MA359 Glossary

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1 Week 1: Outer Measures

Outer Measure An Outer measure is a function $\mu^* : P(X) \to [0, \infty]$ such that the following three axioms are satisfied:
(i) $\mu^*(\emptyset) = 0$
(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
(iii) if $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets of X, then we have

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

2 Week 2: Sigma Algebra

Measurability Let μ^* be an outer measure on X. We say B such that $B \subset X$ is measurable if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ for all A such that $A \subseteq X$.

$\sigma-$**algebra** A collection of sets $\mathcal{A} \subseteq P(X)$ is a σ-algebra if it has the following properties:
(i) $X \in \mathcal{A}$
(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
(iii) For every countable sequence of sets $\{A_i\}_{i=1}^{\infty}$, the union of these sets is also in $\mathcal{A}$. i.e. $\mu^*(\bigcup\limits_{n=1}^{\infty} A_n) \in \mathcal{A}$

3 Week 3: Measures

Measure In general a measure μ is a function of a σ-algebra $\mathcal{A}$ such that $\mu(\emptyset) = 0$ and μ is countably additive, i.e. $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ when A_n are all disjoint sets.

Measure Space and Measurable Space $(X, \mathcal{A}, \mu)$ is a measure space, $(X, \mathcal{A})$ is a measurable space. (Domain set X, σ-algebra $\mathcal{A}$ and measure μ).

Measure Continuity If A_k is an increasing sequence of sets in $\mathcal{A}$ then $\mu(\bigcup\limits_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$
If A_k is an decreasing sequence of sets in $\mathcal{A}$ then $\mu(\bigcap\limits_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$

Translation Invariance A measure is translation invariant: $\mu(A+h) = \mu(A)$

4 Week 4: Measureable Functions

Measurable Function If $A \subset X$ and $A \in \mathcal{A}$ function $f : A \to [-\infty, +\infty]$, then the conditions
(a) for all $t \in \mathbb{R}$ the set $\{x \in A : f(x) \leq t\}$
(b) for all $t \in \mathbb{R}$ the set $\{x \in A : f(x) \leq t\}$
(c) for all $t \in \mathbb{R}$ the set $\{x \in A : f(x) \geq t\}$
(d) for all $t \in \mathbb{R}$ the set $\{x \in A : f(x) \leq t\}$
are equivalent

Simple Functions A simple function is a function of the form $f = \sum\limits_{i=0}^{N} a_i \chi_{A_i}$ where a_i are non-negative real numbers and A_i are disjoint subsets of X.

Integral of a simple Functions The integral of a simple function with respect to measure μ is $\int f\,d\mu = \sum\limits_{i=0}^{N} a_i \mu(A_i)$

For any non-negative function, there is a sequence of simple functions such that the limit of the sequence is our original function

5 Week 5: Integrable Functions

Integral of a non-negative function $\int f\,d\mu = \sup\{\int g\,d\mu : g$ is simple and $g \leq f\}$

Integral of a function $\int f\,d\mu = \int f^+\,d\mu + \int f^-\,d\mu$ where $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$

The integral of simple functions, the integral of a non-negative function and the integral of a function are all linear functionals

BTEC Monotone convergence theorem if we have a monotone sequence of simple functions f_i that converge pointwise to a measurable function f, then the sequence of the integrals of f_i converges to the integral of f (integral with respect to μ

6 Week 6: Integration Theorems

Monotone Convergence Theorem Let f_i be a monotone (increasing) sequence of measurable functions that converge to measurable function f. Then $\int f\,d\mu = \lim_{i\to\infty}\int f_i\,d\mu$

Beppo Levi's Theorem Let $(X, \mathcal{A}, \mu)$ be a measure space, then

$$\int \sum_{i=1}^{\infty} f_i\,d\mu = \sum_{i=1}^{\infty} \int f_i\,d\mu$$

Fatou's Lemma Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_i\}$ be a sequence of $[-\infty, \infty]$ valued $\mathcal{A}$-measurable functions on X. Then

$$\int \liminf_{i\to\infty} f_i(x) \le \liminf_{i\to\infty} \int f_i(x)$$

Dominated Convergence Theorem Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be an integrable function on X and let f_i be a sequence of $[-\infty, \infty]$ valued $\mathcal{A}$-measurable functions such that the sequence converges to f and $|f_i(x)| \le g(x)$ for all $n \in \mathbb{N}$. Then f and f_i are integrable and $\int f\,d\mu = \lim_{i\to\infty}\int f_i\,d\mu$.

Markow's inequalty If we have the set $A_t = \{x \in X : f(x) \ge t\}$ then $\mu(A_k) \le \frac{1}{t}\int_{A_t} f d\mu \le \frac{1}{t}\int f d\mu$.

Link to Reimann Integrals Let $[a, b]$ be a closed and bounded interval and let f be a bounded real valued function on $[a, b]$. Then
(a) f is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.
(b) if f is Riemann integrable then f is Lebesgue integrable and the integrals coincide.

7 Week 7: Modes of Convergence

Convergence in Measure Sequence of $\mathcal{A}$-measurable functions valued on $[-\infty, \infty]$ f_i converge to f in measure if for any $\epsilon > 0$, $\lim_{i\to\infty}\mu(\{x \in X : |f_i(x) - f(x)| > \epsilon\}) = 0$

Pointwise Convergence A sequence of $\mathcal{A}$-measurable functions valued on $[-\infty, \infty]$ f_i converge to f pointwise if for every $x \in X$, $\lim_{i\to\infty} f_i(x) = f(x)$

Pointwise Convergence almost everywhere Same as above but the set of points that fail make up a set of measure zero.

Convergence in mean A sequence of $\mathcal{A}$-measurable functions valued on $[-\infty, \infty]$ f_i converge to f in mean if $\lim_{i\to\infty} \int |f_i - f|\, d\mu = 0$

Egeroff's Theorem If f_i converges pointwise to f then for any $\epsilon > 0$ there is a subset B of X such that f_i restricted to B converges uniformally to f restricted to B and $\mu(B^c) < \epsilon$

8 Week 8: L^p spaces

L^p spaces The vector space $L^p(X, \mathcal{A}, \mu)$ is the space of functions such that $(\int_X |f|^p\, d\mu)^{\frac{1}{p}}$, factored by the equivalence classes of functions that are equal a.e. i.e. $f \sim g$ iff $f(x) = g(x)$ almost everywhere.

(NB, if $|f - g|_p = 0$, this only implies $f = g$ almost everywhere due to the definition of this norm. L^p is also complete, so any Cauchy sequence of functions will have a limit in L^p

Young's Inequality If $\frac{1}{p} + \frac{1}{q} = 1$ $x^{\frac{1}{p}} y^{\frac{1}{q}} \le \frac{x}{p} + \frac{y}{q}$

Holder's Inequality Assume $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, If $f \in L^p$ and $g \in L^q$, then $\int fg\, d\mu \le |f|_p |g|_q$

Minkowski's Inequality Basically traingle rule for the L^p norm

9 Week 9: Product Sigma Algebras

Dynkin Systems A family $\mathcal{D}$ of sets is a d-system if it satisfies axioms (i) and (ii) of a $\sigma-$algebra and (iii) if A_n is an increasing sequence of sets in $\mathcal{D}$, then $\bigcup\limits_{n=1}^{\infty} A_n$ is in $\mathcal{D}$ as well.

π-system Same as a σ-algebra but with finite union instead of countable union for axiom (iii).

Dynkin's Theorem Let X be a set and $\mathcal{C}$ be a π-system on X. Then the σ-algebra generated by $\mathcal{C}$ coincides with the d-system generated by $\mathcal{C}$.

Agreement on Generators Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{C}$ be a π-system such that $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν are measures on $(X, \mathcal{A})$ that satisfy $\mu(X) = \nu(X) < \infty$ and $\mu(C) = \nu(C)$ for $C \in \mathcal{C}$ then $\mu = \nu$.

Product Sigma Algebra Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be two measurable spaces. Then, the family of sets $\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathcal{A}_1 \text{ and } A_2 \in \mathcal{A}_2\})$. is called the product σ-algebra.

Product Measure (Note - this is a theorem and not a definition Given two measure spaces $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ where the measure spaces are $\sigma-$finite. Then there exists a unique measure $\mu_1 \otimes \mu_2$: $(\mathcal{A}_1, \mathcal{A}_2) \to [0, \infty]$ such that $(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for all sets A_1, A_2 in $\mathcal{A}_\infty$, $\mathcal{A}_\in$ respectively. This is called the product measure.

Futhermore, for each $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the function $x \to \mu_2(E_x)$ where $E_x := \{y \in X_2 : (x, y) \in E\}$ is measurable
and the function $y \to \mu_1(E_y)$ where $E_x := \{x \in X_1 : (x, y) \in E\}$ is measurable.
We also have: $(\mu_1 \otimes \mu_2)(E) = \int_{X_1} \mu_2(E_x)\, d\mu_1(x) = \int_{X_2} \mu_1(E_y)\, d\mu_2(x)$ for all $E \in \mathcal{A}_1 \times \mathcal{A}_2$.

10 Week 10: Fubini's Theorem

Fubini's Theorem Part 1 Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be measure spaces which are $\sigma-$finite.
let $f : X_1 \times X_2 \to [0, \infty]$ be an $\mathcal{A}_1 \times \mathcal{A}_2$ measurable function. Then $x \to \int f_x\, d\mu_2$ is $\mathcal{A}_1$ measurable and $y \to \int f_y\, d\mu_1$ is $\mathcal{A}_2$ measurable.
Furthermore, $\int_{X \times Y} f\, d(\mu_1 \times \mu_2) = \int_X (\int_Y f_X\, d\mu_2)\, d\mu_1$ and $\int_{X \times Y} f\, d(\mu_1 \times \mu_2) = \int_Y (\int_X f_Y\, d\mu_1)\, d\mu_2$

Fubini's Theorem Part 2 - what is listed as Fubini in Cohn Same hypotheses in part 1.
(a) μ_1-a.e. $x \in X$ the section f_X is μ_2-integrable and for μ_2-a.e. the section f_Y is μ_1-integrable

(b) The fuctions $I_f(x) = \begin{cases} \int_Y f_X\, d\mu_2 & \text{if } f_X \text{ is } \mu_2\text{-integrable} \\ 0 & \text{else} \end{cases}$
and $J_f(x) = \begin{cases} \int_X f_Y\, d\mu_1 & \text{if } f_Y \text{ is } \mu_1\text{-integrable} \\ 0 & \text{else} \end{cases}$
then $I_f \in \mathcal{L}^1(X, \mu_1)$ and $J_f \in \mathcal{L}^1(Y, \mu_2)$

(c) $\int_{X \times Y} f\, d(\mu_1 \times \mu_2) = \int_Y J_f\, d\mu_2 = \int_X I_f\, d\mu_1$