Derivation of Viscosity

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Abstract

I derive a single PDE that describes the dynamics of fluids in state space. Then the similarity between that PDE and equations of fluid mechanics is demonstrated by using it to deriving a set of three equations analogous to the mass, internal energy and Navier-Stokes equations. Finally I demonstrate that for a fluid with particles following the Maxwell-Boltzmann distribution the set of analogous equations reduces to the equations of inviscid flow.

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I. INTRODUCTION

Fluids consist of a large number of interacting particles, presumably the fluid mechanics observed at a macroscopic level is a result of the interactions occurring at a microscopic level. If the dynamics of each particle was know it should be possible in principle to determine the macroscopic dynamics of the fluid from the collective motion of the particles.

II. STATE SPACE DYNAMICS

Assuming the dynamics a particle is defined by the acceleration and that the acceleration of the particles is given by the potential ϕ such that $\vec{a} = \vec{\nabla} \cdot \phi$, then the equations of motion become

$$\frac{d}{dt}\begin{pmatrix} \vec{x} \\ \dot{\vec{x}} \end{pmatrix} = \begin{pmatrix} \dot{\vec{x}} \\ -\vec{\nabla}\phi \end{pmatrix}$$

For a system of n particles where the i^{th} particle is located at $\vec{x}_i$, the potential can in principle depend on the location of all of the particles, such that the potential for the i^{th} particle is $\phi_i = \phi(\vec{x}_i, \vec{x}_1, \vec{x}_2, ...\vec{x}_i..., \vec{x}_{n-1}, \vec{x}_n)$. Then for this system the equations of motion for each particle is given by the system of equations

$$\frac{d}{dt}\begin{pmatrix} \vec{x}_i \\ \dot{\vec{x}}_i \end{pmatrix} = \begin{pmatrix} \dot{\vec{x}}_i \\ -\partial_{\vec{x}_i}\phi_i \end{pmatrix} \tag{1}$$

Moving over to a state space description of the system, the state space has six coordinates given by the orthogonal coordinate vectors $\vec{x}$ and $\dot{\vec{x}}$. Given a time dependent density distribution defined over state space $\sigma = \sigma(\vec{x}, \dot{\vec{x}}, t)$, such that the total mass at the time t is $M(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(\vec{x}, \dot{\vec{x}}, t) d^3x d^3\dot{x}$. The system of discrete particles described by equation (1) then be described by the distribution

$$\sigma(\vec{x}, \dot{\vec{x}}, t) = m \sum_{i=0}^{n} \delta(\vec{x} - \vec{x}_i) \cdot \delta(\dot{\vec{x}} - \dot{\vec{x}}_i)$$

where m is the mass of the particles, and $\delta(\vec{x})$ is the Dirac delta distribution in 3-space. For this system the six component state space velocity is $\vec{v} = \left\langle \dot{\vec{x}}, -\vec{\nabla}\phi(\vec{x}, \rho, t) \right\rangle$, where the potential ϕ is defined such that at $\vec{x}_i$ the potential is $\phi(\vec{x}_i, \rho(\vec{x}, t), t) = \phi_i$ and $\rho(\vec{x}, t) = \int_{-\infty}^{\infty} \sigma(\vec{x}, \dot{\vec{x}}, t) d^3\dot{x}$. Assuming the number of particles in the distribution is constant, then

$\frac{dM}{dt} = 0$. Since there are no sources or sinks for the particles, the density distribution is constrained by the continuity equation

$$\frac{d}{dt}\int_V \sigma(\vec{x},\dot{\vec{x}},t)d^3xd^3\dot{x} + \oint_{\partial V} \sigma(\vec{x},\dot{\vec{x}},t)\vec{v}\cdot d\vec{a} = 0$$

where V is an arbitrary volume in state space, ∂V is the surface of the arbitrary volume, $\vec{v}$ is the state space velocity and $d\vec{a}$ is a surface element in state space. Rearranging the terms and using the divergence theorem the continuity equation can be rewritten in the form

$$\int_V \left(\partial_t\sigma + \vec{\nabla}\cdot(\sigma\vec{v})\right) d^3xd^3\dot{x} = 0$$

Since the integral equals zero over any arbitrary volume V, then the integrand must be zero

$$\partial_t\sigma + \vec{\nabla}\cdot(\sigma\vec{v}) = 0$$

Finally the independence of $\vec{x}$ and $\dot{\vec{x}}$ can be used to rewrite the continuity equation in the final form, in this case it is written using Einstein notation

$$\partial_t\sigma + \dot{x}_i\partial_{x_i}\sigma - (\partial_{x_i}\phi)\,\partial_{\dot{x}_i}\sigma = 0 \tag{2}$$

While the derivation of equation (2) was motivated using a discrete collection of particles the equation is not restricted to systems of discrete particles. As long as the dynamics in state space is determined by $\vec{v} = \left\langle\dot{\vec{x}}, -\vec{\nabla}\phi(\vec{x},\rho,t)\right\rangle$ and there are no sources or sinks for the state space density, then any state space density function or distribution is described by equation (2).

Assuming the motion of individual particles, in a fluid described by fluid mechanics, is described by equation (1) and assuming that the number and mass of the particles is invariant, then the dynamics of the state space distribution is described by equation (2). If these assumptions are true for any fluid, then equation (2) must be capable of reproducing the behavior of fluid mechanics.

III. FLUID MECHANICS MASS EQUATION: CONSERVATION OF MASS

The state space density function σ is defined such that the total mass M is $M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\vec{x},\dot{\vec{x}},t)d^3xd^3\dot{x}$. Define the mass density ρ as $\rho(\vec{x},t) = \int_{-\infty}^{\infty}\sigma(\vec{x},\dot{\vec{x}},t)d^3\dot{x}$. Also define the bulk momentum $\rho\vec{u}$ as $\rho(\vec{x},t)u_i(\vec{x},t) = \int_{-\infty}^{\infty} \dot{x}_i\sigma(\vec{x},\dot{\vec{x}},t)d^3\dot{x} = \int_{-\infty}^{\infty} \dot{x}_i\sigma d^3\dot{x}$. The integral of equation (2) over velocity space is

$$\int_{-\infty}^{\infty} \left(\partial_t\sigma + \dot{x}_i\partial_{x_i}\sigma - \left(\partial_{x_i}\phi\right)\partial_{\dot{x}_i}\sigma\right) d^3\dot{x} = 0$$

Splitting up the integral, pulling out factors and operations that are independent of $\dot{x}_i$ puts the equation into a form where some terms can be evaluated. Then applying the divergence theorem and evaluating the simplified integrals results in the equation

$$\partial_t\rho + \partial_{x_i}\left(u_i\rho\right) - \left(\partial_{x_i}\phi\right)\oint \sigma da_i = 0$$

where da_i is the i^{th} component of the surface element, and $\oint \sigma da_i$ is the surface integral over all of velocity space. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^2$ then the surface integral converges to zero. Given the surface integral does converge to zero, the resulting equation is,

$$\partial_t\rho + \partial_{x_i}\left(u_i\rho\right) = 0 \tag{3}$$

which is the conservation of mass equation from fluid mechanics.

IV. CONSERVATION OF MOMENTUM

To get an equation for the conservation of momentum, multiply equation (2) by $\dot{\vec{x}}$ before integrating over velocity space.

$$\int_{-\infty}^{\infty} \dot{x}_i\left(\partial_t\sigma + \dot{x}_j\partial_{x_j}\sigma - \left(\partial_{x_j}\phi\right)\partial_{\dot{x}_j}\sigma\right) d^3\dot{x} = 0$$

After simplifying the equation, applying the product rule and divergence theorem it can be written in the form

$$\begin{aligned}\partial_t\left(u_i\rho\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty} \dot{x}_i\dot{x}_j\sigma d^3\dot{x}\right)\\ - \left(\partial_{x_j}\phi\right)\oint \dot{x}_i\sigma da_j + \left(\partial_{x_j}\phi\right)\rho\delta_{ij} = 0\end{aligned}$$

where $\oint \dot{x}_i\sigma da_j$ is a surface integral over all of velocity space and δ_{ij} is the Kronecker delta function. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^3$ then the surface integral converges to zero. Given the surface integral does converge to zero then the result is,

$$\partial_t (u_i\rho) + \partial_{x_j}\left(\int_{-\infty}^{\infty} \dot{x}_i\dot{x}_j\sigma d^3\dot{x}\right) + \rho\partial_{x_i}\phi = 0 \tag{4}$$

If we pulled out a term of $\rho u_i u_j$ from $\int_{-\infty}^{\infty} \dot{x}_i\dot{x}_j\sigma d^3\dot{x}$ and simplified then equation (4) can be put in the form the Navier-Stokes equations.

V. CONSERVATION OF ENERGY

The total energy of the system is $E_{\text{tot}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}_i^2 + \phi(\vec{x},\rho,t)\right)\sigma(\vec{x},\dot{\vec{x}},t)d^3x d^3\dot{x}$. In order to derive an equation for the conservation of energy equation in position space, equation (2) must be multiplied by $\frac{1}{2}\dot{x}_i^2 + \phi$ before integrating. To make the derivation of the conservation of energy simpler we can use the linearity of integration to break the conservation of energy equation into a kinetic energy component and a potential energy component.

A. The kinetic energy component

Multiplying equation (2) by $\frac{1}{2}\dot{x}_i^2$ then integrating over velocity space, the resulting relation is

$$\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2 \left(\partial_t\sigma + \dot{x}_j\partial_{x_j}\sigma - \left(\partial_{x_j}\phi\right)\partial_{\dot{x}_j}\sigma\right) d^3\dot{x} = 0$$

After simplifying the equation, applying the product rule and divergence theorem, it can be written in the form

$$\begin{aligned}\partial_t\left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2\sigma d^3\dot{x}\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2\dot{x}_j\sigma d^3\dot{x}\right)\\ - \left(\partial_{x_j}\phi\right)\left(\oint\left(\frac{1}{2}\dot{x}_i^2\sigma\right)da_j - \int_{-\infty}^{\infty}\sigma\dot{x}_j d^3\dot{x}\right) = 0\end{aligned}$$

where $\oint\left(\frac{1}{2}\dot{x}_i^2\sigma\right)da_j$ is a surface integral over all of velocity space. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^4$ then the surface integral converges to zero. Evaluating integrals and rearranging the equation given the surface integral does converge to zero, results in the equation,

$$\partial_t\left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2\sigma d^3\dot{x}\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2\dot{x}_j\sigma d^3\dot{x}\right) + u_j\rho\partial_{x_j}\phi = 0 \tag{5}$$

B. The potential energy component

Multiplying equation (2) by ϕ and then integrating over velocity space, the resulting relation is

$$\int_{-\infty}^{\infty} \phi \left(\partial_t \sigma + \dot{x}_j \partial_{x_j} \sigma - \left(\partial_{x_j} \phi\right) \partial_{\dot{x}_j} \sigma\right) d^3\dot{x} = 0$$

After rearranging and simplifying use the result that $\int_{-\infty}^{\infty} \partial_{\dot{x}_j} \sigma d^3\dot{x} = \oint \sigma da_j = 0$, from the derivation for the conservation of mass equation, to get

$$\phi \partial_t \rho + \phi \partial_{x_j} \left(u_j \rho\right) = 0$$

Applying the product rule produces the form

$$\partial_t \left(\phi \rho\right) - \rho \partial_t \phi + \partial_{x_j} \left(u_j \rho \phi\right) - u_j \rho \partial_{x_j} \phi = 0 \tag{6}$$

C. Total energy density dynamics

The total energy density E is $E = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}_i^2 + \phi(\vec{x}, \rho, t)\right) \sigma(\vec{x}, \dot{\vec{x}}, t) d^3\dot{x}$. Adding the two energy components from equation (5) and equation (6) produces the equation

$$\begin{aligned} \partial_t \left(\int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}_i^2 + \phi\right) \sigma d^3\dot{x}\right) - \rho \partial_t \phi \\ + \partial_{x_j} \left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2 \dot{x}_j \sigma d^3\dot{x} + u_j \rho \phi\right) = 0 \end{aligned}$$

Using the definition of energy density we can rewrite this equation as,

$$\partial_t E + \partial_{x_j} \left(\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2 \dot{x}_j \sigma d^3\dot{x} + u_j \rho \phi\right) - \rho \partial_t \phi = 0 \tag{7}$$

Equation (7) describes the dynamics of the total energy density. This equaiton can be written in a more standard form if we pull out the terms $u_i \int_{-\infty}^{\infty} \dot{x}_i \dot{x}_j \sigma d^3\dot{x} - \frac{1}{2}u_i^2 u_j \rho$ from the integral $\int_{-\infty}^{\infty} \frac{1}{2}\dot{x}_i^2 \dot{x}_j \sigma d^3\dot{x}$ and then simplify.

VI. FLUID MODEL: GAUSSIAN DISTRIBUTION

Lets assume that the velocity distribution σ'/ρ' is simply a normalized Gaussian distribution. Then the state space distribution σ' can be written as,

$$\sigma'(\vec{x},\dot{\vec{x}}) = \rho' \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m\left(x_i - u'_i\right)^2/2kT} \tag{8}$$

Where m is the mass of the particles the fluid is made from, k is Boltzmann's constant and $T = T(\vec{x}, t)$ is the temperature. Also assume that the potential ϕ only explicitly depends on $\vec{x}$ and ρ'. So in this model $\phi = \phi(\vec{x}, \rho')$. All of the integrals in this derivation of fluid mechanics can be expressed as moments of the state space distribution. Before we calculate the moments of this distribution let us define some notation to simplify the expressions and integrals.

A. Notation

Let $(a^n)_{i_1 i_2 \cdots i_n}$ represent the repeated tensor product of n copies of the tensor a_i such that $(a^n)_{i_1 i_2 \cdots i_n} = a_{i_1} a_{i_2} \cdots a_{i_{n-1}} a_{i_n}$. And let $[a^n]_{i_1 i_2 \cdots i_n}$ represent the sum of every permutation of indecies applied to the tensor $(a^n)_{i_1 i_2 \cdots i_n}$ such that $[a^n]_{i_1 i_2 \cdots i_n} = a_{i_1} [a^{n-1}]_{i_2 i_3 \cdots i_n} + a_{i_2} [a^{n-1}]_{i_1 i_3 \cdots i_n} + \cdots + a_{i_n} [a^{n-1}]_{i_1 i_2 \cdots i_{n-1}}$. For example if u is a one index and δ is the Chroniker delta symbol then using this notation $[u\delta]_{ijk} = 2\left(u_i \delta_{jk} + u_j \delta_{ik} + u_k \delta_{ij}\right)$

B. Moments of a Gaussian

Using this notation we can express the tensor describing the n-th moment $A_{i_1 i_2 \cdots i_n}$ of a Gaussian state space distribution $\sigma = \rho\,(a\pi)^{-3/2} e^{-(\dot{x}_i - u_i)^2/a}$ as $A_{i_1 i_2 \cdots i_n} = \int_{-\infty}^{\infty} (\dot{x}^n)_{i_1 i_2 \cdots i_n} \sigma d^3\dot{x}$. This integral can be solved by using the substitution $w_i = x_i - u_i$ and solving for one term of A by pairing indices and using that to reconstruct the full tensor. Solving for the moments of the distribution results in the equation,

$$A_{i_1 i_2 \cdots i_n} = \sum_{p=0}^{n} \rho \frac{a^{(n-p)/2}}{2^{n-p} p!\left((n-p)/2\right)!} \left[u^p \delta^{(n-p)/2}\right]_{i_1 i_2 \cdots i_n} \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{else} \end{cases} \tag{9}$$

C. Dynamical equation for a Gaussian

Using equation (9) to evaluate the moments of the state space distribution (8) produces the equations; $\int_{-\infty}^{\infty} \sigma' d^3\dot{x} = \rho'$, $\int_{-\infty}^{\infty} \dot{x}_i \sigma' d^3\dot{x} = \rho' u'_i$, $\int_{-\infty}^{\infty} (\dot{x}^2)_{ij} \sigma' d^3\dot{x} = \frac{\rho' kT}{m}\delta_{ij} +$

$\rho' \left(u'^2\right)_{ij}$ and $\int_{-\infty}^{\infty} \left(\dot{x}^3\right)_{ijk} \sigma' d^3\dot{x} = \frac{\rho' kT}{2m} \left[u'\delta\right]_{ijk} + \rho' \left(u'^3\right)_{ijk}$. Then the energy density $E' = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}_i^2 + \phi\right) \sigma' d^3\dot{x} = \rho' \frac{3kT}{2m} + \rho' \frac{1}{2} u_i'^2 + \rho' \phi$. Now that all of the nessisary integrals have been evaluated the conservation equations for the statespace distribution (8) are then,

$$\partial_t \rho' + \partial_{x_i} \left(\rho' u_i'\right) = 0$$

$$\partial_t \left(\rho' u_i'\right) + \partial_{x_i} \left(\rho' \frac{kT}{m}\right) + \partial_{x_j} \left(\rho' u_i' u_j'\right) + \rho' \partial_{x_i} \phi = 0$$

$$\partial_t E' + \partial_{x_i} \left(E' u_i' + \rho' \frac{kT}{m} u_i'\right) - \rho' \partial_t \phi = 0$$

So the conservation equation resulting from using a Gaussian distribution as the statespace distribution produces the equations of fluid dynamics destribing a compressible invicid fluid without thermal conduction.

VII. FLUID MODEL: PERTURBED GAUSSIAN DISTRIBUTION

When constringint the statespace distribution to a guassian distribution it captures some aspects of fluid mechanics but lacks higher order effects because of the symmetry of the distribution. This symmetry can be broken by perturbing the inital distribution sllightyly. The fundimental dynamics of the system is described by equation (2), so let us use that equation to perturb the initial statespace distrbution (8) by evolving the equation forward in time for some short time Δt. Let the new statespace distribution σ be $\sigma = \sigma' + \Delta t \partial_t \sigma'$, where $\partial_t \sigma' = -\dot{x}_i \partial_{x_i} \sigma' + \left(\partial_{x_i} \phi\right) \partial_{\dot{x}_i} \sigma'$. Now to simplify this problem let us drop the terms $\partial_{x_i} \rho'$, $\partial_{x_i} T$ and $\partial_{x_i} \phi$ when computing $\partial_t \sigma'$. Then the new statespace distribution can be written as,

$$\sigma = \sigma' \left(1 - \Delta t \frac{m}{kT} \left(\partial_{x_i} u_j'\right) \dot{x}_i \left(\dot{x}_j - u_j'\right)\right) \tag{10}$$

Note that all of the moments of this new distribution can be found using equation (9). The moments of this distribution are $\int_{-\infty}^{\infty} \sigma' d^3\dot{x} = \rho' \left(1 - \Delta t \partial_{x_i} u_i'\right)$, $\int_{-\infty}^{\infty} \dot{x}_i \sigma' d^3\dot{x} = \rho' u_i' - \rho' \Delta t \partial_{x_j} \left(u_i' u_j'\right)$, $\int_{-\infty}^{\infty} \left(\dot{x}^2\right)_{ij} \sigma' d^3\dot{x} = \frac{\rho' kT}{m} \delta_{ij} + \rho' \left(u'^2\right)_{ij} - \rho' \frac{kT}{m} \Delta t \left(\partial_{x_k} u_k' \delta_{ij} + \partial_{x_i} u_j' + \partial_{x_j} u_i'\right) - \rho' \Delta t \partial_{x_k} \left(u'^3\right)_{ijk}$ and $\int_{-\infty}^{\infty} \left(\dot{x}^3\right)_{ijk} \sigma' d^3\dot{x} = \frac{\rho' kT}{2m} \left[u'\delta\right]_{ijk} + \rho' \left(u'^3\right)_{ijk} - \rho' \frac{kT}{m} \Delta t \left(\delta_{ij} \partial_{x_l} \left(u_k' u_l'\right) + \delta_{ik} \partial_{x_l} \left(u_j' u_l'\right) + \delta_{jk} \partial_{x}\right.$ $\rho' \Delta t \partial_{x_l} \left(u'^4\right)_{ijkl}$. The conservation equations in this case are,

$$\partial_t \rho + \partial_{x_i} (\rho u_i) = 0$$

$$\partial_t (\rho u_i) + \partial_{x_j} A_{ij} + \rho \partial_{x_i} \phi = 0$$

$$\partial_t E + \partial_{x_i} (B_i + \rho \phi u_i) - \rho \partial_t \phi = 0$$

Where $\rho = \rho' (1 - \Delta t \partial_{x_i} u'_i)$, $\rho u_i = \rho' u'_i - \rho' \Delta t \partial_{x_j} \left(u'_i u'_j\right)$, $E = E' - \rho' \frac{5kT}{2m} \Delta t \partial_{x_i} u'_i - \frac{1}{2} \rho' \Delta t \partial_{x_j} \left(u'^2_i u'_j\right) - \rho' \phi \Delta t \partial_{x_i} u'_i$, $A_{ij} = \int_{-\infty}^{\infty} (\dot{x}^2)_{ij} \sigma' d^3 \dot{x}$ and $B_i = \frac{5\rho' kT}{2m} u'_i + \frac{1}{2} \rho' u'^2_j u'_i - \rho' \frac{kT}{2m} \Delta t \left(7 \partial_{x_i} \left(u'_i u'_j\right) + \partial_{x_j} u'^2_i\right) - \frac{1}{2} \rho' \Delta t \partial_{x_k} \left(u'^2_i u'_j u'_k\right)$.

Now these equations can be simplified by making the folowing assumptions. Let us assume that $\rho' \partial_t \partial_{x_i} u'_i \ll \partial_t \rho' \partial_{x_i} u'_i$ and that u'_i is small and the relative fluctuations in ρ', u'_i and T are small. Given these assumptions after dropping the higher order terms the conservation equations can be written as,

$$(1 - \Delta t \partial_{x_i} u'_i) \partial_t \rho' + \partial_{x_i} (\rho' u'_i) = 0$$

$$\partial_t (\rho' u'_i) + \partial_{x_i} \left(\rho' \frac{kT}{m}\right) + \partial_{x_j} (\rho' u'_i u'_j) - \partial_{x_j} \left(\rho' \frac{kT}{m} \Delta t \left(\partial_{x_k} u'_k \delta_{ij} + \partial_{x_i} u'_j + \partial_{x_j} u'_i\right)\right) + \rho' \left(1 - \Delta t \partial_{x_j} u'_j\right) \partial_{x_i} \phi = 0$$

$$\partial_t E' - \partial_t \left(\rho' \frac{5kT}{2m}\right) \Delta t \partial_{x_i} u'_i - \phi \partial_t \rho' \Delta t \partial_{x_i} u'_i + \partial_{x_i} \left(E' u'_i + \frac{\rho' kT}{m} u'_i\right) - \rho' \partial_t \phi = 0$$

Finally if we assume $\partial_{x_i} u'_i = 0$, the equations can be simplified again into the form,

$$\partial_t \rho' + u'_i \partial_{x_i} \rho' = 0$$

$$\rho' \left(\partial_t u'_i + u'_j \partial_{x_j} u'_i\right) + \partial_{x_i} \left(\rho' \frac{kT}{m}\right) - \partial_{x_j} \left(\rho' \frac{kT}{m} \Delta t \left(\partial_{x_i} u'_j + \partial_{x_j} u'_i\right)\right) + \rho' \partial_{x_i} \phi = 0$$

$$\partial_t E' + u'_i \partial_{x_i} \left(E' + \frac{\rho' kT}{m}\right) - \rho' \partial_t \phi = 0$$

So perturbing to the statespace distribution (8) can produce conservation equations which include higher order terms. For the paticular perturbation (10) given some assumptions the conservation equations describe an incompressible fluid.

VIII. COMPLETE FIRST ORDER PERTURBATION

I have demenstrated that perturbing a Gaussian distribution with only the $\partial_{x_i} u'_j$ component of $\partial_t \sigma' = -\dot{x}_i \partial_{x_i} \sigma' + (\partial_{x_i} \phi) \partial_{\dot{x}_i} \sigma'$ leads to viscus terms in the resulting conservation equations. Based on my paliminary investigation into how the remaining components ($\partial_{x_i} \rho$, $\partial_{x_i} T$ and $\partial_{x_i} \phi$) effect the pertubation of the Gaussian statespace distribution (8) I expect that a full first order pertubation of the distribution will produce an aditional three sets of terms in the conservation equations. As part of these new I expect to find terms including density diffution terms from $\partial_{x_i} \rho$, thermal conduction terms from $\partial_{x_i} T$ and terms resulting from $\partial_{x_i} \phi$ due to particle interactions mediated by the potential $\phi(\vec{x}, t, \rho)$. For example if $\phi(\vec{x}, t, \rho) = \int_{-\infty}^{\infty} f(|\vec{x} - \vec{x'}|) \rho(\vec{x'}) d^3 x'$ where $f(r)$ models the interaction between particles, then depending on the structure of local minima and maxima in the function $f(r)$ I expect that the net effect of the terms resulting from $\partial_{x_i} \phi$ will be to introduce phase transitions into the conservation equations.

Also note that if the statespace distribution of the form of equation (8) with upto n-th order pertubations as produced by equation (2) then all of the nessisary integrals can be solved using equation (9).