Periodogram
and
Correlogram
Methods

Lecture 2

Periodogram

Recall 2nd definition of $\phi(\omega)$:

$$\phi(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2 \right\}$$

Given : $\{y(t)\}_{t=1}^{N}$

Drop "$\lim_{N\to\infty}$" and "$E\{\cdot\}$" to get

$$\boxed{\hat{\phi}_p(\omega) = \frac{1}{N} \left| \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2}$$

- Natural estimator
- Used by Schuster ($\sim$1900) to determine "hidden periodicities" (hence the name).

Correlogram

Recall 1st definition of $\phi(\omega)$:

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{-i\omega k}$$

Truncate the "$\sum$" and replace "$r(k)$" by "$\hat{r}(k)$":

$$\boxed{\hat{\phi}_c(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-i\omega k}}$$

Covariance Estimators
(or Sample Covariances)

Standard unbiased estimate:

$$\hat{r}(k) = \frac{1}{N-k} \sum_{t=k+1}^{N} y(t) y^*(t-k), \quad k \geq 0$$

Standard biased estimate:

$$\hat{r}(k) = \frac{1}{N} \sum_{t=k+1}^{N} y(t) y^*(t-k), \quad k \geq 0$$

For both estimators:

$$\hat{r}(k) = \hat{r}^*(-k), \quad k < 0$$

Relationship Between $\hat{\phi}_p(\omega)$ and $\hat{\phi}_c(\omega)$

If: the biased ACS estimator $\hat{r}(k)$ is used in $\hat{\phi}_c(\omega)$,

Then:

$$\begin{aligned}\hat{\phi}_p(\omega) &= \frac{1}{N}\left|\sum_{t=1}^{N} y(t)e^{-i\omega t}\right|^2 \\ &= \sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-i\omega k} \\ &= \hat{\phi}_c(\omega)\end{aligned}$$

$$\boxed{\hat{\phi}_p(\omega) = \hat{\phi}_c(\omega)}$$

Consequence:
Both $\hat{\phi}_p(\omega)$ and $\hat{\phi}_c(\omega)$ can be analyzed simultaneously.

Statistical Performance of $\hat{\phi}_p(\omega)$ and $\hat{\phi}_c(\omega)$

Summary:

- Both are asymptotically (for large N) unbiased:

$$E\left\{\widehat{\phi}_p(\omega)\right\} \to \phi(\omega) \text{ as } N \to \infty$$

- Both have "large" variance, even for large N.

Thus, $\widehat{\phi}_p(\omega)$ and $\widehat{\phi}_c(\omega)$ have **poor performance**.

Intuitive explanation:

- $\widehat{r}(k) - r(k)$ may be large for large $|k|$
- Even if the errors $\{\widehat{r}(k) - r(k)\}_{|k|=0}^{N-1}$ are small, there are "so many" that when summed in $[\widehat{\phi}_p(\omega) - \phi(\omega)]$, the PSD error is large.

Bias Analysis of the Periodogram

$$
\begin{aligned}
E\left\{\hat{\phi}_p(\omega)\right\} &= E\left\{\hat{\phi}_c(\omega)\right\} = \sum_{k=-(N-1)}^{N-1} E\left\{\hat{r}(k)\right\} e^{-i\omega k} \\
&= \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) r(k) e^{-i\omega k} \\
&= \sum_{k=-\infty}^{\infty} w_B(k) r(k) e^{-i\omega k}
\end{aligned}
$$

$$
\begin{aligned}
w_B(k) &= \begin{cases} \left(1 - \frac{|k|}{N}\right), & |k| \leq N-1 \\ 0, & |k| \geq N \end{cases} \\
&= \text{Bartlett, or triangular, window}
\end{aligned}
$$

Thus,

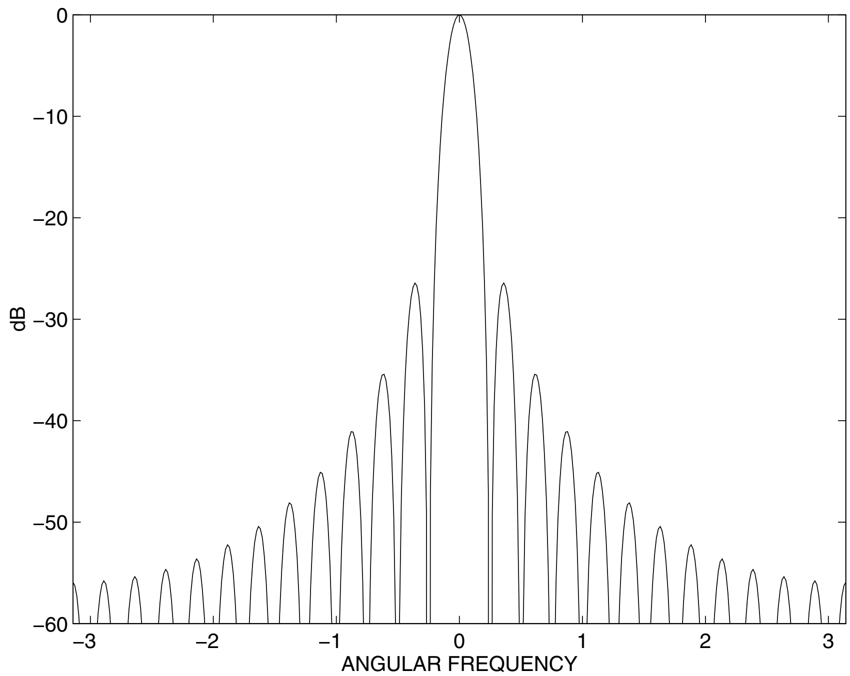
$$
\boxed{E\left\{\hat{\phi}_p(\omega)\right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\zeta) W_B(\omega - \zeta)\, d\zeta}
$$

Ideally: $W_B(\omega) =$ Dirac impulse $\delta(\omega)$.

Bartlett Window $W_B(\omega)$

$$W_B(\omega) = \frac{1}{N}\left[\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right]^2$$

$W_B(\omega)/W_B(0)$, for $N = 25$



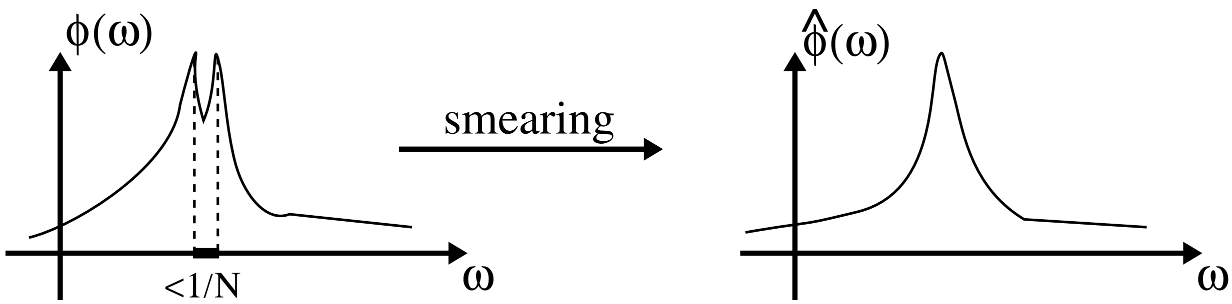
$\boxed{\text{Main lobe 3dB width} \sim 1/N.}$

For "small" N, $W_B(\omega)$ may differ quite a bit from $\delta(\omega)$.

Smearing and Leakage

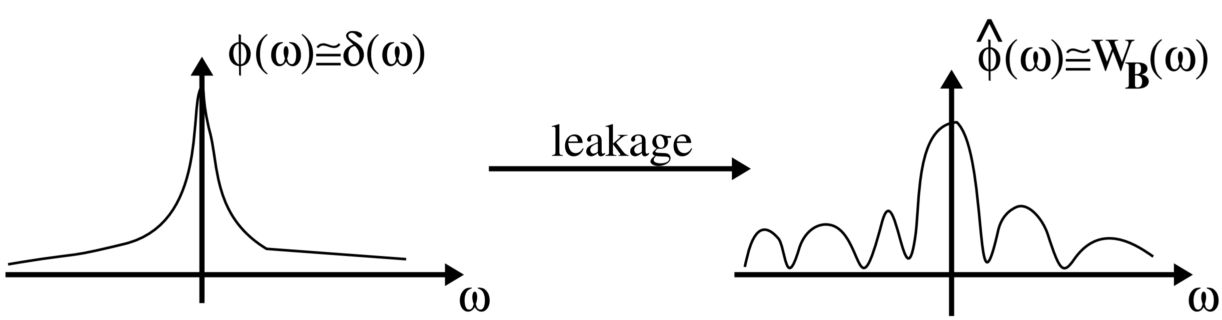
Main Lobe Width: *smearing* or *smoothing*

Details in $\phi(\omega)$ separated in f by less than $1/N$ are not resolvable.



Thus: $\boxed{\text{Periodogram resolution limit} = 1/N.}$

Sidelobe Level: *leakage*



Summary of Periodogram Bias Properties:

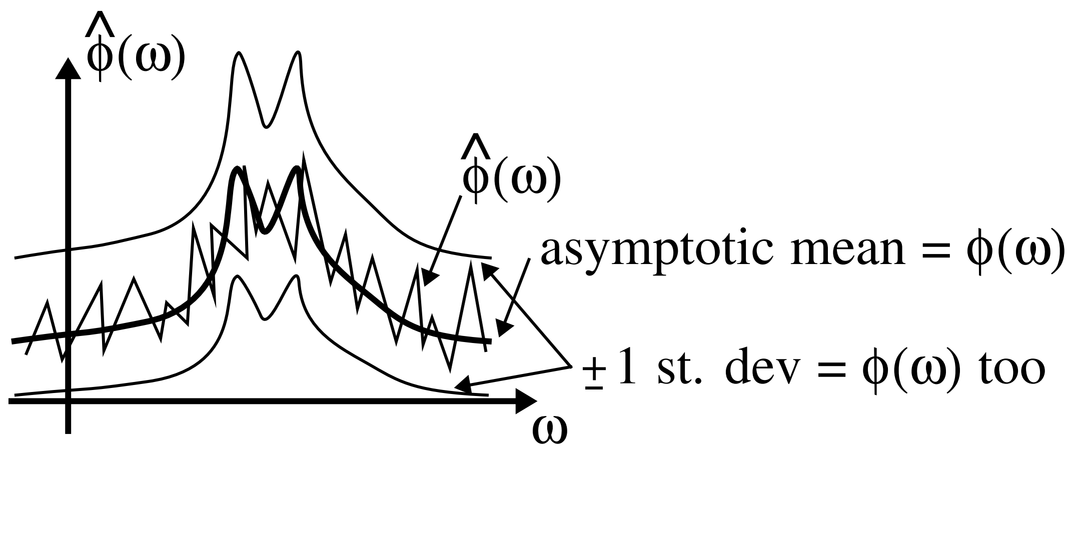
- For “small” N, severe bias
- As $N \to \infty$, $W_B(\omega) \to \delta(\omega)$,
  so $\widehat{\phi}(\omega)$ is asymptotically unbiased.

Periodogram Variance

As $N \to \infty$

$$\boxed{\begin{aligned} & E\left\{\left[\hat{\phi}_p(\omega_1) - \phi(\omega_1)\right]\left[\hat{\phi}_p(\omega_2) - \phi(\omega_2)\right]\right\} \\ & \quad = \begin{cases} \phi^2(\omega_1), & \omega_1 = \omega_2 \\ 0, & \omega_1 \neq \omega_2 \end{cases} \end{aligned}}$$

- Inconsistent estimate
- Erratic behavior



Resolvability properties depend on *both* bias and variance.

Improved Periodogram-Based Methods

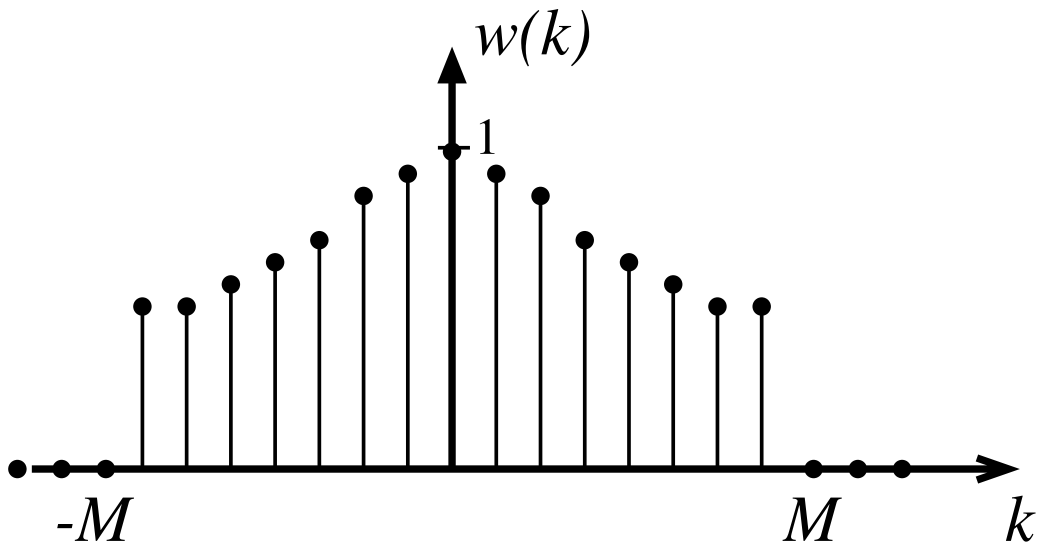
Lecture 3

Blackman-Tukey Method

Basic Idea: Weighted correlogram, with small weight applied to covariances $\hat{r}(k)$ with "large" $|k|$.

$$\boxed{\hat{\phi}_{BT}(\omega) = \sum_{k=-(M-1)}^{M-1} w(k)\hat{r}(k)e^{-i\omega k}}$$

$$\{w(k)\} = \text{ Lag Window}$$



Blackman-Tukey Method, con't

$$\boxed{\hat{\phi}_{BT}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\phi}_p(\zeta) W(\omega - \zeta) d\zeta}$$

$$\begin{aligned} W(\omega) &= \text{DTFT}\{w(k)\} \\ &= \text{Spectral Window} \end{aligned}$$

Conclusion: $\hat{\phi}_{BT}(\omega) =$ "locally" smoothed periodogram

Effect:

- Variance decreases substantially
- Bias increases slightly

By proper choice of M:

$$\text{MSE} = \text{var} + \text{bias}^2 \rightarrow 0 \text{ as } N \rightarrow \infty$$

Window Design Considerations

Nonnegativeness:

$$\hat{\phi}_{BT}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\hat{\phi}_p(\zeta)}_{\geq 0} W(\omega - \zeta) d\zeta$$

If $W(\omega) \geq 0$ ($\Leftrightarrow w(k)$ is a psd sequence)

Then: $\hat{\phi}_{BT}(\omega) \geq 0$ (which is desirable)

Time-Bandwidth Product

$$N_e = \frac{\sum_{k=-(M-1)}^{M-1} w(k)}{w(0)} = \text{equiv time width}$$

$$\beta_e = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega) d\omega}{W(0)} = \text{equiv bandwidth}$$

$$\boxed{N_e \, \beta_e = 1}$$

Window Design, con't

- $\beta_e = 1/N_e = \mathrm{O}(1/M)$
  is the BT resolution threshold.

- As M increases, bias decreases and variance increases.

  $\Rightarrow$ Choose M as a tradeoff between *variance* and *bias*.

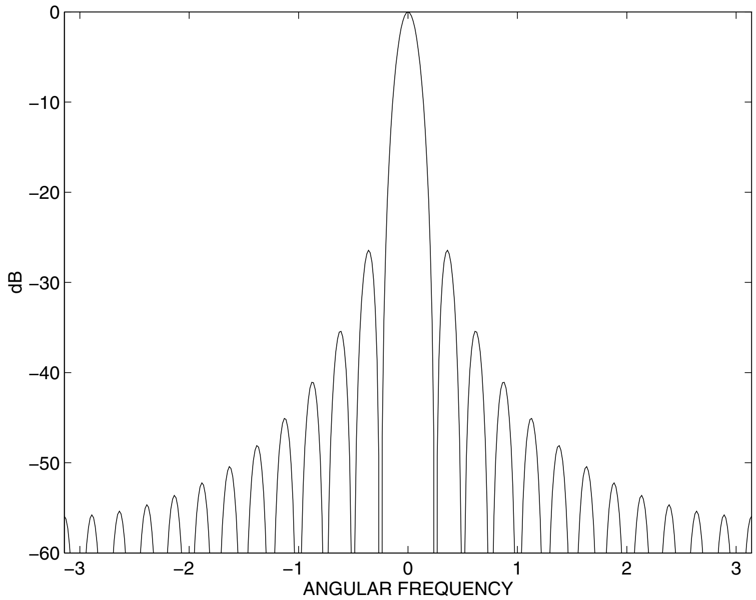
- Once M is given, N_e (and hence β_e) is essentially fixed.

  $\Rightarrow$ Choose window shape to compromise between *smearing* (main lobe width) and *leakage* (sidelobe level).

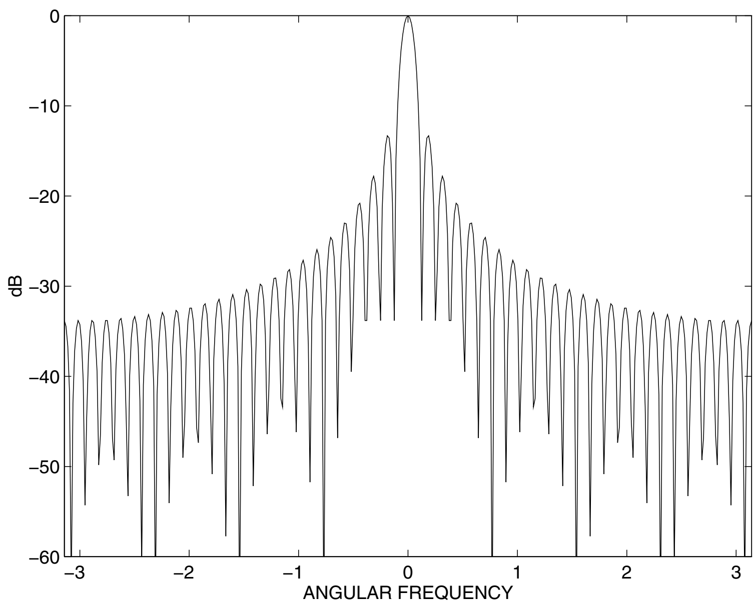
  The energy in the main lobe and in the sidelobes cannot be reduced *simultaneously*, once M is given.

Window Examples

Triangular Window, $M = 25$

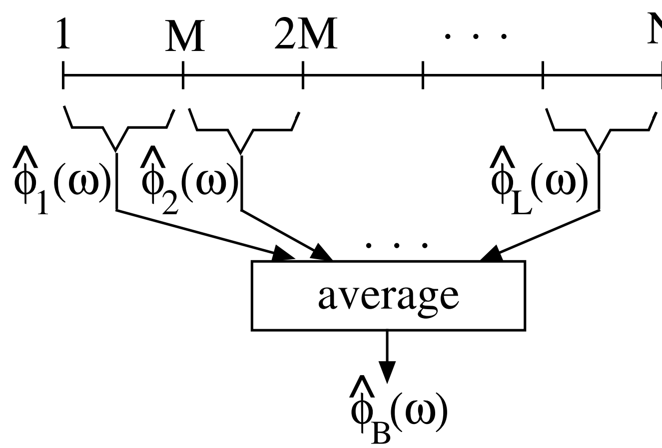


Rectangular Window, $M = 25$



Bartlett Method

Basic Idea:



Mathematically:

$$\begin{aligned} y_j(t) &= y((j-1)M+t) \quad t = 1, \ldots, M \\ &= \text{the } j\text{th subsequence} \\ &\quad (j = 1, \ldots, L \stackrel{\triangle}{=} [N/M]) \end{aligned}$$

$$\hat{\phi}_j(\omega) = \frac{1}{M} \left| \sum_{t=1}^{M} y_j(t) e^{-i\omega t} \right|^2$$

$$\boxed{\hat{\phi}_B(\omega) = \frac{1}{L} \sum_{j=1}^{L} \hat{\phi}_j(\omega)}$$

Comparison of Bartlett and Blackman-Tukey Estimates

$$\begin{aligned}
\hat{\phi}_B(\omega) &= \frac{1}{L} \sum_{j=1}^{L} \left\{ \sum_{k=-(M-1)}^{M-1} \hat{r}_j(k) e^{-i\omega k} \right\} \\
&= \sum_{k=-(M-1)}^{M-1} \left\{ \frac{1}{L} \sum_{j=1}^{L} \hat{r}_j(k) \right\} e^{-i\omega k} \\
&\simeq \sum_{k=-(M-1)}^{M-1} \hat{r}(k) e^{-i\omega k}
\end{aligned}$$

Thus:

$$\boxed{\hat{\phi}_B(\omega) \simeq \hat{\phi}_{BT}(\omega) \text{ with a rectangular lag window } w_R(k)}$$

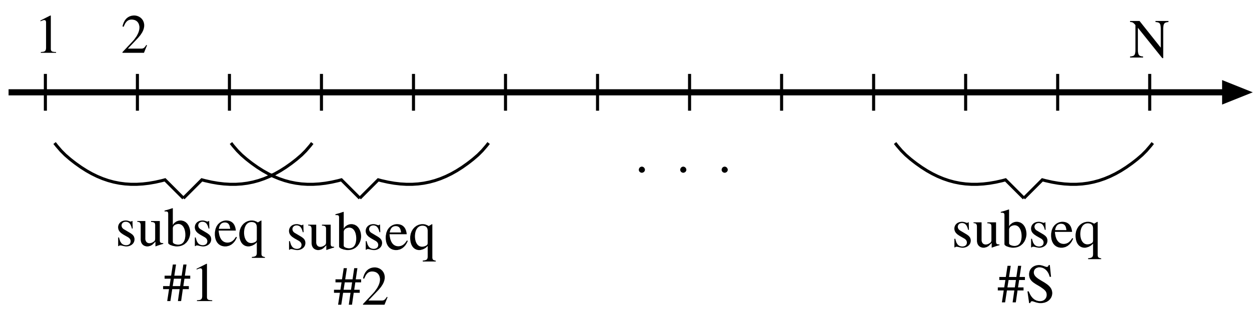
Since $\hat{\phi}_B(\omega)$ implicitly uses $\{w_R(k)\}$, the Bartlett method has

- High resolution (little smearing)
- Large leakage and relatively large variance

Welch Method

Similar to Bartlett method, but

- allow overlap of subsequences (gives more subsequences, and thus "better" averaging)
- use data window for each periodogram; gives mainlobe-sidelobe tradeoff capability



Let $S =$# of subsequences of length M.
(Overlapping means $S > [N/M] \Rightarrow$ "better averaging".)

Additional flexibility:

The data in each subsequence are weighted by a *temporal* window

Welch is approximately equal to $\hat{\phi}_{BT}(\omega)$ with a non-rectangular lag window.

Daniell Method

By a previous result, for $N \gg 1$,

$\{\hat{\phi}_p(\omega_j)\}$ are (nearly) uncorrelated random variables for

$$\left\{\omega_j = \frac{2\pi}{N}\, j\right\}_{j=0}^{N-1}$$

Idea: "Local averaging" of $(2J+1)$ samples in the frequency domain should reduce the variance by about $(2J+1)$.

$$\boxed{\hat{\phi}_D(\omega_k) = \frac{1}{2J+1} \sum_{j=k-J}^{k+J} \hat{\phi}_p\left(\omega_j\right)}$$

Daniell Method, con't

As J increases:

- Bias increases (more smoothing)
- Variance decreases (more averaging)

Let $\beta = 2J/N$. Then, for $N \gg 1$,

$$\boxed{\hat{\phi}_D(\omega) \simeq \frac{1}{2\pi\beta} \int_{-\beta\pi}^{\beta\pi} \hat{\phi}_p(\overline{\omega}) d\overline{\omega}}$$

Hence: $\hat{\phi}_D(\omega) \simeq \hat{\phi}_{BT}(\omega)$ with a *rectangular spectral* window.

Summary of Periodogram Methods

- **Unwindowed periodogram**
  - reasonable bias
  - unacceptable variance

- **Modified periodograms**
  - Attempt to reduce the variance at the expense of (slightly) increasing the bias.

- **BT periodogram**
  - Local smoothing/averaging of $\hat{\phi}_p(\omega)$ by a suitably selected *spectral* window.
  - Implemented by truncating and weighting $\hat{r}(k)$ using a *lag* window in $\hat{\phi}_c(\omega)$

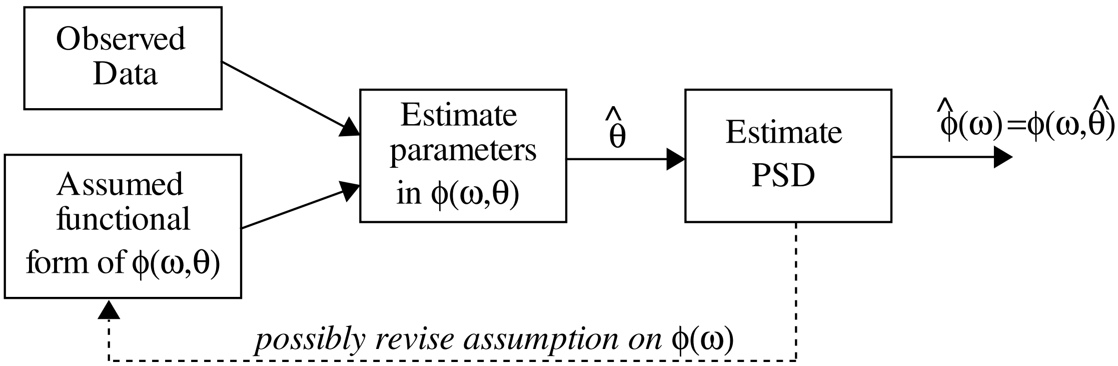
- **Bartlett, Welch periodograms**
  - Approximate interpretation: $\hat{\phi}_{BT}(\omega)$ with a suitable *lag* window (rectangular for Bartlett; more general for Welch).
  - Implemented by averaging subsample periodograms.

- **Daniell Periodogram**
  - Approximate interpretation: $\hat{\phi}_{BT}(\omega)$ with a rectangular *spectral* window.
  - Implemented by local averaging of periodogram values.

Parametric Methods for Rational Spectra

Lecture 4

Basic Idea of Parametric Spectral Estimation



Rational Spectra

$$\phi(\omega) = \frac{\sum_{|k|\leq m} \gamma_k e^{-i\omega k}}{\sum_{|k|\leq n} \rho_k e^{-i\omega k}}$$

$\phi(\omega)$ is a *rational function* in $e^{-i\omega}$.

By *Weierstrass theorem*, $\phi(\omega)$ can approximate arbitrarily well *any continuous PSD*, provided m and n are chosen sufficiently large.

Note, however:

- choice of m and n is not simple
- some PSDs are *not* continuous

AR, MA, and ARMA Models

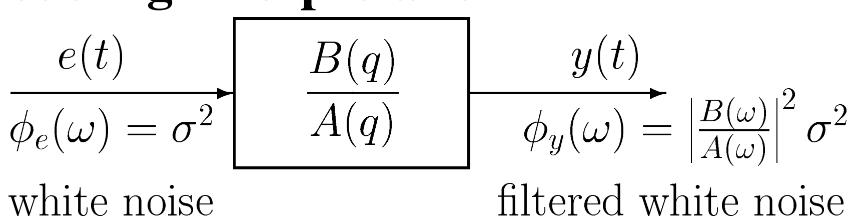
By *Spectral Factorization* theorem, a rational $\phi(\omega)$ can be factored as

$$\phi(\omega) = \left| \frac{B(\omega)}{A(\omega)} \right|^2 \sigma^2$$

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\ B(z) &= 1 + b_1 z^{-1} + \cdots + b_m z^{-m} \end{aligned}$$

and, e.g., $A(\omega) = A(z)|_{z=e^{i\omega}}$

Signal Modeling Interpretation:



ARMA:	$A(q)y(t) = B(q)e(t)$
AR:	$A(q)y(t) = e(t)$
MA:	$y(t) = B(q)e(t)$

ARMA Covariance Structure

ARMA signal model:

$$y(t) + \sum_{i=1}^{n} a_i y(t-i) = \sum_{j=0}^{m} b_j e(t-j), \qquad (b_0 = 1)$$

Multiply by $y^*(t-k)$ and take $E\{\cdot\}$ to give:

$$\begin{aligned} r(k) + \sum_{i=1}^{n} a_i r(k-i) &= \sum_{j=0}^{m} b_j E\{e(t-j) y^*(t-k)\} \\ &= \sigma^2 \sum_{j=0}^{m} b_j h_{j-k}^* \\ &= 0 \text{ for } k > m \end{aligned}$$

where $H(q) = \frac{B(q)}{A(q)} = \sum_{k=0}^{\infty} h_k q^{-k}, \quad (h_0 = 1)$

AR Signals: Yule-Walker Equations

AR: $m = 0$.

Writing covariance equation in matrix form for $k = 1 \dots n$:

$$\begin{bmatrix} r(0) & r(-1) & \dots & r(-n) \\ r(1) & r(0) & & \vdots \\ \vdots & & \ddots & r(-1) \\ r(n) & \dots & & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$R \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

These are the **Yule–Walker (YW) Equations**.

MA Signals

MA: $n = 0$

$$\begin{aligned} y(t) &= B(q)e(t) \\ &= e(t) + b_1 e(t-1) + \cdots + b_m e(t-m) \end{aligned}$$

Thus,

$$\boxed{r(k) = 0 \text{ for } |k| > m}$$

and

$$\phi(\omega) = |B(\omega)|^2 \sigma^2 = \sum_{k=-m}^{m} r(k) e^{-i\omega k}$$

MA Spectrum Estimation

Two main ways to Estimate $\phi(\omega)$:

1. Estimate $\{b_k\}$ and σ^2 and insert them in

$$\phi(\omega) = |B(\omega)|^2\sigma^2$$

   - nonlinear estimation problem
   - $\hat{\phi}(\omega)$ is guaranteed to be ≥ 0

2. Insert sample covariances $\{\hat{r}(k)\}$ in:

$$\phi(\omega) = \sum_{k=-m}^{m} r(k)e^{-i\omega k}$$

   - This is $\hat{\phi}_{BT}(\omega)$ with a rectangular lag window of length $2m+1$.
   - $\hat{\phi}(\omega)$ is *not* guaranteed to be ≥ 0

Both methods are special cases of ARMA methods described below, with AR model order $n = 0$.

ARMA Signals

ARMA models can represent spectra with both peaks (AR part) and valleys (MA part).

$$A(q)y(t) = B(q)e(t)$$

$$\phi(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2 = \frac{\sum_{k=-m}^{m} \gamma_k e^{-i\omega k}}{|A(\omega)|^2}$$

where

$$\begin{aligned}
\gamma_k &= E\left\{[B(q)e(t)][B(q)e(t-k)]^*\right\} \\
&= E\left\{[A(q)y(t)][A(q)y(t-k)]^*\right\} \\
&= \sum_{j=0}^{n} \sum_{p=0}^{n} a_j a_p^* \, r(k+p-j)
\end{aligned}$$

ARMA Spectrum Estimation

Two Methods:

1. Estimate $\{a_i, b_j, \sigma^2\}$ in $\phi(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2$

- nonlinear estimation problem; can use an approximate linear *two-stage least squares* method

- $\hat{\phi}(\omega)$ is guaranteed to be ≥ 0

2. Estimate $\{a_i,\ r(k)\}$ in $\phi(\omega) = \frac{\sum_{k=-m}^{m} \gamma_k e^{-i\omega k}}{|A(\omega)|^2}$

- linear estimation problem (the Modified Yule-Walker method).

- $\hat{\phi}(\omega)$ is *not* guaranteed to be ≥ 0

Modified Yule-Walker Method

ARMA Covariance Equation:

$$r(k) + \sum_{i=1}^{n} a_i r(k-i) = 0, \quad k > m$$

In matrix form for $k = m+1, \ldots, m+M$

$$\begin{bmatrix} r(m) & \ldots & r(m-n+1) \\ r(m+1) & & r(m-n+2) \\ \vdots & \ddots & \vdots \\ r(m+M-1) & \ldots & r(m-n+M) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = - \begin{bmatrix} r(m+1) \\ r(m+2) \\ \vdots \\ r(m+M) \end{bmatrix}$$

Replace $\{r(k)\}$ by $\{\hat{r}(k)\}$ and solve for $\{a_i\}$.

If $M = n$, fast Levinson-type algorithms exist for obtaining $\{\hat{a}_i\}$.

If $M > n$ *overdetermined* YW system of equations; least squares solution for $\{\hat{a}_i\}$.

Note: For narrowband ARMA signals, the accuracy of $\{\hat{a}_i\}$ is often better for $M > n$

Summary of Parametric Methods for Rational Spectra

Method	Computational Burden	Accuracy	Guarantee $\hat{\phi}(\omega) \geq 0$?	Use for
AR: YW or LS	low	medium	Yes	Spectra with (narrow) peaks but no valley
MA: BT	low	low-medium	No	Broadband spectra possibly with valleys but no peaks
ARMA: MYW	low-medium	medium	No	Spectra with both peaks and (not too deep) valleys
ARMA: 2-Stage LS	medium-high	medium-high	Yes	As above

Parametric Methods for Line Spectra — Part 1

Lecture 5

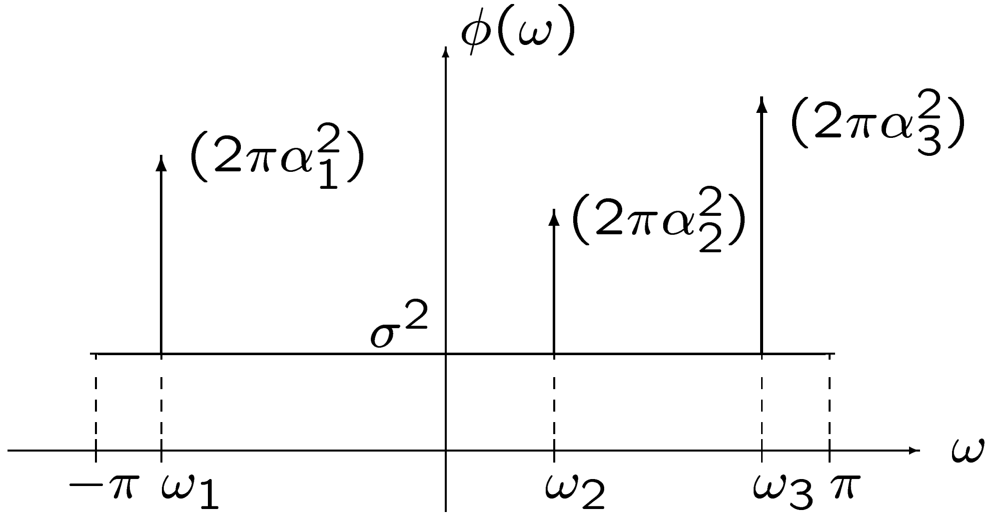
Line Spectra

Many applications have signals with (near) sinusoidal components. Examples:

- communications
- radar, sonar
- geophysical seismology

ARMA model is a *poor approximation*

Better approximation by *Discrete/Line Spectrum Models*



An “Ideal” line spectrum

Line Spectral Signal Model

Signal Model: Sinusoidal components of frequencies $\{\omega_k\}$ and powers $\{\alpha_k^2\}$, superimposed in white noise of power σ^2.

$$
\begin{aligned}
y(t) &= x(t) + e(t) \quad t = 1, 2, \ldots \\
x(t) &= \sum_{k=1}^{n} \underbrace{\alpha_k e^{i(\omega_k t + \phi_k)}}_{x_k(t)}
\end{aligned}
$$

Assumptions:

A1: $\alpha_k > 0 \quad \omega_k \in [-\pi, \pi]$
(prevents model ambiguities)

A2: $\{\varphi_k\} =$ independent rv's, uniformly distributed on $[-\pi, \pi]$
(realistic and mathematically convenient)

A3: $e(t) =$ circular white noise with variance σ^2

$$E\left\{e(t)e^*(s)\right\} = \sigma^2 \delta_{t,s} \quad E\left\{e(t)e(s)\right\} = 0$$

(can be achieved by "slow" sampling)

Covariance Function and PSD

Note that:

- $E\left\{e^{i\varphi_p}e^{-i\varphi_j}\right\} = 1$, for $p = j$

- $E\left\{e^{i\varphi_p}e^{-i\varphi_j}\right\} = E\left\{e^{i\varphi_p}\right\} E\left\{e^{-i\varphi_j}\right\}$
  $= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\varphi}\, d\varphi\right|^2 = 0$, for $p \neq j$

Hence,

$$E\left\{x_p(t)x_j^*(t-k)\right\} = \alpha_p^2\, e^{i\omega_p k}\, \delta_{p,j}$$

$$\boxed{\begin{aligned} r(k) &= E\{y(t)y^*(t-k)\} \\ &= \textstyle\sum_{p=1}^{n} \alpha_p^2 e^{i\omega_p k} + \sigma^2\delta_{k,0} \end{aligned}}$$

and

$$\boxed{\phi(\omega) = 2\pi \sum_{p=1}^{n} \alpha_p^2 \delta(\omega - \omega_p) + \sigma^2}$$

Parameter Estimation

Estimate either:

- $\{\omega_k, \alpha_k, \varphi_k\}_{k=1}^n,\ \sigma^2$ (Signal Model)
- $\{\omega_k, \alpha_k^2\}_{k=1}^n,\ \sigma^2$ (PSD Model)

Major Estimation Problem: $\{\hat{\omega}_k\}$

Once $\{\hat{\omega}_k\}$ are determined:

- $\{\hat{\alpha}_k^2\}$ can be obtained by a least squares method from
$$\hat{r}(k) = \sum_{p=1}^{n} \alpha_p^2 e^{i\hat{\omega}_p k} + \text{residuals}$$

OR:

- Both $\{\hat{\alpha}_k\}$ and $\{\hat{\varphi}_k\}$ can be derived by a least squares method from
$$y(t) = \sum_{k=1}^{n} \beta_k e^{i\hat{\omega}_k t} + \text{residuals}$$
with $\beta_k = \alpha_k e^{i\varphi_k}$.

Nonlinear Least Squares (NLS) Method

$$\min_{\{\omega_k,\alpha_k,\varphi_k\}} \underbrace{\sum_{t=1}^{N} \left| y(t) - \sum_{k=1}^{n} \alpha_k e^{i(\omega_k t + \varphi_k)} \right|^2}_{F(\omega,\alpha,\varphi)}$$

Let:

$$\begin{aligned}
\beta_k &= \alpha_k e^{i\varphi_k} \\
\beta &= [\beta_1 \ldots \beta_n]^T \\
Y &= [y(1) \ldots y(N)]^T \\
B &= \begin{bmatrix} e^{i\omega_1} & \ldots & e^{i\omega_n} \\ \vdots & & \vdots \\ e^{iN\omega_1} & \ldots & e^{iN\omega_n} \end{bmatrix}
\end{aligned}$$

Nonlinear Least Squares (NLS) Method, con't

Then:

$$\begin{aligned} F &= (Y - B\beta)^*(Y - B\beta) = \|Y - B\beta\|^2 \\ &= [\beta - (B^*B)^{-1}B^*Y]^*[B^*B] \\ &\qquad [\beta - (B^*B)^{-1}B^*Y] \\ &\qquad + Y^*Y - Y^*B(B^*B)^{-1}B^*Y \end{aligned}$$

This gives:

$$\boxed{\hat{\beta} = (B^*B)^{-1}B^*Y\Big|_{\omega=\hat{\omega}}}$$

and

$$\boxed{\hat{\omega} = \arg\max_{\omega} Y^*B(B^*B)^{-1}B^*Y}$$

NLS Properties

Excellent Accuracy:

$$\text{var}\ (\hat{\omega}_k) = \frac{6\sigma^2}{N^3\alpha_k^2} \quad (\text{for } N \gg 1)$$

Example: $N = 300$

$$\text{SNR}_k = \alpha_k^2/\sigma^2 = 30 \text{ dB}$$

Then $\sqrt{\text{var}(\hat{\omega}_k)} \sim 10^{-5}$.

Difficult Implementation:

The NLS cost function F is multimodal; it is difficult to avoid convergence to local minima.

Parametric Methods for Line Spectra — Part 2

Lecture 6

The Covariance Matrix Equation

Let:

$$\begin{aligned} a(\omega) &= [1 \; e^{-i\omega} \; \ldots \; e^{-i(m-1)\omega}]^T \\ A &= [a(\omega_1) \; \ldots \; a(\omega_n)] \quad (m \times n) \end{aligned}$$

Note: $\boxed{\text{rank}(A) = n}$ (for $m \geq n$)

Define

$$\tilde{y}(t) \stackrel{\triangle}{=} \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-m+1) \end{bmatrix} = A\tilde{x}(t) + \tilde{e}(t)$$

where

$$\begin{aligned} \tilde{x}(t) &= [x_1(t) \ldots x_n(t)]^T \\ \tilde{e}(t) &= [e(t) \ldots e(t-m+1)]^T \end{aligned}$$

Then

$$\boxed{R \stackrel{\triangle}{=} E\left\{\tilde{y}(t)\tilde{y}^*(t)\right\} = APA^* + \sigma^2 I}$$

with

$$P = E\left\{\tilde{x}(t)\tilde{x}^*(t)\right\} = \begin{bmatrix} \alpha_1^2 & & 0 \\ & \ddots & \\ 0 & & \alpha_n^2 \end{bmatrix}$$

Eigendecomposition of R and Its Properties

$$R = APA^* + \sigma^2 I \quad (m > n)$$

Let:

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$: eigenvalues of R

$\{s_1, \ldots s_n\}$: orthonormal eigenvectors associated with $\{\lambda_1, \ldots, \lambda_n\}$

$\{g_1, \ldots, g_{m-n}\}$: orthonormal eigenvectors associated with $\{\lambda_{n+1}, \ldots, \lambda_m\}$

$$\begin{aligned} S &= [s_1 \ldots s_n] & & (m \times n) \\ G &= [g_1 \ldots g_{m-n}] & & (m \times (m-n)) \end{aligned}$$

Thus,

$$R = [S\ G] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \begin{bmatrix} S^* \\ G^* \end{bmatrix}$$

Eigendecomposition of R and Its Properties, con't

As $\operatorname{rank}(APA^*) = n$:

$$\begin{array}{ll} \lambda_k > \sigma^2 & k = 1, \ldots, n \\ \lambda_k = \sigma^2 & k = n+1, \ldots, m \end{array}$$

$$\mathring{\Lambda} = \begin{bmatrix} \lambda_1 - \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n - \sigma^2 \end{bmatrix} = \text{ nonsingular}$$

Note:

$$RS = APA^*S + \sigma^2 S = S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\boxed{S = A(PA^*S\mathring{\Lambda}^{-1}) \stackrel{\triangle}{=} AC}$$

with $\boxed{|C| \neq 0}$ (since $\operatorname{rank}(S) = \operatorname{rank}(A) = n$).
Therefore, since $S^*G = 0$,

$$\boxed{A^*G = 0}$$

MUSIC Method

$$A^* G = \begin{bmatrix} a^*(\omega_1) \\ \vdots \\ a^*(\omega_n) \end{bmatrix} G = 0$$

$$\Rightarrow \ \{a(\omega_k)\}_{k=1}^n \perp \mathcal{R}(G)$$

Thus,

$\{\omega_k\}_{k=1}^n$ are the *unique solutions* of

$$a^*(\omega) G G^* a(\omega) = 0.$$

Let:

$$\begin{aligned} \hat{R} &= \frac{1}{N} \sum_{t=m}^{N} \tilde{y}(t) \tilde{y}^*(t) \\ \hat{S}, \hat{G} &= S, G \text{ made from the} \\ &\quad\ \text{eigenvectors of } \hat{R} \end{aligned}$$

Spectral and Root MUSIC Methods

Spectral MUSIC Method:

$\{\hat{\omega}_k\}_{k=1}^{n} =$ the locations of the n highest peaks of the "pseudo-spectrum" function:

$$\boxed{\frac{1}{a^*(\omega)\hat{G}\hat{G}^*a(\omega)}, \quad \omega \in [-\pi, \pi]}$$

Root MUSIC Method:

$\{\hat{\omega}_k\}_{k=1}^{n} =$ the angular positions of the n roots of:

$$\boxed{a^T(z^{-1})\hat{G}\hat{G}^*a(z) = 0}$$

that are closest to the unit circle. Here,

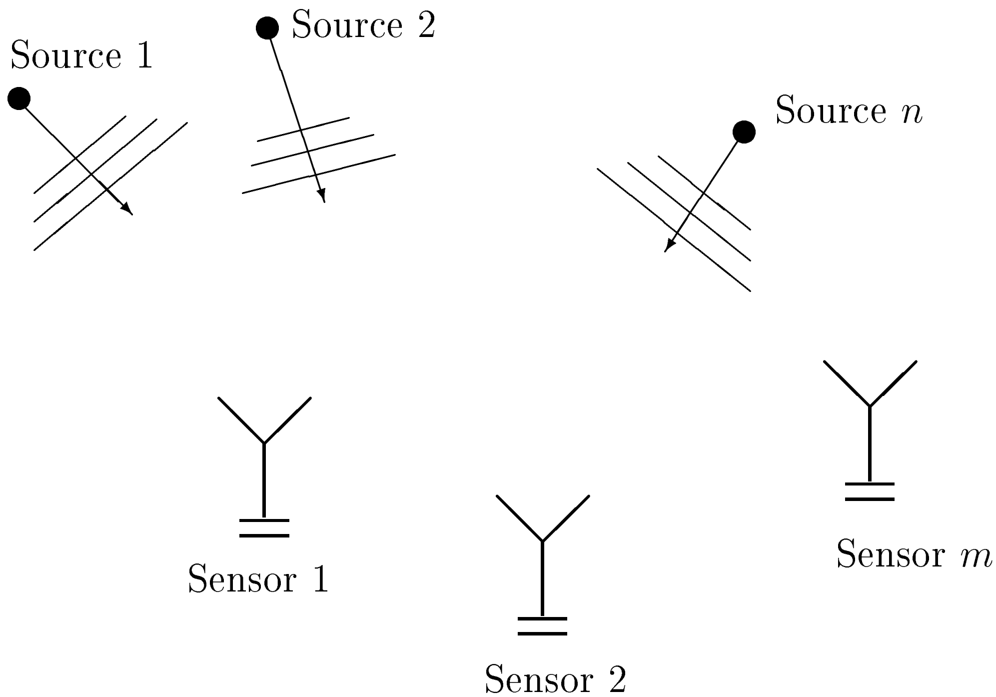
$$a(z) = [1, z^{-1}, \ldots, z^{-(m-1)}]^T$$

Note: Both variants of MUSIC may produce spurious frequency estimates.

Spatial Methods — Part 1

Lecture 8

The Spatial Spectral Estimation Problem



Problem: *Detect* and *locate* n radiating sources by using an array of m passive sensors.

Emitted energy: Acoustic, electromagnetic, mechanical

Receiving sensors: Hydrophones, antennas, seismometers

Applications: Radar, sonar, communications, seismology, underwater surveillance

Basic Approach: Determine energy distribution over *space* (thus the name "spatial spectral analysis")

Simplifying Assumptions

- Far-field sources in the same plane as the array of sensors
- Non-dispersive wave propagation

Hence: The waves are planar and the only location parameter is **direction of arrival (DOA)** (or angle of arrival, AOA).

- The number of sources n is known. (We do not treat the detection problem)
- The sensors are linear dynamic elements with *known transfer characteristics* and *known locations*

  (That is, the array is *calibrated.*)

Array Model — Single Emitter Case

$x(t) =$ the signal waveform as measured at a reference point (*e.g.*, at the "first" sensor)

$\tau_k =$ the delay between the reference point and the kth sensor

$h_k(t) =$ the impulse response (weighting function) of sensor k

$\bar{e}_k(t) =$ "noise" at the kth sensor (*e.g.*, thermal noise in sensor electronics; background noise, etc.)

Note: $t \in \mathcal{R}$ (continuous-time signals).

Then the output of sensor k is

$$\bar{y}_k(t) = h_k(t) * x(t - \tau_k) + \bar{e}_k(t)$$

($* =$ convolution operator).

Basic Problem: Estimate the *time delays* $\{\tau_k\}$ with $h_k(t)$ known but $x(t)$ unknown.

This is a *time-delay estimation problem* in the unknown input case.

Narrowband Assumption

Assume: The emitted signals are narrowband with known carrier frequency ω_c.

Then: $x(t) = \alpha(t) \cos[\omega_c t + \varphi(t)]$

where $\alpha(t),\ \varphi(t)$ vary "slowly enough" so that

$$\alpha(t - \tau_k) \simeq \alpha(t), \qquad \varphi(t - \tau_k) \simeq \varphi(t)$$

Time delay is now $\simeq$ to a *phase shift* $\omega_c \tau_k$:

$$x(t - \tau_k) \simeq \alpha(t) \cos[\omega_c t + \varphi(t) - \omega_c \tau_k]$$

$$\begin{aligned} & h_k(t) * x(t - \tau_k) \\ & \quad \simeq |H_k(\omega_c)| \alpha(t) \cos[\omega_c t + \varphi(t) - \omega_c \tau_k + \arg\{H_k(\omega_c)\}] \end{aligned}$$

where $H_k(\omega) = \mathcal{F}\{h_k(t)\}$ is the kth sensor's transfer function

Hence, the kth sensor output is

$$\begin{aligned} \bar{y}_k(t) = & |H_k(\omega_c)| \alpha(t) \\ & \cdot \cos[\omega_c t + \varphi(t) - \omega_c \tau_k + \arg H_k(\omega_c)] + \bar{e}_k(t) \end{aligned}$$

Complex Signal Representation

The noise-free output has the form:

$$
\begin{aligned}
z(t) &= \beta(t)\cos\left[\omega_c t + \psi(t)\right] = \\
&= \frac{\beta(t)}{2}\left\{e^{i[\omega_c t+\psi(t)]} + e^{-i[\omega_c t+\psi(t)]}\right\}
\end{aligned}
$$

Demodulate $z(t)$ (translate to baseband):

$$
2z(t)e^{-\omega_c t} = \beta(t)\{\underbrace{e^{i\psi(t)}}_{\text{lowpass}} + \underbrace{e^{-i[2\omega_c t+\psi(t)]}}_{\text{highpass}}\}
$$

Lowpass filter $2z(t)e^{-i\omega_c t}$ to obtain $\beta(t)e^{i\psi(t)}$

Hence, by low-pass filtering and sampling the signal

$$
\begin{aligned}
\tilde{y}_k(t)/2 &= \bar{y}_k(t)e^{-i\omega_c t} \\
&= \bar{y}_k(t)\cos(\omega_c t) - i\bar{y}_k(t)\sin(\omega_c t)
\end{aligned}
$$

we get the **complex representation**: (for $t \in \mathcal{Z}$)

$$
y_k(t) = \underbrace{\alpha(t)\; e^{i\varphi(t)}}_{s(t)}\; \underbrace{|H_k(\omega_c)|\; e^{i\arg[H_k(\omega_c)]}}_{H_k(\omega_c)}\; e^{-i\omega_c\tau_k} + e_k(t)
$$

or

$$
\boxed{y_k(t) = s(t)H_k(\omega_c)\; e^{-i\omega_c\tau_k} + e_k(t)}
$$

where $s(t)$ is the *complex envelope* of $x(t)$.

Vector Representation for a Narrowband Source

Let

$$\begin{aligned} \theta &= \text{the emitter DOA} \\ m &= \text{the number of sensors} \\ a(\theta) &= \begin{bmatrix} H_1(\omega_c)\, e^{-i\omega_c\tau_1} \\ \vdots \\ H_m(\omega_c)\, e^{-i\omega_c\tau_m} \end{bmatrix} \end{aligned}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \qquad e(t) = \begin{bmatrix} e_1(t) \\ \vdots \\ e_m(t) \end{bmatrix}$$

Then

$$\boxed{y(t) = a(\theta)s(t) + e(t)}$$

NOTE: θ enters $a(\theta)$ via both $\{\tau_k\}$ and $\{H_k(\omega_c)\}$. For *omnidirectional* sensors the $\{H_k(\omega_c)\}$ do not depend on θ.

Multiple Emitter Case

Given n emitters with

- received signals: $\{s_k(t)\}_{k=1}^n$
- DOAs: θ_k

Linear sensors $\Rightarrow$

$$y(t) = a(\theta_1)s_1(t) + \cdots + a(\theta_n)s_n(t) + e(t)$$

Let

$$A = [a(\theta_1) \ldots a(\theta_n)], \ (m \times n)$$

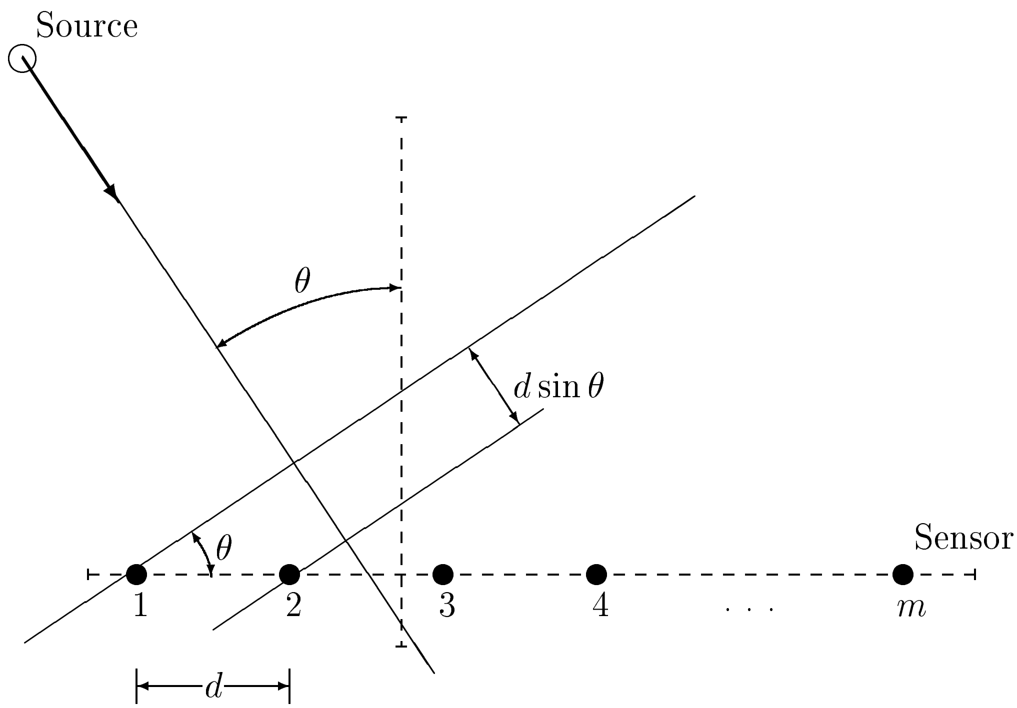
$$s(t) = [s_1(t) \ldots s_n(t)]^T, \ (n \times 1)$$

Then, the **array equation** is:

$$\boxed{y(t) = As(t) + e(t)}$$

Use the *planar wave* assumption to find the dependence of τ_k on θ.

Uniform Linear Arrays



ULA Geometry

Sensor #1 = time delay reference

Time Delay for sensor k:

$$\tau_k = (k-1)\,\frac{d\sin\theta}{c}$$

where c = wave propagation speed

Spatial Frequency

Let:

$$\omega_s \stackrel{\triangle}{=} \omega_c \frac{d \sin \theta}{c} = 2\pi \frac{d \sin \theta}{c/f_c} = 2\pi \frac{d \sin \theta}{\lambda}$$

$$\lambda = c/f_c = \textit{signal wavelength}$$

$$a(\theta) = [1, e^{-i\omega_s} \; \ldots \; e^{-i(m-1)\omega_s}]^T$$

By direct analogy with the vector $a(\omega)$ made from uniform samples of a *sinusoidal time series*,

$$\omega_s = \text{ spatial frequency}$$

The function $\omega_s \mapsto a(\theta)$ is *one-to-one* for

$$|\omega_s| \leq \pi \leftrightarrow \frac{d|\sin \theta|}{\lambda/2} \leq 1 \leftarrow \boxed{d \leq \lambda/2}$$

As

$$d = \text{ spatial sampling period}$$

$d \leq \lambda/2$ is a **spatial** Shannon sampling theorem.

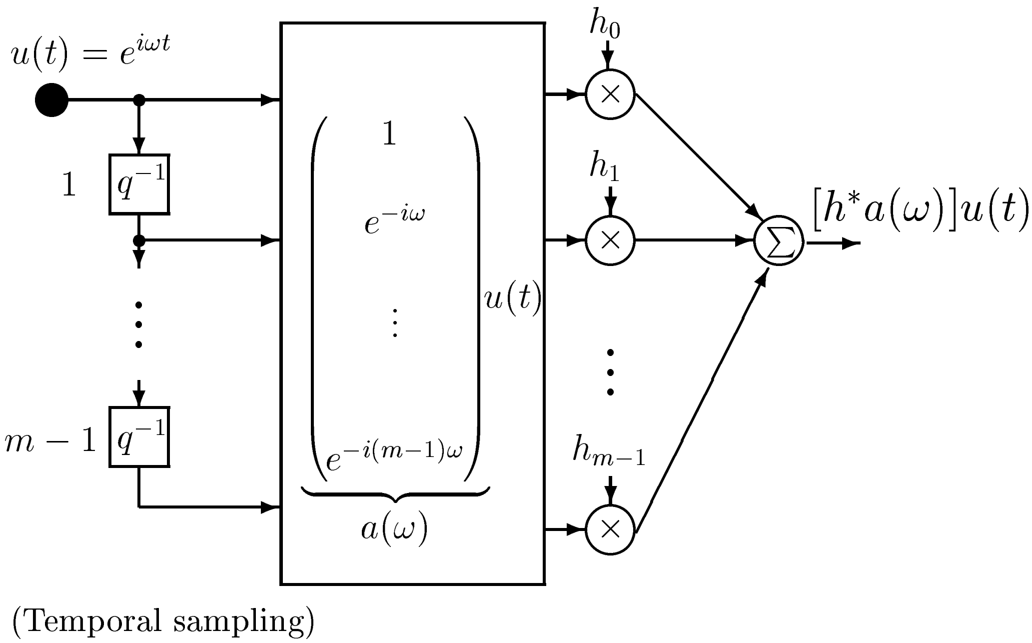
Spatial Filtering

Spatial filtering useful for

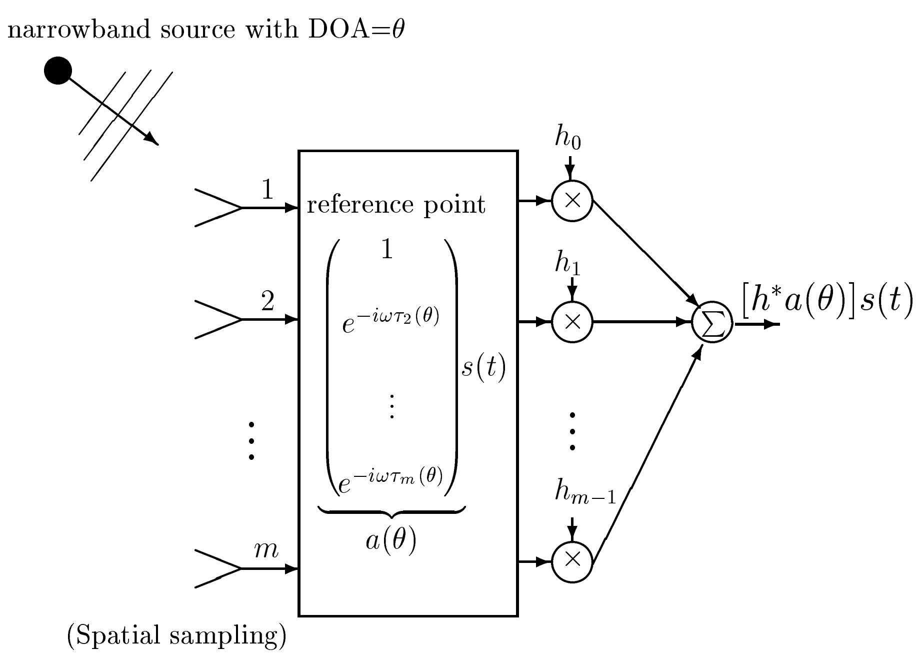
- DOA discrimination (similar to frequency discrimination of time-series filtering)
- Nonparametric DOA estimation

There is a strong analogy between temporal filtering and spatial filtering.

Analogy between Temporal and Spatial Filtering



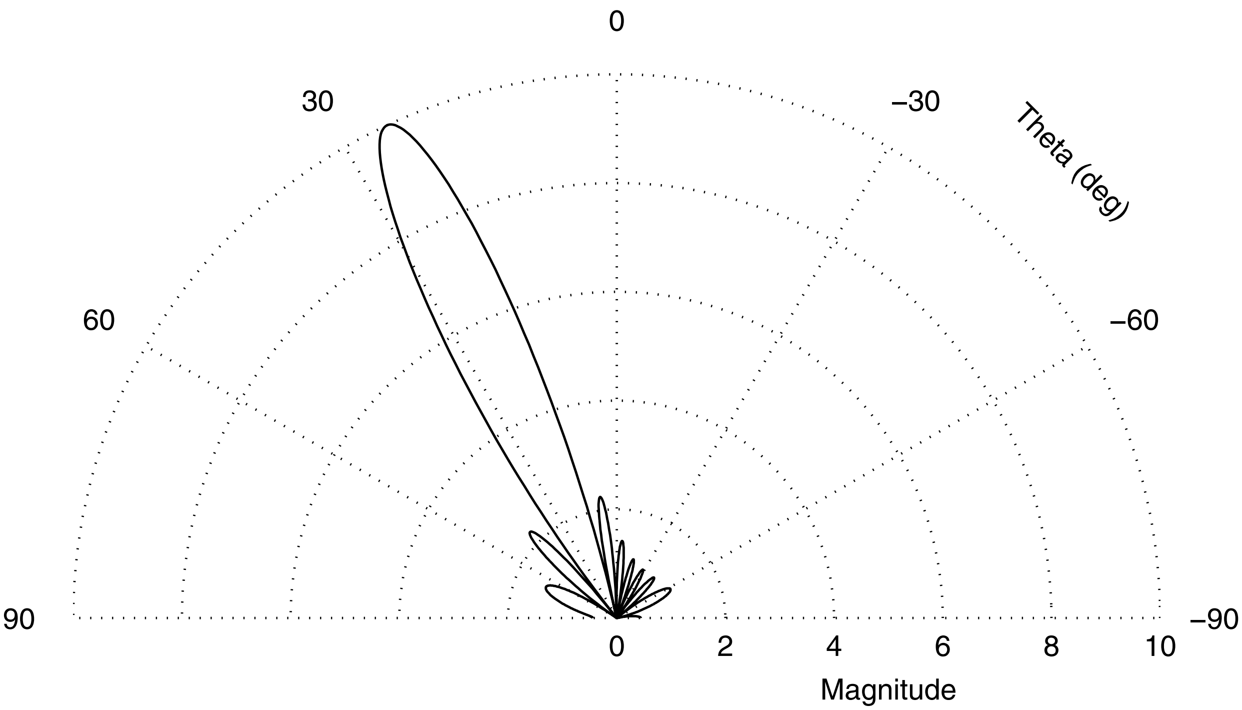
(a) Temporal filter



(b) Spatial filter

Spatial Filtering, con't

Example: The response magnitude $|h^* a(\theta)|$ of a spatial filter (or beamformer) for a 10-element ULA. Here, $h = a(\theta_0)$, where $\theta_0 = 25^\circ$



Spatial Filtering Uses

Spatial Filters can be used

- To pass the signal of interest only, hence filtering out interferences located outside the filter's beam (but possibly having the same temporal characteristics as the signal).
- To locate an emitter in the field of view, by sweeping the filter through the DOA range of interest (“goniometer”).

Nonparametric Spatial Methods

A *Filter Bank Approach* to DOA estimation.

Basic Ideas

- Design a filter $h(\theta)$ such that for each θ
  - It passes undistorted the signal with DOA = θ
  - It attenuates all DOAs $\neq \theta$

- Sweep the filter through the DOA range of interest, and evaluate the powers of the filtered signals:

$$\begin{aligned} E\left\{|y_F(t)|^2\right\} &= E\left\{|h^*(\theta)y(t)|^2\right\} \\ &= h^*(\theta)Rh(\theta) \end{aligned}$$

  with $R = E\{y(t)y^*(t)\}$.

- The (dominant) peaks of $h^*(\theta)Rh(\theta)$ give the DOAs of the sources.

Beamforming Method

Assume the array output is *spatially white*:

$$R = E\{y(t)y^*(t)\} = I$$

Then: $\quad E\left\{|y_F(t)|^2\right\} = h^*h$

Hence: In direct analogy with the temporally white assumption for filter bank methods, $y(t)$ can be considered as impinging on the array from *all* DOAs.

Filter Design:

$$\boxed{\min_h \; (h^*h) \;\text{ subject to }\; h^*a(\theta) = 1}$$

Solution:

$$h = a(\theta)/a^*(\theta)a(\theta) = a(\theta)/m$$

$$E\left\{|y_F(t)|^2\right\} = a^*(\theta)Ra(\theta)/m^2$$

Implementation of Beamforming

$$\hat{R} = \frac{1}{N} \sum_{t=1}^{N} y(t)y^*(t)$$

The beamforming DOA estimates are:

$$\boxed{\begin{array}{rl} \{\hat{\theta}_k\} = & \text{the locations of the } n \text{ largest peaks of} \\ & a^*(\theta)\hat{R}a(\theta). \end{array}}$$

This is the direct spatial analog of the Blackman-Tukey periodogram.

Resolution Threshold:

$$\begin{aligned} \inf \; |\theta_k - \theta_p| \;\; &> \;\; \frac{\text{wavelength}}{\text{array length}} \\ &= \;\; \text{array beamwidth} \end{aligned}$$

Inconsistency problem:
Beamforming DOA estimates are consistent if $n = 1$, but inconsistent if $n > 1$.

Capon Method

Filter design:

$$\boxed{\min_h (h^* R h) \text{ subject to } h^* a(\theta) = 1}$$

Solution:

$$\begin{aligned} h &= R^{-1} a(\theta) / a^*(\theta) R^{-1} a(\theta) \\ E\left\{ |y_F(t)|^2 \right\} &= 1 / a^*(\theta) R^{-1} a(\theta) \end{aligned}$$

Implementation:

$$\boxed{\begin{aligned} \{\hat{\theta}_k\} = \ & \text{the locations of the } n \text{ largest peaks of} \\ & 1/a^*(\theta) \hat{R}^{-1} a(\theta). \end{aligned}}$$

Performance: Slightly superior to Beamforming.

Both Beamforming and Capon are *nonparametric* approaches. They do not make assumptions on the covariance properties of the data (and hence do not depend on them).

Assumptions:

- The array is described by the equation:

$$y(t) = As(t) + e(t)$$

- The noise is spatially white and has the same power in all sensors:

$$E\{e(t)e^*(t)\} = \sigma^2 I$$

- The signal covariance matrix

$$P = E\{s(t)s^*(t)\}$$

is nonsingular.

Then:

$$R = E\{y(t)y^*(t)\} = APA^* + \sigma^2 I$$

Thus: The NLS, YW, MUSIC, MIN-NORM and ESPRIT methods of frequency estimation can be used, almost without modification, for DOA estimation.