A NECESSARY AND SUFFICIENT CONDITION FOR A FUNCTION TO BE SEPARABLE

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ABSTRACT

In this paper, we show that a real-valued function h defined on a rectangle of $\mathbb{R}^2$ is separable if and only if any matrix $[h(x_i, y_j)]_{ij}$ has rank at most 1.

1. DEFINITION

Let I_x and I_y be two nonempty intervals of $\mathbb{R}$.
Define $U = I_x \times I_y$ a rectangle of $\mathbb{R}^2$.
A function $h : U \to \mathbb{R}$ is *separable* if there exist two functions $f : I_x \to \mathbb{R}$ and $g : I_y \to \mathbb{R}$ such that, for all $(x, y) \in U$:

$$h(x, y) = f(x)g(y).$$

2. MAIN RESULT

Theorem 2.1. *Let $U = I_x \times I_y$ be a rectangle of $\mathbb{R}^2$ and $h : U \to \mathbb{R}$ be a function.*
h is separable if and only if for any $n, m \in \mathbb{N}^*$, and for any $[x_i]_{i=1,\dots,n} \in I_x^n$ and $[y_j]_{j=1,\dots,m} \in I_y^m$, the matrix $[h(x_i, y_j)]_{i=1,\dots,n, j=1,\dots,m}$ has rank at most 1.

Proof. Assume that h is separable. Then, for any i, j, $h(x_i, y_j) = f(x_i)g(y_j)$. Thus, we can write the matrix $[h(x_i, y_j)]_{ij}$ as:

$$[h(x_i, y_j)]_{ij} = \begin{bmatrix} f(x_1) & \dots & f(x_n) \end{bmatrix}^T \begin{bmatrix} g(y_1) & \dots & g(y_m) \end{bmatrix},$$

which has rank 1 if there are some i, j such that $f(x_i)g(y_j) \neq 0$ and rank 0 otherwise.

Now, assume that $[h(x_i, y_j)]_{ij}$ has rank at most 1 for any $[x_i]$ and $[y_j]$. If h is identically 0, then we can choose f and g to be identically 0, and h is separable. Assume that h is not identically 0. Let $(x_0, y_0) \in U$ such that $h(x_0, y_0) \neq 0$. Let (x, y) be an arbitrary point of U. Then, (x_0, y) and (x, y_0) also belong to U. Thus, the matrix:

$$\begin{bmatrix} h(x_0, y_0) & h(x_0, y) \\ h(x, y_0) & h(x, y) \end{bmatrix}$$

has rank 1. Hence, there exist $\alpha(x) \in \mathbb{R}$ and $\beta(y) \in \mathbb{R}$ such that:

$$[h(x, y_0), h(x, y)] = \alpha(x)[h(x_0, y_0), h(x_0, y)],$$
$$[h(x_0, y), h(x, y)] = \beta(y)[h(x_0, y_0), h(x, y_0)].$$

Thus, we have $h(x, y) = \alpha(x)\beta(y)h(x_0, y_0)$. For all $x \in I_x$ and $y \in I_y$, define:

$$f(x) = \alpha(x),$$
$$g(y) = \beta(y)h(x_0, y_0).$$

Then, $h = fg$ and h is separable. □