

# MA3G7 Functional Analysis 1

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## 1 Vector Spaces

**Theorem 1.** *Every vector space has a Hamel basis.*

You don't need to prove the above theorem.

**Theorem 2.** *If a vector space has a finite Hamel basis then every Hamel basis for this vector space has the same number of elements.*

**Prop. 1.** *Any  $n$ -dimensional vector space over a field  $\mathbb{F}$  is isomorphic to  $\mathbb{F}^n$ .*

**Lemma 1** (Young's inequality). *If  $a, b > 0$ ,  $1 < p, q < \infty$ , and  $\frac{1}{p} + \frac{1}{q} = 1$ , then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Lemma 2** (Hölder's inequality). *If  $1 \leq p, q \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $x \in \ell^p(\mathbb{F})$  and  $y \in \ell^q(\mathbb{F})$ , then*

$$\sum |x_j y_j| \leq \|x\|_p \|y\|_q.$$

**Lemma 3** (Minkowski's inequality). *Let  $1 \leq p < \infty$ . If  $x, y \in \ell^p(\mathbb{F})$  then  $x + y \in \ell^p(\mathbb{F})$  and*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

## 2 Normed spaces

**Prop. 2.** *Let  $(W, \|\cdot\|_W)$  be a normed space,  $V$  be a vector space, and  $L : V \rightarrow W$  a linear isomorphism. Then the function*

$$V \ni x \mapsto \|L(x)\|_W \in \mathbb{R}$$

*defines a norm on  $V$ .*

**Prop. 3.** *Any finite dimensional vector space can be equipped with a norm. Therefore, any  $n$ -dimensional vector space over  $\mathbb{F}$  is isometrically isomorphic to  $\mathbb{F}^n$  equipped with a suitable norm.*

**Theorem 3.** *All norms on a finite dimensional space are equivalent.*

**Prop. 4.** *If  $f_n \in C[0, 1]$  for all  $n \in \mathbb{N}$  and  $f_n \rightarrow f$  in the sup-norm, then  $f_n \rightarrow f$  in the  $L^1$ -norm.*

**Prop. 5.** *If  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent norms on a vector space, then for any sequence  $x_n$  in the vector space and for any  $x \in V$ ,*

$$\|x_n - x\|_1 \rightarrow 0 \iff \|x_n - x\|_2 \rightarrow 0.$$

**Lemma 4.** *In a normed space  $(V, \|\cdot\|)$ , a sequence  $x_n \rightarrow x$  if and only if for any open neighbourhood  $X$  of  $x$  there is an  $N \in \mathbb{N}$  such that  $x_n \in X$  for all  $n > N$ .*

**Prop. 6.** *On a vector space  $V$ , two norms are equivalent if and only if they induce the same topology.*

**Lemma 5.** *A subset  $X \subset V$  is closed if and only if every convergent sequence with elements in  $X$  has its limit in  $X$ .*

**Prop. 7.** *A finite dimensional linear subspace  $W$  of a normed space  $V$  is closed.*

**Theorem 4** (Heine-Borel). *A subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded. Therefore, a subset of a finite-dimensional vector space is closed if and only if it is bounded.*

You don't need to know the proof of this - but you should, as you studied it in Year 2.

**Lemma 6** (Riesz' Lemma). *Let  $X$  be a normed vector space and  $Y$  be a closed linear subspace of  $X$  such that  $Y \neq X$  and  $\alpha \in (0, 1)$ . Then there is  $x_\alpha \in X$  such that  $\|x_\alpha\| = 1$  and  $\|x_\alpha - y\| > \alpha$  for all  $y \in Y$ .*

**Theorem 5.** *A normed space is finite dimensional if and only if the unit sphere is compact.*

## 3 Banach spaces

**Theorem 6.** *A sequence of real numbers converges if and only if it is Cauchy.*

Proved in Year 1.

**Theorem 7.** *Every finite-dimensional normed space is complete. In particular,  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are complete.*

**Theorem 8.** *The space  $\ell^p(\mathbb{F})$  equipped with the  $\ell^p$ -norm is complete.*

**Theorem 9.** *The space  $C[0, 1]$  equipped with the sup-norm is complete.*

**Theorem 10** (Bernstein polynomials). *Consider the polynomials*

$$B_{np}(x) = \binom{n}{p} x^p (1-x)^{n-p}.$$

*Then the polynomials  $B_{np}$  are of degree  $n$  and we have that*

$$(i) \sum_{p=0}^n B_{np}(x) = 1$$

$$(ii) \sum_{p=0}^n p B_{np}(x) = nx$$

$$(iii) \sum_{p=0}^n (p-nx)^2 B_{np}(x) = nx(1-x).$$

**Theorem 11.** *If a function  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous then the sequence of polynomials*

$$P_n(x) = \sum_{p=0}^n f\left(\frac{p}{n}\right) \binom{n}{p} x^p (1-x)^{n-p}$$

*converges uniformly to  $f$  on  $[0, 1]$ .*

Therefore,  $C[0, 1]$  is separable.

**Theorem 12.** *Any normed space is isometrically isomorphic to a dense subset of a Banach space.*

## 4 Lebesgue spaces

**Theorem 13** (Monotone Convergence Theorem). *Suppose that  $f_n$  are integrable functions,  $f_n(x) \leq f_{n+1}(x)$  almost everywhere, and there is a constant  $K$  such that for all  $n$*

$$\int_X f_n(x) \, dx < K.$$

*Then there is an integrable function  $f$  such that  $f_n \rightarrow f$  almost everywhere and*

$$\int_X g(x) \, dx = \lim_{n \rightarrow \infty} \int_X f_n(x) \, dx.$$

**Prop. 8.** *If  $f$  is integrable and  $\int_X |f(x)| \, dx = 0$  then  $f(x) = 0$  almost everywhere.*

**Theorem 14** (Dominated Convergence Theorem). *Suppose that  $f_n : X \rightarrow \mathbb{R}$  are integrable functions and  $f_n(x) \rightarrow f(x)$  almost everywhere. If there is an integrable function  $g$  such that  $|f_n(x)| \leq g(x)$  for every  $n$  and almost every  $x$ , then  $f$  is integrable and*

$$\int_X f(x) \, dx = \lim_{n \rightarrow \infty} \int_X f_n(x) \, dx.$$

The Dominated Convergence theorem holds for complex-valued functions, and is easily proved from the above version for real-valued functions.

**Theorem 15.**  $L^1(X)$  is a Banach space.

**Lemma 7.** If  $(V, \|\cdot\|)$  is a normed space in which  $\sum_{j=1}^{\infty} \|y_j\| < \infty$  implies that  $\sum_{j=1}^{\infty} y_j$  converges, then  $V$  is complete.

**Lemma 8.** If  $f_k$  is a sequence of Lebesgue-integrable functions such that

$$K := \sum_{j=1}^{\infty} \|f_k\|_1 < \infty,$$

then

- (i) there is an integrable function  $g$  such that  $\sum_{k=1}^n |f_k(x)| \rightarrow g(x)$  almost everywhere;
- (ii) there is an integrable function  $h$  such that  $\sum_{k=1}^n f_k(x) \rightarrow h(x)$  almost everywhere;
- (iii)  $\|h(x) - \sum_{k=1}^n f_k(x)\|_1 \rightarrow 0$ .

**Theorem 16.** The space  $C[0,1]$  is dense in  $L^1[0,1]$ . Therefore,  $L^1[a,b]$  is isometrically isomorphic to the completion of  $C[a,b]$  in the  $L^1$ -norm.

**Theorem 17.**  $L^p(X)$  is a Banach space for any  $p \in [1, \infty)$ .

## 5 Inner product spaces

**Lemma 9** (Cauchy-Schwarz inequality). If  $V$  is an inner product space and  $\|x\| = \sqrt{\langle x, x \rangle}$  for all  $x \in V$ , then for all  $x, y \in V$ ,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

**Prop. 9.** If  $V$  is an inner product space, then the equation  $\|x\| = \sqrt{\langle x, x \rangle}$  defines a norm on  $V$ .

**Lemma 10** (Continuity of the inner product). Let  $V$  be an inner product space equipped with the natural norm. If  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$ .

**Lemma 11** (Parallelogram law). If  $V$  is an inner product space with the natural norm  $\|\cdot\|$ , then for all  $x, y \in V$ ,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

**Lemma 12** (Polarisation identity). Let  $V$  be an inner product space with the natural norm  $\|\cdot\|$ . If the vector space  $V$  is real, then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2.$$

If the vector space  $V$  is complex, then

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

**Prop. 10.** If  $V$  is a real normed space with the norm  $\|\cdot\|$  satisfying the parallelogram law, then

$$\langle x, y \rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4}$$

defines an inner product on  $V$ .

**Theorem 18 (Pythagoras).** If  $x \perp y$  then  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

**Lemma 13.** If  $\{e_1, \dots, e_n\}$  is an orthonormal set in an inner product space  $V$ , then for any  $\alpha_j \in \mathbb{F}$  we have

$$\left\| \sum_{j=1}^n \alpha_j e_j \right\|^2 = \sum_{j=1}^n |\alpha_j|^2.$$

**Lemma 14 (Bessel's inequality).** If  $V$  is an inner product space and  $E = (e_k)$  is an orthonormal sequence, then for every  $x \in V$

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2.$$

**Prop. 11.** If  $E$  is an orthonormal set in an inner product space  $V$  then for any  $x \in V$  the set

$$\mathcal{E}_x := \{e \in E : \langle x, e \rangle \neq 0\}$$

is at most countable.

**Lemma 15 (Gram-Schmidt orthonormalisation).** Let  $V$  be an inner product space and  $v_k$  be a sequence of linearly independent vectors in  $V$ . Then there is an orthonormal sequence  $(e_k)$  such that for every  $k$ ,

$$\text{span}\{v_1, \dots, v_k\} = \text{span}\{e_1, \dots, e_k\}.$$

**Prop. 12.** Any infinite-dimensional inner product space contains an orthonormal sequence.

**Prop. 13.** Any finite-dimensional inner product space has an orthonormal basis.

**Prop. 14.** Any finite-dimensional inner product space is isometrically isomorphic to  $\mathbb{C}^n$  or  $\mathbb{R}^n$  (if the space is complex or real respectively) equipped with the standard inner product.

## 6 Hilbert space

**Lemma 16.** Let  $H$  be a Hilbert space and  $E = (e_k)$  be an orthonormal sequence in  $H$ . The series  $\sum_{k=1}^{\infty} \alpha_k e_k$  converges if and only if  $\sum_{k=1}^{\infty} |\alpha_k|^2 < +\infty$ . In this case, we have

$$\left\| \sum_{k=1}^{\infty} \alpha_k e_k \right\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2.$$

**Prop. 15.** If  $H$  is a Hilbert space and  $E = (e_k)$  is an orthonormal sequence, then for every  $x \in H$  the series  $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$  converges.

**Prop. 16.** Let  $E = e_k$  be an orthonormal set in a Hilbert space  $H$ . Then the following are equivalent:

- for every  $x \in H$  there are  $\alpha_k \in \mathbb{F}$  such that  $x = \sum_{k=1}^{\infty} \alpha_k e_k$ ;
- $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$  for all  $x \in H$ ;
- $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$  for all  $x \in H$ ;
- if  $\langle x, e_k \rangle = 0$  for all  $k \in \mathbb{N}$  then  $x = 0$ ;
- the linear span of  $E$  is dense in  $H$ .

**Theorem 19.** An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.

**Theorem 20.** Any infinite-dimensional separable Hilbert space over  $\mathbb{F}$  is isometrically isomorphic to  $\ell^2(\mathbb{F})$ .

**Lemma 17.** If  $A$  is a non-empty closed convex subset of a Hilbert space  $H$ , then for any  $x \in H$  there is a unique  $a^* \in A$  such that

$$\|x - a^*\| = \inf_{a \in A} \|x - a\|.$$

**Prop. 17.** If  $X \subset H$ , then  $X^\perp$  is a closed linear subspace of  $H$ .

**Prop. 18.** If  $E \subset H$  then  $E^\perp = (\text{span}(E))^\perp = (\overline{\text{span}}(E))^\perp$ .

**Theorem 21.** If  $U$  is a closed linear subspace of a Hilbert space  $H$  then

1.  $H = U \oplus U^\perp$ ;
2.  $u$  is the closest point to  $x$  in  $U$ ;
3. the map  $P_U : H \rightarrow U$  defined by  $P_U(x) = u$  is linear, and for every  $x \in H$  we have that  $P_U^2(x) = P_U(x)$  and  $\|P_U(x)\| \leq \|x\|$ .

**Prop. 19.** The map  $P_U$  is an orthogonal projection onto  $U$ .

**Prop. 20.** If  $U$  is a closed linear subspace of a Hilbert space  $H$  and  $E$  is an at most countable orthonormal subset in  $U$  such that  $\text{span}(E)$  is dense in  $U$ , then

$$P_U(x) = \sum_{e_k \in E} \langle x, e_k \rangle e_k.$$

## 7 Dual Spaces

**Prop. 21.** Let  $V$  be a normed space over  $\mathbb{F}$ . A linear functional  $f : V \rightarrow \mathbb{F}$  is continuous if and only if it is bounded.

**Prop. 22.** If  $V$  is a normed space, then its dual space  $V^*$  is a Banach space with the norm defined by  $\|f\| = \sup_{\|x\|=1} |f(x)|$ .

**Theorem 22** (Riesz Representation Theorem). *If  $H$  is a Hilbert space over  $\mathbb{F}$ , then for any bounded linear functional  $f : H \rightarrow \mathbb{F}$  there is a unique  $y \in H$  such that for all  $x \in H$ ,*

$$f(x) = \langle x, y \rangle.$$

*Moreover, we have  $\|f\|_{H^*} = \|y\|_H$ .*

**Theorem 23.** *If  $X$  is a compact metric space and  $w : C(X) \rightarrow \mathbb{R}$  is a continuous linear functional, then there are two measures  $\mu, \nu \in \mathcal{M}(X)$  such that for every  $\phi \in C(X)$ ,*

$$w(\phi) = w_\mu(\phi) - w_\nu(\phi),$$

*where  $w_\mu(\phi) = \int_X \phi(x) d\mu(x)$  and  $w_\nu(\phi) = \int_X \phi(x) d\nu(x)$ .*

## 8 Linear operators

**Lemma 18.** *A linear operator  $A : V \rightarrow U$  is continuous if and only if it is bounded.*

**Theorem 24.** *If  $V$  is a normed space and  $U$  is a Banach space, then  $B(V, U)$  is a Banach space.*

**Prop. 23.** *If  $A \in B(U, V)$  then  $\ker(A)$  is a closed linear subspace of  $V$ .*

**Prop. 24.** *If  $A : X \rightarrow X$  is bounded and  $\lambda$  is its eigenvalue then  $|\lambda| \leq \|A\|_{op}$ .*

**Lemma 19.** *If  $A : V \rightarrow U$  and  $B : U \rightarrow V$  are linear operators such that  $AB = I_U$  and  $BA = I_V$ , then  $A$  and  $B$  are both invertible and  $B = A^{-1}$ .*

**Prop. 25.** *Let  $V$  and  $U$  be normed spaces and  $A : V \rightarrow U$  be an invertible linear operator. The inverse operator  $A^{-1} : U \rightarrow V$  is bounded if and only if there is a constant  $c > 0$  such that  $\|Ax\| \geq c\|x\|$  for all  $x \in V$ .*

**Lemma 20.** *Let  $X$  be a Banach space and  $T \in B(X, X)$ . If  $\|T\| < 1$ , then  $I - T$  is invertible,  $(I - T)^{-1} \in B(X, X)$ ,  $(I - T)^{-1} = I + T + T^2 + T^3 + \dots$ , and*

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

**Prop. 26.** *Let  $X$  be a Banach space, let  $A : X \rightarrow X$  be an invertible linear operator and let  $A, A^{-1}, B \in B(X, X)$ . If  $\|B\| \|A^{-1}\| < 1$ , then the operator  $A + B$  is invertible, its inverse operator is bounded, and*

$$\|(A + B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|B\|}.$$

**Theorem 25.** *If  $A : H \rightarrow H$  is a bounded linear operator on a Hilbert space  $H$ , then there is a unique bounded linear operator  $A^* : H \rightarrow H$  such that for all  $x, y \in H$ ,*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle.$$

*Moreover,  $\|A^*\|_{op} = \|A\|_{op}$ .*

**Lemma 21** (properties of adjoint operators). *If  $A, B : H \rightarrow H$  are bounded operators on a Hilbert space  $H$  and  $\alpha, \beta \in \mathbb{C}$ , then*

1.  $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$ ;
2.  $(AB)^* = B^*A^*$ ;
3.  $(A^*)^* = A$ ;
4.  $\|A^*A\| = \|AA^*\| = \|A\|^2$ .

**Theorem 26.** *Let  $A$  be a self-adjoint operator on a Hilbert space  $H$ . Then all eigenvalues of  $A$  are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.*

**Prop. 27.** *If  $H$  is a separable Hilbert space and  $A : H \rightarrow H$  is a self-adjoint operator, then  $\sigma_p(A)$  is at most countable.*

**Theorem 27.** *If  $A$  is a bounded self-adjoint operator, then*

1.  $\langle Ax, x \rangle$  is real for all  $x \in H$ ;
2.  $\|A\|_{op} = \sup_{\|x\|=1} |\langle Ax, x \rangle|$ .

## 9 Introduction to Spectral Theory

**Prop. 28.** *If a linear operator  $A : X \rightarrow X$  is bounded and  $\lambda \in \sigma(A)$ , then  $|\lambda| \leq \|A\|_{op}$ .*

**Prop. 29.** *If a linear operator  $A : X \rightarrow X$  is bounded, then its resolvent set  $R(A)$  is open and its spectrum  $\sigma(A)$  is closed.*

**Lemma 22.** *If  $A : H \rightarrow H$  is a bounded linear operator on a Hilbert space  $H$ , then  $\lambda \in \sigma(A)$  if and only if  $\bar{\lambda} \in \sigma(A^*)$ .*

**Theorem 28.** *If  $X$  is a normed space and  $Y$  is a Banach space, then  $K(X, Y)$  is a closed linear subspace in  $B(X, Y)$ .*

**Lemma 23.** *If  $T : H \rightarrow H$  is a compact self-adjoint operator on a Hilbert space  $H$ , then at least one of the real numbers  $\|T\|$  and  $-\|T\|$  is an eigenvalue of  $T$ .*

**Prop. 30.** *If  $T : H \rightarrow H$  is a compact self-adjoint operator on a Hilbert space  $H$ , then any eigenspace corresponding to a non-zero eigenvalue is finite-dimensional.*

**Theorem 29** (Spectral Theorem for a compact self-adjoint operator). *Let  $H$  be a Hilbert space and  $T : H \rightarrow H$  be a compact self-adjoint operator.*

- (a) *If  $\dim \text{range}(T) = N$  then  $T$  has  $N$  non-zero real eigenvalues  $\lambda_1, \dots, \lambda_N$  (not necessarily distinct). If  $\dim H > N$ , then 0 is also an eigenvalue. Moreover, there is an orthonormal set of eigenvectors  $e_1, \dots, e_N$  such that  $Te_k = \lambda_k e_k$ , and for all  $x \in H$*

$$Tx = \sum_{k=1}^N \lambda_k \langle x, e_k \rangle e_k.$$



(b) If  $\dim \text{range}(T) = \infty$ , then  $T$  has an infinite sequence  $\lambda_k$  of non-zero real eigenvalues such that  $\lim_k \lambda_k = 0$ . Moreover, there is an orthonormal sequence  $e_k$  of eigenvectors such that  $Te_k = \lambda_k e_k$  and for all  $x \in H$ ,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

**Prop. 31.** Let  $H$  be an infinite-dimensional separable Hilbert space and let  $T : H \rightarrow H$  be a compact self-adjoint operator. Then there is an orthonormal basis  $E = \{e_k : k \in \mathbb{N}\}$  in  $H$  such that  $Te_k = \lambda_k e_k$  for all  $k \in \mathbb{N}$  and for all  $x \in H$ ,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k.$$

Note that it is possible that  $\lambda_k = 0$  for some  $k$ .

**Prop. 32.** Finite rank self-adjoint operators on a Hilbert space  $H$  are dense in the space of compact self-adjoint operators on  $H$ .

**Theorem 30.** If  $H$  is a Hilbert space and  $T : H \rightarrow H$  is a compact self-adjoint operator, then  $\sigma(T) = \overline{\sigma_p(T)}$ .

## 10 Sturm-Liouville problems

Let  $[a, b] \subset \mathbb{R}$ ,  $p \in C^1[a, b]$  and  $q \in C[a, b]$ . Find all  $\lambda \in \mathbb{R}$  such that the following linear ODE has a non-zero solution  $u$  satisfying the boundary conditions  $u(a) = u(b) = 0$

$$-\frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) + q(x)u(x) = \lambda u(x) \quad (1)$$

This section is devoted to proving the following theorem:

**Theorem 31.** If  $p \in C^1[a, b]$ ,  $q \in C[a, b]$  are such that  $p(x) > 0$  and  $q(x) \geq 0$  for all  $x \in [a, b]$ , then

- the eigenvalues  $\lambda_k$  of the Sturm-Liouville problem (1) form an unbounded monotone sequence of positive numbers;
- the eigenvalues are simple;
- the eigenfunctions form an orthonormal basis  $e_k$  in  $L^2[a, b]$ .

**Lemma 24.** Let  $p \in C^1[a, b]$  and  $q \in C^0[a, b]$ . If  $u_1, u_2 \in C^2[a, b]$  satisfy the linear ODE

$$-(pu')' + qu = 0,$$

then  $W_p(u_1, u_2) = p(u_1' u_2 - u_1 u_2')$  is constant in  $x \in [a, b]$ .

**Lemma 25.** If  $p \in C^1[a, b]$ ,  $q \in C^0[a, b]$ , and  $p(x) > 0$  for all  $x \in [a, b]$ , then the ODE

$$-(pu')' + qu = 0$$

has two solutions  $u_1, u_2 \in C^2[a, b]$  such that  $u_1(a) = 0$  and  $u_1'(a) = 1$ , and  $u_2(b) = 0$  and  $u_2'(b) = 1$ . Moreover, if  $q(x) \geq 0$  for all  $x \in [a, b]$ , then  $W_p(u_1, u_2) \neq 0$ .

**Lemma 26** (Green's function). *If  $p \in C^1[a, b]$ ,  $q \in C^0[a, b]$ ,  $p(x) > 0$  for all  $x \in [a, b]$ , and  $u_1, u_2$  are solutions of the ODE*

$$-(pu')' + qu = 0$$

*such that  $u_1(a) = u_2(b) = 0$  and  $W_p(u_1, u_2) \neq 0$  and*

$$G(x, y) = \frac{1}{W_p(u_1, u_2)} \begin{cases} u_1(x)u_2(y), & \text{for } a \leq x < y \leq b \\ u_1(y)u_2(x), & \text{for } a \leq y \leq x \leq b, \end{cases}$$

*then for any  $f \in C^0[a, b]$  the function  $u(x) = \int_a^b G(x, y)f(y) dy$  is  $C^2[a, b]$  and satisfies both the ODE*

$$-(pu')' + qu = f$$

*and the boundary conditions  $u(a) = u(b) = 0$ .*

**Theorem 32** (the inverse operator  $L^{-1} : C^0 \rightarrow C_0^2$ ). *If  $p \in C^1[a, b]$  and  $q \in C[a, b]$  are such that  $p(x) > 0$  and  $q(x) \geq 0$  for all  $x \in [a, b]$ , then the linear operator  $L : C_0^2 \rightarrow C^0$ ,  $L(u) = -(pu')' + qu$  is bijective. Moreover, there is a continuous function  $G(x, y)$  such that for all  $f \in C^0$*

$$L^{-1}(f) = \int_a^b G(x, y)f(y) dy.$$

**Theorem 33.** *If  $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$  is a continuous function of two variables, then the linear operator  $A : L^2[a, b] \rightarrow L^2[a, b]$  defined by the equation*

$$(Af)(x) = \int_a^b G(x, y)f(y) dy$$

*is compact. Moreover,  $A(f)$  is a continuous function.*

**Prop. 33** (properties of  $A$ ). *The above operator  $A : L^2[a, b] \rightarrow L^2[a, b]$  has the following properties:*

1.  $A$  is a bounded self-adjoint operator;
2.  $\text{range}(A)$  is dense in  $L^2(a, b)$ ;
3.  $\ker(A)$  is trivial;
4.  $A(C^0) = C_0^2$ .

**Prop. 34.** *A function  $u$  is an eigenfunction of the Sturm-Liouville problem if and only if it is an eigenfunction of the operator  $A$ :*

$$Lu = \lambda u \iff Au = \lambda \inf u.$$