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Solving the Moving Obstacle Path Planning Problem using Embedded Variational Methods

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Abstract

An analytically derived algorithm to solve a simple two dimensional robot path planning problem subject to moving obstacle constraints is presented. Normally a variational formulation is intractable since an obstacle-cluttered environment will present multiple trajectories that are locally optimal solutions. To derive an algorithm which produces a unique solution, an embedding method commonly found in homotopic methods is used. A fictitious third dimension is added to the two-dimensional formulation; local (but nonglobal solutions) in the original problem become saddle-point trajectories in the embedded formulation, allowing for convergence of a numerical algorithm to continue along a descent direction. The computational algorithm becomes globally convergent, i.e. convergence to the global solution is achieved regardless of the choice of initial trajectory used to start the algorithm. Simulation results will demonstrate the effectiveness of the algorithm.

I. Introduction

The off-line robot path planning problem using a priori information about the robot and its workspace has traditionally been classified into two methodologies: The first methodology [1]-[4] belongs to the computational geometric method where an approximate solution to the path planning problem is derived by accounting for geometric constraints alone. This particular method derives an optimal path for a complicated workspace cluttered with obstacles or computes an optimal (sometimes any feasible) path for robots with complex kinematic constraints. On the other hand, since the dynamics of the robot are ignored, only optimal *trajectories* (as opposed to control histories) are computed. The second methodology [5]-[7] focuses on computing the optimum control history for robot manipulators described by multi-dimensional, nonlinear dynamics. Some integral cost function is defined and the optimal control is computed by employing optimal control methods derived from variational calculus. Robot dynamics and obstacle constraints are incorporated as equality/inequality constraints and/or penalty terms in the cost function. The optimal control is computed by minimizing some Hamiltonian function.[8] The method is exact since the actuating controls is computed as a solution. These approaches however, solves problems where the desired trajectory is already defined or where the feasible workspace is an uncluttered, convex workspace. Exact approaches are not used to solve a global path planning problem where the 'obstacle complexity' is significant.

Why do two separate methodologies exist? The difficulty lies in the fact that exact methods derived from variational calculus solve only local optimization problems based on the minimization of functional differentials. Numerical algorithms that compute the optimal control are iterative and need an initial nominal control or trajectory to start the algorithm; the optimal solution that is computed is dependent on the choice of this initial guess. Consequently, if an exact variational method were used to solve the robot path planning problem, two steps would be involved: first, to derive an approximate solution which can be used to start the numerical algorithm; second, the implementation of the numerical algorithm itself to the compute the optimal solution. The derivation of the approximate solution is not trivial and must be intelligently chosen so that it is within the convergent manifold of the globally optimal solution.

The limitation of global path planning problems described above is a consequence of the complexity involved in solving any global/nonconvex optimization problem. Direct application of global optimization methods have been limited but includes, for example [9] where simulated annealing, a probabilistic optimization method, is used to solve a path planning problem for a manipulator with redundancies. Another methodology is given in [10] where exact methods to solve a path planning problem in an obstacle-cluttered workspace are given. Their method derives potential fields whose unique minimum represents the final (goal) state. From this model a closed-loop control law with bounded torque constraints can be derived to transfer a robot from any initial starting point. In comparison, the path planning problem treated in this paper is formulated as a mathematical optimization problem where the solution control or trajectory extremizes some explicit cost function, subject to constraints.

To solve the associated global optimization problem, the problem is reformulated such that the computational algorithm becomes globally convergent regardless of the choice of the initial nominal solution. The first subproblem of computing an approximate solution then becomes an unnecessary step. Global convergence is achieved by *embedding* the problem formulation in a higher dimensional parameter space. Similar approaches to solving a global optimization problems can be found in global homotopy methods [11][12] developed to solve large-scale nonlinear equations. In the same context, invariant imbedding methods have been created to solve boundary value problems associated with nonlinear differential equations.[13][14] Although the application presented in this paper is somewhat different to these other methods, it shares the same goal of deriving an algorithm whose computed solution is *robust with respect to initial conditions*. To achieve this robustness, *the topology of the admissible space is manipulated during the computational algorithm.*

In a preceding paper [15] the embedded formulation given above had been applied to a path planning problem in a workspace with stationary obstacles. In this paper the obstacles (with the same simplifying constraint that they are circular) are assumed to be moving with some constant velocity. A growing amount of attention [16] is focusing on the moving obstacle problem but the approaches to date have been geometric. The embedded formulation produces actuating controls consistent with the the dynamics of the robot.

The organization of the paper will be as follows: In section 2, the problem will be formulated as a conventional optimization problem. Numerical algorithms derived from this formulation will produce nonunique local solutions. In section 3 the embedding method will be presented. A unique solution to the global optimization problem will be derived. In section 4 some analytical justification to the method will be given. The paper will conclude with section 5.

2. Formulation of the Path Planning Problem with Moving Obstacles

In this section the robot path planning problem as formulated as a variational optimization problem will be given in detail. Computational results will show that the solution is only locally optimal and dependent on the initial choice of the nominal control history. The path planning problem is stated as follows:

Given the dynamic characteristics of a cartesian-axis robot

$$\ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

and N obstacles defined by

$$\Omega_i(\mathbf{x}), \quad i = 1,.., N$$

find the control history that minimizes the minimum-time cost functional

$$J(\mathbf{u}) = \int_{t_0}^{t_f} dt$$

subject to the constraints

$$\mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_f) = \mathbf{x}_f$$

and the obstacle constraints

$$\mathbf{x}(t) \notin \Omega_i(\mathbf{x}), \ i = 1,..,N, \ t \in [t_0,t_f].$$

Notice that the minimum-time criterion is appropriate for moving-obstacle problems, since a criterion such as minimum-distance may always produce a straight-line trajectory (by waiting for the obstacles to 'clear out'). For N circular obstacles, the obstacle avoidance constraints can be expressed as N inequality constraints

$$\mathbf{f}(\mathbf{x}(t),t) = \begin{bmatrix} f_1(\mathbf{x}(t),t) \\ f_2(\mathbf{x}(t),t) \\ \cdot \\ \cdot \\ f_N(\mathbf{x}(t),t) \end{bmatrix} \geq \mathbf{0}$$

where for circular obstacles in a two-dimensional space

$$f_i(\mathbf{x}(t),t) = (x_1(t) - (obs_{i}\_x+v_{i}\_x\cdot t))^2 + (x_3(t) - (obs_{i}\_y+v_{i}\_y\cdot t))^2 - obs\_rad\_i^2 \geq 0, \ i=1,..,N$$

The inequality constraints are converted to an equality constraint using a method described in [8]: an additional state variable $x_a(t)$ is appended, whose state equation is given by

$$\dot{x}_a(t) = \sum_{i=1}^{N} [f_i(\mathbf{x}(t),t)]^2 \, U(-f_i)$$

with the boundary condition $x_a(t_0) = 0$; $x_a(t_f) = 0$. $U(\cdot)$ denotes a step function. The state $x_a(t)$ is nondecreasing with time; since the boundary conditions at the initial and final times are specified to be zero, $x_a(t) = 0$ for the entire interval $[t_0,t_f]$. This forces the inequality constraint $f_i(\mathbf{x}(t),t) \geq 0$ to be satisfied.

The complete formulation of the optimal control problem is given as follows:

Given the state equations with initial conditions

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(t_0) &= x_{10} \\ \dot{x}_2(t) &= u_1(t), & x_2(t_0) &= 0 \\ \dot{x}_3(t) &= x_4(t), & x_3(t_0) &= x_{30} \\ \dot{x}_4(t) &= u_2(t), & x_4(t_0) &= 0 \\ \dot{x}_5(t) &= \sum_{i=1}^{N} [f_i(\mathbf{x}(t))]^2 \, U(-f_i), & x_5(t_0) &= 0 \end{aligned}$$

Find the control history that minimizes the cost functional

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{x}(t_f)^T \mathbf{G} \mathbf{x}(t_f) + \int_{t_0}^{t_f} \Big[1+B_1[u_1(t)]^2+B_2[u_2(t)]^2 \Big] \, dt$$

where B_1, B_2 are constant weighting coefficients. The term outside of the integral $\frac{1}{2}\mathbf{x}(t_f)^T\mathbf{G}\mathbf{x}(t_f)$ forces the solution to satisfy the controllability problem of bringing the trajectory to a set of desired states.

2.1 Computational Results

The constrained minimization problem is solved using an augmented Lagrangian formulation. The optimal control history is derived by solving the first order necessary conditions for optimal control. The Hamiltonian is given by

$$\mathcal{H} = 1 + [B_1\ \ B_2]\begin{bmatrix} u_1(t)^2 \\ u_2(t)^2 \end{bmatrix} + \lambda_1(t)x_2(t) + \lambda_2(t)u_1(t) + \lambda_3(t)x_4(t) + \lambda_4(t)u_2(t) + \lambda_5(t)\sum_{i=1}^{N}[f_i(\mathbf{x}(t))]^2\, U(-f_i)$$

The necessary conditions for a locally extremal control are

(1) satisfaction of state equations;

(2) satisfaction of the costate equations $\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial \mathbf{x}}$; for an N-obstacle problem, the costate equations are given by:

$$\dot{\lambda}_1(t) = -4\lambda_7(t)\sum_{i=1}^{N}[(x_1(t)-obs(i)\_x)f_iU(-f_i)]$$

$$\dot{\lambda}_2(t) = -\lambda_1(t)$$

$$\dot{\lambda}_3(t) = -4\lambda_7(t)\sum_{i=1}^{N}[(x_3(t)-obs(i)\_y)f_iU(-f_i)]$$

$$\dot{\lambda}_4(t) = -\lambda_3(t)$$

$$\dot{\lambda}_5(t) = 0$$

(3) satisfaction of the traversality conditions $\lambda^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f),t_f)$, i.e.

$$\lambda_i(t_f) = g_i\, x_i(t_f), \quad i = 1,..,5$$

(4) Pontryagin's minimum principle:

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda^*(t),t) = 0$$

$$\frac{\partial \mathcal{H}}{\partial u_1} = 2\,B\,u_1(t) + \lambda_2(t); \qquad \frac{\partial \mathcal{H}}{\partial u_2} = 2\,B\,u_2(t) + \lambda_4(t)$$

(5) minimization of final time for given control history found as a zero of $\frac{dJ}{dt_f}$:

$$\frac{dJ}{dt_f} = \mathbf{x}(t_f)^T\mathbf{G}\dot{\mathbf{x}}(t_f) + 1 + B_1[u_1(t)]^2 + B_2[u_2(t)]^2.$$

The following steepest descent algorithm is implemented to compute the optimal control history which satisfies the conditions given above:

(1) Choose a nominal control history.

(2) Integrate state equations from $\mathbf{x}(t_0)$ using 4th-order Runge-Kutta integration. Integrate until $\frac{dJ}{dt_f} = 0$.

(3) Integrate the costate equations downward from $\mathbf{x}(t_f)$ using 4th-order Runge-Kutta.

(4) Calculate

$$\left\| \frac{\partial \mathcal{H}}{\partial \mathbf{u}} \right\|^2 = \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right]^T \left[\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right] dt$$

for the iteration just completed.

(5) If $\left\| \frac{\partial \mathcal{H}}{\partial \mathbf{u}} \right\|^2 \leq \varepsilon$ then

terminate the algorithm

else

adjust the control history by the steepest descent method

$$\mathbf{u}^{(i+1)}(k) = \mathbf{u}^{(i)}(k) - \tau\, \frac{\partial \mathcal{H}^{(i)}}{\partial \mathbf{u}}(k)$$

($i \equiv$ current iteration #;
$k \equiv$ discrete time step, k (Δtime increment) $\in [t_0,t_f^*]$ where t_f^* is the final time determined from step (2).)

(6) Loop to step (2) to begin new iteration.

Some computed paths, along with initial trajectories (used to begin the descent algorithm) is shown in the Figures 1 and 2.

Implementation of the steepest descent algorithm to solve for the optimal trajectory for the path planning problem shows that the solution is not unique and dependent on the choice of the initial control history given in step (1) above. The numerical computations demonstrate the fact that the path planning problem when formulated as variational problem is intractable. Only when the initial control is properly chosen does the algorithm converge to the global solution.

3. The Embedding Formulation

In this section we will attempt a reformulation of the path planning problem such that the a numerical algorithm applied to solve for the optimal solution becomes globally convergent, i.e. robust to choices of initial solutions.

This is accomplished by adding a fictitious third dimension to the two-dimensional workspace. The two-dimensional path planning problem is solved by computing the equivalent solution of a three-dimensional problem. This will be referred to as the *embedding formulation.* To ensure that the solution is on the original two dimensional problem a penalty term is added to the cost function to force the solution onto the original two-dimensional plane. The circular obstacles become spherical in this formulation. The strength of the formulation lies in the fact that the initial choice of a nominal solution does not have to be derived using a separate algorithm. The initial solution is chosen to be a straight-line trajectory which goes over the two-dimensional plane and is only valid for the computational algorithm. This does not present a problem because only the final solution must be valid for the original two-dimensional problem. In this embedded formulation, all nonglobal local trajectories of the two dimensional problem become saddle point trajectories. If a numerical algorithm based on the satisfaction of first order necessary conditions converges to one of these critical trajectories, small perturbations from that point will give the algorithm a new descent direction. The embedding method essentially develops a systematic method for converting nonglobal local solutions into saddle point critical points. The instability of saddle points are then exploited in the numerical algorithm to allow for convergence to continue toward a global solution.

Compared to the two-dimensional formulation, two states and one additional control variable is added to the system description. The embedded problem formulation is as follows:

Given the state equations with initial conditions

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), & x_1(t_0) &= x_{10}\\
\dot{x}_2(t) &= u_1(t), & x_2(t_0) &= 0\\
\dot{x}_3(t) &= x_4(t), & x_3(t_0) &= x_{30}\\
\dot{x}_4(t) &= u_2(t), & x_4(t_0) &= 0\\
\dot{x}_5(t) &= x_6(t), & x_5(t_0) &= x_{50}\\
\dot{x}_6(t) &= u_3(t), & x_6(t_0) &= 0\\
\dot{x}_7(t) &= \sum_{i=1}^{N}[f_i(\mathbf{x}(t),t)]^2\, U(-f_i), & x_7(t_0) &= 0
\end{aligned}$$

Find the control history that minimizes the cost functional

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{x}(t_f)^T\mathbf{G}\mathbf{x}(t_f) + \int_{t_f}^{t_0}\Big[1+C\,[x_5(t)]^2+B_1[u_1(t)]^2+B_2[u_2(t)]^2\Big]\,dt$$

where B_1, B_2 are constant weighting coefficients.

The penalty term $C[x_5(t)]^2$ in the cost functional recovers the two dimensional solution from the three dimensional problem formulation. The term forces the optimal solution to lie on the original two dimensional plane. The fictitious control variable $u_3(t)$ is not constrained.

The Hamiltonian is given by

$$\mathcal{H} = 1 + C[x_5(t)]^2 + [B_1\;\; B_2]\begin{bmatrix}u_1(t)^2\\ u_2(t)^2\end{bmatrix} + \lambda_1(t)x_2(t) + \lambda_2(t)u_1(t) + \lambda_3(t)x_4(t) + \lambda_4(t)u_2(t) + \lambda_5(t)x_6(t) + \lambda_6(t)u_3(t) + \lambda_7(t)\sum_{i=1}^{N}[f_i(\mathbf{x}(t),t)]^2\,U(-f_i)$$

As in the two-dimensional problem, the necessary conditions for a locally extremal control are

(1) satisfaction of state equations;

(2) satisfaction of the costate equations $\dot{\lambda} = \frac{\partial\mathcal{H}}{\partial\mathbf{x}}$;

$$\dot{\lambda}_1(t) = -4\lambda_7(t)\sum_{i=1}^{N}[(x_1(t)-obs(i)\_x)f_iU(-f_i)]$$

$$\dot{\lambda}_2(t) = -\lambda_1(t)$$

$$\dot{\lambda}_3(t) = -4\lambda_7(t)\sum_{i=1}^{N}[(x_3(t)-obs(i)\_y)f_iU(-f_i)]$$

$$\dot{\lambda}_4(t) = -\lambda_3(t)$$

$$\dot{\lambda}_5(t) = -2\,C\,x_5(t) - 4\lambda_7(t)\sum_{i=1}^{N}[f_iU(-f_i)]$$

$$\dot{\lambda}_6(t) = -\lambda_5(t)$$

$$\dot{\lambda}_7(t) = 0$$

(3) satisfaction of the traversality conditions $\lambda^*(t_f) = \frac{\partial h}{\partial\mathbf{x}}(\mathbf{x}^*(t_f),t_f)$, i.e.

$$\lambda_i(t_f) = g_i\,x_i(t_f), \quad i = 1,..,7$$

(4) Pontryagin's minimum principle:

$$\frac{\partial\mathcal{H}}{\partial\mathbf{u}}(\mathbf{x}^*(t),\mathbf{u}^*(t),\lambda^*(t),t) = 0$$

$$\frac{\partial\mathcal{H}}{\partial u_1} = 2\,B\,u_1(t) + \lambda_2(t); \quad \frac{\partial\mathcal{H}}{\partial u_2} = 2\,B\,u_2(t) + \lambda_4(t);$$

$$\frac{\partial\mathcal{H}}{\partial u_3} = \lambda_6(t)$$

(5) minimization of final time for given control history found as a zero of $\frac{dJ}{dt_f}$.

$$\frac{dJ}{dt_f} = \mathbf{x}(t_f)^T\mathbf{G}\dot{\mathbf{x}}(t_f)+1+C[x_5(t)]^2+B_1[u_1(t)]^2+B_2[u_2(t)]^2.$$

The necessary conditions are implemented using the steepest descent algorithm described at the end of section 2. Figure 3 shows that the global solution can be computed using a generic initial control history (straight and over the plane) that is independent of the specific obstacle configuration.

4. Justification of the Embedding Algorithm

Although no rigorous proof is given, this section gives analytical background to show the effectiveness of the embedding approach:

The existence of multiple local solutions from the formulation given in section 2 can be shown by analyzing the topology of the admissible vector space of the path planning problem. The vector space that defines the solution of the two-dimensional path planning problem is given by the Cartesian product $C[t_0,t_f] \times C[t_0,t_f]$ where $C[t_0,t_f]$ is the vector space of continuous functions defined over the interval $[t_0,t_f]$. Obstacles, however, are represented as inadmissible regions or 'holes' in the defined vector space and makes the admissible subspace over which the cost functionals are defined a nonconvex subset of $C[t_0,t_f] \times C[t_0,t_f]$. The admissible subspace is given by the difference of two cartesian products (Ω_x, Ω_y denote the subspace of inadmissible vectors)

$$(C[t_0,t_f] \times C[t_0,t_f]) - (\Omega_x[t_0,t_f] \times \Omega_y[t_0,t_f])$$

where $(C[t_0,t_f] \times C[t_0,t_f]) \supset (\Omega_x[t_0,t_f] \times \Omega_y[t_0,t_f])$;

$(\Omega_x[t_0,t_f] \times \Omega_y[t_0,t_f]) = \Omega_i(x,y) = \Omega_i(\mathbf{x})$

for any $i = 1,..,N$; and for any $t \in [t_0,t_f]$.

Notice that although the inadmissible region $(\Omega_x[t_0,t_f] \times \Omega_y[t_0,t_f])$ for the moving obstacle problem is different than that of the stationary obstacle problem, the region, as defined as the subspace of the cartesian product $C[t_0,t_f] \times C[t_0,t_f]$ is *time-independent*. This makes the analysis of the embedding methodology of the moving obstacle problem equivalent to that of the stationary obstacle problem. Algorithms that are effective for stationary obstacle problems should equally be effective for moving obstacle problems. *Topologically, the stationary and moving obstacle problems are equivalent*.

To be precise, convexity of both cost functional and the space of functions over which the functionals are defined is only a sufficiency condition for an optimal solution to be unique and hence global[17]; hence nonconvexity does not prove the lack of uniqueness. However, intractability can be shown from a fundamental necessary condition of iterative numerical algorithms. These convergence algorithms produce candidate solutions every iteration which continuously "deform" to the optimal solution. Only simply connected spaces guarantee continuous deformations of vector elements of the given vector space to the unique optimizing vector; path homotopy (which is realized in a simply connected space) is a necessary condition for the algorithm to be globally convergent. The variational formulation of the two-dimensional path planning problem does not satisfy this necessary condition. This is true for moving obstacle problems as well as for problems where the obstacle configuration is fixed. Thus local optimization techniques will produce initial choice-dependent solutions as shown in the steepest descent implementation given in section 2.

However, in a three dimensional path planning problem whose vector space is defined by the Cartesian triple product $C[t_0,t_f] \times C[t_0,t_f] \times C[t_0,t_f]$ path homotopy exists even with regions of 'inadmissible' holes defined by the obstacles. Then by virtue of definition of path homotopy, there is a continuous deformation from one trajectory (an element of the vector space) to another. This provides the necessary condition for global convergence. In addition, however, a minimization problem requires an iterative process such that there is a *decrease in cost function* for every iteration of the algorithm. A globally convergent algorithm exhibits this property regardless of choice of the initial nominal solution. We will exploit this property apparent in the three-dimensional problem when the inadmissible regions (obstacles) are spherical.

As mentioned in section 3, non-global local solutions of the two-dimensional problem become saddle-point trajectories in the embedded three-dimensional formulation. The addition of an extra parameter due to the fictitious dimension creates a gradient direction such that numerical convergence towards the global solution can continue. The embedding formulation greatly enlarges the convergent manifold containing the global solution, allowing the associated algorithm to be globally convergent and robust to choices of initial nominal solutions.

5. Conclusion

A global path planning algorithm amongst moving obstacles was presented. The topological equivalency of the stationary and moving obstacle problems was exploited to allow a previously developed algorithm for stationary obstacle problems to be directly applied to the moving obstacle problem just presented. A globally convergent algorithm was derived by embedding the problem into three-dimensions and solving the equivalent optimization problem. Despite the simplifying assumptions, a direct method was found to remove the intractability of the variational formulation and produce an analytically exact method to solve the path planning problem without the used of 'intelligent' initial conditions. The justification of the algorithm is determined from the trade-off between the added computational complexity of the embedded approach and the amount of computational / human effort that is required to produce an approximate solution which can be used as an initial guess.

Natural extensions to the material presented can be given by more complicated robot dynamics, more arbitrarily-shaped obstacles, solving the three-dimensional path planning problem (by mapping the problem into four dimensions), and accommodating more general obstacle movement. The derivation of an analytical proof relating the embedding formulation to necessary and sufficient conditions for globally optimal solutions is also being pursued.

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Figure 1 Solution & Obstacle Plots Equally Spaced in Time

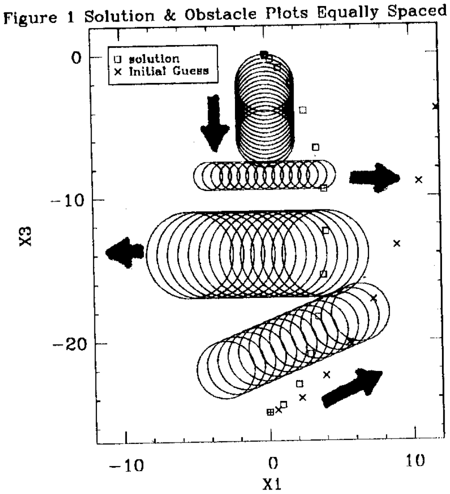

Figure 2 Solution and Obstacle Plots Equally Spaced in Time

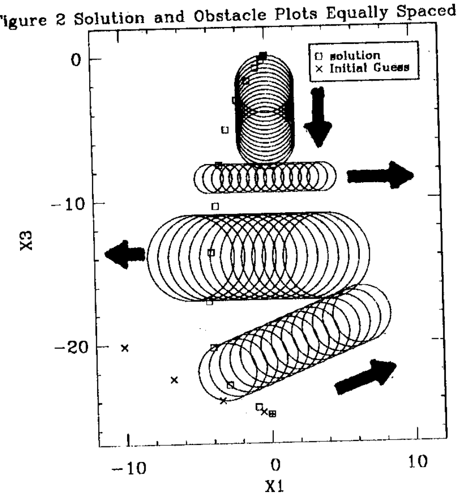

Figure 3 Embedded Solution Independent of Initial Guess