

Transformée de Fourier

Soit:

$$s(t) = \left(\exp \left\{ -\frac{\alpha}{2} (t - t_k)^2 \right\} + \exp \left\{ -\frac{\alpha}{2} (t - \Delta_t - t_k)^2 \right\} \right)^2,$$

et soit $S(f)$ sa transformée de Fourier.

On développe le carré:

$$s(t) = \exp(-\alpha(t - t_k)^2) + \exp(-\alpha(t - \Delta_t - t_k)^2) + 2 \exp \left(-\frac{\alpha}{2} \{ (t - t_k)^2 + (t - \Delta_t - t_k)^2 \} \right).$$

Soit $a(t) = -\frac{\alpha}{2} \{ (t - t_k)^2 + (t - \Delta_t - t_k)^2 \}$. On a:

$$\begin{aligned} a(t) &= -\frac{\alpha}{2} (t^2 + t_k^2 - 2tt_k + (t - \Delta_t)^2 + t_k^2 - 2t_k(t - \Delta_t)), \\ &= -\frac{\alpha}{2} (t^2 + t_k^2 - 2tt_k + t^2 + \Delta_t^2 - 2t\Delta_t + t_k^2 - 2t_k t + 2t_k \Delta_t), \\ &= -\frac{\alpha}{2} (2t^2 + 2t_k^2 - 4tt_k + \Delta_t^2 + 2\Delta_t(t_k - t)), \\ &= -\alpha \left(t^2 + t_k^2 - 2tt_k + \frac{\Delta_t^2}{2} + \Delta_t(t_k - t) \right), \\ &= -\alpha \left((t - t_k)^2 + \Delta_t(t_k - t) + \frac{\Delta_t^2}{2} \right), \\ &= -\alpha \left(\left(t - t_k - \frac{\Delta_t}{2} \right)^2 + \frac{\Delta_t^2}{4} \right). \end{aligned}$$

On a :

$$\text{TF}\{\exp(-\alpha(t - t_k)^2)\} = \exp(-j2\pi f t_k) \text{TF}\{\exp(-\alpha t^2)\} = \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 - j2\pi f t_k \right).$$

De même:

$$\text{TF}\{\exp(-\alpha(t - \Delta_t - t_k)^2)\} = \exp(-j2\pi f(\Delta_t + t_k)) \text{TF}\{\exp(-\alpha t^2)\} = \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 - j2\pi f(\Delta_t + t_k) \right).$$

Et enfin:

$$\text{TF}\{2 \exp(a(t))\} = 2 \exp \left(-j2\pi f \left(t_k + \frac{\Delta_t}{2} \right) - \alpha \frac{\Delta_t^2}{4} \right) \text{TF}\{\exp(-\alpha t^2)\} = 2 \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 - j2\pi f \left(t_k + \frac{\Delta_t}{2} \right) - \alpha \frac{\Delta_t^2}{4} \right)$$

Finalement:

$$\begin{aligned} S(f) &= \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 - j2\pi f t_k \right) \left\{ 1 + \exp(-j2\pi f \Delta_t) + 2 \exp \left(-j\pi f \Delta_t - \alpha \frac{\Delta_t^2}{4} \right) \right\}, \\ &= \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 - j2\pi f t_k \right) \left\{ \exp(-j\pi f \Delta_t) \left(\exp(j\pi f \Delta_t) + \exp(-j\pi f \Delta_t) + 2 \exp \left(-\alpha \frac{\Delta_t^2}{4} \right) \right) \right\}, \\ &= 2 \sqrt{\frac{\pi}{\alpha}} \exp \left(-\frac{\pi}{\alpha} f^2 \right) \left\{ \cos(\pi f \Delta_t) + \exp \left(-\alpha \frac{\Delta_t^2}{4} \right) \right\} \exp \left(-j2\pi f \left(t_k + \frac{\Delta_t}{2} \right) \right). \end{aligned}$$