

Frakcionalno Brownovo gibanje i frakcionalna integracija

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- 1 Samosličnost i FBM
- 2 Frakcionalna integracija

Definicija. Slučajni proces $\{X_t\}$ na \mathbb{R}^d je *samosličan* ako za sve $a > 0, t \geq 0$ i neki $H \geq 0$

$$\{X_{at}\} \stackrel{d}{=} \{a^H X_t\}$$

Parametar H zove se Hurstov parametar/eksponent.

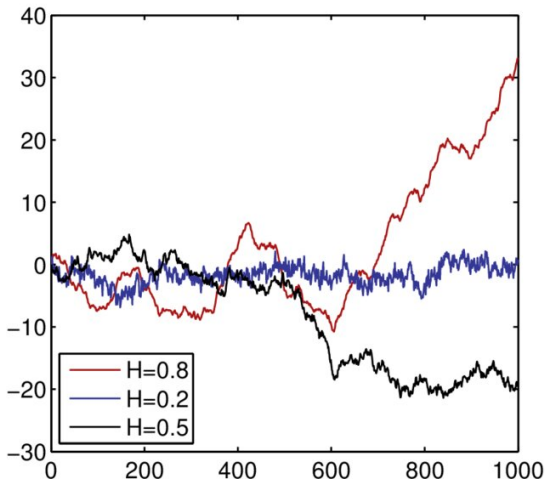
Definicija. *Frakcionalno Brownovo gibanje* (FBM) s parametrom $H \in \langle 0, 1 \rangle$ je gaussovki proces $\{B_t^H : t \in \mathbb{R}\}$ u \mathbb{R} definiran s

$$\mathbb{E}B_t^H = 0, \quad t \in \mathbb{R}$$

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2} \mathbb{E} \left[(B_1^H)^2 \right] \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad t, s \in \mathbb{R}.$$

- jedinstveni centr. gauss. H -ss. proces sa stacionarnim prirastima
- prirasti korelirani

Parametar H i glatkoća trajektorija



Slika: simulacije FBM za različite H (Bovet 2015.)

Teorem. Neka je

$$B_t^H = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}} d\widehat{B}_\xi, \quad t \in \mathbb{R},$$

ψ Meyerov matični valić i $\varepsilon_{j,k} \stackrel{\text{n.j.d.}}{\sim} N(0, 1)$ definirane sa

$$\varepsilon_{j,k} = \int_{\mathbb{R}} 2^{-j/2} e^{ik2^{-j}\xi} \widehat{\psi}(-2^{-j}\xi) d\widehat{B}_\xi = \int_{\mathbb{R}} 2^{j/2} \psi(2^j t - k) dB_t.$$

Tada je

$$B_t^H = \sum_{j,k \in \mathbb{Z}} 2^{-jH} (\psi_H(2^j t - k) - \psi_H(-k)) \varepsilon_{j,k}, \quad t \in \mathbb{R}$$

pri čemu konvergencija vrijedi i g.s. uniformno po t na svakom kompaktnom podskupu od \mathbb{R} .

Integralne reprezentacije FBM

Definiramo

$$\int_{\mathbb{R}} \widehat{f}(\xi) d\widehat{B}_{\xi} = \int_{\mathbb{R}} f(x) dB_x, \quad f \in L^2(\mathbb{R}).$$

Sjetimo se

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \widehat{f}(\xi), \quad f \in C^n(\mathbb{R})$$

i očito

$$B_t = \int_{\mathbb{R}} 1_{[0,t]}(u) dB_u.$$

Mandelbrot–van Ness:

$$B_t^H \stackrel{d}{=} \int_{\mathbb{R}} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] dB_u, \quad t \in \mathbb{R}.$$

Harmonizabilna reprezentacija:

$$B_t^H \stackrel{d}{=} \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}} d\widehat{B}_{\xi}, \quad t \in \mathbb{R}.$$

Neka je ψ Meyerov matični valić. Definiramo

$$\begin{aligned}\Psi_H(x) &= \int_{\mathbb{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{(i\xi)^{H+1/2}} d\xi, \\ \Psi_{-H}(x) &= \int_{\mathbb{R}} e^{ix\xi} \widehat{\psi}(\xi) (-i\xi)^{H+1/2} d\xi, \quad x \in \mathbb{R}.\end{aligned}$$

Vrijedi

$$\begin{aligned}\int_{\mathbb{R}} \Psi_{-H}(t) dt &= \int_{\mathbb{R}} \Psi_H(t) dt = 0. \\ 2^{(j'+j)/2} \int_{\mathbb{R}} \Psi_H(2^{j'} t - k') \Psi_{-H}(2^j t - k) dt &= \begin{cases} 1, & (j, k) = (j', k') \\ 0, & \text{inače.} \end{cases}\end{aligned}$$

Riemann-Liouvilleov frakcionalni integral

Cauchyjeva formula za opetovanu integraciju: za

$$f^{(-n)}(t) = \int_a^t \int_a^{\sigma_1} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n \cdots d\sigma_1,$$

vrijedi

$$f^{(-n)}(t) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

Definiraju se lijevi i desni R-L frakcionalni integral

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad I_{a-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (t-x)^{\alpha-1} f(t) dt$$

za $f \in L^1(\mathbb{R})$, $\alpha > 0$ i $x > a$ ili $x < a$ redom.

Weylov frakcionalni integral

Lijevi i desni Weylov frakcionalni integral

$$I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt, \quad I_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) dt$$

Fourierova transformacija (Samko, Kilbas i Marichev 1993., § 7.1):

$$\mathcal{F}(I_\pm^\alpha f)(\xi) = \frac{\widehat{f}(\xi)}{(\pm i\xi)^\alpha},$$

gdje $0 < \operatorname{Re} \alpha < 1$ i $f \in L^1(\mathbb{R})$.

(uz konvenciju $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx$)

Iz Meyer, Sellan i Taqqu 1999.; krenimo od

$$\Delta f(k) = f(k) - f(k-1) = (I - \tau)f(k), \quad \tau f(k) = f(k-1).$$

Zato Δ^d za $d \in \mathbb{R}$ definiramo preko

$$(I - r\tau)^d = \sum_{k=0}^{\infty} (-1)^k \binom{d}{k} r^k \tau^k =: \sum_{k=0}^{\infty} \gamma_k^{(d)} r^k \tau^k, \quad 0 < r < 1,$$

tj.

$$\Delta^d f(t) = \sum_{k=0}^{\infty} \gamma_k^{(d)} f(t-k)$$

s konvergencijom za $f \in \mathcal{S}(\mathbb{R})$ zbog $\gamma_k^{(d)} = O(k^{-1-d})$.

U Fourierovoj domeni $\widehat{\tau f}(\xi) = e^{-i\xi} \widehat{f}(\xi)$ pa

$$\widehat{\Delta^d f}(\xi) = (1 - e^{-i\xi})^d \widehat{f}(\xi),$$

pri čemu

$$(1 - e^{-i\xi})^d := \lim_{r \uparrow 1} (1 - re^{-i\xi})^d.$$

Neprekidno frakcionalno deriviranje II

Definiramo $\tau_h f(t) = f(t - h)$,







$$D^d = \lim_{h \downarrow 0} \left(\frac{I - \tau_h}{h} \right)^d,$$








onda u Fourierovoj domeni

$$\widehat{D^d} = \lim_{h \downarrow 0} \left(\frac{1 - e^{-i\xi h}}{h} \right)^d = \lim_{h \downarrow 0} \left(\frac{1 - e^{-ih\xi}}{ih\xi} \right)^d (i\xi)^d = (i\xi)^d$$

tj.

$$\widehat{D^d f}(\xi) = (i\xi)^d \widehat{f}(\xi).$$

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