MRI Simulator Notes

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Tasks

- I looked at the simulation method for excitation and for free precession using rotations and scalings.
- I wrote a proposal for the basic structure of our simulation.

Simulating excitation

Bloch equation in rotating frame

Let $R_z(\theta)$ be the rotation matrix about the z-axis by the angle θ. Assume the magnetic field during excitation is given by

$$\mathbf{B}(z,t) = R_z(\omega t) \underbrace{\begin{bmatrix} B_{1,x}(t) \\ B_{1,y}(t) \\ B_0 + G_z(t)z \end{bmatrix}}_{\mathbf{B}_{\mathrm{rot}}(z,t)},$$

which corresponds to an excitation pulse with carrier frequency ω in the transverse plane in the presence of the main field and a time-varying gradient in the longitudinal direction.

Consider the Bloch equation in a frame of reference rotating about the z axis at frequency ω. Define the rotating magnetization vector, $\mathbf{M}_{\mathrm{rot}}(z,t)$, through the relation

$$\mathbf{M}(z,t) = R_z(\omega t)\mathbf{M}_{\mathrm{rot}}(z,t)\,.$$

In this rotating frame of reference,

$$\frac{d\mathbf{M}_{\mathrm{rot}}}{dt} = \mathbf{M}_{\mathrm{rot}} \times \gamma \mathbf{B}_{\mathrm{eff}}\,,$$

where

$$\mathbf{B}_{\mathrm{eff}} = \mathbf{B}_{\mathrm{rot}} - \frac{\omega}{\gamma}\hat{\mathbf{z}}\,.$$

Assume $\omega = \gamma B_0$. Then

$$\mathbf{B}_{\mathrm{eff}}(z,t) = B_{1,x}(t)\hat{\mathbf{x}} + B_{1,y}(t)\hat{\mathbf{y}} + G_z(t)z\hat{\mathbf{z}}\,.$$

Analytical solution for constant B_1 and G_z

Assume the RF pulse modulation and gradient are constant, so $B_{1,x}(t) \equiv B_{1,x}$, $B_{1,y}(t) \equiv B_{1,y}$, $G_z(t) \equiv G_z$, and

$$\mathbf{B}_{\mathrm{eff}}(z) = B_{1,x}\hat{\mathbf{x}} + B_{1,y}\hat{\mathbf{y}} + G_z z\hat{\mathbf{z}}\,.$$

Solving the Bloch equation in the rotating frame under this assumption, we find $\mathbf{M}_{\mathrm{rot}}$ precesses about the axis

$$\mathbf{u}(z) := \frac{1}{\|\mathbf{B}_{\mathrm{eff}}(z)\|}\mathbf{B}_{\mathrm{eff}}(z)$$

with rotational frequency

$$\gamma\|\mathbf{B}_{\mathrm{eff}}(z)\| = \gamma\sqrt{B_{1,x}^2 + B_{1,y}^2 + (G_z z)^2}\,.$$

Numerical solution for time-varying B_1 and G_z

- We no longer assume the RF pulse modulation and the gradient are constant. Suppose, however, that these signals are well-approximated by piecewise constant functions with time steps Δt. Let t_0 be the time of excitation and let $t_i := t_0 + i\Delta t$ for $i = 0, 1, 2, \ldots, N$ be the set of times spanning the RF pulse.
- During the time interval $[t_i, t_{i+1}]$, $\mathbf{M}_{\mathrm{rot}}$ rotates about the axis

$$\mathbf{u}^i(z) := \frac{1}{\|\mathbf{B}_{\mathrm{eff}}(z, t_i)\|}\mathbf{B}_{\mathrm{eff}}(z, t_i)$$

by the angle

$$\begin{aligned}\theta^i(z) &:= \gamma\|\mathbf{B}_{\mathrm{eff}}(z, t_i)\|\Delta t \\ &= \gamma\Delta t\sqrt{(B_{1,x}(t_i))^2 + (B_{1,y}(t_i))^2 + (G_z(t_i)z)^2}\,.\end{aligned}$$

Therefore, we may compute $\mathbf{M}_{\mathrm{rot}}(z, t_N)$ by applying a sequence of rotations to $\mathbf{M}_{\mathrm{rot}}(z, t_0)$.

Implementation using quaternions

- For each z position, define an initial orientation quaternion $m_0(z)$ with imaginary part $\mathbf{M}_{\mathrm{rot}}(z, t_0)$ and zero real part.
- Define the rotation quaternion for each time interval $[t_i, t_{i+1}]$

$$q_i(z) := \exp\left(\frac{\theta_i(z)}{2}(u_x^i(z)\mathbf{i} + u_y^i(z)\mathbf{j} + u_z^i(z)\mathbf{k})\right) .$$

- The total effect of the RF pulse is the rotation quaternion q_{RF} given by the product of the individual rotation quaternions:

$$q_{\mathrm{RF}}(z) := q_N(z) \cdots q_0(z) .$$

- The final magnetization is the imaginary part of the orientation quaternion

$$m_N(z) := q_{\mathrm{RF}}(z) m_0(z) q_{\mathrm{RF}}(z)^{-1} .$$

Simulating free precession

Bloch equation for free precession

During free precession, the Bloch equation in the laboratory frame reads

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B} - \frac{(M_z - M_0)}{T_1}\hat{\mathbf{z}} - \frac{(M_x\hat{\mathbf{x}} + M_y\hat{\mathbf{y}})}{T_2} \, ,$$

where M_0 is the equilibrium magnetization along the z direction. Let $\mathbf{M} = M_x\hat{\mathbf{x}} + M_y\hat{\mathbf{y}} + M_z\hat{\mathbf{z}}$. Assume that the magnetic field only has a component in the z direction so that $\mathbf{B} = B_z\hat{\mathbf{z}}$.

Define the transverse magnetization as the complex number $M := M_x + jM_y$. The Bloch equation divides into two uncoupled equations

$$\frac{dM}{dt} = -\left(\frac{1}{T_2} + i\gamma B_z\right) M$$
$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}\,.$$

One may recover the x and y components of the magnetization as $M_x = \mathrm{Re}(M)$ and $M_y = \mathrm{Im}(M)$.

Solving these equations, we find

$$M(\mathbf{r},t) = M(\mathbf{r},0)\exp\left(-\frac{t}{T_2(\mathbf{r})}\right)\exp\left(-i\gamma\int_0^t B_z(\mathbf{r},t)dt\right)$$

and

$$M_z(\mathbf{r},t) = M_0 + (M_z(\mathbf{r},0) - M_0)\exp\left(-\frac{t}{T_1(\mathbf{r})}\right)\,.$$

The received MR signal is

$$s(t) = \int M(\mathbf{r},t)d^3\mathbf{r}\,,$$

for t in the readout interval.

Basic structure of our simulation

Basic setup

(Ignore metabolic conversion for now.)

- Define a set of "em"s.
- Each em has (at least) two data attributes: a magnetization vector $\boldsymbol{\mu} \in \mathbb{R}^3$ and a position vector $\mathbf{r} \in \mathbb{R}^3$.
- The simulation instantiates a large number of ems.
- The simulation reads in a pulse sequence of piecewise constant signals, a set of readout times $\mathrm{T}_{\mathrm{readout}}$, and a set of physical parameters $\{T_1(\mathbf{r}), T_2(\mathbf{r})\}$.
- The simulation steps through time. At each time step, the simulation can be in one of two states: excitation or free precession.

Excitation

- During excitation, an RF pulse and a z-gradient are applied.
- Excitation will cause a rotation of each em's magnetization vector in 3D space.
- I propose it will be most efficient to compute this rotation using quaternions.

- Assign a rotation quaternion $q_{\mathrm{RF}} := 1 + \mathbf{0}$ to each em.
- For each time t_i in the excitation interval and each em:
  - Compute the position $\mathbf{r}(t_i + \Delta t)$ according to the physics (blood flow, Brownian motion, etc.).
  - Define
$$\mathbf{u} := \frac{1}{\|\mathbf{B}_{\mathrm{eff}}(\mathbf{r}^*, t_i)\|}\mathbf{B}_{\mathrm{eff}}(\mathbf{r}^*, t_i)$$
$$\theta := \gamma\|\mathbf{B}_{\mathrm{eff}}(\mathbf{r}^*, t_i)\|\Delta t\,.$$
    where $\mathbf{r}^* = (\mathbf{r}(t_i) + \mathbf{r}(t_i + \Delta t))/2$.
  - Update the rotation quaternion using
$$q_{\mathrm{RF}} := \exp(\theta/2(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}))q_{\mathrm{RF}}\,.$$

- At the end of the excitation interval (of length T seconds) update the rotation quaternion to account for the rotation in the laboratory frame:

$$q_{\mathrm{RF}} := \exp(\omega T/2(1\mathbf{k}))q_{\mathrm{RF}}$$

- Update the magnetization of each em using

$$\boldsymbol{\mu} := \operatorname{Im}\left\{q_{\mathrm{RF}}(0+\boldsymbol{\mu})q_{\mathrm{RF}}^{-1}\right\}\ .$$

Free precession

- During free precession, no RF pulse is applied; the only magnetic field present is along the longitudinal direction.
- During free precession, rotation occurs around the z-axis only (i.e. rotation occurs in the 2D transverse plane alone).
- I propose it will be most efficient to represent this rotation using complex numbers.

Free precession

- Assign a transverse magnetization $m := \mu_x + i\mu_y$ to each em.
- For each time t_i in the free precession interval and each em:
  - If $t_i \in \mathrm{T}_{\mathrm{readout}}$ then compute the MR signal sample

    $$s(t_i) := \sum_{k \in (\text{set of ems})} m_k$$

    and store in memory.
  - Compute the position $\mathbf{r}(t_i + \Delta t)$ according to the physics (blood flow, Brownian motion, etc.).
  - Update the transverse magnetization as

    $$m := m \exp\left(-\frac{\Delta t}{T_2(\mathbf{r}^*)}\right) \exp\left(-i\gamma B_z(\mathbf{r}^*, t_i)\Delta t\right) ,$$

    and update the longitudinal magnetization as

    $$\mu_z := \mu_0 + (\mu_z - \mu_0) \exp(-\Delta t / T_1(\mathbf{r}^*)) ,$$

    where $\mathbf{r}^* = (\mathbf{r}(t_i) + \mathbf{r}_i(t_i + \Delta t))/2$.

- At the end of the free precession interval, for each em, assign

$$\mu_x := \mathrm{Re}(m)$$
$$\mu_y := \mathrm{Im}(m)\,.$$