Problem Set 3.3　吴雨涵　医01班　2020012288

1. $A=\begin{bmatrix}2&4&6&4\\2&5&7&6\\2&3&5&2\end{bmatrix}$　$b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}4\\3\\5\end{bmatrix}$

(1) $[A\ b]=\left[\begin{array}{cccc|c}2&4&6&4&4\\2&5&7&6&3\\2&3&5&2&5\end{array}\right]\xrightarrow[(3)-(1)]{(2)-(1)}\left[\begin{array}{cccc|c}2&4&6&4&4\\0&1&1&2&-1\\0&-1&-1&-2&1\end{array}\right]\xrightarrow{(3)+(2)}\left[\begin{array}{cccc|c}2&4&6&4&4\\0&1&1&2&-1\\0&0&0&0&0\end{array}\right]$

(2) $b_3+b_2-2b_1=0$

(3) ① A的列空间是A的主元列 $\begin{bmatrix}2\\2\\2\end{bmatrix}\begin{bmatrix}4\\5\\3\end{bmatrix}$ 线性组合成的平面;

② A的列空间包括所有向量 $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ 满足 $b_3+b_2-2b_1=0$

(4) 进一步消元为R: $\xrightarrow{(1)-(2)\times 4}\left[\begin{array}{cccc|c}2&0&2&-4&8\\0&1&1&2&-1\\0&0&0&0&0\end{array}\right]\xrightarrow{(1)\times\frac{1}{2}}\left[\begin{array}{cccc|c}1&0&1&-2&4\\0&1&1&2&-1\\0&0&0&0&0\end{array}\right]$

$\therefore x=x_3\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$.

(5) 由(4)中，$x_1=-x_3+2x_4+4$

$x_2=-x_3-2x_4-1$

$\therefore x_p=\begin{bmatrix}4\\-1\\0\\0\end{bmatrix}$

(6) ① $Ax=(0,6,-6)$

$\left[\begin{array}{cccc|c}2&4&6&4&0\\2&5&7&6&6\\2&3&5&2&-6\end{array}\right]\xrightarrow[(3)-(1)]{(2)-(1)}\left[\begin{array}{cccc|c}2&4&6&4&0\\0&1&1&2&6\\0&-1&-1&-2&-6\end{array}\right]\xrightarrow{(3)+(2)}\left[\begin{array}{cccc|c}2&4&6&4&0\\0&1&1&2&6\\0&0&0&0&0\end{array}\right]\xrightarrow{(1)-(2)\times 4}\left[\begin{array}{cccc|c}2&0&2&-4&-24\\0&1&1&2&6\\0&0&0&0&0\end{array}\right]$

$\xrightarrow{(1)\times\frac{1}{2}}\left[\begin{array}{cccc|c}1&0&1&-2&-12\\0&1&1&2&6\\0&0&0&0&0\end{array}\right]$

$\therefore x=\begin{bmatrix}-12\\6\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$

② $Ax=(4,3,5)$

由(4)(5)结果知 $x=\begin{bmatrix}4\\-1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}2\\-2\\0\\1\end{bmatrix}$

4. $\begin{bmatrix}1&3&1&2\\2&\cancel{4}6&4&8\\0&0&2&4\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}$

$\left[\begin{array}{cccc|c}1&3&1&2&1\\2&6&4&8&3\\0&0&2&4&1\end{array}\right]\xrightarrow[(3)-(2)]{(2)-2\times(1)}\left[\begin{array}{cccc|c}1&3&1&2&1\\0&0&2&4&1\\0&0&0&0&0\end{array}\right]\xrightarrow{(1)-(2)\times\frac{1}{2}}\left[\begin{array}{cccc|c}1&3&0&0&\frac{1}{2}\\0&0&2&4&1\\0&0&0&0&0\end{array}\right]\xrightarrow{(2)\times\frac{1}{2}}\left[\begin{array}{cccc|c}1&3&0&0&\frac{1}{2}\\0&0&1&2&\frac{1}{2}\\0&0&0&0&0\end{array}\right]$

$$\therefore x = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

9\. (a) $L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & -1 & 1 \end{bmatrix}$

$$LU = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & 2 & 8 & 12 \\ 3 & -2 & 7 & 13 \end{bmatrix} = A$$

$$Lc = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix} = b$$

(b) $-9 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix} = b$

因为 x_p 是 $Ax=b$ 的特解. 故 $Ax_p = [c_1 \ c_2 \ c_3 \ c_4] \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} = -9c_1 + 3c_3 = b$

其中 c_1、c_3 分别为A的2个主元列.

18\. $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \xrightarrow[(3)+(1)]{(2)-(1)\times 2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{bmatrix} \xrightarrow{(3)-(2)\times 2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore r(A) = 2$

$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 11 & 2 \\ 0 & 5 & 10 \end{bmatrix} \xrightarrow{(2)-(1)\times 4} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix} \xrightarrow{(3)-\frac{5}{3}(2)} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$r(A^T) = 2$

$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \xrightarrow[(3)-(1)]{(2)-(1)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \xrightarrow{(3)-(2)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

若 $q=2$ 则 $r(A) = 2$

$q \neq 2$ 则 $r(A) = 3$

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & q \end{bmatrix} \xrightarrow{(3)-(1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \xrightarrow{(3)-(2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

结果同上

24\. (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

30\. $Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \xrightarrow[(3)-2(1)]{(2)-(1)} \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{(2)+(3)} \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \xrightarrow[(3)\times\frac{1}{3}]{(2)\times\frac{1}{3}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$ $\quad \begin{cases} x_1 + 3x_4 = 2 \\ x_2 = 3 \\ x_4 = 2 \end{cases}$

$x_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

34. (a) $r(A) = 3$ $\quad x_n = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

(b) $R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) 因为 $C(A) \in R^3$，而A中有4个列向量，$\operatorname{rank} A = 3 < 4 = n$ A 不是列满秩，但 A中的列向量已充满 R^3，b一定在A的列空间中，故 $Ax=b$ 一定有解

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6. $U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ column 124, 134, 234 $\quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}$ 5U 相同

11. (a) a line $\quad$ (c) R^3

(b) a plane $\quad$ (d) R^3

16. (a) (1, 1, 1, 1)

(b) (1 −1 0 0)
(0 1 −1 0)
(0 0 1 −1)

(c) (2 −2 −1 −1)
(2 −2 −2 0)

(d) column space: (1 0 0 0)
(0 1 0 0)
(0 0 1 0)
(0 0 0 1)

nullspace $\{0\}$ 即 Z

25. $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$ $\begin{cases} c = 0 \\ d = 2 \end{cases}$ $\quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ $c \neq \pm d$ 即可

29. (a)

(b)

$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) 对角线矩阵 $D = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$ 其中 $d_1 = d_2 = d_3$