Likelyhood Of Shapes In Moment Space Documentation

Release

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CHAPTER

ONE

WEIGHTED MOMENTS

Define the weighted moments as measured from the image. Let r_i be a vector $[x_i, y_i]$.

$$R_0 = \sum_i z(r_i)w(r_i) \tag{1.1}$$

$$R_1 = \sum_i z(r_i)w(r_i)r_i \tag{1.2}$$

$$R_2 = \sum_i z(r_i)W(r_i)r_i r_i^T \tag{1.3}$$

1.1 Weight Function

We will define the weight function used in calculating moments to be an elliptical Gaussian:

$$w(r_i) = \frac{e^{-\frac{1}{2}r_i^T C_2^{-1} r_i}}{C_0}$$
$$C_0 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}r^T C_2^{-1} r} d^k r : label : weightFunction$$

1.2 Debiasing

The weight function used in calculating the moments is useful for suppressing extranious contributions from noise, but it also biases the measurement. To correct for this, we can calculate an approximate debiasing factor by assuming the source $z(r_i)$ is itself a Gaussian with moments Q_i of the form:

$$z(r_i) = \frac{e^{-\frac{1}{2}(r_i - Q_1)^T Q_2^{-1}(r_i - Q_1)}}{z_0}$$
$$z_0(Q_2) = \int_{-\infty}^{\infty} e^{-1\frac{1}{2}r^T Q_2^{-1} r} d^k r : label : debias$$

1.3 Zeroth Moment

With this assumption the calculation of R_0 becomes:

$$R_0 = \frac{1}{W_0}\frac{Q_0}{z_0(Q_2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(r-Q_1)^T Q_2^{-1}(r-Q_1)} e^{-\frac{1}{2}(r-C_1)^T C_2^{-1}(r-C_1)} d^k r \tag{1.4}$$

Using the Matrix cook book, we recognize the integral can be re-expressed as:

$$R_0 = Q_0 N \int_{-\infty}^{\infty} e^{-\frac{1}{2}(r-\alpha)^T \beta^- 1(r-\alpha)} d^k r \tag{1.5}$$

Where:

$$\begin{aligned} N &= \frac{e^{\frac{-1}{2}(Q_1-C_1)^T (Q_2+C_2)^{-1}(Q_1-C_1)}}{\sqrt{\det(2\pi(Q_2+C_2))}} \\ \alpha &= (Q_2^{-1}+C_2^{-1})^{-1}(Q_2^{-1}Q_1+C_2^{-1}C_1) \\ \beta &= (Q_2^{-1}+C_2^{-1})^{-1} : label : zero_t wo \end{aligned}$$

If we make the assumption that $\beta_x\beta_y - \beta_{xy}^2 > 0$ the above integral evaluates to:

$$R_0 = 2\pi Q_0 \sqrt{\det(\beta)} N \tag{1.6}$$

1.4 First Moment

Following the procedure for the zeroth moment, the first moment is simply:

$$R_1 = 2\pi Q_0 \det(\beta) N\alpha \tag{1.7}$$

1.5 Second Moment

Likewise the second moment is:

$$R_2 = 2\pi Q_0 \det(\beta) N[\beta + \alpha\alpha^T] \tag{1.8}$$

CHAPTER

TWO

UNCERTANTIES IN MOMENTS

Because the data used to calculate moments is uncertain, the moments themselves have uncertanty. Also, because each of the moments is calculated using the same underlying data, the uncertanty in each moment will be correlated every other moment.

To calculate the uncertanties in the moments, we will express the transformation from pixel space to moment space as the following linear algebra equation,

$$\vec{R} = A\vec{Z} \tag{2.1}$$

where $\vec{R}$ is the vector of moments, A is a number of pixels by number of moments matrix of the coefficients in calculation moments, and $\vec{Z}$ is the vector of pixels in the image.

The A matrix can be visualized with the first row contaning the value of the weight function $W(r_i)$ in each column. The second row would be the value of the weight function times the position in x, the third the value of the weight funciton times y, so on and so fourth.

The matrix A can then be used to transfrom the diagonal matrix of image pixel uncertainties denoated by Σ_Z according to the standard transformation:

$$\Sigma_R = A\Sigma_Z A^T \tag{2.2}$$

CHAPTER

THREE

LIKELIHOOD OF MOMENTS

We seek to find what true moments, $\vec{Q}$ are maximumly likely given moments $\vec{R}$ measured on a noisy image. We can express this as the probability of moments $\vec{Q}$ given $\vec{R}$, i.e. $P(\vec{Q}|\vec{R})$. Bayes formula can be used to re-express this in the following way:

$$P(\vec{Q}|\vec{R}) \propto P(\vec{R}|\vec{Q}) * P(\vec{Q}) \tag{3.1}$$

We can then determine the maximumly probable moments, $\vec{Q}$ by finding the arguments of $\vec{Q}$ that maximize the likelyhood of the right hand side of the inequality.

3.1 Probability of Measured Moments Given Real Moments

The first step in maximizing the likelyhood is finding an expression for $P(\vec{R}|\vec{Q})$. This expression says that an object that has real moments $\vec{Q}$ will produce moments $\vec{R}$ when measured from and image that contains noise. This relation can be expressed as a Gaussian random variable distributed about a function of the vector $\vec{Q}$:

$$P(\vec{R}|\vec{Q}) = \frac{1}{a} e^{\frac{-1}{2}(\vec{R}-f(\vec{Q}))^T \Sigma_R^{-1} (\vec{R}-f(\vec{Q}))} \tag{3.2}$$

In this equation a is the normalization constant for a Gaussian and is does not contribute to in finding the maximum likelyhood. $\vec{R}$ is the vector of moments measured from the image,

$$\vec{R} =< R_0, R_{1x}, R_{1y}, R_{2x}, R_{2y}, R_{2xy} > \tag{3.3}$$

The mean of the Gaussian is a vector of functions where each component is the expression used to calcuate the corresponding weighted moment given $\vec{Q}$ as an input. These are the expressions derived in *weighted-moments*.

Finally, the convariance of this Gaussian Σ_R is as derived in the uncertanty of moments section *uncertanties-in-moments*.

3.2 Probability of True moments

The probability of "true" moments $P(\vec{Q})$ can be interpreted as a prior probability distribution on the parameters $\vec{Q}$. When constructing this prior we must take into account that objects measured in an image are not heterogenious, that is objects may either be a star or a galaxy. The form the prior takes will differ depending on which type of object is being measured. We can express a single prior by making it a linear combination of both types weighted by the probability of the object being either type:

$$P(\vec{Q}) = P(\vec{Q}|Q_0)_{gal}[1 - P(*|Q_0)] + P(Q|\vec{Q}_0)_{star}P(*|Q_0) \tag{3.4}$$

where $P(*|Q_0)$ is the probability of an object being a star, and Q_0 is the zeroth moment, aka the flux of an object. This probabiliy can be tuned to different areas of the sky, and is most likely to be a funciton of flux. At this point we will not specify a specific form.

Specifying $P(\vec{Q})$ then becomes an exercise in specifying $P(\vec{Q})_{gal}$ and $P(\vec{Q})_{star}$.

The form of $P(\vec{Q})_{star}$ is strait forward, as stars should always have moments which correspond to the PSF, making $P(\vec{Q})_{star}$ a delta function in parameters space, at the location corresponsing to the PSF. Using a delta funciton can cause issues when evalutating the likelyhood with numeric solvers, as the delta function will contain infinite derivatives. We therefore adopt a delta function convolved by a narrow Gaussian which serves to "fuzz" the delta function in parameter space. The width of this Gaussian can be tuned at evaluation time.

The last peice needed to specifiy $P(\vec{Q})$ is an expression for the distribution of true moments for Galaxies. This is hard to determine in an analytically, as it relies on knowing "true" information about the properties of the universe, which are also what we are attempting to determine. As such, we adopt a distribution that is constructed from past measurements of galaxy moments. This distribution will serve as a prior on what parameters are likely and help regularize the search though the entire space of possible moments. For the pourposes of this document, we use a Gaussian Mixture model with XX components fit to shape parameters determined from the HSC survey. By using a Guassian mixture model we have an analytic model that makes it possible to evaluate first and second derivatives.