Probability Theory

Cheat Sheet & Study materials*

These materials...

Contents

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1 Set structures

1.1 Definitions

Let S be a set and $\mathcal{S} \subseteq \mathcal{P}(S)$ a family of subsets. We define a number of terms: $\mathcal{S}$ is a

(a) **semiring** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- if $A, B \in \mathcal{S}$ then $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$

(b) **ring** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- $A, B \in \mathcal{S} \implies A \setminus B \in \mathcal{S}$

(c) **semialgebra** if
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- $A \in \mathcal{S} \implies A^c$ is a finite disjoint union of sets in $\mathcal{S}$

(d) **algebra** if it is closed under complements and **finite** unions.

(e) **monotone class** if $A_j \uparrow A$ and $A_j \downarrow A$ imply $A \in \mathcal{S}$

(f) **σ-algebra** if...

(g) **π-system** if $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$.

(h) **λ-system** if
- $S \in \mathcal{S}$
- $A \subseteq B,\ A, B \in \mathcal{S} \implies B \setminus A \in \mathcal{S}$
- $A_j \uparrow A \implies A \in \mathcal{S}$

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Clearly, a σ-algebra is all of the above. Also:

- if $\mathcal{S}$ is both a monotone class and an algebra, it is a σ-algebra,
- if $\mathcal{S}$ is an algebra closed with respect to disjoint unions, it is a σ-algebra,
- if $\mathcal{S}$ is both a π-system and λ-system, it is a σ-algebra.

1.2 Equivalent definition of λ-system

A λ-system can also be defined by:

- $S \in \mathcal{S}$
- $A \in \mathcal{S} \implies A^c \in \mathcal{S}$
- if $A_j \in \mathcal{S}$ with $j \in \mathbb{N}$ are disjoint, then $\cup_j A_j \in \mathcal{S}$.

Suppose the first definition and let us prove the other. We simply get complements from $A^c = S \setminus A$ and $S \in \mathcal{S}$. Suppose A_1 and A_2 are disjoint. Then $A_1 \subseteq A_2^c$ so $A_2^c \setminus A_1 \in \mathcal{S}$. This means we have finite disjoint unions because

$$A_2^c \setminus A_1 = A_2^c \cap A_1^c = (A_1 \cup A_2)^c .$$

With finite disjoint unions,

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \left[\bigcup_{k=1}^{j} A_j \right] \in \mathcal{S}$$

as this is a monotone union.

Now let us prove the other direction. If $A \subseteq B$ then $B \setminus A = B \cap A^c = (B^c \cup A)^c$. Then suppose A_j are monotone increasing. We have the disjoint union

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup \bigcup_{j=2}^{\infty} \left[A_j \setminus \bigcup_{k=1}^{j-1} A_k \right] \in \mathcal{S}$$

as all the set differences are proper.

1.3 Dynkin–Sierpinski π-λ lemma

Let S be a set, $\mathcal{C}$ a π-system and $\mathcal{D}$ a λ-system. Then

$$\mathcal{C} \subseteq \mathcal{D} \implies \sigma(\mathcal{C}) \subseteq \mathcal{D}.$$

The proof is from [3, p. 10]. Clearly we can assume that $\mathcal{D}$ is the smallest possible λ-system, this being denoted $\lambda(\mathcal{D})$. Taking into account that π and λ together make a σ-algebra, the statement becomes:

> If $\mathcal{C}$ is a π-system, then $\lambda(\mathcal{C})$ is also a π-system.

Therefore, we need to prove $A \cap B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$. For an arbitrary $B \in \mathcal{C}$ define

$$\mathcal{S}_B = \{A \in \mathcal{D} \colon A \cap B \in \mathcal{D}\} .$$

Clearly $\mathcal{S}_B$ is a λ-system containing $\mathcal{C}$ (because $\mathcal{C}$ is itself a π-system). Since $\mathcal{D} = \lambda(\mathcal{C})$, it follows that $\mathcal{D} \subseteq \mathcal{S}_B$. That is, $A \cap B \in \mathcal{D}$ whenever $B \in \mathcal{C}$ and $A \in \mathcal{D}$.

Now for any $A \in \mathcal{D}$ define

$$\mathcal{S}'_A = \{B \in \mathcal{D} \colon A \cap B \in \mathcal{D}\} .$$

Similarly, and also using the previous result, we get $\mathcal{D} \subseteq \mathcal{S}'_A$ and we conclude that $A \cap B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$.

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Suppose $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{F}$ where $\mathcal{F}$ is a σ-algebra on a probability space. Let $\mathcal{D}$ be the set of all events with a certain property, this property naturally making it a λ-system. Suppose also $\sigma(\mathcal{C}) = \mathcal{F}$. By the lemma, $\mathcal{D} = \mathcal{F}$, that is, all events have the desired property.

1.4 Notable exercises

1.4.1 Any open set of reals is a countable union of intervals

See [1, ex. 1.1.2]. Define

$$\mathcal{S}_d = \left\{ \prod_{j=1}^{d} \langle a_j, b_j] : -\infty \leqslant a < b \leqslant +\infty \right\} \subseteq \mathbb{R}^d.$$

Note that $\mathcal{S}_d$ is a semialgebra (automatically, also a π-system). We claim that $\sigma(\mathcal{S}_d) = \mathcal{B}(\mathbb{R}^d)$. It suffices to prove that every open set in $\mathbb{R}^d$ can be written as the countable union of sets in $\mathcal{S}_d$.

Take $d = 1$ with a simple extension afterwards. Let $A \subseteq \mathbb{R}$ be open, so that for each $x \in A$ there is ε_x such that $B(x, \varepsilon_x) \subseteq A$. Also,

$$A = \bigcup_{x \in A} B(x, \varepsilon_x).$$

Note that $A \cap \mathbb{Q}$ is dense in A. Namely, each x is within its ball, and within the ball is a sequence of rationals converging to x. For each rational y such that $|x - y| < \varepsilon_x/2$ we can clearly choose an ε_y such that $x \in B(y, \varepsilon_y) \subseteq A$. This gives us the countable union[1]

$$A = \bigcup_{y \in A \cap \mathbb{Q}} B(y, \varepsilon_y)$$

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1.4.2 Infinite unions of algebras/σ-algebras

See [1, ex. 1.1.4]. Suppose $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ and $\mathcal{F} = \cup_j \mathcal{F}_j$.

If $\mathcal{F}_j$ are algebras, then $\mathcal{F}$ is also an algebra because

$$\bigcup_{j \in N} A_j \in \mathcal{F}_{\max N}, \quad A_j \in \mathcal{F}_j.$$

where $N \subseteq \mathbb{N}$ is finite.

The analogous claim is not true if $\mathcal{F}_j$ are σ-algebras. Let $S = \mathbb{R}$ and

$$\mathcal{F}_j = \sigma\left(\left[\frac{n}{2^j}, \frac{n+1}{2^j} \right\rangle : n \in \mathbb{N} \right), \quad j \in \mathbb{N}.$$

If $\mathcal{F}$ is a σ-algebra,

$$\{0\} = \bigcap_{j \in \mathbb{N}} \left[0, \frac{1}{2^j} \right\rangle \in \mathcal{F},$$

but there is no j such that $\{0\} \in \mathcal{F}_j$, giving us a contradiction.

[1] while different x can give different ε_y, these clearly have a finite supremum

2 Measure and integration

2.1 Products of σ-algebras

2.1.1 Defintions of product σ-algebra

If $(S_\alpha)_{\alpha\in A}$ are measurable spaces with σ-algebras $\mathcal{F}_\alpha$ and $S = \prod_\alpha S_\alpha$, then S has the **product σ-algebra** $\mathcal{F} = \prod_\alpha \mathcal{F}_\alpha = \bigotimes_\alpha \mathcal{F}_\alpha$ defined as

$$\mathcal{F} = \sigma\left(\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{F}_\alpha\right),$$

i.e. **generated by one-dimensional measurable cylinders**.

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If A is countable, then $\mathcal{F}$ is **generated by cuboids** $\prod_\alpha E_\alpha$, $\alpha \in A$ (see [2, p. 23]). To prove it, define

$$\mathcal{F}_1 = \sigma\left(\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{F}_\alpha\right), \quad \mathcal{F}_2 = \sigma\left(\prod_{\alpha\in A} E_\alpha \colon E_\alpha \in \mathcal{F}_\alpha\right).$$

Firstly, $\pi_\alpha^{-1}(E_\alpha) = \prod_\alpha E_\alpha$ where $E_\beta = S_\beta$ for $\beta \neq \alpha$, so $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Secondly,

$$\prod_{\alpha\in A} E_\alpha = \bigcap_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha) \in \mathcal{F}_1,$$

so that $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

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Instead of taking E_α's from the σ-algebras, we can restrict ourselves to **sets of generators**. Suppose $\sigma(\mathcal{E}_\alpha) = \mathcal{F}_\alpha$, then (the second part again if A is countable, and proven the same way)

$$\bigotimes_{\alpha\in A} \mathcal{F}_\alpha = \sigma\left(\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{E}_\alpha\right) = \sigma\left(\prod_{\alpha\in A} E_\alpha \colon E_\alpha \in \mathcal{E}_\alpha\right).$$

To prove the first part, consider the sets

$$\mathcal{D} = \left\{\pi_\alpha^{-1}(E_\alpha) \colon \alpha \in A,\ E_\alpha \in \mathcal{E}_\alpha\right\}, \quad \mathcal{C}_\alpha = \left\{E \in F_\alpha \colon \pi_\alpha^{-1}(E) \in \sigma(\mathcal{D})\right\}.$$

Then C_α is a σ-algebra containing $\mathcal{E}_\alpha$, which means that $C_\alpha = \mathcal{F}_\alpha$ and $\sigma(\mathcal{D}) = \bigotimes_\alpha \mathcal{F}_\alpha$.

2.1.2 Borel σ-algebra on product metric space

Suppose S_j are separable metric spaces ($j \in \mathbb{N}$) with Borel σ-algebras $\mathcal{B}_j$. Then (see [3, p. 11])

$$\mathcal{B}(S_1 \times S_2 \times \cdots) = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \cdots \tag{2.1}$$

Consider the sets

$$\mathcal{C}_1 = \left\{A \colon A \text{ open in } \bigtimes_j S_j\right\}, \quad \mathcal{C}_2 = \left\{S_1 \times \cdots \times B_j \times S_{j+1} \times \cdots \colon B_j \in \mathcal{B}_j\right\},$$

and $\mathcal{C}$ defined similar to $\mathcal{C}_2$ but where B_j has to be open instead. The claim is then $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$.

Clearly $\mathcal{C} \subseteq \mathcal{C}_1, \mathcal{C}_2$. But it is also clear that $\sigma(\mathcal{C}) \supseteq \sigma(\mathcal{C}_2)$, so $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_2)$. This proves that the $\supseteq$ part in (2.1) holds without separability.

If S_j are separable, $\mathcal{C}$ is a topological basis of $\bigtimes_j S_j$, so that $\sigma(\mathcal{C}) \supseteq \sigma(\mathcal{C}_1)$. Then $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_1)$ as well.

2.1.3 Measurability in coordinate functions

See [3, p. 15]. Suppose that $(\Omega, \mathcal{A})$ and $(S_j, \mathcal{S}_j)$ are measurable spaces for $j \in \mathbb{N}$, denoting $S = \bigtimes_j S_j$. Let $f\colon \Omega \to S$ be a function and define f_j as its **coordinate functions** $f_j = \pi_j \circ f$. Then

> f is measurable if and only if all f_j are measurable.

The "only if" part is trivial, because each f_j is a composition of measurable f and π_j. For the "if" part, note that f satisfies the definition of measurability on a generating subset, those being measurable cuboids:

$$f^{-1}(B) = \bigcap_j f_j^{-1}(B_j), \quad B = B_1 \times B_2 \times \cdots, \tag{2.2}$$

where B and all B_j are measurable sets.

Note that (2.2) indeed holds specifically for cuboid sets B and not generally. Instead of these cuboids, we could have also chosen the simpler generating set of one-dimensional cylinders $\pi_j^{-1}(B_j)$. Note that in all cases pointwise convergence can be replaced with a.e. convergence, which is weaker but without affecting integration.

2.2 Convergence theorems

We consider whether $\int f_n \to \int f$ if $f \to f_n$ pointwise, where the functions are defined on a σ-finite measure space X. Following [1, §1.4], we present the convergence theorem in order: bounded, Fatou's lemma, monotone, dominated.

2.2.1 Bounded convergence theorem

Suppose the f_n have

- **bounded domain**: f_n vanishes on E^c for some E with $\mu(E) < \infty$,
- **bounded range**: $|f_n| \leqslant M$ uniformly,
- $f_n \to f$ **in measure**.

Then $\int f_n \to \int f$.

To **prove** it, we have:

$$\begin{aligned}\left|\int f - f_n\right| &\leqslant \int |f - f_n| \\ &= \int\limits_{\{|f-f_n|\geqslant\varepsilon\}} |f - f_n| + \int\limits_{\{|f-f_n|<\varepsilon\}} |f - f_n| \\ &\leqslant 2M\mu\left(|f - f_n| \geqslant \varepsilon\right) + \varepsilon\mu(E) \to 0, \quad n \to \infty,\ \varepsilon \downarrow 0.\end{aligned}$$

We implicitly used that $|f| \leqslant M$ as well. We obtain this because $|f| \geqslant M + \varepsilon$ implies $|f - f_n| \geqslant \varepsilon$, giving us $\mu\left(|f| > M\right) = 0$ after some simple work.

2.2.2 Fatou's lemma

Suppose $f_n \geqslant 0$, then

$$\liminf_{n\to\infty} \int f_n \geqslant \int \liminf_{n\to\infty} f_n.$$

Recall that

$$\liminf_{n\to\infty} f_n = \sup_m \inf_{n\geqslant m} f_n,$$

thus we define $g_n = \inf_{m\geqslant n} f_m$ so that $g_n \uparrow g := \liminf_n f_n$.

It then suffices to show (note $f_n \geqslant g_n$)

$$\liminf_{n\to\infty} \int g_n \geqslant \int g.$$

Let $E_m \uparrow X$ be measurable sets with $\mu(E_m) < \infty$. For fixed m:

$$(g_n \wedge m) \cdot 1_{E_m} \to (g \wedge m) \cdot 1_{E_m}.$$

Finally,

$$\liminf_{n\to\infty} \int g_n \geqslant \int_{E_m} g_n \wedge m \to \int_{E_m} g \wedge m \to \int g.$$

The inequality holds on a subsequence, the first convergence ($n \to \infty$) is the bounded convergence theorem, and the second ($m \to \infty$) is the below lemma.

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In the proof we used the following **lemma** (see [1, p. 22]): let $E_n \uparrow X$ be measurable sets with $\mu(E_n) < \infty$. Then

$$\int_{E_n} f \wedge n \to \int f.$$

Clearly the left side is increasing and below $\int f$. By definition of the Lebesgue integral, it suffices to show that for every simple nonnegative φ, there exists n with

$$\int \varphi \leqslant \int_{E_n} f \wedge n \leqslant \int f.$$

Let N be such that $\mu(\varphi > N) = 0$ and choose $n \geqslant N$. Then

$$\begin{aligned}\int_{E_n} f \wedge n \geqslant \int_{E_n} \varphi &= \int \varphi - \int_{E_n^c} \varphi \\ &\geqslant \int \varphi - N\mu(E_n^c),\end{aligned}$$

so that

$$\liminf_{n\to\infty} \int_{E_n} f \wedge n \geqslant \int \varphi,$$

since $\mu(E_n^c) \to 0$.

2.2.3 Monotone convergence theorem

Suppose $f_n \geqslant 0$ and $f_n \uparrow f$, then

$$\int f_n \to \int f.$$

Since $f_n \leqslant f$ we have $\limsup_n \int f_n \leqslant \int f$, and by Fatou's lemma we have $\liminf_n \int f_n \geqslant \int f$.

2.2.4 Dominated convergence theorem

Let f_n be such that $|f_n| \leqslant g$ uniformly, for some integrable g. If $f_n \to f$, then

$$\int f_n \to \int f.$$

The condition $|f_n| \leqslant g$ can be written as $g \pm f_n \geqslant 0$. With $g \pm f_n \to f$, we apply Fatou's lemma twice. The $+$ and $-$ parts respectively give $\int f \geqslant \limsup_n \int f_n$ and $\int f \leqslant \liminf_n \int f_n$.

2.3 Measure extension

2.4 Product measure

2.5 Fubini's theorem

2.6 Notable exercises

References

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