Anomalous Diffusion in One-Dimensional Disordered Systems: A Discrete Fractional Laplacian Method

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Abstract

This work extends the study of Anderson-type Hamiltonians to include transport characterized by anomalous diffusion. Herein, we investigate the transport properties of a one-dimensional disordered system that employs the discrete fractional Laplacian, $(-\Delta)^s$, $s \in (0, 2)$, in combination with results from spectral and measure theory **(changed per Eva's suggestion)**. It is a classical mathematical result that the standard Anderson model exhibits localization of energy states for all nonzero disorder in one-dimensional systems. Numerical simulations of our proposed model demonstrates that this localization effect is enhanced for sub-diffusive realizations of the operator, $s \in (1, 2)$, and that the super-diffusive realizations of the operator, $s \in (0, 1)$, can exhibit delocalized energy states. These results agree with recent experimental predictions and the proposed method can be used to examine anomalous diffusion in physical systems where strong interactions, structural defects, and correlated effects are present.

Keywords: Anderson localization, anomalous diffusion, discrete fractional Laplacian, spectral approach

1. Introduction

The concept of localization was first studied by P. W. Anderson in 1958, when he suggested that sufficiently large impurities in a semiconductor could lead to the localization of electrons. This phenomenon is known as *Anderson localization* and has motivated various mathematical and physical studies in the past 60 years. The classical localization problem in one-dimension is well understood from a physical perspective, yet there are numerous open mathematical questions. **Herein, we propose a non-local model for studying one-dimensional anomalous transport which provides new perspectives on numerous physical and mathematical problems, This new model allows for the consideration of systems which exhibit more exotic transport properties and interactions.** In particular, we consider a generalized notion of diffusion known as *anomalous diffusion*.

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Diffusion is a persistent random walk characteristic of diverse phenomena, such as neutrons in nuclear reactors [1], stock market prices [2], and pollen particles suspended in fluids [3]. In the standard diffusion regime, the mean square displacement of an ensemble of moving particles increases linearly in time, *i.e.* $\langle x^2 \rangle \sim t^\beta$, where $\beta = 1$. However, nonlinear mean square displacement, characterized by exponents $\beta \neq 1$, is also possible, yielding two distinct examples of *anomalous transport*: *subdiffusion*, when $\beta \in (0, 1)$, and *superdiffusion*, when $\beta > 1$. **It is worth noting that the fractional power β, while related, differs from the fractional power of the discrete Laplace operator, s. For the particular operator considered in this exploration, one has that β is proportional to s^{-1} (I need to clarify further).** Anomalous diffusion has been analyzed theoretically and observed experimentally in various physical systems, such as amorphous semiconductors, porous media, glasses, granular matter, ionic liquids, polymers, and plasmas [4–9]. Such diffusive processes are also characteristic of systems modeling turbulence, superconductivity, self-diffusion in two-dimensional fluids, biological cell motility, reaction-diffusion, and Nuclear Magnetic Resonance diffusion [9–15]. Many of the aforementioned systems are known to exhibit localization properties strongly dependent on the physical system parameters. The study of the localization phenomenon in the presence of anomalous diffusion and the determination of the critical parameters needed to induce localization are the primary motivators of the current study.

1. **Should we add a section discussing how the power of the fractional Laplacian relates to the power of mean square displacement?**

The main object of interest is the one-dimensional *random discrete fractional Schrödinger operator*, formally given by

$$H_{s,\epsilon} := (-\Delta)^s + \sum_{i \in \mathbb{Z}} \epsilon_i \langle \cdot, \delta_i \rangle \delta_i, \tag{1}$$

for some $s \in (0, 2)$, with $\langle \cdot, \cdot \rangle$ being the $\ell^2(\mathbb{Z})$ inner product, δ_i the standard basis of $\mathbb{Z}$, and ϵ_i random variables taken to be independently and identically distributed (i.i.d.) according to the uniform distribution on $[-W/2, W/2]$, with $W > 0$. The operator $(-\Delta)^s$ is the *discrete fractional Laplacian* and will be defined in the following section. The operator $(-\Delta)^s$ describes the *nonlocal* movement of an electron in a one-dimensional chain with atoms located at all integer lattice points in $\mathbb{Z}$. When $s = 1$, the operator in (1) reduces to the classical random discrete Schrödinger operator studied in [16]. However, when $s \neq 1$, this operator considers the possibility of electrons jumping to non-neighboring lattice points, which corresponds to an *anomalous-type diffusion* process. The perturbation $\sum_{i \in \mathbb{Z}} \epsilon_i \langle \cdot, \delta_i \rangle \delta_i$ represents random displacements of the atoms located at the lattice points. This perturbation is almost surely a non-compact operator, which means that classical perturbation theory cannot be applied (for more details see [17, 18]). Finally, the parameter W may be interpreted as the strength of the disorder at the lattice points.

In the classical setting, $s = 1$, it has been shown via the spectral method (see Section 3 for details) that localization is expected for all disorders $W > 0$ [19]. However, delocalization behavior in one-dimensional chains has been experimentally observed which differs from these classical mathematical results [20]. These results are due to non-local transport behavior, which can be modeled using the operator given in (1). By considering various fractional powers of the Laplacian, we demonstrate enhanced localized behavior for $s \in (1, 2)$ and delocalized energy states for $s \in (0, 1)$. These observations have interesting implications both mathematically and physically, thus yielding exciting new avenues of research.

Herein, we numerically investigate localization properties of the operator (1). This research is the first of its kind and combines the spectral approach developed in [19] and the known discrete

fractional Laplacian results from [21]. The spectral method was introduced in [19], where it was also used to numerically confirm the existence of extended states for the two-dimensional discrete random Schrödinger operator for weak disorder. Applications to other underlying geometries, such as the square, hexagonal, triangular lattice in two-dimensions, and the three-dimensional square lattice, were explored in [22–26]. The unperturbed operator used in the aforementioned papers was always the classical Laplacian ($s = 1$). While the spectral approach is based on a weaker definition of localization, it allows for the development of efficient computational techniques that can be employed to further the analytic results in the field. Moreover, the physical interpretation of this approach has been recently established [23]. The current study will undoubtedly motivate the consideration of localization in the presence of anomalous diffusion with respect to the more restrictive definitions, such as *dynamic localization*.

This article is organized as follows. Section 2 provides relevant theoretical background regarding the discrete fractional Laplacian and its associated nonlocal weights. Section 3 details the employed spectral approach for studying localization behavior of the newly proposed model. Section 4 outlines in detail the numerical method used in our simulations. We provide computational results that validate the proposed method and elucidate numerous analytical properties of the random discrete fractional Schrödinger operator. Finally, concluding remarks and a discussion of possible future projects are provided in Section 5.

2. Theoretical Background

In the following sub-sections we present the necessary theoretical background for the discrete fractional Laplacian. Section 2.1 introduces the discrete fractional Laplacian and outlines some necessary results for the construction of the numerical method. Novel results regarding higher fractional powers of the Laplacian are provided. Section 2.2 provides a brief description of a particular physical interpretation of the discrete fractional Laplacian relevant to the current work.

I will update this more...

2.1. Discrete Fractional Laplacian

The fractional Laplacian has been studied in mathematics for nearly a century. Understood as the classical Laplacian raised to positive powers, this operator has received attention in potential theory, fractional calculus, harmonic analysis, and probability theory [27–30]. However, only recently has the operator garnered attention in the fields of differential equations and physics. Due to this newfound interest, the fractional Laplacian has become one of the most researched mathematical objects of the past decade. Classically, only fractional powers $s \in (0, 1)$ have been considered, and in this case, one can define the fractional Laplacian on $\mathbb{R}^d$ as the hyper-singular integral given by

$$(-\Delta)^s u(x) := c_{d,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x) - u(\xi)}{|x - \xi|^{d+2s}} \, d\xi, \tag{2}$$

where $x \in \mathbb{R}^d$, $B_\varepsilon(x)$ is the $d-$dimensional ball of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^d$, and $c_{d,s}$ is some normalization constant. The operator can also be defined as a pseudo-differential operator via its Fourier transform, *i.e.*,

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi). \tag{3}$$

The recent interest in the fractional Laplacian has been a direct consequence of the revolutionary work by Caffarelli and Silvestre, who demonstrated in [31] that one may study $(-\Delta)^s$, $s \in$

$(0,1)$, via the Dirichlet-to-Neumann operator associated with a particular extension problem. The Dirichlet-to-Neumann operator is a particular example of the Poincaré-Steklov operator and maps the values of a harmonic function on the boundary of some domain to the normal derivative values of the same function on the same boundary. Caffarelli and Silvestre's approach provided an extension of a well-known result regarding the square root of the Laplacian as it arose in fluid dynamics and finance (see [32] and the references therein). That is, for $s \in (0,1)$, they showed that

$$(-\Delta)^s u(x) = c_s \lim_{t\to 0^+} t^{1-2s} v_t(x,t), \tag{4}$$

where $v(x,t) : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}_+$ is the solution to the following Bessel-type problem

$$\begin{cases} v_{tt}(x,t) + \dfrac{1-2s}{t} v_t(x,t) + \Delta v(x,t) &= 0, & x \in \mathbb{Z},\ t > 0, \\ v(x,0) &= u(x), & x \in \mathbb{Z}, \end{cases} \tag{5}$$

and $c_s := 2^{2s-1}\Gamma(s)/\Gamma(1-s)$. Thus, one may study the highly nonlocal fractional Laplacian operator by considering the local problem (5). While (5) is posed in one higher dimension and exhibits either a singular or degenerate nature depending on the value of s, it is amenable to classical analytical and numerical techniques. Recent work by Chen, Lei, and Wei has demonstrated that similar extension problems may be derived for higher fractional powers of the Laplacian, though these extensions have not yet been employed in approximating solutions to higher-order problems [33].

Herein, we investigate the fractional powers of the discrete Laplacian for exponents $s \in (0,2)$. While numerous definitions of such an operator exist (for instance, see [34]), we begin by introducing existing results for $s \in (0,1)$ (see [21, 35]) and then expand them to the case where $s \in (0,2)$. Let $u : \mathbb{Z} \to \mathbb{R}$ with $u_n := u(n),\ n \in \mathbb{Z}$. We then define the discrete Laplacian on $\mathbb{Z}$ as

$$\Delta u_n := u_{n+1} - 2u_n + u_{n-1}. \tag{6}$$

For the discrete Laplacian, (5) can be solved uniquely and it is known that the bounded solution is given by

$$v(x,t) = \frac{1}{\Gamma(s)} \int_0^\infty z^{s-1} e^{-t^2/4z} e^{-z\Delta} (-\Delta)^s u(x)\, dz, \tag{7}$$

where $e^{-z\Delta}$ is the standard semigroup generated by the discrete Laplacian on $\mathbb{Z}$ [35–38]. Explicit calculation then yields, via (4), the following representation

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty z^{-s-1} \left(e^{-z\Delta} - I\right) u(x)\, dz, \tag{8}$$

where I is the identity operator, which is often used as the definition of fractional powers of the Laplacian [21, 35]. From (8) and standard results regarding the discrete Laplacian semigroup (see [39] and references therein), the following theorem has been developed [21].

Theorem 1 [21]**.** *For $s \in (0,1)$, we define*

$$\ell_s := \left\{ u : \mathbb{Z} \to \mathbb{R} \ :\ \|u\|_{\ell_s} := \sum_{n\in\mathbb{Z}} \frac{|u_n|}{(1+|m|)^{1+2s}} < \infty \right\}.$$

i. For $u \in \ell_s$ we have

$$(-\Delta)^s u_n = \sum_{m\in\mathbb{Z};\, m\neq n} (u_n - u_m)\, K_s(n-m), \tag{9}$$

where the discrete kernel is given by

$$K_s(m) := \begin{cases} \dfrac{4^s\Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|} \cdot \dfrac{\Gamma(|m|-s)}{\Gamma(|m|+1+s)}, & m \in \mathbb{Z}\backslash\{0\}, \\ 0, & m = 0. \end{cases} \tag{10}$$

ii. For $s \in (0,1)$ there exists constants $0 < c_s \leq C_s$ such that, for any $m \in \mathbb{Z}\backslash\{0\}$,

$$\frac{c_s}{|m|^{1+2s}} \leq K_s(m) \leq \frac{C_s}{|m|^{1+2s}}. \tag{11}$$

iii. If $u \in \ell_0$, then $\lim_{s\to 0^+}(-\Delta)^s u_n = u_n$.

iv. If u is bounded, then $\lim_{s\to 1^-}(-\Delta)^s u_n = -\Delta u_n$.

From Theorem 1, we are able to discern many useful properties which are important for both the theoretical analysis and computational procedures employed in problems involving the fractional Laplacian. In the ensuing experiments, our calculations will be based on (9) and its various approximations, the details of which are contained in Section 3.

It now remains to discuss the case when $s \in (1,2)$. In [40] it was shown that for $s > 1$, we have

$$(-\Delta)^s u_n = (-\Delta)^{s-1}(-\Delta)u_n. \tag{12}$$

Using (12), we have the following theorem for $s \in (1,2)$. The proof for Theorem 2, parts *ii.-iv.*, are similar to those in [35] but we include them for completeness.

Theorem 2. *For $s \in (1,2)$, we define*

$$\ell_s := \left\{ u \,:\, \mathbb{Z} \to \mathbb{R} \,:\, \|u\|_{\ell_s} := \sum_{n\in\mathbb{Z}} \frac{|u_n|}{(1+|m|)^{1+2s}} < \infty \right\}.$$

i. For $u \in \ell_s$ we have $(-\Delta)^s u_n$ is given by (9)*, where the discrete kernel is also given by* (10)*.* ***(Eva, the newest code does not use different codes for each parameter set)***

ii. For $s \in (1,2)$ there exists constants $0 < c_s \leq C_s$ such that, for any $m \in \mathbb{Z}\backslash\{0\}$, the discrete kernel K_s satisfies (11)*.*

iii. If u is bounded, then $\lim_{s\to 1^+}(-\Delta)^s u_n = -\Delta u_n$.

iv. If u is bounded, then $\lim_{s\to 2^-}(-\Delta)^s u_n = (-\Delta)^2 u_n$, where $(-\Delta)^2$ is the classical bi-harmonic operator.

I moved the proofs back into this section, in case we submit to a math journal...

Proof. *i.* Let $s \in (1,2)$ and define $v_n := (-\Delta)u_n$. Then, for $n \in \mathbb{Z}$, we have

$$\begin{aligned}
(-\Delta)^s u_n &= (-\Delta)^{s-1}(-\Delta)u_n \\
&= (-\Delta)^{s-1} v_n \\
&= \sum_{m\in\mathbb{Z};\, m\neq n} (v_n - v_m) K_{s-1}(n-m) \\
&= v_n \sum_{m\in\mathbb{Z};\, m\neq n} K_{s-1}(n-m) - \sum_{m\in\mathbb{Z};\, m\neq n} v_m K_{s-1}(n-m) \\
&= A_{s-1} v_n - \sum_{m\in\mathbb{Z};\, m\neq n} v_m K_{s-1}(n-m), \qquad (13)
\end{aligned}$$

where

$$A_s := \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}\,\Gamma(1+s)}. \qquad (14)$$

Employing the definition of v_n in (13) yields

$$\begin{aligned}
(-\Delta)^s u_n &= A_{s-1}[2u_n - u_{n-1} - u_{n+1}] - \sum_{m\in\mathbb{Z};\, m\neq n} [2u_m - u_{m-1} - u_{m+1}] K_{s-1}(n-m) \\
&= A_{s-1}[2u_n - u_{n-1} - u_{n+1}] - [2u_{n-1} - u_{n-2} - u_n] K_{s-1}(1) \\
&\quad - [2u_{n+1} - u_n - u_{n+2}] K_{s-1}(-1) - \sum_{m\in\mathbb{Z};\, m\neq n, n\pm 1} [2u_m - u_{m-1} - u_{m+1}] K_{s-1}(n-m) \\
&= [2A_{s-1} + 2K_{s-1}(1)] u_n - \sum_{m\in\mathbb{Z};\, m\neq 0} u_{n-m} [2K_{s-1}(m) - K_{s-1}(m-1) - K_{s-1}(m+1)] \\
&= \gamma^{(1)}_{s-1} u_n - \sum_{m\in\mathbb{Z};\, m\neq 0} u_{n-m} \gamma^{(2)}_{s-1}(m), \qquad (15)
\end{aligned}$$

where

$$\gamma^{(1)}_{s-1} := 2A_{s-1} + 2K_{s-1}(1) \quad \text{and} \quad \gamma^{(2)}_{s-1}(m) := 2K_{s-1}(m) - K_{s-1}(m-1) - K_{s-1}(m+1)$$

and we have used the fact that $K_{s-1}(-1) = K_{s-1}(1)$. In order to obtain the desired result, we must show that $\gamma^{(1)}_{s-1} = A_s$ and $\gamma^{(2)}_{s-1}(m) = K_s(m),\ m \in \mathbb{Z}\backslash\{0\}$. We proceed by direct calculation. First, we note that

$$\begin{aligned}
K_{s-1}(1) &= \frac{4^{s-1}\Gamma(s-1/2)\Gamma(2-s)}{\sqrt{\pi}|\Gamma(1-s)|\Gamma(1+s)} \\
&= \frac{4^{s-1}\Gamma(s-1/2)(1-s)\Gamma(1-s)}{\sqrt{\pi}|\Gamma(1-s)|s\Gamma(s)} \\
&= \frac{4^{s-1}\Gamma(s-1/2)(s-1)}{\sqrt{\pi}s\Gamma(s)},
\end{aligned}$$

since $s \in (1,2)$, which yields

$$\begin{aligned}
\gamma_{s-1}^{(1)} &= 2A_{s-1} + 2K_{s-1}(1) \\
&= 2\frac{4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)} + 2\left[\frac{4^{s-1}\Gamma(s-1/2)(s-1)}{\sqrt{\pi}s\Gamma(s)}\right] \\
&= \frac{2\cdot 4^{s-1}\Gamma(s-1/2)}{\sqrt{\pi}\Gamma(s)}\left[1+\frac{s-1}{s}\right] \\
&= \frac{4^{s}\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(1+s)} \\
&= A_s. \qquad (16)
\end{aligned}$$

In order to prove the remaining equality, we note that we can rewrite (10) as

$$K_s(m) = \frac{(-1)^{m+1}\Gamma(2s+1)}{\Gamma(1+s+m)\Gamma(1+s-m)}, \qquad (17)$$

by employing the duplication and Euler reflection formula to each Gamma function, as was shown in [21]. Thus, we have

$$\begin{aligned}
\gamma_{s-1}^{(2)}(m) &= 2K_{s-1}(m) - K_{s-1}(m-1) - K_{s-1}(m+1) \\
&= (-1)^m\Gamma(2s-1)\left[\frac{-2}{\Gamma(s+m)\Gamma(s-m)}\right. \\
&\qquad \left. -\frac{s+m-1}{\Gamma(s+m)\Gamma(s-m)(s-m)} - \frac{s-m-1}{(s+m)\Gamma(s+m)\Gamma(s-m)}\right] \\
&= \frac{(-1)^m\Gamma(2s-1)}{\Gamma(s+m)\Gamma(s-m)}\left[-2 - \frac{s+m-1}{s-m} - \frac{s-m-1}{s+m}\right] \\
&= \frac{(-1)^m\Gamma(2s-1)}{\Gamma(s+m+1)\Gamma(s+m-1)}\left[-4s^2+2s\right] \\
&= \frac{(-1)^{m+1}\Gamma(2s+1)}{\Gamma(1+s+m)\Gamma(1+s-m)} \\
&= K_s(m), \qquad (18)
\end{aligned}$$

by (17). Combining (16) with (18) yields the desired result.

ii. This result follows from the the application of Lemma 9.2, from [35], to (10).

iii. Following the ideas from *i.*, we can write

$$(-\Delta)^s u_n = P_1 + P_2,$$

where

$$P_1 := (-u_{n-1} + 2u_n - u_{n+1})K_s(1) \quad \text{and} \quad P_2 := \sum_{m\in\mathbb{Z};\, m\neq 0,1} (u_n - u_{n-m})K_s(m).$$

We obtain the desired result if we show $K_s(1) \to 1$ and $P_2 \to 0$, as $s \to 1^+$. To that end, we have

$$\lim_{s\to 1^+} K_s(1) = \lim_{s\to 1^+} \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|}\cdot\frac{\Gamma(1-s)}{\Gamma(2+s)} = \frac{4\Gamma(3/2)}{\sqrt{\pi}\Gamma(3)} = 1.$$

Now, by the assumption that u is bounded—that is, $\|u\|_{\ell^\infty} < \infty$, where $\|\cdot\|_{\ell^\infty}$ is the norm on $\ell^\infty(\mathbb{Z})$—we have

$$\|P_2\|_{\ell^\infty} \leq 2\|u\|_{\ell^\infty} \sum_{m\in\mathbb{Z};\, m\neq 0,1} K_s(m) = 2\|u\|_{\ell^\infty}\left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} - 1\right].$$

Since $\lim_{s\to 1^+} 4^s\pi^{-1/2}\Gamma(1/2+s)/\Gamma(1+s) = 1$, we have

$$\|P_2\|_{\ell^\infty} \to 0, \quad \text{as } s \to 1^+,$$

which is the desired result.

iv. We begin by recalling that the discrete biharmonic operator is given by

$$(-\Delta)^2 u_n = u_{n-2} - 4u_{n-1} + 6u_n - 4u_{n+1} + u_{n+2}. \tag{19}$$

Just as before, by symmetry we can write

$$(-\Delta)^s u_n = S_1 + S_2 + S_3,$$

where

$$S_1 := (-u_{n-1} + 2u_n - u_{n+1})K_s(1), \quad S_2 := (-u_{n-2} + 2u_n - u_{n+2})K_s(2),$$

and

$$S_3 := \sum_{m\in\mathbb{Z};\, m\neq 0,1,2} (u_n - u_{n-m})K_s(m).$$

Similar to before, we show that $K_s(1) \to 4$, $K_s(2) \to -1$, and $S_3 \to 0$, as $s \to 2^-$. Similar to before, we have

$$\lim_{s\to 2^-} K_s(1) = \lim_{s\to 2^-} \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|}\cdot\frac{\Gamma(1-s)}{\Gamma(2+s)} = \frac{32\Gamma(5/2)}{\sqrt{\pi}\Gamma(4)} = 4$$

and

$$\lim_{s\to 2^-} K_s(2) = \lim_{s\to 2^-} \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|}\cdot\frac{\Gamma(2-s)}{\Gamma(3+s)} = \frac{-16\Gamma(5/2)}{\sqrt{\pi}\Gamma(5)} = -1.$$

Once again, by the assumption that u is bounded, we have

$$\|S_3\|_{\ell^\infty} \leq 2\|u\|_{\ell^\infty} \sum_{m\in\mathbb{Z};\, m\neq 0,1,2} K_s(m) = 2\|u\|_{\ell^\infty}\left[\frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1+s)} - 4 + 1\right].$$

Since $\lim_{s\to 2^-} 4^s\pi^{-1/2}\Gamma(1/2+s)/\Gamma(1+s) = 3$, we have

$$\|S_3\|_{\ell^\infty} \to 0, \quad \text{as } s \to 2^-,$$

which is the desired result. □

Thus, we may use (9) and (10) for numerical approximations of the random discrete fractional Schrödinger operator given in (1).

I will add descriptions for the weight plots...

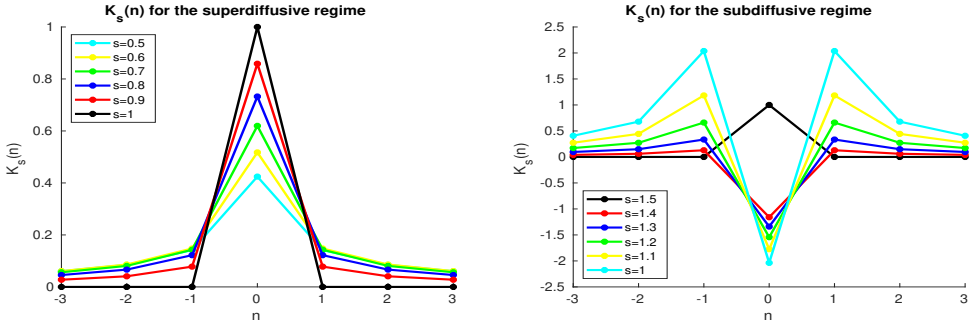

Figure 1

2.2. *Physical Interpretation*

A physical interpretation of the discrete fractional Laplacian has been provided in [35]. We include that description for completeness and expand it to our newly developed situation. Let u be a discrete harmonic function on $\mathbb{Z}$, that is, $-\Delta u = 0$. Then u also satisfies the following discrete mean value property:

$$u_n = \frac{1}{2}u_{n-1} + \frac{1}{2}u_{n+1}. \tag{20}$$

This classical result provides the physical intuition that is well understood for the classical discrete Laplacian in one-dimension. That is, by (20), we see that a discrete harmonic function represents a physical situation in which a particle will jump to either of the two adjacent nodes with equal probability, namely, one-half. This intuition may be generalized to the current situation to provide a physical interpretation of the fractional Laplacian.

Assume that u is a fractional discrete harmonic function, that is, $(-\Delta)^s u = 0$. Then by (9) we have that u satisfies the following discrete fractional mean value property:

$$u_n = \sum_{m\in\mathbb{Z}, m\neq n} u_m P_s(n-m), \tag{21}$$

where

$$P_s(m) := \frac{1}{A_s}K_s(m), \tag{22}$$

$A_s := \sum_{m\in\mathbb{Z}} K_s(m)$, and $P_s(0) = 0$. In this representation, $P_s(m)$ is a probability distribution on $\mathbb{Z}$, allowing one to interpret the fractional case in a similar fashion to the classical result given by (20). That is, by (21), it follows that a fractional discrete harmonic function describes a particle which may jump to *any* point in $\mathbb{Z}$ and the probability that the particle jumps from point n to point m is given by $P_s(n-m)$.

> **Eva, it is quite possible (and, in fact, very likely) that the fractional power and the disorder are related. However, from a modeling perspective, it is very difficult to construct such a model. We are using a drastically simplified model as the first step, in the right direction, to constructing a physically accurate model.**

If $s \in (0,1)$, Theorem 1 implies that the probability of jumping from point n to point m is proportional to $|n-m|^{-(1+2s)}$. In this situation, as $s \to 1^-$, the probability of jumping from n to

an adjacent point tends to one, while the probability of jumping to a nonadjacent point tends to zero. Further, as $s \to 0^+$, the probability of jumping from the point n to any point in $\mathbb{Z}$ tends to zero, resulting in no jumps.

Lorin, I suppose that this is technically localization, but really it is due the action of the operator. We are studying the outcome of the action

$$(-A)^s u_n,$$

and when $s = 0$, we have

$$(-A)^0 u_n = I u_n = u_n,$$

so there is no movement. This is why there is no jumping. I am not sure I would still call this localization, as we are no longer even considering a diffusive process, though. However, that would be more of a physicist question, I think.

Theorem 2 allows for a similar physical interpretation to hold for $s \in (1, 2)$. The probability of jumping from a point n to a point m is still proportional to $|n - m|^{-(1+2s)}$. As before, when $s \to 1^+$, the probability of jumping from n to an adjacent point tends to one, while the probability of jumping to a nonadjacent point tends to zero. However, as $s \to 2^-$, we have that the probability of jumping from n to an adjacent point is two-thirds, and the probability of jumping from n to the *next* adjacent point is one-third.

I still need to add the description of the relationship between the fractional power s and the mean-square displacement...

3. Spectral Approach to Transport in Disordered Systems

We have successfully employed the spectral approach in numerous settings and have demonstrated it to be quite effective in the study of localization behavior – see, for instance, [19, 22–25]. In this section we provide a brief overview of the spectral method as it is employed in [19].

3.1. Overview of Spectral Approach

Should I remove this subsection heading?

We present a heavily streamlined version of the spectral approach developed in [19]. On $\ell^2(\mathbb{Z})$ – the space of two-sided square-summable sequences – we consider the random discrete fractional Schrödinger operator defined in (1). That is, we consider

$$H_{s,\epsilon} := (-\Delta)^s + \sum_{i \in \mathbb{Z}} \epsilon_i \langle \cdot, \delta_i \rangle \delta_i$$

where $\langle \cdot, \cdot \rangle$ denotes the $\ell^2(\mathbb{Z})$ inner product, δ_i is the standard basis of $\mathbb{Z}$, and ϵ_i are random variables taken to be i.i.d. according to the uniform distribution on $[-W/2, W/2]$. We are most interested in the change in localization behavior as we vary the diffusion parameter s.

Lorin, "clos span" means to take the span of the set of vectors and then take the closure of that set (include its limit points).

A vector φ is cyclic for a (bounded) operator T on a separable Hilbert space X, if the forward orbit of φ under T has dense linear span (*i.e.*, $X = \operatorname{clos}\operatorname{span}\{T^n\varphi : n \in \mathbb{N} \cup \{0\}\}$). The central result (see Corollary 3.2 of [19]) behind the spectral method can be formulated as follows:

> *If we can find a (non-trivial) vector that is not cyclic for $(H_{s,\epsilon})$ with positive probability, then almost surely there are scattering states.*

As a side note, we mention that transport and strong delocalization then follow by the RAGE theorem, see *e.g.*, Section 1.2 of [41].

The non-cyclicity of a vector follows if its forward orbit stays away from a particular direction. In other words, if we can find a vector v that remains at a positive distance from $\operatorname{span}\{H_{s,\epsilon}^k\varphi : k \in \mathbb{N}, k \leq n\}$ as $n \to \infty$, then the vector is non-cyclic. With some linear algebra (see Proposition 3.1 of [19]) this distance can be expressed explicitly by

$$D_{s,\epsilon}^n := \sqrt{1 - \sum_{k=0}^{n} \frac{\langle v, m_k\rangle}{\langle m_k, m_k\rangle}},$$

where $\{m_k\}$ is the orthogonal sequence of $\ell^2(\mathbb{Z})$ vectors obtained from applying the Gram–Schmidt algorithm to $\{\varphi, H_{s,\epsilon}\varphi, H_{s,\epsilon}^2\varphi, \ldots\}$.

In the current study, we choose a v that is a linear combination of basis vectors, in order to account for the nonlocality of the action of $H_{s,\epsilon}$. However, it is worth noting that spectral theory allows for any $v \in \ell^2(\mathbb{Z})$ to be an appropriate choice. With this choice, we investigate the dependence of (de)localization on the diffusion parameter s. To emphasize this concern, we will write $D_{s,\epsilon}^n$. Summarizing the theory, our numerical investigations employ the following tool:

$$\lim_{n\to\infty} D_{s,\epsilon}^n \neq 0 \qquad \Rightarrow \qquad \text{delocalization.}$$

We also note that the operator $H_{s,\epsilon}$ is self-adjoint in H. This follows immediately from the spectral theorem and the fact that the discrete Laplacian in self-adjoint in H. This fact allows us to apply the efficient computational techniques outlined in [19]. We omit these details for brevity, but use this fact in the numerical experiments in the ensuing section.

4. Numerical Experiments

This section outlines the numerical method and provides numerical simulations which verify the proposed approach. In Section 4.1, the formulation of the fractional Laplacian given in Theorems 1 and 2 is used to justify the numerical method. Section 4.2 then demonstrates the application of these techniques to one-dimensional disordered systems, characterized by various parameter choices. We discuss possible limitations to our computational exploration in Section 4.3. Finally, an orthogonality check via the Lanczos algorithm is outlined and performed in Section 4.4.

It is worth noting that the experiments presented do not provide rigorous justification for the occurrence of localized or extended states. Rather, they provide a qualitative analysis of the transport behavior of the modeled systems as the fractional power s is varied. Rigorous justifications and studies will be provided in forthcoming works.

4.1. Motivation of Computational Method

As outlined in Section 3.1, the spectral approach employed in this study requires the examination of the forward orbit of arbitrarily chosen initial vectors under the action of the random discrete fractional Schrödinger operator. Thus, our computational method must accurately and efficiently apply the operator given by (1). The case $s = 1$ is relatively straightforward to implement due to the definition of the discrete Laplacian given in (6). This definition will be used as a special case of our current approach and has been considered in more detail in [19, 24, 25].

We consider the one-dimensional lattice $\mathbb{Z}$ and an arbitrary function $u : \mathbb{Z} \to \mathbb{R}$. Then by (9), we have

$$(-\Delta)^s u_n = \sum_{m\in\mathbb{Z};\, m\neq n} (u_n - u_m) K_s(n-m),$$

for $s \in (0, 2)$ and the kernel K_s given by (10). Due to the symmetry of the kernel, we may rewrite the above and obtain

$$(-\Delta)^s u_n = \sum_{m\in\mathbb{N}} (2u_n - u_{n-m} - u_{n+m}) K_s(m) \tag{23}$$

Lorin, the symbol $\mathbb{N}$ represents the natural numbers: $\mathbb{N} = \{1, 2, ...\}$.

for $s \in (0, 2)$. Thus, for some $1 \ll M \in \mathbb{N}$, we have

$$(-\Delta)^s u_n = \sum_{m=1}^{M} (2u_n - u_{n-m} - u_{n+m}) K_s(m) + R_M(u_n), \tag{24}$$

Lorin, the remainder arises from a simple rearrangement. Remainder may be a poor choice of terminology because it does not arise from any particular calculations. Due to the decay of the weights as $n \to \infty$, I simply choose a number of terms to "keep," in this case M terms, and then we want to justify disregarding any terms after that (hence, the study of the remainder).

where

$$R_M(u_n) := \sum_{m=M+1}^{\infty} (2u_n - u_{n-m} - u_{n+m}) K_s(m). \tag{25}$$

For simplicity, our computational method will disregard the remainder term, $R_M(u_n)$. A similar truncation was employed for the simulations in [21, 35], as well. The remainder is guaranteed to be bounded and well-controlled by our choice of M, due to (11). In fact, we have

$$\begin{aligned}
|R_M(u_n)| &\leq 4 \max_{m>M} |u_m| \int_M^\infty K_s(x)\, dx \\
&= B_s \int_M^\infty \frac{\Gamma(x-s)}{\Gamma(x+1+s)}\, dx \\
&= \tilde{B}_s \int_M^\infty \int_0^\infty e^{-(x-s)y} (1-e^{-y})^{2s}\, dy\, dx \\
&\leq \tilde{B}_s \int_M^\infty \int_0^\infty e^{-(x-s)y}\, dy\, dx \\
&= \frac{\tilde{B}_s}{(M-s)^2},
\end{aligned}$$

where

$$B_s := 4 \max_{m>M} |u_m| \cdot \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|} \quad \text{and} \quad \tilde{B}_s := \frac{B_s}{\Gamma(1+2s)},$$

and we have employed the property

$$\frac{\Gamma(x-s)}{\Gamma(x+1+s)} = \frac{1}{\Gamma(1+2s)} \int_0^\infty e^{-(x-s)y}(1-e^{-y})^{2s}\, dy, \tag{26}$$

which is valid for $x - s > 0$ [42]. Moreover, we have that

$$\max_{s\in(0,2)} \tilde{B}_s \approx 1.27324 \times \max_{m>M} |u_m|,$$

occurring when $s = 1/2, 3/2$, which yields

$$R_M(u_m) \sim \frac{1}{M^2}. \tag{27}$$

I thought the previous bit may clarify the type of "control" we have over this remainder...

Thus, dropping the term $R_M(u_n)$ in (24) appears reasonable. By doing so, we obtain

$$H_{s,\epsilon} u_n \approx \sum_{m=1}^{M} (2u_n - u_{n-m} - u_{n+m}) K_s(m) + \epsilon_n u_n, \tag{28}$$

where the ϵ_n are i.i.d. according to the uniform distribution on $[-W/2, W/2]$, for some fixed $W > 0$. The approximation given by (28) is employed in the following computations. Since the goal of the current project is to explore the behavior of (1), we leave detailed error and convergence analysis of our approximation given by (28) for future work.

4.2. Numerical Simulations in One-Dimensional Disordered System

I will wrap the figures with valid descriptions, I mostly wanted everyone to provide input on the figures and whether I should make other runs...

One thing that I will obviously add is that the slight difference in the graphs are due to differences in the realizations of the random perturbations...

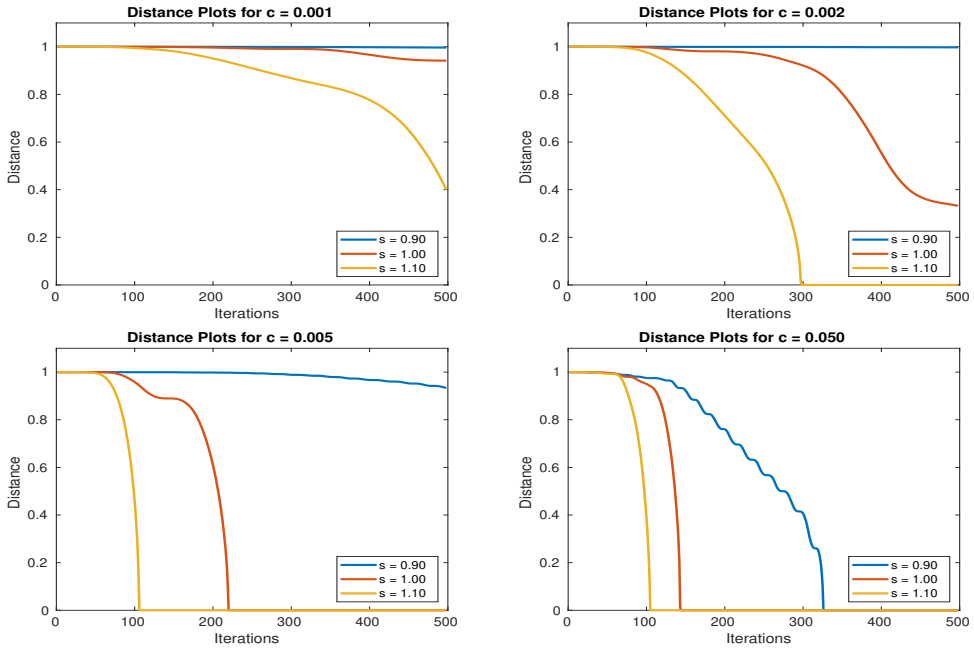

Figure 2

4.3. Interpretations and Limitations

This will include a brief discussion on the "influence" issues that we have discussed in great length...

4.4. Orthogonality Check

I will briefly include a description of the Lanczos algorithm here. There will also be a table depicting the various outputs of the algorithm (which are all outputs very close to zero–on the scale of 10^{13} to 10^{14}...

5. Conclusions and Future Work

I will update this later...

5.1. Connection with Lie Algebraic Formulations and Quantum Mechanics

We haven't discussed any ideas in this direction, so we should maybe remove this section...

5.2. Application to Strongly Coupled Coulomb Systems

This needs double checking...

Strongly correlated systems (SCSs) are systems that cannot be effectively described by the physics of free-particle ensembles. Instead, these systems exhibit anomalous (often technologically useful) collective behavior, which results from strong interactions between the involved

entities [43, 44]. Strong correlations characterize important observed phenomena in the fields of particle physics (hadron-meson interaction and quark-gluon plasma formation), materials science (electrical resistance of metallic magnets and high-temperature superconductivity), and biology (muscle cell excitation and protein folding) [45–48]. It has been recognized that the development of a comprehensive theoretical model for strong correlations which scales appropriately across various size and time scales is one of the most difficult and yet fundamental problems in modern physics [43, 47, 49]. Of particular interest to both the science community and industry is the dynamical behavior of unconventional superconductors such as heavy-fermion compounds [50], organic superconductors [51], and iron cuprates [52]. Despite the remarkable advancements in the experimental characterization of such materials, the theoretical explanation of unconventional superconductivity has proven challenging. One reason is that the experimentally studied strongly correlated materials are generally very complex, multi-layered crystals (for instance, BSCCO), which makes numerical modeling difficult and the growth of high quality crystalline samples expensive. Such difficulties have motivated the development of simplified SCSs, which exhibit analogous interactions in a controlled environment. One possible approach is the use of strongly correlated ultracold atoms (fermions) trapped in optical lattices, which have been shown to exhibit anomalous diffusion effects [53–55]. An important disadvantage of cold-atom systems is the effective lack of lattice disorder, which plays an important role in the transport properties of most crystals [56]. Another solution is offered by the fast-growing field of complex (dusty) plasmas, where the interparticle interaction is often characterized by a Yukawa (or shielded-Coulomb) potential. Superdiffusion has been investigated numerically in two-dimensional [57–59] and quasi-two-dimensional [60] Yukawa liquids. Experimentally, deviations from classical diffusion have been observed in driven-dissipative two-dimensional dusty plasma [61].

Besides the need for optimization of simplified crystal analogues, the theory of superconductivity (and, in general, strong correlations) lacks a sound mathematical formulation that scales appropriately for the variety of SCSs found in nature. The implementation of the fractional Laplacian model for anomalous diffusion due to strong correlations is an important step in the development of such theory. The focus of our future work will be optimization and generalizations of the numerical techniques presented herein. The numerical predictions will be experimentally verified using a SCSs analogue system and then compared to experimental studies of superconductor materials.

5. **It would be nice if we could break this first paragraph in this section up since it is a little long, but this is in no way necessary.**

6. **I will need the physics crowd to proofread the above, as this is a bit outside of my expertise...**

Acknowledgments

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