习题4.3

4.(1) $y=1+\frac{2x}{1-x+x^2}$

$=1+2x+\frac{-2x^3+2x^2}{1-x+x^2}$

$=1+2x+2x^2+\frac{-2x^4}{1-x+x^2}$

$=1+2x+2x^2-2x^4+\frac{-2x^5+2x^6}{1-x+x^2}$

$=1+2x+2x^2-2x^4+o(x^4)$

上下次数最小的拿出来

4.(6) $y=\frac{x-2}{x^2-4x}=\frac{x-2}{(x-2)^2-4}$

令 $x-2=t$. 则 $y=\frac{t}{t^2-4}$

$=-\frac{t}{4}+\frac{t^3/4}{t^2-4}$

$=-\frac{t}{4}-\frac{t^3}{16}+\frac{t^5/16}{t^2-4}$

$=-\frac{t}{4}-\frac{t^3}{16}-\frac{t^5}{64}\cdots-\frac{t^{2n-1}}{4^n}+o(t^{2n})$

$\therefore y=-\frac{x-2}{4}-\frac{(x-2)^3}{16}-\cdots-\frac{(x-2)^{2n-1}}{4^n}+$
$o((x-2)^{2n})$

(8) $y=\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+$
$(-1)^{n-1}\cdot\frac{x^n}{n}+o(x^n)$

$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$
$-\frac{x^n}{n}+o(x^n)$

$\therefore y=\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)$

$=\frac{1}{2}(2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots+\frac{(-1)^{n-1}+1}{n}x^n)$
$+o(x^n)$

$=x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\frac{[(-1)^{n-1}+1]\cdot x^n}{2n}+o(x^n)$

(9) $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+o(x^n)$

$-\frac{1}{x^2}$ 在 $x=0$ 处无展开

$\lim\limits_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x^n}=\lim\limits_{x\to0}\frac{\frac{1}{x^n}}{e^{\frac{1}{x^2}}}=\lim\limits_{x\to0}\frac{-n\cdot\frac{1}{x^{n+1}}}{-2\cdot\frac{1}{x^3}\cdot e^{\frac{1}{x^2}}}$

$=\frac{n}{2}\cdot\lim\limits_{x\to0}\frac{\frac{1}{x^{n-2}}}{\frac{1}{e^{x^2}}}=\frac{n}{2}\cdot\frac{n-2}{2}\lim\limits_{x\to0}\frac{\frac{1}{x^{n-4}}}{e^{\frac{1}{x^2}}}$

$=\cdots\cdots=0$. 故 $e^{-\frac{1}{x^2}}=o(x^n)$

$\therefore y=o(x^n)$ 订正在后

5.(1) $\ln(1+\frac{1}{x})=\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})$

$\lim\limits_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})]$

$=\lim\limits_{x\to\infty}[x-x^2(\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2}))]$

$=\lim\limits_{x\to\infty}[x-x+\frac{1}{2}+o(1)]$

$=\frac{1}{2}$

(2) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$

$\sin x = x + o(x)$

$\sin^2 x = x^2 + o(x^4)$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4)$

$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$

$$\lim_{x\to 0} \frac{1}{[x^2+o(x^4)]} \cdot \frac{\frac{x^2}{2}+1-1-\frac{1}{2}x^2+\frac{1}{8}x^4+o(x^4)}{(1-\frac{x^2}{2}+\frac{x^4}{24}-1-x^2-\frac{x^4}{2}+o(x^4))}$$

$$=\lim_{x\to 0} \frac{\frac{1}{8}x^4+o(x^4)}{\quad}$$

不一定上下均展到 x^4

$$\lim_{x\to 0} \frac{1}{x^2+o(x^2)} \cdot \frac{\frac{1}{8}x^4+o(x^4)}{-\frac{3}{2}x^2+o(x^2)}$$

$$= \frac{\frac{1}{8}x^4+o(x^4)}{-\frac{3}{2}x^4+o(x^4)} = -\frac{1}{12}$$

7.(2) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^2}{4} + \cdots$

$+ (-1)^{n-1}\frac{x^n}{n} + o(x^n)$

$\ln(1+0.02) = 0.02 - \frac{0.02^2}{2} + \frac{0.02^3}{3} - \cdots$

$= 0.019802626 \approx 0.019803$

四章复习题

9. $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(\xi_1)}{2!}x^2$

$f(1) = f(0) + f'(0)\cdot 1 + \frac{f''(\xi_1)}{2}$

$\therefore f'(0) = -\frac{f''(\xi_1)}{2}$ $\therefore |f'(0)| \le \frac{M}{2}$

同理: $f(x) = f(1) + f'(1)(x-1) + \frac{f''(\xi_2)}{2}(x-1)^2$

$\therefore f'(1) = -\frac{f''(\xi_2)}{2}$, $|f'(1)| \le \frac{M}{2}$

$\forall x_0 \in [0,1]$

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$

$f(0) = f(x_0) - f'(x_0)\cdot x_0 + f''(\xi_3)\cdot\frac{x_0^2}{2}$

$f(1) = f(x_0) + f'(x_0)(1-x_0) + f''(\xi_4)\cdot\frac{(x_0-1)^2}{2}$

$-f'(x_0)\cdot x_0 + f''(\xi_3)\cdot\frac{x_0^2}{2} = f'(x_0)(1-x_0) + f''(\xi_4)\cdot\frac{(x_0-1)^2}{2}$

$\therefore f'(x_0) = f''(\xi_3)\cdot\frac{x_0^2}{2} - f''(\xi_4)\cdot\frac{(x_0-1)^2}{2}$

$\therefore |f'(x_0)| \le |f''(\xi_3)|\cdot\frac{x_0^2}{2} + |f''(\xi_4)|\cdot\frac{(x_0-1)^2}{2}$

$\le M\cdot\left[\frac{x_0^2+(x_0-1)^2}{2}\right] \le \frac{M}{2}$

$x_0 \in [0,1]$

证毕

11. 设 $f(x_0) = -1$.

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$

$f(1) = f(x_0) + f'(x_0)(1-x_0) + \frac{f''(\xi_1)}{2}(1-x_0)^2$

$f(0) = f(x_0) + f'(x_0)(-x_0) + \frac{f''(\xi_2)}{2}(x_0)^2$

$\therefore f'(x_0) = f''(\xi_2)\frac{x_0^2}{2} - f''(\xi_1)\cdot\frac{(1-x_0)^2}{2}$

$\therefore f''(\xi_2)x_0^2 = f''(\xi_1)(1-x_0)^2$

又 $0 = -1 + \frac{f''(\xi_1)}{2}(1-x_0^2)$

$0 = -1 + \frac{f''(\xi_2)}{2}(x_0)^2$

$\therefore f''(\xi_1) = \frac{2}{(1-x_0)^2} \quad f''(\xi_2) = \frac{2}{(x_0)^2}$

$\frac{f''(\xi_1) + f''(\xi_2)}{2} = \frac{1}{(1-x_0)^2} + \frac{1}{(x_0)^2} \geq 8$

由导函数达布定理.

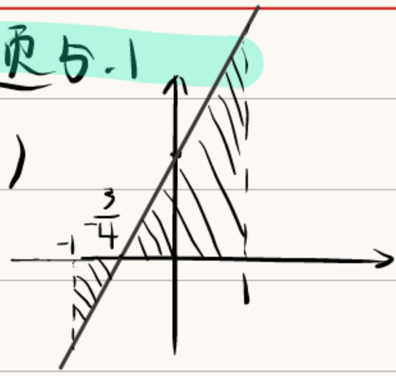
$\exists\, \xi \in (\xi_2, \xi_1)$ s.t.

$f''(\xi) = \frac{f(\xi_1) + f''(\xi_2)}{2} \geq 8$

证毕

习题5.1

(3)

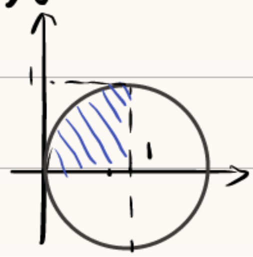


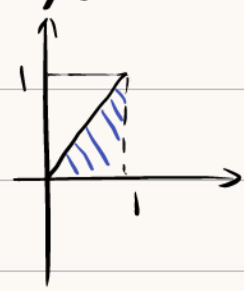
$\int_{-1}^{1}(3+4x)dx = \frac{7}{4} \times 7/2 - \frac{1}{4}\cdot 1/2$

$= 6$

(4) $\int_0^1 (\sqrt{2x-x^2} - x)dx$

$= \int_0^1 (\sqrt{2x-x^2})dx - \int_0^1 x\,dx$





$\therefore$ 原式 $= \frac{\pi}{4} - \frac{1}{2}$

订正：4.3的4.(9)

记 $g(x)=e^{-\frac{1}{x^2}}$

当 $x\neq0$ 时，$f'(x)=\frac{2}{x^3}\cdot e^{-\frac{1}{x^2}}$

但 $g(x)$ 在 $x=0$ 处不连续，故

$f'(0)=\lim\limits_{x\to0}\frac{f(x)}{x}=\lim\limits_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x}$ 令 $\frac{1}{x}=t$

$(t\to\infty)$，则 $f'(0)=\lim\limits_{t\to\infty}\frac{t}{e^{t^2}}=0$

以下断言，$x\neq0$ 时，

$f^{(n)}(x)=P_n(\frac{1}{x})\cdot e^{-\frac{1}{x^2}}$，其中 P_n 为

多项式。可知，$n=1$ 时，断

言成立，假设 $n=k$ 时成立，

$f^{(k+1)}(x)=P_n'(\frac{1}{x})\cdot(-\frac{1}{x^2})e^{-\frac{1}{x^2}}$

$+P_n(\frac{1}{x})\cdot\frac{2}{x^3}\ e^{-\frac{1}{x^2}}$

取 $P_{n+1}(u)=2u^3P_n(u)-u^2P_n'(u)$

即可。

$f^{(2)}(0)=\lim\limits_{x\to0}\frac{f^{(1)}(x)-f^{(1)}(0)}{x-0}=0$

……

$f^{(n)}(0)=\lim\limits_{x\to0}\frac{f^{(n-1)}(x)-f^{(n-1)}(0)}{x-0}=\lim\limits_{x\to0}\frac{\frac{1}{x}P_{(n-1)}(\frac{1}{x})}{e^{\frac{1}{x^2}}}=0$

$\therefore f(x)$ 的泰勒展开为

$f(x)=o(x^n)$

补证 $\lim\limits_{t\to\infty}\frac{t^n}{e^{t^2}}=0$

令原式 $=M$. $M\le\lim\limits_{t\to+\infty}\frac{t^n}{e^{t^2}}$

记右者为 N.

$\ln N=n\ln t-t^2$. $t>n$ 时，

$\ln N<n(\ln t-t)=n(-t-\ln\frac{1}{t})$

$\lim\limits_{t\to+\infty}-t-\ln\frac{1}{t}=-\lim\limits_{t\to+\infty}(t+\ln\frac{1}{t})$

$=-\lim\limits_{t\to+\infty}(\ln e^t\frac{1}{t})=-\infty$

$\because\ln N\to-\infty\quad\therefore N\to0$

又 $0<M<N\quad\therefore\lim\limits_{t\to\infty}M=0$

定理 5.1.4 改正在后

$f(x)$在$[a,b]$上有界,且只有有限个不连续点.则 $f\in R[a,b]$.

假设f在$[a,b]$上仅一个间断点 $x=b$.则f在$[a,b]$上有界.
$|f(x)|\le M$ ($a\le x\le b$).取
$c\in(a,b)$ s.t. $b-c<\frac{\varepsilon}{4M}$
则 $f\in C[a,c]$. 则 $f\in R[a,c]$
则$\exists[a,c]$的一个分割:
$T_1: a=x_0<x_1<x_2\cdots<x_n=c$. s.t.
$U(f,T_1)-L(f,T_1)<\frac{\varepsilon}{2}$
现考虑$[a,b]$的分割T_2:
$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=c<b$.
$\sup\{f(x)\mid c\le x\le b\}-\inf\{f(x)\mid c\le x\le b\}$
$\le 2M$ 于是
$0\le U(f,T_2)-L(f,T_2)$
$\le[U(f,T_1)-L(f,T_1)]+2M\cdot\frac{\varepsilon}{4M}$
$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$
$\therefore f\in[a,b]$

即当$f(x)$有有限个间断点时
记为 $b_1,b_2\cdots b_n$.则
$f(x)$在$[a,b_1],[b_1,b_2]\cdots[b_n,b]$上可积,则 $f(x)$在$[a,b]$上可积

改正见后

习题5.1.2题

由$f(x)\le g(x)$.故$\forall[a,b]$的分割$T: a=x_0<x_1<x_2<\cdots<x_n=b$
$\sup f(x)\le\sup g(x)$ (对于所有 $x_i<x<x_{i+1}$, $i\le n-1$且$i\in N^*$)
同样 $\inf f(x)\le\inf g(x)$ ($\forall x\in(x_i,x_{i+1})$, $i\le n-1$且$i\in N^*$)
由极限保序性.
可知 $T\to0$时, $L(f,T)=U(f,T)$
$L(g,T)=U(g,T)$ 而
$L(f,T)\le L(g,T)$ $U(f,T)\le U(g,T)$
$\therefore\int_a^b f(x)dx\le\int_a^b g(x)dx$

反之并不成立

4. $\forall \varepsilon>0$, $\exists |T|<\delta$,

$U(f,T)-L(f,T)<d^2\varepsilon$

故 $U(\frac{1}{f},T)-L(\frac{1}{f},T)$

$=\sum_{i=1}^{n}\sup_{\eta,\xi\in[x_{i-1},x_i]}\left(\frac{1}{f(\eta)}-\frac{1}{f(\xi)}\right)(x_i-x_{i-1})$

$=\sum_{i=1}^{n}\sup_{\eta,\xi\in[x_{i-1},x_i]}\frac{f(\xi)-f(\eta)}{f(\eta)f(\xi)}(x_i-x_{i-1})\le$

$\frac{1}{d^2}\left(U(f,T)-L(f,T)\right)<\varepsilon$

故得证

(6) $F(x)$ 可积.

$\forall x_1,x_2\in[a,b]$. $x_1\geqslant x_2$.

有 $F(x_1)-F(x_2)=\sup_{x\in[a,x_1]}f(x)-\sup_{[a,x_2]}f(x)\geqslant 0$

$\therefore F(x)$ 单调不减

由定理 5.1.5. $f(x)\in R[a,b]$

改正：定理5.1.4：

不妨设 f 在 $[a,b]$ 上仅有一个间断点 b. f 在 $[a,b]$ 上有界; $|f(x)|\le M\ (a\le x\le b)$

$\forall\varepsilon>0$, 取 $c\in(a,b)$ 且 $b-c<\frac{\varepsilon}{4M+1}$; $f\in C[a,c]$

$\therefore f\in R[a,c]$ $\therefore$ $\exists$分割:

$T_1: a=x_0<x_1<x_2<\cdots<x_n=c$

s.t $U(f,T_1)-L(f,T_1)<\frac{\varepsilon}{2}$

$\exists [a,b]$间分割: $T: a=x_0<x_1<\cdots<x_{n-1}<x_n=c<b$

则 $0\le U(f,T)-L(f,T)=$

$U(f,T_1)-L(f,T_1)+\sup_{x\in[c,b]}f(x)(b-c)-\inf_{x\in[c,b]}f(x)(b-c)\le U(f,T_1)-L(f,T_1)+2M(b-c)$

$<\frac{\varepsilon}{2}+2M\cdot\frac{\varepsilon}{4M+1}<\varepsilon$

$\therefore f\in R[a,b]$

改正：习题5.1.2

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$, $[a,b]$分割 $T: a = x_0 < x_1 < \cdots < x_n = b$.

$|T| < \delta_1$时, $\forall \xi_i \in [x_{i-1}, x_i]$有

$\int_a^b f(x)dx - \varepsilon < \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$

$< \int_a^b f(x)dx + \varepsilon$;

$\exists \delta_2$, $[a,b]$分割 $T: a = x_0 < x_1 < \cdots < x_n = b$, $|T| < \delta_2$时, $\forall \xi_i \in [x_{i-1}, x_i]$

$\int_a^b g(x)dx - \varepsilon < \sum_{i=1}^n g(\xi_i)\cdot(x_i - x_{i-1})$

$< \int_a^b g(x)dx + \varepsilon$

任取$[a,b]$一分割：

$T: a = x_0 < x_1 < \cdots < x_n = b$

$|T| < \min\{\delta_1, \delta_2\}$, 则：

$\int_a^b f(x)dx - \varepsilon < \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$

$\leq \sum_{i=1}^n g(\xi_i)(x_i - x_{i-1}) < \int_a^b g(x)dx + \varepsilon$

$\therefore \int_a^b f(x)dx - \varepsilon < \int_a^b g(x)dx + \varepsilon$

$\int_a^b f(x)dx - \int_a^b g(x)dx < 2\varepsilon$

如果 $\int_a^b f(x)dx - \int_a^b g(x)dx = m > 0$

则与 ε 任意性矛盾

由 ε 任意性：

$\int_a^b f(x)dx \leq \int_a^b g(x)dx$