Ordinary Least Squares Estimation

2023148056 Seohyun Lee

Introduction

Ordinary Least Squares (OLS) is a method used to estimate the parameters of a linear regression model by minimizing the sum of squared residuals. In this document, we derive the OLS estimators for both Simple Linear Regression and Multiple Linear Regression.

1. Simple Linear Regression

The model for simple linear regression is given by:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where:

- y_i is the dependent variable,
- x_i is the independent variable,
- β_0 and β_1 are the parameters to be estimated,
- ϵ_i is the error term.

Objective

The objective of OLS is to minimize the sum of squared residuals:

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 .$$

Derivation of the Estimators

1. Take the partial derivatives of $S(\beta_0, \beta_1)$ with respect to β_0 and β_1:

$$\frac{\partial S}{\partial \beta_0} = -2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i),$$
$$\frac{\partial S}{\partial \beta_1} = -2\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i).$$

2. Set the partial derivatives to zero to find the critical points:

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0,$$
$$\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0.$$

3. Solve the system of equations to obtain:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$

where $\bar{x}$ and $\bar{y}$ are the sample means of x and y, respectively.

2. Multiple Linear Regression

The model for multiple linear regression is:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where:

- $\mathbf{y}$ is an $n \times 1$ vector of observations,
- $\mathbf{X}$ is an $n \times p$ matrix of predictors,
- $\boldsymbol{\beta}$ is a $p \times 1$ vector of parameters,
- $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of errors.

Objective

The objective is to minimize the sum of squared residuals:

$$S(\boldsymbol{\beta}) = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^T(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Derivation of the Estimator

1. Expand $S(\boldsymbol{\beta})$:
$$S(\boldsymbol{\beta}) = \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}.$$

2. Take the gradient with respect to $\boldsymbol{\beta}$ and set it to zero:
$$\frac{\partial S}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} = 0.$$

3. Solve for $\boldsymbol{\beta}$:
$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}.$$

Conclusion

In this document, we derived the OLS estimators for both simple and multiple linear regression. The solutions involve minimizing the sum of squared residuals, leading to closed-form solutions for the parameter estimates.