those constants are equal, we can actually take c out of the sum:

$$c \sum_{n=0}^{\infty} r^n = c(1 + r + r^2 + r^3 \dots)$$

So, the convergence of this sum only depends on what the common ratio, r is. Well, the terms need to continue to get smaller as $n \to \infty$, by the divergence test, so when $|r| < 1$ is the only time where this can converge.

Hopefully by this point in a calculus course, you've already encountered what this sum converges to. For the sake of keeping things consistency, I'm just gonna use x, instead of r. Given that $|r| < 1$:

Definition.

A geometric series is a series that can be written as:

$$\sum_{n=0}^{\infty} cx^n \tag{5}$$

Given that $|x| < 1$, it converges to:

$$c\left(\frac{1}{1-x}\right).$$

Alright, so, we can write a function of x as a polynomial. This is great, since polynomials are really easy to differentiate and integrate, since all we need is the power rule! Let's see an example.

Example.

Find the power series representation of $g(x) = \frac{1}{1+x^3}$ and find its interval of convergence.

So, we're trying to relate the function we have here, to the one that we had in our formula, $\frac{1}{1-x}$. Well, first we have an x^3 in the spot where there should be an x. Secondly, we need to have a subtraction in the denominator where there is now an addition. How can I rewrite addition as subtraction? Well, remember that subtracting a negative number is the same as adding a positive number; i.e., $x + y = x - (-y)$. So, I can "force" the addition to become subtraction by rewriting $\frac{1}{1+x^3}$ as $\frac{1}{1-(-x^3)}$. So, let's just call our original function, $f(x) = \frac{1}{1-x}$, then $g(x)$ will have a x^3, wherever $f(x)$ has an x, so: $g(x) = f((-x)^3)$. So, since $f(x)$ has a power series of $\sum_{n=0}^{\infty} x^n$, $g(x)$ has a power series of $f((-x)^3) = \sum_{n=0}^{\infty}(-x)^{3n}$. We can rewrite this a little, by separating out the (-1).

$$\begin{aligned}
\frac{1}{1+x^3} &= \sum_{n=0}^{\infty}(-1)^{3n}(x^{3n}) \\
&= \sum_{n=0}^{\infty}\left((-1)^3\right)^n(x^{3n}) \\
&= \sum_{n=0}^{\infty}(-1)^n(x^{3n})
\end{aligned}$$

Alright, we can go a little further with this by differentiating or integrating the power series, but it still has the same problem: we *must* relate the function to $\frac{1}{1-x}$ for this to work. How can we extend this out more? Well, we needed all of the constants to be equal to each other. Note that that other condition, where $a = 0$ wasn't necessary, we could always plug in $(x - a)$ for x, but the constants being equal was necessary, since we needed it to set up the relation $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. So, maybe we can extend out the power series, if we can somehow let the constants be different?

4 The Taylor and Maclaurin Series

So, we're trying to find a way to get a power series where the constants can be different, so we can find the power series of any function, $f(x)$. So, we hope that there's some power series that is equal to our function, meaning that:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

and like we said, at least one of the constants has to be different, otherwise we end up in the same case as before.

Here's the problem: how can we even find out what those constants are? Here's a generally good piece of advice when dealing with these series: write out the first few terms so you can see what's actually happening.

$$f(x) = c_0(x-a)^0 + c_1(x-a)^1 + c_2(x-a)^2 \dots$$

Alright, so how can we find c_0? Well, of course, $(x-a)^0 = 1$, so the first term is c_0. But what about all those other terms, how can we make them 'go away' so we can isolate c_0? All of those terms have those $(x-a)^n$ terms, so if $x = a$, then they all become 0, right? So, we now have a way to find c_0, just plug in $x = a$; i.e.,

$$f(a) = c_0$$

Now, how do we find c_1? Our problem is that if we try to plug in $x = a$, then we also get rid of c_1. We really want c_1 to be in the spot that c_0 is in, how can we do that? Well, how can we get rid of the $x - a$ being multiplied by c_1?

How can we get rid of x terms? Well, if we do the power rule, then it reduces the power of a polynomial by 1, right? That is: $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, the power goes down by 1. Likewise, if we had $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$. So, we just got rid of an x term, right? Even better, the derivative of a constant is 0, so that c_0 will also disappear from the series, when we differentiate; if c is a constant, $\frac{\mathrm{d}}{\mathrm{d}x}c = 0$.

Let's try this with our series:

$$f'(x) = 0 + c_1 + 2c_2(c-a) + 3c_3(x-a)^2 \ldots$$

Now, like we did before, if we set $x = a$, all of the terms disappear, except for the constant, so we can use it to isolate c_1.

$$f'(a) = c_1$$

Now, using the same process for c_2, notice that there's an extra 2 that we have to divide out.

$$\begin{aligned} f''(a) &= 2c_2 \\ \implies \frac{f''(a)}{2} &= c_2 \end{aligned}$$

We had to divide out the 2, since we are gonna keep doing the power rule, which causes the exponent to 'come down'. How will this trend continue? Let's do c_3.

$$\begin{aligned} f'''(a) &= 3 \cdot 2c_3 \\ \implies \frac{f'''(a)}{3 \cdot 2} &= c_3 \end{aligned}$$

Notice that if we continue, we need to differentiate once to get c_1, twice to get c_2, etc. (and I suppose, 0 times for c_0.) So, we need to differentiate n times to find c_n. However, when we do this, the power rule causes $n(n-1)(n-2)\ldots 1$ to 'come down', so we need to divide by those terms. This sequence of numbers may look familiar, it's the **factorial**!

Actually, this is a result of the fact that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = (n(n-1)(n-2)\cdots 1)x = n!x$. In our case, this means that $f^{(n)}(a) = n!c_n$, so $\frac{f^{(n)}(a)}{n!} = c_n$.

So, we actually have an expression to find *any* c_n! This means, we're done, this is the power series for any function (provided that it's differentiable). It's just the power series, where $c_n = \frac{f^{(n)}(a)}{n!}$. We call this the **Taylor series**, or the **Taylor series expansion of** f.

Definition.

The **Taylor series** of some function $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a = 0$, we have a special name for this: the **Maclaurin series.**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Again this is very practical, we can basically rewrite (almost) any function as a polynomial.

Now, of course, there's a problem with the polynomial with infinite terms, but we can approximate the function by using some of the terms. Actually, a cool observation is that the first two terms of the Taylor series is the linear approximation of a function near a:

$$f(x) \approx f'(a)(x-a) + f(a)$$

near a.

So the Taylor series approximates the function near $x = a$, by just using a terms. Of course, if we use the whole series, then the function will equal the series. So, the first few terms of the Maclaurin series is an approximation of $f(x)$ near $x = 0$. Let's try some examples.

5 Some Applications of Taylor Series

Now, let's look at the Taylor/Maclaurin series of some common functions. A word of advice: you're going to be trying to find some pattern in these sums, so it's usually easier to see the pattern if you don't simplify anything.

Example.

Find the Maclaurin Series of e^x

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e^x is a pretty complicated function, so this one may be pretty helpful. First, let's find our first few constants to see if we can find a pattern.

$$c_0 = \frac{f(0)}{0!} = e^0 = 1$$
$$c_1 = \frac{f'(0)}{1!} = e^1 = 1$$
$$c_2 = \frac{f''(0)}{2!} = \frac{e^2}{2!}$$

So, this nice thing about e^x is that it is its own derivative, so that numerator isn't going to change. That is: $\frac{f^{(n)}(0)}{n!} = \frac{e^0}{n} = \frac{1}{n!}$. So, now we can write the whole series:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Example.

Find the Taylor series of $\ln(x)$ centered at $x = 2$.

I suppose it's worth mentioning that the Maclaurin series of $\ln(x)$ is undefined, since $\ln(0)$ is undefined. Let's look at the first few terms. Remember that the derivative of $\ln(x) = \frac{1}{x}$.

$$\begin{aligned}
c_0 &= \frac{f(2)}{0!} = ln(2) \\
c_1 &= \frac{f'(2)}{1!} = \frac{1/2}{1!} = \frac{1}{2 \cdot 1!} \\
c_2 &= \frac{f''(2)}{2!} = -\frac{1/(2^2)}{2!} = -\frac{1}{2^2 \cdot 2!} \\
c_3 &= \frac{f'''(2)}{3!} = \frac{2/(2)^3}{3!} = \frac{2}{2^3 \cdot 3!} \\
c_4 &= \frac{f^{(4)}(2)}{4!} = -\frac{(2 \cdot 3)/(2)^4}{4!} = -\frac{2 \cdot 3}{2^4 \cdot 4!}
\end{aligned}$$

Notice that the numerator is gonna keep getting new terms, from differentiating the $1/x^n$ term over and over again, but it's always one less than n. Since we're multiplying, the numerator is $(n-1)!$.
Also, notice that c_0 doesn't really fit into our pattern, so let's just take it out of the sum, but doing this removes the first term of our series, so it's probably easier to start the sum at $n = 1$, instead of making a whole new index.
Also, notice that the even n values (except c_0) are gonna be negative, since we keep bringing down a negative power, so we need to account for that in our indexes. So, if we wanted the even terms to be negative, all we have to do is raise -1 to the power n, since if n is even, then $(-1)^n = 1$, and if it's odd, then $(-1)^n = -1$. But since we're starting the series at $n = 1$, we just need to add 1 to n, since we kinda just switched what the even and odd terms of the series are. So, we must multiply by $(-1)^{n+1}$.
Also, remember that the Taylor series has an $x - a$ term, so we need to have $x - 2$ after our constant term.

$$\begin{aligned}
\ln(x) &= \ln(2) + \frac{1}{2 \cdot 1!}(x-2) - \frac{2}{2^2 \cdot 2!}(x-2)^2 + \frac{2}{2^3 \cdot 3!}(x-2)^3 - \frac{2 \cdot 3}{2^4 \cdot 4!}(x-2)^4 \ldots \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n+1}}{2^n \cdot n!}(x-2)^n
\end{aligned}$$

We can simplify this a little more, since we have a fraction with two factorial terms. All the terms in $n-1$ are in $n!$, right? The only difference is that $n!$ is multiplied by n. That is: $n! = n(n-1)(n-2)\ldots 1$ and $(n-1)! = (n-1)(n-2)\ldots 1$, so $n! = n(n-1)!$. Thus, $\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$. So, finally,

$$\ln(x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(x-2)^n(-1)^{n+1}}{2^n \cdot n}$$

Example.

Find the Maclaurin Series for $\sin(x)$.

Once again, let's begin by looking at some constant terms.

$$\begin{aligned} c_0 &= \frac{f(2)}{0!} = \sin(0) = 0 \\ c_1 &= \frac{f'(2)}{1!} = \frac{cos(0)}{1!} = \frac{1}{1!} \\ c_2 &= \frac{f''(2)}{2!} = \frac{\sin(0)}{2!} = 0 \\ c_3 &= \frac{f'''(2)}{3!} = \frac{-\cos(0)}{3!} = -\frac{1}{3!} \end{aligned}$$

So, we can already notice a pattern here. If n is even, then $c_n = 0$, meaning those terms disappear from the series. How can we only get odd terms in our series? What's an odd number? Well, it's all the numbers not divisible by 2. Since ever two numbers is divisible by 2 (2,4,6,8 ...), it's all the numbers in between the even numbers. That is, it's an even number (which is a multiple of 2, $2k$) plus (or minus) 1. So, we can write all of the odd numbers as $2k+1$, where k is any positive, whole number, or 0. [a] So, we need the constants to only include odd n values, so we gotta replace n by $2n+1$, remember that $2n+1$ is all the odd numbers, when we plug in the n values.
Also, we have one more issue to address: our series is alternating in sign, so we have to add an extra term. It starts positive, so we have to use $(-1)^{n+1}$, so that the first -1 happens on the second term.

$$\begin{aligned} sin(x) &= 0 + \frac{1}{1!}x + -\frac{1}{3!}x^3 \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1} \end{aligned}$$

[b]

[a] You could also write this as $2k-1$, but you have to start k at 1, whereas $2k+1$ starts k at 0.

[b] If you decided to use $2n-1$, the series would just start at $n=1$. Also, of course, you would have replaced the n with $2n-1$, instead. This is just an index shift, so it's the same as the one above.

Example.

Find the Maclaurin series of $\cos(x)$.

Let's compute some constants, again.

$$\begin{aligned}
c_0 &= \frac{cos(0)}{0!} = \frac{1}{0!} \\
c_1 &= \frac{-sin(0)}{1!} = 0 \\
c_2 &= \frac{-cos(0)}{2!} = \frac{-1}{2!} = -\frac{1}{2!} \\
c_3 &= \frac{sin(0)}{3!} = \frac{0}{3!} = 0
\end{aligned}$$

So, this is the opposite trend to what we saw before: there's only even values of c_n. So, we can deal with the same way we dealt with the Maclaurin series of $\sin(x)$. This time, we need to replace n with $2n$, since $2n$ is all the even numbers. Also, like with $sin(x)$, we need to have the sign alternating, again. Then, we can rewrite the series, just like we did with $sin(x)$.

$$\begin{aligned}
\cos(x) &= \frac{0}{0!} + 0 - \frac{1}{2!} + 0 \dots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n}
\end{aligned}$$

6 Closing Remarks

Specifically, the e^x, $\cos(x)$ and $\sin(x)$ Taylor series are quite important, since we use them to derive **Euler's Formula**: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. From this, we can find the renowned **Euler's Identity**: $e^{i\pi} + 1 = 0$. But more on that in the next article (I'll link it here when complete).

Of course, in practice, we would much rather use the Taylor series of a function, since it's easy to differentiate and integrate (that's why we did this whole thing!). However, we cannot actually add infinitely many things together, so we use some of the terms of the series. Using certain methods, we can actually limit how far our approximation is to the real answer, but again, this will be a topic for another paper (which will be linked here).