MA3G7 Functional Analysis 1

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Contents

1 Vector Spaces

Theorem 1. *Every vector space has a Hamel basis.*

You don't need to prove the above theorem.

Theorem 2. *If a vector space has a finite Hamel basis then every Hamel basis for this vector space has the same number of elements.*

Prop. 1. *Any n-dimensional vector space over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.*

Lemma 1 (Young's inequality)**.** *If $a, b > 0$, $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 2 (Hölder's inequality)**.** *If $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \ell^p(\mathbb{F})$ and $y \in \ell^q(\mathbb{F})$, then*

$$\sum |x_j y_j| \leq \|x\|_p \|y\|_q .$$

Lemma 3 (Minkowski's inequality)**.** *Let $1 \leq p < \infty$. If $x, y \in \ell^p(\mathbb{F})$ then $x + y \in \ell^p(\mathbb{F})$ and*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p .$$

Exercise 1. $C(0,1)$ is not a subset of $L^1_c(0,1)$.

Exercise 2. A set E is linearly independent iff for every subset $\{e_1, \ldots, e_n\} \subset E$, if $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ and $\sum e_i \alpha_i = 0$, then $\alpha_1 = \cdots = \alpha_n = 0$.

Exercise 3. The functions

$$f_\alpha(x) = \begin{cases} x(\alpha - x) & \text{for } 0 \le x \le \alpha \\ 0 & \text{for } \alpha \le x \le 1 \end{cases}$$

Exercise 4. If $L : V \to W$ is a linear isomorphism and $E \subset V$ is a Hamel basis for V, then $L(E)$ is a Hamel basis for W.

2 Normed spaces

Prop. 2. *Let $(W, \|\cdot\|_W)$ be a normed space, V be a vector space, and $L : V \to W$ a linear isomorphism. Then the function*

$$V \ni x \mapsto \|L(x)\|_W \in \mathbb{R}$$

defines a norm on V.

Prop. 3. *Any finite dimensional vector space can be equipped with a norm. Therefore, any n-dimensional vector space over $\mathbb{F}$ is isometrically isomorphic to $\mathbb{F}^n$ equipped with a suitable norm.*

Theorem 3. *All norms on a finite dimensional space are equivalent.*

Prop. 4. *If $f_n \in C[0,1]$ for all $n \in \mathbb{N}$ and $f_n \to f$ in the sup-norm, then $f_n \to f$ in the L^1-norm.*

Prop. 5. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space, then for any sequence x_n in the vector space and for any $x \in V$,*

$$\|x_n - x\|_1 \to 0 \iff \|x_n - x\|_2 \to 0.$$

Lemma 4. *In a normed space $(V, \|\cdot\|)$, a sequence $x_n \to x$ if and only if for any open neighbourhood X of x there is an $N \in \mathbb{N}$ such that $x_n \in X$ for all $n > N$.*

Prop. 6. *On a vector space V, two norms are equivalent if and only if they induce the same topology.*

Lemma 5. *A subset $X \subset V$ is closed if and only if every convergent sequence with elements in X has its limit in X.*

Prop. 7. *A finite dimensional linear subspace W of a normed space V is closed.*

Theorem 4 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore, a subset of a finite-dimensional vector space is closed if and only if it is bounded.*

You don't need to know the proof of this - but you should, as you studied it in Year 2.

Lemma 6 (Riesz' Lemma)**.** *Let X be a normed vector space an dY be a closed linear subspace of X such that $Y \neq X$ and $\alpha \in (0,1)$. Then there is $x_\alpha \in X$ such that $\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ for all $y \in Y$.*

Theorem 5. *A normed space is finite dimensional if and only if the unit sphere is compact.*

Exercise 5. Prove these things from Year 1:

(i) $x_n \to x$ if and only if $\|x_n - x\| \to 0$;

(ii) the limit of a convergent sequence is unique;

(iii) any convergent sequence is bounded;

(iv) if $x_n \to x$ then $\|x_n\| \to \|x\|$;

(v) any convergent sequence is Cauchy.

Exercise 6. A (sequentially) compact subset of a normed space is compact if and only if it is closed and bounded.

3 Banach spaces

Theorem 6. *A sequence of real numbers converges if and only if it is Cauchy.*

Proved in Year 1.

Theorem 7. *Every finite-dimensional normed space is complete. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete.*

Theorem 8. *The space $\ell^p(\mathbb{F})$ equipped with the ℓ^p-norm is complete.*

Theorem 9. *The space $C[0,1]$ equipped with the sup-norm is complete.*

Theorem 10 (Bernstein polynomials)**.** *Consider the polynomials*

$$B_{np}(x) = \binom{n}{p} x^p (1-x)^{n-p}.$$

Then the polynomials B_{np} are of degree n and we have that

(i) $\sum_{p=0}^{n} B_{np}(x) = 1$

(ii) $\sum_{p=0}^{n} pB_{np}(x) = nx$

(iii) $\sum_{p=0}^{n} (p-nx)^2 B_{np}(x) = nx(1-x)$.

Theorem 11. *If a function $f : [0,1] \to \mathbb{R}$ is continuous then the sequence of polynomials*

$$P_n(x) = \sum_{p=0}^{n} f\left(\frac{p}{n}\right) \binom{n}{p} x^p (1-x)^{n-p}$$

converges uniformly to f on $[0,1]$.

Therefore, $C[0,1]$ is separable.

Theorem 12. *Any normed space is isometrically isomorphic to a dense subset of a Banach space.*

4 Lebesgue spaces

Theorem 13 (Monotone Convergence Theorem). *Suppose that f_n are integrable functions, $f_n(x) \le f_{n+1}(x)$ almost everywhere, and there is a constant K such that for all n*

$$\int_X f_n(x)\,\mathrm{d}x < K.$$

Then there is an integrable function f such that $f_n \to f$ almost everywhere and

$$\int_X g(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}x.$$

Prop. 8. *If f is integrable and $\int_X |f(x)|\,\mathrm{d}x = 0$ then $f(x) = 0$ almost everywhere.*

Theorem 14 (Dominated Convergence Theorem). *Suppose that $f_n : X \to \mathbb{R}$ are integrable functions and $f_n(x) \to f(x)$ almost everywhere. If there is an integrable function g such that $|f_n(x)| \le g(x)$ for every n and almost every x, then f is integrable and*

$$\int_X f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_X f_n(x)\,\mathrm{d}x.$$

The Dominated Convergence theorem holds for complex-valued functions, and is easily proved from the above version for real-valued functions.

Theorem 15. *$L^1(X)$ is a Banach space.*

Lemma 7. *If $(V, \|\cdot\|)$ is anormed space in which sum$_{j=1}^{\infty} \|y_j\| < \infty$ implies that $\sum_{j=1}^{\infty} y_j$ converges, then V is complete.*

Lemma 8. *If f_k is a sequence of Lebesgue-integrable functions such that*

$$K := \sum_{j=1}^{\infty} \|f_k\|_1 < \infty,$$

then

(i) there is an integrable function g such that $\sum_{k=1}^{n} |f_k(x)| \to g(x)$ almost everywhere;

(ii) there is an integrable function h such that $\sum_{k=1}^{n} f_k(x) \to h(x)$ almost everywhere;

(iii) $\|h(x) - \sum_{k=1}^{n} f(x)\|_1 \to 0$.

Theorem 16. *The space $C[0,1]$ is dense in $L^1[0,1]$. Therefore, $L^1[a,b]$ is isometrically isomorphic to the completion of $C[a,b]$ in the L^1-norm.*

Theorem 17. *$L^p(X)$ is a Banach space for any $p \in [1, \infty)$.*

5 Inner product spaces

Lemma 9 (Cauchy-Schwarz inequality)**.** *If V is an inner product space and $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in V$, then for all $x, y \in V$,*

$$|\langle x.y\rangle| \leq \|x\| \, \|y\|$$

Prop. 9. *If V is an inner product space, then the equation $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on V.*

Lemma 10 (Continuity of the inner product)**.** *Let V be an inner product space equipped ith the natural norm. If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Lemma 11 (Parallelogram law)**.** *If V is an inner product space with the natural norm $\|\cdot\|$, then for all $x, y \in V$,*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Lemma 12 (Polarisation identity)**.** *Let V be an inner product space with the natural norm $\|\cdot\|$. If the vector space V is real, then*

$$4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 .$$

If the vector spcae V is complex, then

$$4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2 .$$

Prop. 10. *If V is a real normed space with the norm $\|\cdot\|$ satisfying the parallelogram law, then*

$$\langle x, y\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}$$

defines an inner product on V.

Theorem 18 (Pythagoras)**.** *If $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.*

Lemma 13. *If $\{e_1, \ldots, e_n\}$ is an orthonormal set in an inner product space V, then for any $\alpha_j \in \mathbb{F}$ we have*

$$\left\| \sum_{j=1}^{n} \alpha_j e_j \right\|^2 = \sum_{j=1}^{n} |\alpha_j|^2.$$

Lemma 14 (Bessel's inequality)**.** *If V is an inner product spcae and $E = (e_k)$ is an orthonormal sequence, then for every $x \in V$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2 .$$

Prop. 11. *If E is an orthonormal set in an inner product space V then for any $x \in V$ the set*

$$\mathcal{E}_x := \{e \in E : \langle x, e\rangle \neq 0\}$$

is at most countable.

Lemma 15 (Gram-Schmidt orthonormalisation). *Let V be an inner product space and v_k be a sequence of linearly independent vectors in V. Then there is an orthonormal sequence (e_k) such that for every k,*

$$\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{e_1, \ldots, e_k\}.$$

Prop. 12. *Any infinite-dimensional inner product space contains an orthonormal sequence.*

Prop. 13. *Any finite-dimensional inner product space has an orthonormal basis.*

Prop. 14. *Any finite-dimensional inner product space is isometrically isomorphic to $\mathbb{C}^n$ or $\mathbb{R}^n$ (if the space is complex or real respectively) equipped with the standard inner product.*

6 Hilbert space

Lemma 16. *Let H be a Hilbert space and $E = (e_k)$ be an orthonormal sequence in H. The series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges if and only if $\sum_{k=1}^{\infty} |\alpha_k|^2 < +\infty$. In this case, we have*

$$\left\| \sum_{k=1}^{\infty} \alpha_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2.$$

Prop. 15. *If H is a Hilbert space and $E = (e_k)$ is an orthonormal sequence, then for every $x \in H$ the series $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges.*

Prop. 16. *Let $E = e_k$ be an orthonormal set in a Hilbert space H. Then the following are equivalent:*

- *for every $x \in H$ there are $\alpha_k \in \mathbb{F}$ such that $x = \sum_{k=1}^{\infty} \alpha_k e_k$;*
- *$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ for all $x \in H$;*
- *$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$ for all $x \in H$;*
- *if $\langle x, e_k \rangle = 0$ for all $k \in \mathbb{N}$ then $x = 0$;*
- *the linear span of E is dense in H.*

Theorem 19. *An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.*

Theorem 20. *Any infinite-dimensional separable Hilbert space over $\mathbb{F}$ is isometrically isomorphic to $\ell^2(\mathbb{F})$.*

Lemma 17. *If A is a non-empty closed convex subset of a Hilbert space H, then for any $x \in H$ there is a unique $a^* \in A$ such that*

$$\|x - a^*\| = \inf_{a \in A} \|x - a\|.$$

Prop. 17. *If $X \subset H$, then $X^\perp$ is a closed linear subspace of H.*

Prop. 18. *If $E \subset H$ then $E^{\perp} = (\mathrm{span}(E))^{\perp} = (\overline{\mathrm{span}}(E))^{\perp}$.*

Theorem 21. *If U is a closed linear subspace of a Hilbert space H then*

1. *$H = U \oplus U^{\perp}$;*
2. *u is the closest point to x in U;*
3. *the map $P_U : H \to U$ defined by $P_U(x) = u$ is linear, and for every $x \in H$ we have that $P_U^2(x) = P_U(x)$ and $\|P_U(x)\| \leq \|x\|$.*

Prop. 19. *The map P_U is an orthogonal projection onto U.*

Prop. 20. *If U is a closed linear subspace of a Hilbert space H and E is an at most countable orthonormal subset in U such that $\mathrm{span}(E)$ is dense in U, then*

$$P_U(x) = \sum_{e_k \in E} \langle x, e_k \rangle e_k.$$

7 Dual Spaces

Prop. 21. *Let V be a normed space over $\mathbb{F}$. A linear functional $f : V \to \mathbb{F}$ is continuous if and only if it is bounded.*

Prop. 22. *If V is a normed space, then its dual space V^* is a Banach space with the norm defined by $\|f\| = \sup_{\|x\|=1} |f(x)|$.*

Theorem 22 (Riesz Representation Theorem). *If H is a Hilbert space over $\mathbb{F}$, then for any bounded linear functional $f : H \to \mathbb{F}$ there is a unique $y \in H$ such that for all $x \in H$,*

$$f(x) = \langle x, y \rangle.$$

Moreover, we have $\|f\|_{H^*} = \|y\|_H$.

Theorem 23. *If X is a compact metric space and $w : C(X) \to \mathbb{R}$ is a continuous linear functional, then there are two measures $\mu, \nu \in \mathcal{M}(X)$ such that for every $\phi \in C(X)$,*

$$w(\phi) = w_\mu(\phi) - w_\nu(\phi),$$

where $w_\mu(\phi) = \int_X \phi(x) \,\mathrm{d}\mu(x)$ and $w_\nu(\phi) = \int_X \phi(x) \,\mathrm{d}\nu(x)$.

Lemma 18. *A linear operator $A : V \to U$ is continuous if and only if it is bounded.*

Theorem 24. *If V is a normed space and U is a Banach space, then $B(V, U)$ is a Banach space.*

Prop. 23. *If $A \in B(U, V)$ then $\ker(A)$ is a closed linear subspace of V.*

Prop. 24. *If $A : X \to X$ is bounded and λ is its eigenvalue then $|\lambda| \leq \|A\|_{op}$.*

Lemma 19. *If $A : V \to U$ and $B : U \to V$ are linear operators such that $AB = I_U$ and $BA = I_V$, then A and B are both invertible and $B = A^{-1}$.*

Prop. 25. *Let V and U be normed spaces and $A : V \to U$ be an invertible linear operator. The inverse operator $A^{-1} : U \to V$ is bounded if and only if there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in V$.*

Lemma 20. *Let X be a Banach space and $T \in B(X,X)$. If $\|T\| < 1$, then $I - T$ is invertible, $(I-T)^{-1} \in B(X,X)$, $(I-T)^{-1} = I + T + T^2 + T^3 + \dots$, and*

$$\left\|(I-T)^{-1}\right\| \leq \frac{1}{1 - \|T\|}.$$

Prop. 26. *Let X be a Banach space, let $A : X \to X$ be an invertible linear operator and let $A, A^{-1}, B \in B(X,X)$. If $\|B\| \left\|A^{-1}\right\| < 1$, then the operator $A+B$ invertible, its inverse operator is bounded, and*

$$\left\|(A+B)^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1 - \left\|A^{-1}\right\| \|B\|}.$$

Theorem 25. *If $A : H \to H$ is a bounded linear operator on a Hilbert space H, then there is a unique bounded linear operator $A^* : H \to H$ such that for all $x, y \in H$,*

$$\langle Ax, y \rangle = \langle x, A^*, y \rangle.$$

Moreover, $\|A^*\|_{op} = \|A\|_{op}$.

Lemma 21 (properties of adjoint operators)**.** *If $A, B : H \to H$ are bounded operators on a Hilbert space H and $\alpha, \beta \in \mathbb{C}$, then*

1. $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$;
2. $(AB)^* = B^* A^*$;
3. $(A^*)^* = A$;
4. $\|A^* A\| = \|AA^*\| = \|A\|^2$.

Theorem 26. *Let A be a self-adjoint operator on a Hilbert space H. Then all eigenvalues of A are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Prop. 27. *If H is a separable Hilbert space and $A : H \to H$ is a self-adjoint operator, then $\sigma_p(A)$ is at most countable.*

Theorem 27. *If A is a bounded self-adjoint operator, then*

1. $\langle Ax, x \rangle$ *is real for all* $x \in H$;
2. $\|A\|_{op} = \sup_{\|x\|=1} |\langle Ax, x \rangle|$.

8 Introduction to Spectral Theory

Prop. 28. *If a linear operator $A : X \to X$ is bounded and $\lambda \in \sigma(A)$, then $|\lambda| \leq \|A\|_{op}$.*

Prop. 29. *If a linear operator $A : X \to X$ is bounded, then its resolvent set $R(A)$ is open and its spectrum $\sigma(A)$ is closed.*

Lemma 22. *If $A : H \to H$ is a bounded linear operator on a Hilbert space H, then $\lambda \in \sigma(A)$ if and only if $\overline{\lambda} \in \sigma(A^*)$.*

Theorem 28. *If X is a normed space and Y is a Banach space, then $K(X, Y)$ is a closed linear subspace in $B(X, Y)$.*

Lemma 23. *If $T : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then at least one of the real numbers $\|T\|$ and $-\|T\|$ is an eigenvalue of T.*

Prop. 30. *If $T : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then any eigenspace corresponding to a non-zero eigenvalue is finite-dimensional.*

Theorem 29 (Spectral Theorem for a compact self-adjoint operator)**.** *Let H be a Hilbert space and $T : H \to H$ be a compact self-adjoint operator.*

(a) If $\dim \operatorname{range}(T) = N$ then T and N non-zero real eigenvalues $\lambda_1, \dots, \lambda_N$ (not necessarily distinct). If $\dim H > N$, then 0 is also an eigenvalue. Moreover, there is an orthonormal set of eigenvectors $e_1, \dots, e_N$ such that $Te_k = \lambda_k e_k$, and for all $x \in H$

$$Tx = \sum_{k=1}^{N} \lambda_k \langle x, e_k \rangle e_k.$$

(b) If $\dim \operatorname{range}(T) = \infty$, then T has an infinite sequence λ_k of non-zero real eigenvalues such that $\lim_k \lambda_k = 0$. Moreover, there is an orthonormal sequence e_k of eigenvectors such that $Te_k = \lambda_k e_k$ and for all $x \in H$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Prop. 31. *Let H be an infinite-dimensional separable Hilbert space and let $T : H \to H$ be a compact self-adjoint operator. Then there is an orthonormal basis $E = \{e_k : k \in \mathbb{N}\}$ in H such that $Te_k = \lambda_k e_k$ for all $k \in \mathbb{N}$ and for all $x \in H$,*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_l \rangle e_k.$$

Note that it is possible that $\lambda_k = 0$ for some k.

Prop. 32. *Finite rank self-adjoint operators on a Hilbet space H are dense in the space of compact self-adjoint operators on H.*

Theorem 30. *If H is a Hilbert space and $T : H \to H$ is a compact self-adjoint operator, then $\sigma(T) = \overline{\sigma_p(T)}$.*