Linear Algebra

For Machine Learning

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What is **machine learning** ?

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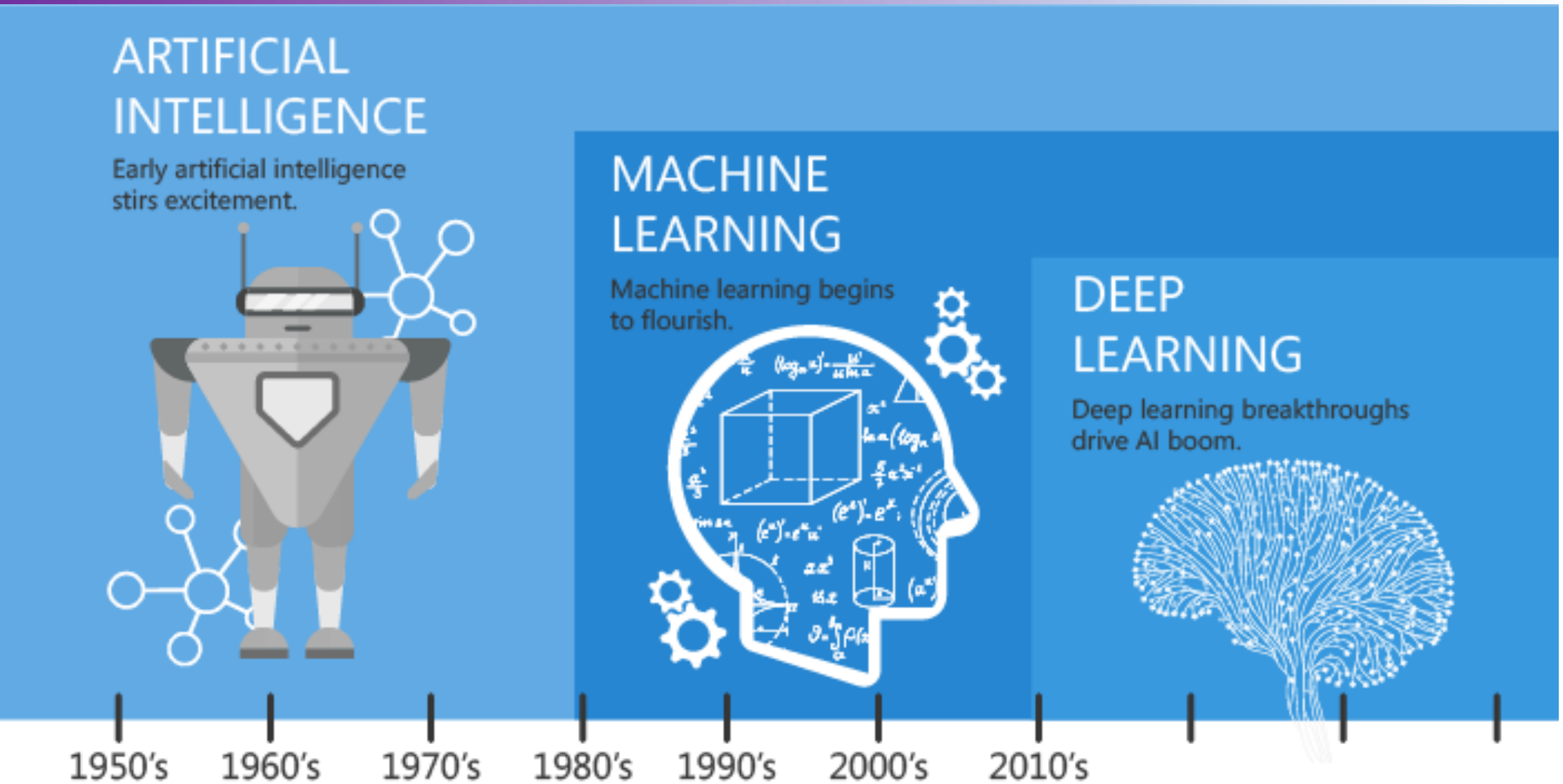

What is **linear algebra** ?

Why do we need to learn **linear algebra** in **machine learning** ?

Curriculum

- Matrices and vectors
- Addition and scalar multiplication
- Matrix multiplication
- Inverse and transpose

Matrices and vectors

Matrices and Vectors

- Matrix : A matrix is a rectangular array of numbers.

$$\mathbb{R}^{4X2} \qquad \begin{bmatrix} 396 & 34 \\ 61 & 63 \\ 565 & 830 \\ 47 & 16 \end{bmatrix} \qquad \begin{bmatrix} 12 & 35 & 8 \\ 43 & 87 & 49 \end{bmatrix} \qquad \mathbb{R}^{2X3}$$

4 X 2 matrix

2 X 3 matrix

- Dimension of matrix: number of rows X Number of Columns.

Matrices and Vectors

- Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 352 & 461 \\ 671 & 361 \\ 89 & 72 \\ 147 & 88 \end{bmatrix}$$

A_{ij}=" i, j entry" in the ith row, j^{th} column.

Matrices and Vectors

Vector: An n X 1 matrix.

$$y = \begin{bmatrix} 760 \\ 22 \\ 415 \\ 778 \end{bmatrix}$$

$$y_{i=\ i^{th}\ \mathrm{element}}$$

1-indexed vs 0-indexed:

$$y = \begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \end{bmatrix} \qquad y = \begin{bmatrix} y0 \\ y1 \\ y2 \\ y3 \end{bmatrix}$$

Addition and scalar multiplication

Addition and scalar multiplication

- Matrix Addition

$$\begin{bmatrix} 5 & 3 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.7 \\ 2 & 5 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

Addition and scalar multiplication

- Matrix Addition

$$\begin{bmatrix} 5 & 3 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.7 \\ 2 & 5 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 3.7 \\ 4 & 10 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}$$ error!

Addition and scalar multiplication

- Scalar Multiplication

$$3 \times \begin{bmatrix} 1 & 0 \\ 3 & 5 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} / 4 = \begin{bmatrix} & \\ & \end{bmatrix}$$

Addition and scalar multiplication

- Scalar Multiplication

$$3 \times \begin{bmatrix} 1 & 0 \\ 3 & 5 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 9 & 15 \\ 18 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 8 & 16 \end{bmatrix} / 4 = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$$

Addition and scalar multiplication

- Combination of Operands

$$3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Addition and scalar multiplication

- Combination of Operands

$$3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} /3 = \begin{bmatrix} 2 \\ 12 \\ 10\frac{1}{3} \end{bmatrix}$$

Matrix multiplication

Matrix - vector multiplication

Matrix - matrix multiplication

Matrix multiplication properties

Matrix - vector multiplication

- Example

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}$$

Matrix - matrix multiplication

Details:

$$A \times X = Y$$

$$\begin{bmatrix} \quad \end{bmatrix} \times \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

m x n matrix
(m rows, n columns)

n x 1 matrix
(n- dimensional vector)

m- dimensional vector

To get y_i , multiply $A's\ i^{th}$ row with elements of vector x , and add them up.

Matrix - vector multiplication

- Example

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ -1 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}$$

Matrix - vector multiplication

- House size:

  24

  146

  534

  752

$$h_\theta(x) = -60 + 0.73\ \ x$$

Matrix - matrix multiplication

- Example

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Matrix – matrix multiplication

- Details:

$$\underset{\substack{\text{m x n matrix} \\ \text{(m rows, n columns)}}}{\overset{A}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}} \overset{\times}{\times} \underset{\substack{\text{n x o matrix} \\ \text{(n rows, o columns)}}}{\overset{B}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}} \overset{=}{=} \underset{\text{m x o matrix}}{\overset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}}$$

The ith column of matric C is obtained by multiplying $\boldsymbol{A}$ with $\boldsymbol{i^{th}}$ column of $\boldsymbol{B}$. (for i = 1,2,3⋯o)

Matrix – matrix multiplication

- Example

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

Matrix - matrix multiplication

- House sizes:
  68
  16
  24
  52

Have 3 competing hypotheses:

1. $h_\theta(x) = -30 + 0.5x$
2. $h_\theta(x) = 50 + 3x$
3. $h_\theta(x) = -150 + 4x$

Matrix Matrix

$$\begin{bmatrix} 1 & 68 \\ 1 & 16 \\ 1 & 24 \\ 1 & 52 \end{bmatrix} \times \begin{bmatrix} -30 & 50 & -150 \\ 0.5 & 3 & 4 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix}$$

Matrix multiplication properties

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices. Then in general,

$$A \times B \neq B \times A. \quad \text{(not commutative.)}$$

E.g.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$A \times B \times C$

Let $\boldsymbol{D} = \boldsymbol{B} \times \boldsymbol{C}$ compute $A \times D$

Let $\boldsymbol{E} = \boldsymbol{A} \times \boldsymbol{B}$ compute $\boldsymbol{E} \times \boldsymbol{C}$

Identity Matrix

- Denoted I ($or\ I_{n \times n}$).
- Example of identity matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \qquad \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

2 x 2 3 x 3 4 x 4 n x n

For any matrix ***A,***

$$A \cdot I = I \cdot A = A$$

Inverse and transpose

Inverse and transpose

- Not all numbers have an inverse.

- **Matrix inverse:**

  If A is an m x m matrix, and if it has an inverse,

$$\boldsymbol{A} \quad \boldsymbol{A}^{-1} \quad = \quad \boldsymbol{A}^{-1}\,\mathbf{A} \quad = \quad \boldsymbol{I}$$

Matrices that don't have an inverse are "singular" or "degenerate".

Matrix Transpose

Example:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} \qquad \boldsymbol{B} = \quad A^T = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 9 \end{bmatrix}$$

Let $\boldsymbol{A}$ be an m x n matrix, and let $\boldsymbol{B} = A^T$

Then $\boldsymbol{B}$ is an n x m matrix, and

$$B_{ij} = A_{ji}$$

Thanks !

Linear Algebra for Machine Learning

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