1 Notations

Let $d, c \in \mathbb{N}^*$ such that $d \geq c > 1$. Let $m = c - 1$.
Let $\mathcal{M}_{m,d}(\mathbb{R})$ be the set of real matrices of dimension $m \times d$.
Let $\mathcal{S}_m^+(\mathbb{R})$ be the set of symmetric positive-definite real matrices of dimension $m \times m$.
Let $\mathcal{S}_d(\mathbb{R})$ be the set of symmetric positive-semidefinite real matrices of dimension $d \times d$.
Let $\mathrm{GL}_m(\mathbb{R})$ be the set of non-singular real matrices of dimension $m \times m$.
Let $\mathcal{O}_m(\mathbb{R})$ be the set of real orthogonal matrices of dimension $m \times m$.
The range of a matrix M is denoted $\mathrm{rg}(M)$, its rank is denoted $\mathrm{rk}(M)$, and its spectrum is denoted $\mathrm{sp}(M)$.
The Euclidean norm is denoted $\|\cdot\|$. We use the notation $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
Following the convention in differential geometry, we denote the components of a vector v by v^i with a superscript.

2 Definitions

Definition 2.1 (Probability simplex)**.** Define the *probability simplex* of dimension m by:

$$\Delta^m = \left\{ \theta \in \mathbb{R}^c : \forall k \in \{1, \dots, c\}, \theta^k > 0 \text{ and } \sum_{i=1}^{c} \theta^i = 1 \right\}.$$

Δ^m is a smooth submanifold of $\mathbb{R}^c$ of dimension $m = c - 1$.
When we write $\theta \in \Delta^m$, we see θ as having m coordinates: $\theta = (\theta^1, \dots, \theta^m)$. Then, we define $\theta^c = 1 - \sum_{i=1}^{m} \theta^i$.

Definition 2.2 (Fisher information metric)**.** We endow Δ^m with the *Fisher information metric* (FIM) g. For each $\theta \in \Delta^m$, the FIM defines a *symmetric positive-definite bilinear form* g_θ over the tangent space $T_\theta \Delta^m$. In the *standard coordinates* of $\mathbb{R}^c$, we have, for all $\theta \in \Delta^m$ and for all *tangent vectors* $v, w \in T_\theta \Delta^m$:

$$g_\theta(v, w) = v^T G_\theta w,$$

where G_θ is the *Fisher information matrix* for parameter $\theta \in \Delta^m$ defined by:

$$G_{\theta,ij} = \frac{\delta_{ij}}{\theta^i} + \frac{1}{\theta^c}. \tag{1}$$

For any $\theta \in \Delta^m$, the matrix G_θ is *symmetric positive-definite and non-singular* (Proposition 1.6.2 in [1]). The FIM induces a distance on Δ^m called the *Fisher-Rao distance* denoted $d(\theta_1, \theta_2)$ for any $\theta_1, \theta_2 \in \Delta^m$.

Definition 2.3 (Euclidean metric)**.** We consider the *Euclidean space* $\mathbb{R}^d$ endowed with the *Euclidean metric* $\overline{g}$. It is defined in the standard coordinates of $\mathbb{R}^d$ for all $x \in \mathbb{R}^d$ and for all tangent vectors $v, w \in T_x \mathbb{R}^d$ by:

$$\overline{g}_x(v, w) = v^T w,$$

thus its matrix is the identity matrix of dimension d denoted I_d. The Euclidean metric induces a distance on $\mathbb{R}^d$ that we will denote with the l_2-norm: $\|x_1 - x_2\|_2$ for any $x_1, x_2 \in \mathbb{R}^d$.

From now on, we fix:

- a smooth map $f : (\mathbb{R}^d, \overline{g}) \to (\Delta^m, g)$. We denote by f^i the i-th component of f in the standard coordinates of $\mathbb{R}^c$.
- a point $x \in \mathbb{R}^d$.
- a positive real number $\epsilon > 0$.

Definition 2.4 (Euclidean ball)**.** Define the Euclidean open ball centered at x with radius ϵ by:

$$\overline{b}(x, \epsilon) = \left\{ z \in \mathbb{R}^d : \|z - x\|_2 < \epsilon \right\}.$$

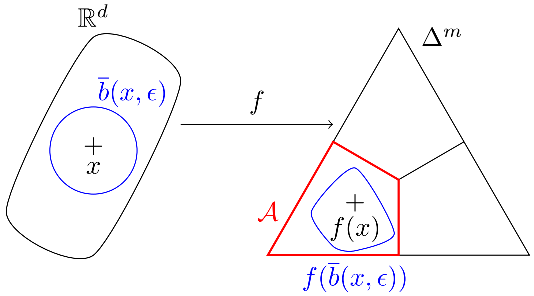

Figure 1: ϵ-robustness at x is enforced if and only if $f(\bar{b}(x, \epsilon)) \subseteq \mathcal{A}$.

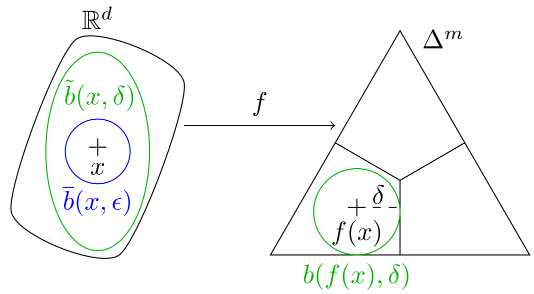

Figure 2: ϵ-robustness at x is enforced if $\bar{b}(x, \epsilon) \subseteq \tilde{b}(x, \delta)$.

Definition 2.5. Define the set $\mathcal{A} = \left\{\theta \in \Delta^m : \arg\max_i \theta^i = \arg\max_i f^i(x)\right\}$ (Figure 1). For simplicity, assume that $f(x)$ is not on the "boundary" of $\mathcal{A}$, such that $\arg\max_i f^i(x)$ is well defined.

Definition 2.6 (Geodesic ball of the FIM)**.** Let $\delta > 0$ be the Fisher-Rao distance between $f(x)$ and $\Delta^m \setminus \mathcal{A}$ (Figure 2).
Define the geodesic ball centered at $f(x) \in \Delta^m$ with radius δ by:

$$b(f(x), \delta) = \{\theta \in \Delta^m : d(f(x), \theta) \leq \delta\}.$$

In section 8, we propose a efficient approximation of δ.

Definition 2.7 (Pullback metric)**.** On $\mathbb{R}^d$, define the *pullback metric* $\tilde{g}$ of g by f. In the standard coordinates of $\mathbb{R}^d$, $\tilde{g}$ is defined for all tangent vectors $v, w \in T_x\mathbb{R}^d$ by:

$$\tilde{g}_x(v, w) = v^T J_x^T G_{f(x)} J_x w,$$

where J_x is the Jacobian matrix of f at x (in the standard coordinates of $\mathbb{R}^d$ and $\mathbb{R}^c$). Define the matrix of $\tilde{g}_x$ is the standard coordinates of $\mathbb{R}^d$ by:

$$\widetilde{G}_x = J_x^T G_{f(x)} J_x. \tag{2}$$

Definition 2.8 (Geodesic ball of the pullback metric)**.** Let $\tilde{d}$ be the distance induced by the pullback metric $\tilde{g}$ on $\mathbb{R}^d$. We can define the geodesic ball centered at x with radius δ by:

$$\tilde{b}(x, \delta) = \left\{z \in \mathbb{R}^d : \tilde{d}(x, z) \leq \delta\right\}.$$

3 Robustness condition

Definition 3.1 (Robustness)**.** We say that f is ϵ-robust at x if:

$$\forall z \in \mathbb{R}^d, \|z - x\|_2 < \epsilon \Rightarrow f(z) \in \mathcal{A}. \tag{3}$$

Equivalently, we can write (Figure 1):

$$f(\bar{b}(x, \epsilon)) \subseteq \mathcal{A}. \tag{4}$$

Proposition 3.2 (Sufficient condition for robustness)**.** *If $\bar{b}(x, \epsilon) \subseteq \tilde{b}(x, \delta)$, then f is ϵ-robust at x (Figure 2).*

Our goal is to start from Proposition 3.2 and make several assumptions in order to derive a condition that can be efficiently implemented.

Working with geodesic balls $\bar{b}(x, \eta)$ and $\tilde{b}(x, \delta)$ is intractable, so our first assumption consists in using an "infinitesimal" condition by restating Proposition 3.2 in the tangent space $T_x\mathbb{R}^d$ instead of working directly on $\mathbb{R}^d$.

Definition 3.3. In $T_x\mathbb{R}^d$, define the Euclidean ball of radius ϵ by:

$$\overline{\mathcal{B}}_x(0,\epsilon) = \left\{ v \in T_x\mathbb{R}^d : \overline{g}_x(v,v) = v^T v \leq \epsilon^2 \right\}.$$

Definition 3.4. In $T_x\mathbb{R}^d$, define the $\tilde{g}_x$-ball of radius δ by:

$$\widetilde{\mathcal{B}}_x(0,\delta) = \left\{ v \in T_x\mathbb{R}^d : \tilde{g}_x(v,v) = v^T \tilde{G}_x v \leq \delta^2 \right\}.$$

Assumption 1. We replace Proposition 3.2 by:

$$\overline{\mathcal{B}}_x(0,\epsilon) \subseteq \widetilde{\mathcal{B}}_x(0,\delta). \tag{5}$$

For small enough δ, Equation (5) implies ϵ-robustness at x. However, contrary to Proposition 3.2, Equation (5) does not offer any guarantee on the ϵ-robustness at x for arbitrary δ.

Proposition 3.5. *Equation (5) is equivalent to:*

$$\forall v \in T_x\mathbb{R}^d, \quad \tilde{g}_x(v,v) \leq \frac{\delta^2}{\epsilon^2}\overline{g}_x(v,v). \tag{6}$$

Since $m < d$, the Jacobian matrix J_x has rank smaller or equal to m. Thus, since $G_{f(x)}$ has full rank, $\widetilde{G}_x = J_x^T G_{f(x)} J_x$ has rank at most m (when J_x has rank m).

Assumption 2. The Jacobian matrix J_x has full rank equal to m.

4 Isometry condition

In order to simplify the notations, we replace:

- J_x by $J \in \mathcal{M}_{m,d}(\mathbb{R})$,
- $G_{f(x)}$ by $G \in \mathcal{S}_m^+(\mathbb{R})$,
- $\widetilde{G}_x$ by $\widetilde{G} \in \mathcal{S}_d(\mathbb{R})$,

We define $D = (\ker(\widetilde{G}))^\perp$. We will use the two following facts.

Fact 4.1.

$$D = \mathrm{rg}(J^T) = (\ker(J))^\perp = \left(\ker(J^T G J)\right)^\perp$$

Fact 4.2. $J^T G J$ is symmetric positive semidefinite. Thus, by the spectral theorem, the eigenvectors associated to its nonzero eigenvalues are all in $D = \mathrm{rg}(J^T)$.
In particular, since $\mathrm{rk}(J) = m$, there exists an orthonormal basis of $T_x\mathbb{R}^d$, denoted $\mathcal{B} = (e_1, \ldots, e_m, e_{m+1}, \ldots, e_d)$, such that each e_i is an eigenvector of $J^T G J$ and such that $(e_1, \ldots, e_m)$ is a basis of $D = \mathrm{rg}(J^T)$ and $(e_{m+1}, \ldots, e_d)$ is a basis of $\ker(J)$.

The set $D = \mathrm{rg}(J^T)$ is a m-dimensional subspace of $T_x\mathbb{R}^d$. $\tilde{g}_x$ does not define an inner product[1] on $T_x\mathbb{R}^d$ because $\widetilde{G}$ has a nontrivial kernel of dimension $d - m$. However, when restricted to D, $\tilde{g}_x|_D$ defines an inner product.

Definition 4.3. We define the restriction of $\widetilde{\mathcal{B}}_x(0,\delta)$ to D:

$$\widetilde{\mathcal{B}}_D(0,\delta) = \left\{ v \in D : v^T \widetilde{G} v \leq \delta \right\}$$

Definition 4.4. We define the restriction of $\overline{\mathcal{B}}_x(0,\epsilon)$ to D:

$$\overline{\mathcal{B}}_D(0,\epsilon) = \left\{ v \in D : v^T v \leq \epsilon^2 \right\}.$$

[1]In particular, the set $\widetilde{\mathcal{B}}_x(0,\delta)$ is not bounded, i.e., it is a cylinder rather than a ball.

Assumption 3. We replace Equation (5) with:

$$\overline{\mathcal{B}}_D(0,\epsilon) = \widetilde{\mathcal{B}}_D(0,\delta). \tag{7}$$

Equation 7 is the limit case of Equation 5, in the sense that if Equation 7 holds, then $\widetilde{\mathcal{B}}_x(0,\delta)$ is the smallest possible $\tilde{g}_x$-ball (for the inclusion) such that Equation 5 holds.

Proposition 4.5. *Equation (7) is equivalent to:*

$$\forall v \in D, \quad \tilde{g}_x(v,v) = \frac{\delta^2}{\epsilon^2}\overline{g}_x(v,v). \tag{8}$$

We can rewrite Equation (8) in a matrix form:

$$\forall v \in D, \quad v^T \widetilde{G} v = \frac{\delta^2}{\epsilon^2} v^T v. \tag{9}$$

In section 7, we show how to exploit the properties of the FIM to derive a closed-form expression for a matrix $P \in \mathrm{GL}_m(\mathbb{R})$ such that $G = P^T P$. For now, we assume that we can easily access such a P and we are looking for a condition on P and J that is equivalent with Equation 9.

Proposition 4.6. *The following statements are equivalent:*

$$\begin{aligned} &(i) \qquad \forall u \in D, \quad u^T J^T G J u = \frac{\delta^2}{\epsilon^2} u^T u, \\ &(ii) \qquad P J J^T P^T = \frac{\delta^2}{\epsilon^2} I_m, \end{aligned}$$

where I_m is the identity matrix of dimension $m \times m$.

Finally, we can define a regularization term:

$$\alpha(x,\epsilon,f) = \frac{1}{m^2} \left|\left|\left| P J J^T P^T - \frac{\delta^2}{\epsilon^2} I_m \right|\right|\right|, \tag{10}$$

where $|||\cdot|||$ is any matrix norm, such as the Frobenius norm or the spectral norm. The loss function is:

$$L(y,x,\epsilon,f) = l(y,f(x)) + \lambda\,\alpha(x,\epsilon,f), \tag{11}$$

where l is the cross-entropy loss and $\lambda > 0$.

5 Randomized isometry condition

The backpropagation algorithm is applied by computing the gradient $\nabla_x f(x)^T v$ where v is a vector of $\mathbb{R}^m$. Thus, in order to compute the Jacobian matrix J, we need to apply the backpropagation algorithm m times, by computing each row $\nabla_x f(x)^T e_i$, where $e_i = (\delta_{ij})_{j=1,\ldots,m}$ is the canonical basis of $\mathbb{R}^m$.

To reduce the computational cost, we propose a randomized version of the isometry condition. Let u and v be two vectors sampled uniformly on the sphere of radius 1 in $\mathbb{R}^m$. Consider the bilinear form:

$$(u,v) \mapsto F(u,v) = \left(1 - \sqrt{f^c(x)}\right)^2 \nabla((\tau \circ f(x))^T u)\nabla((\tau \circ f(x))^T v)) - \frac{\delta^2}{\epsilon^2} u^T v.$$

Then, we can use $F(u,v)$ as a regularization term.

6 Jacobian bound condition

Let us use the same notations simplifications introduced at the beginning of section 4. Let $P \in \mathrm{GL}_m(\mathbb{R})$ be as in Proposition 7.3.

Proposition 6.1. *Let λ_{max} be the largest eigenvalue of PJJ^TP^T. The following statements are equivalent:*

$$
\begin{aligned}
&(i) \qquad \overline{\mathcal{B}}_x(0,\epsilon) \subseteq \widetilde{\mathcal{B}}_x(0,\delta),\\
&(ii) \qquad \lambda_{max} \leq \frac{\delta^2}{\epsilon^2},
\end{aligned}
$$

We use Hölder's inequality to upper-bound the spectral norm: $\|PJ\|_2^2 \leq \|PJ\|_1 \|PJ\|_\infty$ with $\|M\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^m |M_{ij}|$ and $\|M\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |M_{ij}|$. Now, we can define a regularization term:

$$
\alpha(x,\epsilon,f) = \max\left\{\sqrt{\|PJ\|_1 \|PJ\|_\infty} - \frac{\delta}{\epsilon}, 0\right\}. \tag{12}
$$

7 Coordinate change

In this section, we show how to compute the matrix P introduced for Corollary 4.6. To this end, we isometrically embed Δ^m into the Euclidean space $\mathbb{R}^c$ using the following inclusion map:

$$
\begin{aligned}
\mu : \Delta^m &\longrightarrow \mathbb{R}^c\\
\left(\theta^1,\ldots,\theta^m\right) &\longmapsto 2\left(\sqrt{\theta^1},\ldots,\sqrt{\theta^m},\sqrt{1-\sum_{i=1}^m \theta^i}\right)
\end{aligned}
$$

We can easily see that μ is an embedding. If $\mathcal{S}^m(2)$ is the sphere of radius 2 centered at the origin in $\mathbb{R}^c$, then $\mu\left(\Delta^m\right)$ is the subset of $\mathcal{S}^m(2)$ where all coordinates are strictly positive (using the standard coordinates of $\mathbb{R}^c$).

Proposition 7.1. *Let g be the Fisher information metric on Δ^m (Definition 2.2), and $\overline{g}$ be the Euclidean metric on $\mathbb{R}^c$. Then μ is an isometric embedding of (Δ^m, g) into $(\mathbb{R}^c, \overline{g})$.*

Now, we use the stereographic projection to embed Δ^m into $\mathbb{R}^m$:

$$
\begin{aligned}
\tau : \mu(\Delta^m) &\longrightarrow \mathbb{R}^m\\
\left(\mu^1,\ldots,\mu^m,\mu^c\right) &\longmapsto 2\left(\frac{\mu^1}{2-\mu^c},\ldots,\frac{\mu^m}{2-\mu^c}\right),
\end{aligned}
$$

with $\mu^c = 2\sqrt{1-\sum_{i=1}^m \theta^i}$.

Proposition 7.2. *In the coordinates τ, the FIM is:*

$$
G_{\tau,ij} = \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\delta_{ij}. \tag{13}
$$

Let $\widetilde{J}$ be the Jacobian matrix of $\tau \circ \mu : \Delta^m \to \mathbb{R}^m$ at $f(x)$. Then we have:

$$
G = \widetilde{J}^T G_\tau \widetilde{J} = \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\widetilde{J}^T\widetilde{J}.
$$

Thus, we can choose:

$$
P = \frac{2}{1+\|\tau/2\|^2}\widetilde{J}. \tag{14}
$$

Write $f(x) = \theta = (\theta^1,\ldots,\theta^m)$ and $\theta_c = 1-\sum_{i=1}^m \theta^i$. For simplicity, write $\tau^i(\theta) = \tau^i(\mu(\theta)) = 2\sqrt{\theta^i}/(1-\sqrt{\theta^c})$ for $i = 1,\ldots,m$. More explicitly, we have:

Proposition 7.3. *For $i,j = 1,\ldots,m$:*

$$
P_{ij} = \frac{\delta_{ij}}{\sqrt{\theta^i}} - \frac{\tau^i(\theta)}{2\sqrt{\theta^c}}. \tag{15}
$$

8 The Fisher-Rao distance

As stated in Proposition 7.1, the probability simplex Δ^m endowed with the FIM can be isometrically embedded into the m-sphere of radius 2. Thus, the angle β between two distributions of coordinates θ_1 and θ_2 in Δ^m with $\mu_1 = \mu(\theta_1)$ and $\mu_2 = \mu(\theta_2)$ is:

$$\cos(\beta) = \frac{1}{4}\sum_{i=1}^{c} \mu_1^i \mu_2^i = \sum_{i=1}^{c} \sqrt{\theta_1^i \theta_2^i}.$$

The Riemannian distance between these two points is the arc length on the sphere:

$$d(\theta_1, \theta_2) = 2 \arccos \sum_{i=1}^{c} \sqrt{\theta_1^i \theta_2^i}.$$

In the regularization terms, we replace δ by the following upper bound:

$$\delta = d\left(f(x), \Delta^m \setminus \mathcal{A}\right) \leq d\left(f(x), O\right),$$

where $O = \frac{1}{c}(1, \ldots, 1)$ is the center of the simplex Δ^m. Thus:

$$\delta \leq 2 \arccos \sum_{i=1}^{c} \sqrt{\frac{f(x)^i}{c}}. \tag{16}$$

9 Proofs

Proof of Proposition 3.5. (6) $\Rightarrow$ (5]). Assume (6). Let $v \in \overline{\mathcal{B}}_x(0, \epsilon)$. Thus $\overline{g}_x(v, v) \leq \epsilon^2$. We have:

$$\tilde{g}_x(v, v) \leq \frac{\delta^2}{\epsilon^2} \overline{g}_x(v, v) \leq \frac{\delta^2}{\epsilon^2}\epsilon^2 = \delta^2.$$

Thus $v \in \widetilde{\mathcal{B}}_x(0, \delta)$.
(5) $\Rightarrow$ (6]). Assume (5). Let $v \in T_x\mathbb{R}^d$. Define $w = \epsilon\, v/\sqrt{\overline{g}_x(v, v)}$. Then $\overline{g}_x(w, w) = \epsilon^2$. Thus, $w \in \overline{\mathcal{B}}_x(0, \epsilon)$. Hence, $w \in \widetilde{\mathcal{B}}_x(0, \delta)$. Thus, $\tilde{g}_x(w, w) < \delta^2$. Finally, we have:

$$\tilde{g}_x(w, w) = \frac{\epsilon^2}{\overline{g}_x(v, v)} \tilde{g}_x(v, v) < \delta^2.$$

We obtain Equation (6) by multiplying by $\overline{g}_x(v, v)/\epsilon^2$. □

Proof of Fact 4.1. We prove the third equality (the second equality is a well-known fact of linear algebra).
Let $u \in \ker J$. Then $J^T G J u = 0$, thus $u \in \ker(J^T G J)$. Hence $\left(\ker(J^T G J)\right)^\perp \subseteq (\ker(J))^\perp$.
Let $v \in \ker J^T G J$. Since G is symmetric positive-definite, the function $w \mapsto N(w) = \sqrt{w^T G w}$ is a norm. We have $0 = v^T J^T G J v = N(Jv)^2$. The positive-definiteness of the norm N implies $Jv = 0$. Thus, $v \in \ker J$. Hence $(\ker(J))^\perp \subseteq \left(\ker(J^T G J)\right)^\perp$. □

Proof of Proposition 4.5. The implication (8) $\Rightarrow$ (7]) is immediate (by double inclusion).
Now, assume (7]) holds. Let $v \in D$. Define $w_1 = \epsilon\, v/\sqrt{\overline{g}_x(v, v)}$ and $w_2 = \epsilon\, v/\sqrt{\tilde{g}_x(v, v)}$. Then, with a similar argument as in the proof of Proposition 3.5, we can obtain Equation (8). Note that w_2 is well defined because $v \notin \ker(J)$. □

Proof of Proposition 4.6. **Let us first introduce the polar decomposition**.
Let A be a $m \times d$ matrix.
Define the absolute value[2] of A by $|A| = (A^T A)^{\frac{1}{2}}$.
Define the linear map $u : \mathrm{rg}(|A|) \to \mathrm{rg}(A)$ by $u(|A|x) = Ax$ for any $x \in \mathbb{R}^d$.

[2]The square root of $A^T A$ is well defined because it is a positive semidefinite matrix.

Using the fact that $|A|$ is symmetric, we have that $\|Ax\|^2 = x^T A^T Ax = (A^T Ax)^T x = (|A|^2 x)^T x = x^T |A|^T |A| x = \||A|x\|^2$, thus u is an isometry[3].
Let U be the matrix associated to u in the canonical basis.
We now prove the main result.
Let $A = PJ$. Using the polar decomposition, we have

$$PJ = U|PJ|,$$

where U is an isometry from $\mathrm{rg}(|PJ|) = (\ker|PJ|)^\perp = (\ker(PJ))^\perp = (\ker(J))^\perp = D$ to $\mathrm{rg}(PJ) = \mathbb{R}^m$ (using our assumption that $\mathrm{rk}(J) = m$). Transposing this relation, we obtain:

$$J^T P^T = |PJ| U^T.$$

Hence, by multiplying both relations, we have:

$$PJJ^T P^T = U|PJ|^2 U^T = UJ^T P^T PJU^T$$

Assume that (ii) holds, i.e., $PJJ^T P = I_m$. Then:

$$J^T GJ = J^T P^T PJ = U^T PJJ^T P^T U = U^T U.$$

Since U is an isometry from D to $\mathbb{R}^m$, then $U^T U$ is the projection onto D, denoted Π_D. Thus, we have $J^T GJ = \Pi_D$ which is (i).
Now, assume that (i) holds, i.e., $J^T P^T PJ = \Pi_D$ where Π_D is the projection onto D. We have:

$$PJJ^T P^T = UJ^T P^T PJU^T = U\Pi_D U^T.$$

Since $\mathrm{rg}(U^T) = D$, then $\Pi_D U^T = U^T$. Since U is an isometry from D to $\mathbb{R}^m$, then $UU^T = I_m$. Thus, $PJJ^T P^T = I_m$ which is (ii). $\square$

Proof of Proposition 7.1. We need to show that $\mu^* \overline{g} = g$. Using the coordinates θ on Δ^m (Definition 2.1) and the standard coordinates on $\mathbb{R}^c$, and writing $f(x) = \theta_0 = (\theta_0^1, \ldots, \theta_0^m)$ we have:

$$G_{ij} = G_{\theta_0,ij} = \sum_{\alpha=1}^{c} \sum_{\beta=1}^{c} \frac{\partial \mu^\alpha(\theta_0)}{\partial \theta^i} \frac{\partial \mu^\beta(\theta_0)}{\partial \theta^j} \delta_{\alpha\beta} = \sum_{\alpha=1}^{c} \frac{\partial \mu^\alpha(\theta_0)}{\partial \theta^i} \frac{\partial \mu^\alpha(\theta_0)}{\partial \theta^j}.$$

For $i = 1, \ldots, m$ and $\alpha = 1, \ldots, m$ we have:

$$\frac{\partial \mu^\alpha(\theta_0)}{\partial \theta^i} = \frac{\delta_{i\alpha}}{\sqrt{\theta_0^i}},$$

and for $\alpha = c$:

$$\frac{\partial \mu^c(\theta_0)}{\partial \theta^i} = -\frac{1}{\sqrt{\theta_0^c}},$$

with $\theta_0^c = \sqrt{1 - \sum_{i=1}^m \theta_0^i}$. Thus:

$$G_{\theta,ij} = \frac{\delta_{ij}}{\theta_0^i} + \frac{1}{\theta_0^c},$$

which is the FIM as defined in Definition 2.2. $\square$

Proof of Proposition 7.2. For $i = 1, \ldots, m$, the inverse transformation of $\tau(\mu)$ is (proof below):

$$\mu^i(\tau) = \frac{2\tau^i}{1 + \|\tau/2\|^2}, \tag{17}$$

[3] We can arbitrarily extend u on the entire $\mathbb{R}^d$, e.g., by setting $\ker(u) = \ker(|A|)$.

and:

$$\mu^c(\tau) = 2\frac{\|\tau/2\|^2 - 1}{\|\tau/2\|^2 + 1}. \tag{18}$$

Moreover, according to Proposition 7.1, the FIM in the coordinates $(\mu^1, \ldots, \mu^m)$ is the metric induced on $\mu(\Delta^m)$ by the identity matrix (i.e., the Euclidean metric) of $\mathbb{R}^c$. Hence, we have:

$$G_{\tau,ij} = \sum_{\alpha=1}^{c}\sum_{\beta=1}^{c} \frac{\partial \mu^\alpha(\tau)}{\partial \tau^i}\frac{\partial \mu^\beta(\tau)}{\partial \tau^j}\delta_{\alpha\beta} = \sum_{\alpha=1}^{c} \frac{\partial \mu^\alpha(\tau)}{\partial \tau^i}\frac{\partial \mu^\alpha(\tau)}{\partial \tau^j}.$$

For $i = 1, \ldots, m$ and $\alpha = 1, \ldots, m$ we have:

$$\frac{\partial \mu^\alpha(\tau)}{\partial \tau^i} = \frac{2}{1 + \|\tau/2\|^2}\left(\delta_{i\alpha} - \frac{\tau^\alpha \tau^i}{2\left(1 + \|\tau/2\|^2\right)}\right),$$

and for $\alpha = c$:

$$\frac{\partial \mu^c(\tau)}{\partial \tau^i} = \frac{2\tau^i}{\left(1 + \|\tau/2\|^2\right)^2},$$

Thus:

$$\begin{aligned}
G_{\tau,ij} &= \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\left(\sum_{\alpha=1}^{m}\left\{\delta_{i\alpha}\delta_{j\alpha} - \frac{\delta_{i\alpha}\tau^j\tau^\alpha}{2\left(1+\|\tau/2\|^2\right)} - \frac{\delta_{j\alpha}\tau^i\tau^\alpha}{2\left(1+\|\tau/2\|^2\right)} + \frac{\tau^i\tau^j\left(\tau^\alpha\right)^2}{4\left(1+\|\tau/2\|^2\right)^2}\right\} + \frac{\tau^i\tau^j}{\left(1+\|\tau/2\|^2\right)^2}\right) \\
&= \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\left(\delta_{ij} - \frac{\tau^i\tau^j}{1+\|\tau/2\|^2} + \frac{\tau^i\tau^j\|\tau/2\|^2}{\left(1+\|\tau/2\|^2\right)^2} + \frac{\tau^i\tau^j}{\left(1+\|\tau/2\|^2\right)^2}\right) \\
&= \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\left(\delta_{ij} - \frac{\tau^i\tau^j}{1+\|\tau/2\|^2} + \frac{\tau^i\tau^j}{1+\|\tau/2\|^2}\right) \\
&= \frac{4}{\left(1+\|\tau/2\|^2\right)^2}\delta_{ij}
\end{aligned}$$

□

Proof of Equations 17 and 18. We have $\tau^i(\mu) = \lambda\mu^i$ with $\lambda = 2/(2-\mu^c)$. Let us express μ^c as a function of τ. We have:

$$\|\tau\|^2 = \sum_{i=1}^{m}(\tau^i)^2 = \lambda^2\|\mu\|^2.$$

Since μ belongs to the sphere of radius 2, we have $\|\mu\|^2 + (\mu^c)^2 = 4$. Thus:

$$\|\tau\|^2 = \lambda^2\left(4 - (\mu^c)^2\right) = 4\frac{4-(\mu^c)^2}{(2-\mu^c)^2} = 4\frac{2+\mu^c}{2-\mu^c}.$$

Isolating μ^c, we get:

$$\mu^c(\tau) = \frac{2\|\tau\|^2 - 8}{\|\tau\|^2 + 4} = 2\frac{\|\tau/2\|^2 - 1}{\|\tau/2\|^2 + 1}.$$

Now, we can replace μ^c into the expression of λ. We obtain $\lambda = \left(1 + \|\tau/2\|^2\right)/2$, and thus:

$$\mu^i(\tau) = \frac{\tau^i}{\lambda} = \frac{2\tau^i}{1 + \|\tau/2\|^2}$$

□

Proof of Proposition 7.3. We have $\tau^i(\theta) = 2\sqrt{\theta^i}/\left(1-\sqrt{\theta^c}\right)$. Thus:

$$\left\|\frac{\tau(\theta)}{2}\right\|^2 = \sum_{i=1}^{m} \frac{\tau^i(\theta)^2}{4} = \frac{\sum_{i=1}^{m}\theta^i}{\left(1-\sqrt{\theta^c}\right)^2} = \frac{1-\theta^c}{\left(1-\sqrt{\theta^c}\right)^2} = \frac{1+\sqrt{\theta^c}}{1-\sqrt{\theta^c}}.$$

Hence, for any $i = 1, \ldots, m$:

$$\frac{2}{1+\|\tau(\theta)/2\|^2} = 1-\sqrt{\theta^c} = \frac{2\sqrt{\theta^i}}{\tau^i(\theta)}. \tag{19}$$

Now, we compute $\widetilde{J}$. Let i and j in $\{1, \ldots, m\}$:

$$\frac{\partial \tau^i(\theta)}{\partial \theta^j} = \frac{\delta_{ij}}{\sqrt{\theta^i}\left(1-\sqrt{\theta^c}\right)} - \frac{\sqrt{\theta^i}}{\sqrt{\theta^c}\left(1-\sqrt{\theta^c}\right)^2} = \frac{\tau^i(\theta)}{2}\left(\frac{\delta_{ij}}{\theta^i} - \frac{\tau^i(\theta)}{2\sqrt{\theta^i\theta^c}}\right) \tag{20}$$

Replacing Equations 19 and 20 into Equation 14 yields the result. □

Proof of Proposition 6.1. TBD □

References

[1] O. Calin and C. Udrişte, *Geometric Modeling in Probability and Statistics*. Springer International Publishing, 2014.