**CS2402 Assignment 1**

# Lecturer: Keda Ma and Shuaicheng Li Due on Mar 9, 2025, 23:59

# You should submit your solution to canvas before the deadline. Late submission halves the score.

***For all the problems in this* assignment*, you SHOULD present your solution step by step instead of giving the answers only.***

Question 1:

Bill is devising a security code that consists of 5 characters. Each character can be either a letter (considering uppercase and lowercase as distinct) or a digit.

(a) If characters can be repeated, calculate the probability of security codes that contain exactly two digits.

(b) Suppose each character is a letter, a digit, or a special character (assume there are 10 possible special characters to choose from). If no repeated characters are allowed, find the probability that the password contains at least one special character.

Solution:

(a) There are 26 uppercase letters, 26 lowercase letters, and 10 digits, totalling 62 characters.

Total number of security codes: Since characters can be repeated, the total number of possible security codes is .

Number of codes with exactly two digits: We choose 2 positions out of 5 for the digits:.

There are 10 choices for each digit, and 52 choices (letters only) for the remaining 3 positions.

So, the number of such codes is .

Since probability = ,

Probability = .

(b) Total number of possible security codes: 52 (26 uppercase letters + 26 lowercase letters) + 10 (digits) + 10 (special characters) = 72 characters.

Number of security codes that do not contain any special characters: .

Probability of not containing any special characters = .

Probability of containing at least one special character =.

**Question 2:**

In a game, you draw four cards at random from a standard deck of 52 cards. Write down numerical expressions for the following probabilities:

(a) The probability that the first card drawn is a King.

(b) The probability that the cards drawn are all from different suits (*i.e.*, ♠, ♣, ♥, ♦).

(c) The probability of drawing exactly three face cards (Jacks, Queens, or Kings).

Solution:

(a) The probability that the first card is a king is given by the ratio of kings to total cards:

(b) The probability that all four cards are from different suits can be calculated by sequentially ensuring each new card is from a different suit than the previous ones:

(c) The probability of drawing exactly three face cards (J, Q, K) involves choosing 3 face cards from 12 and 1 non-face card from 40:

**Question 3:**

We have two same boxes. One box has 10 balls numbered 1 to 10 and another one has 100 balls numbered 1 to 100. You first pick a box at random, then pick a ball at random, and the ball's number is 5 (exactly 5, not a number including 5).

(a) What is the probability that it came from the second box?

(b) If we mix all 110 balls and randomly choose a ball. The ball's number is 5. What is the probability in (a)?

Solution:

We use *B* to represent balls and *U* to represent boxes.

(a) = =

(b) = =

**Question 4:**

A new virus test is 90% accurate when the virus is present. It has a 15% false positive rate when the virus is not present. If 2% of the population is infected with this virus, what is the probability that someone has the virus given that their test result is positive?

Solution:

The probability of a positive test result can be calculated as follows:

p(positive) = p(positive, virus) + p(positive, no virus )

= p(positive | virus) p(virus) + p(positive | no virus ) p(no virus)

= 0.9 × 0.02 + 0.15 × 0.98

= 0.018 + 0.147

= 0.165

p(virus | positive ) = p(virus , positive) / p(positive )

= 0.018 / 0.165

≈ 0.1091

**Question 5:**

Suppose your utility function is , where 𝑤 is the wealth, and you are offered a bet on flipping a coin:

Heads, you win $50. Tails, you lose $100.

Please compute your expected change in utility when your wealthy is

(a) w=100;

(b) w =200;

(c) w =1000;

Hint: change in utility = U(𝑤 + 𝑔𝑎𝑖𝑛) − U(𝑤). The gain can be positive or negative. For

each outcome, multiply the probability with the change in utility. Then, take the sum of all these values to get the expected change in utility.

Solution:

(a)

|  |  |  |
| --- | --- | --- |
| Outcome | H | T |
| Prob. | 0.5 | 0.5 |
| Gain | +50 | -100 |
| Utility change | =2.0840 | =-6.4823 |
| Prob. X utility change | 1.042 | -3.24115 |
| Expected utility change | -2.1992 | |

(b)

|  |  |  |
| --- | --- | --- |
| Outcome | H | T |
| Prob. | 0.5 | 0.5 |
| Gain | +50 | -100 |
| Utility change | =1.5993 | =-3.8779 |
| Prob. X utility change | 0.79965 | -1.93895 |
| Expected utility change | -1.1393 | |

(c)

|  |  |  |
| --- | --- | --- |
| Outcome | H | T |
| Prob. | 0.5 | 0.5 |
| Gain | +50 | -100 |
| Utility change | =0.7734 | =-1.6059 |
| Prob. X utility change | 0.3867 | -0.8030 |
| Expected utility change | -0.4163 | |

**Question 6:**

Two independent fair six-sided dice are rolled. Let X be the larger number rolled, and Y be the smaller number. X=Y if the two numbers rolled are equal.  
(a) Determine the probability that the sum of the numbers is prime.  
(b) Find the distribution of Z = X - Y.  
(c) Compute E(X), Var(X), E(Z), and Var(Z).

Solution:

(a) Probability that the sum is prime

Possible sums of the two dice: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

Prime numbers in this range are: 2, 3, 5, 7, 11.

Outcomes for each prime sum:

- Sum = 2: (1,1),

- Sum = 3: (1,2), (2,1)

- Sum = 5: (1,4), (2,3), (3,2), (4,1)

- Sum = 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1)

- Sum = 11: (5,6), (6,5)

Total prime outcomes = 15.

Total possible outcomes = 6 × 6 = 36.

Probability = 15/36 .

(b) Distribution of Z = X - Y:

Z can take values from 0 to 5:

- P(Z = 0): Both dice show the same number, 6 outcomes. P(Z = 0) = 6/36 = 1/6.

- P(Z = 1): (2,1),(1,2), (3,2),(2,3). (4,3),(3,4), (5,4),(4,5), (6,5),(5,6) 10 outcomes. P(Z = 1) = 10/36.

Similarly

- P(Z = 2): 8 outcomes. P(Z = 2) = 8/36

- P(Z = 3): 6 outcomes. P(Z = 3) = 6/36.

- P(Z = 4): 4 outcomes. P(Z = 4) = 4/36.

- P(Z = 5): 2 outcomes. P(Z = 5) = 2/36.

(c) E(X), Var(X), E(Z), and Var(Z)

**E(X)** = Σ [x \* P(X = x)] = 1/36 \* (1 + 3\*2 + 5\*3 + 7\*4 + 9\*5 + 11\*6) = 161/36 ≈ 4.472

**Var(X):**

Var(X) = E(X^2) - (E(X))^2

E(X^2) = 1/36 \* (1^2 + 3\*2^2 + 5\*3^2 + 7\*4^2 + 9\*5^2 + 11\*6^2) = 791/36

Var(X) = 791/36 - (161/36)^2 =2555/1296

**E(Z):**

E(Z) = Σ [z \* P(Z = z)] = 0(6/36) + 1(10/36) + 2(8/36) + 3(6/36) + 4(4/36) + 5(2/36) = 35/18

**Var(Z):**

Var(Z) = E(Z^2) - (E(Z))^2 = 665/324

**Question 7:**

In *n+m* independent Bernoulli(*p*) trials, let be the number of successes in the first *n* trials, the number of successes in the last *m* trials.

(a) What is the distribution of ? Why?

*Requirement: Write down your formula. One or two sentences to explain why.*

(b) What is the distribution of ? Why?

*Requirement: Write down your formula. One or two sentences to explain why.*

(c) What is the distribution of ? Why?

*Requirement: Write down your formula. One or two sentences to explain why*

(d) Are and independent? Why?

*Requirement: Give your answer and one or two sentences to explain why. You don’t need to prove it mathematically. Use of the common sense should be ok.*

Solution:

(a) We can ignore the last m trials. follows the definition of Binomial Distribution.

Then P()=

Proof:

The number of all the possible outcomes in n independent Bernoulli trials is

The number of outcomes with r successes is

The probability of each case is

Therefore, P()=

(b) We can ignore the first n trials. follows the definition of Binomial Distribution.

P()=

(c) follows the definition of Binomial Distribution.

P()=

(d)Yes, the function of disjoint blocks of independent variables is independent.

**Question 8:** Let X and Y be independent random variables with non-negative integer values, show that P(X+Y=n)=

If you feel difficult to prove for general k. You can give a proof for a special case, say, n=4. Your mark will be halved.

**Answer 8:**

Since X and Y are random variables with non-negative integer, we have

P(X+Y=n) = P(X=0,Y=n)+P(X=1,Y=n-1)+ P(X=2,Y=n-2)+…+ P(X=n,Y=0)

As X and Y are independent random variables,

P(X=i,Y=j)= P(X=i)P(Y=j), for any

Therefore,

P(X+Y=n)

= P(X=0)P(Y=n)+P(X=1)P(Y=n-1)+ P(X=2)P(Y=n-2)+…+ P(X=n)P(Y=0)

=