Empirical Welfare Maximization with Constraints*

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Abstract

Empirical Welfare Maximization (EWM) is a framework that can be used to select welfare program eligibility policies based on data. This paper extends EWM by allowing for uncertainty in estimating the budget needed to implement the selected policy, in addition to its welfare. Due to the additional estimation error, I show there exist no rules that achieve the highest welfare possible while satisfying a budget constraint uniformly over a wide range of DGPs. This differs from the setting without a budget constraint where uniformity is achievable. I propose an alternative trade-off rule and illustrate it with Medicaid expansion, a setting with imperfect take-up and varying program costs.

Keywords: empirical welfare maximization, heterogeneous treatment effects and costs, cost-benefit analysis, local asymptotics

JEL classification codes: C14, C44, C52

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1 Introduction

When a welfare program induces varying benefits across individuals, and when resources are scarce, policymakers naturally want to prioritize eligibility to individuals who will benefit the most. Based on experimental data, cost-benefit analysis can inform policymakers on which subpopulations to prioritize, but these subpopulations might not align with any available eligibility policy such as an income threshold. Kitagawa and Tetenov (2018) propose a statistical rule, Empirical Welfare Maximization (EWM), that can directly select an eligibility policy from a set of available policies based on the experimental data. For example, if available policies take the form of income thresholds, EWM considers the problem of maximizing the expected benefits in the population

$$\max_{t \leq \bar{t}} \ \mathbb{E}[\text{benefit} \cdot \mathbf{1}\{\text{income} \leq t\}]$$

and approximates the optimal threshold based on benefits estimated from experimental data. Recent work has demonstrated that the EWM approach performs well across a broad range of data distributions. As the sample size grows, the average of benefits obtained under the eligibility policy selected by EWM converges to the highest attainable level, a property I refer to as as *uniform asymptotic welfare efficiency*.

I follow this line of work in focusing on settings where the eligibility policy for welfare programs must be determined ex ante and cannot be easily adjusted during implementation, such as Food Stamp and Medicaid in the United States.[1] In practice policymakers often face budget constraints, but only have imperfect information about whether a given eligibility policy satisfies the budget constraint. First, there may be imperfect take-up: eligible individuals might not participate in the welfare program, resulting in zero cost to the government, e.g. Finkelstein and Notowidigdo (2019). Second, costs incurred by eligible individuals who participate

[1] There are also settings in which eligibility is implemented sequentially until the budget is exhausted, such as anti-poverty programs in Chile as analyzed by Carneiro et al. (2019). My analysis does not extend to those cases, as it does not account for the possibility that the benefits and costs may vary depending on the order in which individuals enroll.

in the welfare program may vary considerably, largely driven by individuals' different needs but also many other factors, e.g. Finkelstein et al. (2017). Both considerations are hard to predict ex ante, implying that the potential cost of providing eligibility to any given individual is unknown at the time of designing the eligibility policy. Unobservability of the potential cost requires estimation based on experimental data, contributing to uncertainty in the budget estimate of a given eligibility policy.

For this empirically relevant setting where the budget needed to implement an eligibility policy involves an unknown cost, this paper introduces a new property of statistical rules, namely *asymptotic feasibility*. Policies are *feasible* if they satisfy the budget constraint in the target population; otherwise, they are referred to as *infeasible*. A statistical rule is *asymptotically feasible* if given a large enough experimental sample, the statistical rule is very likely to select feasible eligibility policies. While budget overruns are generally tolerated in countercyclical programs like unemployment insurance and Medicaid in the United States, in other settings, such as subsidized health and education programs in developing countries, a potential budget overrun due to an underestimated budget can be highly undesirable for policymakers, as securing additional funding may be difficult.[2] In this context, ensuring *asymptotic feasibility* is particularly desirable, especially for policymakers who are highly risk-averse to budget overruns.

This paper answers three questions in the current setting with unknown cost: is there any statistical rule that achieves uniformly good performance for a wide range of data distributions in terms of both welfare efficiency and feasibility, whether the obvious extension of the existing EWM statistical rule remains attractive, and what are some alternative statistical rules.

Firstly as a novel theoretical contribution, I quantify a class of reasonable data distributions that is particularly challenging for statistical rules. Specifically, no statistical rule can be uniformly welfare-efficient and feasible simultaneously over this class of data distributions. A notable example within this class is the expansion

[2]For example, international and national funding for malaria control has fallen short of what is estimated to be needed in recent years. Motivated by the fact that Kenya can afford to distribute bed net subsidies to only 50% of its target population in 2007, Bhattacharya and Dupas (2012) analyzed the constrained optimal allocation to increase take-up.

of welfare programs, such as tax credits to incentivize labor force participation, which can "pay for themselves," as demonstrated in Hendren and Sprung-Keyser (2020), because they have zero net cost to the government. This impossibility result provides theoretical characterization of constrained settings that need to be ruled out for any statistical rule to achieve uniformity.

Second, I show the direct extension of the existing EWM statistical rule is not appealing in the setting with unknown cost. The reason is that this *sample-analog rule* ignores the estimation error in the estimated budget needed to implement a given policy, which has non-negligible consequences even when the sample size is large. For data distributions mentioned earlier where the budget constraint is exactly binding, the welfare loss does not vanish with sample size. The probability of selecting infeasible policies also does not vanish with sample size. Intuitively, this latter issue can be mitigated by using a slightly downward-biased version of the budget constraint, effectively making the budget estimate more conservative.[3] I formalize this idea by setting the degree of conservativeness proportional to the standard error of the budget estimate, effectively selecting only policies with an upper confidence bound below the budget constraint. This modification to the *sample-analog rule* ensures that feasible policies are selected with high probability.

So far, the discussion assumes that even minor budget violations are unacceptable. However, in practice, exceeding the budget constraint may be desirable if the welfare gains outweigh the borrowing costs.[4] To account for this situation, I introduce a new objective function that maximizes welfare gains but imposes some penalty once spending exceeds the budget, thereby giving the new objective function the intuitive interpretation of a *trade-off* function. I propose the *trade-off rule*, which optimizes the sample-analog version of the trade-off function. I show the trade-off rule is uniformly asymptotically welfare efficient.

To illustrate the trade-off rule, I apply it to data from the Oregon Health Insur-

[3]Similar intuition arises in optimal prediction under asymmetric loss (Christoffersen and Diebold, 1997), where positive prediction errors are more costly than negative ones, making a downward-biased predictor optimal.

[4]What happens when a welfare program goes over budget is highly context-specific, and there are various ways to model it. In this paper, I focus specifically on the scenario where the government borrows money to cover the excess cost while other models-such as those involving rationing as in Kitagawa and Tetenov (2018) and poorer services are certainly possible.

ance Experiment (OHIE) to select a more flexible Medicaid expansion policy than the current one. Medicaid is a government-sponsored health insurance program intended for the low-income population in the United States. The current Medicaid expansion policy determines eligibility solely based on household income. I examine whether health can be improved by allowing the income threshold to vary by the number of children in the household, setting the budget constraint equal to the cost of the current policy. Imposing a reasonable penalty for exceeding the budget in this context, that any overrun needs to be repaid in full, the trade-off rule selects an eligibility policy that expands eligibility for many households above the current threshold, especially those with children. This occurs because, based on the OHIE data, the additional health benefit from extending eligibility to these households outweigh the penalty of having to repay the overrun in full.

The rest of the paper proceeds as follows. Section 1.1 discusses related work in more detail. Section 2 presents theoretical results. Sections 3 and 4 discuss properties of two statistical rules, illustrated by an empirical example of designing a more flexible Medicaid expansion policy for the low-income population in Oregon. Section 5 conducts a simulation study and Section 6 concludes. Proofs can be found in the Appendix. Supporting lemmas, additional results and computational details can be found in the Online Appendix.

1.1 Literature review

This paper is related to the traditional literature on cost-benefit analysis, e.g. Dhailiwal et al. (2013), and to the recent literature on EWM, e.g. Kitagawa and Tetenov (2018), Rai (2019), Athey and Wager (2021) and Mbakop and Tabord-Meehan (2021). More broadly, this paper contributes to a growing literature on statistical rules in econometrics, including Manski (2004), Dehejia (2005), Hirano and Porter (2009), Stoye (2009), Chamberlain (2011), Bhattacharya and Dupas (2012), Demirer et al. (2019), Yata (2021) and Kitagawa et al. (2022), among others.

The traditional cost-benefit analysis compares the cost and benefit of a given welfare program. The effect of program eligibility is first estimated based on a randomized control trial (RCT), and then converted to a monetary benefit for calculating the cost-benefit ratio. For example, Gelber et al. (2016) and Heller et al. (2017) compare the efficiency of various crime prevention programs based on their cost-benefit ratios. However, the cost-benefit ratio is only informative for whether this welfare program should be implemented with the fixed eligibility policy as implemented in the RCT.

The literature on statistical rules in econometrics has also developed a definition for optimality of statistical rules. Manski (2004) considers the regret, defined to be loss in expected welfare achieved by the statistical rule relative to the welfare achieved by the theoretically optimal eligibility policy. In the absence of any constraint, under the theoretically optimal eligibility policy, anyone with positive benefit from the welfare program would be assigned with eligibility. The minimax regret rule minimizes the upper bound on the regret that results from not knowing the data distribution. Stoye (2009) shows that with continuous covariates and no functional form restrictions on the set of policies, minimax regret does not converge to zero with the sample size because the theoretically optimal policy can be too difficult to approximate by a statistical rule. Kitagawa and Tetenov (2018) avoid this issue by imposing functional form restrictions. They propose the EWM rule, which starts with functional form restrictions on the class of available policies, and then selects the policy with the highest estimated benefit (empirical welfare) based on an RCT sample. They prove the optimality of EWM in the sense that its regret converges to zero at the minimax rate. Importantly, the regret is defined to be loss in expected welfare relative to the maximum achievable welfare in the constrained class, which avoids the negative results of Stoye (2009). Athey and Wager (2021) propose doubly-robust estimation of the average benefit, which leads to an optimal rule even with quasi-experimental data. Mbakop and Tabord-Meehan (2021) propose a Penalized Welfare Maximization rule which relaxes restrictions of the policy class.

The existing EWM literature has not addressed budget constraints with an unknown cost. Kitagawa and Tetenov (2018) consider a capacity constraint, which they enforce using random rationing. Random rationing is not ideal as it uses the limited resource less efficiently than accounting for the cost of providing the welfare program to an individual. When there is no restriction on the functional form of the eligibility policy, Bhattacharya and Dupas (2012) demonstrate that given a capacity constraint, the optimal eligibility policy is based on a threshold on the benefit of the welfare program to an individual. When the cost of providing the welfare program to an individual is heterogeneous, however, budget constraints can be more complicated than capacity constraints, and require estimation. Carneiro et al. (2020) considers the optimal choice of covariate collection in order to maximize the precision of the estimation for the average treatment effect. While they also consider a constrained decision problem, the budget constraint can be verified directly. They also allow for a more complicate trade-off between additional covariates and additional observations. Sun et al. (2021) propose a framework for estimating the optimal rule under a budget constraint when there is no functional form restriction. The main contribution of this paper is to characterize theoretical properties of statistical rules when allowing both functional form restrictions and budget constraints with an unknown cost.

As explained in a prior version of this paper (Sun, 2021), the current setting of EWM with constraint shares the same mathematical structure as another important setting: fairness constraints across sensitive subgroups. Viviano and Bradic (2024) cast welfare of sensitive subgroups as multiple objective functions for policymakers, and solve the constrained optimal policy via the Pareto frontier. Kock and Preinerstorfer (2024) consider a penalized objective function that penalizes violations to the constraint. The trade-off rule proposed in this paper takes a similar form by penalizing budget overrun, but is tailored to linear objectives and constraints.

2 Theoretical results

2.1 Motivation and setup

I begin by setting up a general constrained optimization problem, which depends on the following attributes of an individual:

$$A = (\tau, C, X) \in \mathcal{A} \subseteq \mathbb{R}^{2+p}.$$

Here τ is benefit from the treatment for the individual, C is the cost to the policymaker of providing the individual with the treatment, and $X \in \mathcal{X} \subset \mathbb{R}^p$ denotes their p-dimensional characteristics. The individual belongs to a population that can be characterized by the joint distribution P on the attributes A. The unknown distribution $P \in \mathcal{P}$ is from a class of distributions $\mathcal{P}$.

A policy $g(X) \in \{0, 1\}$ determines the treatment status for an individual with observed characteristics X, where 1 is treatment and 0 is no treatment. Let $\mathcal{G}$ denote the class of policies policymakers can choose from. The optimization problem is to find a policy with maximal benefit while subject to a constraint on its cost:[5]

$$\max_{g \in \mathcal{G}} \mathbb{E}_P[\tau \cdot g(X)] \text{ subject to } \mathbb{E}_P[C \cdot g(X)] \leq k. \tag{1}$$

If the policymaker does not have a fixed budget but still wants to account for cost, the scalar τ can be the difference in benefit and cost. I impose a harsh constraint at known k to model a fixed budget.

The benefit-cost attributes (τ, C) of any given individual may be unobserved in practice. The focus of this paper is the setting where policymakers can construct their estimates (τ_i^*, C_i^*) in a random sample of sample size n along with the characteristics X_i from an experiment or quasi-experiment that satisfy Assumption 3.1, as discussed later.

Applying the Law of Iterated Expectation, the constrained optimization problem (1) can be written as

$$\max_{g \in \mathcal{G}} \mathbb{E}_P[\mathbb{E}_P[\tau \mid X] \cdot g(X)] \text{ subject to } \mathbb{E}_P[\mathbb{E}_P[C \mid X] \cdot g(X)] \leq k.$$

When the eligibility policy can be based on any characteristics whatsoever, the class of available policies is unrestricted i.e. $\mathcal{G} = 2^{\mathcal{X}}$. In this unrestricted class, when the cost is non-negative, the above expression makes clear that the optimal eligibility policy is based on thresholding by the benefit-cost ratio $\mathbb{E}_P[\tau \mid X]/\mathbb{E}_P[C \mid X]$ where the numerator and the denominator are respectively the average effects

[5]Following Kitagawa and Tetenov (2018), I implicitly assume the maximizer exists in $\mathcal{G}$ with the notation in (1).

conditional on the observed characteristics (CATE) and the conditional average resource required. Online Appendix B provides a formal statement.

Given a random sample, to approximate the optimal eligibility policy g_P^*, one can estimate the benefit-cost ratio based on the estimated CATE and the estimated conditional average resource required. The resulting statistical rule selects eligibility policies that are thresholds based on the estimated benefit-cost ratio. The challenge is that the selected eligibility policy can be hard to implement when the estimated benefit-cost ratio is a complicated function of X. Restrictions on the policy class $\mathcal{G}$ address this issue. A common restriction is to consider thresholds based directly on X, e.g. assigning eligibility when an individual's income is below a certain value.

Restrictions on the policy class $\mathcal{G}$ mean that there might not be closed-form solutions to the population problem (1). In particular, the constrained optimal eligibility policy g_P^* might not be an explicit function of the CATE and the conditional average resource required. Therefore it might be difficult to directly approximate g_P^* based on the estimated CATE and the estimated conditional average resource required. However, this is not an obstacle to deriving guarantees for the statistical rules. As I demonstrate later, the derivation does not require the knowledge of the functional form of the constrained optimal eligibility policy g_P^*.

I next specialize the constrained optimization problem to selecting eligibility policy for welfare programs with the example of Medicaid expansion. In the example of implementing welfare programs, policies take the form of eligibility policies. I restrict attention to non-randomized policies as in the leading example of welfare programs, deterministic policies such as income thresholds are more relevant. Theoretically oriented readers may proceed directly to Section 2.2.

Example 2.1. Welfare program eligibility policy under budget constraint

Suppose the government wants to implement some welfare program. The treatment in this example is eligibility for such welfare program. Due to a limited budget, the government cannot make eligibility universal and can only provide eligibility to a subpopulation. To use the budget efficiently, policymakers consider the constrained optimization problem (1). In this example, the policy $g(X)$ assigns an individual to eligibility based on their observed characteristics X, and is usually referred to as an eligibility policy. I denote τ to be the benefit experienced by an individual after receiving eligibility for the welfare program. Specifically, let (Y_1, Y_0) denote the potential outcomes that would have been observed if an individual were assigned with and without eligibility, respectively. The benefit from eligibility policy is therefore defined as $\tau := Y_1 - Y_0$. Note that maximizing benefit is equivalent to maximizing the outcomes (welfare) under the utilitarian social welfare function: $\mathbb{E}_P[Y_1 \cdot g(X) + Y_0(1 - g(X))]$. I denote C to be the potential cost from providing an individual with eligibility for the welfare program. Both τ and C are unobserved at the time of assignment and will need to be estimated.

Policymakers might be interested in multiple outcomes for an in-kind transfer program. Hendren and Sprung-Keyser (2020) capture benefits by the willingness to pay (WTP). Assuming eligible individuals make optimal choices across multiple outcomes, the envelope theorem allows policymakers to focus on benefit in terms of one particular outcome.

Medicaid Expansion Medicaid is a government-sponsored health insurance program intended for the low-income population in the United States. Up till 2011, many states provided Medicaid eligibility to able-bodied adults with income up to 100% of the federal poverty level. The 2011 Affordable Care Act (ACA) provided resources for states to expand Medicaid eligibility for all adults with income up to 138% of the federal poverty level starting in 2014.

Suppose policymakers want to maximize the health benefit of Medicaid by adopting a more flexible expansion policy. Specifically, they relax the uniform income threshold of 138% and allow the income thresholds to vary with the number of children in the household. Once the income thresholds are set, they must be codified in state legislation and publicly announced.[6] Therefore, in this example, it is reasonable to assume that the eligibility policy must be determined ex ante.

The policy class in this example includes income thresholds that can vary with

[6]The legislated eligibility policy is publicly available on federal websites such as MACPAC.

the number of children in the household:

$$\mathcal{G} = \left\{ g(x) = \begin{cases} \mathbf{1}\{\text{income} \le \beta_0\} & , \text{ numchild} = 0 \\ \vdots & \\ \mathbf{1}\{\text{income} \le \beta_j\} & , \text{ numchild} = j \end{cases} \right\} \tag{2}$$

for characteristics $x = (\text{income}, \text{ numchild})$ and $\beta_j \geq 0$.

Convincing policymakers to adopt a more flexible expansion policy as in (2) may still require specifying a clear budget target to ensure that the new policy does not exceed the expenditure level of the current expansion with the uniform income threshold of 138%. Given that Medicaid is a countercyclical program and has some flexibility to accommodate budget overruns, later in Section 4, I develop a new rule to allow explicit trade-off between welfare gains from additional spending and a penalty for budget overruns.

Correspondingly, the constrained optimization problem (1) sets τ to be the health benefit from Medicaid, C to be the cost to Medicaid, and the appropriate threshold k to be the average cost to Medicaid under the current expansion policy with the uniform income threshold of 138%. The characteristics X include both income and number of children in the household.

2.2 Desirable properties for statistical rules

To simplify the notation, I define the *welfare* function and the *budget* function:

$$W(g; P) = \mathbb{E}_P[\tau \cdot g(X)], \ B(g; P) = \mathbb{E}_P[C \cdot g(X)].$$

and the constrained optimal policy g_P^* is therefore the solution to

$$\max_{g \in \mathcal{G}} W(g; P) \text{ subject to } B(g; P) \leq k.$$

As explained in Example 2.1 from Section 2, under a utilitarian social welfare function, maximizing the benefit with respect to eligibility policy is equivalent to maximizing the welfare, which is why I refer to $W(g; P)$ as the welfare function. The welfare function and the budget function are both deterministic functions from $\mathcal{G} \to \mathbb{R}$. The index by the distribution P highlights that welfare and budget of policy g vary with P, and in particular, whether a policy g satisfies the budget constraint depends on which distribution P is of interest.

When the benefit-cost attributes (τ, C) are unobserved and the distribution P is unknown, both the welfare function and the budget function are unknown functions. Denote by $\widehat{g}$ a statistical rule that selects an eligibility policy after observing some experimental data of sample size n distributed according to P^n. This section provides formal definitions for two desirable properties of $\widehat{g}$.

Definition 2.1. A statistical rule $\widehat{g}$ is *pointwise asymptotically welfare-efficient* under the data distribution P if for any $\epsilon > 0$

$$\lim_{n\to\infty} \Pr_{P^n} \{W(\widehat{g}; P) - W(g^*_P; P) < -\epsilon\} = 0,$$

and *uniformly asymptotically welfare-efficient* over the class of distributions $\mathcal{P}$ if for any $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n} \{W(\widehat{g}; P) - W(g^*_P; P) < -\epsilon\} = 0.$$

A statistical rule is *pointwise asymptotically feasible* under the data distribution P if

$$\lim_{n\to\infty} \Pr_{P^n}\{B(\widehat{g}; P) > k\} = 0,$$

and *uniformly asymptotically feasible* over the class of distributions $\mathcal{P}$ if

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n}\{B(\widehat{g}; P) > k\} = 0.$$

■

The above two properties build on the existing EWM literature. For the first property, the current EWM literature evaluates statistical rules by whether they attain at least $W(g^*_P; P)$ in expectation over repeated sample draws as $n \to \infty$. Instead, I focus on convergence in probability, where the probability of $\widehat{g}$ selecting eligibility policies that achieve strictly lower welfare than g^*_P approaches zero as $n \to$

∞. In the setting of the existing EWM literature, the constrained optimal policy g_P^* is also the unconstrained optimal policy $\mathcal{G}$. Therefore it is impossible for any statistical rule $\widehat{g}$ to select a policy that achieves higher value than g_P^*. In my setting, however, the constrained optimal policy g_P^* is not necessarily the unconstrained optimal policy. Thus, I allow the statistical rule $\widehat{g}$ to select a policy that achieves higher welfare than g_P^* for all data distributions, albeit at the cost of violating the budget constraint.

The second property is new to the EWM literature. It imposes that given a large enough sample size, the statistical rule $\widehat{g}$ is unlikely to select infeasible eligibility policies that violate the budget constraint, so that it is "asymptotically feasible". Asymptotic feasibility of statistical rules is specific to the current setting where the budget constraint involves unknown cost. Exactly satisfying a fixed budget constraint without the smallest violation is the most conservative way to articulate policy makers' preferences.

While both are desirable properties, the next section shows a negative result that it is impossible for a statistical rule to satisfy both properties when the data distribution is unknown and belongs to a sufficiently rich class of distributions $\mathcal{P}$.

2.3 An impossibility result

Uniformity is a desirable property for a statistical rule, ensuring that its performance guarantees hold uniformly over a class of DGPs, thereby providing robustness to uncertainty about the true DGP. However, in this section, I prove an impossibility result that no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible in a sufficiently rich class of distributions $\mathcal{P}$ described in the following Assumptions 2.1-2.3. The intuition is that there exists a sequence P_{h_n} that converges to $P_0 \in \mathcal{P}$, but along the sequence, their corresponding constrained optimum $g_{h_n}^*$ does not converge to the constrained optimum $g_{P_0}^*$ under P_0. Assumptions 2.2 and 2.3 characterize such point of discontinuity P_0.

Assumption 2.1. Contiguity. *There exists a distribution $P_0 \in \mathcal{P}$ under which the set $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$ of eligibility policies satisfying the constraint exactly is* *non-empty. Furthermore, the class of distributions $\mathcal{P}$ includes a sequence of data distributions $\{P_{h_n}\}$ contiguous to P_0, under which for all $g \in \mathcal{G}_0$, there exists some $c > 0$ such that*

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > c.$$

Assumption 2.1 assumes there exists a sequence of data distributions $\{P_{h_n}\}$ that differ only marginally relative to P_0. Moreover, the budget function $B(g; P_{h_n})$, evaluated at polices that meet the budget constraint exactly under P_0, approaches $B(g; P_0)$ from above. In Online Appendix B.2, I give more primitive assumptions under which Assumption 2.1 is guaranteed to hold, requiring the sequence to be differentiable in quadratic mean at P_0, and along the sequence, the budget function $B(g; P_{h_n})$ is twice continuously differentiable at P_0 with positive derivatives. These primitive assumptions are relatively weak, and have also been considered in the literature to construct the local parametrization around P_0, e.g. Hirano and Porter (2009).

Assumption 2.2. Binding constraint. *Under the data distribution P_0, the constraint is satisfied exactly at the constrained optimum i.e. $B(g^*_{P_0}; P_0) = k$.*

Assumption 2.3. Separation. *Under the data distribution P_0, $\exists \epsilon > 0$ such that for any feasible policy g, whenever*

$$\left| B(g; P_0) - B(g^*_{P_0}; P_0) \right| > 0,$$

we have

$$W(g^*_{P_0}; P_0) - W(g; P_0) > \epsilon.$$

Equivalently, $W(g^*_{P_0}; P_0)$ is separated from that of other feasible policies with different $B(g; P_0)$.

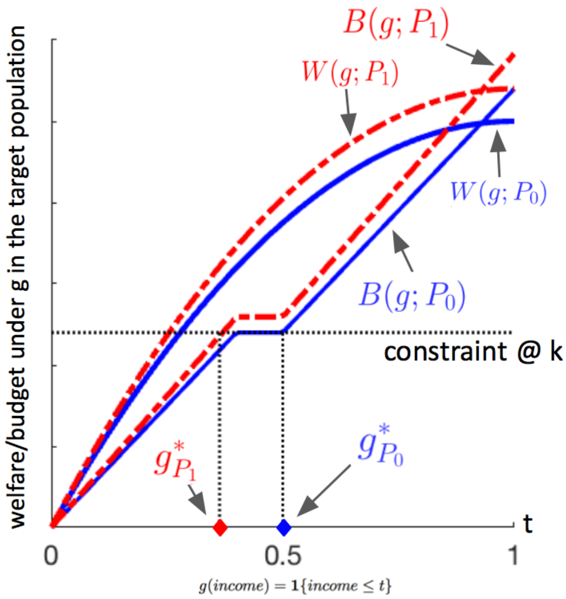
Consider a one-dimensional policy class $\mathcal{G}$, e.g. income thresholds. Figure 2.1 illustrates a distribution P_0 that satisfies both Assumptions 2.2 and 2.3, while both the welfare function $W(g; P_0)$ and the budget function $B(g; P_0)$ are still continuous in g, satisfying Assumption 2.1. Importantly, the constrained optimal policy $g^*_{P_0}$ satisfies the budget constraint exactly, i.e. $B(g^*_{P_0}; P_0) = k$, but is separated from the rest of feasible eligibility policies such that there exists a neighborhood around $g^*_{P_0}$ where feasible policies can achieve welfare gain without any effect on the budget.

Theorem 2.1. *Suppose Assumption 2.1 holds for the class of data distributions $\mathcal{P}$. For $P_0 \in \mathcal{P}$ considered in Assumption 2.1, suppose it also satisfies Assumptions 2.2 and 2.3. Then no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible. In particular, if a statistical rule $\widehat{g}$ is pointwise asymptotically welfare-efficient and pointwise asymptotically feasible under P_0, then it is not uniformly asymptotically feasible.*

Figure 2.1 provides some intuition for Theorem 2.1. Note that if a statistical rule $\widehat{g}$ is pointwise asymptotically welfare-efficient and pointwise asymptotically feasible under P_0, then it has to select eligibility policies close to $g^*_{P_0}$ with high probability over repeated sample draws distributed according to P_0^n as $n \to \infty$.

Under Assumption 2.1, the class of distributions $\mathcal{P}$ is sufficiently rich so that along a sequence of data distributions $\{P_{h_n}\}$ that is contiguous to P_0 as $n \to \infty$, the budget functions $B(g; P_{h_n})$ converge to $B(g; P_0)$ while $B(g^*_{P_0}; P_{h_n}) > k$, i.e. $g^*_{P_0}$ is not feasible under P_{h_n}. Figure 2.1 showcases P_1 as one distribution from this sequence. The contiguity between $\{P_{h_n}\}$ and P_0 implies that the statistical rule $\widehat{g}$ must select policies close to $g^*_{P_0}$ with high probability under $P^n_{h_n}$ as well. However, the policy $g^*_{P_0}$ is infeasible under P_{h_n}, and therefore the statistical rule $\widehat{g}$ cannot be asymptotically feasible under P_{h_n}.

Figure 2.1: Illustration for Assumptions 2.1-2.3 underlying Theorem 2.1



Notes: This figure plots welfare functions $W(g; P)$ and budget functions $B(g; P)$ for populations distributed according to P_0 (blue solid lines) or P_1 (red dashed lines), where P_1 is a distribution from the sequence of distributions $\{P_{h_n}\}$ that is contiguous to P_0 under Assumption 2.1. The distribution P_0 satisfies Assumptions 2.2 and 2.3. The x-axis indexes a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$ for a one-dimensional continuous characteristic X_i with support on $[0, 1]$, e.g. eligibility policies based on income thresholds. The black dotted line marks the budget threshold k. The bold blue dot marks $g^*_{P_0}$, the constrained optimal eligibility policy under P_0. The bold red dot marks $g^*_{P_1}$, the constrained optimal eligibility policy under P_1.

Figure 2.1 also highlights the importance of Assumptions 2.2 and 2.3 in driving the impossibility result. Importantly, under P_0, the budget constraint is binding at the constrained optimal threshold (Assumption 2.2), and increasing the eligibility threshold here has a strictly positive impact on welfare but zero impact on budget (Assumption 2.3). If, instead, increasing the threshold strictly raises both welfare and budget, then Assumption 2.3 is violated. In Section 3.4, I demonstrate that relaxing either Assumption 2.2 or 2.3 gives rise to statistical rules that achieve uniformity within certain subclasses of DGPs. Nonetheless, the impossibility result remains relevant for some real-world policy settings where such assumptions may approximately hold. While the nominal cost of welfare program eligibility may be positive, the Marginal Value of Public Funds (MVPF) framework emphasizes that the relevant cost is the net cost, which incorporates fiscal externalities, such as increased tax revenue if the program increases individuals' incomes. Specifically, Hendren and Sprung-Keyser (2020) estimate fourteen welfare programs (out of 133) to have negative or zero net cost to the government, which implies these programs "pay for themselves", aligning with Assumptions 2.2 and 2.3. As a result, the impossibility result suggests that statistical rules designed for such cases may be highly sensitive to small changes in the underlying distributions even in large samples.[7]

3 Sample-analog rule

The previous negative result implies that no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible. Thus, policymakers might want to consider statistical rules that satisfy one of these two properties. A direct extension to the existing approach in the EWM literature, the sample-analog rule, might be a natural candidate. In this section, I show the direct extension is neither uniformly asymptotically welfare-efficient nor uniformly asymptotically feasible.

I first describe the EWM approach (Kitagawa and Tetenov, 2018) and its direct extension. Since (τ, C) involves potential outcomes, they are often unobserved and require estimation based on RCT that introduces estimation errors in addition to sampling errors. Section 3.2 describes how to construct individuals' benefit and cost estimates (τ_i^*, C_i^*). To highlight the drawback of the direct extension to EWM, I first consider settings where we observe an experimental data of sample size n where (τ, C) is directly observable, i.e. $(\tau_i^*, C_i^*) = (\tau_i, C_i)$. The goal of the simplification is to highlight that the non-uniformity I show below can arise from sampling errors alone.

One can estimate the welfare function and the budget function using their

[7] As estimated by Hendren and Sprung-Keyser (2020), expanding Medicaid, as in Example 2.1, entails a strictly positive net cost. The impossibility result discussed in this section therefore does not apply to this example. Instead, I use this example to illustrate a new trade-off rule proposed later in Section 4.

sample-analog versions:

$$\widehat{W}_n(g) := \frac{1}{n}\sum_i \tau_i^* \cdot g(X_i),\ \widehat{B}_n(g) := \frac{1}{n}\sum_i C_i^* \cdot g(X_i). \tag{3}$$

A direct extension to the existing approach in the EWM literature is a statistical rule that solves the sample version of the population constrained optimization problem (1):

$$\widehat{g}_{\text{sample}} \in \arg\max_{\widehat{B}_n(g)\le k} \widehat{W}_n(g). \tag{4}$$

The subscript "sample" emphasizes how this approach verifies whether a policy satisfies the constraint by comparing the *sample analog* $\widehat{B}_n(g)$ with k directly, i.e. imposes a sample-analog constraint. If no policy satisfies the constraint, then I set $\widehat{g}_{\text{sample}}$ to not assign any eligibility, i.e. $\widehat{g}_{\text{sample}}(x) = 0$ for all $x \in \mathcal{X}$.

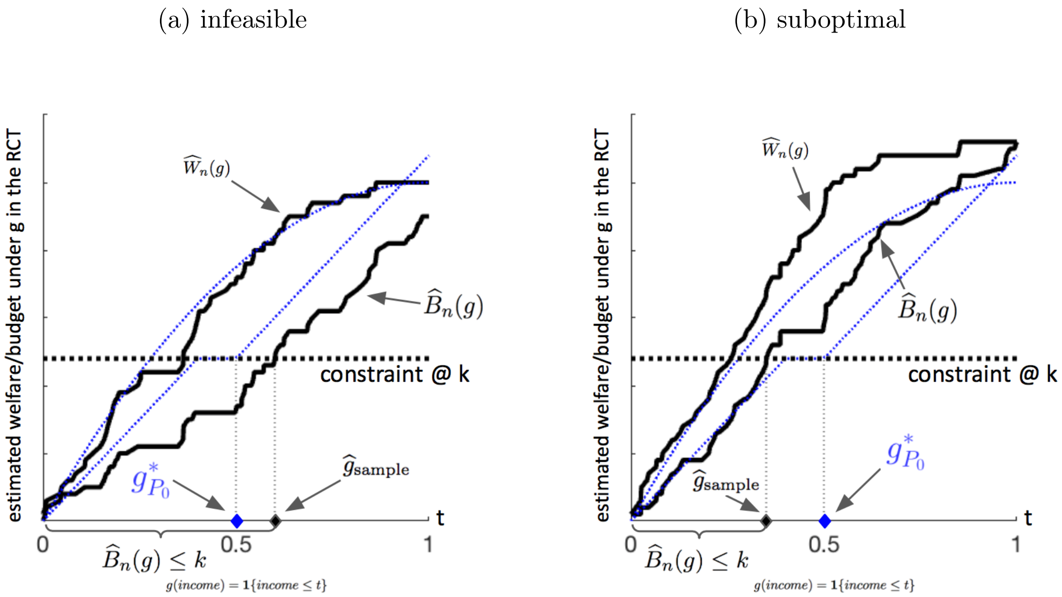
3.1 Welfare inefficiency of the sample-analog rule

A key insight from Kitagawa and Tetenov (2018) is that without a constraint, the sample-analog rule is uniformly asymptotically welfare-efficient. Unfortunately this intuition does not extend to the current setting where the constraint involves an unknown cost. There are common data distributions under which the amount of welfare loss does not vanish even as the sample size gets larger. Consider a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \le t\}\}$, which is based on thresholds of a one-dimensional continuous characteristic X. Suppose the policymakers know benefit is positive for everyone so that welfare function is strictly increasing, and only need to estimate whether a given threshold satisfies a capacity constraint due to imperfect take-up. Furthermore, suppose the experiment sample observes take-up C_i so that the only uncertainty arises from sampling errors. Proposition 3.1 shows settings with some zero take-ups satisfy Assumptions 2.2 and 2.3, and the sample-analog rule is neither pointwise asymptotically welfare efficient nor pointwise asymptotically feasible.

Proposition 3.1. One-dimensional threshold and imperfect take-up. *Consider a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \le t\}\}$, which includes thresholds* *for a one-dimensional continuous characteristic X. Consider the special case where under distribution P, the benefit $\tau > 0$ almost surely and the cost $C \in \{0, 1\}$ is binary. Suppose $C = 0$ for $X \in [\underline{t}, \overline{t}]$, but $\Pr_P\{C = 1 \mid X\} \in (0, 1)$ otherwise. Suppose further the budget constraint is at $k = \mathbb{E}_P[C \cdot \mathbf{1}\{X \leq \underline{t}\}]$. Then P satisfies Assumptions 2.2 and 2.3. As $n \to \infty$, the sample-analog rule $\widehat{g}_{sample}$ violates the budget constraint with probability approaching 50%, and there exists an $\epsilon > 0$ such that it incurs at least an ϵ welfare loss with probability approaching 50%:*

$$\Pr_{P^n}\{W(g_P^*; P) - W(\widehat{g}_{sample}; P) \geq \epsilon\} \to 50\%.$$

Figure 3.1: Illustration for Proposition 3.1



Notes: This figure plots the budget function $B(g; P)$ (blue dotted line) and its sample-analogs $\widehat{B}_n(g)$ (black wriggly line) based on two different observed samples in panel (a) and (b) respectively. The x-axis indexes a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$ for a one-dimensional continuous characteristic X_i with support on $[0, 1]$. The black dotted line marks the budget threshold k. The constrained optimal threshold g_P^* is $t = 0.5$. The sample-analog rule $\widehat{g}_{\text{sample}}$ selects an infeasible threshold in panel (a) and selects a suboptimal threshold in panel (b).

Figure 3.1 illustrates the setup of Proposition 3.1, where the sampling uncertainty can be particularly problematic for the sample-analog rule. Since policymakers know the benefit is positive for everyone, $\widehat{g}_{\text{sample}}$ takes a simple form of the highest threshold where the sample-analog constraint is satisfied exactly. The driving force behind the failure of $\widehat{g}_{\text{sample}}$ as described in Proposition 3.1 is that due to sampling uncertainty, whether a policy satisfies the sample-analog constraint is an imperfect measure of whether it satisfies the constraint in the population.

Since the sample-analog rule $\widehat{g}_{\text{sample}}$ restricts attention to policies that satisfy the sample-analog constraint, there is no guarantee the selected policy is actually feasible. This is very likely to happen when there is welfare gain in exceeding the budget constraint as in the setup of Proposition 3.1 where $W(g; P)$ is strictly increasing in g. Therefore, the sample-analog rule $\widehat{g}_{\text{sample}}$ is not asymptotically feasible under P. As illustrated in Figure 3.1, after observing a sample depicted in panel (a), the sample-analog rule picks an infeasible threshold because the sample-analog constraint is still satisfied there. However, as argued later in Corollary 3.1, the probability of large budget violations vanishes as the sample size increases.

The more problematic case is illustrated in Figure 3.1 panel (b), where the sample-analog rule picks a suboptimal threshold because the sample-analog constraint is violated at the constrained optimum g_P^*. In the setup of Proposition 3.1, when the sample-analog rule $\widehat{g}_{\text{sample}}$ misses g_P^*, it is guaranteed to select a suboptimal policy and therefore it is welfare-inefficient under P even asymptotically. This result relies on the assumption that the welfare function strictly increases when the budget function remains constant in the neighborhood of the budget constraint. This aligns with Assumption 2.3, which, as discussed in Section 2.3, reflect real-world scenarios where some welfare programs have zero or negative net cost to the government. I demonstrate that the same issue extends to broader contexts by employing a simulation calibrated to a real-world DGP from the OHIE in Section 5. If instead the budget function is strictly increasing and violates Assumption 2.3, I show the sample-analog rule is asymptotically welfare-efficient in Proposition C.1 of Appendix C.

3.2 Estimates for benefit and cost

In the remainder of the paper, I present constructive results that modify the sample-analog rule and propose a new rule. To lay the groundwork, this section outlines the benefit and cost estimates. The appropriate expressions for these estimates depend on the type of observed data. Below I state the estimates formed based an RCT that randomly assigns the eligibility, which is the leading case of Kitagawa and Tetenov (2018). The observed data $\{A_i^*\}_{i=1}^n$ consists of i.i.d. observations $A_i^* = (Y_i, Z_i, D_i, X_i) \in \mathcal{A}^*$. The distribution of A_i^* is induced by the distribution of (Y_1, Y_0, C, X) as in the population, as well as the sampling design of the RCT. Here D_i is an indicator for being in the eligibility arm of the RCT, Y_i is the observed outcome and Z_i is the observed cost of providing eligibility to an individual participating in the RCT. The observed cost is mechanically zero if an individual is not randomized into the eligibility arm. The estimates for (τ, C) are

$$\tau_i^* = \alpha(X_i, D_i) \cdot Y_i,\ C_i^* = \frac{D_i}{p(X_i)} \cdot Z_i, \tag{5}$$

where $\alpha(X_i, D_i) = \frac{D_i}{p(X_i)} - \frac{1-D_i}{1-p(X_i)}$ and $p(X_i)$ is the propensity score, the probability of receiving eligibility conditional on the observed characteristics. Since the sampling design of an RCT is known, the propensity score is a known function of the observed characteristics.

Assumption 3.1. Estimation quality. *The recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ defined in (3) converge to mean-zero Gaussian processes G_P^W and G_P^B uniformly over $g \in \mathcal{G}$, with covariance functions $\Sigma_P^W(\cdot,\cdot)$ and $\Sigma_P^B(\cdot,\cdot)$ respectively:*

$$\left\{\sqrt{n} \cdot \left(\frac{1}{n}\sum_i \tau_i^* \cdot g(X_i) - W(g;P)\right)\right\}_{g\in\mathcal{G}} \to_d G_P^W$$
$$\left\{\sqrt{n} \cdot \left(\frac{1}{n}\sum_i C_i^* \cdot g(X_i) - B(g;P)\right)\right\}_{g\in\mathcal{G}} \to_d G_P^B$$

Moreover, the convergence holds uniformly over $P \in \mathcal{P}$. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. There is a uniformly consistent estimator $\widehat{\Sigma}^B(\cdot,\cdot)$ of the covariance function $\Sigma_P^B(\cdot,\cdot)$.

Online Appendix B.3 gives primitive assumptions under which Assumption 3.1 is guaranteed to hold for $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ constructed using an RCT such as in

(5) or an observational study, assuming unconfoundedness and strong overlap. As standard in the literature, I need to restrict the complexity of the policy class $\mathcal{G}$. The policy class of income thresholds considered in this paper is in fact a rather simple class with VC-dimension $d+1$ where d is the number of different thresholds as in (2).

3.3 Modification that ensures feasibility

Let $\hat{\mathcal{G}} = \{g \in \mathcal{G} : \widehat{B}_n(g) \leq k\}$ denote the set of policies that $\widehat{g}_{\text{sample}}$ can choose from, which contains policies that do not violate the sample-analog of the budget constraint. Unsurprisingly, for any finite sample, $\hat{\mathcal{G}}$ can always contain infeasible policies, sometimes of sizable budget violations. The probability that $\hat{\mathcal{G}}$ contains a policy that violates the population budget constraint by a fixed amount c is as follows:

$$\Pr_{P^n}\left\{\exists g : g \in \hat{\mathcal{G}} \text{ and } B(g;P) > k + c\right\} \tag{6}$$

The corollary stated below shows the chance that a large amount of budget violation of $c > 0$ occurs is smaller when there is less variability in $\widehat{B}_n(g)$. The chance also vanishes to zero as the sample size gets larger.

Corollary 3.1. *Under Assumption 3.1, an upper bound of the probability (6) can be approximated by the CDF of* $\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$ *evaluated at* $\frac{-\sqrt{n}\cdot c}{\max_{g\in\mathcal{G}} \Sigma^B(g,g)^{1/2}}$.

A simple modification to the sample-analog rule can reduce the probability of selecting infeasible policies. Instead of $\hat{\mathcal{G}}$, let $\widehat{g}_\alpha$ choose policies from a subset $\hat{\mathcal{G}}_\alpha$ defined in Theorem 3.1 and maximize $\widehat{W}_n(g)$ as before. Then as a direct consequence of Theorem 3.1, with probability at least $1-\alpha$ this modified rule is guaranteed to not mistakenly choose infeasible eligibility policies. In practice, α may be set at the conventional level, e.g. 5%. Then effectively this modification first forms a uniform upper confidence band for the costs of all the policies with asymptotic coverage at least 95% and $\hat{\mathcal{G}}_\alpha$ collects only the policies for which the upper bound on cost is below the threshold.

Theorem 3.1. *Suppose Assumption 3.1 holds for the class of data distributions $\mathcal{P}$.*

Collect eligibility policies in

$$\hat{\mathcal{G}}_\alpha = \left\{ g : g \in \mathcal{G} \text{ and } \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \le c_\alpha \right\}, \tag{7}$$

where c_α *is the* α*-quantile from* $\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$ *for* G_P^B *the Gaussian process defined in Assumption 3.1, and* $\widehat{\Sigma}^B(\cdot,\cdot)$ *the consistent estimator for its covariance function. Then*

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n}\{\hat{\mathcal{G}}_\alpha \cap \mathcal{G}^+ \neq \emptyset\} < \alpha,$$

where $\mathcal{G}^+ = \{g : B(g;P) > k\}$ *is the set of infeasible eligibility policies.*

Note that in (7) the sample-analog constraint is tightened by $c_\alpha \cdot \frac{\widehat{\Sigma}^B(g,g)^{1/2}}{\sqrt{n}}$ where c_α is negative, which means the class $\hat{\mathcal{G}}_\alpha$ only includes eligibility policies where the constraint is slack in the sample. The sample-analog constraint is tightened proportionally to the standard error to reflect that $\widehat{B}_n(g)$ might be particularly noisy for some g. The tightening therefore shrinks inversely proportional to the (square root of) sample size because intuitively larger sample size reduces the sampling uncertainty.

3.4 Modification that ensures uniformity in subclasses of DGPs

Although $\widehat{g}_\alpha$ achieves asymptotic feasibility uniformly over a wide class of DGPs, it may lead to welfare inefficiencies under certain DGPs. Below I show that if one lets $\alpha_n \to 0$ as sample size increases, at a rate such that $c_{\alpha_n} = o(n^{1/2})$, then the modified sample-analog rule $\widehat{g}_{\alpha_n}$ is both asymptotically welfare-efficient and asymptotically feasible uniformly over two subclasses of distributions.

Corollary 3.2. *Consider a family of distributions* $\mathcal{P}$ *such that for all* $P \in \mathcal{P}$*, Assumption 3.1 holds and the budget constraint is uniformly slack under the constrained optimal policy. That is, there exists* $\delta > 0$ *such that*

$$B(g_P^*; P) \le k - \delta \quad \text{for all } P \in \mathcal{P},$$

so that Assumption 2.2 is violated for every $P \in \mathcal{P}$*. Then the modified sample-analog* *rule*

$$\widehat{g}_{\alpha_n} \in \arg\max_{g \in \hat{\mathcal{G}}_{\alpha_n}} \widehat{W}_n(g) \tag{8}$$

is asymptotically welfare-efficient and asymptotically feasible uniformly under $\mathcal{P}$ for $\alpha_n \to 0$ at a rate such that $c_{\alpha_n} = o(n^{1/2})$.

Corollary 3.3. *Let the policy class $\mathcal{G}$ be based on continuously distributed covariates X and parameterized by a finite-dimensional threshold parameter θ. Consider a family of distributions $\mathcal{P}$ such that for all $P \in \mathcal{P}$, Assumption 3.1 holds. Moreover, for every $P \in \mathcal{P}$, both the conditional benefit $\mathbb{E}_P[\tau \mid X]$ and the conditional cost $\mathbb{E}_P[C \mid X]$ are strictly positive. In particular, there exists $\underline{c} > 0$ such that*

$$\mathbb{E}_P[C \mid X] \geq \underline{c} \quad \textit{for all } P \in \mathcal{P},$$

so that Assumption 2.3 is violated for every $P \in \mathcal{P}$. Then the modified sample-analog rule $\widehat{g}_{\alpha_n}$, as defined in (8)*, is asymptotically welfare-efficient and asymptotically feasible uniformly under $\mathcal{P}$.*

Note that the above results do not contradict the impossibility result in Section 2.3, which shows lack of uniformity over a broader class of distributions, including those satisfying Assumptions 2.2 and 2.3. Specifically, Corollary 3.2 establishes uniformity within a subclass of distributions that violate Assumption 2.2, while Corollary 3.3 establishes uniformity within a subclass of distributions that violate Assumption 2.3.

4 Trade-off rule

Up to this point, the discussion has assumed that any budget violation is undesirable. If policymakers are willing to borrow, potentially at some penalty, to exceed the budget constraint in order to achieve higher welfare, then alternative rules might be preferable. Section 4.1 first formalizes this setting as a trade-off problem. Section 4.2 derives a statistical rule that implements such trade-off in the sample, and is shown to be uniformly asymptotically welfare-efficient.

4.1 Form of the trade-off

Exceeding the budget can have negative economic consequences, such as increased borrowing costs or reduced funding for other programs. However, it may not be severe enough to entirely outweigh the perceived welfare gains. Consider a new objective function in which the policymaker, while operating within the budget constraint, seeks to allocate resources efficiently to maximize welfare, but once the budget is exceeded, must trade off the welfare gains from additional spending against the penalty associated with the overrun. Let $\overline{\lambda} > 0$ denote this penalty. Using the notation $(x)_+$ to represent the positive part of $x \in \mathbb{R}$, the objective function can be written as[8]

$$\max_{g \in \mathcal{G}} V(g; P) \text{ where } V(g; P) := W(g; P) - \frac{\overline{\lambda}}{r} \cdot (B(g; P) - k)_+. \tag{9}$$

Here $r > 0$ denotes the rate at which monetary units are converted into welfare units, since welfare is often not measured in monetary terms whereas the budget is.

This objective function (9) is a non-smooth but piecewise linear optimization problem. Denote its solution to be:

$$\widetilde{g}_P \in \arg\max_{g \in \mathcal{G}} \; V(g; P), \tag{10}$$

and note that the solution $\widetilde{g}_P$ can relax the budget constraint and achieves weakly higher welfare than the constrained optimal policy g^*_P for any data distribution P. The next lemma formalizes this observation.

Lemma 4.1. *For any $\overline{\lambda} \in [0, \infty)$, the solution to the trade-off problem (10) achieves weakly higher welfare than the constrained optimum: $W(\widetilde{g}_P; P) \geq V(\widetilde{g}_P; P) \geq W(g^*_P; P)$ for any distribution P. Moreover, the violation to the budget constraint is upper bounded by $\frac{r \cdot (W(\widetilde{g}_P; P) - W(g^*_P; P))}{\overline{\lambda}}$.*

[8] This objective function can also be motivated as policymakers might be willing to trade off violations of the constraint against gains in welfare only to a certain extent, bounding the marginal gain of relaxing the constraint $\lambda \in [0, \overline{\lambda}]$:

$$\max_{g \in \mathcal{G}} \min_{\lambda \in [0, \overline{\lambda}]} \; W(g; P) - \frac{\lambda}{r} \cdot (B(g; P) - k).$$

4.2 Welfare efficiency of the trade-off rule

Given a new objective function (10) that trades off the gain and the cost from violating the constraint, the goal is to derive a statistical rule that is likely to select eligibility policies that maximize the new objective function $V(g;P)$. Consider the trade-off statistical rule defined as

$$\widehat{g}_{\text{tradeoff}} \in \arg\max_{g\in\mathcal{G}} \ \widehat{V}_n(g) \text{ where } \widehat{V}_n(g) := \widehat{W}_n(g) - \frac{\overline{\lambda}}{r}\cdot(\widehat{B}_n(g)-k)_+, \tag{11}$$

where the subscript "tradeoff" highlights that this statistical rule is able to relax the constraint by trading off the gain and the cost from violating the constraint. Since $\widehat{g}_{\text{tradeoff}}$ solves a sample-analog version of (10), I show it consistently achieves the maximal value of $V(g;P)$ under weak conditions in Lemma 4.2. Theorem 4.1 further verifies that it consistently achieves welfare that is weakly higher than $W(g_P^*;P)$. I leave to future research to verify whether the trade-off rule is minimax rate optimal.

Lemma 4.2. *Suppose Assumption 3.1 holds for the class of data distributions $\mathcal{P}$. Then the trade-off rule $\widehat{g}_{tradeoff}$ defined in (11) consistently achieves the maximized value of $V(g;P)$, namely $V(\widetilde{g}_P;P)$.*

Theorem 4.1. *For the class of data distributions $\mathcal{P}$ satisfying Assumption 3.1, the trade-off rule $\widehat{g}_{tradeoff}$ consistently achieves welfare that is weakly higher than $W(g_P^*;P)$ for all $P\in\mathcal{P}$ and is therefore uniformly asymptotically welfare-efficient under $\mathcal{P}$. Moreover, with probability approaching one, the violation to the budget constraint is upper bounded by $\frac{r\cdot(W(\widehat{g}_{tradeoff};P)-W(g_P^*;P))}{\bar{\lambda}}$ uniformly over $\mathcal{P}$.*

To gain intuition for the above results, note that Lemma 4.2 shows the trade-off rule $\widehat{g}_{\text{tradeoff}}$ uniformly consistently achieves $V(\widetilde{g}_P;P)$, which by Lemma 4.1 is weakly higher than the welfare achieved by the constrained optimal policy g_P^* for any data distribution P. Therefore, the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is asymptotically welfare-efficient uniformly over $\mathcal{P}$. At the same time, larger $\bar{\lambda}$ implies smaller violation to the budget constraint, relative to the welfare gain $W(\widehat{g}_{\text{tradeoff}};P)-W(g_P^*;P)$.

4.3 Medicaid expansion: empirical illustration

Example 2.1 of Section 2 explains a more flexible Medicaid expansion policy that would allow the income thresholds to vary with the number of children. In this example, while the budget constraint is set equal to the cost of the current policy, policymakers would be willing to exceed the budget constraint, potentially at some penalty, in order to achieve higher welfare. This subsection uses this example, together with data from the Oregon Medicaid Health Insurance Experiment (OHIE), to illustrate the trade-off rule and compare it with the sample-analog rule and its modification.

4.3.1 Data

I use the experimental data from the OHIE, where Medicaid eligibility (D_i) was randomized in 2007 among Oregon residents who were low-income adults, but previously ineligible for Medicaid, and who expressed interest in participating in the experiment. Finkelstein et al. (2012) include a detailed description of the experiment and an assessment of the average effects of Medicaid on health and health care utilization. I include a cursory explanation here for completeness.

The original OHIE sample consists of 74,922 individuals (representing 66,385 households). Of these, 26,423 individuals responded to the initial mail survey, which collects information on income as percentage of the federal poverty level and number of children, which are the characteristics of interest for targeting (X_i).[9] After one year, the main survey collects data related to health (Y_i), health care utilization (H_i) and actual enrollment in Medicaid (M_i), which allows me to construct estimates for the benefit and cost of Medicaid eligibility (τ, C). Therefore I further exclude individuals who did not respond to the main survey from my sample.

For health (Y_i), I follow the binary measurement in Finkelstein et al. (2012)

[9] More accurately, I follow Sacarny et al. (2020) to approximate number of children by the number of family members under age 19 living in house as reported on the initial mail survey. I exclude individuals who did not respond to the initial survey from my sample, which differs from the sample analyzed in Finkelstein et al. (2012) as I focus on individuals who responded both to the initial and the main surveys from the OHIE. Due to this difference, the expansion policies selected using my sample do not directly carry their properties to the population underlying the original OHIE sample, as the distributions of X differ.

based on self-reported health, where an answer of "poor/fair" is coded as $Y_i = 0$ and "excellent/very good/good" is coded as $Y_i = 1$. For health care utilization (H_i), the study collected measures of utilization of prescription drugs, outpatient visits, ER visits, and inpatient hospital visits. Finkelstein et al. (2012) annualize these utilization measures to turn these into spending estimates, weighting each type by its average cost (expenditures inflated with the CPI-U to 2007 dollars) among low-income publicly insured non-elderly adults in the Medical Expenditure Survey (MEPS). Note that health and health care utilization are not measured at the same scale, which requires rescaling when I consider the trade-off between the two. I address this issue in Section 4.3.2. Lastly, since the enrollment in Medicaid still requires an application, not everyone eligible in the OHIE eventually enrolled in Medicaid, which implies $M_i \leq D_i$.

Given the setup of the OHIE, Medicaid eligibility (D_i) is random conditional on household size (number of adults in the household) entered on the lottery sign-up form and survey wave. While the original experimental setup would ensure randomization given household size, the OHIE had to adjust randomization for later waves of survey respondents (see the Appendix of Finkelstein et al. (2012) for more details). Denote the confounders (household size and survey wave) with V_i, and define the propensity score as $p(V_i) = \Pr\{D_i = 1 \mid V_i\}$. If the propensity score is known, then the construction of the estimates follows directly from the formula (5). However, the adjustment for later survey waves means I need to estimate the propensity score, and I adapt the formula (5) following Athey and Wager (2021) to account for the estimated propensity score.

Specifically, define the conditional expectation function (CEF) of a random variable U_i as $\gamma^U = \mathbb{E}[U_i \mid V_i, D_i]$. Since V_i in my case is discrete, I use a fully saturated model to estimate the propensity score $\widehat{p}(V_i)$ and the CEF $\widehat{\gamma}^U(V_i, D_i)$. I then form the estimated Horvitz-Thompson weight with the estimated propensity score as $\widehat{\alpha}(V_i, D_i) = \frac{D_i}{\widehat{p}(V_i)} - \frac{1-D_i}{1-\widehat{p}(V_i)}$. For health benefit due to Medicaid eligibility, define the estimate $\tau_i^* = \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \big(Y_i - \widehat{\gamma}^Y(V_i, D_i)\big)$. For the cost due to Medicaid eligibility, define the estimate $C_i^* = \widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(W_i)} \cdot \big(Z_i - \widehat{\gamma}^Z(V_i, D_i)\big)$ where $Z_i = M_i \cdot H_i$. Since an eligible individual only incurs cost to Medicaid if enrolled, I need to account for imperfect take-up in forming C_i^*.

Table 4.1: Summary statistics of the OHIE sample by number of children

Number of children	Sample size	Sample mean of τ_i^*	Sample mean of C_i^*
0	5,758	3.16%	\$1,974
		(0.01)	(110)
1	1,736	10.34%	\$1,615
		(0.02)	(195)
≥ 2	2,641	1.55%	\$1,451
		(0.02)	(129)
Full sample	10,135	3.97%	\$1,776
		(0.01)	(79)

Notes: This table presents summary statistics on the sample of individuals who responded to both the initial and the main surveys from the Oregon Health Insurance Experiment (the OHIE sample). The first three rows represent individuals living with different number of children (family members under age 19), and the last row is the aggregate. The estimate for benefit τ_i^* is an estimate for the increase in the probability of an individual reporting "excellent/very good/good" on self-reported health (as opposed to "poor/fair") after receiving Medicaid eligibility. The estimate for cost C_i^* is an estimate for individual's health care expenditure that needs to be reimbursed by Medicaid. Standard errors are shown in parentheses below.

Table 4.1 presents the summary statistics. While Online Appendix B.3 argues the estimation errors in τ_i^* and C_i^* are asymptotically negligible, in finite samples, the cost estimates are highly variable, resulting in noisy estimate $\widehat{B}_n(g)$.

To formalize the budget constraint requiring that the average cost of any proposed policy does not exceed that of the status quo 2014 Medicaid expansion, I calibrate the per capita cost under the 2014 policy. Following Finkelstein et al. (2012), who cite Wallace et al. (2008), Medicaid spending among individuals comparable to the Oregon Health Insurance Experiment (OHIE) participants was approximately \$3,000 per enrollee in Oregon in 2004, which corresponds to about \$3,300 in 2007 dollars. Under the 2014 policy, I adjust for imperfect take-up by defining the per capita cost as

$$k = \mathbb{E}_P[M(1)g_{2014}(X)] \cdot \$3,300 = \$1,377 \tag{12}$$

where $M(1)$ denotes enrollment status if offered Medicaid eligibility and $g_{2014}(x) = \mathbf{1}\{\text{income} \leq 138\%\}$ represents the status quo 2014 expansion policy of providing eligibility to all adults with income up to 138%. The enrollment rate under the status quo 2014 policy $\mathbb{E}_P[M(1)g_{2014}(X)]$ is based on point estimate from the OHIE.[10]

4.3.2 Budget-constrained Medicaid expansion

Figure 4.1 summarizes the selected expansion policies, which are income thresholds specific to the number of children. The sample-analog rule $\widehat{g}_{\text{sample}}$ chooses to restrict Medicaid eligibility, especially lowering the income threshold for childless individuals far below the current level, and the estimated welfare is 3.79% increase in reporting good subjective health. The budget estimate for the selected policy is \$1,311, slightly below the threshold of $k = \$1,377$, as $\widehat{g}_{\text{sample}}$ imposes the sample-analog version of the budget constraint. However, due to the large variation in the cost estimates as illustrated in Table 4.1, meeting the sample budget constraint still involves uncertainty about whether the selected policy meets the budget constraint in the population as argued in Proposition 3.1. After taking into account of estimation uncertainty, the modified sample-analog rule $\widehat{g}_{\alpha=5\%}$ is much more conservative than the sample-analog rule $\widehat{g}_{\text{sample}}$. The budget estimate of the selected policy \$1,174, with a standard error of 66, making it statistically significantly below the budget constraint at the conventional 5% level. The welfare estimate of the selected policy is lower at 3.43%. One can reduce the conservativeness by increasing α and therefore lowering the statistical guarantee that the selected policy meets the budget constraint in the population. I examine the results under higher values of α in Online Appendix D.1.

To construct the trade-off rule $\widehat{g}_{\text{tradeoff}}$ as proposed in Section 4.1, I need to specify both the penalty parameter, $\overline{\lambda}$ and the conversion rate, r. For illustration, I assume that exceeding the budget incurs a full repayment of the overrun, implying $\overline{\lambda} = 1$. In my empirical illustration, the budget constraint is in terms of monetary value. The objective function, however, is measured based on self-reported health, which does not directly translate to a monetary value. Finkelstein et al. (2019) converts self-reported health into value of a statistical life year (VSLY) based on

[10] The theoretical results in this paper are developed for a fixed k. Addressing the estimation error in k is left to future work.

existing estimates. Specifically, a conservative measure for the increase in quality-adjusted life year (QALY) when self-reported health increases from "poor/fair" to "excellent/very good/good" is roughly 0.6. The "consensus" estimate for the VSLY for one unit of QALY from Cutler (2004) is $100,000 for the general US population. I therefore follow Finkelstein et al. (2019) and set $r = (0.6 \cdot \$100,000) = \$60,000$. The trade-off objective function is therefore

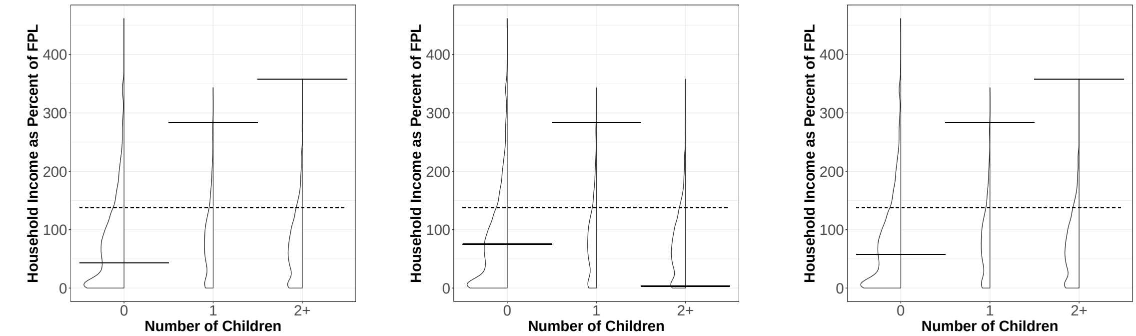
$$\max_{g \in \mathcal{G}} V(g; P) \text{ where } V(g; P) := W(g; P) - \frac{1}{\$60,000} \cdot (B(g; P) - k)_+. \tag{13}$$

The trade-off rule $\widehat{g}_{\text{tradeoff}}$ based on (13) chooses to assign Medicaid eligibility to more individuals, and to raise the income thresholds above the current level for those with children. Therefore the welfare estimate for the selected policy is higher at 3.86% and budget estimate for the selected policy is $1,402, slightly above the budget constraint. The higher level occurs because on average the benefit estimates are positive, and the trade-off rule finds that the additional health benefit from violating the budget constraint exceeds the cost of doing so, especially those with children. While a traditional cost-benefit analysis would also recommend prioritizing those with children based on Table 4.1, the trade-off rule offers a more interpretable recommendation by directly selecting income thresholds.

However, setting $\overline{\lambda} = 1$ assumes that the penalty is limited to repaying the budget overrun, potentially understating the true penalty for policymakers, as it does not account for additional economic costs associated with exceeding the budget, such as reduced funding for other programs or lower quality of Medicaid services. Accordingly, in Online Appendix D.1, I explore how the results change when $\overline{\lambda}$ is increased, which is equivalent to making budget overruns more costly to the policymaker. For example, with $\overline{\lambda} = 1.68$, the welfare and budget estimates for the selected policy essentially drop to those of $\widehat{g}_{\text{sample}}$.

Figure 4.1: More flexible Medicaid expansion policies

(a) policy selected by the sample-analog rule $\widehat{g}_{\text{sample}}$ (b) policy selected by the modified rule $\widehat{g}_{\alpha=5\%}$ (c) policy selected by the trade-off rule $\widehat{g}_{\text{tradeoff}}$



Notes: This figure plots the more flexible Medicaid expansion policies selected by statistical rules based on results from the OHIE. The horizontal dashed line marks the income thresholds under the current expansion policy, which is 138% regardless of the number of children in a household. The horizontal solid lines mark the more flexible policy selected by various statistical rules, i.e. income thresholds that can vary with number of children. For each number of children, I also plot the underlying income distribution to visualize individuals below the thresholds. Panel (a) plots the policy selected by the sample-analog rule $\widehat{g}_{\text{sample}}$. Panel (b) plots the policy selected by the modified sample-analog rule $\widehat{g}_{\alpha=5\%}$. Panel (c) plots the policy selected by the trade-off rule $\widehat{g}_{\text{tradeoff}}$.

In this example, the policies chosen by different rules differ substantially. A natural question is how policymakers should choose the rules. In this Medicaid example, it is reasonable to assume that policymakers can borrow, potentially at some penalty, to exceed the budget constraint in order to achieve higher welfare. Therefore the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is theoretically attractive. It selects policies that achieve at least the maximum feasible welfare, while accounting for the penalty $\overline{\lambda}$ from violating the budget constraint. Because the amount of budget violations depend on the choice of $\overline{\lambda}$ as shown in Theorem 4.1, one can assess this trade-off for their particular settings by varying $\overline{\lambda}$. However, in other context where policymakers are financially conservative and even minor violation to the budget constraint is unacceptable, then the modified sample-analog rule $\widehat{g}_{\alpha}$, with a pre-specified significance level α, is theoretically attractive. It offers a statistical guarantee that the budget constraint will be met in the population with high confidence. However, the selected eligibility policy may appear very conservative, as illustrated in the Medicaid expansion example above.

5 Monte Carlo Simulations

To ensure the practical relevance of the simulation, I calibrate to the distribution of the data from the OHIE, which is also used for empirical illustration in Section 4.3, based on Example 2.1 of Section 2.

For the purpose of this simulation study, the OHIE represents the population P, and I take the estimates (τ_i^*, C_i^*) constructed in Section 4.3.1 as the true benefit and cost (τ, C). Under this simulation design, I can solve for the constrained optimal policy as

$$g_P^* \in \arg \max_{g \in \widetilde{\mathcal{G}},\ B(g;P) \leq k} W(g;P)$$

where $(W(g;P), B(g;P))$ are the sample analogs in the OHIE sample. The policy class $\widetilde{\mathcal{G}}$ includes income thresholds that can vary with the number of children as described in Equation (19) of Appendix D.2, which is slightly coarsened than the class used in Section 4.3.2. The maximum feasible welfare is given by $W(g_P^*; P) = 3.76\%$, an increase of 3.76% in reporting good subjective health. The cost associated with the constrained optimal policy is $B(g_P^*; P) = \$1,340$, slightly below the constraint $k = \$1,377$ as defined in Equation (12).

5.1 Simulation results

Table 5.1 compares the performance of various statistical rules $\widehat{g}$ through 500 Monte Carlo iterations. At each iteration, I randomly draw observations from the OHIE sample to form a random sample. I simulate with the same sample size as the original sample to hold the amount of sampling uncertainty constant. Given the random sample, I collect eligibility policies chosen by each of the following statistical rules:

- sample-analog rule $\widehat{g}_{\text{sample}}$,
- modified sample-analog rule $\widehat{g}_{\alpha}$ described in Theorem 3.1 with α set to 5%,
- trade-off rule $\widehat{g}_{\text{tradeoff}}$ with $\bar{\lambda} = 1$ and $r = \$60,000$, same as the empirical illustration as explained in Equation (13).

I evaluate the welfare function and the budget function $(W(g; P), B(g; P))$ for a given policy in the original OHIE sample. Averages over 500 iterations provide simulation evidence on the properties of the above statistical rules, as shown in Table 5.1.

Table 5.1: Simulation results

Statistical rule	sample-analog $\widehat{g}_{\text{sample}}$	modified $\widehat{g}_{\alpha=5\%}$	trade-off $\widehat{g}_{\text{tradeoff}}$
Prob. of selecting infeasible policies	10.2%	0.2%	45.2%
Prob. of selecting suboptimal policies	94.0%	99.4%	65.6%
Average welfare loss	0.06	0.17	0.01
Average budget overrun	-$70	-$278	$62

Notes: This table reports properties of statistical rules $\widehat{g}$, as averaged over 500 simulations. Row 1 reports the probability that the rule selects an eligibility policy that violates the budget constraint, i.e. $\Pr_{P^n}\{B(\widehat{g}; P) > 0\}$. Row 2 reports the probability that the rule achieves strictly less welfare than the constrained optimal policy g_P^*, i.e. $\Pr_{P^n}\{W(\widehat{g}; P) < W(g_P^*; P)\}$. Row 3 reports the average welfare loss of the rule relative to the maximum feasible welfare, i.e. $\frac{E_{P^n}\left[W(g_P^*;P)-W(\widehat{g};P)\right]}{W(g_P^*;P)}$. Row 4 reports the average budget overrun of the policies selected by the rule, i.e. $E_{P^n}\left[B(\widehat{g}; P)\right] - k$.

Row 1 of Table 5.1 illustrates that it is possible for all three statistical rule $\widehat{g}$ to select infeasible policies. A lower probability of selecting infeasible policies suggests the rule is closer to achieving asymptotic feasibility. In the distribution calibrated to the OHIE sample, the original sample-analog rule $\widehat{g}_{\text{sample}}$ might not be asymptotically feasible as it can select infeasible eligibility policies in 10.2% of the draws. In contrast, Theorem 3.1 guarantees that a simple modification $\widehat{g}_{\alpha=5\%}$ selects infeasible eligibility policies in less than 5% of the draws, regardless of the distribution. Simulation confirms such guarantee as the mistakes only happen 0.2% of the time.

Row 2 of Table 5.1 illustrates that it is possible for all three statistical rule $\widehat{g}$ to achieve weakly higher welfare than the constrained optimal policy g_P^*. This can happen when $\widehat{g}$ selects an infeasible policy. A lower probability of selecting suboptimal policies suggests the rule is closer to achieving asymptotic welfare-efficiency. Theorem 4.1 implies that the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is uniformly asymptotically welfare efficient while there is no such guarantee for the sample-analog rule $\widehat{g}_{\text{sample}}$.

In the distribution calibrated to the OHIE, the trade-off rule $\widehat{g}_{\text{tradeoff}}$ on average achieves higher welfare than the sample-analog rule $\widehat{g}_{\text{sample}}$. As shown in row 3 of Table 5.1, the welfare loss of $\widehat{g}_{\text{tradeoff}}$ is 1% of the maximum feasible welfare $W(g_P^*; P)$, compared to 6% for $\widehat{g}_{\text{sample}}$. However, its improvement can be at the cost of violating the budget constraint more often than $\widehat{g}_{\text{sample}}$, at a rate of 45.2%. Though as shown in row 4 of Table 5.1, on average the violation is limited, which confirms Theorem 4.1.

6 Conclusion

In this paper, I focus on properties of statistical rules when the cost of implementing any given policy needs to be estimated. The existing EWM rule selects an eligibility policy that maximizes a sample analog of the social welfare function, and only accounts for constraints that can be verified with certainty in the population. However, in some cases, the cost of providing eligibility to any given individual might be unknown ex-ante due to imperfect take-up and heterogeneity. Therefore, in addition to asymptotic welfare-efficiency that has been studied by the EWM literature, I introduce a new desirable property of statistical rules in the setting of unknown cost, namely asymptotic feasibility, which requires the selected policy to satisfy a budget constraint when the sample size is large enough. Unlike the setting of known cost, I prove an impossibility result that no statistical rule can be uniformly asymptotically welfare-efficient and feasible. The direct extension to the existing EWM approach is no longer asymptotically welfare efficient nor asymptotically feasible for certain real-world relevant data distributions. As an alternative, I propose the trade-off rule that guarantees asymptotic welfare efficiency while ensuring any budget violations are bounded above by welfare gains. I illustrate the theoretical results using experimental data from the OHIE. A promising avenue for future research is to verify whether the trade-off rule maintains minimax rate optimality in the constrained setting, just as how the EWM rule is in the unconstrained setting.

References

Athey, Susan and Stefan Wager, "Policy Learning With Observational Data," *Econometrica*, 2021, *89* (1), 133–161.

Bhattacharya, Debopam and Pascaline Dupas, "Inferring welfare maximizing treatment assignment under budget constraints," *Journal of Econometrics*, March 2012, *167* (1), 168–196.

Carneiro, Pedro, Emanuela Galasso, and Rita Ginja, "Tackling social exclusion: evidence from Chile," *The Economic Journal*, 2019, *129* (617), 172–208.

_ , Sokbae Lee, and Daniel Wilhelm, "Optimal data collection for randomized control trials," *The Econometrics Journal*, January 2020, *23* (1), 1–31.

Chamberlain, Gary, "Bayesian Aspects of Treatment Choice," in "The Oxford Handbook of Bayesian Econometrics," New York: Oxford University Press, September 2011, pp. 11–39.

Christoffersen, Peter F and Francis X Diebold, "Optimal prediction under asymmetric loss," *Econometric theory*, 1997, *13* (6), 808–817.

Cutler, David, *Your Money or Your Life: Strong Medicine for America's Health Care System*, New York, New York: Oxford University Press, 2004.

Dehejia, Rajeev H., "Program evaluation as a decision problem," *Journal of Econometrics*, 2005, *125* (1-2), 141–173.

Demirer, Mert, Vasilis Syrgkanis, Greg Lewis, and Victor Chernozhukov, "Semi-Parametric Efficient Policy Learning with Continuous Actions," *arXiv:1905.10116 [cs, econ, math, stat]*, July 2019.

Dhailiwal, Iqbal, Esther Duflo, Rachel Glennerster, and Caitlin Tulloch, "Comparative Cost-Effectiveness Analysis to Inform Policy in Developing Countries: A General Framework with Applications for Education," in "Education Policy in Developing Countries," University of Chicago Press, December 2013.

Finkelstein, Amy and Matthew J. Notowidigdo, "Take-Up and Targeting: Experimental Evidence from SNAP," *The Quarterly Journal of Economics*, August 2019, *134* (3), 1505–1556.

_ , Matthew Gentzkow, Peter Hull, and Heidi Williams, "Adjusting Risk Adjustment — Accounting for Variation in Diagnostic Intensity," *New England Journal of Medicine*, February 2017, *376* (7), 608–610.

_ , Nathaniel Hendren, and Erzo F. P. Luttmer, "The Value of Medicaid: Interpreting Results from the Oregon Health Insurance Experiment," *Journal of Political Economy*, December 2019, *127* (6), 2836–2874.

__ , Sarah Taubman, Bill Wright, Mira Bernstein, Jonathan Gruber, Joseph P. Newhouse, Heidi Allen, and Katherine Baicker, "The Oregon Health Insurance Experiment: Evidence from the First Year," *The Quarterly Journal of Economics*, August 2012, *127* (3), 1057–1106.

Gelber, Alexander, Adam Isen, and Judd B. Kessler, "The Effects of Youth Employment: Evidence from New York City Lotteries," *The Quarterly Journal of Economics*, February 2016, *131* (1), 423–460.

Heller, Sara B., Anuj K. Shah, Jonathan Guryan, Jens Ludwig, Sendhil Mullainathan, and Harold A. Pollack, "Thinking, Fast and Slow? Some Field Experiments to Reduce Crime and Dropout in Chicago," *The Quarterly Journal of Economics*, February 2017, *132* (1), 1–54.

Hendren, Nathaniel and Ben Sprung-Keyser, "A Unified Welfare Analysis of Government Policies," *Quarterly Journal of Economics*, 2020, *135* (3), 1209–1318.

Hirano, Keisuke and Jack R. Porter, "Asymptotics for Statistical Treatment Rules," *Econometrica*, 2009, *77* (5), 1683–1701.

Kitagawa, Toru and Aleksey Tetenov, "Who Should Be Treated? Empirical Welfare Maximization Methods for Treatment Choice," *Econometrica*, 2018, *86* (2), 591–616.

__ , Sokbae Lee, and Chen Qiu, "Treatment Choice with Nonlinear Regret," July 2022. arXiv:2205.08586 [econ, stat].

Kock, Anders Bredahl and David Preinerstorfer, "Regularizing Discrimination in Optimal Policy Learning with Distributional Targets," 2024.

Manski, Charles F., "Statistical Treatment Rules for Heterogeneous Populations," *Econometrica*, 2004, *72* (4), 1221–1246.

Mbakop, Eric and Max Tabord-Meehan, "Model Selection for Treatment Choice: Penalized Welfare Maximization," *Econometrica*, 2021, *89* (2), 825–848. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.3982/ECTA16437.

Rai, Yoshiyasu, "Statistical Inference for Treatment Assignment Policies," 2019.

Sacarny, Adam, Katherine Baicker, and Amy Finkelstein, "Out of the Woodwork: Enrollment Spillovers in the Oregon Health Insurance Experiment," Working Paper 26871, National Bureau of Economic Research March 2020.

Stoye, Jörg, "Minimax regret treatment choice with finite samples," *Journal of Econometrics*, 2009, *151* (1), 70–81.

Sun, Hao, Evan Munro, Georgy Kalashnov, Shuyang Du, and Stefan Wager, "Treatment Allocation under Uncertain Costs," *arXiv:2103.11066 [stat]*, March 2021. arXiv: 2103.11066.

Sun, Liyang, "Empirical Welfare Maximization with Constraints," 2021. arXiv:2103.15298v1 [econ].

van der Vaart, A. W., "Asymptotic Statistics by A. W. van der Vaart," October 1998.

van der Vaart, Aad W. and Jon A. Wellner, *Weak Convergence and Empirical Processes* Springer Series in Statistics, New York: Springer, 1996.

Viviano, Davide and Jelena Bradic, "Fair Policy Targeting," *Journal of the American Statistical Association*, 2024, *119* (545), 730–743.

Wainwright, Martin J., "High-Dimensional Statistics by Martin J. Wainwright," February 2019.

Wallace, Neal T, K John McConnell, Charles A Gallia, and Jeanene A Smith, "How effective are copayments in reducing expenditures for low-income adult Medicaid beneficiaries? Experience from the Oregon health plan," *Health services research*, 2008, *43* (2), 515–530.

Yata, Kohei, "Optimal Decision Rules Under Partial Identification," November 2021. arXiv:2111.04926 [econ, math, stat].

A Proofs of theorems

Proof. **Proof of Theorem 2.1.** The population is a probability space $(\Omega, \mathcal{A}, P)$, which induces the sampling distribution P^n that governs the observed sample. A statistical rule $\widehat{g}$ is a mapping $\widehat{g}(\cdot) : \Omega \to \mathcal{G}$ that selects a policy from the policy class $\mathcal{G}$ based on the observed sample. Note that the selected policy $\widehat{g}(\omega)$ is still deterministic because the policy class $\mathcal{G}$ is restricted to be deterministic policies. When no confusion arises, I drop the reference to event ω for notational simplicity.

Suppose $\widehat{g}$ is asymptotically welfare-efficient and asymptotically feasible under P_0. We want to prove there is non-vanishing chance that $\widehat{g}$ selects policies that are infeasible some other distributions:

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n} \{\omega : B(\widehat{g}(\omega); P) > k\} > 0.$$

Firstly, asymptotical welfare-efficiency of $\widehat{g}$ under P_0 implies for any $\epsilon > 0$ we have

$$\limsup_{n\to\infty} \Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) > \epsilon\right\} = 0.$$

Asymptotical feasibility under P_0 implies $\Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) \leq k\} \to 1$.

Consider the event ω' where $|W(\widehat{g}(\omega'); P) - W(g^*_P; P)| < \epsilon$ and $\widehat{g}(\omega')$ is feasible. Asymptotic welfare-efficiency and asymptotic feasibility imply $\Pr_{P_0^n} \{\omega'\} \to 1$. To see this, note the probability of such event has an asymptotic lower bound of one

$$\begin{aligned}
&\Pr_{P_0^n} \left\{\omega' : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega'); P_0) \leq \epsilon \text{ and } B(\widehat{g}(\omega'); P_0) \leq k\right\} \\
\geq &\Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) \leq \epsilon\right\} + \Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) \leq k\} - 1
\end{aligned}$$

where the first two terms converge to one as $n \to \infty$ respectively under asymptotic welfare-efficiency and asymptotic feasibility.

Now consider P_0 that satisfies Assumptions 2.2 and 2.3. For ϵ in Assumption 2.3, we have $B(\widehat{g}; P_0) = B(g^*_{P_0}; P_0) = k$ under the event ω', since the constraint is exactly satisfied at $g^*_{P_0}$ under Assumption 2.2. By Law of Total Probability, we have

$$\Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) = k\} \geq \Pr_{P_0^n} \{\omega'\} \cdot \Pr_{P_0^n} \{B(\widehat{g}; P_0) = k \mid \omega'\}$$

Then the above argument shows $\Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) = k\} \to 1$ as $n \to \infty$.

Following the notation in Assumption 2.1, denote the set of policies where the constraints bind exactly under the limit distribution P_0 by

$$\mathcal{G}_0 = \{g \in \mathcal{G} : B(g; P_0) = k\}.$$

Under Assumption 2.1, the sequence $P_{h_n}^n$ is contiguous with respect to the sequence P_0^n , which means $P_0^n(A_n) \to 0$ implies $P_{h_n}^n(A_n) \to 0$ for every sequence of measurable sets A_n on $\mathcal{A}^n$. Then $\Pr_{P_0^n}\{A_n : B(\widehat{g}(A_n); P_0) = k\} \to 1$ implies there exists an $N(u)$ such that for all $n \geq N(u)$, we have $\Pr_{P_{h_n}^n}\{A_n : B(\widehat{g}(A_n); P_0) = k\} \geq 1 - u$. That is, with high probability, the statistical rule $\widehat{g}$ selects policies from $\mathcal{G}_0$ based on the observed sample distributed according to $P_{h_n}^n$. Recall Assumption 2.1 implies for all $g \in \mathcal{G}_0$, for any sample size n, we have $B(g; P_{h_n}) - k > c/\sqrt{n}$. Thus this statistical rule cannot uniformly satisfy the constraint since with sample size $n \geq N(u)$, we have

$$\sup_{P \in \mathcal{P}} \Pr_{P^n}\{B(\widehat{g}; P) > k\} \geq \Pr_{P_{h_n}^n}\{B(\widehat{g}; P_{h_n}) > k\} \geq 1 - u$$

□

Proof. **Proof of Proposition 3.1.**

By assumption, the probability an individual takes up the treatment for $X \in [\underline{t}, \overline{t}]$ is zero but between zero and one otherwise. Then the budget function $B(t; P) := \mathbb{E}_P[C \cdot \mathbf{1}\{X \leq t\}]$ is flat in the interval $[\underline{t}, \overline{t}]$ but strictly increasing otherwise. Since $\tau > 0$ almost surely, the welfare function $W(t; P) := \mathbb{E}_P[\tau \cdot \mathbf{1}\{X \leq t\}]$ is strictly increasing in t and the constrained optimal policy satisfies Assumption 2.2 and is binding. Let $\inf_{B(g;P)<k} W(g_P^*; P) - W(g; P) = \mathbb{E}_P[\tau \cdot \mathbf{1}\{\underline{t} \leq X \leq \overline{t}\}] = \epsilon$ be the smallest amount of welfare loss from missing the binding solution g_P^*. Then $\epsilon > 0$ and satisfies Assumption 2.3.

The population problem is

$$\max_t \; W(t; P) \text{ subject to } B(t; P) \leq k,$$

where $B(t;P) = k$ for $t \in [\underline{t}, \overline{t}]$ by assumption. The constrained optimal threshold is therefore the highest threshold where the constraint is satisfied exactly i.e. $t^* = \overline{t}$. This also implies $C \cdot \mathbf{1}\{X \leq t\} \sim Bernoulli(k)$ for $t \in [\underline{t}, \overline{t}]$.

The sample-analog rule solves the following sample problem

$$\max_t \frac{1}{n} \sum_i \tau_i \cdot \mathbf{1}\{X_i \leq t\} \text{ subject to } \widehat{B}_n(t) \leq k$$

for $\widehat{B}_n(t) := \frac{1}{n}\sum_i C_i \cdot \mathbf{1}\{X_i \leq t\}$. Given that $\tau > 0$ almost surely, the sample-analog rule equivalently solves $\max_t \widehat{B}_n(t)$ subject to $\widehat{B}_n(t) \leq k$. However, the solution is not unique because $\widehat{B}_n(t)$ is a step function. To be conservative, let the sample-analog rule be the smallest possible threshold to maximize $\widehat{B}_n(t)$:

$$\widehat{t} = \min\left\{\arg\max_t \{\widehat{B}_n(t) \text{ subject to } \widehat{B}_n(t) \leq k\}\right\}.$$

Note that we can also write $\widehat{B}_n(t) = \frac{1}{n}\sum_{C_i=1} \mathbf{1}\{X_i \leq t\}$, which makes it clear that $\widehat{t}$ corresponds to ranking X_i among individuals with $C_i = 1$, and then picking the lowest threshold such that we assign treatment to the first $\lfloor k \cdot n \rfloor$ individuals. This also means if in the sample few individuals take up the treatment such that $\frac{1}{n}\sum_i C_i \leq k$, we can have a sample-analog rule that treats everyone up to $\max_{C_i=1} X_i$. Taken together both scenarios, we note the sample-analog rule implies the treated share in the sample is equal to

$$\widehat{B}_n(\widehat{t}) := \frac{1}{n}\sum_i C_i \cdot \mathbf{1}\{X_i \leq \widehat{t}\} = \min\left\{\frac{1}{n}\sum_i C_i, \frac{\lfloor k \cdot n \rfloor}{n}\right\}.$$

Note that

$$\begin{cases} \widehat{t} > \overline{t} \Leftrightarrow & B(\widehat{t};P) > B(\overline{t};P) \\ \widehat{t} < \underline{t} \Leftrightarrow & B(\widehat{t};P) < B(\underline{t};P) \Leftrightarrow W(\widehat{t};P) < W(\underline{t};P) \end{cases}$$

which means whenever $\widehat{t} > \overline{t}$, the sample-analog rule violates the constraint in the population as $B(\overline{t};P) = k$; whenever $\widehat{t} < \underline{t}$, the sample-analog rule achieves strictly less welfare than t^* in the population because $W(\underline{t};P)$ is strictly less than $W(t^*;P)$. We next derive the limit probability for these two events. Applying Law of Total

Probability, we have

$$
\begin{aligned}
&\Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \widehat{B}_n(\widehat{t})\right\} \\
=&\Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k\cdot n\rfloor}{n} \text{ and } \frac{\lfloor k\cdot n\rfloor}{n} < \frac{1}{n}\sum_i C_i\right\} \\
&+\Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{1}{n}\sum_i C_i \text{ and } \frac{\lfloor k\cdot n\rfloor}{n} \geq \frac{1}{n}\sum_i C_i\right\} \\
\geq&\Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k\cdot n\rfloor}{n} \text{ and } \frac{\lfloor k\cdot n\rfloor}{n} < \frac{1}{n}\sum_i C_i\right\} \\
\geq&\Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k\cdot n\rfloor}{n}\right\} + \Pr_{P^n}\left\{\frac{\lfloor k\cdot n\rfloor}{n} < \frac{1}{n}\sum_i C_i\right\} - 1
\end{aligned} \tag{14}
$$

For the first term in (14), we have the following lower bound

$$
\begin{aligned}
&\Pr_{P^n}\left\{\frac{1}{n}\sum_i C_i\cdot\mathbf{1}\{X_i\leq\bar{t}\} \leq \frac{k\cdot n-1}{n}\right\} \\
=&\Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i C_i\cdot\mathbf{1}\{X_i\leq\bar{t}\} - k\right) \leq -\frac{1}{\sqrt{n}}\right\} \to 0.5
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $-\frac{1}{\sqrt{n}}$ converges to zero. Denote $p_C = \Pr\{C=1\}$. For the second term in (14), we have the following lower bound

$$
\begin{aligned}
&\Pr_{P^n}\left\{\frac{1}{n}\sum_i C_i \geq \frac{k\cdot n}{n}\right\} \\
=&\Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i C_i - p_C\right) \geq \sqrt{n}\cdot(k-p_C)\right\} \to 1
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $\sqrt{n}\cdot(k-p_C)$ diverges to $-\infty$ for $p_C > k$. We thus conclude

$$
\lim_{n\to\infty}\Pr_{P^n}\left\{B(\widehat{t};P) > B(\bar{t};P)\right\} \geq 0.5
$$

which proves $\widehat{t}$ is not pointwise asymptotically feasible under the distribution P.

Similar argument shows $\Pr_{P^n}\left\{\widehat{B}_n(\underline{t}) > \widehat{B}_n(\widehat{t})\right\}$ has a limit of at least one half.

We thus conclude

$$\lim_{n\to\infty} \Pr_{P^n}\left\{B(\widehat{t};P) < B(\underline{t};P)\right\} \geq 0.5 \Leftrightarrow \lim_{n\to\infty} \Pr_{P^n}\left\{W(\widehat{t};P) < W(\underline{t};P)\right\} \geq 0.5$$

which proves that $\widehat{t}$ is not pointwise asymptotically welfare-efficient under the distribution P. Since $B(\underline{t};P) = B(\bar{t};P)$, we actually have

$$\lim_{n\to\infty} \Pr_{P^n}\left\{B(\widehat{t};P) > B(\bar{t};P)\right\} = \lim_{n\to\infty} \Pr_{P^n}\left\{W(\widehat{t};P) < W(\underline{t};P)\right\} = 0.5$$

□

Proof. **Proof of Corollary 3.1.** The event in (6) is equivalent to the event that $\hat{\mathcal{G}}$ includes at least one criteria that violates the budget constraint by c. The derivation for the bound therefore is based on such an event:

$$\begin{aligned}
&\Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-k\right)}{\Sigma^B(g,g)^{1/2}} \leq 0\right\} \\
=&\Pr_{P^n}\left\{\min_{B(g;P)>k+c}\left\{\frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} + \frac{\sqrt{n}\left(B(g;P)-k\right)}{\Sigma^B(g,g)^{1/2}}\right\} \leq 0\right\} \\
\leq&\Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq -\min_{B(g;P)>k+c}\frac{\sqrt{n}\left(B(g;P)-k\right)}{\Sigma^B(g,g)^{1/2}}\right\} \\
\leq&\Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq -\frac{\min_{B(g;P)>k+c}\sqrt{n}\left(B(g;P)-k\right)}{\max_{B(g;P)>k+c}\Sigma^B(g,g)^{1/2}}\right\} \\
<&\Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq \frac{-\sqrt{n}\cdot c}{\max_{B(g;P)>k+c}\Sigma^B(g,g)^{1/2}}\right\} \\
<&\Pr_{P^n}\left\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq \frac{-\sqrt{n}\cdot c}{\max_{g\in\mathcal{G}}\Sigma^B(g,g)^{1/2}}\right\}
\end{aligned}$$

Under Assumption 3.1, the empirical process $\left\{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)\right\}$ converges to a Gaussian process G_P^B for $G_P^B(\cdot) \sim \mathcal{GP}(0,\Sigma_P^B(\cdot,\cdot))$ and we have a consistent covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$, which proves the corollary.

□

Proof. **Proof of Theorem 3.1.** By construction, the limit probability for any policy in $\hat{\mathcal{G}}_\alpha$ to violate the budget constraint is

$$
\begin{aligned}
&\Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_\alpha \text{ and } B(g;P) > k\} = \Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
&\leq \Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
&\leq \Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\}
\end{aligned}
$$

Under Assumption 3.1, uniformly over $P \in \mathcal{P}$, the empirical process $\left\{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)\right\}$ converges to a Gaussian process G_P^B for $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$ and we have a consistent covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$. Then by the definition of c_α, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
=&\sup_{P\in\mathcal{P}} \Pr_{P^n}\{\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}} \leq c_\alpha\} = \alpha.
\end{aligned}
$$

□

Proof. **Proof of Corollary 3.2** Uniformly asymptotic feasibility follows from Theorem 3.1, replacing α with α_n in its proof. Specifically, by construction we have

$$
\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n}\{B(\hat{g}_{\alpha_n};P) > k\} \leq \limsup_{n\to\infty} \sup_{P\in\mathcal{P}} \Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_{\alpha_n} \text{ and } B(g;P) > k\} \leq \alpha_n.
$$

For the second part, we decompose the welfare loss into

$$
\begin{aligned}
&W(g_P^*;P) - W(\hat{g}_{\alpha_n};P) \\
=&W(g_P^*;P) - \widehat{W}_n(\hat{g}_{\alpha_n}) + \widehat{W}_n(\hat{g}_{\alpha_n}) - W(\hat{g}_{\alpha_n};P) \\
\leq&\sup_{g\in\mathcal{G}} 2|W(g;P) - \widehat{W}_n(g)| + \widehat{W}_n(g_P^*) - \widehat{W}_n(\hat{g}_{\alpha_n})
\end{aligned}
$$

Under the event that $g_P^* \in \hat{\mathcal{G}}_{\alpha_n}$, we are guaranteed that the last term is non-positive.

This happens with probability

$$\Pr_{P^n}\{\frac{\sqrt{n}\left(\widehat{B}_n(g_P^*)-k\right)}{\widehat{\Sigma}^B(g_P^*,g_P^*)^{1/2}}\leq c_{\alpha_n}\}\overset{a}{\sim}\Phi\left(c_{\alpha_n}+\frac{\sqrt{n}\,(k-B(g_P^*;P))}{\Sigma_P^B(g_P^*,g_P^*)^{1/2}}\right)\to 1$$

for $B(g_P^*;P)$ strictly below the threshold. Since c_{α_n} diverges to $-\infty$ at a rate slower $\sqrt{n}$, the term in the parenthesis diverges to ∞ as $n\to\infty$. We then apply Lemma B.4 and B.6 in the Online Appendix (proved in Section B.3 under primitive conditions that lead to Assumption 3.1), which shows $\sup_{g\in\mathcal{G}}|W(g;P)-\widehat{W}_n(g)|$ converges to zero in probability uniformly. We then conclude $\hat{g}_{\alpha_n}$ is asymptotically welfare-efficient uniformly under $\mathcal{P}$. □

Proof. **Proof of Corollary 3.3** Uniformly asymptotic feasibility follows exactly as in the Proof of Corollary 3.2. To prove uniformity in asymptotic welfare-efficiency, I slightly abuse notation by indexing the policy class by θ in the welfare and budget functions below. Since the policy class includes thresholding on continuously distributed X, and by assumption the conditional benefit and cost are positive, the welfare function $W(\theta;P)$ and the budget function $B(\theta;P)$ are both continuous and increases in θ. That is, whenever $\theta\prec\theta'$ where $\prec$ denotes element-wise inequality, we have $W(\theta;P)<W(\theta';P)$ and $B(\theta;P)<B(\theta';P)$. Therefore, at the constrained optimal threshold θ_P^*, the constraint is binding $B(\theta_P^*;P)=k$. By the continuity and strict monotonicity of $W(\theta;P)$, for any $\epsilon>0$, there exists $\theta_{P,\epsilon}\prec\theta_P^*$ such that $W(\theta_{P,\epsilon};P)=W(\theta_P^*;P)-\epsilon$.

Construct a sequence $b_n=-2c_{\alpha_n}/\sqrt{n}$, which decreases to zero since c_{α_n} diverges to $-\infty$ at a rate slower $\sqrt{n}$. Consider $B(\theta_P^*;P)-B(\theta_{P,\epsilon};P)=k-B(\theta_{P,\epsilon};P)$, which is strictly positive for all $P\in\mathcal{P}$ because by assumption the conditional cost is bounded away from zero. Therefore there exists N large enough such that for all $P\in\mathcal{P}$ we have

$$k-B(\theta_{P,\epsilon};P)\geq b_n\cdot\Sigma_P^B(\theta_{P,\epsilon},\theta_{P,\epsilon})^{1/2}>0,\ \forall n\geq N. \tag{15}$$

We decompose the welfare loss into

$$
\begin{aligned}
&W(\theta_P^*;P) - W(\hat{\theta}_{\alpha_n};P) \\
=&W(\theta_P^*;P) - W(\theta_{P,\epsilon};P) + W(\theta_{P,\epsilon};P) - W(\hat{\theta}_{\alpha_n};P) \\
=&\epsilon + W(\theta_{P,\epsilon};P) - \widehat{W}_n(\theta_{P,\epsilon}) + \widehat{W}_n(\theta_{P,\epsilon}) - \widehat{W}_n(\hat{\theta}_{\alpha_n}) + \widehat{W}_n(\hat{\theta}_{\alpha_n}) - W(\hat{\theta}_{\alpha_n};P) \\
\le&\epsilon + \sup_{g\in\mathcal{G}} 2|W(g;P) - \widehat{W}_n(g)| + \widehat{W}_n(\theta_{P,\epsilon}) - \widehat{W}_n(\hat{\theta}_{\alpha_n})
\end{aligned}
$$

Under the event that $\theta_{P,\epsilon} \in \hat{\mathcal{G}}_{\alpha_n}$, we are guaranteed that the last term is non-positive. This happens with probability

$$
\Pr_{P^n}\{\frac{\sqrt{n}\left(\widehat{B}_n(\theta_{P,\epsilon}) - k\right)}{\widehat{\Sigma}^B(\theta_{P,\epsilon},\theta_{P,\epsilon})^{1/2}} \le c_{\alpha_n}\} \overset{a}{\sim} \Phi\left(c_{\alpha_n} + \frac{\sqrt{n}\left(k - B(\theta_{P,\epsilon};P)\right)}{\Sigma_P^B(\theta_{P,\epsilon},\theta_{P,\epsilon})^{1/2}}\right).
$$

From (15), we have

$$
\lim_{n\to\infty}\inf_{P\in\mathcal{P}} \Phi\left(c_{\alpha_n} + \frac{\sqrt{n}\left(k - B(\theta_{P,\epsilon};P)\right)}{\Sigma_P^B(\theta_{P,\epsilon},\theta_{P,\epsilon})^{1/2}}\right) \ge \lim_{n\to\infty}\Phi\left(c_{\alpha_n} - 2c_{\alpha_n}\right) = \lim_{n\to\infty}\Phi\left(-c_{\alpha_n}\right) = 1
$$

as c_{α_n} diverges to $-\infty$ as $n \to \infty$. We then apply Lemma B.4 and B.6 in the Online Appendix (proved in Section B.3 under primitive conditions that lead to Assumption 3.1), which shows $\sup_{g\in\mathcal{G}}|W(g;P) - \widehat{W}_n(g)|$ converges to zero in probability uniformly over $P \in \mathcal{P}$. We then conclude $\hat{g}_{\alpha_n}$ is uniformly asymptotically welfare-efficient under $\mathcal{P}$. □

Proof. **Proof of Lemma 4.1.** By definition

$$
\begin{aligned}
W(g_P^*;P) = \max_{B(g;P)\le k} W(g;P) &= \max_{B(g;P)\le k} W(g;P) - \frac{\overline{\lambda}}{r}\cdot\left(B(g;P) - k\right)_+ \\
&\le \max_{g\in\mathcal{G}} W(g;P) - \frac{\overline{\lambda}}{r}\cdot\left(B(g;P) - k\right)_+ \\
&= W(\widetilde{g}_P;P) - \frac{\overline{\lambda}}{r}\cdot\left(B(\widetilde{g}_P;P) - k\right)_+ \le W(\widetilde{g}_P;P)
\end{aligned}
$$

and suppose $B(\widetilde{g}_P;P) > k$, we have the following upper bound for violations to the budget constraint

$$B(\widetilde{g}_P; P) - k \leq \frac{r \cdot (W(\widetilde{g}_P; P) - W(g_P^*; P))}{\overline{\lambda}}.$$

$\square$

Proof. **Proof of Lemma 4.2**

Recall the new objective function with $V(g;P) = W(g;P) - \frac{\overline{\lambda}}{r} \cdot (B(g;P) - k)_+$, whose maximizer is $\widetilde{g}_P$. Recall $\widehat{V}_n(g) = \widehat{W}_n(g) - \frac{\overline{\lambda}}{r} \cdot (\widehat{B}_n(g) - k)_+$ is the sample-analog, whose maximizer is $\widehat{g}_{\text{tradeoff}}$. Apply uniform deviation bound to the difference between the value under $\widehat{g}_{\text{tradeoff}}$ and $\widetilde{g}_P$, we have

$$\begin{aligned}
&V(\widetilde{g}_P; P) - V(\widehat{g}_{\text{tradeoff}}; P) \\
=&V(\widetilde{g}_P; P) - \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) + \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) - V(\widehat{g}_{\text{tradeoff}}; P) \\
\leq&V(\widetilde{g}_P; P) - \widehat{V}_n(\widetilde{g}_P) + \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) - V(\widehat{g}_{\text{tradeoff}}; P) \\
\leq&2\sup_g \left| V(g;P) - \widehat{V}_n(g) \right| \\
=&2\sup_g \left| W(g;P) - \widehat{W}_n(g) - \frac{\overline{\lambda}}{r} \cdot (\max\{B(g;P) - k, 0\} - \max\{\widehat{B}_n(g) - k, 0\}) \right| \\
\leq&2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\frac{\overline{\lambda}}{r} \cdot \sup_g \left| \max\{\widehat{B}_n(g) - k, 0\} - \max\{B(g;P) - k, 0\} \right| \\
\leq&2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\frac{\overline{\lambda}}{r} \cdot \sup_g \left| \widehat{B}_n(g) - B(g;P) \right|
\end{aligned}$$

The last line uses the fact that $|\max\{a,0\} - \max\{b,0\}| \leq |a-b|$. Both terms, $\sup_g \left| \widehat{W}_n(g) - W(g;P) \right|$ and $\sup_g \left| \widehat{B}_n(g) - B(g;P) \right|$ converge to zero in probability under Assumption 3.1. Specifically, Lemma B.4 and B.6 in the Online Appendix (proved in Section B.3 under primitive conditions that lead to Assumption 3.1) imply the uniform convergence in probability

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \widehat{W}_n(g) - W(g;P) \right| \to_p 0, \ \sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \widehat{B}_n(g) - B(g;P) \right| \to_p 0.$$

At the same time, by definition we have $V(\widetilde{g}_P; P) - V(\widehat{g}_{\text{tradeoff}}; P) \geq 0$. We thus conclude

$$\sup_{P\in\mathcal{P}} |V(\widehat{g}_{\text{tradeoff}};P) - V(\widetilde{g}_P;P)| \to_p 0.$$

□

Proof. **Proof of Theorem 4.1.**

By Lemma 4.1, we have $W(g_P^*;P) \leq V(\widetilde{g}_P;P) \leq W(\widetilde{g}_P;P)$ for any $P \in \mathcal{P}$. Putting these together with Lemma 4.2, we have

$$\begin{aligned}
&\inf_{P\in\mathcal{P}} \{W(\widehat{g}_{\text{tradeoff}};P) - W(g_P^*;P)\} \\
&\geq \inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}};P) - W(g_P^*;P)\} \\
&\geq \inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}};P) - V(\widetilde{g}_P;P)\} + \inf_{P\in\mathcal{P}} \{V(\widetilde{g}_P;P) - W(g_P^*;P)\} \\
&\geq \inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}};P) - V(\widetilde{g}_P;P)\} \to_p 0.
\end{aligned}$$

which proves uniform asymptotic welfare-efficiency.

By the uniform deviation bound used in the proof for Lemma 4.2, applied to the difference between the value under $\widehat{g}_{\text{tradeoff}}$ and g_P^*, we have that uniformly with probability approaching one

$$V(g_P^*;P) - V(\widehat{g}_{\text{tradeoff}};P) \leq 0$$

which is equivalent to $(B(\widehat{g}_{\text{tradeoff}};P) - k)_+ \leq \frac{r\cdot(W(\widehat{g}_{\text{tradeoff}};P) - W(g_P^*;P))}{\bar{\lambda}}$. □

Online Appendix to Empirical Welfare Maximization with Constraints

This Online Appendix contains proofs of supporting lemmas and additional results stated in the paper.

B Primitive assumptions and auxiliary lemmas

I first prove the optimal rule that solves the population constrained optimization problem takes the form of threshold. In Section B.2, I first provide primitive assumptions on the class of DGPs. I then prove in Lemma B.1, which establishes that these primitive assumptions imply Assumption 2.1.

In Section B.3, I verify Assumption 3.1 for settings where the observed sample comes from an RCT or an observational study, and the propensity score can be estimated efficiently based on parametric regressions.

B.1 Constrained optimal rule without functional form restriction

The population problem is to find rules based on X_i that solves

$$\max_{g:\mathcal{X}\to\{0,1\}} \mathbb{E}[\tau_i g(X_i)] \text{ subject to } \mathbb{E}[C_i g(X_i)] < k$$

By Law of Iterated Expectation, we can write the constrained optimization problem as

$$\max_{g:\mathcal{X}\to\{0,1\}} \mathbb{E}[\gamma(X_i) g(X_i)] \text{ subject to } \mathbb{E}[r(X_i) g(X_i)] < k$$

where $\gamma(X_i) = \mathbb{E}[\tau_i \mid X_i]$ and $r(X_i) = \mathbb{E}[C_i \mid X_i]$.

Claim B.1. Let $d\mu = r(x) f(x) dx$ denote the positive measure. The constrained optimization problem is equivalent to

$$\max_{g:\mathcal{X}\to\{0,1\}} \int \frac{\gamma(x)}{r(x)} g(x) d\mu \text{ subject to } \int g(x) d\mu = k$$

Let X^* be the support of the solution g^*. It will take the form of $X^* = \{x : \frac{\gamma(x)}{r(x)} > c\}$ where c is chosen so that $\mu(X^*) = k$.

Proof. Let X be the support of any $g \neq g^*$ with $\mu(X) = k$. Then the objective function associated g is

$$\begin{aligned} \int_{X^*} \frac{\gamma}{r} d\mu - \int_X \frac{\gamma}{r} d\mu &= \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X^*\} d\mu - \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X\} d\mu \\ &= \int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \end{aligned}$$

By definition of X^*, we have $\frac{\gamma(x)}{r(x)} > c$ for $x \in X^* \backslash X$ and $\frac{\gamma(x)}{r(x)} < c$ for $x \in X \backslash X^*$. Also note that μ is a positive measure. Then the above difference is lower bounded by

$$\int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq c \int \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq 0$$

as by construction, we have

$$\int \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu = \int \left(\mathbf{1}\{x \in X^*\} - \mathbf{1}\{x \in X\}\right) d\mu = 0$$

since $\mu(X^*) = \mu(X) = k$. $\square$

B.2 Primitive assumptions for contiguity

Assumption B.1. *Assume the class of DGPs $\{P_\theta : \theta \in \Theta\}$ has densities p_θ with respect to some measure μ. Assume P_θ is DQM at P_0 i.e. $\exists \dot{\ell}_0$ s.t. $\int [\sqrt{p_h} - \sqrt{p_0} - \frac{1}{2} h' \dot{\ell}_0 \sqrt{p_0}]^2 d\mu = o(\|h^2\|)$ for $h \to 0$.*

Assumption B.2. *For all policies g, $B(g; P_\theta)$ is twice continuously differentiable in θ at 0, and the derivatives are bounded from above and away from zero within an open neighborhood $\mathcal{N}_\theta$ of zero uniformly over $g \in \mathcal{G}$.*

Lemma B.1. *Under Assumption B.1, the class $\mathcal{P}$ includes a sequence of data distribution $\{P_{h_n}\}$ that is contiguous to P_0 for every h_n satisfying $\sqrt{n}h_n \to h$ e.g. take $h_n = h/\sqrt{n}$. This proves the first part of Assumption 2.1. Suppose further Assumption B.2 holds, then there exists some h for the second part of Assumption 2.1 to hold,.*

Proof. **Proof of Lemma B.1.** By Theorem 7.2 of Vaart (1998), the log likelihood ratio process converges under P_0 (denoted with $\overset{p_0}{\rightsquigarrow}$) to a normal experiment

$$\begin{aligned}\log \prod_{i=1}^{n} \frac{p_{h_n}}{p_0}(A_i) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h' \dot{\ell}_0(A_i) - \frac{1}{2} h' I_0 h + o_{P_0}(1) \\ &\overset{p_0}{\rightsquigarrow} \mathcal{N}\left(-\frac{1}{2} h' I_0 h, h' I_0 h\right)\end{aligned} \tag{16}$$

where $\dot{\ell}_0$ is the score and $I_{P_0} = E_{P_0}[\dot{\ell}_0(A_i)\dot{\ell}_0(A_i)']$ exists. The convergence in distribution of the log likelihood ratio to a normal with mean equal to $-\frac{1}{2}$ of its variance in (16) implies mutual contiguity $P_0^n \lhd\rhd P_{h_n}^n$ by Le Cam's first lemma (see Example 6.5 of Vaart (1998)). This proves the first part of the lemma.

By Taylor's theorem with remainder we have for each policy g

$$B(g; P_{h/\sqrt{n}}) - B(g; P_0) = \frac{h'}{\sqrt{n}} \frac{\partial B(g; P_0)}{\partial \theta} + \frac{1}{2} \frac{1}{n} h' \frac{\partial^2 B(g; P_{\tilde{\theta}_n})}{\partial \theta \partial \theta'} h$$

where $\tilde{\theta}_n$ is a sequence of values with $\tilde{\theta}_n \in [0, h/\sqrt{n}]$ that can depend on g. Take h so that the first term is positive for policies with $B(g; P_0) = k$. Such h exists because we assume $\frac{\partial B(g;P_0)}{\partial \theta}$ is bounded away from zero. For $g \in \mathcal{G}_0$ where $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$, the constraints are violated under P_{h_n} and furthermore (multiplying by $\sqrt{n}$)

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > h' \frac{\partial B(g; P_0)}{\partial \theta} > 0$$

for every n. This proves the second part of the lemma for $c = \inf_{g \in \mathcal{G}_0} \left| h' \frac{\partial B(g;P_0)}{\partial \theta} \right|$. $\square$

B.3 Primitive assumptions and proofs for estimation quality

In this section, I verify that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ satisfy Assumption 3.1 under primitive assumptions on the policy class and the OHIE.

Assumption B.3. *VC-class: The policy class $\mathcal{G}$ has a finite VC-dimension $v < \infty$.*

To introduce the assumptions on the OHIE, I first recall the definitions of components in $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$:

$$\widehat{W}_n(g) := \frac{1}{n}\sum_i \tau_i^* \cdot g(X_i), \;\; \widehat{B}_n(g) := \frac{1}{n}\sum_i C_i^* \cdot g(X_i)$$

for the doubly-robust scores

$$\begin{aligned}
\tau_i^* &= \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \big(Y_i - \widehat{\gamma}^Y(V_i, D_i)\big) \\
C_i^* &= \widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(V_i)} \cdot \big(Z_i - \widehat{\gamma}^Z(V_i, D_i)\big)
\end{aligned}$$

and observed characteristics X_i. Here V_i collects the confounders in OHIE, namely household size (number of adults entered on the lottery sign-up form) and survey wave. Note that while X_i can overlap with V_i, the policy $g(X_i)$ needs not vary by V_i. In the OHIE example, the policy is based on number of children and income. However, conditional on household size and survey wave, income and number of children is independent of the lottery outcome in OHIE.

Recall that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ are supposed to approximate net benefit τ and net excess cost C of Medicaid eligibility. I provide more precise definitions for τ and C as the primitive assumptions are stated in terms of their components.

Let $Y(1)$ be the (potential) subjective health when one is given Medicaid eligibility, and $Y(0)$ be the (potential) subjective health when one is not given Medicaid eligibility. Recall the definition for

$$\tau = Y(1) - Y(0)$$

We only observe the actual subjective health Y_i.

Let $M(1)$ be the (potential) enrollment in Medicaid when one is given Medicaid eligibility, and $M(0)$ be the (potential) enrollment in Medicaid when one is not given Medicaid eligibility.

Let $H(1)$ be the (potential) health care utilization when one is given Medicaid eligibility, and $H(0)$ be the (potential) health care utilization when one is not given Medicaid eligibility. Even when given eligibility, one might not enroll and thus incur zero cost to the government. So the per capita cost of Medicaid eligibility policy $g(X)$ is

$$\mathbb{E}_P[C \cdot g(X)] \text{ where } C = M(1) \cdot H(1).$$

We only observe the actual health care utilization ($H_i = D_i H_i(1) + (1 - D_i) H_i(0)$) and the actual Medicaid enrollment (M_i) in OHIE. We calculate $Z_i = M_i \cdot H_i$.

Assumption B.4. *Suppose for all $P \in \mathcal{P}$, the following statements hold for the OHIE:*

Independent characteristics: $\Pr\{D_i = 1 \mid V_i, X_i\} = \Pr\{D_i = 1 \mid V_i\}$

Unconfoundedness: $(Y(1), Y(0), H(1), M(1)) \perp D_i | V_i$.

Bounded attributes: the support of variables X_i, Y_i and Z_i are bounded.

Strict overlap: There exist $\kappa \in (0, 1/2)$ such that the propensity score satisfies $p(v) \in [\kappa, 1-\kappa]$ for all $v \in \mathcal{V}$.

B.3.1 Uniform convergence of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$

We want to show the recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ converge to mean-zero Gaussian processes G_P^W and G_P^B with covariance functions $\Sigma_P^W(\cdot,\cdot)$ and $\Sigma_P^B(\cdot,\cdot)$ respectively uniformly over $P \in \mathcal{P}$. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. Take $\widehat{W}_n(\cdot)$ for example, the recentered empirical processes is

$$\sqrt{n}\left(\frac{1}{n}\sum_i \tau_i^* \cdot g(X_i) - \mathbb{E}_P[\tau \cdot g(X_i)]\right)$$

and can be expressed as the sum of two terms

$$\frac{1}{\sqrt{n}}\sum_i(\tau_i^* - \widetilde{\tau}_i)\cdot g(X_i) + \sqrt{n}\left(\frac{1}{n}\sum_i \widetilde{\tau}_i \cdot g(X_i) - \mathbb{E}_P[\tau \cdot g(X_i)]\right).$$

Here $\widetilde{\tau}_i$ are the theoretical analogs

$$\widetilde{\tau}_i = \gamma^Y(V_i,1) - \gamma^Y(V_i,0) + \alpha(V_i,D_i)\cdot\left(Y_i - \gamma^Y(V_i,D_i)\right)$$

which is doubly-robust score with the theoretical propensity score and the CEF. A similar expansion holds for $\widehat{B}_n(\cdot)$ involving the theoretical analog

$$\widetilde{C}_i = \gamma^Z(V_i,1) + \frac{D_i}{p(V_i)}\cdot\left(Z_i - \gamma^Z(V_i,D_i)\right).$$

The following lemmas prove the uniform convergence of $\widehat{W_n}(\cdot)$ and $\widehat{B}_n(\cdot)$.

The last part of the assumption is that we have a uniformly consistent estimate for the covariance function. I argue the sample analog

$$\widehat{\Sigma}^B(g,g') = \frac{1}{n}\sum_i(\widetilde{C}_i)^2\cdot g(X_i)\cdot g'(X_i) - \left(\frac{1}{n}\sum_i \widetilde{C}_i\cdot g(X_i)\right)\cdot\left(\frac{1}{n}\sum_i \widetilde{C}_i\cdot g'(X_i)\right)$$

is a pointwise consistent estimate for $\widehat{\Sigma}^B(g,g')$. To see this, note that the covariance function $\Sigma_P^B(\cdot,\cdot)$ maps $\mathcal{G}\times\mathcal{G}$ to $\mathbb{R}$. Under Assumption B.4 that $\mathcal{G}$ is a VC-class, the product $\mathcal{G}\times\mathcal{G}$ is also a VC-class (see e.g. Lemma 2.6.17 of van der Vaart and Wellner (1996)). By a similar argument leading to Lemma B.4 and B.6 , we conclude the uniform consistency of $\widehat{\Sigma}^B(\cdot,\cdot)$.

Lemma B.2. *Let $\mathcal{G}$ be a VC-class of subsets of $\mathcal{X}$ with VC-dimension $v<\infty$. The following sets of functions from $\mathcal{A}$ to $\mathbb{R}$*

$$\mathcal{F}^W = \{\widetilde{\tau}_i\cdot g(X_i) : g\in\mathcal{G}\}$$
$$\mathcal{F}^B = \{\widetilde{C}_i\cdot g(X_i) : g\in\mathcal{G}\}$$

are VC-subgraph class of functions with VC-dimension less than or equal to v for all $P\in\mathcal{P}$. For notational simplicity, we suppress the dependence of $\mathcal{F}$ on P.

Lemma B.3. *For all P in the family $\mathcal{P}$ of distributions satisfying Assumption B.4, for all $g \in \mathcal{G}$ we have*

$$\mathbb{E}_P[\widetilde{\tau}_i \cdot g(X_i)] = W(g; P)$$
$$\mathbb{E}_P[\widetilde{C}_i \cdot g(X_i)] = B(g; P)$$

Lemma B.4. *Let $\mathcal{G}$ satisfy Assumption B.4 (VC). Let U_i be a mean-zero bounded random vector of fixed dimension i.e. there exists $M < \infty$ such that i.e. $U_i \in [-M/2, M/2]$ almost surely under all $P \in \mathcal{P}$. Then the uniform deviation of the sample average of vanishes*

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_i U_i \cdot g(X_i) \right| \to_{a.s.} \mathbf{0}.$$

Lemma B.5. *Let $\mathcal{P}$ be a family of distributions satisfying Assumption B.4. Let $\mathcal{G}$ satisfy Assumption B.4 (VC). Then $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker for all $P \in \mathcal{P}$. That is, the empirical process indexed by $g \in \mathcal{G}$*

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{\tau}_i \cdot g(X_i) - W(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^W(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$, and the empirical process indexed by $g \in \mathcal{G}$

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{C}_i \cdot g(X_i) - B(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^B(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$.

Lemma B.6. *Let $\mathcal{P}$ be a family of distributions satisfying Assumption B.4. Let $\mathcal{G}$ satisfy Assumption B.4 (VC). Then the estimation errors vanishes*

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{\sqrt{n}} \sum_i (\tau_i^* - \widetilde{\tau}_i) \cdot g(X_i) \right| \to_p 0$$
$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{\sqrt{n}} \sum_i (C_i^* - \widetilde{C}_i) \cdot g(X_i) \right| \to_p 0$$

B.3.2 Proofs of auxiliary lemmas

Proof. **Proof of Lemma B.2.** This lemma follows directly from Lemma A.1 of Kitagawa and Tetenov (2018). □

Proof. **Proof of Lemma B.3.** Under Assumption B.4, we prove each $\widetilde{\tau}_i$ is an conditionally unbiased estimate for τ:

$$\mathbb{E}_P[\widetilde{\tau}_i g(X_i)] = \mathbb{E}_P[\mathbb{E}_P[\widetilde{\tau}_i \mid X_i]g(X_i)] = \mathbb{E}_P[\mathbb{E}_P[\tau \mid X_i]g(X_i)] = W(g; P).$$

We focus on $\mathbb{E}_P[\widetilde{\tau}_i \mid X_i] = \mathbb{E}_P[\mathbb{E}_P[\widetilde{\tau}_i \mid V_i, X_i] \mid X_i]$. Dropping the subscript P for simplicity, conditional on V_i, by unconfoundedness and strict overlap we have

$$\begin{aligned}\mathbb{E}[\widetilde{\tau}_i \mid V_i, X_i] &= \mathbb{E}[Y_i \mid V_i, X_i, D_i = 1] - \mathbb{E}[Y_i \mid V_i, X_i, D_i = 0] \\ &= \mathbb{E}[Y_i(1) - Y_i(0) \mid V_i, X_i] = \mathbb{E}[\tau_i \mid V_i, X_i]\end{aligned}$$

Specifically

$$\begin{aligned}\mathbb{E}[\widetilde{\tau}_i \mid V_i, X_i] &= \mathbb{E}\left[\gamma^Y(V_i, 1) - \gamma^Y(V_i, 0) \mid V_i, X_i\right] + \mathbb{E}\left[\alpha(V_i, D_i) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \mid V_i, X_i\right] \\ &= \mathbb{E}\left[Y_i \mid V_i, D_i = 1\right] - \mathbb{E}\left[Y_i \mid V_i, D_i = 0\right] + \\ &\mathbb{E}\left[Y_i(1) - Y_i(0) \mid V_i, X_i\right] - \left(\mathbb{E}\left[Y_i \mid V_i, D_i = 1\right] - \mathbb{E}\left[Y_i \mid V_i, D_i = 0\right]\right) \\ &= \mathbb{E}[\tau \mid V_i, X_i]\end{aligned}$$

Specifically, we expand $\mathbb{E}[\alpha(V_i, D_i)Y_i \mid V_i, X_i]$

$$\begin{aligned}&= \mathbb{E}\left[\frac{Y_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot \Pr\{D_i = 1 \mid V_i, X_i\} - \mathbb{E}\left[\frac{Y_i}{1 - p(V_i)} \mid D_i = 0, X_i\right] \cdot \Pr\{D_i = 0 \mid V_i, X_i\} \\ &= \mathbb{E}\left[D_iY_i(1) + (1 - D_i)Y_i(0) \mid D_i = 1, V_i, X_i\right] - \mathbb{E}\left[D_iY_i(1) + (1 - D_i)Y_i(0) \mid D_i = 0, V_i, X_i\right] \\ &= \mathbb{E}\left[Y_i(1) \mid V_i, X_i\right] - \mathbb{E}\left[Y_i(0) \mid V_i, X_i\right]\end{aligned}$$

where the second line holds by independent characteristic such that

$$\Pr\{D_i = 1 \mid V_i, X_i\} = \Pr\{D_i = 1 \mid V_i\} =: p(V_i)$$

and the last line follows from unconfoundedness. Similarly, we show

$$\mathbb{E}\left[\alpha(V_i, D_i) \cdot \gamma^Y(V_i, D_i) \mid V_i, X_i\right] = \mathbb{E}\left[Y_i \mid V_i, D_i = 1\right] - \mathbb{E}\left[Y_i \mid V_i, D_i = 0\right]$$

Similar argument holds for $\mathbb{E}_P[\widetilde{C}_i \cdot g(X_i)]$, with the only modification:

$$\begin{aligned}
&\mathbb{E}[\frac{D_i}{p(V_i)} H_i \mid V_i, X_i] = \mathbb{E}\left[\frac{H_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot \Pr\{D_i = 1 \mid V_i, X_i\} \\
&= \mathbb{E}[H_i(1) \mid V_i, X_i]
\end{aligned}$$

□

Proof. **Proof of Lemma B.4.** Denote the following set of functions from $\mathcal{U}$ to $\mathbb{R}$

$$\mathcal{F}^U = \{U_i \cdot g(X_i) : g \in \mathcal{G}\}$$

and it has uniform envelope $\bar{F} = M/2$ since U_i is bounded. This envelop function is bounded uniformly over $\mathcal{P}$. Also, by Assumption B.4 (VC) and Lemma B.2, $\mathcal{F}^U$ is VC-subgraph class of functions with VC-dimension at most v. By Lemma 4.14 and Proposition 4.18 of Wainwright (2019), we conclude that $\mathcal{F}^U$ has Rademacher complexity $2\sqrt{M^2 \frac{v}{n}}$. Then by Proposition 4.12 of Wainwright (2019) we conclude that $\mathcal{F}^U$ are P-Glivenko–Cantelli for each $P \in \mathcal{P}$, with an $O(\sqrt{\frac{v}{n}})$ rate of convergence. Note that this argument does not use any constants that depend on P but only M and v, so we can actually get uniform convergence over $\mathcal{P}$. □

Proof. **Proof of Lemma B.5.** Note that Assumption B.4 imply that $\mathcal{F}^W$ and $\mathcal{F}^B$ have uniform envelope $\bar{F} = M/(2\kappa)$. $\mathcal{F}^W$ and $\mathcal{F}^B$ thus have square integrable envelop functions uniformly over $\mathcal{P}$. Also, by Assumption B.4 (VC) and Lemma B.2, $\mathcal{F}^W$ and $\mathcal{F}^B$ are VC-subgraph class of functions with VC-dimension at most v. Even though both $\mathcal{F}^W$ and $\mathcal{F}^B$ depend on P, a similar argument for Theorem 1 in Rai (2019) show that $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker uniformly in $P \in \mathcal{P}$. □

Proof. **Proof of Lemma B.6.** We focus on the deviation in $\tau_i^* - \widetilde{\tau}_i$. The deviation in $C_i^* - \widetilde{C}_i$ can be proven to vanish in a similar manner. Denote $\Delta\gamma^Y(V_i) = \gamma^Y(V_i, 1) -$

$\gamma^Y(V_i,0)$. For any fixed policy g, we expand the deviation $\frac{1}{\sqrt{n}}\sum_i(\tau_i^* - \widetilde{\tau}_i)g(X_i)$ into three terms

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\alpha}(V_i,D_i)-\alpha(V_i,D_i)\right)\cdot\left(Y_i-\gamma^Y(V_i,D_i)\right) \qquad (17)\\
&+\frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\Delta\widehat{\gamma}^Y(V_i)-\Delta\gamma^Y(V_i)-\alpha(V_i,D_i)\cdot\left(\widehat{\gamma}^Y(V_i,D_i)-\gamma^Y(V_i,D_i)\right)\right)\\
&-\frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\gamma}^Y(V_i,D_i)-\gamma^Y(V_i,D_i)\right)\cdot\left(\widehat{\alpha}(V_i,D_i)-\alpha(V_i,D_i)\right)\cdot
\end{aligned}$$

Denote these three summands by $D_1(g)$, $D_2(g)$ and $D_3(g)$. We will bound all three summands separately. Recall we use the full sample to estimate the propensity score and the CEF with a saturated model. The purpose of the above expansion is to separately bound the estimation error from the estimated CEF and propensity score, and the deviation from taking sample averages. For cross-fitted estimators for the propensity score and the CEF, a similar bound can be found in Athey and Wager (2021).

Uniform consistency of the estimated CEF and propensity score Denote with $b(V_i,D_i)$ the dictionary that spans (V_i,D_i), and $b(V_i,)$ the dictionary that spans V_i. The saturated models are therefore parameterized as $\gamma^Z(V_i,D_i)=\gamma' b(V_i,D_i)$ and $p(V_i)=\beta' b(V_i)$. Under standard argument, the OLS estimators $\widehat{\gamma}$ and $\widehat{\beta}$ are asymptotically normal uniformly over $P\in\mathcal{P}$:

$$\sqrt{n}\cdot(\widehat{\gamma}-\gamma)=O_P(1),\ \sqrt{n}\cdot\left(\widehat{\beta}-\beta\right)=O_P(1).$$

Furthermore, the in-sample L_2 errors from the estimated CEF and propensity score vanish. Consider

$$\frac{1}{n}\sum_i\left(\widehat{\gamma}^Z(V_i,D_i)-\gamma^Z(V_i,D_i)\right)^2=\frac{1}{n}\sum_i\left((\widehat{\gamma}-\gamma)' b(V_i,D_i)\right)^2=(\widehat{\gamma}-\gamma)'\widehat{M}(\widehat{\gamma}-\gamma)$$

where $\widehat{M}=\frac{1}{n}\sum_i b(V_i,D_i)b(V_i,D_i)'$. It converges in probability to a fixed matrix $M=\mathbb{E}[b(V_i,D_i)b(V_i,D_i)']$. So the in-sample L_2 error from the estimated CEF vanishes at the rate of $n^{-1/2}$.

Similarly, consider expanding $\frac{1}{n}\sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2$ as

$$\frac{1}{n}\sum_i \left(\frac{1}{\widehat{\beta}'b(V_i)} - \frac{1}{\beta' b(V_i)}\right)^2 D_i^2 + \left(\frac{1}{1-\widehat{\beta}'b(V_i)} - \frac{1}{1-\beta' b(V_i)}\right)^2 (1-D_i)^2$$

With a first-order Taylor approximation, for each term in the summand, the dominating term would be

$$\frac{1}{n}\sum_i \left(\left(\widehat{\beta}-\beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2}\right)^2 D_i^2 = \left(\widehat{\beta}-\beta\right)' \left(\frac{1}{n}\sum_i \frac{b(V_i)b(V_i)'}{(\beta' b(V_i))^2} D_i^2\right)\left(\widehat{\beta}-\beta\right)$$

where the middle term converges to a fixed matrix as implied by $\beta' b(V_i)$ being bounded away from zero and one. So the in-sample L_2 error from the estimated propensity score also vanishes at the rate of $n^{-1/2}$.

Bounding the deviation We now bound each term in (17). Plugging in the first-order Taylor approximation with a remainder term to the estimated propensity score, we have

$$\begin{aligned}
D_1(g) =& \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(Y_i - \gamma^Y(V_i, D_i)\right)\cdot \\
& \left(\left(\frac{1}{\widehat{\beta}'b(V_i)} - \frac{1}{\beta' b(V_i)}\right) D_i + \left(\frac{1}{1-\widehat{\beta}'b(V_i)} - \frac{1}{1-\beta' b(V_i)}\right)(1-D_i)\right) \\
=& \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\beta}-\beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2}\cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) + \\
& \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\beta}-\beta\right)' \frac{b(V_i)b(V_i)'}{(\widetilde{\beta}' V_i)^3}\left(\widehat{\beta}-\beta\right)\cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \\
=& \underbrace{\sqrt{n}\left(\widehat{\beta}-\beta\right)'}_{O_P(1)} \underbrace{\frac{1}{n}\sum_i g(X_i)\frac{-b(V_i)}{(\beta' b(V_i))^2}\cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)}_{o_P(1)} + o_P(1)
\end{aligned}$$

where $\widetilde{\beta}$ is a sequence between $\widehat{\beta}$ and β. This remainder term therefore converges to zero. Uniform convergence of the sample average follows from Lemma B.4: the random vector $\frac{-b(V_i)}{(\beta' b(V_i))^2}\cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)$ is mean-zero, has fixed dimension, and bounded.

We can decompose $D_2(g)$ into the product of two terms

$$D_2(g) = \underbrace{\sqrt{n}\,(\widehat{\gamma}^{'} - \gamma)}_{O_p(1)} \frac{1}{n}\sum_i g(X_i)\left(\Delta b(V_i, D_i) - \alpha(V_i, D_i)\cdot b(V_i, D_i)\right)$$

Uniform convergence of the sample average again follows from Lemma B.4. We thus conclude $D_1(g)$ and $D_2(g)$ vanish uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$.

For $D_3(g)$, we apply the Cauchy-Schwarz inequality to note that

$$D_3(g) \leq \sqrt{n}\cdot\sqrt{\frac{1}{n}\sum_i \left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right)^2}\cdot\sqrt{\frac{1}{n}\sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2}$$

The terms in the square root are the in-sample L_2 errors from the estimated CEF and propensity score, which vanish at the rate of $n^{-1/2}$ uniformly over $P \in \mathcal{P}$ as shown in the paragraph above. We thus conclude $D_3(g)$ vanishes uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$. $\square$

C Additional Theoretical Results and Discussion

Proposition C.1. *Consider the same setting as in Proposition 3.1, except that we have* $\Pr_P\{C = 1 \mid X\} \in (0,1)$ *almost surely such that the budget function strictly increases for* $t \in [\underline{t}, \overline{t}]$. *Similarly, suppose the budget constraint is at* $k = \mathbb{E}_P[C \cdot \mathbf{1}\{X \leq \underline{t}\}]$. *Then the sample-analog rule* $\widehat{g}_{sample}$ *is asymptotically welfare efficient.*

Proof. **Proof of Proposition C.1.** Since $\Pr_P\{C = 1|X\} > 0$ almost surely, we have $B(t; P) = \mathbb{E}_P[C \cdot \mathbf{1}\{X \leq t\}] = \Pr_P\{X \leq t\} \cdot \Pr_P\{C = 1|X \leq t\}$ is strictly increasing in t. The constrained optimal threshold is therefore the highest threshold where the constraint is satisfied exactly i.e. $t^* = \underline{t}$.

Since X is one-dimensional and continuously distributed, the welfare function $W(t; P)$ is continuous in t. Since $W(t; P)$ is also strictly increasing in t, for all $\epsilon > 0$, there exists $t_\epsilon < t^*$ such that

$$W(t_\epsilon; P) = W(t^*; P) - \epsilon.$$

Since $C_i \cdot \mathbf{1}\{X_i \le t\}$ is bounded for all t, the uniform law of large numbers applies and there exists a sequence $b_n \to 0$ such that

$$\lim_{n\to\infty} \Pr_{P^n}\left\{\sup_t \left|\widehat{B}_n(t) - B(t;P)\right| \le b_n\right\} = 1.$$

Since $t_\epsilon < t^*$ and $B(t;P)$ continuous and strictly increasing, for n large enough (and b_n small enough), we must have

$$B(t_\epsilon;P) \le B(t^*;P) - b_n = k - b_n < k = B(t^*;P).$$

Taken together, we have

$$\lim_{n\to\infty} \Pr_{P^n}\left\{\widehat{B}_n(t_\epsilon) \le k - b_n + b_n\right\} \ge \lim_{n\to\infty} \Pr_{P^n}\left\{\widehat{B}_n(t_\epsilon) \le B(t_\epsilon;P) + b_n\right\} = 1.$$

As in the setting of Proposition 3.1, the policymakers know the welfare function is strictly increasing, and the sample-analog rule solves $\max_t \widehat{B}_n(t)$ subject to $\widehat{B}_n(t) \le k$. However, the solution is not unique because $\widehat{B}_n(t)$ is a step function. For the proof, let the sample-analog rule be the largest possible threshold to maximize $\widehat{B}_n(t)$:

$$\widehat{t} = \max\left\{\arg\max_t\{\widehat{B}_n(t) \text{ subject to } \widehat{B}_n(t) \le k\}\right\}.$$

This implies that $\lim_{n\to\infty} \Pr_{P^n}\left\{t_\epsilon \le \hat{t}\right\} = 1$ and since $W(t;P)$ is strictly increasing in t, we have

$$\lim_{n\to\infty} \Pr_{P^n}\left\{W(\hat{t};P) \ge W(t^*;P) - \epsilon\right\} = \lim_{n\to\infty} \Pr_{P^n}\left\{W(\hat{t};P) \ge W(t_\epsilon;P)\right\} = 1$$

which proves the asymptotic welfare efficiency. $\square$

D Additional Empirical Results and Discussion

D.1 Robustness checks

In this section, I present additional empirical results as I vary the significance level α and the penalty level $\bar{\lambda}$, relative to the benchmark values considered in Section 4.3.2.

I explore how the modified rule $\widehat{g}_\alpha$ changes as α varies from 5% (as in the main text) to 50% in Figure D.1 below. Each point on the curve represents the policy chosen by $\widehat{g}_\alpha$ for a corresponding statistical significance level α. The vertical axis shows the estimated welfare and the horizontal axis shows the estimated budget of the selected policy. As α increases, the statistical test becomes less conservative and the upper confidence band becomes wider. So the feasible region expands and the selected policy achieves higher estimated welfare.

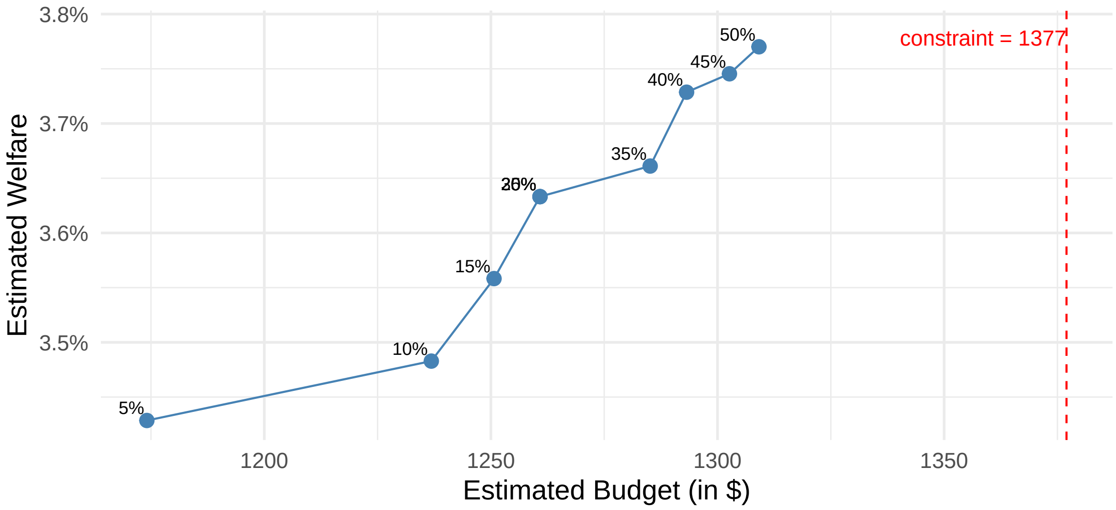


Figure D.1: Frontier of $\widehat{g}_\alpha$ across statistical significance levels α

I also explore how $\widehat{g}_{\text{tradeoff}}$ changes for a range of values for $\overline{\lambda}$ as illustrated in the Figure D.2 below. Each point on the curve represents the policy chosen by $\widehat{g}_{\text{tradeoff}}$ for a corresponding $\overline{\lambda}$. The vertical axis shows the estimated welfare and the horizontal axis shows the estimated budget achieved by $\widehat{g}_{\text{tradeoff}}$. For example, if each dollar of budget overrun required a repayment of 1.68 times the amount, rather than just 1 as in the main text, $\widehat{g}_{\text{tradeoff}}$ selects a policy that entails no estimated budget violation, aligning with $\widehat{g}_{\text{sample}}$. Because the optimization is discrete in the sample, intermediate values of $\overline{\lambda}$ often result in the same selected policy, leading to gaps in the curve.

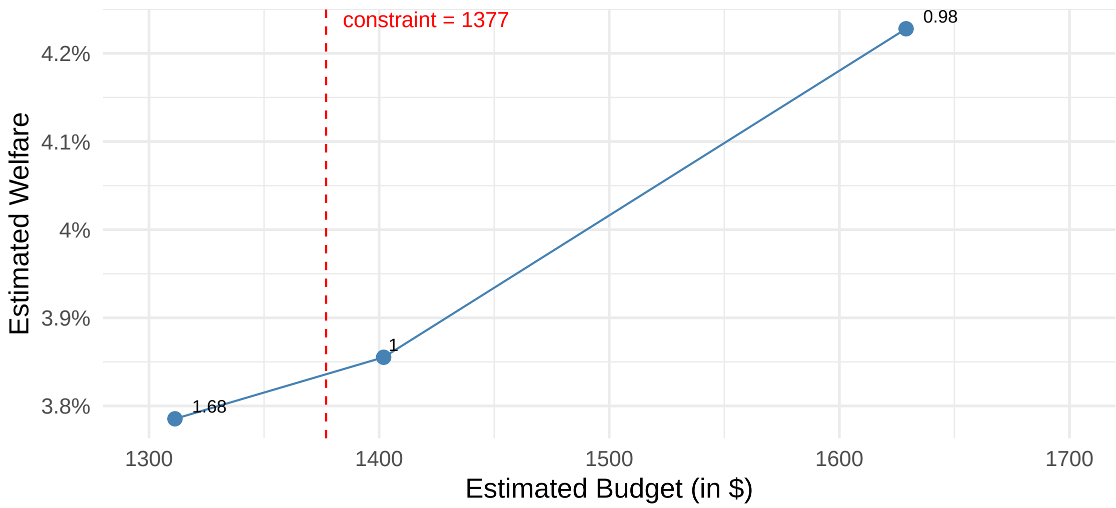


Figure D.2: Frontier of $\widehat{g}_{\text{tradeoff}}$ across different penalty levels $\overline{\lambda}$

D.2 Computation

The policy class considered in this paper $g(x) = \mathbf{1}\{\beta' x \geq 0\}$ consists of linear eligibility score. Therefore I follow (Kitagawa and Tetenov, 2018, Online Appendix C) to set up the problems of sample-analog rule (4) and trade-off rule (11) as a Mixed Integer Linear Programming (MILP), which is more efficient than solving them through a search over x. Due to the estimated constraint, I add linear constraints to the version of Kitagawa and Tetenov (2018) and instead solve for the trade-off rule in two parts:

$$
\begin{array}{rlrl}
 & \text{constrained} & & \text{unconstrained} \\
 & \min_{\beta\in\mathbf{B}, z\in\mathbb{R}^n} -\sum_{i=1}^{n} \tau_i^* z_i & & \min_{\beta\in\mathbf{B}, z\in\mathbb{R}^n} -\sum_{i=1}^{n} \tau_i^* z_i + \overline{\lambda}(\sum_{i=1}^{n} C_i^* z_i - n\cdot k) \\
\text{subject to} & \frac{X_i'\beta}{\bar{C}_i} \le z_i \le 1 + \frac{X_i'\beta}{\bar{C}_i}, & \text{subject to} & \frac{X_i'\beta}{\bar{C}_i} \le z_i \le 1 + \frac{X_i'\beta}{\bar{C}_i}, \\
 & z_i \in \{0,1\}, & & z_i \in \{0,1\}, \\
 & \sum_{i=1}^{n} C_i^* z_i < n\cdot k. & & \sum_{i=1}^{n} C_i^* z_i \ge n\cdot k.
\end{array}
\tag{18}
$$

where constants $\bar{C}_i = \sup_{\beta\in B} |X_i'\beta|$. The additional binary parameters $(z_1, \dots, z_n)$ replace the policy functions $g(X_i) = \mathbf{1}\{\beta' X_i \ge 0\}$. The sample-analog rule $\widehat{g}_{\text{sample}}$ is the constrained solution on the left hand side. The trade-off rule $\widehat{g}_{\text{tradeoff}}$ is the better solution across constrained and unconstrained policies in (18).

To modify the sample-analog rule to control the probability of selecting infeasible policies by $\widehat{g}_{\alpha}$ as proposed in Section 3.3, I introduce quadratic constraints to (18). First, for a conventional level of α, constructing the tightened constraint $\hat{\mathcal{G}}_{\alpha}$ requires an estimate for the critical value c_{α}, the α%-quantile from $\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$, the infimum of the Gaussian process $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$. In practice, I construct a grid on $\mathcal{G}$ as

$$
\widetilde{\mathcal{G}} = \{g(x) = \mathbf{1}\{\text{income}\cdot 1\{\text{numchild} = j\} \le y_j\} : j \in \{0, 1, \ge 2\}, y_j \in \{0, 50, 100, \dots, 500\}\}
\tag{19}
$$

for characteristics $x = (\text{income}, \text{ numchild})$. This grid thus consists of income thresholds every 50% of the federal poverty level, and the thresholds can vary with number of children. I then approximate the infimum over infinite-dimensional $\mathcal{G}$ by the minimum over $\widetilde{\mathcal{G}}$ with estimated covariance i.e. $\min_{g\in\widetilde{\mathcal{G}}} \frac{\widetilde{h}(g)}{\widehat{\Sigma}^B(g,g)^{1/2}}$. Here $\widetilde{h}(\cdot) \sim \mathcal{N}(0, \widehat{\Sigma}^B)$ is a Gaussian vector indexed by $g \in \widetilde{\mathcal{G}}$, with $\widehat{\Sigma}^B$ is sub-matrix of the covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$ for $g \in \widetilde{\mathcal{G}}$. Based on 10,000 simulation draws I estimate c_{α} to be -2.56 for $\alpha = 5\%$. The validity of this approximation is given by the uniform consistency of the covariance estimator under Assumption 3.1.

Second, to select policies in $\hat{\mathcal{G}}_\alpha$ is equivalent to invert a test: $\frac{\sqrt{n}\left(\widehat{B}_n(g)-k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha$. This test inversion can be written as two constraint (note that $c_\alpha < 0$). The first constraint is linear: $\sum_{i=1}^n C_i^* z_i \leq n \cdot k$ and the second is $n \cdot \left(\widehat{B}_n(g) - k\right)^2 > c_\alpha^2 \cdot \widehat{\Sigma}^B(g,g)$ which translates to a quadratic constraint

$$
\begin{aligned}
& n\left(\frac{\sum_{i=1}^n C_i^* z_i}{n} - k\right)^2 > c_\alpha^2 \left(\left(\frac{1}{n}\sum_{i=1}^n (C_i^*)^2 z_i\right) - \left(\frac{\sum_{i=1}^n C_i^* z_i}{n}\right)^2\right) \\
\Rightarrow & \frac{c_\alpha^2}{n}\sum_{i=1}^n (C_i^*)^2 z_i + 2k\sum_{i=1}^n C_i^* z_i - \left(\frac{1}{n} + \frac{c_\alpha^2}{n^2}\right)(\sum_{i=1}^n C_i^* z_i)^2 < nk^2
\end{aligned}
$$