TO: Dr. Sivakumar Rathinam
FROM: Laura Smith
Subject: MEEN 363 Project 1 with additional honors problem
Date: December 10, 2020

SUMMARY OF THIS MEMO

This memorandum presents the solution to project 1: motion of a slider-crank and the additional work for the honors project. It uses rigid-body kinematics to determine position, velocity, and acceleration at two points along the mechanism. It then graphs the findings and analyzes them. It also uses rigid body kinematics and the energy method to solve for theta. The honors project begins on page 5.

METHOD

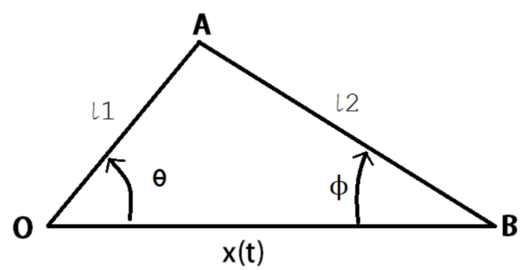

Figure 1: Diagram of Mechanism

The solution to this project determines expressions for the position, velocity, and acceleration at point A, based on vectors $\vec{r_{OA}}$ and $\vec{r_{BA}}$. The position of A calculated using $\vec{r_{OA}}$ is set equal to the position calculated using $\vec{r_{BA}}$, so that the unknown values of $x(t)$ and ϕ can be determined. The same procedure is used for the two velocity equations to determine $\dot{x}(t)$ and $\dot{\phi}$, and for acceleration to determine $\ddot{x}(t)$ and $\ddot{\phi}$

The following equations are used to determine velocity and acceleration at point A:

$$\vec{v_2} = \vec{v_1} + \vec{\omega_1} \times \vec{r_{12}}$$
$$\vec{a_2} = \vec{a_1} + \vec{\alpha_1} \times \vec{r_{12}} + \vec{\omega_1} \times (\vec{\omega_1} \times \vec{r_{12}})$$

This solution writes the vectors in terms of a single global coordinate system ($\hat{I}$ $\hat{J}$ and $\hat{K}$) where the origin is placed at point O.

PROCEDURE

Relevant position vectors:

$$\vec{r_{OA}} = l_1(cos\theta\hat{I} + sin\theta\hat{J})$$
$$\vec{r_{OB}} = x(t)\hat{I}$$
$$\vec{r_{BA}} = l_2(-cos\phi\hat{I} + sin\phi\hat{J})$$
$$\vec{r_{BG}} = \frac{1}{2}l_2(-cos\phi\hat{I} + sin\phi\hat{J})$$

Calculation of position ($x(t)$ and ϕ)
along $\overline{OA}$: $\vec{r_1} = \vec{r_{OA}}$
along $\overline{OBA}$: $\vec{r_2} = \vec{r_{OB}} + \vec{r_{BA}} = (x(t) - l_2 * cos\phi)\hat{I} + (l_2 * sin\phi)\hat{J}$

$$\vec{r_1} = \vec{r_2}$$
$$l_1(cos\theta\hat{I} + sin\theta\hat{J}) = (x(t) - l_2 * cos\phi)\hat{I} + (l_2 * sin\phi)\hat{J}$$
$$[l1 * cos\theta = x(t) - l_2 * cos\phi]\hat{I}$$
$$[l1 * sin\theta = l_2 * sin\phi]\hat{J}$$
$$\phi = asin(\frac{l_1}{l_2}sin\theta)$$
$$x(t) = l_1 * cos\theta + l_2 * cos\phi$$

Calculation of velocity ($\dot{x}(t)$ and $\dot{\phi}$)

$$\vec{\omega_1} = \dot{\theta}\hat{K}$$
$$\dot{\vec{r_A}} = \vec{v_o} + \vec{\omega_1} \times \vec{r_{OA}}$$
$$= 0 + (\dot{\theta}\hat{K}) \times l_1(cos\theta\hat{I} + sin\theta\hat{J})$$
$$= (-\dot{\theta}l_1 sin\theta)\hat{I} + (\dot{\theta}l_1 cos\theta)\hat{J}$$
$$\vec{\omega_2} = -\dot{\phi}\hat{K}$$
$$\dot{\vec{r_A}} = \vec{v_B} + \vec{\omega_2} \times \vec{r_{BA}}$$
$$= (\dot{x}\hat{I}) + (-\dot{\phi}\hat{K}) \times l_2(-cos\phi\hat{I} + sin\phi\hat{J})$$
$$= \dot{x}\hat{I} + \dot{\phi}l_2 sin\phi\hat{I} + \dot{\phi}l_2 cos\phi\hat{J}$$
$$= (\dot{x} + \dot{\phi}l_2 sin\phi)\hat{I} + (\dot{\phi}l_2 cos\phi)\hat{J}$$

$$\dot{\vec{r_A}} = \dot{\vec{r_A}}$$
$$(-\dot{\theta}l_1 sin\theta)\hat{I} + (\dot{\theta}l_1 cos\theta)\hat{J} = (\dot{x} + \dot{\phi}l_2 sin\phi)\hat{I} + (\dot{\phi}l_2 cos\phi)\hat{J}$$
$$[-\dot{\theta}l_1 sin\theta = \dot{x} + \dot{\phi}l_2 sin\phi]\hat{I}$$
$$[\dot{\theta}l_1 cos\theta = \dot{\phi}l_2 cos\phi]\hat{J}$$

$$\dot{\phi} = \dot{\theta}\frac{l_1 cos\theta}{l_2 cos\phi}$$
$$\dot{x}(t) = -(\dot{\theta}l_1 sin\theta + \dot{\phi}l_2 sin\phi)$$

Calculation of acceleration ($\ddot{x}(t)$ and $\ddot{\phi}$)

$$\vec{\alpha_1} = \ddot{\theta}\hat{K}$$
$$\ddot{\vec{r_A}} = \vec{a_o} + \vec{\alpha_1} \times \vec{r_{OA}} + \vec{\omega_1} \times (\vec{\omega_1} \times \vec{r_{OA}})$$
$$= 0 + (\ddot{\theta}\hat{K}) \times l_1(cos\theta\hat{I} + sin\theta\hat{J}) + (\dot{\theta}\hat{K}) \times (-\dot{\theta}l_1 sin\theta\hat{I} + \dot{\theta}l_1 cos\theta\hat{J})$$
$$= (-\ddot{\theta}l_1 sin\theta)\hat{I} + (\ddot{\theta}l_1 cos\theta)\hat{J} - (\dot{\theta}^2 l_1 cos\theta)\hat{I} - (\dot{\theta}^2 l_1 sin\theta)\hat{J}$$
$$= (-\ddot{\theta}l_1 sin\theta - \dot{\theta}^2 l_1 cos\theta)\hat{I} + (\ddot{\theta}l_1 cos\theta - \dot{\theta}^2 l_1 sin\theta)\hat{J}$$
$$\vec{\alpha_2} = -\ddot{\phi}\hat{K}$$
$$\ddot{\vec{r_A}} = \vec{a_B} + \vec{\alpha_2} \times \vec{r_{BA}} + \vec{\omega_2} \times (\vec{\omega_2} \times \vec{r_{BA}})$$
$$= \ddot{x}\hat{I} + (-\ddot{\phi}\hat{K}) \times l_2(-cos\phi\hat{I} + sin\phi\hat{J}) + (-\dot{\phi}\hat{K}) \times (\dot{\phi}l_2 sin\phi\hat{I} + \dot{\phi}l_2 cos\phi\hat{J})$$
$$= (\ddot{x})\hat{I} + (\ddot{\phi}l_2 sin\phi)\hat{I} + (\ddot{\phi}l_2 cos\phi)\hat{J} + (\dot{\phi}^2 l_2 cos\phi)\hat{I} - (\dot{\phi}^2 l_2 sin\phi)\hat{J}$$
$$= (\ddot{x} + \ddot{\phi}l_2 sin\phi + \dot{\phi}^2 l_2 cos\phi)\hat{I} + (\ddot{\phi}l_2 cos\phi - \dot{\phi}^2 l_2 sin\phi)\hat{J}$$

$\ddot{\vec{r_A}} = \ddot{\vec{r_A}}$
$(-\ddot{\theta} l_1 sin\theta - \dot{\theta}^2 l_1 cos\theta)\hat{I} + (\ddot{\theta} l_1 cos\theta - \dot{\theta}^2 l_1 sin\theta)\hat{J} = (\ddot{x} + \ddot{\phi} l_2 sin\phi + \dot{\phi}^2 l_2 cos\phi)\hat{I} + (\ddot{\phi} l_2 cos\phi - \dot{\phi}^2 l_2 sin\phi)\hat{J}$
$[-\ddot{\theta} l_1 sin\theta - \dot{\theta}^2 l_1 cos\theta = \ddot{x} + \ddot{\phi} l_2 sin\phi + \dot{\phi}^2 l_2 cos\phi]\hat{I}$
$[\ddot{\theta} l_1 cos\theta - \dot{\theta}^2 l_1 sin\theta = \ddot{\phi} l_2 cos\phi - \dot{\phi}^2 l_2 sin\phi]\hat{J}$

$$\ddot{\phi} = \frac{\ddot{\theta} l_1 cos\theta - \dot{\theta}^2 l_1 sin\theta + \dot{\phi}^2 l_2 sin\phi}{l_2 cos\phi}$$
$$\ddot{x}(t) = -[\ddot{\theta} l_1 sin\theta + \dot{\theta}^2 l_1 cos\theta + \ddot{\phi} l_2 sin\phi + \dot{\phi}^2 l_2 cos\phi]$$

Motion of Point G:

$$\vec{r_G} = \vec{r_{OB}} + \vec{r_{BG}}$$
$$= (x - \frac{1}{2} l_2 cos\phi)\hat{I} + (\frac{1}{2} l_2 sin\phi)\hat{J}$$
$$\dot{\vec{r_G}} = \vec{v_B} + \vec{\omega_2} \times \vec{r_{BG}}$$
$$= (\dot{x}\hat{I}) + (-\dot{\phi}\hat{K}) \times \frac{1}{2} l_2(-cos\phi\hat{I} + sin\phi\hat{J})$$
$$= \dot{x}\hat{I} + \dot{\phi}\frac{1}{2} l_2 sin\phi\hat{I} + \frac{1}{2}\dot{\phi} l_2 cos\phi\hat{J}$$
$$= (\dot{x} + \frac{1}{2}\dot{\phi} l_2 sin\phi)\hat{I} + (\frac{1}{2}\dot{\phi} l_2 cos\phi)\hat{J}$$
$$\ddot{\vec{r_G}} = \vec{a_B} + \vec{\alpha_2} \times \vec{r_{BG}} + \vec{\omega_2} \times (\vec{\omega_2} \times \vec{r_{BG}})$$
$$= \ddot{x}\hat{I} + (-\ddot{\phi}\hat{K}) \times frac12 l_2(-cos\phi\hat{I} + sin\phi\hat{J}) + (-\dot{\phi}\hat{K}) \times (frac12\dot{\phi} l_2 sin\phi\hat{I} + \frac{1}{2}\dot{\phi} l_2 cos\phi\hat{J})$$
$$= (\ddot{x})\hat{I} + (\frac{1}{2}\ddot{\phi} l_2 sin\phi)\hat{I} + (\frac{1}{2}\ddot{\phi} l_2 cos\phi)\hat{J} + (\frac{1}{2}\dot{\phi}^2 l_2 cos\phi)\hat{I} - (\frac{1}{2}\dot{\phi}^2 l_2 sin\phi)\hat{J}$$
$$= (\ddot{x} + \frac{1}{2}\ddot{\phi} l_2 sin\phi + \frac{1}{2}\dot{\phi}^2 l_2 cos\phi)\hat{I} + (\frac{1}{2}\ddot{\phi} l_2 cos\phi - \frac{1}{2}\dot{\phi}^2 l_2 sin\phi)\hat{J}$$

The lengths associated with Q are:

Q	l_1	l_2
4	2.4	9.6
3	3	9
2	4	8

RESULTS and DISCUSSION

Definition of Unknowns:

$$\phi = asin(\frac{l_1}{l_2} sin\theta)$$
$$\dot{\phi} = \dot{\theta}\frac{l_1 cos\theta}{l_2 cos\phi}$$
$$\ddot{\phi} = \frac{\ddot{\theta} l_1 cos\theta - \dot{\theta}^2 l_1 sin\theta + \dot{\phi}^2 l_2 sin\phi}{l_2 cos\phi}$$

Equations of motion at point B:

$$\vec{r_b} = x(t)\hat{I} = (l_1 * cos\theta + l_2 * cos\phi)\hat{I}$$
$$\dot{\vec{r_b}} = \dot{x}(t)\hat{I} = -(\dot{\theta}l_1 sin\theta + \dot{\phi}l_2 sin\phi)\hat{I}$$
$$\ddot{\vec{r_b}} = \ddot{x}(t)\hat{I} = -[\ddot{\theta}l_1 sin\theta + \dot{\theta}^2 l_1 cos\theta + \ddot{\phi}l_2 sin\phi + \dot{\phi}^2 l_2 cos\phi]\hat{I}$$

Equations of motion at point G:

$$\vec{r_G} = (x - \frac{1}{2}l_2 cos\phi)\hat{I} + (\frac{1}{2}l_2 sin\phi)\hat{J}$$
$$\dot{\vec{r_G}} = (\dot{x} + \frac{1}{2}\dot{\phi}l_2 sin\phi)\hat{I} + (\frac{1}{2}\dot{\phi}l_2 cos\phi)\hat{J}$$
$$\ddot{\vec{r_G}} = (\ddot{x} + \frac{1}{2}\ddot{\phi}l_2 sin\phi + \frac{1}{2}\dot{\phi}^2 l_2 cos\phi)\hat{I} + (\frac{1}{2}\ddot{\phi}l_2 cos\phi - \frac{1}{2}\dot{\phi}^2 l_2 sin\phi)\hat{J}$$

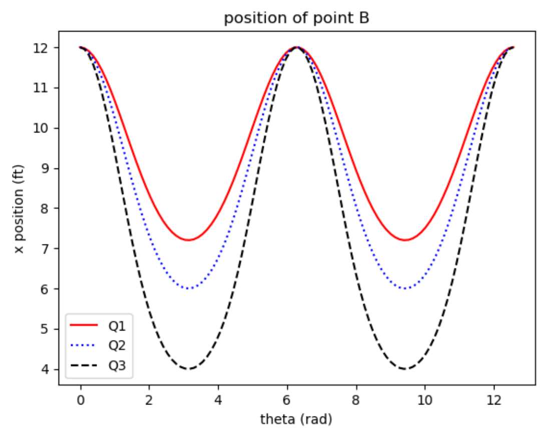

Figure 2: Position of point B

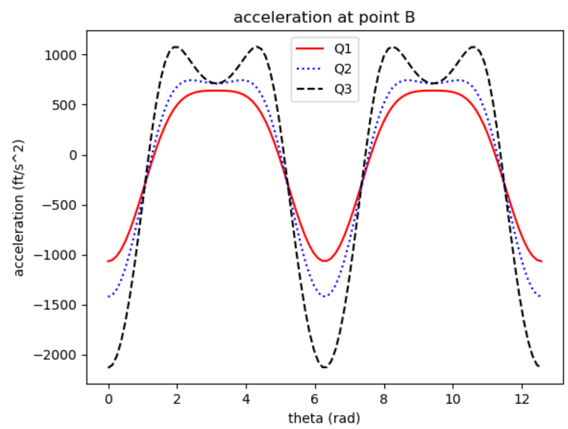

Figure 3: Acceleration at point B

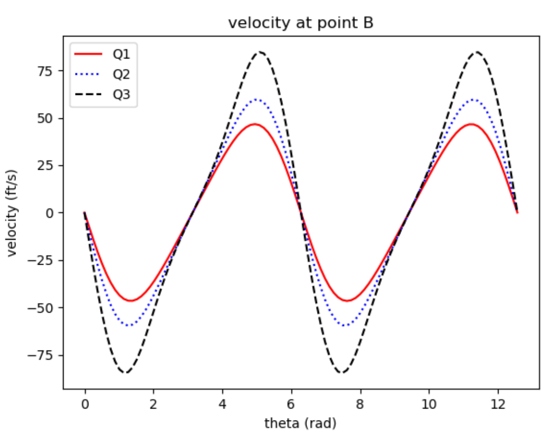

Figure 4: Velocity at point B

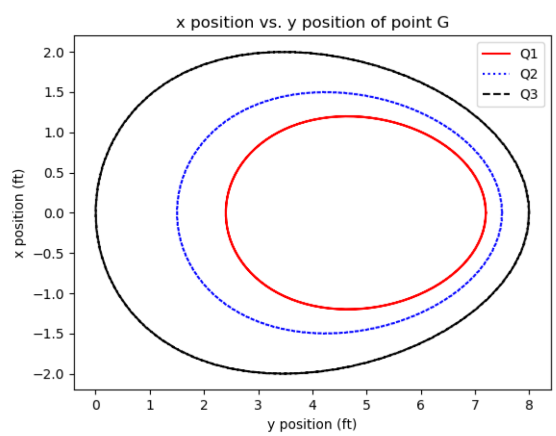

Figure 5: x vs. y Position of point G

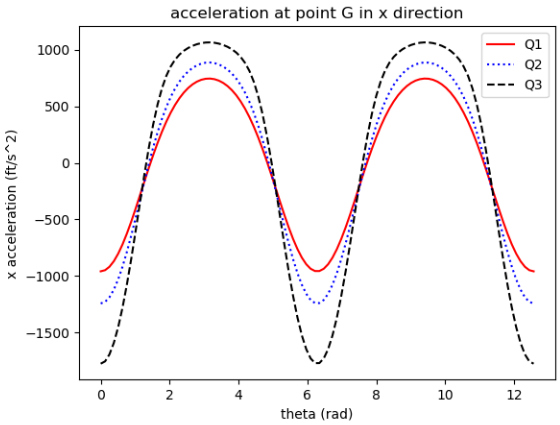

Figure 6: x Acceleration at point G

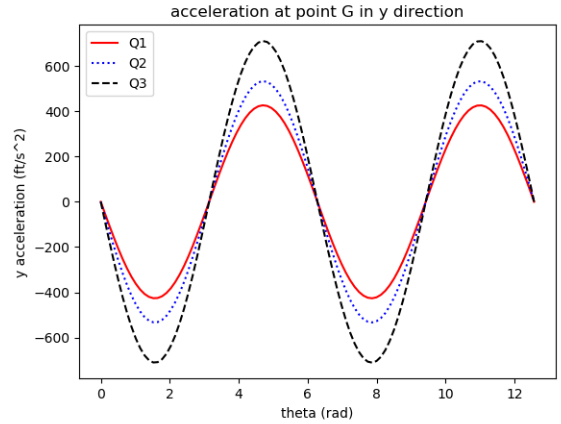

Figure 7: y Acceleration at point G

Honors Project

From above, the velocity of the center of mass of bar AB is:

$$\begin{aligned}\vec{v_G} &= (\dot{x} + \frac{1}{2}\dot{\phi}l_2 sin\phi)\hat{I} + (\frac{1}{2}\dot{\phi}l_2 cos\phi)\hat{J} \\ |\vec{v_G}|^2 &= (\dot{x} + \frac{1}{2}\dot{\phi}l_2 sin\phi)^2 + (\frac{1}{2}\dot{\phi}l_2 cos\phi)^2 \\ &= \dot{x}^2 + \frac{1}{4}\dot{\phi}^2 l_2^2 (sin\phi)^2 + \dot{x}\dot{\phi}l_2 sin\phi + \frac{1}{4}\dot{\phi}^2 l_2^2 (cos\phi)^2 \\ &= \dot{x}^2 + \frac{1}{4}\dot{\phi}^2 l_2^2 + \dot{x}\dot{\phi}l_2 sin\phi\end{aligned}$$

Translational kinetic energy of bar AB:

$$\begin{aligned}KE &= \frac{1}{2}m\,|\vec{v_G}|^2 \\ &= \frac{1}{2}m_2\left(\dot{x}^2 + \frac{1}{4}\dot{\phi}^2 l_2^2 + \dot{x}\dot{\phi}l_2 sin\phi\right)\end{aligned}$$

Rotational kinetic energy of bar AB:

$$\begin{aligned}KE &= \frac{1}{2}I_{cm}\omega^2 \\ &= \frac{1}{2}\left(\frac{1}{12}m_2 l_2^2\right)\dot{\phi}^2 \\ &= \frac{1}{24}m_2 l_2^2\dot{\phi}^2\end{aligned}$$

Potential energy of bar AB:

$$\delta y = \frac{1}{2}l_2 sin\phi$$

$$PE = mg\delta y = \frac{1}{2}m_2 g l_2 sin\phi$$

Because point O is fixed, the translational and rotational kinetic energy of bar OA can be decribed with the following formula:

$$\begin{aligned}KE &= \frac{1}{2}I_o\omega^2\\ &= \frac{1}{2}\left(\frac{1}{3}m_1l_1^2\right)\dot{\theta}^2\\ &= \frac{1}{6}m_1l_1^2\dot{\theta}^2\end{aligned}$$

Potential energy of bar OA:

$$\delta y = \frac{1}{2}l_1 sin\theta$$

$$PE = mg\delta y = \frac{1}{2}m_1gl_1sin\theta$$

Translational kinetic energy of slider:

$$\begin{aligned}KE &= \frac{1}{2}m\left|\vec{v_B}\right|^2\\ &= \frac{1}{2}m_B\dot{x}^2\end{aligned}$$

Summing the energy:

$$\begin{aligned}\sum E &= \frac{1}{2}m_2\left(\dot{x}^2+\frac{1}{4}\dot{\phi}^2l_2^2+\dot{x}\dot{\phi}l_2sin\phi\right)+\frac{1}{24}m_2l_2^2\dot{\phi}^2+\frac{1}{6}m_1l_1^2\dot{\theta}^2+\frac{1}{2}m_B\dot{x}^2+\frac{1}{2}mgl_2sin\phi+\frac{1}{2}mgl_1sin\theta\\ &= \frac{1}{6}m_2l_2^2\dot{\phi}^2+\frac{1}{6}m_1l_1^2\dot{\theta}^2+\frac{1}{2}(m_2+m_B)\dot{x}^2+\frac{1}{2}m_2\dot{x}\dot{\phi}l_2sin\phi+\frac{1}{2}m_2gl_2sin\phi+\frac{1}{2}m_1gl_1sin\theta\end{aligned}$$

This simplifies to:

$$E = \frac{1}{6}m_2l_2^2\dot{\phi}^2+\frac{1}{6}m_1l_1^2\dot{\theta}^2+\frac{1}{2}(m_2+m_B)\dot{x}^2+\frac{1}{2}m_2\dot{x}\dot{\phi}l_2sin\phi+\frac{1}{2}m_2gl_2sin\phi+\frac{1}{2}m_1gl_1sin\theta \quad = const.$$

The relations between theta, phi, and x are:

$$\phi = asin(\frac{l_1}{l_2}sin\theta)$$

$$\dot{\phi} = \dot{\theta}\frac{l_1cos\theta}{l_2cos\phi} = \dot{\theta}\frac{l_1cos\theta}{l_2cos(asin(\frac{l_1}{l_2}sin\theta))}$$

$$\dot{x}(t) = -(\dot{\theta}l_1sin\theta+\dot{\phi}l_2sin\phi) = -\dot{\theta}l_1(sin\theta+cos\theta tan\phi) = -\dot{\theta}l_1(sin\theta+cos\theta*tan(asin(\frac{l_1}{l_2}sin\theta)))$$

Plugging in:

$$\begin{aligned}E = &\left[\frac{1}{6}m_2\left(\frac{l_1cos\theta}{cos(asin(\frac{l_1}{l_2}sin\theta))}\right)^2+\frac{1}{6}m_1l_1^2+\frac{1}{2}(m_2+m_B)\left(l_1sin\theta+l_1cos\theta*tan(asin(\frac{l_1}{l_2}sin\theta))\right)^2\right.\\ &\left.-\frac{1}{2}m_2l_1^3sin\theta(sin\theta+cos\theta*tan(asin(\frac{l_1}{l_2}sin\theta)))*\frac{cos\theta}{l_2cos(asin(\frac{l_1}{l_2}sin\theta))}\right]\dot{\theta}^2+\frac{1}{2}(m_1+m_2)gl_1sin\theta\end{aligned}$$

Isolating $\dot{\theta}$:

$$\begin{aligned}\dot{\theta} = \pm\, sqrt&\left(\left(E-\frac{1}{2}(m_1+m_2)gl_1sin\theta\right)\Big/\left[\frac{1}{6}m_2\left(\frac{l_1cos\theta}{cos(asin(\frac{l_1}{l_2}sin\theta))}\right)^2\right.\right.\\ &+\frac{1}{6}m_1l_1^2+\frac{1}{2}(m_2+m_B)\left(l_1sin\theta+l_1cos\theta*tan(asin(\frac{l_1}{l_2}sin\theta))\right)^2\\ &\left.\left.-\frac{1}{2}m_2l_1^3sin\theta(sin\theta+cos\theta*tan(asin(\frac{l_1}{l_2}sin\theta)))*\frac{cos\theta}{l_2cos(asin(\frac{l_1}{l_2}sin\theta))}\right]\right)\end{aligned}$$

The Runge Kutta method can be used to solve from here. A fourth order Runge Kutta was used as follows:

```
t = 0, h = 0.01, theta = pi/4
while theta > - (pi + pi/4):
    k1 = h*theta_dot(x,t)
    k2 = h*theta_dot(x+ 0.5*k1,t+0.5*h)
    k3 = h*theta_dot(x+ 0.5*k2,t+0.5*h)
    k4 = h*theta_dot(x+k3,t+h)
    theta = theta + (k1 +2*k2 + 2*k3+ k4)/6
    t += h
```

Note: theta_dot(x,t) is the function defined on the previous page

The following constants were plugged in:

Variable	Value
l_1	3 m
l_2	9 m
m_1	1 kg
m_2	1 kg
m_B	10 kg

The initial conditions used were $\theta = \frac{\pi}{4}$ rad and $\dot{\theta}$ = -0.1 rad/s. This gave PE = 20.8 W and KE = 0.398 W, so the total energy of the system is 21.2 W. With these initial conditions, I expect crank OA to turn clockwise at least until it reaches -(180 + 45) degrees, its starting height. The $\dot{\theta}$ equation has a $\pm$ sign in front, so I made this a negative sign and solved for θ while 45 > θ > -225 degrees. The results are shown below.

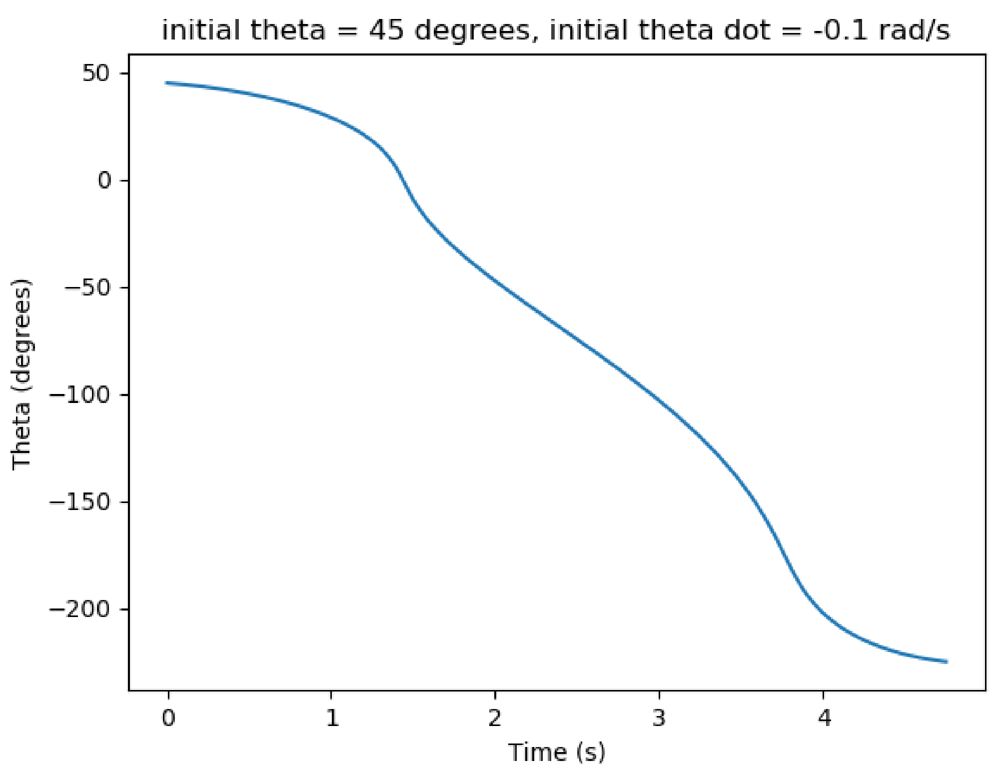

Figure 8: θ vs. Time

This solution has two limitations. First, the Runge Kutta method calculates successive estimates of θ based on $\dot{\theta}$. Because $\dot{\theta}$ is a function of θ only, and not time, then $\Delta\theta = 0$ when $\dot{\theta} = 0$. This happens when the crank changes direction. When this happens, all successive estimates of theta are the same, creating a horizontal line. The numerical solution does not predict the behavior of the system correctly over larger intervals.

Second, the $\dot{\theta}$ equation has a $\pm$ sign in front. θ cannot be calculated numerically over a larger interval without knowing the correct sign of $\dot{\theta}$, and this changes based on θ.

The solution to these problems is to integrate the energy equation. By creating a second order differential equation, successive estimates of θ now also rely on $\ddot{\theta}$, so $\Delta\theta$ does not become zero when the crank changes directions. In addition, integrating the energy equation removes the squared terms, so $\pm$ signs are not a problem. The second order equation can be solved by isolating $\ddot{\theta}$, turning it into two coupled first order equations, and solving the system with the Runge Kutta method. An example of this method is shown below.

```
t=0, h=0.01, x1=pi/4, x2=0,
for i in range(100):
    k11 = h*f1(x1,x2,t)
    k21 = h*f2(x1,x2,t)
    k12 = h*f1(x1+0.5*k11,x2+0.5*k21,t+0.5*h)
    k22 = h*f2(x1+0.5*k11,x2+0.5*k21,t+0.5*h)
    k13 = h*f1(x1+0.5*k12,x2+0.5*k22,t+0.5*h)
    k23 = h*f2(x1+0.5*k12,x2+0.5*k22,t+0.5*h)
    k14 = h*f1(x1+k13,x2+k23,t+h)
    k24 = h*f2(x1+k13,x2+k23,t+h)
    x1 += (k11+2*k12+2*k13+k14)/6
    x2 += (k21+2*k22+2*k23+k24)/6
    t += h
```

Where θ is x1, $\dot{\theta}$ is x2, f1() returns x2 and f2() returns $\ddot{\theta}$, as calculated by the energy equation. This project could be expanded further by performing these calculations. This would allow theta to be calculated over a larger interval.

References

[1] Sivakumar Rathinam. "Kinematics of crankshaft and connecting rod". 2020.

[2] Sivakumar Rathinam. "Writing Technical Memos". 2011.