Probability Theory

Cheat Sheet & Study materials*

These materials...

Contents

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Ovo sluzi samo za template.

1 Introduction

1.1 Set structures

1.1.1 Definitions

Let S be a set and $\mathcal{S} \subseteq \mathcal{P}(S)$ a family of subsets. We define a number of terms: $\mathcal{S}$ is a

(a) **semiring** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- if $A, B \in \mathcal{S}$ then $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$

(b) **ring** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- $A, B \in \mathcal{S} \implies A \setminus B \in \mathcal{S}$

(c) **semialgebra** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- $A \in \mathcal{S} \implies A^c$ is a finite disjoint union of sets in $\mathcal{S}$

(d) **algebra** if it is closed under complements and **finite** unions.

(e) **monotone class** if $A_j \uparrow A$ and $A_j \downarrow A$ imply $A \in \mathcal{S}$

(f) **σ-algebra** if...

(g) **π-system** if $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$.

(h) **λ-system** if
- $S \in \mathcal{S}$
- $A \subseteq B,\ A, B \in \mathcal{S} \implies B \setminus A \in \mathcal{S}$
- $A_j \uparrow A \implies A \in \mathcal{S}$

* * *

Clearly, a σ-algebra is all of the above. Also:

- if $\mathcal{S}$ is both a monotone class and an algebra, it is a σ-algebra,
- if $\mathcal{S}$ is an algebra closed with respect to disjoint unions, it is a σ-algebra,
- if $\mathcal{S}$ is both a π-system and λ-system, it is a σ-algebra.

1.1.2 Equivalent definition of λ-system

A λ-system can also be defined by:

- $S \in \mathcal{S}$
- $A \in \mathcal{S} \implies A^c \in \mathcal{S}$
- if $A_j \in \mathcal{S}$ with $j \in \mathbb{N}$ are disjoint, then $\cup_j A_j \in \mathcal{S}$.

Suppose the first definition and let us prove the other. We simply get complements from $A^c = S \setminus A$ and $S \in \mathcal{S}$. Suppose A_1 and A_2 are disjoint. Then $A_1 \subseteq A_2^c$ so $A_2^c \setminus A_1 \in \mathcal{S}$. This means we have finite disjoint unions because

$$A_2^c \setminus A_1 = A_2^c \cap A_1^c = (A_1 \cup A_2)^c .$$

With finite disjoint unions,

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \left[\bigcup_{k=1}^{j} A_j \right] \in \mathcal{S}$$

as this is a monotone union.

Now let us prove the other direction. If $A \subseteq B$ then $B \setminus A = B \cap A^c = (B^c \cup A)^c$. Then suppose A_j are monotone increasing. We have the disjoint union

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup \bigcup_{j=2}^{\infty} \left[A_j \setminus \bigcup_{k=1}^{j-1} A_k \right] \in \mathcal{S}$$

as all the set differences are proper.

1.1.3 Dynkin–Sierpinski π-λ lemma

Let S be a set, $\mathcal{C}$ a π-system and $\mathcal{D}$ a λ-system. Then

$$\mathcal{C} \subseteq \mathcal{D} \implies \sigma(\mathcal{C}) \subseteq \mathcal{D}.$$

The proof is from [3, p. 10]. Clearly we can assume that $\mathcal{D}$ is the smallest possible λ-system, this being denoted $\lambda(\mathcal{D})$. Taking into account that π and λ together make a σ-algebra, the statement becomes:

> If $\mathcal{C}$ is a π-system, then $\lambda(\mathcal{C})$ is also a π-system.

Therefore, we need to prove $A \cap B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$. For an arbitrary $B \in \mathcal{C}$ define

$$\mathcal{S}_B = \{A \in \mathcal{D} \colon A \cap B \in \mathcal{D}\} .$$

Clearly $\mathcal{S}_B$ is a λ-system containing $\mathcal{C}$ (because $\mathcal{C}$ is itself a π-system). Since $\mathcal{D} = \lambda(\mathcal{C})$, it follows that $\mathcal{D} \subseteq \mathcal{S}_B$. That is, $A \cap B \in \mathcal{D}$ whenever $B \in \mathcal{C}$ and $A \in \mathcal{D}$.

Now for any $A \in \mathcal{D}$ define

$$\mathcal{S}'_A = \{B \in \mathcal{D} \colon A \cap B \in \mathcal{D}\} .$$

Similarly, and also using the previous result, we get $\mathcal{D} \subseteq \mathcal{S}'_A$ and we conclude that $A \cap B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$.

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Suppose $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{F}$ where $\mathcal{F}$ is a σ-algebra on a probability space. Let $\mathcal{D}$ be the set of all events with a certain property, this property naturally making it a λ-system. Suppose also $\sigma(\mathcal{C}) = \mathcal{F}$. By the lemma, $\mathcal{D} = \mathcal{F}$, that is, all events have the desired property.

1.1.4 Notable exercises

Any open set of reals is a countable union of intervals. See [1, ex. 1.1.2]. Define

$$\mathcal{S}_d = \left\{ \prod_{j=1}^{d} \langle a_j, b_j] : -\infty \leqslant a < b \leqslant +\infty \right\} \subseteq \mathbb{R}^d.$$

Note that $\mathcal{S}_d$ is a semialgebra (automatically, also a π-system). We claim that $\sigma(\mathcal{S}_d) = \mathcal{B}(\mathbb{R}^d)$. It suffices to prove that every open set in $\mathbb{R}^d$ can be written as the countable union of sets in $\mathcal{S}_d$.

Take $d = 1$ with a simple extension afterwards. Let $A \subseteq \mathbb{R}$ be open, so that for each $x \in A$ there is ε_x such that $B(x, \varepsilon_x) \subseteq A$. Also,

$$A = \bigcup_{x \in A} B(x, \varepsilon_x).$$

Note that $A \cap \mathbb{Q}$ is dense in A. Namely, each x is within its ball, and within the ball is a sequence of rationals converging to x. For each rational y such that $|x - y| < \varepsilon_x/2$ we can clearly choose an ε_y such that $x \in B(y, \varepsilon_y) \subseteq A$. This gives us the countable union[1]

$$A = \bigcup_{y \in A \cap \mathbb{Q}} B(y, \varepsilon_y)$$

* * *

Infinite unions of algebras/σ-algebras. See [1, ex. 1.1.4]. Suppose $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ and $\mathcal{F} = \cup_j \mathcal{F}_j$.

If $\mathcal{F}_j$ are algebras, then $\mathcal{F}$ is also an algebra because

$$\bigcup_{j \in N} A_j \in \mathcal{F}_{\max N}, \quad A_j \in \mathcal{F}_j.$$

where $N \subseteq \mathbb{N}$ is finite.

The analogous claim is not true if $\mathcal{F}_j$ are σ-algebras. Let $S = \mathbb{R}$ and

$$\mathcal{F}_j = \sigma\left(\left[\frac{n}{2^j}, \frac{n+1}{2^j} \right\rangle : n \in \mathbb{N} \right), \quad j \in \mathbb{N}.$$

If $\mathcal{F}$ is a σ-algebra,

$$\{0\} = \bigcap_{j \in \mathbb{N}} \left[0, \frac{1}{2^j} \right\rangle \in \mathcal{F},$$

but there is no j such that $\{0\} \in \mathcal{F}_j$, giving us a contradiction.

[1] while different x can give different ε_y, these clearly have a finite supremum

1.2 Measure and integration

1.2.1 Products of σ-algebras

If $(S_\alpha)_{\alpha \in A}$ are measurable spaces with σ-algebras $\mathcal{F}_\alpha$ and $S = \prod_\alpha S_\alpha$, then S has the **product σ-algebra** $\mathcal{F} = \prod_\alpha \mathcal{F}_\alpha = \bigotimes_\alpha \mathcal{F}_\alpha$ defined as

$$\mathcal{F} = \sigma\left(\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{F}_\alpha\right),$$

i.e. **generated by one-dimensional measurable cylinders**.

* * *

If A is countable, then $\mathcal{F}$ is **generated by cuboids** $\prod_\alpha E_\alpha$, $\alpha \in A$ (see [2, p. 23]). To prove it, define

$$\mathcal{F}_1 = \sigma\left(\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{F}_\alpha\right), \quad \mathcal{F}_2 = \sigma\left(\prod_{\alpha \in A} E_\alpha \colon E_\alpha \in \mathcal{F}_\alpha\right).$$

Firstly, $\pi_\alpha^{-1}(E_\alpha) = \prod_\alpha E_\alpha$ where $E_\beta = S_\beta$ for $\beta \neq \alpha$, so $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Secondly,

$$\prod_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \in \mathcal{F}_1,$$

so that $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

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