

# Construction of a likelihood function for the moments of an image

## Weighted Moments

Define the weighted moments as measured from the image. Let  $r_i$  be a vector  $[x_i, y_i]$ .

$$R_0 = \sum_i z(r_i)w(r_i)$$

$$R_1 = \sum_i z(r_i)w(r_i)r_i$$

$$R_2 = \sum_i z(r_i)W(r_i)r_i r_i^T$$

## Normalized Weighted Moments

The normalized weighted moments can be calculated as follows:

$$M_0 = R_0$$

$$M_1 = \frac{R_1}{M_0}$$

$$M_2 = \frac{R_2}{M_0} - M_1 M_1^T$$

## Weight Function

We will define the weight function used in calculating moments to be an elliptical Gaussian:

$$w(r_i) = \frac{e^{-\frac{1}{2}r_i C_2^{-1} r_i^T}}{w_0}$$

$$w_0 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}r C_2^{-1} r^T} d^k r$$

## Debiasing

The weight function used in calculating the moments is useful for suppressing extraneous contributions from noise, but it also biases the measurement. To correct for this, we can calculate an approximate debiasing factor by assuming the source  $z(r_i)$  is itself a Gaussian with moments  $Q_i$  of the form:

$$z(r_i) = \frac{e^{-\frac{1}{2}(r_i - Q_1) Q_2^{-1} (r_i - Q_1)^T}}{z_0}$$

$$z_0(Q_2) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}r Q_2^{-1} r^T} d^k r$$

## Zeroth Moment

With this assumption the calculation of  $R_0$  becomes:

$$R_0 = \frac{1}{W_0} \frac{Q_0}{z_0(Q_2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(r - Q_1) Q_2^{-1} (r - Q_1)^T} e^{-\frac{1}{2}(r - C_1) C_2^{-1} (r - C_1)^T} d^k r$$

Using the Matrix cook book, we recognize the integral can be re-expressed as:

$$= \frac{1}{N} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(r - \alpha) \beta^{-1} (r - \alpha)^T} d^k r$$

$$N = \sqrt{\det(2\pi(Q_2^{-1} + C_2^{-1}))}$$

$$\alpha = (Q_2^{-1} + C_2^{-1})^{-1} (Q_2^{-1} Q_1 + C_2^{-1} C_1)$$

$$\beta = (Q_2^{-1} + C_2^{-1})^{-1}$$

If we make the assumption that  $\beta_x \beta_y - \beta_{xy}^2 > 0$  the above integral evaluates to:

$$= \frac{1}{N} \frac{2\sqrt{2}\sqrt{\frac{1}{2}\pi}}{\sqrt{\beta_x}\sqrt{-\frac{\beta_{xy}^2 - \beta_x\beta_y}{\beta_x}}}$$

### First Moment

see eq 1 test euler\_\_