LYAPUNOV CHARACTERISTIC EXPONENTS FOR SMOOTH DYNAMICAL SYSTEMS AND FOR HAMILTONIAN SYSTEMS; A METHOD FOR COMPUTING ALL OF THEM. PART 1: THEORY.

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SOMMARIO. Da diversi anni gli esponenti caratteristici di Lyapunov sono divenuti di notevole interesse nello studio dei sistemi dinamici al fine di caratterizzare quantitativamente le proprietà di stocasticità, legate essenzialmente alla divergenza esponenziale di orbite vicine. Si presenta dunque il problema del calcolo esplicito di tali esponenti, già risolto solo per il massimo di essi. Nel presente lavoro si dà un metodo per il calcolo di tutti tali esponenti, basato sul calcolo degli esponenti di ordine maggiore di uno, legati alla crescita di volumi. A tal fine si dà un teorema che mette in relazione gli esponenti di ordine uno con quelli di ordine superiore. Il metodo numerico e alcune applicazioni saranno date in un sucessivo articolo.

SUMMARY. Since several years Lyapunov Characteristic Exponents are of interest in the study of dynamical systems in order to characterize quantitatively their stochasticity properties, related essentially to the exponential divergence of nearby orbits. One has thus the problem of the explicit computation of such exponents, which has been solved only for the maximal of them. Here we give a method for computing all of them, based on the computation of the exponents of order greater than one, which are related to the increase of volumes. To this end a theorem is given relating the exponents of order one to those of greater order. The numerical method and some applications will be given in a forthcoming paper.

1. Introduction.

1.1. Lyapunov Characteristic Exponents (briefly LCEs, also called Lyapunov Characteristic Numbers or simply Characteristic Exponents) play an important role when the stochasticity properties of dynamical systems are studied both from a theoretical and a numerical point of view. Roughly speaking, the LCEs of a trajectory of a dynamical system measure the mean exponential rate of divergence of trajectories surrounding it.

The use of such exponents (and the related notion of regularity) in the study of ordinary differential equations goes back to Lyapunov and his so-called first method (see his classical treatise [1] and, for a modern exposition, [2], [3] and [4]). On the other hand the theory of LCEs in a form adapted to the needs of the theory of dynamical systems and of ergodic theory was given only in the year 1968 in the paper by Oseledec [5]. The main result of such paper, namely the general Noncommutative Ergodic Theorem (briefly, Oseledec Theorem) is at the basis of the present paper.

LCEs began to play an important role in numerical studies of stochasticity in dynamical systems since about ten years, as it came to be realized that there exists a very simple algorithm allowing to compute numerically the maximal LCE. See [6] for a detailed discussion of such algorithm and [7] for a minor improvement: see also the papers [8], [9] and [10], where such algorithm was used with no explicit mention of LCEs.

While the maximal LCE was easily computable, no method was known to compute the others. Now, also apart from its intrinsic interest, the problem of the computation of all LCEs necessarily arises if one is interested in the computation of the metric entropy of non-linear dynamical systems (see [11] for the theory, and [6, 12] for the numerical aspect). The latter problem actually brought us to the one discussed in the present paper.

In the short note [13] we formulated a simple theorem which leads to an algorithm for computing all LCEs for a large class of dynamical systems and gave a numerical application to a non-trivial example. The same technique was also applied by one of us in [14]. In the present paper and in a forthcoming one [15] the results announced in the note [13] are expounded and described in detail and several numerical examples are reported.

More precisely this paper is divided in two chapters, *A* and *B*. In *A* we present the theoretical results which are necessary for the numerical computation of all LCEs, while the proofs are given in *B*. A description of the numerical scheme and of the numerical results for some examples is given instead in the forthcoming paper [15], which constitutes Part II of this paper.

This work, which is intended particularly for people interested in numerical studies of stochasticity, is self-contained, so that no preliminary knowledge of LCEs is assumed.

1.2. The main theoretical result which leads to the algo-

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rithm for the computation of all LCEs and was announced in the note [13] is formulated here as Theorem 2, and is an easy consequence of Theorem 1. The first part of the latter is classical, and its proof can be found for example in the Oseledec's paper [5]. On the other hand, as we learned recently from the paper [16], Theorem 1 has also been obtained, by a different technique, in an unpublished paper by Raghunathan [17].

In the study of the stochasticity properties of dynamical systems, the case of Hamiltonian flows restricted to surfaces of constant energy is of great importance. The direct application of our Theorem 2, however, is not sufficient to obtain a numerical algorithm in such case. To the necessary supplementary work Sec. 4 will be devoted.

1.3. We will use the standard notations **Z**, $\mathbf{Z}^+$ and **R** for the integers, the nonnegative integers and the real numbers respectively, and $\mathbf{R}^n$ for the n-dimensional real cartesian space; also, ln will denote the natural logarithm.

All vector spaces will be real and finite dimensional; the dimension of a vector space E will be denoted by $\dim E$. If $(\,,)$ is a scalar product defined on E, then, as usual, for $x \in E$ we will write $\|x\| = \sqrt{(x, x)}$.

If e^p is a p-dimensional vector subspace of a vector space E and A a measurable subset of e^p, then by $\mathrm{Vol}^p(A)$ we will denote the p-dimensional volume of A induced by the scalar product $(\,,)$.

A slight extension of the standard notion of the absolute value of the determinant, usually referred to a linear mapping of a linear space onto itself, will be required for the case of a linear mapping S of a linear space E onto another linear space F of the same dimension. Let E, F be two p-dimensional linear spaces, each endowed with a particular scalar product, and S a linear mapping of E onto F; then we set by definition

$$|\det S| = \frac{\mathrm{Vol}^p(S(U))}{\mathrm{Vol}^p(U)},$$

where U is an arbitrary bounded open set of E. The definition is clearly independent of U and reduces to the standard one for $E = F$. The formulae are numbered independentley in each Section.

1.4. We thank Dr. G. Casati (Milano) for drawing our attention to the possible revelance of LCEs of higher order for the computation of all LCEs. We also thank Dr. M. Hénon (Nice) for suggesting the behaviour of LCEs in the symplectic case and Dr. Ya. G. Sinai (Moscow) for indicating the result of Sec. 4.1. as well as the method of proving it.

A. ON LYAPUNOV CHARACTERISTIC EXPONENTS.

2. Lyapunov Characteristic Exponents; elementary properties and basic theorems.

2.1. We recall here the classical notions on LCEs and formulate the first theorems on which the numerical computations of all LCEs are based.

The typical situation we have to deal with is that we are given a differentiable mapping T or flow $\{T^t\}$ defined on a differentiable manifold, and we are interested in the asymptotic behaviour of the differential $\{\mathrm{d}T^t_x\}$, x being a point of the manifold. Thus we are led to the following general framework, which we consider in the present Section.

Let $\{E_t\}$, with t a nonnegative integer or real number, be a family of n-dimensional real vector spaces, each endowed with a scalar product $(\,,)_t$. For any t, let a_t be a linear mapping of E_0 onto E_t. The condition

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|a_t\| < \infty,$$

$\|a_t\|$ denoting as usual the norm of a_t, will be assumed to be satisfied throughout this paper; then, for any nonzero vector $g \in E_0$ one defines

$$\chi(a_t, g) \overset{\text{def}}{=} \limsup_{t\to\infty} \frac{1}{t} \ln \|a_t g\| < \infty$$

(the existence of a nonzero g with $\chi(a_t, g) = -\infty$ is not excluded). The number $\chi(a_t, g)$ is called the LCE of vector g with respect to the family $\{a_t\}$. It is convenient to extend the definition to the zero vector 0 of E_0 by $\chi(a_t, 0) = -\infty$.

Actually, from the very definition of the LCEs it follows that one has

$$\chi(a_t, f+g) \leqslant \max\{\chi(a_t, f), \chi(a_t, g)\} \tag{1}$$

$$\chi(a_t, cg) = \chi(a_t, g) \tag{2}$$

for any two vectors $f, g \in E_0$ and for any $c \in \mathbf{R}$, $c \neq 0$. By (2) one can consider the LCE as a function defined on the set of 1-dimensional subspaces of E_0. This fact naturally leads to the concept of LCEs of higher order.

Let e^p be a p-dimensional subspace of E_0 $(1 \leqslant p \leqslant n)$ and U a bounded open subset of e^p. Then the quantity

$$\chi(a_t, e^p) \overset{\text{def}}{=} \limsup_{t\to\infty} \frac{1}{t} \ln \mathrm{Vol}^p(a_t(U)),$$

which is clearly independent of the choice of the open set $U \in e^p$, is called the LCE (of order p) of subspace e^p with respect to the family $\{a_t\}$. As usual, by Vol^p we have denoted the p-dimensional volume induced in E_t by the scalar product $(\,,)_t$.

Again one has $\chi(a_t, e^p) < \infty$ for any subspace $e^p \in E_0$. This is easily seen by choosing as U the open parallelepiped generated by p linearly independent vectors of e^p and recalling the Hadamard inequality (see for example [18]), according to which the euclidean volume of a parallelepiped does not exceed the product of the lengths of its sides. In particular, with the definition of $|\det a_t|$ given in Sec. 1.3, for $p = n$ one obtains

$$\limsup_{t\to\infty} \frac{1}{t} \ln |\det a_t| < \infty.$$

When no ambiguity arises, we will simply write $\chi(g)$ for $\chi(a_t, g)$ and $\chi(e^p)$ for $\chi(a_t, e^p)$.

By (1) and (2) one easily sees that for any $r \in \mathbf{R}$ the set of vectors $\{g \in E_0\,;\, \chi(g) \leqslant r\}$ is a vector subspace of E_0. It then easily follows that the numbers

$$\{\chi(g)\}_{\substack{g \in E_0 \\ g \neq 0}}$$

take at most $n = \dim E_0$ different values ($-\infty$ is not excluded), say $\nu_1 > \ldots > \nu_s$, with $1 \leqslant s \leqslant n$. Let $\{L_i = \{g \in E_0; \chi(g) \leqslant \nu_i\}$. Clearly, for the subspaces $\{L_i\}_{1 \leqslant i \leqslant s}$ one has

$$\{0\} \stackrel{\text{def}}{=} L_{s+1} \subset \cdots \subset L_1 = E_0, \tag{3}$$

with $L_{i+1} \neq L_i$, and $\chi(g) = \nu_i$ if and only if $g \in L_i \backslash L_{i+1}$, $1 \leqslant i \leqslant s$.

A basis $f_1, \ldots, f_n$ of E_0 is called a normal basis (with respect to the family $\{a_t\}$) if

$$\sum_{i=1}^{n} \chi(f_i) \leqslant \sum_{i=1}^{n} \chi(g_i),$$

where $g_1, \ldots, g_n$ is any basis of E_0. A normal basis never is unique, but the set of numbers $\{\chi(f_i)\}_{1 \leqslant i \leqslant n}$ depends only on the family $\{a_t\}$ and not on the choice of the normal basis. By a possible permutation of the vectors of the given normal basis, one can always suppose that $\chi(f_1) \geqslant \chi(f_2) \geqslant \ldots \geqslant \chi(f_n)$. As these numbers just depend on the family $\{a_t\}$, the notation $\chi_i = \chi(f_i)$, $1 \leqslant i \leqslant n$, is justified. The numbers $\chi_1, \ldots, \chi_n$ are called the LCEs of the family $\{a_t\}$. The set of all LCEs is called the spectrum of the family $\{a_t\}$, and denoted $Sp(a_t)$. The number of repetitions of ν_i will be denoted by k_i; it is easy to see that $k_i = \dim L_i - \dim L_{i+1}, 1 \leqslant i \leqslant s$ (1).

2.2. The Hadamard lemma recalled above implies that, for any basis $g_1, \ldots, g_n$ of E_0, one has the inequality

$$\limsup_{t \to \infty} \frac{1}{t} \ln |\det a_t| \leqslant \sum_{i=1}^{n} \chi(g_i).$$

The family $\{a_t\}$ is called *regular* if all the mappings a_t are invertible, if

$$\lim_{t \to \infty} \frac{1}{t} \ln |\det a_t|$$

exists and is finite, and if there exists a basis $f_1, \ldots, f_n$ of E_0 such that

$$\lim_{t \to \infty} \frac{1}{t} \ln |\det a_t| = \sum_{i=1}^{n} \chi(f_i). \tag{4}$$

Any such basis $f_1, \ldots, f_n$ is clearly a normal one. Thus (4) implies that the LCEs $\{\chi(f_i)\}_{1 \leqslant i \leqslant n}$ of a regular family $\{a_t\}$ are all finite.

When in the definition of LCEs

$$\limsup_{t \to \infty}$$

(1) Properly speaking, by LCEs of the family $\{a_t\}$ one usually understands the set $\{\nu_i\}_{1 \leqslant i \leqslant s}$ and by its spectrum the set $\nu_i, k_i \; {}_{1 \leqslant i \leqslant s}$ (see [5]).

can be replaced by

$$\lim_{t \to \infty}$$

then we will say that *exact* LCEs exist. We can now formulate the first theorem on LCEs, which is proven in Sec. 5.

THEOREM 1. Let $\{a_t\}$ be a regular family. Then:

a. The exact LCEs of any order exist: in particular, for any $0 \neq g \in E_0$,

$$\chi(g) = \lim_{t \to \infty} \frac{1}{t} \ln \|a_t g\|;$$

b. For any p-dimensional subspace $e^p \subset E_0$ one has

$$\chi(e^p) = \sum_{k=1}^{p} \chi_{i_k},$$

with a suitable sequence $1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n$;

c. For any p-dimensional subspace $e^p \subset E_0$ one has

$$\chi(e^p) = \min \sum_{i=1}^{p} \chi(g_i),$$

where the minimum is taken over all bases $\{g_1, \ldots, g_p\}$ of e^p.

Remark that, if one denotes by $b_t : e^p \to a_t(e^p)$ the restriction of a_t to the subspace e^p, then point c of Theorem 1 is equivalent to the fact that the regularity of the family $\{a_t\}$ implies the regularity of the family $\{b_t\}$. Moreover, notice that for $e^p = E_0$ (*i. e.* $p = n$), point b reads

$$\chi(E_0) = \sum_{i=1}^{n} \chi_i$$

and is an immediate consequence of regularity, because

$$\chi(E_0) = \lim_{t \to \infty} \frac{1}{t} \ln |\det a_t|.$$

Point a is explicitly stated in Sec. 1 of [5]; points b and c, which are not explicitly formulated there, can be rather casily proven as a byproduct of the classical proof of a; points b and c, in a slightly different form, are stated in [17] and also reported in Remark 2.2.6 of [16] and in Sec. 2.1 of [19]. The proof of Theorem 1 reported in Sec. 5 extends the proof of point a given in [5].

2.3. Let us now add some comments in connection with the computation of the LCEs.

If one could know a priori the sequence (3) of subspaces $L_1, L_2, \ldots, L_s$ of E_0 and perform exact computations, then one would in principle be able to estimate all LCEs, i.e. $\nu_1, \ldots, \nu_s$; indeed, by taking an initial vector ν in $L_i \setminus L_{i+1}$ one would obtain

$$\nu_i = \lim_{t \to \infty} \frac{1}{t} \ln \|a_t g\|, \; 1 \leqslant i \leqslant s.$$

Now, apart from L_1, all the remaining subspaces $L_2, \ldots, L_s$ have positive codimension and thus vanishing Lebesgue mesaure. It is then clear that by a 'random' choice of $g \in E_0$ in the usual sense one will obtain ν_1; actually, this remak is just at the basis of the numerical technique for computing the maximal LCE ν_1 as shown in [6].

By the way, notice that, even if in some special examples one could happen to know a priori the subspaces $L_2, \ldots, L_s$ and so one could choose for example $g \in L_2$, then the computational errors would nevertheless lead to the actual computation of ν_1 and not of ν_2; this will be shown on an example in Sec. 6.2 of Part II of this work [15].

Thus we are confronted with the problem of computing all LCEs. Now, as the random choice of a 1-dimensional subspace leads to the computation of the maximal LCE of order 1, we will see below that the random choice of a p--dimensional subspace leads to the computation of the maximal LCE of order $p, 1 \leqslant p \leqslant n$, and thus, by Theorem 1, it will then be possible to compute all LCEs.

Indeed, by Theorem 1, it is very easy to see that, given a p-dimensional subspace $e^p \subset E_0$, the necessary and sufficient condition to be satisfied in order to have $\chi(e^p) = \chi_1 + \ldots + \chi_p$, is the following

Condition R (at random): for all j, $2 \leqslant j \leqslant s$ one has

$$\dim (e^p \cap L_j) = \max \left(0, p - \sum_{i=1}^{j-1} k_i\right).$$

In the following we will also say that p vectors spanning e^p satisfy Condition R if e^p does. On the other hand, it is clear that such condition will be satisfied if the subspace e^p is generated by p vectors chosen at random in E_0 in the usual sense.

Thus we obtain the following theorem, which is at the basis of the numerical algorithm for the computation of all LCEs.

THEOREM 2. Let $\{a_t\}$ be a regular family and e^p a p-dimensional subspace of $E_0 (1 \leqslant p \leqslant n)$ satisfying Conditition R with respect to the family $\{a_t\}$. Then one has

$$\chi(e^p) = \sum_{i=1}^{p} \chi_i.$$

3. The Oseledec Noncommutative Ergodic Theorem and Lyapunov Characteristic Exponents for differentiable dynamical system.

3.1. Consider now the case in which one is given a diffeomorphism T of class C^1 of a compact connected Riemannian manifold M onto itself (here and in the following all manifolds may be with border). Given $x \in M$, let T_xM be the tangent space to M at x and denote by $\mathrm{d}T_x^t : T_xM \to T_{Tx}M$, the tangent mapping of T^t, where t is a nonnegative integer. Then the problem naturally arises whether the family of linear mappings $\{\mathrm{d}T_x^t\}$ is regular.

An analogous problem occurs for differentiable flows, and all considerations of this section are also valid in such case. The same can be said for other classes of mappings or flows, as for example billard flows in domains of finite volume with a sufficiently regular border (see [5] and [12]). As a matter of fact, the following considerations hold in any case in which the general Oseledec Noncommutative Ergodic Theorem (see Sec. 6) can be applied. However, for definiteness, we will restrict ourselves in this section to the framework indicated above.

Nevetheless, also in that framework a slight generalization is of interest, namely the case in which the diffeomorphism T is defined on a manifold N, not necessarily compact, which contains a compact differentiable submanifold M invariant under T (i.e. with $TM = M$). This is indeed the case occurring in Hamiltonian systems when an energy surface has a compact invariant component. In particular, the case of a diffeomorphism of a compact manifold onto itself just corresponds to the case $N = M$.

Precisely, let N be an n-dimensional Riemannian manifold of classe C^1, with Riemannian metric ρ, and let T_xN denote the tangent space to N at $x \in N$, with the induced scalar product $(\ ,\ )_x$ and corresponding norm $\|\cdot\|$. Let T be a C^1 diffeomorphism of N onto itself, and M a compact m-dimensional C^1 submanifold of N, invariant under T. Obviously, $T_xM \subset T_xN$ for every $x \in M$.

The well known Krylov-Bogoliubov theorem (see for example [20]) implies that the set of all T-invariant Borel probability (i.e. nonnegative and normed) measures on M is non empty. Let μ be one of such measures.

3.2. The following theorem contains the heart of the Oseledec Noncommutative Ergodic Theorem in the present framework.

The passage from the general Oseledec Noncommutative Ergodic Theorem to the theorem formulated below is sketched in Sec. 6.

THEOREM 3. Let M, N, T, μ etc. be as above. Then there exist a measurable subset $M_1 \subset M$, $\mu(M_1) = 1$, such that for every $x \in M_1$ the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_xN \to T_{T^tx}N$, is regular.

The points $x \in M$ for which the above family is regular will be called T-regular points (an analogous terminology will be used for flows). Then, from point c of Theorem 1 one immediately deduces the regularity of the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_xM \to T_{T^tx}M$, for all T-regular points x.

This theorem, together with the results of Sec. 2 and the above remark, immediately implies the following one (usually called the Oseledec Noncommutative Ergodic Theorem, when $N = M$).

THEOREM 4. Let M, N, T, μ etc. be as above. Then there exists a measurable subset $M_1 \subset M$, $\mu(M_1) = 1$, such that, for every $x \in M_1$, one has:

1) considering the family $\mathrm{d}T_x^t$, where $\mathrm{d}T_x^t : T_xM \to T_{T^tx}M$,

a. The LCEs of any order exist. In particular, for any $g \in T_xM, g \neq 0$, the finite limit

$$\chi(x, g) \stackrel{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \ln \| \mathrm{d}T_x^t g \|$$

exists. As $g \neq 0$ varies in T_xM, $\chi(x, g)$ takes $s \leqslant m$ different values $\nu_1(x) > \ldots > \nu_s(x)$, where $s = s(x)$;

b. The set of vectors $g \in T_xM$ such that $\chi(x, g) \leqslant \nu_i(x)$,

$1 \leqslant i \leqslant s$, is a linear subspace of $T_x M$, denoted $L_i(x)$, and one has

$$\{0\} \overset{\text{def}}{=} L_{s+1}(x) \subset L_s(x) \subset \ldots \subset L_1(x) = T_x M;$$

if $g \in L_i \setminus L_{i+1}$, then $\chi(x, g) = \nu_i$, $1 \leqslant i \leqslant s$;

c. Denoting $k_i(x) = \dim L_i(x) - \dim L_{i+1}(x)$, then one has

$$\sum_{i=1}^{s} k_i(x)\, \nu_i(x) = \lim_{t \to \infty} \frac{1}{t} \ln |\det \mathrm{d}T_x^t|.$$

2) Considering the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x N \to T_{T^t x} N$, analogous properties a, b and c hold.

Clearly, as M is compact, the LCEs defined above do not depend on the choice of the Riemannian metric ρ of N, neither does

$$\lim_{t \to \infty} \frac{1}{t} \ln |\det \mathrm{d}T_x^t|.$$

It can be of interest to remark that, by taking simultaneously into consideration the LCEs for diffeomorphisms T and T^{-1}, one could produce a more detailed formulation of the points 1b and 2b of Theorem 4 (see [5], [16], [17] and [21]). Moreover, the Oseledec Theorem and its Raghunathan improvement are in fact applicable to mappings which are only almost everywhere differentiable. Notice (see Sec. 6) that in [5] only the condition

$$\int_M \ln^+ \| \mathrm{d}T_x^{\pm 1} \| \,\mathrm{d}\mu < \infty,$$

where $\ln^+ a = \max(\ln a, 0)$, is assumed instead of our stronger condition that T be a diffeomorphism of class C^1. Finally, from the improvement of Raghunathan one can deduce Theorem 4 for any mapping T not necessarily invertible satisfying the condition

$$\int_M \ln^+ \| \mathrm{d}T_x \| \,\mathrm{d}\mu < \infty\ ;$$

but in such case one cannot guarantee the finiteness of all LCEs. However, the formulation given in Theorem 4 suffices for the purposes of the present paper.

Let us denote the spectrum of the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x M \to T_{T^t x} M$, by $Sp_M(x)$ and the spectrum of the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x N \to T_{T^t x} N$, by $Sp_N(x)$. The only general relation between the two spectra is $Sp_M(x) \subset Sp_N(x)$. Clearly one has, for any t, $Sp_M(x) = Sp_M(T^t x)$ and $Sp_N(x) = Sp_N(T^t x)$.

We can now also formulate in the present framework the theorem corresponding to Theorem 2, which is the relevant one for the numerical computation of all LCEs.

THEOREM 5. Let M, N, T, μ etc. be as above. Then there exists a measurable subset $M_1 \subset M$, $\mu(M_1) = 1$, such that, for every $1 \leqslant p \leqslant n$, if $g_1, \ldots, g_p \in T_x N$ satisfy condition R with respect to the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x N \to T_{T^t x} N$, one has

$$\lim_{t \to \infty} \frac{1}{t} \ln \mathrm{Vol}^p([\mathrm{d}T_x^t g_1, \ldots, \mathrm{d}T_x^t g_p]) = \sum_{i=1}^{p} \chi_i(x),$$

where $[\mathrm{d}T_x^t g_1, \ldots, \mathrm{d}T_x^t g_p]$ denotes the open parallelepiped generated by the vectors $\mathrm{d}T_x^t g_1, \ldots, \mathrm{d}T_x^t g_p$.

An analogous statement could be given for the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x M \to T_{T^t x} M$. Our concern with $Sp_N(x)$ arises from the fact that, although being interested in $Sp_M(x)$, in many cases we can only handle the mappings $\mathrm{d}T_x^t : T_x N \to T_{T^t x} N$; this occurs for example in the Hamiltonian case. More special relations between the two spectra $Sp_M(x)$, $Sp_N(x)$ in the symplectic and in the Hamiltonian cases will be worked out in the next section.

4. Lyapunov Characteristic Exponents in the presence of a Liouville mesaure. The symplectic and Hamiltonian cases.

4.1. Denote by ν_ρ the normalized volume induced on M by the Riemannian metric ρ (for details, see for example [22]). In addition to the hypotheses concerning the manifolds M, N and the diffeomorphism T made in Sec. 3, we will now suppose that T admits a Liouville mesaure on M, i.e. that there exists a T-invariant probability measure μ_L on M which is absolutely continuous with respect to the mesaure ν_ρ (examples with no Liouville measure can be trivially found already among the diffeomorphisms of the circle). Denote by $\mathrm{d}\mu_L \setminus \mathrm{d}\nu_\rho$ the Radon-Nikodym derivative (see for example [23]) of μ_L with respect to ν_ρ; we will also write $\mathrm{d}\mu_L \setminus \mathrm{d}\nu_\rho(x) = \phi(x)$.

The following results are well known. For a proof see Sec. 7.

PROPOSITION 6. In the above assumptions, if

$$\int_M |\ln \phi| \,\mathrm{d}\mu_L < \infty, \tag{1}$$

then one has, μ_L-almost everywhere on M,

$$\lim_{t \to \infty} \frac{1}{t} \ln |\det \mathrm{d}T_x^t| = 0,$$

where $\mathrm{d}T_x^t : T_x M \to T_{T^t x} M$.

Obviously, condition (1) is always satisfied if ϕ is a positive continuous function on M.

From point 1.c of Theorem 4 one then immediately deduces

COROLLARY 7. If the diffeomorphism T admits a Liouville measure μ_L and condition (1) is satisfied, then μ_L-almost everywhere on M one has

$$\sum_{i=1}^{m} \chi_i(x) = 0.$$

4.2. Let us now recall some basic notions on symplectic diffeomorphisms and on Hamiltonian systems. For more details, see [24].

The coordinates of a vector of $\mathbf{R}^{2n}$ will be denoted, as usual in this context, by $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$; if A is a linear mapping of $\mathbf{R}^{2n}$ onto itself, by the corresponding script letter $\mathscr{A}$ we will denote the matrix of A in the standard basis $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ etc. of $\mathbf{R}^{2n}$. Let

$$\mathscr{T} = \begin{pmatrix} 0 & -\mathscr{E}_n \\ \mathscr{E}_n & 0 \end{pmatrix}$$

be the $2n \times 2n$ matrix in which $\mathscr{E}_n$ is the $n \times n$ identity matrix.

A linear mapping A of $\mathbf{R}^{2n}$ onto itself is called symplectic if $\mathscr{A}'\mathscr{T}\mathscr{A} = \mathscr{T}$, where $\mathscr{A}'$ is the transposed matrix of $\mathscr{A}$. The linear symplectic mappings on $\mathbf{R}^{2n}$ form a group, denoted $Sp(2n, \mathbf{R})$. If $A \in Sp(2n, \mathbf{R})$, then $\det A = 1$.

If U is an open subset of $\mathbf{R}^{2n}$, a diffeomorphism T of U onto itself is called symplectic if for every $x \in U$ one has $\mathrm{d}T_x \in Sp(2n, R)$. As $\det \mathrm{d}T_x = 1$ for every $x \in U$, then the Lebesgue measure on U is T-invariant. In an evident manner this definition makes sense also for diffeomorphisms of the torus $\mathbf{T}^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$. Examples of symplectic diffeomorphisms of the torus will be considered in Part II [15].

4.3. We come now to the case of Hamiltonian systems. Let U be an open subset of $\mathbf{R}^{2n}$ and $H \in C^2(U)$. Consider in U the Hamiltonian equations with Hamiltonian H, i.e.

$$\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H(q, p)}{\partial q_i} \quad 1 \leqslant i \leqslant n.$$

We will suppose once for all that any solution $(q(t), p(t))$ of such equations is defined for all $t \in \mathbf{R}$, so that the Hamiltonian flow $\{T^t\}$ on U is well defined and is of class C^1. One has then that $\mathrm{d}T^t_x \in Sp(2n, \mathbf{R})$ for any $x \in U$ and $t \in \mathbf{R}$.

The surface of constant energy E, $\{(q, p) \in U;\ H(q, p) = E\}$, is $\{T^t\}$-invariant. Suppose it contains a compact connected $\{T^t\}$-invariant component Γ_E and that grad $H(q, p) \neq 0$ for $(q, p) \in \Gamma_E$; then Γ_E is a $(2n-1)$-dimensional differentiable manifold of class C^2. It is well known that in a standard way (see for example [25]), Chap. III, Sec. 4) one can define a canonical Liouville mesaure μ_L on Γ_E. This mesaure is given by the formula

$$\mu_L(A) = \frac{1}{C} \int_A \frac{\mathrm{d}\sigma}{\|\operatorname{grad} H\|},$$

where σ denotes the measure induced on Γ_E by the Lebesgue measure on $\mathbf{R}^{2n}$, $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^{2n}$, and

$$C = \int_A \frac{\mathrm{d}\sigma}{\|\operatorname{grad} H\|}.$$

For this Liouville measure condition (1) is automatically satisfied, because in such case ϕ is a positive continuous function on Γ_E. Moreover, since grad $H \neq 0$, the Hamiltonian flow $\{T^t\}$ on Γ_E has no stationary points.

4.4. We come now to the formulation of the results which constitute the main object of the present section and are proven in Sec. 8. We have first

THEOREM 8.

a. Let U be an open subset of $\mathbf{R}^{2n}$ and T a symplectic diffeomorphism of U onto itself. If $x \in \mathrm{U}$ is a T-regular point, then one has $Sp_U(x) = -Sp_U(x)$, i.e.

$$Sp_U(x) = \{\chi_1(x), \chi_2(x), \ldots, \chi_n(x), -\chi_n(x), \ldots, -\chi_2(x), -\chi_1(x)\}.$$

b. Exactly the same holds for the symplectic diffeomorphisms of $\mathbf{T}^{2n}$.

Suppose now furthermore that $\Gamma \subset U$ is a $(2n-1)$-dimensional compact submanifold of class C^1, T-invariant and such that $T: \Gamma \to \Gamma$ preserves a Liouville measure μ_L. Then from Theorem 3 we know that μ_L-almost all points $x \in \Gamma$ are T-regular. Thus, if condition (1) is satisfied, we have the following corollary, for the formulation of which we use the same terminology as in connection with Theorems 3 and 4 of Sec. 3 (with $M = \Gamma$ and $N = U$):

COROLLARY 9. For μ_L-almost all points $x \in \Gamma$ one has

$$Sp_\Gamma(x) = \{\chi_1(x), \chi_2(x), \ldots, \chi_{n-1}(x), 0, -\chi_{n-1}(x), \ldots, -\chi_2(x), -\chi_1(x)\},$$

$$Sp_U(x) = \{\chi_1(x), \chi_2(x), \ldots, \chi_{n-1}(x), 0, -\chi_{n-1}(x), \ldots, -\chi_2(x), -\chi_1(x)\}.$$

The same result also holds for Hamiltonian flows $\{T^t\}$ on Γ_E. Indeed, in such case T^t is, for any t, a symplectic diffeomorphism, and on the other hand the LCEs of the flow $\{T^t\}$ are evidently the same as those for the diffeomorphism T^1.

As a final remark, it is very easy to show that, if one has a differentiable flow on a compact manifold without stationary points, as in our assumptions of Sec. 4.3, then the LCE of a vector tangent to the orbit of the flow vanishes.

Notice that, in the proof of Theorem 8, given in Sec. 8, we will make use of a lemma which goes back to Perron (see [26] and also Sec. 3 of [11]) and is of some interest by itself.

B. PROOFS OF THE RESULTS OF CHAPTER A.

5. Proof of Theorem 1.

5.1. We are concerned with a regular family $\{a_t\}$, t being a nonnegative integer or real number, of linear mappings $a_t: E_0 \to E_t$, where $\{E_t\}$ is a family of real n-dimensional vector spaces, each endowed with a scalar product $(,)_t$. We recall that the family $\{a_t\}$ is called regular if all the mappings a_t are invertible, if

$$\lim_{t\to\infty} \frac{1}{t} \ln|\det a_t|$$

exists and is finite, and if there eixsts a basis $f_1, \ldots, f_n$ of E_0 such that

$$\lim_{t\to\infty} \frac{1}{t} \ln|\det a_t| = \sum_{i=1}^{n} \chi(f_i). \tag{1}$$

Any such basis is a normal basis for the family $\{a_t\}$.

First of all we remark that the problem of proving Theorem 1 can be reduced trivially to the same problem for a regular family $\{b_t\}$ of mappings of a fixed n-dimensiona real vector space E onto itself endowed with a scalar product $(,)$. Indeed, consider any isomorphism h_t of the space $(E_t, (,)_t)$ onto the space $(E, (,))$. Denote $b_t = h_t a_t h_0^{-1}$; for any t this is a linear mapping of E onto itself. As h_t preserves volumes and lenghts, it is clear that $\{a_t\}$ is regular if and only if $\{b_t\}$ is. It is then sufficient to prove Theorem 1 for the family $\{b_t\}$, as we will do.

In the following, when speaking of a space E we will mean an n-dimensional space E endowed with a fixed scalar product $(,)$.

5.2. In a space E fix an orthonormal basis $e_1, \ldots, e_n$ and consider the corresponding sequence of subspaces $F_1 \subset F_2 \subset \subset \ldots \subset F_n = E$ defined by the property that $e_1, \ldots, e_i$ span F_i for any $1 \leqslant i \leqslant n$. If a linear invertible mapping k of E onto itself has the property that $k(F_i) = F_i$ for all $1 \leqslant i \leqslant n$ we will say that k is uppertriangular with respect to the given basis, because such is the matrix $\mathscr{K}$ of the mapping k in the basis $e_1, \ldots, e_n$, i.e. one has

$$\mathscr{K} = \begin{pmatrix} K_1, & K_{12}, & K_{13}, \ldots, K_{1n} \\ & K_2, & K_{23}, \ldots, K_{2n} \\ & 0 & \ddots \quad \vdots \\ & & K_n \end{pmatrix}.$$

The diagonal elements $K_1, \ldots, K_n$ are clearly the eigenvalues of k.

PROPOSITION A. Let $e_1, \ldots, e_n$ be a fixed orthonormal basis in a space E. Then any invertible linear mapping b of E onto itself has a unique factorization $b = uk$, where u and k are linear mappings of E onto itself, u being orthogonal and k uppertriangular (with respect to the given basis) with positive eigenvalues.

Proof. Define the subspaces $G_1 \subset G_2 \subset \ldots \subset G_n = E$ by $G_i = b(F_i)$ for $1 \leqslant i \leqslant n$ and let $g_1, \ldots, g_n$ be an orthonormal basis such that $g_1, \ldots, g_i$ span G_i for $1 \leqslant i \leqslant n$. Such basis is unique up to orientation of the axes. Define the mappings v and k by $vg_i = e_i$ for $1 \leqslant i \leqslant n$ and $k = vb$. Clearly v is orthogonal and k is uppertriangular with respect to the original basis. By a suitable unique orientation of $g_1, \ldots, g_n$ one can always obtain a mapping k with positive eigenvalues.

Then the proposition is proven with $u = v^{-1}$.

PROPOSITION B. Let $\{b_t\}$ be a family of linear mappings of a space E onto itself, such that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \| b_t \| < \infty.$$

Fix a sequence of linear subspaces $F_1 \subset F_2 \subset \ldots \subset F_n = E$ such that $\dim F_i = i$ for any $1 \leqslant i \leqslant n$. Then there exists a basis $f_1, \ldots, f_n$ of E such that, for any $1 \leqslant p \leqslant n$, $f_1, \ldots, f_p$ is a basis of F_p and

$$\sum_{i=1}^{p} \chi(f_i) \leqslant \sum_{i=1}^{p} \chi(g_i)$$

for any other basis $g_1, \ldots, g_p$ of F_p. In particular $f_1, \ldots, f_n$ is a normal basis.

Proof. Such a basis is constructed as follows: take any $0 \neq f_1 \in F_1$ and, for $2 \leqslant i \leqslant n$, any $f_i \in F_i \backslash F_{i-1}$ such that $\chi(f_i) \leqslant \chi(g)$ for any $g \in F_i \backslash F_{i-1}$ (at least one such vector exists because the set of LCEs of the family $\{b_t\}$ is finite). Then $f_i, \ldots, f_p$ is clearly a basis of F_p for all $1 \leqslant p \leqslant n$.

The lemma is proven by induction with respect to p. For $p = 1$ the statement is trivial. Let $1 \leqslant p \leqslant n-1$ and suppose that one has

$$\sum_{i=1}^{p} \chi(f_i) \leqslant \sum_{i=1}^{p} \chi(g_i)$$

for any basis $g_1, \ldots, g_p$ of F_p. We have to prove that one has

$$\sum_{i=1}^{p+1} \chi(f_i) \leqslant \sum_{i=1}^{p+1} \chi(g_i)$$

for any basis $g_1, \ldots, g_{p+1}$ of F_{p+1}.

As the proposition is assumed to be true for p, then the latter inequality is evident for those bases $g_1, \ldots, g_{p+1}$ of F_{p+1} such that $g_1, \ldots, g_p \in F_p$. When this is not the case, we construct another basis $\tilde{g}_1, \ldots, \tilde{g}_{p+1}$ of F_{p+1} such that $\tilde{g}_1, \ldots, \tilde{g}_p \in F_p$ and

$$\sum_{i=1}^{p+1} \chi(\tilde{g}_i) \leqslant \sum_{i=1}^{p+1} \chi(g_i),$$

so that the inequality is again evident, as

$$\sum_{i=1}^{p} \chi(f_i) \leqslant \sum_{i=1}^{p} \chi(\tilde{g}_i)$$

and $\chi(f_{p+1}) \leqslant \chi(\tilde{g}_{p+1})$.

The construction of the vectors $\tilde{g}_1, \ldots, \tilde{g}_{p+1}$ is the following. Given a basis $g_1, \ldots, g_{p+1}$ of F_{p+1}, there exists an integer s, $0 \leqslant s \leqslant p$, such that s vectors of the basis belong to F_p and the remaining ones to $F_{p+1} \backslash F_p$. With no restriction of generality we can suppose that $g_1, \ldots, g_s \in F_p$, that $g_{s+1}, \ldots, g_{p+1} \in F_{p+1} \backslash F_p$ and that $\chi(g_{p+1}) \leqslant \chi(g_i)$ for all i with $s < i \leqslant p$. Define now $\tilde{g}_i = g_i$ for $1 \leqslant i \leqslant s$ and for $i = p+1$. For $s < i \leqslant p$ define instead $\tilde{g}_i = g_i + c_i g_{p+1}$ with the unique real c_i such that $\tilde{g}_i \in F_p$ (i.e. define $\tilde{g}_i$ as the projection of g_i onto F_p along the direction of g_{p+1}). From the inequality $\chi(h+g) \leqslant \max(\chi(h), \chi(g))$, true for any $h, g \in E$ (see Sec. 2.1), one has then $\chi(\tilde{g}_i) \leqslant \leqslant \max(\chi(g_i), \chi(g_{p+1})) = \chi(g_i)$ for any i with $s \leqslant i \leqslant p$. Then $\tilde{g}_1, \ldots, \tilde{g}_{p+1}$ is the required basis and this concludes the proof of the proposition.

5.3. We come now to the heart of the proof ot Theorem 1.

For any p-dimensional subspace e^p we have to show that the exact limit $\chi(e^p)$ exists. Let us consider an arbitrary sequence of subspaces $F_1 \subset F_2 \subset \ldots \subset F_n = E$, $\dim F_i = i$, $1 \leqslant i \leqslant n$, such that $F_p = e^p$, and correspondingly an orthonormal basis $e_1, \ldots, e_n$ such that $e_1, \ldots, e_i$ span F_i, $1 \leqslant i \leqslant n$.

By Proposition A we then have the unique factorization $b_t = u_t k_t$, where u_t is orthogonal and k_t uppertriangular (with respect to the basis $e_1, \ldots, e_n$) with positive eigenvalues $K_i(t)$, $1 \leqslant i \leqslant n$. Denote

$$\lambda_i = \limsup_{t \to \infty} \frac{1}{t} \ln K_i(t). \tag{2}$$

We first show that for any normal basis $f_1, \ldots, f_n$ corresponding to the sequence $F_1, \ldots, F_n$ according to Proposition B one has $\chi(f_i) = \lambda_i$, $1 \leqslant i \leqslant n$, where both $\chi(f_i)$ and λ_i are exact limits.

Indeed, as u_t is orthogonal, then

$$\| b_t f_i \| = \| k_t f_i \| \geqslant |c_i| K_i(t), \qquad 1 \leqslant i \leqslant n, \tag{3}$$

where $c_1, \ldots, c_n$ are nonvanishing real numbers independent of t; this is seen because $f_i = c_i e_i + h_{i-1}$ with $h_{i-1} \in F_{i-1}$, $k_t(F_{i-1}) = F_{i-1}$ and e_i is orthogonal to F_{i-1}. Thus, from (3) one gets

$$\chi(f_i) \geqslant \lambda_i, \qquad 1 \leqslant i \leqslant n. \tag{4}$$

Using now the regularity of the family $\{b_t\}$, we will prove that

$$\chi(f_i) = \lambda_i \qquad 1 \leqslant i \leqslant n. \tag{5}$$

Indeed, from

$$|\det b_t| = \prod_{i=1}^{n} K_i(t), \tag{6}$$

one has by (4) and (1)

$$\begin{aligned} \lim_{t \to \infty} \frac{1}{t} \ln |\det b_t| &\leqslant \sum_{i=1}^{n} \lambda_i \leqslant \\ &\leqslant \sum_{i=1}^{n} \chi(f_i) = \lim_{t \to \infty} \frac{1}{t} \ln |\det b_t| \end{aligned} \tag{7}$$

i.e.

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \chi(f_i),$$

so that (5) follows from (4). Notice in particular that from (7) we have furthermore

$$\sum_{i=1}^{n} \lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln |\det b_t|, \tag{8}$$

which will be used below.

We show that the exact limits

$$\lim_{t \to \infty} \frac{1}{t} \ln K_i(t)$$

exist. By (6),

$$\frac{1}{t} \ln K_i(t) = \frac{1}{t} \ln |\det b_t| - \sum_{j \neq i} \frac{1}{t} \ln K_j(t). \tag{9}$$

Given any $\epsilon > 0$, for all sufficiently large t and all $1 \leqslant j \leqslant n$ one has

$$\frac{1}{t} \ln K_j(t) \leqslant \lambda_j + \epsilon$$

and thus, from (9),

$$\begin{aligned} \frac{1}{t} \ln K_i(t) &\geqslant \frac{1}{t} \ln |\det b_t| - \sum_{j \neq i} (\lambda_j + \epsilon) = \\ &= \frac{1}{t} \ln |\det b_t| - \lim_{t \to \infty} \frac{1}{t} \ln |\det b_t| + \lambda_i - (n-1)\epsilon, \end{aligned}$$

where (8) has been used. Thus we have

$$\liminf_{t \to \infty} \frac{1}{t} \ln K_i(t) \geqslant \lambda_i,$$

and, in virtue of (2), we obtain the existence of the exact limits

$$\lim_{t \to \infty} \frac{1}{t} \ln K_i(t) = \lambda_i, \qquad 1 \leqslant i \leqslant n. \tag{10}$$

The fact that the exact limits

$$\lim_{t \to \infty} \frac{1}{t} \ln \| b_t f_i \| = \lambda_i$$

exist for all $1 \leqslant i \leqslant n$ immediately follows from (3), (5) and (10).

As a corollary we also obtain that

$$\left\{ \lim_{t \to \infty} \frac{1}{t} \ln K_i(t) \right\}_{1 < i < n} = \{\chi_j\}_{1 < j < n},$$

where $\{\chi_j\}_{1 < j < n}$ denotes the LCEs of the family $\{b_t\}$.

Moreover, as

$$\begin{aligned} \mathrm{Vol}^p(b_t(e^p)) &= \mathrm{Vol}^p([b_t e_1, \ldots, b_t e_p]) = \\ &= \mathrm{Vol}^p([k_t e_1, \ldots, k_t e_p]) = \prod_{i=1}^{p} K_i(t), \end{aligned}$$

we obtain

$$\begin{aligned} \chi(e^p) &= \lim_{t \to \infty} \frac{1}{t} \ln \mathrm{Vol}^p(b_t(e^p)) = \\ &= \sum_{i=1}^{p} \lim_{t \to \infty} \frac{1}{t} \ln K_i(t) = \sum_{i=1}^{p} \lambda_i = \sum_{r=1}^{p} \chi_{i_r} \end{aligned}$$

for a suitable set of indices $1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n$. Thus we get that the exact LCE $\chi(e^p)$ exists and that

$$\chi(e^p) = \sum_{r=1}^{p} \chi_{i_r}.$$

So we have explicitly proven points a and b of Theorem 1.

Point c is also a direct consequence of the demonstration, as $\{\chi_{i_r}\}_{1<r<p} = \{\chi(f_i)\}_{1<i<p}$.

6. General form of the Oseledec Noncommutative Ergodic Theorem and application to the differentiable case.

6.1. Let X be a set, $\mathscr{A}$ a σ-algebra of subsets of X and μ a probability measure defined on $\mathscr{A}$. By an endomorphism of the measure space $(X, \mathscr{A}, \mu)$ one understands a measurable and measure preserving mapping of X into itself, i.e. a mapping $T: X \to X$ such that for any $A \in \mathscr{A}$ one has $T^{-1}(A) \in \mathscr{A}$ and $\mu(A) = \mu(T^{-1}(A))$. An endomorphism admitting a measurable inverse is called an automorphism. For more details see for example [27] or [28].

Let n be a natural number. A measurable mapping $X \times \mathbf{Z} \ni (x, t) \mapsto \mathscr{N}(t, x) \in GL(n, \mathbf{R})$ is called a multiplicative cocycle with respect to T with values in $GL(n, \mathbf{R})$ if, for all $s, t \in \mathbf{Z}$ and μ-almost all $x \in X$ one has

$$\mathscr{N}(t+s, x) = \mathscr{N}(s, T^t x)\, \mathscr{N}(t, x).$$

A standard example of a multiplicative cocycle is $\mathscr{N}(t, x) = \mathrm{d}T_x^t$, where T is a diffeomorphism of an open subset of $\mathbf{R}^n$ onto itself.

Let $\|\cdot\|$ be a fixed norm on $\mathbf{R}^n$. The heart of the Oseledec Noncommutative Ergodic Theorem (see [5]) is contained in the following

THEOREM C. Let $\mathscr{N}$ be a multiplicative cocycle with respect to an automorphism T with values in $GL(n, \mathbf{R})$, such that

$$\int_X \ln^+ \|\mathscr{N}(1, x)\| \,\mathrm{d}\mu < \infty \quad \text{and}$$

$$\int_X \ln^+ \|\mathscr{N}(1, x)^{-1}\| \,\mathrm{d}\mu < \infty. \tag{1}$$

Then, for μ-almost all points $x \in X$ the family $\{\mathscr{N}(t, x)\}_{t\in Z}$ of linear mappings of $\mathbf{R}^n$ onto $\mathbf{R}^n$ is regular.

In particular, one thus deduces that Theorem 1 of Sec. 2 applies to the family $\{\mathscr{N}(t, x)\}_{t\in Z}$ for μ-almost all points $x \in X$.

6.2. We sketch here how one deduces Theorem 3 of Sec. 3.2 from Theorem C (see also [16]).

Let N be a C^1 Riemannian manifold, T a C^1 diffeomorphism of N onto itself such that $T(M) = M$, where M is a C^1 compact submanifold of N. Let μ be a T-invariant Borel probability measure defined on M. To every point $x \in M$ we associate the family $\{\mathrm{d}T_x^t\}$, where $\mathrm{d}T_x^t : T_x N \to T_{T^t x} N$. It is clear that the condition

$$\int_X \ln^+ \|\mathrm{d}T_x^{\pm 1}\| \,\mathrm{d}\mu < \infty \tag{2}$$

is satisfied.

The unique obstruction to apply Theorem C to the family $\{\mathrm{d}T_x^t\}$ is the fact that the spaces $\{T_x N\}_{x\in M}$ are not, in general, canonically isomorphic. But this difficulty is very easly overcome; indeed, by using local coordinates one can canonically indentify all the tangent spaces to N in any open set of N where such coordinates are defined.

Accordingly, in virtue of the compactness of M, one can define a family of linear mappings $\{\psi_x\}_{x\in M}$, $\psi_x : T_x N \to \mathbf{R}^n$ such that ψ_x is piecewise continuous with respect to $x \in M$ and

$$c_1 \|f\| \leqslant \|\psi_x(f)\| \leqslant c_2 \|f\| \tag{3}$$

for any $x \in M$ and $f \in T_x N$, c_1 and c_2 being positive numbers independent of $x \in M$, and $\|\psi_x(f)\|$ being the euclidean norm of $\psi_x(f)$.

Let $\mathscr{N}(t, x) = \psi_{T^t x} \circ \mathrm{d}T_x^t \circ \psi_x^{-1}$; $\mathscr{N}(t, x)$ is clearly a measurable cocycle defined on M with values in $GL(n, \mathbf{R})$. By (2) and (3), condition (1) is satisfied. Thus one can apply Theorem C to the cocycle $\mathscr{N}$, and this proves Theorem 3.

6.3. A theorem completely analogous to Theorem C holds for a measurable, measure preserving flow $\{T^t\}_{t\in\mathbf{R}}$ defined on a measure space $(X, \mathscr{A}, \mu)$, if condition

$$\int_X \sup_{|t|\leqslant 1} \ln^+ \|(t, x)\| \,\mathrm{d}\mu < \infty$$

is satisfied (see [5]).

As in Sec. 6.2, this implies that for differentiable flows of class C^1 one has a theorem analogous to Theorem 3 (and consequently one also has analogues of Theorems 4 and 5).

7. Proof of Proposition 6.

7.1. Let $x \in M$. A sequence $\{A_r\}_{r\geqslant 1}$ of open balls of M will be called an x-sequence if $x \in A_r$ for any $r \geqslant 1$ and if

$$\lim_{r\to\infty} \operatorname{diam}(A_r) = 0,$$

where

$$\operatorname{diam} B = \sup_{u,v\in B} d(u, v),$$

d denoting the distance induced on M by the Riemannian metric ρ. It is easy to see that, for any $x \in M$ and for any x-sequence $\{A_r\}_{r\geqslant 1}$, one has

$$\lim_{r\to\infty} \frac{\nu_\rho(T(A_r))}{\nu_\rho(A_r)} = |\det \mathrm{d}T_x| \tag{1}$$

From the well known Lebesgue-Vitali theorem on differentiation (see for example [23], Chap. 10) we have that

$$\phi(x) = \frac{\mathrm{d}\mu_L}{\mathrm{d}\nu_\rho}(x) = \lim_{r\to\infty} \frac{\mu_L(A_r)}{\nu_\rho(A_r)}$$

for ν_ρ-almost all points $x \in M$ and for all x-sequences $\{A_r\}_{r\geqslant 1}$.

Consequently, in virtue of (1) we obtain that, for any $t \geqslant 0$ and for μ_L-almost all points $x \in M$, one has

$$|\det \mathrm{d}T_x^t| = \lim_{r\to\infty} \frac{\nu_\rho(T^t(A_r))}{\nu_\rho(A_r)} =$$

$$= \lim_{r\to\infty} \frac{\dfrac{\nu_\rho(T^t(A_r))}{\mu_L(T^t(A_r))}}{\dfrac{\nu_\rho(A_r)}{\mu_L(A_r)}} = \frac{\lim\limits_{r\to\infty} \dfrac{\nu_\rho(T^t(A_r))}{\mu_L(T^t(A_r))}}{\lim\limits_{r\to\infty} \dfrac{\nu_\rho(A_r)}{\mu_L(A_r)}} = \frac{\phi(T^t x)}{\phi(x)},$$

because $\mu_L(A_r) = \mu_L(T^t(A_r))$. Thus, for any t, $\ln|\det dT_x^t| = \ln\phi(T_x^t) - \ln\phi(x)$ μ_L-almost everywhere. This, for any t, implies that

$$\frac{1}{t}\ln|\det dT_x^t| = \frac{1}{t}\sum_{s=1}^{t}\ln\phi(T^s x) -$$

$$-\frac{1}{t}\sum_{s=0}^{t-1}\ln\phi(T^s x).$$

As we suppose that $\ln\phi\in L^1(M,\mu_L)$ and μ_L is an invariant probability measure, from Birkhoff's ergodic theorem applied to the function $\ln\phi(T_x) - \ln\phi(x)$ one ontains that

$$\lim_{t\to\infty}\frac{1}{t}\ln|\det dT_x^t| =$$

$$= \lim_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1}[\ln\phi(T^{s+1}x) - \ln\phi(T^s x)] = 0$$

μ_L-almost everywhere on M. Proposition 6 is thus proven.

8. Proof of Theorem 8 and of Corollary 9.

8.1. The proof of Theorem 8 is an easy consequence of the following lemma which goes back to Perron (see [26] and also Sec. 3 of [11]). This lemma has the same degree of generality as Theorem 1.

Let us recall that, to every family of linear mappings $\{a_t\}$, where $a_t : E_0 \to E_t$ is invertible, one can associate the family $\{(a_t^*)^{-1}\}$, where a_t^* denotes the mapping adjoint to a_t. As all spaces E_t are endowed with a scalar product, then E_t is canonically conjugate to its dual E_t^*, and thus $(a_t^*)^{-1} : E_0 \to E_t$.

We will use the following notations. If $f_1, \ldots, f_n$ is a basis of E_0, then $\tilde f_1, \ldots, \tilde f_n$ is the dual basis, i.e. the unique basis of E_0 such that

$$(\tilde f_i, f_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

If $C = \{c_1, \ldots, c_n\}$, where $c_1, \ldots, c_n$ are real numbers, then $-C = \{-c_1, \ldots, -c_n\}$.

LEMMA D. Let $\{a_t\}$ be a regular family. Then $\{(a_t^*)^{-1}\}$ is also regular and $Sp((a_t^*)^{-1}) = -Sp(a_t)$. More precisely, if $f_1, \ldots, f_n$ is a normal basis for the family $\{a_t\}$, then the basis $\tilde f_1, \ldots, \tilde f_n$ is normal for the family $\{(a_t^*)^{-1}\}$, and furthermore one has $\chi((a_t^*)^{-1}, \tilde f_i) = -\chi(a_t, f_i)$.

Proof. As for the proof of Theorem 1, we can without loss of generality suppose that the family $\{a_t\}$ is defined on a unique real vector space E endowed with a fixed scalar product (,).

For any p-dimensional subspace e^p, we will denote by $(e^p)^\perp$ its orthogonal complement in E. As the family $\{a_t\}$ is regular, then the limit

$$\lim_{t\to\infty}\frac{1}{t}\ln|\det a_t| = \delta$$

exists and is finite, and the exact LCE $\chi(a_t, e^p)$ exists.

First of all we will show that the exact LCE $\chi((a_t^*)^{-1}, (e^p)^\perp)$ exists, and

$$\chi((a_t^*)^{-1}, (e^p)^\perp) = \chi(a_t, e^p) - \delta. \tag{1}$$

To show this, take any orthonormal basis $v_1, \ldots, v_n$ such that $v_1, \ldots, v_p$ span e^p; then $v_{p+1}, \ldots, v_n$ span $(e^p)^\perp$. According to Proposition A of Sec. 5, a_t admits the unique factorization $a_t = u_t k_t$, where u_t is orthogonal and the matrix of k_t in the basis $v_1, \ldots, v_n$, denoted $\mathscr{K}_t$, is uppertriangular;

$$\mathscr{K}_t = \begin{pmatrix} K_1(t) & K_{12}(t) & K_{13}(t) & \ldots & K_{1n}(t) \\ & K_2(t) & K_{23}(t) & \ldots & K_{2n}(t) \\ & O & \ddots & & \vdots \\ & & & & K_n(t) \end{pmatrix}.$$

From the factorization $a_t = u_t k_t$, as u_t is orthogonal, we deduce immediately that $(a_t^*)^{-1} = u_t (k_t^*)^{-1}$. The matrix of $(k_t^*)^{-1}$ in the basis $v_1, \ldots, v_n$ is the matrix $(\mathscr{K}_t')^{-1}$, where ′ denotes transposition. Such matrix is lowertriangular and its diagonal elements are

$$\frac{1}{K_1(t)}, \ldots, \frac{1}{K_n(t)}.$$

It is then clear that one has

$$\mathrm{Vol}^p([a_t v_1, \ldots, a_t v_p]) = K_1(t)\ldots K_p(t) =$$

$$= (\det k_t)(K_{p+1}(t)\ldots K_n(t))^{-1} =$$

$$= |\det a_t|\,\mathrm{Vol}^{n-p}([(a_t^*)^{-1}v_{p+1}, \ldots, (a_t^*)^{-1}v_n]).$$

This equality immediately implies the existence of the exact LCE $\chi((a_t^*)^{-1}, (e^p)^\perp)$ and the equality (1).

In particular, from (1), for any f one has

$$\chi((a_t^*)^{-1}, f) = \chi(a_t, f^\perp) - \delta. \tag{2}$$

Let now a normal basis $f_1, \ldots, f_n$ for the family $\{a_t\}$ be given, and consider the corresponding basis $\tilde f_1, \ldots, \tilde f_n$. From (2) and point c of Theorem 1 we have, for any $1 \leqslant i \leqslant n$,

$$\chi((a_t^*)^{-1}, \tilde f_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \chi(a_t, f_j) - \delta. \tag{3}$$

As from the regularity of $\{a_t\}$ we have

$$\delta = \sum_{j=1}^{n}\chi(a_t, f_j), \tag{4}$$

then (3) implies

$$\chi((a_t^*)^{-1}, \tilde f_i) = -\chi(a_t, f_i). \tag{5}$$

Finally, from (5) and (4), recalling $\det((a_t^*)^{-1}) = (\det a_t)^{-1}$, one has

$$\sum_{i=1}^{n} \chi((a_t^*)^{-1}, \tilde{f}_i) = -\delta = \lim_{t\to\infty} \frac{1}{t} \ln|\det (a_t^*)^{-1}|,$$

and thus one sees that the family $\{(a_t^*)^{-1}\}$ is regular and that $\tilde{f}_1, \ldots, \tilde{f}_n$ is a normal basis for it. Lemma D is then proven.

Consider now the space $\mathbf{R}^{2n}$ with its canonical basis $e_1, \ldots, e_{2n}$ (see Sec. 4.2) and its canonical scalar product (,). If a is a linear mapping of $\mathbf{R}^{2n}$ into itself, by a^* we denote its adjoint linear mapping with respect to the canonical scalar product (,). To prove Theorem 8 it suffices now to demonstrate the following

PROPOSITION E. If $\{a_t\}$ is a regular family of symplectic mappings of $\mathbf{R}^{2n}$, then one has $Sp((a_t^*)^{-1}) = Sp(a_t)$.

Proof. Denote by $\mathscr{A}_t$ the matrix of the mapping a_t in the baisis $e_1, \ldots, e_{2n}$. Recall that a_t is symplectic if one has $\mathscr{A}_t' \mathscr{T} \mathscr{A}_t = \mathscr{T}$, or equivalently

$$(\mathscr{A}_t')^{-1} = -\mathscr{T} \mathscr{A}_t \mathscr{T}, \tag{6}$$

where denotes transposition, and the matrix $\mathscr{T}$ is the one defined in Sec. 4.2. Remark that $(\mathscr{A}_t')^{-1}$ is the matrix of the mapping $(a_t^*)^{-1}$ in the basis $e_1, \ldots, e_{2n}$. As $\mathscr{T}$ is norm preserving, it then immediately follows from (6) that

$$\chi((\mathscr{A}_t')^{-1}, f) = \chi(\mathscr{A}_t, \mathscr{T} f)$$

for any vector $f \in \mathbf{R}^{2n}$. $\mathscr{T}$ being invertible, the proposition then follows.

By Lemma D and Proposition E, i.e. $Sp((a_t^*)^{-1}) = -Sp(a_t)$ and $Sp((a_t^*)^{-1}) = Sp(a_t)$, one has then, for a regular family $\{a_t\}$ of symplectic mappings of $\mathbf{R}^{2n}$,

$$Sp(a_t) = -Sp(a_t),$$

so that Theorem 8 is proven.

8.2. We now come to the proof of Corollary 9. Let $x \in \Gamma$ be any T-regular point. Denote $Sp_U(x) = \{\phi_1(x), \ldots, \phi_{2n}(x)\}$ and $Sp_\Gamma(x) = \{\psi_1(x), \ldots, \psi_{2n-1}(x)\}$. T being symplectic, we have

$$\sum_{i=1}^{2n} \phi_i(x) = 0.$$

Moreover, as $T : \Gamma \to \Gamma$ preserves a Liouville measure μ_L, then in virtue of Corollary 7 and Theorem 3 we have that

$$\sum_{j=1}^{2n-1} \psi_j(x) = 0$$

for a set of T-regular points x of μ_L-measure 1.

For such points, recalling that $Sp_\Gamma(x) \subset Sp_U(x)$, one obtains that at least one element of $Sp_U(x)$ vanishes, and then by the antisymmetry of $Sp_U(x)$ (Theorem 8) at least two elements of $Sp_U(x)$ vanish. Consequently, at least one element of $Sp_\Gamma(x)$ too vanishes. Corollary 9 is thus proven.

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