

LYAPUNOV CHARACTERISTIC EXPONENTS FOR SMOOTH DYNAMICAL SYSTEMS AND FOR HAMILTONIAN SYSTEMS; A METHOD FOR COMPUTING ALL OF THEM. PART 2: NUMERICAL APPLICATION

Giancarlo Benettin,* Luigi Galgani,**
Antonio Giorgilli,** Jean-Marie Strelcyn***

SOMMARIO. Questo articolo, insieme con il precedente (Parte 1: Teoria, pubblicato in questa stessa rivista) è inteso a fornire un metodo esplicito per il calcolo di tutti gli esponenti caratteristici di Lyapunov per un sistema dinamico. Dopo la teoria generale su tali esponenti sviluppata nella prima parte, qui si illustra il metodo di calcolo (Capitolo A) e si danno esempi numerici per applicazioni di varietà in sé e per sistemi Hamiltoniani (Capitolo B).

SUMMARY. The present paper, together with the previous one (Part 1: Theory, published in this journal) is intended to give an explicit method for computing all Lyapunov Characteristic Exponents of a dynamical system. After the general theory on such exponents developed in the first part, in the present paper the computational method is described (Chapter A) and some numerical examples for mappings on manifolds and for Hamiltonian systems are given (Chapter B).

1. Introduction.

1.1. Lyapunov Characteristic Exponents (briefly LCEs) are of particular interest in the study of stochasticity properties of dynamical systems, being essentially related to the exponential divergence of nearby orbits; one then often meets with the problem of their explicit numerical computation. The method for computing the maximal one was already given in ref [1] (see also [2] and [3]), while the method for computing all of them, based on the possibility of computing LCEs of order greater than one which are related to the exponential increase of volumes, was announced in the brief note [4], where a numerical application was also given.

In the paper [5], which constitutes the first part of the present work, the general theory of LCEs was recalled and in particular the relation between the exponents of order one and those of greater order, which is at the basis of the numerical method for the computation of all LCEs, was given. Now, in this second part, we illustrate the computational method (Chapter A) and give some numerical examples for

mappings on manifolds and for Hamiltonian systems (Chapter B). Preliminarily, in Sec. 2, some general problems on the reliability of numerical computations are discussed.

1.2. The present Part II is largely independent of Part I, and we hope that it can be readable even without a detailed knowledge of the latter. In order to concentrate on the essential problems, we will consider the case of a diffeomorphism T of a differentiable compact connected manifold M onto itself. Indeed the case of a differentiable flow $\{T^t\}$ essentially reduces to the previous one because, as already remarked in Sec. I.4.4 (1), the LCEs of a flow $\{T^t\}$ are the same as those of the mapping T^1 . However, by the method described below one can also consider differentiable mappings more general than diffeomorphisms (see Sec. I.3.2). Z^+ will denote the nonnegative integers and \ln the natural logarithm.

2. Some problems on numerical studies of stochasticity.

2.1. In the present section for convenience we will use the following simplified notation: having fixed $x \in M$ we denote $x_t = T^t x$ (in particular $x_0 = x$) and $E_t = T_t^* x^* M$.

A first general problem concerns the numerical errors, which in principle could be particularly relevant in our case, because we are interested in computing limits when $t \rightarrow \infty$. Precisely, given a point $x \in M$, the «true» orbit $\{x_t\}_{t \in Z^+}$ is defined; analogously, given a vector $v \in E_0$, the sequence of evolved vectors $\{dT_x^t v\}_{t \in Z^+}$, where $dT_x^t v \in E_t$, is also defined.

In principle, these sequences are computed by iterations, using the properties $T^t = T \circ T^{t-1}$ and $dT_x^t = dT_{T^{t-1}x} \circ dT_x^{t-1}$. Actually, due to unavoidable errors, one computes sequences $\{x'_t\}_{t \in Z^+}$ and $\{v'_t\}_{t \in Z^+}$ which are only some approximations to $\{x_t\}_{t \in Z^+}$ and $\{dT_x^t v\}_{t \in Z^+}$ respectively; precisely, one can only control that the error be small at each iteration step, while nothing is known a priori for large t . The problem of the relations between true orbits $\{x_t\}$ and «pseudoorbits» $\{x'_t\}$ was studied by means of numerical experiments in [6]. From the results reported there and inspired by the so-called Anosov-Bowen theorem it appears that, if the orbit has a sufficiently stochastic behaviour and if the precision of the computation at each step is high enough, the time averages of continuous functions on M computed along a pseudoorbit are practically reliable. One can conjecture that the same

* Istituto di Fisica dell'Università, Via Marzolo 8, 35100 Padova.

** Istituto di Fisica e Istituto di Matematica dell'Università, Via Celoria 16, 20133 Milano.

*** Département de Mathématiques, Centre Scientifique et Polytechnique, Université de Paris-Nord, 93430 Villeneuve, France.

(1) Sec. I.4.4 means Sec. 4.4 of Part I; analogously, for example, Theorem I.5 will denote Theorem 5 of Part I.

happens for what concerns the computation of LCEs, although nothing is known at the moment.

2.2. Another important problem is related to the role played by the T -invariant Borel probability measure μ on M appearing in Theorem I.5 if one takes into consideration the effect of numerical errors. This problem too has been studied in an experimental way in the slightly different context of computations of time averages in [6, 7]. The conclusion there drawn is that, if T admits a Liouville measure μ_L , this one appears to play a privileged role in numerical computations, at least if the system has sufficiently strong stochastic properties. One can then again conjecture that the same should happen in the computations of LCEs.

Now, many dynamical systems of interest are non conservative and thus do not admit a Liouville measure; such are for example the systems studied by Lorenz (see [8, 9]) and Hénon [10].

In particular, the maximal LCEs for the Lorenz and the Hénon models have been numerically computed in [11] and [12] respectively.

The problem of a privileged measure for numerical experiments on such systems, in the sense explained in [6, 7], is also a delicate one, which is completely open to our knowledge (see however [13, 14]).

2.3. Finally, another problem is related to the circumstance that T might not be explicitly defined on the manifold M ; for example, M could be a compact component Γ_E of a surface of constant energy, with $\dim \Gamma_E = 2n - 1$, and the flow be defined by differential equations on an open subset U of \mathbb{R}^{2n} . Then, due to numerical errors, the pseudoorbit will not lie exactly on M and the computed vectors $\{v'_i\}$ will not be tangent to M , even if one had initially $x \in M$ and v tangent to M . In such case, this difficulty is overcome in virtue of Corollary I.9, which insures that the LCEs computed for the flow on U are the same as those for the induced flow on M , apart from a vanishing one.

A. THE NUMERICAL METHOD.

3. The method for the maximal exponent.

3.1. Neglecting all the difficulties related to the general problems discussed in Sec. 2, we come now to the specific problems that one encounters in the numerical application of Theorem I.5.

Let M be a compact connected m -dimensional Riemannian manifold of class C^1 , and T a diffeomorphism of M into itself. One wants to compute the LCEs $\chi_1(x), \dots, \chi_m(x)$ as defined in Sec. I.2. Given m initial vectors $v_1, \dots, v_m \in E_0$, chosen at random in the usual sense of the term, one has thus to evaluate, according to Theorem I.5, the quantities

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \text{Vol}^p([dT_x^t v_1, \dots, dT_x^t v_p]) = \quad (1)$$

$$= \chi_1(x) + \dots + \chi_p(x)$$

for $p = 1, \dots, m$, where $[v_1, \dots, v_p]$ denotes the open parallelepiped defined by the vectors $v_1, \dots, v_p \in E_0$ and Vol^p denotes the p -dimensional (Riemannian) volume in the tangent spaces to M ; in particular, $\text{Vol}^1([v_1]) = \|v_1\|$, where $\|\cdot\|$ denotes the norm induced in E_0 by the Riemannian metric on M .

However, the direct application of this formula meets with two difficulties. Precisely, if v is chosen at random one has in general that $\|dT_x^t v\|$ increases exponentially with increasing t , so that it rapidly exceeds the possibilities of ordinary numerical computations; this is indeed the case if $\chi_1(x) > 0$, in virtue of Theorem I.5. Furthermore, it occurs that, if v_1 and v_2 are chosen at random, the angle between the directions of $dT_x^t v_1$ and $dT_x^t v_2$ in general becomes rapidly so small that again the possibilities of ordinary numerical computations are exceeded; indeed this can be easily shown as a consequence of Theorem I.5 if one has $\chi_1(x) > \chi_2(x)$.

3.2. In the computation of the maximal LCE χ_1 only the first difficulty occurs, and it is overcome by a well known procedure that takes profit of the linearity and of the composition law $dT_x^{t+s} v = dT_x^t (dT_x^s v)$ for the tangent mappings $\{dT_x^t\}$. Roughly speaking, having fixed a number s of iterations, one replaces after each s iterations the evolved tangent vector $dT_x^{ts} v$ by another vector having the same direction and a fixed norm, for example norm 1. More precisely, having fixed $s \geq 1$, one can choose at random $v \in E_0$ with $\|v\| = 1$ and define and compute recursively

$$w_0 = v \in E_0$$

$$\alpha_k = \|dT_{x_{(k-1)s}}^s w_{k-1}\|$$

$$w_k = \frac{dT_{x_{(k-1)s}}^s w_{k-1}}{\alpha_k} \in E_{k_s}$$

for $k \geq 1$. One then easily sees that

$$\|dT_x^{ks} v\| = \alpha_1 \cdot \dots \cdot \alpha_k$$

for $k \geq 1$, so that obviously one has

$$\chi_1 = \lim_{k \rightarrow \infty} \frac{1}{ks} \sum_{i=1}^k \ln \alpha_i.$$

Now, if s is not too large, all of the quantities $\alpha_i (i \geq 1)$ are uniformly bounded and no difficulty arises. A minor variant of this method is presented in [1] (see also [2]).

4. The method for all exponents.

4.1. The second difficulty occurs when at least two vectors are involved, i.e. already in the computation of $\chi_2(x)$, and in general in the computation of $\chi_p(x)$ for $2 \leq p \leq m$, where the evaluation of $\text{Vol}^p([dT_x^t v_1, \dots, dT_x^t v_p])$ is required. Indeed, if the vectors v_1, \dots, v_p are chosen at random, the angles between the directions of $dT_x^t v_i$ and $dT_x^t v_j (i \neq j)$ become in general too small for numerical computations. This difficulty can be overcome on the basis of the following simple remark: an invertible linear mapping, as our tangent mapping is, maps linear p -dimensional subspaces

onto linear subspaces of the same dimension, and the 'rate of growth' of p -dimensional volumes under the action of any linear mapping A , i.e.

$$\frac{\text{Vol}^p(A(U))}{\text{Vol}^p(U)}$$

where U is an open set in the considered p -dimensional subspace, does not depend on the choice of the open set U . Indeed, for a given linear mapping this is just an intrinsic quantity of the subspace involved.

Thus one can replace after each s iterations all the p evolved tangent vectors by a set of p other vectors spanning the same p -dimensional subspace and suitably chosen, for example p vectors which form an orthonormal set. More precisely, one can choose at random the vectors $v_1, \dots, v_p \in E_0$ with $(v_i, v_j) = \delta_{ij}$, where (\cdot) is the scalar product induced in E_0 by the Riemannian metric on M and, as usual

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then one defines recursively

$$w_0^{(j)} = v_j \in E_0, \quad j = 1, \dots, p,$$

$$\beta_k^{(p)} = \text{Vol}^p([dT_{x(k-1)s}^s w_{k-1}^{(1)}, \dots, dT_{x(k-1)s}^s w_{k-1}^{(p)}]),$$

where $\{w_k^{(j)}\}_{1 \leq j \leq p}$ is an arbitrary orthonormalization of $\{dT_{x(k-1)s}^s w_{k-1}^{(j)}\}_{1 \leq j \leq p}$, and $w_k^{(j)} \in E_{ks}$, for $k \geq 1$.

One then easily sees that

$$\text{Vol}^p([dT_{x(k-1)s}^{ks} v_1, \dots, dT_{x(k-1)s}^{ks} v_p]) = \beta_1^{(p)} \dots \beta_k^{(p)}. \quad (2)$$

In such way, by relation (1) one can compute $\chi_1(x) + \dots + \chi_p(x)$ for $1 \leq p \leq m$. The computation of $\chi_1(x), \dots, \chi_m(x)$ thus requires the computation of suitable p -dimensional volumes for all $1 \leq p \leq m$

4.2. As a matter of fact, a particular choice of the orthonormalization procedure allows one to evaluate all such volumes at once by computing the evolution of just m vectors. Indeed, for any $k \geq 1$, choose the Gram-Schmidt orthonormalization procedure defined by the recursive formulae

$$\alpha_k^{(1)} = \|dT_{x(k-1)s}^s w_{k-1}^{(1)}\|$$

$$w_k^{(1)} = \frac{dT_{x(k-1)s}^s w_{k-1}^{(1)}}{\alpha_k^{(1)}}$$

and

$$\alpha_k^{(j)} = \|dT_{x(k-1)s}^s w_{k-1}^{(j)} - \sum_{i=1}^{j-1} (w_k^{(i)}, dT_{x(k-1)s}^s w_{k-1}^{(j)}) w_k^{(i)}\|$$

$$w_k^{(j)} = \frac{dT_{x(k-1)s}^s w_{k-1}^{(j)} - \sum_{i=1}^{j-1} (w_k^{(i)}, dT_{x(k-1)s}^s w_{k-1}^{(j)}) w_k^{(i)}}{\alpha_k^{(j)}}$$

for $j = 2, \dots, p$.

One then easily sees that

$$\beta_k^{(p)} = \alpha_k^{(1)} \dots \alpha_k^{(p)},$$

so that, by (2),

$$\begin{aligned} \text{Vol}^p([dT_{x(k-1)s}^{ks} v_1, \dots, dT_{x(k-1)s}^{ks} v_p]) &= \\ &= (\alpha_1^{(1)} \dots \alpha_k^{(1)}) \dots (\alpha_1^{(p)} \dots \alpha_k^{(p)}) \end{aligned} \quad (3)$$

where $1 \leq p \leq m$, and all the required volumes are thus given as a byproduct of the orthonormalization procedure.

In such way one also obtains a direct formula for $\chi_1(x), \dots, \chi_m(x)$; indeed by (1) and (3) one immediately gets

$$\chi_p(x) = \lim_{k \rightarrow \infty} \frac{1}{ks} \sum_{i=1}^k \ln \alpha_i^{(p)}$$

for $1 \leq p \leq m$. This is the relation that was actually applied in our computations.

A completely analogous algorithm also holds for the computation of all LCEs $\chi_1(x), \dots, \chi_n(x)$ of the family dT_x^t where $x \in M$ and $dT_x^t : T_x N \rightarrow T_{T_x^t N}$ (see Sec. 1.3).

B. NUMERICAL TESTS AND EXAMPLES.

5. Generalities.

5.1. According to what explained in the previous section, we then compute the quantities

$$\chi_p^{(k,s)}(x) = \frac{1}{ks} \sum_{i=1}^k \ln \alpha_i^{(p)}$$

for $1 \leq p \leq m$ and expect that, as k increases, $\chi_p^{(k,s)}(x)$ approaches a sufficiently well defined limit to be identified with the LCE $\chi_p(x)$.

From the very definition of $\chi_p^{(k,s)}(x)$ it follows that it depends on k and s only through their product $r = ks$. Thus we will write in the following $\chi_p^{(r)}(x)$ instead of $\chi_p^{(k,s)}(x)$. Moreover we may expect that:

$$1) \lim_{r \rightarrow \infty} \chi_p^{(r)}$$

is independent of the choice of the initial vectors v_1, \dots, v_p if they are chosen at random, and that such limit is equal to $\chi_p(x)$ for $1 \leq p \leq m$;

$$2) \lim_{r \rightarrow \infty} \sum_{p=1}^m \chi_p^{(r)}(x) = 0$$

if T admits a Liouville measure (see Corollary I.7);

$$3) \lim_{r \rightarrow \infty} (\chi_p^{(r)}(x) + \chi_{m-p+1}^{(r)}(x)) = 0$$

for all $1 \leq p \leq m$, if T is a symplectic diffeomorphism (see Theorem I.8; m is necessarily even in such case); this implies in particular property 2).

These properties provide significative tests for the consistency of our numerical computations. Actually we will check properties 1) and 3).

First, in Sec. 6, we consider two simple algebraic automor-

phisms of the torus T^6 , where the LCEs are known analytically and can thus be compared with our numerical results. Then, in Sec. 7, we report the results of the computation of all LCEs on two examples of symplectic nonlinear diffeomorphisms of the tori T^4 and T^6 respectively, studied previously by C. Froeschlé and J. - P. Scheidecker [15 - 21]. Finally, in Sec. 8 we report the results on an Hamiltonian system with three degrees of freedom of interest in astronomy (see [3]). All the computations were performed on a CDC - CY76, with a precision of 14 digits.

6. Algebraic automorphisms of the tori T^6 and T^2 .

6.1. Consider the algebraic automorphisms of the torus T^6 defined by the matrices

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 5 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 9 & 0 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix}$$

For such automorphisms it is easily seen that all the

LCEs are constant on T^6 , being given by the logarithms of the moduli of the eigenvalues $\lambda_1, \dots, \lambda_6$ of the corresponding matrices. The eigenvalues of both matrices are real and positive with moreover $\lambda_1 = \lambda_6^{-1}$, $\lambda_2 = \lambda_5^{-1}$ and $\lambda_3 = \lambda_4^{-1}$, where $\lambda_1 \geq \dots \geq \lambda_6$. For matrix A one has

$$\lambda_1 = 5 + \sqrt{24}, \quad \lambda_2 = 4 + \sqrt{15}, \quad \lambda_3 = \frac{3 + \sqrt{5}}{2};$$

for matrix B one has

$$\lambda_1 = \frac{11 + \sqrt{117}}{2}, \quad \lambda_2 = 4 + \sqrt{15}, \quad \lambda_3 = 1.$$

In the particular case of algebraic automorphisms of the torus, the computation scheme described above essentially trivializes, because the knowledge of an orbit is irrelevant, as dT_x^r is independent of x .

Coming now to the numerical results, in Table I we report, for matrices A and B respectively, the values of $\chi_p^{(r)}$ at several values of r (with $s = 2$), and correspondingly the exact values of $\chi_p = \ln \lambda_p$ for $1 \leq p \leq 6$. As one sees, the agreement is excellent; in particular, notice that $\chi_3^{(r)}$ and $\chi_4^{(r)}$ appear to tend to zero with increasing r in case of matrix B , as expected.

6.2. In this subsection we intend to illustrate on a very simple example a fact already discussed in Sec. I.2.3 and related to the role of the random choice of the initial vectors in our method. Precisely we want to illustrate how it occurs that, even if one happened to know exactly the subspaces L_1, \dots, L_s described in Sec. I.2.1, one would not in general be able to compute the LCEs different from the maximal one.

Consider the well known Anosov automorphism of the

Table I Values of $\chi_p^{(r)}$ for increasing r and exact values of χ_p ($p = 1, \dots, 6$), for two algebraic diffeomorphisms of the torus T^6 . Up, matrix A ; down, matrix B .

r	$\chi_1^{(r)}$	$\chi_2^{(r)}$	$\chi_3^{(r)}$	$\chi_4^{(r)}$	$\chi_5^{(r)}$	$\chi_6^{(r)}$
2×10	2.284496	2.026829	0.886125	-0.875646	-1.830430	-2.491375
2×10^2	2.291635	2.057780	0.954794	-0.953746	-2.040080	-2.312383
2×10^3	2.292352	2.063071	0.961661	-0.961556	-2.061101	-2.294427
2×10^4	2.292424	2.063401	0.962347	-0.962337	-2.063204	-2.292631
2×10^5	2.292431	2.063433	0.962416	-0.962415	-2.063414	-2.292452
exact val.	2.292432	2.063437	0.962424	-0.962424	-2.063437	-2.292432

r	$\chi_1^{(r)}$	$\chi_2^{(r)}$	$\chi_3^{(r)}$	$\chi_4^{(r)}$	$\chi_5^{(r)}$	$\chi_6^{(r)}$
2×10	2.382273	2.051932	0.061830	-0.052828	-1.052828	-2.562805
2×10^2	2.388801	2.062286	0.016703	-0.015802	-2.045126	-2.406862
2×10^3	2.389453	2.063322	0.002811	-0.002721	-2.061606	-2.391260
2×10^4	2.389519	2.063426	0.000396	-0.000387	-2.063254	-2.389700
2×10^5	2.389526	2.063436	0.000051	-0.000050	-2.063419	-2.389544
exact val.	2.389526	2.063437	0.0	0.0	-2.063437	-2.389526

torus T^2 defined by the matrix

$$C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvectors are

$$e_1 = \left(-1, \frac{\sqrt{5}+1}{2}\right), \quad e_2 = \left(1, \frac{\sqrt{5}-1}{2}\right)$$

with eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{2}, \quad \lambda_2 = \frac{3-\sqrt{5}}{2}$$

respectively; correspondingly one has $\chi_1 = \ln \lambda_1 \simeq 0.96242$ and $\chi_2 = \ln \lambda_2 = -\chi_1$.

The problem is then that of the random choice of the initial vector v in the computation of χ_1 . In this connection we made what a priori is the worst choice, i.e. the choice $v = e_2$ as far as allowed by the precision available in the computer. More precisely, we performed two computations, one with 14 digits (single precision) and the other one with 28 digits (double precision); the graphs for $\chi_1^{(r)}$ as a function of r (with $s = 5$) in semilogarithmic scale for both cases are reported in Fig. 1. As one sees, up to a certain number of iterations ($r = 15$ for single precision) one has with excellent accuracy $\chi_1^{(r)} = \chi_2$, but then $\chi_1^{(r)}$ 'relaxes' to the value of χ_1 , and an excellent agreement is obtained at $r = 5000$. The double precision calculation does not change substantially the picture.

This indicates, on the other hand, that the choice of the initial vector is really unessential for the limit of $\chi_1^{(r)}$ as r increases: the unavoidable computational errors produce in a sense themselves the good random choice.

7. Symplectic diffeomorphisms of T^2 and T^3 .

7.1. We then come to the class of mappings studied by C. Froeschlé and J.-P. Scheidecker [15 - 21]. Given functions $f_1, \dots, f_n, g \in C^1(T^1)$ and real numbers a_1, \dots, a_n, b , define the diffeomorphism $T_n: T^{2n} \rightarrow T^{2n}$ by the formula

$$T_n \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ \vdots \\ q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_1 + a_1 f_1(p_1 + q_1) + b g \left(\sum_{i=1}^n (p_i + q_i) \right) \\ p_1 + q_1 \\ q_2 + a_2 f_2(p_2 + q_2) + b g \left(\sum_{i=1}^n (p_i + q_i) \right) \\ p_2 + q_2 \\ \vdots \\ q_n + a_n f_n(p_n + q_n) + b g \left(\sum_{i=1}^n (p_i + q_i) \right) \\ p_n + q_n \end{pmatrix} \pmod{1}$$

T_n is easily seen to be symplectic. Indeed, it is not difficult to recognize that one can write $T_n = T_n^{(2)} \circ T_n^{(1)}$, with $T_n^{(1)}: (q_1, p_1, \dots, q_n, p_n) \rightarrow (q_1, q_1 + p_1, \dots, q_n, q_n + p_n) \pmod{1}$, and

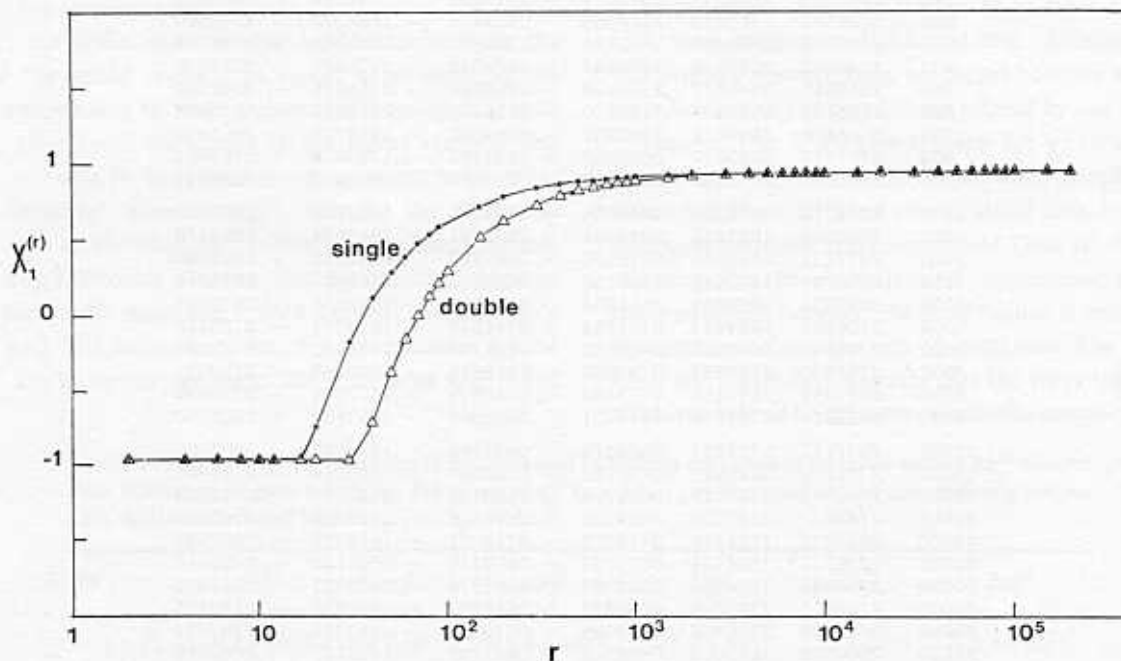


Figure 1. Showing the independence of the computed maximal LCE from the initial vector. Even if the «worst» choice is made, after a sufficiently large number of iterations the computed value relaxes to the value of $\chi_1 = 0.96242$. The number of iterations required for relaxation increases with the accuracy of the computation: dots, single precision (14 digits); triangles, double precision (28 digits).

$$T_n^{(2)} \begin{pmatrix} q_1 \\ p_1 \\ \vdots \\ q_n \\ p_n \end{pmatrix} = \begin{pmatrix} q_1 + \phi_1(p_1, \dots, p_n) \\ p_1 \\ \vdots \\ q_n + \phi_n(p_1, \dots, p_n) \\ p_n \end{pmatrix} \pmod{1},$$

where ϕ_1, \dots, ϕ_n are suitable C^2 functions on T^n such that

$$\frac{\partial \phi_i}{\partial p_j} = \frac{\partial \phi_j}{\partial p_i}, \quad 1 \leq i, j \leq n.$$

Both $T_n^{(1)}$ and $T_n^{(2)}$ are easily seen to be symplectic, so that the assertion follows.

As reported in Sec. 2 of [16], for $n = 1$ and $f_1(x) = \sin(2\pi x)$, this mapping was first studied numerically by J. B. Taylor, and for $n > 1$ the coupling was suggested by V. I. Arnol'd.

For $n > 1$ and $f_1(x) = \dots = f_n(x) = g(x) = \sin(2\pi x)$, these mappings were extensively studied by C. Froeschlé and J.-P. Scheidecker in order to investigate numerically the stochastic behaviour in conservative dynamical systems. As in the present paper, one of their main motivations was to find a quantitative characterization of stochasticity by numerical means.

7.2. As in refs. [15-21], in our computations we considered the particular case with $a_1 = a_2 = \dots = a_n = a$ and $f_1(x) = \dots = f_n(x) = g(x) = \sin(2\pi x)$; we restricted ourselves to the cases $n = 2$ and $n = 3$.

All the properties expected were always well satisfied. As an example, we report here in detail the results for $n = 3$, $a = -1.3$, $b = 0.1$, initial point $(2.02, 0.0, 2.1, 0.0, 2.0, 0.02)$ and a certain choice of the orthonormal initial vectors

Table II. Values of $\chi_p^{(r)}$ ($p = 1, \dots, 6$) for increasing r , for the symplectic diffeomorphism of the torus T^6 described in Sec. 6.4, with $a = -1.3$ and $b = 0.1$. This is a reproduction of the original computer output. N stand for r , and columns 1, ..., 6 give the values of $\chi_1^{(r)}, \dots, \chi_6^{(r)}$. NIT denotes s , MRANF is a parameter determining the choice of the initial vectors, and x is the initial point.

DIFFEOMORFISMO SIMPLETTICO DI T6				A = -1.3000 B = .1000 NIT = 5		
MRANF = 72		X = 2.020000 0.000000		2.100000 0.000000 2.000000 .020000		
N	1	2	3	4	5	6
20	.4121611	.1194656	.2398909	-.0679061	-.3758162	-.3277954
30	.3646772	.1856021	.2360410	-.1052208	-.3515771	-.3295224
40	.3470904	.2088445	.2387275	-.1388676	-.3271262	-.3286685
50	.3289096	.2007709	.1818509	-.1034656	-.1034656	-.3139466
60	.3258403	.1758648	.1555377	-.0884135	-.2557816	-.3130476
70	.3084328	.1610597	.1791196	-.1219399	-.2292455	-.2974266
80	.3223415	.1672005	.1884534	-.1436579	-.2216008	-.3127366
90	.3001481	.1572982	.1807813	-.1396294	-.2069888	-.2916094
100	.3009585	.1493404	.1782684	-.1361945	-.1990987	-.2932743
150	.2888098	.1305598	.1372077	-.1017339	-.1711566	-.2836869
200	.3005112	.1367010	.1212868	-.0953479	-.1664822	-.2966690
300	.3167323	.1788885	.1412835	-.1239887	-.1987448	-.3141709
400	.3240881	.2027679	.1454247	-.1324535	-.2176601	-.3221670
500	.3218351	.2045817	.1372134	-.1268364	-.2164955	-.3202982
600	.3183492	.1973574	.1157330	-.1070856	-.2072856	-.3170685
700	.3104468	.1859672	.1028201	-.0954080	-.1944770	-.3093490
800	.2767328	.1683017	.0896053	-.0831197	-.1757478	-.2757723
900	.2666977	.1579358	.0871346	-.0813696	-.1645546	-.2658439
1000	.2815218	.1668228	.0846656	-.0794771	-.1727797	-.2807533
1500	.2860493	.1623812	.0842781	-.0808191	-.1663524	-.2855370
2000	.2827132	.1612266	.0713370	-.0687427	-.1642050	-.2823290
3000	.2615787	.1472668	.0568258	-.0550964	-.1492524	-.2613266
4000	.2627822	.1562465	.0631354	-.0618383	-.1577358	-.2625901
5000	.2706865	.1661837	.0751892	-.0741515	-.1673751	-.2705328
6000	.2728352	.1660005	.0759935	-.0751288	-.1669933	-.2727071
7000	.2751820	.1612787	.0725960	-.0718548	-.1621297	-.2750722
8000	.2779396	.1670155	.0773464	-.0766979	-.1677601	-.2778436
9000	.2823514	.1726819	.0814731	-.0808966	-.1733437	-.2822660
10000	.2807277	.1727627	.0826372	-.0821184	-.1733584	-.2806509
15000	.2798260	.1629693	.0785996	-.0782537	-.1633664	-.2797748
20000	.2742772	.1562314	.0666724	-.0664139	-.1565293	-.2742388
30000	.2768470	.1587190	.0694155	-.0692426	-.1589176	-.2768214
40000	.2802275	.1621745	.0715218	-.0713921	-.1623235	-.2802083
50000	.2736771	.1564921	.0670153	-.0669115	-.1566112	-.2736617
60000	.2744946	.1534938	.0642834	-.0641970	-.1535931	-.2744818
70000	.2750415	.1549048	.0649279	-.0648537	-.1549899	-.2750306
80000	.2781428	.1596466	.0679999	-.0679350	-.1597211	-.2781332
90000	.2800605	.1620065	.0668312	-.0667735	-.1620727	-.2800520
100000	.2779343	.1596567	.0648798	-.0644280	-.1597163	-.2779267
150000	.2747565	.1542743	.0611924	-.0611578	-.1543140	-.2747514
200000	.2739936	.1555098	.0614087	-.0613827	-.1555396	-.2739898
300000	.2724579	.1527992	.0601472	-.0601299	-.1528190	-.2724553
400000	.2735160	.1537142	.0603894	-.0603765	-.1537290	-.2735141
500000	.2725962	.1532106	.0600644	-.0600541	-.1532225	-.2725946

v_1, \dots, v_6 . The results are reported in Table II, which is a reproduction of the original computer's output. In the Table, NIT denotes $s(=5)$, MRANF is a parameter which determines the choice of the initial vectors, N stands for r and runs from 20 to 5×10^5 and the columns 1, 2, ..., 6 give the values of $\chi_1^{(r)}, \dots, \chi_6^{(r)}$.

First of all one observes that the quantities $\chi_p^{(r)}$ appear to approach rather well defined limits with increasing r . This property is better visualized if the results are plotted in a graph in a log-log scale; such graphs will be shown below in connection with the dependence on the initial data.

From the Table one observes furthermore that property 3) is well satisfied. More precisely, in Table III we report, for increasing values of r , the corresponding values of $\chi_1^{(r)} + \chi_6^{(r)}, \chi_2^{(r)} + \chi_5^{(r)}, \chi_3^{(r)} + \chi_4^{(r)}$ as computed from Table II; one thus observes that they appear to tend to zero, with increasing r , with remarkable regularity (as $1/r$).

Table III. Values of $\chi_1^{(r)} + \chi_6^{(r)}, \chi_2^{(r)} + \chi_5^{(r)}, \chi_3^{(r)} + \chi_4^{(r)}$ for increasing r , obtained from Table II.

r	$\chi_1^{(r)} + \chi_6^{(r)}$	$\chi_2^{(r)} + \chi_5^{(r)}$	$\chi_3^{(r)} + \chi_4^{(r)}$
5×10	1.1×10^{-3}	3.0×10^{-2}	2.9×10^{-2}
5×10^2	1.1×10^{-4}	2.9×10^{-3}	2.8×10^{-3}
5×10^3	1.1×10^{-5}	2.9×10^{-4}	2.8×10^{-4}
5×10^4	1.1×10^{-6}	2.9×10^{-5}	2.8×10^{-5}
5×10^5	$\approx 1 \times 10^{-7}$	2.9×10^{-6}	2.8×10^{-6}

For what concerns the independence of $\chi_p^{(r)}$ from s , we have checked it in several cases; for example, by changing s from 5 to 1 we got a Table analogous to Table II, with *exactly* all the same numbers.

Finally, for what concerns the independence from the choice of the initial vectors, we report as an example the results corresponding to a computation with initial data as in Table II, apart from the choice of the initial vectors. This is shown in Table IV which gives, for increasing values of r , the corresponding values of $\Delta\chi_p^{(r)}$, namely the difference between $\chi_p^{(r)}$ in the first and in the second computation.

From this Table one observes that actually $\Delta\chi_p^{(r)}$ appears to go to zero, with increasing r , with a remarkable regularity (again as $1/r$). The example of Sec. 6.2 also provides a good check of the independence from the choice of the initial vectors.

Table IV. Showing the independence of the computed LCEs from the choice of the initial vectors. $\Delta\chi_p^{(r)}$ denotes the difference, at given r , between the values of $\chi_p^{(r)}$ ($p = 1, \dots, 6$) computed in two cases differing just for the choice of the initial vectors v_1, \dots, v_6 .

r	$\Delta\chi_1^{(r)}$	$\Delta\chi_2^{(r)}$	$\Delta\chi_3^{(r)}$	$\Delta\chi_4^{(r)}$	$\Delta\chi_5^{(r)}$	$\Delta\chi_6^{(r)}$
5×10	8.3×10^{-3}	9.9×10^{-2}	-2.9×10^{-2}	-5.3×10^{-2}	2.5×10^{-2}	-2.4×10^{-2}
5×10^2	8.4×10^{-4}	1.2×10^{-2}	-8.0×10^{-3}	-5.2×10^{-3}	2.4×10^{-3}	-2.5×10^{-3}
5×10^3	8.4×10^{-5}	1.2×10^{-3}	-8.0×10^{-4}	-5.2×10^{-4}	2.4×10^{-4}	-2.5×10^{-4}
5×10^4	8.4×10^{-6}	1.2×10^{-4}	-8.0×10^{-5}	-5.2×10^{-5}	2.4×10^{-5}	-2.5×10^{-5}
5×10^5	$\approx 8 \times 10^{-7}$	1.2×10^{-5}	-7.9×10^{-6}	-5.2×10^{-6}	2.4×10^{-6}	-2.5×10^{-6}

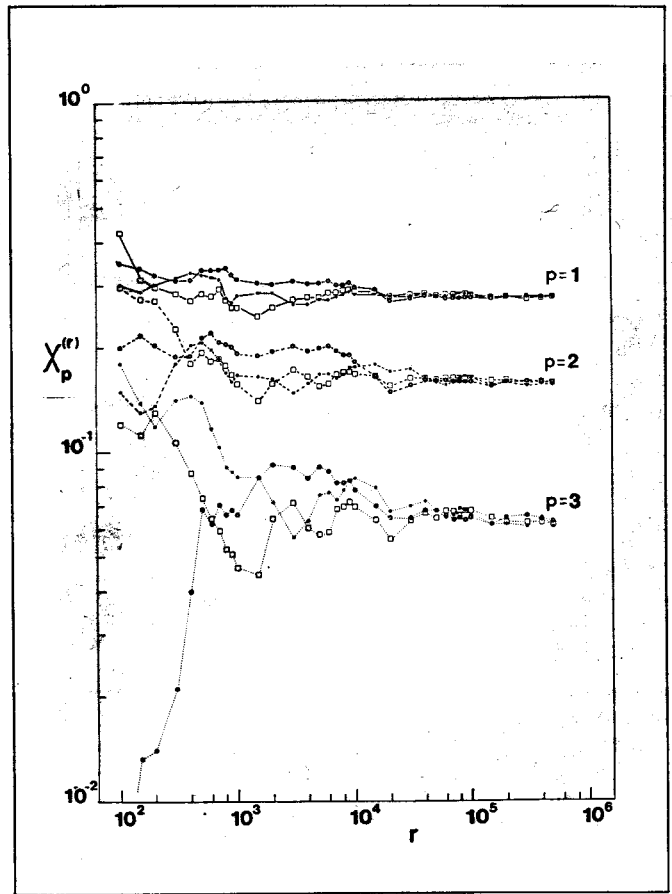


Figure 2. $\chi_p^{(r)}$ as a function of r ($p = 1, 2, 3$) for the symplectic diffeomorphism T_3 of the torus T^6 , with $a = -1.3, b = 0.1$ and three different sets of initial data: dots, (2.02, 0.0, 2.1, 0.0, 2.0, 0.02); squares, (2.02, 0.01, 2.1, 0.0, 2.0, 0.02); circles, (1.4, 0.9, 3.1, 2.6, 0.1, 0.25).

7.3. In general, a point of great interest in the study of a model is how different are the LCEs from each other and how they depend on the initial data. Although this is not our primary concern here, we report however some results as an indication of the possibilities offered by our technique.

Thus in Fig. 2 we give a graph for $\chi_p^{(r)}$ ($p = 1, 2, 3$) as a function of r in log-log scale for the same mapping considered above and three different sets of initial data, indicated in the figure caption (one set is just that of Table II). As anticipated, one sees that the indication of convergence is quite good; the separation between the limit values is evident, as is the independence from the sets of initial data. The independence from the initial data suggests that the three initial points on the torus belong to the same 'stochastic component'. An anal-

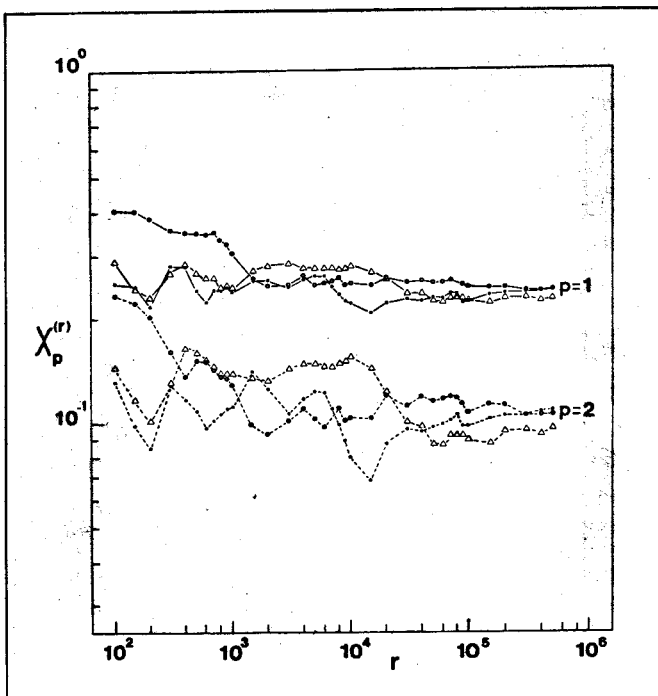


Figure 3. Same as Fig. 2 for the symplectic diffeomorphism T_2 of the torus T^4 , with $a = -1.3, b = 0.2$ and three different sets of initial data: dots, (2.0, 0.08, 2.1, 0.05); squares, (1.4, 1.5, 0.3, 1.9); circles, (2.0, 0.0, 2.1, 0.0).

ogous graph for the mapping $T_2 : T^4 \rightarrow T^4$ is given in Fig. 3; here $a = -1.3, b = 0.2$, and the initial data are indicated in the figure caption.

As an indication of the dependence of the LCEs on the parameters of the mapping T_n , consider for example the mapping T_3 with initial point (2.02, 0.0, 2.1, 0.0, 2.0, 0.02). With $a = 2.5, b = 0.1$ one obtains $\chi_1 = 0.681, \chi_2 = 0.510, \chi_3 = 0.250$, while from Fig. 2 one infers that with $a = -1.3, b = 0.1$ one has 0.273, 0.154, 0.061 respectively for the same initial point.

A. An Hamiltonian system.

8.1. We conclude with the study of an Hamiltonian flow (see Sec. I.4.3). Precisely, we consider the model of three harmonic oscillators coupled by cubic terms in the coordinates, as described by the Hamiltonian

$$H(q, p) = \sum_{i=1}^3 \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3, \quad (1)$$

where $q = (q_1, q_2, q_3), p = (p_1, p_2, p_3) \in \mathbb{R}^3; \omega_1 = 1, \omega_2 = \sqrt{2}, \omega_3 = \sqrt{3}$. This model, suggested by G. Contopoulos, is of some interest in astronomical problems, and has recently been investigated numerically for what concerns both its maximal LCE and its formal integrals of motion [3]. On this example we want to illustrate a possible significance of the additional information provided by knowledge of the LCEs different from the maximal one. For this model the energy surfaces $H(q, p) = h$ have a unique compact component Γ_h for each h in an interval $0 < h < \bar{h} \simeq 0.097$, and all the numerical computations were performed, as in [3], on the compact component Γ_h with $h = 0.09$.

Denote by $x = (q, p) \in \mathbb{R}^6$ a point in the phase space of our system, and let $\{T^t\}$ denote as usual the Hamiltonian flow induced by the Hamiltonian (1) in \mathbb{R}^6 . In this case the tangent space $T_x \mathbb{R}^6$ is canonically isomorphic to \mathbb{R}^6 . Then, given a point $x \in \Gamma_h \subset \mathbb{R}^6$ and an initial vector $v \in \mathbb{R}^6$, one has to compute the evolved points $T^t x$ and the evolved vectors $dT_x^t v$, whose coordinates satisfy the Hamiltonian equations with Hamiltonian H and the corresponding variational (i.e. linearized) equations respectively.

We are thus interested in the LCEs of the flow $\{T_h^t\}$, the restriction to Γ_h of the flow $\{T^t\}$, and in this connection the relevant framework is provided by Corollary I.9. Namely, we can make reference to the flow $\{T^t\}$ on \mathbb{R}^6 and compute its LCEs for a given x . These are known a priori to be of the form

$$\chi_1(x) \geq \chi_2(x) \geq \chi_3(x) = 0 = -\chi_3(x) \geq -\chi_2(x) \geq -\chi_1(x).$$

The Corollary I.9 insures that the LCEs of the flow $\{T_h^t\}$ at x are equal to

$$\chi_1(x) \geq \chi_2(x) \geq \chi_3(x) = 0 \geq -\chi_2(x) \geq -\chi_1(x).$$

For what concerns the numerical method, the Hamiltonian as well as the corresponding variational equations were integrated by a central point method correct up to third order, with an integration step τ in the interval $0.005 \leq \tau \leq 0.1$. Our results were checked to be independent of the integration step in this range.

8.2. Having recalled these general facts, we pass now to describe the properties already established in [3] for this model through numerical computations of the maximal LCE χ_1 . The connected compact component $\Gamma_{0.09}$ of the energy surface with energy $H = 0.09$ was found to be subdivided into three disjoint invariant regions:

- 1) A large stochastic region, with $\chi_1 = 0.03$,
- 2) an 'ordered' region, with $\chi_1 = 0$,
- 3) an 'intermediate' region.

The last region is of stochastic type, because on it one has $\chi_1 > 0$; on the other hand it has an intermediate character, namely is 'less stochastic' than the first one, because on it χ_1 takes values significantly smaller than 0.03. Actually such region appears to be subdivided into several disjoint invariant components with different values of χ_1 ; for example $\chi_1 = 0.005$ and $\chi_1 = 0.002$.

The regions of the types 1), 2), 3) are called, in a descriptive way which generalizes an established use, 'big sea', 'islands' and 'small seas' respectively. The coexistence of stochastic components (seas) and ordered regions (islands) was well known since the first numerical studies on the onset of stochasticity in dynamical systems [22], while the existence of intermediate regions (small seas) was reported only recently [23]. In the paper [3] the latter fact was confirmed and also an attempt was made to characterize such partially stochastic regions in terms of formal integrals of motion. It is then very natural to try to see how these intermediate regions behave with respect to the LCEs.

8.3. As said before, we computed the LCEs for the flow

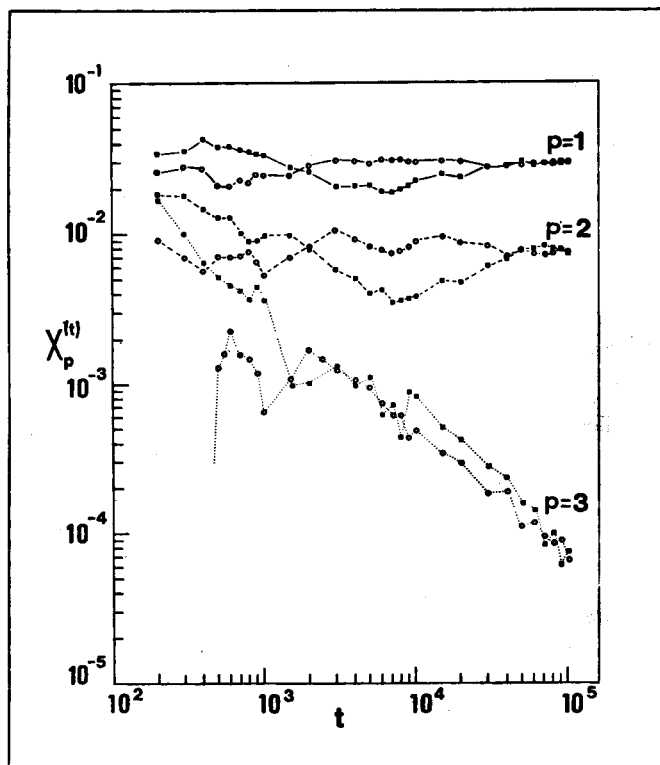


Figure 4. $\chi_p^{(t)}$ ($p = 1, 2, 3$) as a function of t for the Hamiltonian system of three degrees of freedom of Contopoulos. Two orbits are given with initial points in the «big sea». These are determined by the three harmonic energies: dots, (0.01, 0.01, 0.07); circles, (0.03, 0.03, 0.03).

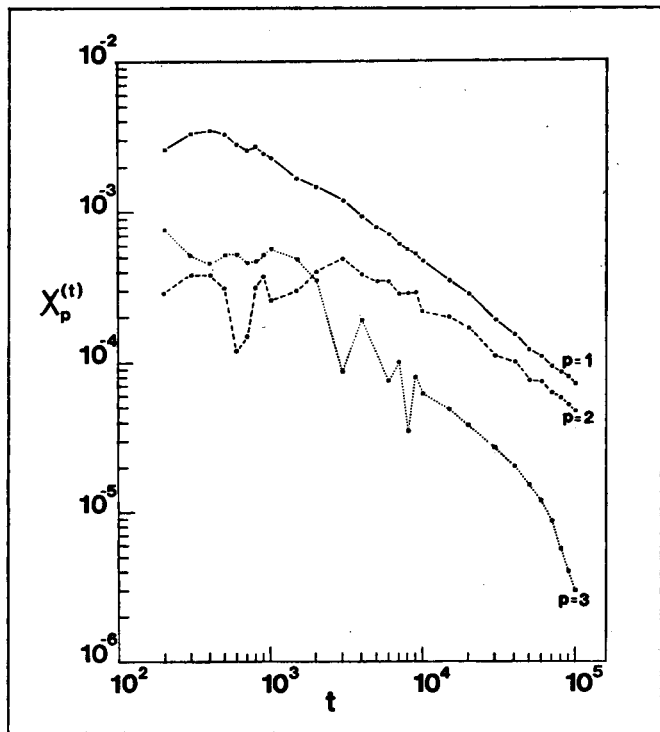


Figure 5. Same as Fig. 4 with initial point in the «ordered region». Harmonic energies (0.0005, 0.0895, 0.0).

defined on R^6 . Because of the Hamiltonian character of the flow, one knows a priori that $\chi_p = -\chi_{6-p+1}$ for $p = 1, 2, 3$; this property was very well satisfied, so that we just report the results for χ_1, χ_2, χ_3 . Moreover, in agreement with the theoretical prediction, we always found $\chi_3 = 0$.

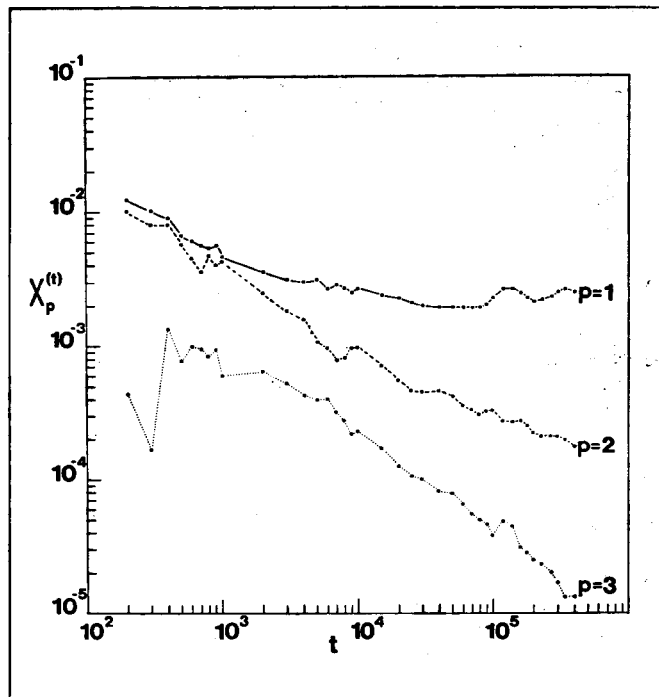


Figure 6. Same as Fig. 4 with initial point in a «small sea». Harmonic energies (0.0085, 0.00815, 0.0).

In Figs. 4, 5, 6 we report typical results obtained for the big sea, the ordered region and a small sea respectively. The graphs give $\chi_p^{(t)}$ as a function of t in log-log scale ($p = 1, 2, 3$), where $t = r\tau$, r having the same meaning than in the previous section and τ being the integration step. The initial conditions were defined assigning arbitrary values to the 'harmonic energies' of the oscillators E_1, E_2, E_3 , with vanishing values for the coordinates, i.e. $q_1 = q_2 = q_3 = 0$,

$$p_i = \sqrt{\frac{2E_i}{\omega_i}} \quad (i = 1, 2, 3),$$

with given values for E_1, E_2, E_3 , indicated in each figure caption.

In the figures, $\chi_3^{(t)}$ appears to tend to zero as expected. For what concerns χ_1 and χ_2 , they appear to vanish in the ordered region (Fig. 5) and to be both positive (with $\chi_1 > \chi_2$ as expected) in the big sea (Fig. 4).

Coming finally to the small sea (Fig. 6), which is the point of main interest, χ_1 appears to be positive there, with a value smaller than in the big sea, as already known, while the behaviour of $\chi_2^{(t)}$ thereby is not completely clear, although one might appear to be justified in guessing it is tending to zero. In any case, it is certain that the behaviour of $\chi_2^{(t)}$ is very different there than in the big sea; thus our results appear to confirm those already obtained in [3] through the study of formal integrals, and indicating that the small seas actually present an intermediate character between stochasticity and order. We hope to return to this intriguing problem in the future.

Received: 5 October 1979.

REFERENCES.

- [1] BENETTIN G., GALGANI L. and STRELCYN J.-M., *Kolmogorov Entropy and Numerical Experiments*, Phys. Rev. A14, p. 2338 - 2345, 1976.
- [2] BENETTIN G. and STRELCYN J.-M., *Numerical Experiments on the Free Motion of a Point Mass Moving in a Plane Convex Region: Stochastic Transition and Entropy*, Phys. Rev. A17, p. 773 - 785, 1978.
- [3] CONTOPOUOS G., GALGANI L. and GIORGILLI A., *On the Number of Isolating Integrals for Hamiltonian Systems*, Phys. Rev. A18, p. 1183 - 1189, 1978.
- [4] BENETTIN G., GALGANI L., GIORGILLI A. and STRELCYN J.-M., *Tous les Nombres Caractéristiques de Lyapounov sont effectivement calculables*, C.R. Acad. Sc. Paris, 286A, p. 431 - 433, 1978.
- [5] BENETTIN G., GALGANI L., GIORGILLI A. and STRELCYN J.-M., *Lyapunov Characteristic Exponents for Smooth Dynamical Systems and for Hamiltonian Systems; a Method for Computing all of them. Part I: Theory*, Meccanica
- [6] BENETTIN G., CASARTELLI M., GALGANI L., GIORGILLI A. and STRELCYN J.-M., *On the Reliability of Numerical Studies of Stochasticity. Part I: Existence of Time Averages*, Nuovo Cimento 44B, p. 183 - 195, 1978.
- [7] BENETTIN G., CASARTELLI M., GALGANI L., GIORGILLI A. and STRELCYN J.-M., *On the Reliability of Numerical Studies of Stochasticity, Part II: Identification of Time Averages*, Nuovo Cimento 50B, p. 211 - 232, 1979.
- [8] LORENZ E. N., *Deterministic Nonperiodic Flow*, J. Atmospheric Sciences 20, p. 130 - 141, 1963.
- [9] LANFORD O. E., *Qualitative and Statistical Theory of Dissipative Systems*. in GALLAVOTTI G. (ed.), *Statistical Mechanics*, Liguori, Napoli, 1978.
- [10] HENON M., *A Two-dimensional Mapping with a Strange Attractor*, Commun. math. Phys. 50, 69 - 77, 1976.
- [11] NAGASHIMA T. and SHIMADA I., *On the C-System-Like Property of the Lorenz System*, Progress of Theor. Phys. 58, p. 1318 - 1319, 1977.
- [12] FEIT S. D., *Characteristic Exponents and Strange Attractors*, Commun. Math. Phys. 61, p. 249 - 260, 1978.
- [13] RUELLE D., *Dynamical Systems with Turbulent Behaviour*, IHES preprint, 1977.
- [14] RUELLE D., *On the Measures which describe Turbulence*, IHES preprint 1978.
- [15] FROESCHLE C., *A Numerical Study of the Stochasticity of Dynamical Systems with two Degrees of Freedom*, Astron. and Astrophys. 9, p. 15 - 23, 1970.
- [16] FROESCHLE C., *Numerical Study of a Four-Dimensional Mapping*, Astron. and Astrophys. 16, p. 172 - 189, 1972.
- [17] FROESCHLE C. and SCHEIDECKER J. P., *Numerical Study of the Stochasticity of Dynamical Systems with more than two Degrees of Freedom*, Journal of Comput. Phys. 11, p. 423 - 439, 1973.
- [18] FROESCHLE C. and SCHEIDECKER J. P., *Numerical Study of a four Dimensional Mapping II*, Astron. and Astrophys. 22, p. 431 - 436, 1973.
- [19] FROESCHLE C. and SCHEIDECKER J. P., *On the Disappearance of Isolating Integrals in Dynamical Systems with more than two Degrees of Freedom*, in KOZAI Y. (ed.) - *The stability of the Solar System and Small Stellar Systems* (IAU 1974). Published also in *Astrophysics and Space Science* 25, p. 373 - 386, 1973.
- [20] FROESCHLE C., *Numerical Study of a Random Dynamical System with two Degrees of Freedom*, *Astrophysics and Space Science* 37, p. 87 - 100, 1975.
- [21] FROESCHLE C., *Connectance of Dynamical Systems with Increasing Number of Degrees of Freedom*, Phys. Rev. A18, p. 277 - 281, 1978.
- [22] HENON M. and HEILES C., *The Applicability of the Third Integral of Motion: Some Numerical Experiments*, Astron. Journal 69, p. 73 - 79, 1964.
- [23] CAROTTA M. C., FERRARIO C., LO VECCHIO G. and GALGANI L., *New Phenomenon in the Stochastic Transition of Coupled Oscillators*, Phys. Rev. A17, p. 786 - 794, 1978.

Note added in proofs: in connection with the problem of Sec. 2.2 see also

RUELLE D., *Sensitive Dependence on Initial Conditions and Turbulent Behavior of Dynamical Systems*, Ann. N.Y. Acad. Sci. 316, p. 468 (1978).

RUELLE D., *What are the Measures Describing Turbulence?*, Suppl. Progr. Theor. Phys. 64, p. 339 - 343 (1978),

RUELLE D., *Measures Describing a Turbulent Flow*, to appear in, Ann. N.Y. Acad. Sci.