

MA3G7 Functional Analysis 1

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1 Vector Spaces

Theorem 1. *Every vector space has a Hamel basis.*

You don't need to prove the above theorem.

Theorem 2. *If a vector space has a finite Hamel basis then every Hamel basis for this vector space has the same number of elements.*

Prop. 1. *Any n -dimensional vector space over a field \mathbb{F} is isomorphic to \mathbb{F}^n .*

Lemma 1 (Young's inequality). *If $a, b > 0$, $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 2 (Hölder's inequality). *If $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \ell^p(\mathbb{F})$ and $y \in \ell^q(\mathbb{F})$, then*

$$\sum |x_j y_j| \leq \|x\|_p \|y\|_q.$$

Lemma 3 (Minkowski's inequality). *Let $1 \leq p < \infty$. If $x, y \in \ell^p(\mathbb{F})$ then $x + y \in \ell^p(\mathbb{F})$ and*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

2 Normed spaces

Prop. 2. *Let $(W, \|\cdot\|_W)$ be a normed space, V be a vector space, and $L : V \rightarrow W$ a linear isomorphism. Then the function*

$$V \ni x \mapsto \|L(x)\|_W \in \mathbb{R}$$

defines a norm on V .

Prop. 3. *Any finite dimensional vector space can be equipped with a norm. Therefore, any n -dimensional vector space over \mathbb{F} is isometrically isomorphic to \mathbb{F}^n equipped with a suitable norm.*

Theorem 3. *All norms on a finite dimensional space are equivalent.*

Prop. 4. *If $f_n \in C[0, 1]$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ in the sup-norm, then $f_n \rightarrow f$ in the L^1 -norm.*

Prop. 5. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space, then for any sequence x_n in the vector space and for any $x \in V$,*

$$\|x_n - x\|_1 \rightarrow 0 \iff \|x_n - x\|_2 \rightarrow 0.$$

Lemma 4. *In a normed space $(V, \|\cdot\|)$, a sequence $x_n \rightarrow x$ if and only if for any open neighbourhood X of x there is an $N \in \mathbb{N}$ such that $x_n \in X$ for all $n > N$.*

Prop. 6. *On a vector space V , two norms are equivalent if and only if they induce the same topology.*

Lemma 5. *A subset $X \subset V$ is closed if and only if every convergent sequence with elements in X has its limit in X .*

Prop. 7. *A finite dimensional linear subspace W of a normed space V is closed.*

Theorem 4 (Heine-Borel). *A subset of \mathbb{R}^n is compact if and only if it is closed and bounded. Therefore, a subset of a finite-dimensional vector space is closed if and only if it is bounded.*

You don't need to know the proof of this - but you should, as you studied it in Year 2.

Lemma 6 (Riesz' Lemma). *Let X be a normed vector space and Y be a closed linear subspace of X such that $Y \neq X$ and $\alpha \in (0, 1)$. Then there is $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ for all $y \in Y$.*

Theorem 5. *A normed space is finite dimensional if and only if the unit sphere is compact.*

3 Banach spaces

Theorem 6. *A sequence of real numbers converges if and only if it is Cauchy.*

Proved in Year 1.

Theorem 7. *Every finite-dimensional normed space is complete. In particular, \mathbb{R}^n and \mathbb{C}^n are complete.*

Theorem 8. *The space $\ell^p(\mathbb{F})$ equipped with the ℓ^p -norm is complete.*

Theorem 9. *The space $C[0, 1]$ equipped with the sup-norm is complete.*

Theorem 10 (Bernstein polynomials). *Consider the polynomials*

$$B_{np}(x) = \binom{n}{p} x^p (1-x)^{n-p}.$$

Then the polynomials B_{np} are of degree n and we have that

$$(i) \sum_{p=0}^n B_{np}(x) = 1$$

$$(ii) \sum_{p=0}^n p B_{np}(x) = nx$$

$$(iii) \sum_{p=0}^n (p-nx)^2 B_{np}(x) = nx(1-x).$$

Theorem 11. *If a function $f : [0, 1] \rightarrow \mathbb{R}$ is continuous then the sequence of polynomials*

$$P_n(x) = \sum_{p=0}^n f\left(\frac{p}{n}\right) \binom{n}{p} x^p (1-x)^{n-p}$$

converges uniformly to f on $[0, 1]$.

Therefore, $C[0, 1]$ is separable.

Theorem 12. *Any normed space is isometrically isomorphic to a dense subset of a Banach space.*

4 Lebesgue spaces

Theorem 13 (Monotone Convergence Theorem). *Suppose that f_n are integrable functions, $f_n(x) \leq f_{n+1}(x)$ almost everywhere, and there is a constant K such that for all n*

$$\int_X f_n(x) \, dx < K.$$

Then there is an integrable function f such that $f_n \rightarrow f$ almost everywhere and

$$\int_X g(x) \, dx = \lim_{n \rightarrow \infty} \int_X f_n(x) \, dx.$$

Prop. 8. *If f is integrable and $\int_X |f(x)| \, dx = 0$ then $f(x) = 0$ almost everywhere.*

Theorem 14 (Dominated Convergence Theorem). *Suppose that $f_n : X \rightarrow \mathbb{R}$ are integrable functions and $f_n(x) \rightarrow f(x)$ almost everywhere. If there is an integrable function g such that $|f_n(x)| \leq g(x)$ for every n and almost every x , then f is integrable and*

$$\int_X f(x) \, dx = \lim_{n \rightarrow \infty} \int_X f_n(x) \, dx.$$

The Dominated Convergence theorem holds for complex-valued functions, and is easily proved from the above version for real-valued functions.

Theorem 15. $L^1(X)$ is a Banach space.

Lemma 7. If $(V, \|\cdot\|)$ is a normed space in which $\sum_{j=1}^{\infty} \|y_j\| < \infty$ implies that $\sum_{j=1}^{\infty} y_j$ converges, then V is complete.

Lemma 8. If f_k is a sequence of Lebesgue-integrable functions such that

$$K := \sum_{j=1}^{\infty} \|f_k\|_1 < \infty,$$

then

- (i) there is an integrable function g such that $\sum_{k=1}^n |f_k(x)| \rightarrow g(x)$ almost everywhere;
- (ii) there is an integrable function h such that $\sum_{k=1}^n f_k(x) \rightarrow h(x)$ almost everywhere;
- (iii) $\|h(x) - \sum_{k=1}^n f_k(x)\|_1 \rightarrow 0$.

Theorem 16. The space $C[0,1]$ is dense in $L^1[0,1]$. Therefore, $L^1[a,b]$ is isometrically isomorphic to the completion of $C[a,b]$ in the L^1 -norm.

Theorem 17. $L^p(X)$ is a Banach space for any $p \in [1, \infty)$.

5 Inner product spaces

Lemma 9 (Cauchy-Schwarz inequality). If V is an inner product space and $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in V$, then for all $x, y \in V$,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Prop. 9. If V is an inner product space, then the equation $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm on V .

Lemma 10 (Continuity of the inner product). Let V be an inner product space equipped with the natural norm. If $x_n \rightarrow x$ and $y_n \rightarrow y$, then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$.

Lemma 11 (Parallelogram law). If V is an inner product space with the natural norm $\|\cdot\|$, then for all $x, y \in V$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Lemma 12 (Polarisation identity). Let V be an inner product space with the natural norm $\|\cdot\|$. If the vector space V is real, then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2.$$

If the vector space V is complex, then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

Prop. 10. If V is a real normed space with the norm $\|\cdot\|$ satisfying the parallelogram law, then

$$\langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}$$

defines an inner product on V .

Theorem 18 (Pythagoras). If $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Lemma 13. If $\{e_1, \dots, e_n\}$ is an orthonormal set in an inner product space V , then for any $\alpha_j \in \mathbb{F}$ we have

$$\left\| \sum_{j=1}^n \alpha_j e_j \right\|^2 = \sum_{j=1}^n |\alpha_j|^2.$$

Lemma 14 (Bessel's inequality). If V is an inner product space and $E = (e_k)$ is an orthonormal sequence, then for every $x \in V$

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

Prop. 11. If E is an orthonormal set in an inner product space V then for any $x \in V$ the set

$$\mathcal{E}_x := \{e \in E : \langle x, e \rangle \neq 0\}$$

is at most countable.

Lemma 15 (Gram-Schmidt orthonormalisation). Let V be an inner product space and v_k be a sequence of linearly independent vectors in V . Then there is an orthonormal sequence (e_k) such that for every k ,

$$\text{span}\{v_1, \dots, v_k\} = \text{span}\{e_1, \dots, e_k\}.$$

Prop. 12. Any infinite-dimensional inner product space contains an orthonormal sequence.

Prop. 13. Any finite-dimensional inner product space has an orthonormal basis.

Prop. 14. Any finite-dimensional inner product space is isometrically isomorphic to \mathbb{C}^n or \mathbb{R}^n (if the space is complex or real respectively) equipped with the standard inner product.

6 Hilbert spaces

Lemma 16. Let H be a Hilbert space and $E = (e_k)$ be an orthonormal sequence in H . The series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges if and only if $\sum_{k=1}^{\infty} |\alpha_k|^2 < +\infty$. In this case, we have

$$\left\| \sum_{k=1}^{\infty} \alpha_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2.$$

Prop. 15. If H is a Hilbert space and $E = (e_k)$ is an orthonormal sequence, then for every $x \in H$ the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges.

Prop. 16. Let $E = e_k$ be an orthonormal set in a Hilbert space H . Then the following are equivalent:

- for every $x \in H$ there are $\alpha_k \in \mathbb{F}$ such that $x = \sum_{k=1}^{\infty} \alpha_k e_k$;
- $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ for all $x \in H$;
- $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ for all $x \in H$;
- if $\langle x, e_k \rangle = 0$ for all $k \in \mathbb{N}$ then $x = 0$;
- the linear span of E is dense in H .

Theorem 19. An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.

Theorem 20. Any infinite-dimensional separable Hilbert space over \mathbb{F} is isometrically isomorphic to $\ell^2(\mathbb{F})$.

Lemma 17. If A is a non-empty closed convex subset of a Hilbert space H , then for any $x \in H$ there is a unique $a^* \in A$ such that

$$\|x - a^*\| = \inf_{a \in A} \|x - a\|.$$

Prop. 17. If $X \subset H$, then X^\perp is a closed linear subspace of H .

Prop. 18. If $E \subset H$ then $E^\perp = (\text{span}(E))^\perp = (\overline{\text{span}}(E))^\perp$.

Theorem 21. If U is a closed linear subspace of a Hilbert space H then

1. $H = U \oplus U^\perp$;
2. u is the closest point to x in U ;
3. the map $P_U : H \rightarrow U$ defined by $P_U(x) = u$ is linear, and for every $x \in H$ we have that $P_U^2(x) = P_U(x)$ and $\|P_U(x)\| \leq \|x\|$.

Prop. 19. The map P_U is an orthogonal projection onto U .

Prop. 20. If U is a closed linear subspace of a Hilbert space H and E is an at most countable orthonormal subset in U such that $\text{span}(E)$ is dense in U , then

$$P_U(x) = \sum_{e_k \in E} \langle x, e_k \rangle e_k.$$

7 Dual Spaces

Prop. 21. Let V be a normed space over \mathbb{F} . A linear functional $f : V \rightarrow \mathbb{F}$ is continuous if and only if it is bounded.

Prop. 22. If V is a normed space, then its dual space V^* is a Banach space with the norm defined by $\|f\| = \sup_{\|x\|=1} |f(x)|$.

Theorem 22 (Riesz Representation Theorem). *If H is a Hilbert space over \mathbb{F} , then for any bounded linear functional $f : H \rightarrow \mathbb{F}$ there is a unique $y \in H$ such that for all $x \in H$,*

$$f(x) = \langle x, y \rangle.$$

Moreover, we have $\|f\|_{H^} = \|y\|_H$.*

Theorem 23. *If X is a compact metric space and $w : C(X) \rightarrow \mathbb{R}$ is a continuous linear functional, then there are two measures $\mu, \nu \in \mathcal{M}(X)$ such that for every $\phi \in C(X)$,*

$$w(\phi) = w_\mu(\phi) - w_\nu(\phi),$$

where $w_\mu(\phi) = \int_X \phi(x) d\mu(x)$ and $w_\nu(\phi) = \int_X \phi(x) d\nu(x)$.

8 Linear operators

Lemma 18. *A linear operator $A : V \rightarrow U$ is continuous if and only if it is bounded.*

Theorem 24. *If V is a normed space and U is a Banach space, then $B(V, U)$ is a Banach space.*

Prop. 23. *If $A \in B(U, V)$ then $\ker(A)$ is a closed linear subspace of V .*

Prop. 24. *If $A : X \rightarrow X$ is bounded and λ is its eigenvalue then $|\lambda| \leq \|A\|_{op}$.*

Lemma 19. *If $A : V \rightarrow U$ and $B : U \rightarrow V$ are linear operators such that $AB = I_U$ and $BA = I_V$, then A and B are both invertible and $B = A^{-1}$.*

Prop. 25. *Let V and U be normed spaces and $A : V \rightarrow U$ be an invertible linear operator. The inverse operator $A^{-1} : U \rightarrow V$ is bounded if and only if there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in V$.*

Lemma 20. *Let X be a Banach space and $T \in B(X, X)$. If $\|T\| < 1$, then $I - T$ is invertible, $(I - T)^{-1} \in B(X, X)$, $(I - T)^{-1} = I + T + T^2 + T^3 + \dots$, and*

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

Prop. 26. *Let X be a Banach space, let $A : X \rightarrow X$ be an invertible linear operator and let $A, A^{-1}, B \in B(X, X)$. If $\|B\| \|A^{-1}\| < 1$, then the operator $A + B$ is invertible, its inverse operator is bounded, and*

$$\|(A + B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|B\|}.$$

Theorem 25. *If $A : H \rightarrow H$ is a bounded linear operator on a Hilbert space H , then there is a unique bounded linear operator $A^* : H \rightarrow H$ such that for all $x, y \in H$,*

$$\langle Ax, y \rangle = \langle x, A^*y \rangle.$$

Moreover, $\|A^\|_{op} = \|A\|_{op}$.*

Lemma 21 (properties of adjoint operators). *If $A, B : H \rightarrow H$ are bounded operators on a Hilbert space H and $\alpha, \beta \in \mathbb{C}$, then*

1. $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$;
2. $(AB)^* = B^*A^*$;
3. $(A^*)^* = A$;
4. $\|A^*A\| = \|AA^*\| = \|A\|^2$.

Theorem 26. *Let A be a self-adjoint operator on a Hilbert space H . Then all eigenvalues of A are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Prop. 27. *If H is a separable Hilbert space and $A : H \rightarrow H$ is a self-adjoint operator, then $\sigma_p(A)$ is at most countable.*

Theorem 27. *If A is a bounded self-adjoint operator, then*

1. $\langle Ax, x \rangle$ is real for all $x \in H$;
2. $\|A\|_{op} = \sup_{\|x\|=1} |\langle Ax, x \rangle|$.

9 Introduction to Spectral Theory

Prop. 28. *If a linear operator $A : X \rightarrow X$ is bounded and $\lambda \in \sigma(A)$, then $|\lambda| \leq \|A\|_{op}$.*

Prop. 29. *If a linear operator $A : X \rightarrow X$ is bounded, then its resolvent set $R(A)$ is open and its spectrum $\sigma(A)$ is closed.*

Lemma 22. *If $A : H \rightarrow H$ is a bounded linear operator on a Hilbert space H , then $\lambda \in \sigma(A)$ if and only if $\bar{\lambda} \in \sigma(A^*)$.*

Theorem 28. *If X is a normed space and Y is a Banach space, then $K(X, Y)$ is a closed linear subspace in $B(X, Y)$.*

Lemma 23. *If $T : H \rightarrow H$ is a compact self-adjoint operator on a Hilbert space H , then at least one of the real numbers $\|T\|$ and $-\|T\|$ is an eigenvalue of T .*

Prop. 30. *If $T : H \rightarrow H$ is a compact self-adjoint operator on a Hilbert space H , then any eigenspace corresponding to a non-zero eigenvalue is finite-dimensional.*

Theorem 29 (Spectral Theorem for a compact self-adjoint operator). *Let H be a Hilbert space and $T : H \rightarrow H$ be a compact self-adjoint operator.*

- (a) *If $\dim \text{range}(T) = N$ then T has N non-zero real eigenvalues $\lambda_1, \dots, \lambda_N$ (not necessarily distinct). If $\dim H > N$, then 0 is also an eigenvalue. Moreover, there is an orthonormal set of eigenvectors e_1, \dots, e_N such that $Te_k = \lambda_k e_k$, and for all $x \in H$*

$$Tx = \sum_{k=1}^N \lambda_k \langle x, e_k \rangle e_k.$$

(b) If $\dim \text{range}(T) = \infty$, then T has an infinite sequence λ_k of non-zero real eigenvalues such that $\lim_k \lambda_k = 0$. Moreover, there is an orthonormal sequence e_k of eigenvectors such that $Te_k = \lambda_k e_k$ and for all $x \in H$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Prop. 31. Let H be an infinite-dimensional separable Hilbert space and let $T : H \rightarrow H$ be a compact self-adjoint operator. Then there is an orthonormal basis $E = \{e_k : k \in \mathbb{N}\}$ in H such that $Te_k = \lambda_k e_k$ for all $k \in \mathbb{N}$ and for all $x \in H$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Note that it is possible that $\lambda_k = 0$ for some k .

Prop. 32. Finite rank self-adjoint operators on a Hilbert space H are dense in the space of compact self-adjoint operators on H .

Theorem 30. If H is a Hilbert space and $T : H \rightarrow H$ is a compact self-adjoint operator, then $\sigma(T) = \overline{\sigma_p(T)}$.

10 Sturm-Liouville problems

Let $[a, b] \subset \mathbb{R}$, $p \in C^1[a, b]$ and $q \in C[a, b]$. Find all $\lambda \in \mathbb{R}$ such that the following linear ODE has a non-zero solution u satisfying the boundary conditions $u(a) = u(b) = 0$

$$-\frac{d}{dx} \left(p(x) \frac{du(x)}{dx} \right) + q(x)u(x) = \lambda u(x) \quad (1)$$

This section is devoted to proving the following theorem:

Theorem 31. If $p \in C^1[a, b]$, $q \in C[a, b]$ are such that $p(x) > 0$ and $q(x) \geq 0$ for all $x \in [a, b]$, then

- the eigenvalues λ_k of the Sturm-Liouville problem (1) form an unbounded monotone sequence of positive numbers;
- the eigenvalues are simple;
- the eigenfunctions form an orthonormal basis e_k in $L^2[a, b]$.

Lemma 24. Let $p \in C^1[a, b]$ and $q \in C^0[a, b]$. If $u_1, u_2 \in C^2[a, b]$ satisfy the linear ODE

$$-(pu')' + qu = 0,$$

then $W_p(u_1, u_2) = p(u_1' u_2 - u_1 u_2')$ is constant in $x \in [a, b]$.

Lemma 25. If $p \in C^1[a, b]$, $q \in C^0[a, b]$, and $p(x) > 0$ for all $x \in [a, b]$, then the ODE

$$-(pu')' + qu = 0$$

has two solutions $u_1, u_2 \in C^2[a, b]$ such that $u_1(a) = 0$ and $u_1'(a) = 1$, and $u_2(b) = 0$ and $u_2'(b) = 1$. Moreover, if $q(x) \geq 0$ for all $x \in [a, b]$, then $W_p(u_1, u_2) \neq 0$.

Lemma 26 (Green's function). *If $p \in C^1[a, b]$, $q \in C^0[a, b]$, $p(x) > 0$ for all $x \in [a, b]$, and u_1, u_2 are solutions of the ODE*

$$-(pu')' + qu = 0$$

such that $u_1(a) = u_2(b) = 0$ and $W_p(u_1, u_2) \neq 0$ and

$$G(x, y) = \frac{1}{W_p(u_1, u_2)} \begin{cases} u_1(x)u_2(y), & \text{for } a \leq x < y \leq b \\ u_1(y)u_2(x), & \text{for } a \leq y \leq x \leq b, \end{cases}$$

then for any $f \in C^0[a, b]$ the function $u(x) = \int_a^b G(x, y)f(y) dy$ is $C^2[a, b]$ and satisfies both the ODE

$$-(pu')' + qu = f$$

and the boundary conditions $u(a) = u(b) = 0$.

Theorem 32 (the inverse operator $L^{-1} : C^0 \rightarrow C_0^2$). *If $p \in C^1[a, b]$ and $q \in C[a, b]$ are such that $p(x) > 0$ and $q(x) \geq 0$ for all $x \in [a, b]$, then the linear operator $L : C_0^2 \rightarrow C^0$, $L(u) = -(pu')' + qu$ is bijective. Moreover, there is a continuous function $G(x, y)$ such that for all $f \in C^0$*

$$L^{-1}(f) = \int_a^b G(x, y)f(y) dy.$$

Theorem 33. *If $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is a continuous function of two variables, then the linear operator $A : L^2[a, b] \rightarrow L^2[a, b]$ defined by the equation*

$$(Af)(x) = \int_a^b G(x, y)f(y) dy$$

is compact. Moreover, $A(f)$ is a continuous function.

Prop. 33 (properties of A). *The above operator $A : L^2[a, b] \rightarrow L^2[a, b]$ has the following properties:*

1. A is a bounded self-adjoint operator;
2. $\text{range}(A)$ is dense in $L^2(a, b)$;
3. $\ker(A)$ is trivial;
4. $A(C^0) = C_0^2$.

Prop. 34. *A function u is an eigenfunction of the Sturm-Liouville problem if and only if it is an eigenfunction of the operator A :*

$$Lu = \lambda u \iff Au = \lambda \inf u.$$