An Intuitive Approach to the Laplace Transform

Liam Donovan

March 2023

Abstract

In most differential equations classes, the Laplace Transform is introduced as this ambiguous, mysterious definition that isn't really explained thoroughly. Despite this, this is a fairly intuitive and amazing transformation, and its derivation should be explain more often.

1 The Goal

Differential equations are tough to solve, in fact, at their core we *cannot* solve them. We need to transform them into algebraic equations, then solve them that way. The Laplace Transform is just another way to accomplish this task. Let's begin with an example:

$$y'(t) + y(t) = 4e^t$$

So our problem is this: we cannot integrate y' by itself, if there was just a y' in the equation, I could easily just integrate both sides to find $y(t)$. However, we can't possibly integrate this without also integrating $y(t)$, which gets us farther from our goal.

So, I want something that takes in some derivative: $y^{(n)}(t)$, by itself (is linear), and outputs something like this: $f(y)$; just a big function of $y(t)$, which I can solve using algebra. A function that takes in a function is called a *transform*. Let's call out transform T for now. Let's think about this: is there any way to solve this in t, i.e., by changing the variable, t to something else. Not really, that's more of an algebraic trick, it doesn't help us as much here. In the end, all the functions will still be derivatives of each other. So we need to change what the functions independent variable is. Let's call this other variable s. So we want the function to take in a function in t and output a function in s, and hopefully, we can do algebra in s to solve, then we can invert the process to get back to the original $y(t)$, i.e.,

$$T : y^{(n)}(t) \rightsquigarrow f(s)$$

How can we do this? Is there a process that allows us to "get rid" of a variable, and put in a new one? Well, a definite integral "gets rid" of all of one variable, and if we put another variable, it won't integrate it, so it'll be treated like a constant, leaving it in the final expression. We call this class of transforms *integral transforms*. They have the form we just discussed:

$$(Tf)(y) = \int_{x_1}^{x_2} f(t)K(t,y)\,\mathrm{d}t$$

We call K the *kernel* function.

2 Finding A Suitable Transform

So all we need to do is find the bounds and kernel function need to be to accomplish our goal. Here's our transform:

$$(Ty^{(n)})(s) = \int_{t_1}^{t_2} y^{(n)}(t)K(s,t)\,\mathrm{d}t$$

Notice that we need K as a function of both s and t, since if it was only a function of t, the integral would be a constant. Now, since we have two functions, thus *must* be computeable by integration by parts.
Recall: Integration by parts (for definite integals):

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = f(t)g(t)\big|_a^b - \int_a^b g(t)f'(t)\,\mathrm{d}t$$

We often often see this as:

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$$

Note that in our case, we have a multivariate function, the kernel, since we, of course, want to integrate the $y^{(n)}(t)$, we need to partial derivatives of K. Remember, our goal is the reduce everything to a function of s, so we want to solve the algebraic equation:

$$(Ty)(s) = \ldots$$

Since, then we can "undo" the transform. So we're gonna have to integrate this a bunch of times, meaning we're gonna be needing to have a repeating integration by parts. After n iterations, we want to see something like this:

$$\underbrace{y^{(n)}(t) \xrightarrow{\int dt} y^{(n-1)} \xrightarrow{\int dt} \ldots y(t)}_{n}$$

So, now we just have to make sure our choice of K doesn't stop the integration. So, we want something that can be differentiated infinitely many times, we call say these functions are *smooth* or *of class C^∞*. Let's quickly write the whole integration by parts:

$$\int_{t_1}^{t_2} y^{(n)}(t)K(s,t)\,dt = K(s,t)y^{(n)}(t)\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} K_t(s,t)y^{(n-1)}(t)\,dt$$

just looking at the $\int_{t_1}^{t_2} K_t(s,t)y^{(n-1)}(t)\,dt$:

$$\int_{t_1}^{t_2} K_t(s,t)y^{(n-1)}(t)\,dt = K_t(s,t)y^{(n-1)}(t)\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} K_t(s,t)y^{(n-2)}(t)\,dt$$

Note that as long as we have bounds, $y^{(n)}$ and all its derivatives are constants. Also notice that if at any point, if $K_t(s,t) = 0$ or any of its integrals equal 0, then the integration by parts breaks.

Let's think about smooth functions, there's two main types, either trig.(sin, cos) and exponential (t^x). The problem with a trig function is that if we do a trig. function, the the integral will repeat, so it will stop too early, also the values would awkward. So we're gonna use an exponential function of K, we also use e because its derivative is itself, so that helps with intermediate calculations. So here's out transform now:

$$e^{st}y^{(t_1)}t_2\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} e^{st}y^{(n-1)}(t)\,dt$$

If t_1, t_2 are "bad" values, we may get some unnecessarily tough calculations. We can make this easier on ourselves. We know that $e^\infty \to 0$ and $e^0 = 1$, and even better, $y^{(n)}(0), y^{(n-1)}(0) \ldots y(0)$ should be pretty easy to calculate. So why not make our bounds $t_1 = -\infty$, $t_2 = 0$? This *would* work, however, Laplace made this transform because he was working with probability distributions, so he wanted his t values to be positive, so he elected to make $K(s,t) = e^{-st}$ and $t_1 = 0$, $t_2 = \infty$. So now we have the *Laplace Transform*. which we denote $\mathscr{L}\{f(t)\}$:

Theorem. *(Laplace Transform)*

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$$

Note that from the first step of integration by parts, we get:

$$e^{-st}f(t)\big|_0^\infty$$

This is only defined if $\lim_{t \to \infty} \frac{f(t)}{e^{st}} = 0$, i.e., $|f(t)| \geq e_0 e^{c_0 t}$. Basically, if $f(t)$ grows faster/declines faster than e^{-st}.

3   Checking Our Work

While we could check our work by tediously doing integration by parts and looking at the pattern, we've already done that in the gamma function derivation, where we discussed a much faster and more rigorous method, proof by induction. We'll use it here:

After doing integraiton by parts a few times, we notice the pattern:

$$\mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(0)\} - s^{n-1}f'(0) - s^{n-2}f''(0) \ldots f^{(n)}(0)$$

If this is correct, then we've succeeded, as the RHS is an algebraic equation, in fact it's a polynomial.

Claim.
$$\mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(0)\} - s^{n-1}f'(0) - s^{n-2}f''(0) \ldots f^{(n)}(0)$$

Proof. By induction:
Base Case: $n = 1$

$$\mathscr{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,\mathrm{d}t$$

by integration by parts: let $u = e^{-st} \implies \mathrm{d}u = -se^{-st}$, $\mathrm{d}v = f'(t)\,\mathrm{d}t \implies v = f(t)$:

$$\mathscr{L}\{f'(t)\} = e^{-st}f(t)\big|_0^\infty - \int_0^\infty -se^{-st}f(t)\,\mathrm{d}t$$

assuming that the first term doesn't diverge:

$$= -f(0) + \int_0^\infty se^{-st}f(t)\,\mathrm{d}t$$

Notice that the integral is just $\mathscr{L}\{f(t)\}$, so;

$$= -f(0) + \mathscr{L}\{f(t)\}$$

Inductive Step:
Assume:
$$\mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(0)\} - s^{n-1}f'(0) - s^{n-2}f''(0) \ldots f^{(n)}(0)$$
Show:
$$\mathscr{L}\{f^{(n+1)}(t)\} = s^{n+1} \mathscr{L}\{f(0)\} - s^n f'(0) - s^{n-1}f''(0) \ldots f^{(n+1)}(0)$$
let by integration by parts: let $u = e^{-st} \implies \mathrm{d}u = -se^{-st}$, $\mathrm{d}v = f^{(n+1)}(t)\,\mathrm{d}t \implies v = f^{(n)}(t)$:

$$\mathscr{L}\{f^{(n+1)}(t)\} = e^{-st}f^{(n+1)}(t)\big|_0^\infty - \int_0^\infty -se^{-st}f^{(n)}(t)\,\mathrm{d}t$$

Note: if $f^{(n)}(t)$ converged, $f^{(n+1)}(t)$ also converges:

$$= -f^{(n+1)}(0) + \int_0^\infty se^{-st}f^{(n)}(t)\,\mathrm{d}t$$

RHS is just $\mathscr{L}\{f(t)\}$

$$\mathscr{L}\{f^{(n+1)}(t)\} = -f^{(n+1)}(0) + \mathscr{L}\{f^{(n)}(t)\}$$

and since $\mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(0)\} - s^{n-1}f'(0) - s^{n-2}f''(0) \ldots f^{(n)}(0)$:

$$\mathscr{L}\{f^{(n+1)}(t)\} = s^{n+1} \mathscr{L}\{f(0)\} - s^n f'(0) - s^{n-1}f''(0) \ldots f^{(n+1)}(0)$$

$$\square$$

4   A Nice Connection

Notice how similar our derivation of the Laplace Transform is the the derivation of the gamma function also how similar they look, perhaps they are related, in some way? Well, we derived the gamma function by differentiating an exponential, what if we made $f(t)$ an exponential function?

$$\mathscr{L}\{t^x\} = \int_0^\infty t^x e^{-st}\,\mathrm{d}t$$

This is already quite close to the gamma function, but we need to integrate over the negative exponent of e, so let's try a substitution:

let $st = \alpha \implies \mathrm{d}\alpha = s\,\mathrm{d}t \implies \frac{\alpha}{2} = t \implies \frac{1}{s}\,\mathrm{d}\alpha = \mathrm{d}t$

$$\mathscr{L}\{t^x\} = \int_0^\infty \left(\frac{\alpha}{s}\right)^x e^{-\alpha}\frac{\mathrm{d}\alpha}{s} \tag{1}$$

We can distribute the x over the numerator, then take all the s out of the integer, since we're integrating over α

$$= \frac{1}{s^x \cdot s}\int_0^\infty \alpha^x e^{-\alpha}\,\mathrm{d}\alpha \tag{2}$$

$$= \frac{1}{s^{x+1}}\int_0^\infty \alpha^x e^{-\alpha}\,\mathrm{d}\alpha \tag{3}$$

Notice that's just $\Gamma(x+1)$, with a dummy variable (which is technically the Pi function, but the gamma function is more well known). So, finally, the Laplace transform of t^x is:

$$\mathscr{L}\{t^x\} = \frac{\Gamma(x+1)}{s^{x+1}}$$

So, if x is a natural number, then So, the Laplace transform is very much related to the gamma function.

Let's plug in some values to apply this formula:

$$\mathscr{L}\{t\} = \frac{1!}{s^2} = \frac{1}{s^2}$$
$$\mathscr{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$
$$\mathscr{L}\{1/2\} = \frac{\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{s^{3/2}}$$

[1]

5   Final Thoughts

So, as it turns out, the Laplace transform is as ambiguous as we thought. It is, in fact, a very clever use of an integral transform, along with some knowledge of common functions and limits. Therefore, its derivation should be covered in more differential equations classes, as it gives a deeper thinking of how differential equations work and is a good mental exercise, which uses many common results from basic calculus.

---

[1] We derived $\Gamma(1/2)$ in the document where we derived the gamma function (link above).