Quaternions: the basics

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Define the matrices

$$
\begin{aligned}
1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
i &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
j &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
k &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\end{aligned}
$$

Define the *set of quaternions* $\mathbb{H}$ to be the real vector space with basis $\{1, i, j, k\}$.

Exercises The following should be routine to check, and are basic/important properties of the quaternions:

1. The quaternions $1, i, j, k$ satisfy the famous identities:

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = k$$

$$ki = -ik = j.$$

The last three identities can be unified in the following diagram:

$$
\begin{tikzcd}
 & i \arrow[dr, bend left] & \\
k \arrow[ur, bend left] & & j \arrow[ll, bend left]
\end{tikzcd}
$$

where following the arrow is multiplication (the result being the quaternion which wasn't in the computation), and if you go against the arrow you need to add a minus sign.

2. For now, we distinguish between the *quaternion* $i \in \mathbb{H}$ and the *complex number* $i \in \mathbb{C}$. **However**, we can embed the $\mathbb{C}$ into $\mathbb{H}$ in the obvious way:
$$\mathbb{C} \ni a + bi \mapsto a + bi \in \mathbb{H}.$$
3. If a quaternion $X = a1 + bi + cj + dk$ then it is represented as a complex matrix by
$$X = \begin{pmatrix} a+bi & c+di \\ -(c-di) & a-bi \end{pmatrix}.$$
Let $\alpha = a + bi$ and $\beta = c + di$. Then $X = \alpha + \beta j$, and
$$X = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$
Also, any complex matrix of the above form is a quaternion.
4. $\mathbb{H}$ is a $\mathbb{C}$-vector space over the basis $\{1, j\}$. Therefore, $\mathbb{H}$ is isomomorphic to $\mathbb{C}^2$ as a vector space (but not as an algebra - $\mathbb{C}^2$ doesn't have a multiplication so it isn't an algebra).
5. $\mathbb{H}$ is a *skew field*, i.e. it is a field except for the fact that it isn't commutative (you can see this from the above exercise).

Define the *pure quaternions* to be those with no "real" part, i.e. those of the form $ai + bj + ck$. There is an obvious embedding of $\mathbb{R}^3 \hookrightarrow \mathbb{H}$ by $(x, y, z) \mapsto xi + yj + zk$. Define the *unit quaternions* to be those quaternions $X \in \mathbb{H}$ such that $\det(X) = 1$.

Exercises Again, the following should be routine:

1. If $X = a1 + bi + cj + dk = \alpha + \beta j$ then $\det(X) = a^2 + b^2 + c^2 + d^2 = |\alpha|^2 + |\beta|^2$.
2. The set of unit quaternions is exactly the set of *special unitary matrices* $\mathrm{SU}(2)$. [1] Not only do the unit quaternions form a group (inheriting the operation of matrix multiplication), we also have that the set of unit quaternions is homeomorphic to S^3! We call $\mathrm{SU}(2)$ the *group of unit quaternions.*
3. Let U be a unitary matrix. While it is true that $|\det(U)| = 1$, it need not be that $\det(U) = 1$.
4. If $X \in \mathrm{SU}(2)$ then $X^{-1} = \overline{X}^t = X^*$.

[1] A matrix X is *special* if $\det(X) = 1$, and it is *unitary* if $XX^* = I_n$, where $X^* := \overline{X}^t$.

For the following, let's embed $\mathbb{R}^3 \hookrightarrow \mathbb{H}$ and have no notational difference between elements of $\mathbb{R}^3$ and pure quaternions. Define an action of $\mathrm{SU}(2)$ on $\mathbb{R}^3$ by congugation; i.e. define the action $\cdot : \mathrm{SU}(2) \times \mathbb{R}^3 \to \mathbb{R}^3$ to be

$$Z \cdot X = ZXZ^{-1} = ZXZ^*.$$

Exercises Quickly check that the above-defined action is indeed an action, and then prove something very important about the quaternions:

1. The above is in fact an action, i.e. $Z \cdot X$ is indeed a pure quaternion for any $Z \in \mathrm{SU}(2)$ and for any $X \in \mathbb{R}^3$. *Hint:* Trivial but tricky calculations; keep your head straight.
2. The action of $\mathrm{SU}(2)$ on $\mathbb{R}^3$, i.e. the map $\rho_Z : \mathbb{R}^3 \ni X \mapsto ZXZ^* \in \mathbb{R}^3$, as described above is in fact a rotation on $\mathbb{R}^3$. In other words, check that $\rho_Z \in \mathrm{SO}(3)$. Try splitting the exercise into the following parts:
   i. Notice that any $Z \in \mathrm{SU}(2)$ is of the form

   $$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & -\overline{\alpha} \end{pmatrix}$$

   for some $\alpha, \beta \in \mathbb{C}$. Notice also that any $X \in \mathbb{R}^3$ is of the form

   $$\begin{pmatrix} xi & \omega \\ -\overline{\omega} & -xi \end{pmatrix}$$

   for some $x \in \mathbb{R}$ and $\omega \in \mathbb{C}$. You might find it useful to use the notation $(x, \omega) \in \mathbb{R} \times \mathbb{C} \equiv \mathbb{R}^3 \hookrightarrow \mathbb{H}$. Now, calculate ZXZ^* explicitly.

   ii. To calculate the matrix of the map ρ_Z, fist see what it does to the basis elements: replace X in part (i) with $(1, 0, 0) = i$, $(0, 1, 0) = j$, and $(0, 0, 1) = k$ (with the above "(x, ω)" notation that I suggested, these basis elements are just $(1, 0)$, $(0, 1)$, and $(0, i)$ respectively).

   iii. For $\alpha = a + bi$ and $\beta = c + di$, calculate the matrix to get

   $$M = \begin{pmatrix} 1 - 2c^2 - 2d^2 & 2bc - 2ad & 2ac + 2bd \\ 2ad + 2bc & 1 - 2b^2 - 2d^2 & 2cd - 2ab \\ 2bd - 2ac & 2ab + 2cd & 1 - 2b^2 - 2c^2 \end{pmatrix}.$$

   iv. Show that this matrix is *special* by showing that its determinant is 1, and show that it is *orthogonal* by showing that $MM^t = I_3$.

   Hint: Some very involved calculations. If you find yourself lost and frustrated in a miasma of symbols then stop writing and breathe: what's the most efficient way of going about this? If you haven't already learned to use an algebra package, consider doing so now.

Define the map $\rho : \mathrm{SU}(2) \ni Z \mapsto \rho_Z \in \mathrm{SO}(3)$.

Exercise Show that ρ is a surjective homomorphism with kernel $\{-1, 1\} \subset \mathrm{SU}(2)$.

References

- *Clifford Algebras, Clifford Groups, and a Generalization of the Quaternions: The Pin and Spin Groups*, notes by Jean Gallier, 2014.