Transformée de Fourier

On cherche la transformée de Fourier de :

$$m_{\Delta_\tau}(t) = \left(e^{-\frac{\alpha}{2}t^2} + e^{-\frac{\alpha}{2}(t-\Delta_t)^2}\right)\left(e^{-\frac{\alpha}{2}(t-\Delta_\tau)^2} + e^{-\frac{\alpha}{2}(t-\Delta_\tau-\Delta_t)^2}\right).$$

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Pour simplifier les notations, on fait les substitutions suivantes :

$$\begin{aligned}\frac{\alpha}{2} &\longleftrightarrow \alpha,\\ \Delta_t &\longleftrightarrow \Delta,\\ \Delta_\tau &\longleftrightarrow \tau.\end{aligned}$$

On cherche maintenant la transformée de Fourier de $m(t)$, notée $M(f)$, avec :

$$m(t) = \left(e^{-\alpha t^2} + e^{-\alpha(t-\Delta)^2}\right)\left(e^{-\alpha(t-\tau)^2} + e^{-\alpha(t-\tau-\Delta)^2}\right).$$

En utilisant les propriétés de la transformée de Fourier ainsi que les transformées de Fourier usuelles, on obtient :

$$M(f) = \frac{\pi}{\alpha}\left[e^{-\frac{\pi f^2}{\alpha}}\left(1 + e^{-j2\pi f\Delta}\right)\right] * \left[e^{-\frac{\pi f^2}{\alpha}}\left(e^{-j2\pi f\tau} + e^{-j2\pi f(\tau+\Delta)}\right)\right]$$

On doit calculer quatre produits de convolution. On va traiter le cas général, avec a, b et c des réels positifs :

$$\begin{aligned}e^{-af^2-jbf} * e^{-af^2-jcf} &= \int_{-\infty}^{\infty} e^{-au^2-jbu}e^{-a(f-u)^2-jc(f-u)du},\\ &= e^{-af^2-jcf}\int_{-\infty}^{\infty} e^{-2au^2+u(2fa+j(c-b))du},\\ &= \exp\left\{-af^2 - jcf\right\}\int_{-\infty}^{\infty}\exp\left\{-2a\left(u - \frac{2fa+j(c-b)}{4a}\right)^2 + \frac{af^2}{2} - \frac{(c-b)^2}{8a} + j\frac{f(c-b)}{2}\right\}du,\\ &= \exp\left\{-\frac{af^2}{2} - \frac{(b-c)^2}{8a} - jf\frac{b+c}{2}\right\}\int_{-\infty}^{\infty}\exp\left\{-2a\left(u - \frac{2fa+j(c-b)}{4a}\right)^2\right\}du,\end{aligned}$$

En faisant le rectangle bien relou dans le plan complexe (cf TD traitement du signal), on peut montrer que :

$$e^{-af^2-jbf} * e^{-af^2-jcf} = \exp\left\{-\frac{af^2}{2} - \frac{(b-c)^2}{8a} - jf\frac{b+c}{2}\right\}\int_{-\infty}^{\infty}\exp\left\{-2au^2\right\}du,$$

Puis en utilisant l'intégrale de Gauß (à lire avec l'élocution bourgeoise de Øljen) :

$$e^{-af^2-jbf} * e^{-af^2-jcf} = \sqrt{\frac{\pi}{2a}}\exp\left\{-\frac{af^2}{2} - \frac{(b-c)^2}{8a} - jf\frac{b+c}{2}\right\}.$$

On applique ce résultat avec $a = \pi/\alpha$, $b \in \{0, 2\pi\Delta\}$, et $c \in \{2\pi\tau, 2\pi(\tau+\Delta)\}$. On obtient :

$$\begin{aligned} M(f) = \frac{\pi}{\sqrt{2\alpha}} \Bigg(& \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha\tau^2}{2} - j\pi f\tau\right\} \\ & + \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha(\tau+\Delta)^2}{2} - j\pi f(\tau+\Delta)\right\} \\ & + \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha(\tau-\Delta)^2}{2} - j\pi f(\tau+\Delta)\right\} \\ & + \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha\tau^2}{2} - j\pi f(\tau+2\Delta)\right\}\Bigg). \end{aligned}$$

On simplifie un peu :

$$\begin{aligned} M(f) = \frac{\pi}{\sqrt{2\alpha}} \Bigg(& \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha\tau^2}{2} - j\pi f\tau\right\} (1 + \exp\{-j2\pi f\Delta\}) \\ & + \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha(\tau^2+\Delta^2)}{2} - j\pi f(\tau+\Delta)\right\} (\exp\{-\pi\alpha\tau\Delta\} + \exp\{\pi\alpha\tau\Delta\})\Bigg) \end{aligned}$$

On simplifie encore plus :

$$M(f) = \pi\sqrt{\frac{2}{\alpha}} \exp\left\{-\frac{\pi f^2}{2\alpha} - \frac{\pi\alpha\tau^2}{2} - j\pi f(\tau+\Delta)\right\} \left(\cos(\pi f\Delta) + \exp\left\{-\frac{\pi\alpha\Delta^2}{2}\right\} \mathrm{ch}(\pi\alpha\tau\Delta)\right)$$

On défait les substitutions :

$$\begin{aligned} \frac{\alpha}{2} &\longleftrightarrow \alpha, \\ \Delta_t &\longleftrightarrow \Delta, \\ \Delta_\tau &\longleftrightarrow \tau, \end{aligned}$$

et on obtient :

$$M_{\Delta_\tau}(f) = \frac{2\pi}{\sqrt{\alpha}} \exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\pi\alpha\Delta_\tau^2}{4} - j\pi f(\Delta_\tau + \Delta_t)\right\} \left(\cos(\pi f\Delta_t) + \exp\left\{-\frac{\pi\alpha\Delta_t^2}{4}\right\} \mathrm{ch}\left(\frac{\pi\alpha\Delta_\tau\Delta_t}{2}\right)\right)$$

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$$\begin{aligned} m_{\Delta_\tau}(t) &= \left(\exp\left\{-\frac{\alpha}{2}t^2\right\} + \exp\left\{-\frac{\alpha}{2}(t-\Delta_t)^2\right\}\right)\left(\exp\left\{-\frac{\alpha}{2}(t-\Delta_\tau)^2\right\} + \exp\left\{-\frac{\alpha}{2}(t-\Delta_\tau-\Delta_t)^2\right\}\right), \\ &= \Bigg(\exp\left\{-\frac{\alpha}{2}\left(2t^2 + \Delta_\tau^2 - 2t\Delta_\tau\right)\right\} + \exp\left\{-\frac{\alpha}{2}\left(2t^2 + (\Delta_\tau+\Delta_t)^2 - 2t(\Delta_\tau+\Delta_t)\right)\right\} \\ &\quad + \exp\left\{-\frac{\alpha}{2}\left(2t^2 + \Delta_t^2 + \Delta_\tau^2 - 2t(\Delta_\tau+\Delta_t)\right)\right\} + \exp\left\{-\frac{\alpha}{2}\left(2t^2 + \Delta_t^2 + (\Delta_\tau+\Delta_t)^2 - 2t(\Delta_\tau+2\Delta_t)\right)\right\}\Bigg), \\ &= \Bigg(\exp\left\{-\alpha\left(t^2 + \frac{\Delta_\tau^2}{2} - t\Delta_\tau\right)\right\} + \exp\left\{-\alpha\left(t^2 + \frac{(\Delta_\tau+\Delta_t)^2}{2} - t(\Delta_\tau+\Delta_t)\right)\right\} \\ &\quad + \exp\left\{-\alpha\left(t^2 + \frac{\Delta_t^2+\Delta_\tau^2}{2} - t(\Delta_\tau+\Delta_t)\right)\right\} + \exp\left\{-\alpha\left(t^2 + \frac{\Delta_t^2+(\Delta_\tau+\Delta_t)^2}{2} - t(\Delta_\tau+2\Delta_t)\right)\right\}\Bigg), \\ &= \Bigg(\exp\left\{-\alpha\left(t - \frac{\Delta_\tau}{2}\right)^2 - \frac{\alpha\Delta_\tau^2}{4}\right\} + \exp\left\{-\alpha\left(t - \frac{\Delta_\tau+\Delta_t}{2}\right)^2 - \frac{\alpha(\Delta_\tau+\Delta_t)^2}{4}\right\} \\ &\quad + \exp\left\{-\alpha\left(t - \frac{\Delta_\tau+\Delta_t}{2}\right)^2 - \frac{\alpha(\Delta_\tau-\Delta_t)^2}{4}\right\} + \exp\left\{-\alpha\left(t - \frac{(\Delta_\tau+2\Delta_t)}{2}\right)^2 - \frac{\alpha\Delta_\tau^2}{4}\right\}\Bigg), \end{aligned}$$

On applique la transformée de Fourier :

$$
\begin{aligned}
M_{\Delta_\tau}(f) =&\sqrt{\frac{\pi}{\alpha}}\Bigg(\exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha\Delta_\tau^2}{4} - j\pi f\Delta_\tau\right\} + \exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha(\Delta_\tau+\Delta_t)^2}{4} - j\pi f(\Delta_\tau+\Delta_t)\right\} \\
&+ \exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha(\Delta_\tau-\Delta_t)^2}{4} - j\pi f(\Delta_\tau+\Delta_t)\right\} + \exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha\Delta_\tau^2}{4} - j\pi f(\Delta_\tau+2\Delta_t)\right\}\Bigg)
\end{aligned}
$$

On simplifie :

$$
\begin{aligned}
M_{\Delta_\tau}(f) =&\sqrt{\frac{\pi}{\alpha}}\exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha\Delta_\tau^2}{4} - j\pi f\Delta_\tau\right\}\Bigg(1 + \exp\left\{-\frac{\alpha(\Delta_t^2+2\Delta_\tau\Delta_t)}{4} - j\pi f\Delta_t\right\} \\
&+ \exp\left\{-\frac{\alpha(\Delta_t^2-2\Delta_\tau\Delta_t)}{4} - j\pi f\Delta_t\right\} + \exp\left\{-j2\pi f\Delta_t\right\}\Bigg).
\end{aligned}
$$

On simplifie encore :

$$
M_{\Delta_\tau}(f) = 2\sqrt{\frac{\pi}{\alpha}}\exp\left\{-\frac{\pi f^2}{\alpha} - \frac{\alpha\Delta_\tau^2}{4} - j\pi f(\Delta_\tau+\Delta_t)\right\}\left(\cos(\pi f\Delta_t) + \exp\left\{-\frac{\alpha\Delta_t^2}{4}\right\}\mathrm{ch}\left(\frac{\alpha\Delta_\tau\Delta_t}{2}\right)\right).
$$