Eigenvalues and Eigenvectors

Consider equations

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \text{or} \quad (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$$

for $\mathbf{x}$ (a vector) and λ (a scalar), and solve for $\mathbf{x}$.

If the matrix $\mathbf{A}-\lambda\mathbf{I}$ is nonsingular, the unique solution to these equations is $\mathbf{x} = 0$. You only get a nontrivial solution when $|\mathbf{A}-\lambda\mathbf{I}| = 0$. If you expand this determinant, the condition becomes a polynomial equation in λ of degree p. This is called the *characteristic equation* of $\mathbf{A}$. Its p roots (which might be real or complex, simple or multiple) are called *eigenvalues* (or proper values, characteristic values, or latent roots) of $\mathbf{A}$. If λ is an eigenvalue, a nonzero vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$ is called an *eigenvector* (or proper vector, characteristic vector, or latent vector) corresponding to λ. It is often convenient to normalize each eigenvector to have a squared length of 1.

Example:

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ has a characteristic equation:

$$\left|\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0; \text{ that is, } \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0.$$

$(1-\lambda)^2 - 36 = 0$, or $\lambda = -5$ or 7

It can be seen that

$$\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = -5\begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 7\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

therefore, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue -5

and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 7.