MA3G7 Functional Analysis 1

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19th August 2021

Contents

1 Vector Spaces

Theorem 1. *Every vector space has a Hamel basis.*

You don't need to prove the above theorem.

Theorem 2. *If a vector space has a finite Hamel basis then every Hamel basis for this vector space has the same number of elements.*

Prop. 1. *Any n-dimensional vector space over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.*

Lemma 1 (Young's inequality)**.** *If $a, b > 0$, $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 2 (Hölder's inequality)**.** *If $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \ell^p(\mathbb{F})$ and $y \in \ell^q(\mathbb{F})$, then*

$$\sum |x_j y_j| \leq \|x\|_p \|y\|_q .$$

Lemma 3 (Minkowski's inequality)**.** *Let $1 \le p < \infty$. If $x, y \in \ell^p(\mathbb{F})$ then $x + y \in \ell^p(\mathbb{F})$ and*

$$\|x + y\|_p \le \|x\|_p + \|y\|_p .$$

2 Normed spaces

Prop. 2. *Let $(W, \|\cdot\|_W)$ be a normed space, V be a vector space, and $L : V \to W$ a linear isomorphism. Then the function*

$$V \ni x \mapsto \|L(x)\|_W \in \mathbb{R}$$

defines a norm on V.

Prop. 3. *Any finite dimensional vector space can be equipped with a norm. Therefore, any n-dimensional vector space over $\mathbb{F}$ is isometrically isomorphic to $\mathbb{F}^n$ equipped with a suitable norm.*

Theorem 3. *All norms on a finite dimensional space are equivalent.*

Prop. 4. *If $f_n \in C[0, 1]$ for all $n \in \mathbb{N}$ and $f_n \to f$ in the sup-norm, then $f_n \to f$ in the L^1-norm.*

Prop. 5. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space, then for any sequence x_n in the vector space and for any $x \in V$,*

$$\|x_n - x\|_1 \to 0 \iff \|x_n - x\|_2 \to 0.$$

Lemma 4. *In a normed space $(V, \|\cdot\|)$, a sequence $x_n \to x$ if and only if for any open neighbourhood X of x there is an $N \in \mathbb{N}$ such that $x_n \in X$ for all $n > N$.*

Prop. 6. *On a vector space V, two norms are equivalent if and only if they induce the same topology.*

Lemma 5. *A subset $X \subset V$ is closed if and only if every convergent sequence with elements in X has its limit in X.*

Prop. 7. *A finite dimensional linear subspace W of a normed space V is closed.*

Theorem 4 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore, a subset of a finite-dimensional vector space is closed if and only if it is bounded.*

You don't need to know the proof of this - but you should, as you studied it in Year 2.

Lemma 6 (Riesz' Lemma)**.** *Let X be a normed vector space and Y be a closed linear subspace of X such that $Y \neq X$ and $\alpha \in (0, 1)$. Then there is $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ for all $y \in Y$.*

Theorem 5. *A normed space is finite dimensional if and only if the unit sphere is compact.*

3 Banach spaces

Theorem 6. *A sequence of real numbers converges if and only if it is Cauchy.*

Proved in Year 1.

Theorem 7. *Every finite-dimensional normed space is complete. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete.*

Theorem 8. *The space $\ell^p(\mathbb{F})$ equipped with the ℓ^p-norm is complete.*

Theorem 9. *The space $C[0,1]$ equipped with the sup-norm is complete.*

Theorem 10 (Bernstein polynomials)**.** *Consider the polynomials*

$$B_{np}(x) = \binom{n}{p} x^p (1-x)^{n-p}.$$

Then the polynomials B_{np} are of degree n and we have that

(i) $\sum_{p=0}^{n} B_{np}(x) = 1$

(ii) $\sum_{p=0}^{n} pB_{np}(x) = nx$

(iii) $\sum_{p=0}^{n} (p-nx)^2 B_{np}(x) = nx(1-x)$.

Theorem 11. *If a function $f : [0,1] \to \mathbb{R}$ is continuous then the sequence of polynomials*

$$P_n(x) = \sum_{p=0}^{n} f\left(\frac{p}{n}\right) \binom{n}{p} x^p (1-x)^{n-p}$$

converges uniformly to f on $[0,1]$.

Therefore, $C[0,1]$ is separable.

Theorem 12. *Any normed space is isometrically isomorphic to a dense subset of a Banach space.*

4 Lebesgue spaces

Theorem 13 (Monotone Convergence Theorem)**.** *Suppose that f_n are integrable functions, $f_n(x) \le f_{n+1}(x)$ almost everywhere, and there is a constant K such that for all n*

$$\int_X f_n(x)\,\mathrm{d}x < K.$$

Then there is an integrable function f such that $f_n \to f$ almost everywhere and

$$\int_X g(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}x.$$

Prop. 8. *If f is integrable and $\int_X |f(x)|\,\mathrm{d}x = 0$ then $f(x) = 0$ almost everywhere.*

Theorem 14 (Dominated Convergence Theorem)**.** *Suppose that $f_n : X \to \mathbb{R}$ are integrable functions and $f_n(x) \to f(x)$ almost everywhere. If there is an integrable function g such that $|f_n(x)| \le g(x)$ for every n and almost every x, then f is integrable and*

$$\int_X f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}x.$$

The Dominated Convergence theorem holds for complex-valued functions, and is easily proved from the above version for real-valued functions.

Theorem 15. *$L^1(X)$ is a Banach space.*

Lemma 7. *If $(V, \|\cdot\|)$ is a normed space in which sum$_{j=1}^{\infty} \|y_j\| < \infty$ implies that $\sum_{j=1}^{\infty} y_j$ converges, then V is complete.*

Lemma 8. *If f_k is a sequence of Lebesgue-integrable functions such that*

$$K := \sum_{j=1}^{\infty} \|f_k\|_1 < \infty,$$

then

(i) there is an integrable function g such that $\sum_{k=1}^{n} |f_k(x)| \to g(x)$ almost everywhere;

(ii) there is an integrable function h such that $\sum_{k=1}^{n} f_k(x) \to h(x)$ almost everywhere;

(iii) $\|h(x) - \sum_{k=1}^{n} f(x)\|_1 \to 0$.

Theorem 16. *The space $C[0,1]$ is dense in $L^1[0,1]$. Therefore, $L^1[a,b]$ is isometrically isomorphic to the completion of $C[a,b]$ in the L^1-norm.*

Theorem 17. *$L^p(X)$ is a Banach space for any $p \in [1, \infty)$.*

5 Inner product spaces

Lemma 9 (Cauchy-Schwarz inequality)**.** *If V is an inner product space and $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in V$, then for all $x, y \in V$,*

$$|\langle x.y\rangle| \leq \|x\| \, \|y\|$$

Prop. 9. *If V is an inner product space, then the equation $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on V.*

Lemma 10 (Continuity of the inner product)**.** *Let V be an inner product space equipped with the natural norm. If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Lemma 11 (Parallelogram law)**.** *If V is an inner product space with the natural norm $\|\cdot\|$, then for all $x, y \in V$,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Lemma 12 (Polarisation identity)**.** *Let V be an inner product space with the natural norm $\|\cdot\|$. If the vector space V is real, then*

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2.$$

If the vector space V is complex, then

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2.$$

Prop. 10. *If V is a real normed space with the norm $\|\cdot\|$ satisfying the parallelogram law, then*

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

defines an inner product on V.

Theorem 18 (Pythagoras)**.** *If $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Lemma 13. *If $\{e_1, \dots, e_n\}$ is an orthonormal set in an inner product space V, then for any $\alpha_j \in \mathbb{F}$ we have*

$$\left\|\sum_{j=1}^{n} \alpha_j e_j\right\|^2 = \sum_{j=1}^{n} |\alpha_j|^2.$$

Lemma 14 (Bessel's inequality)**.** *If V is an inner product space and $E = (e_k)$ is an orthonormal sequence, then for every $x \in V$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

Prop. 11. *If E is an orthonormal set in an inner product space V then for any $x \in V$ the set*

$$\mathcal{E}_x := \{e \in E : \langle x, e\rangle \neq 0\}$$

is at most countable.

Lemma 15 (Gram-Schmidt orthonormalisation)**.** *Let V be an inner product space and v_k be a sequence of linearly independent vectors in V. Then there is an orthonormal sequence (e_k) such that for every k,*

$$\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{e_1, \dots, e_k\}.$$

Prop. 12. *Any infinite-dimensional inner product space contains an orthonormal sequence.*

Prop. 13. *Any finite-dimensional inner product space has an orthonormal basis.*

Prop. 14. *Any finite-dimensional inner product space is isometrically isomorphic to $\mathbb{C}^n$ or $\mathbb{R}^n$ (if the space is complex or real respectively) equipped with the standard inner product.*

6 Hilbert spaces

Lemma 16. *Let H be a Hilbert space and $E = (e_k)$ be an orthonormal sequence in H. The series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges if and only if $\sum_{k=1}^{\infty} |\alpha_k|^2 < +\infty$. In this case, we have*

$$\left\|\sum_{k=1}^{\infty} \alpha_k e_k\right\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2.$$

Prop. 15. *If H is a Hilbert space and $E = (e_k)$ is an orthonormal sequence, then for every $x \in H$ the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges.*

Prop. 16. *Let $E = e_k$ be an orthonormal set in a Hilbert space H. Then the following are equivalent:*

- *for every $x \in H$ there are $\alpha_k \in \mathbb{F}$ such that $x = \sum_{k=1}^{\infty} \alpha_k e_k$;*
- *$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ for all $x \in H$;*
- *$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ for all $x \in H$;*
- *if $\langle x, e_k \rangle = 0$ for all $k \in \mathbb{N}$ then $x = 0$;*
- *the linear span of E is dense in H.*

Theorem 19. *An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.*

Theorem 20. *Any infinite-dimensional separable Hilbert space over $\mathbb{F}$ is isometrically isomorphic to $\ell^2(\mathbb{F})$.*

Lemma 17. *If A is a non-empty closed convex subset of a Hilbert space H, then for any $x \in H$ there is a unique $a^* \in A$ such that*

$$\|x - a^*\| = \inf_{a \in A} \|x - a\| .$$

Prop. 17. *If $X \subset H$, then $X^\perp$ is a closed linear subspace of H.*

Prop. 18. *If $E \subset H$ then $E^\perp = (\mathrm{span}(E))^\perp = (\overline{\mathrm{span}}(E))^\perp$.*

Theorem 21. *If U is a closed linear subspace of a Hilbert space H then*

1. *$H = U \oplus U^\perp$;*
2. *u is the closest point to x in U;*
3. *the map $P_U : H \to U$ defined by $P_U(x) = u$ is linear, and for every $x \in H$ we have that $P_U^2(x) = P_U(x)$ and $\|P_U(x)\| \leq \|x\|$.*

Prop. 19. *The map P_U is an orthogonal projection onto U.*

Prop. 20. *If U is a closed linear subspace of a Hilbert space H and E is an at most countable orthonormal subset in U such that $\mathrm{span}(E)$ is dense in U, then*

$$P_U(x) = \sum_{e_k \in E} \langle x, e_k \rangle e_k.$$

7 Dual Spaces

Prop. 21. *Let V be a normed space over $\mathbb{F}$. A linear functional $f : V \to \mathbb{F}$ is continuous if and only if it is bounded.*

Prop. 22. *If V is a normed space, then its dual space V^* is a Banach space with the norm defined by $\|f\| = \sup_{\|x\|=1} |f(x)|$.*

Theorem 22 (Riesz Representation Theorem)**.** *If H is a Hilbert space over $\mathbb{F}$, then for any bounded linear functional $f : H \to \mathbb{F}$ there is a unique $y \in H$ such that for all $x \in H$,*

$$f(x) = \langle x, y \rangle.$$

Moreover, we have $\|f\|_{H^*} = \|y\|_H$.

Theorem 23. *If X is a compact metric space and $w : C(X) \to \mathbb{R}$ is a continuous linear functional, then there are two measures $\mu, \nu \in \mathcal{M}(X)$ such that for every $\phi \in C(X)$,*

$$w(\phi) = w_\mu(\phi) - w_\nu(\phi),$$

where $w_\mu(\phi) = \int_X \phi(x)\,\mathrm{d}\mu(x)$ and $w_\nu(\phi) = \int_X \phi(x)\,\mathrm{d}\nu(x)$.

8 Linear operators

Lemma 18. *A linear operator $A : V \to U$ is continuous if and only if it is bounded.*

Theorem 24. *If V is a normed space and U is a Banach space, then $B(V, U)$ is a Banach space.*

Prop. 23. *If $A \in B(U, V)$ then $\ker(A)$ is a closed linear subspace of V.*

Prop. 24. *If $A : X \to X$ is bounded and λ is its eigenvalue then $|\lambda| \leq \|A\|_{op}$.*

Lemma 19. *If $A : V \to U$ and $B : U \to V$ are linear operators such that $AB = I_U$ and $BA = I_V$, then A and B are both invertible and $B = A^{-1}$.*

Prop. 25. *Let V and U be normed spaces and $A : V \to U$ be an invertible linear operator. The inverse operator $A^{-1} : U \to V$ is bounded if and only if there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in V$.*

Lemma 20. *Let X be a Banach space and $T \in B(X, X)$. If $\|T\| < 1$, then $I - T$ is invertible, $(I - T)^{-1} \in B(X, X)$, $(I - T)^{-1} = I + T + T^2 + T^3 + \ldots$, and*

$$\left\|(I - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}.$$

Prop. 26. *Let X be a Banach space, let $A : X \to X$ be an invertible linear operator and let $A, A^{-1}, B \in B(X, X)$. If $\|B\| \left\|A^{-1}\right\| < 1$, then the operator $A + B$ invertible, its inverse operator is bounded, and*

$$\left\|(A + B)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \left\|A^{-1}\right\| \|B\|}.$$

Theorem 25. *If $A : H \to H$ is a bounded linear operator on a Hilbert space H, then there is a unique bounded linear operator $A^* : H \to H$ such that for all $x, y \in H$,*

$$\langle Ax, y \rangle = \langle x, A^*, y \rangle.$$

Moreover, $\|A^*\|_{op} = \|A\|_{op}$.

Lemma 21 (properties of adjoint operators)**.** *If $A, B : H \to H$ are bounded operators on a Hilbert space H and $\alpha, \beta \in \mathbb{C}$, then*

1. $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$;
2. $(AB)^* = B^* A^*$;
3. $(A^*)^* = A$;
4. $\|A^* A\| = \|AA^*\| = \|A\|^2$.

Theorem 26. *Let A be a self-adjoint operator on a Hilbert space H. Then all eigenvalues of A are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Prop. 27. *If H is a separable Hilbert space and $A : H \to H$ is a self-adjoint operator, then $\sigma_p(A)$ is at most countable.*

Theorem 27. *If A is a bounded self-adjoint operator, then*

1. *$\langle Ax, x\rangle$ is real for all $x \in H$;*
2. *$\|A\|_{op} = \sup_{\|x\|=1} |\langle Ax, x\rangle|$.*

9 Introduction to Spectral Theory

Prop. 28. *If a linear operator $A : X \to X$ is bounded and $\lambda \in \sigma(A)$, then $|\lambda| \leq \|A\|_{op}$.*

Prop. 29. *If a linear operator $A : X \to X$ is bounded, then its resolvent set $R(A)$ is open and its spectrum $\sigma(A)$ is closed.*

Lemma 22. *If $A : H \to H$ is a bounded linear operator on a Hilbert space H, then $\lambda \in \sigma(A)$ if and only if $\overline{\lambda} \in \sigma(A^*)$.*

Theorem 28. *If X is a normed space and Y is a Banach space, then $K(X, Y)$ is a closed linear subspace in $B(X, Y)$.*

Lemma 23. *If $T : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then at least one of the real numbers $\|T\|$ and $-\|T\|$ is an eigenvalue of T.*

Prop. 30. *If $T : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then any eigenspace corresponding to a non-zero eigenvalue is finite-dimensional.*

Theorem 29 (Spectral Theorem for a compact self-adjoint operator)**.** *Let H be a Hilbert space and $T : H \to H$ be a compact self-adjoint operator.*

(a) If $\dim \operatorname{range}(T) = N$ then T and N non-zero real eigenvalues $\lambda_1, \ldots, \lambda_N$ (not necessarily distinct). If $\dim H > N$, then 0 is also an eigenvalue. Moreover, there is an orthonormal set of eigenvectors $e_1, \ldots, e_N$ such that $Te_k = \lambda_k e_k$, and for all $x \in H$

$$Tx = \sum_{k=1}^{N} \lambda_k \langle x, e_k\rangle e_k.$$

(b) If $\dim \operatorname{range}(T) = \infty$*, then* T *has an infinite sequence* λ_k *of non-zero real eigenvalues such that* $\lim_k \lambda_k = 0$*. Moreover, there is an orthonormal sequence* e_k *of eigenvectors such that* $Te_k = \lambda_k e_k$ *and for all* $x \in H$*,*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Prop. 31. *Let* H *be an infinite-dimensional separable Hilbert space and let* $T : H \to H$ *be a compact self-adjoint operator. Then there is an orthonormal basis* $E = \{e_k : k \in \mathbb{N}\}$ *in* H *such that* $Te_k = \lambda_k e_k$ *for all* $k \in \mathbb{N}$ *and for all* $x \in H$*,*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_l \rangle e_k.$$

Note that it is possible that $\lambda_k = 0$ *for some* k*.*

Prop. 32. *Finite rank self-adjoint operators on a Hilbert space* H *are dense in the space of compact self-adjoint operators on* H*.*

Theorem 30. *If* H *is a Hilbert space and* $T : H \to H$ *is a compact self-adjoint operator, then* $\sigma(T) = \overline{\sigma_p(T)}$*.*

10 Sturm-Liouville problems

Let $[a, b] \subset \mathbb{R}$, $p \in C^1[a, b]$ and $q \in C[a, b]$. Find all $\lambda \in \mathbb{R}$ such that the following linear ODE has a non-zero solution u satisfying the boundary conditions $u(a) = u(b) = 0$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + q(x)u(x) = \lambda u(x) \tag{1}$$

This section is devoted to proving the following theorem:

Theorem 31. *If* $p \in C^1[a, b]$*,* $q \in C[a, b]$ *are such that* $p(x) > 0$ *and* $q(x) \geq 0$ *for all* $x \in [a, b]$*, then*

- *the eigenvalues* λ_k *of the Sturm-Liouville problem (1) form an unbounded monotone sequence of positive numbers;*
- *the eigenvalues are simple;*
- *the eigenfunctions form an orthonormal basis* e_k *in* $L^2[a, b]$*.*

Lemma 24. *Let* $p \in C^1[a, b]$ *and* $q \in C^0[a, b]$*. If* $u_1, u_2 \in C^2[a, b]$ *satisfy the linear ODE*

$$-(pu')' + qu = 0,$$

then $W_p(u_1, u_2) = p(u_1' u_2 - u_1 u_2')$ *is constant in* $x \in [a, b]$*.*

Lemma 25. *If* $p \in C^1[a, b]$*,* $q \in C^0[a, b]$*, and* $p(x) > 0$ *for all* $x \in [a, b]$*, then the ODE*

$$-(pu')' + qu = 0$$

has two solutions $u_1, u_2 \in C^2[a, b]$ *such that* $u_1(a) = 0$ *and* $u_1'(a) = 1$*, and* $u_2(b) = 0$ *and* $u_2'(b) = 1$*. Moreover, if* $q(x) \geq 0$ *for all* $x \in [a, b]$*, then* $W_p(u_1, u_2) \neq 0$*.*

Lemma 26 (Green's function)**.** *If $p \in C^1[a,b]$, $q \in C^0[a.b]$, $p(x) > 0$ for all $x \in [a,b]$, and u_1, u_2 are solutions of the ODE*

$$-(pu')' + qu = 0$$

such that $u_1(a) = u_2(b) = 0$ and $W_p(u_1, u_2) \neq 0$ and

$$G(x,y) = \frac{1}{W_p(u_1,u_2)} \begin{cases} u_1(x)u_2(y), \text{for } a \leq x < y \leq b \\ u_1(y)u_2(x), \text{for } a \leq y \leq x \leq b, \end{cases}$$

then for any $f \in C^0[a,b]$ the function $u(x) = \int_a^b G(x,y)f(y)\,\mathrm{d}y$ is $C^2[a,b]$ and satisfies both the ODE

$$-(pu')' + qu = f$$

and the boundary conditions $u(a) = u(b) = 0$.

Theorem 32 (the inverse operator $L^{-1} : C^0 \to C_0^2$)**.** *If $p \in C^1[a,b]$ and $q \in C[a,b]$ are such that $p(x) > 0$ and $q(x) \geq 0$ for all $x \in [a,b]$, then the linear operator $L : C_0^2 \to C^0$, $L(u) = -(pu')' + qu$ is bijective. Moreover, there is a continuous function $G(x,y)$ such that for all $f \in C^0$*

$$L^{-1}(f) = \int_a^b G(x,y)f(y)\,\mathrm{d}y.$$

Theorem 33. *If $G : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function of two variables, then the linear operator $A : L^2[a,b] \to L^2[a,b]$ defined by the equation*

$$(Af)(x) = \int_a^b G(x,y)f(y)\,\mathrm{d}y$$

is compact. Moreover, $A(f)$ is a continuous function.

Prop. 33 (properties of A)**.** *The above operator $A : L^2[a,b] \to L^2[a,b]$ has the following properties:*

1. *A is a bounded self-adjoint operator;*
2. *$\operatorname{range}(A)$ is dense in $L^2(a,b)$;*
3. *$\ker(A)$ is trivial;*
4. *$A(C^0) = C_0^2$.*

Prop. 34. *A function u is an eigenfunction of the Sturm-Liouville problem if and only if it is an eigenfunction of the operator A:*

$$Lu = \lambda u \iff Au = \lambda \inf u.$$