习题6.1

(2).1,3,7,8

① $\int_1^{+\infty} \frac{1}{x(x+1)} dx$

$\lim\limits_{n\to+\infty} \int_1^N (\frac{1}{x} - \frac{1}{x+1}) dx$

$\lim\limits_{n\to+\infty} (\ln^x |_1^N - \ln^{x+1} |_1^N)$

$= \lim\limits_{N\to+\infty} (\ln^N - \ln^{N+1} + \ln^2)$

$= \lim\limits_{N\to+\infty} \ln \frac{2N}{N+1}$

$= \ln 2$

③ $\int_0^{+\infty} e^{-x} \sin x \, dx$

$= \lim\limits_{N\to+\infty} \int_0^N e^{-x} \sin x \, dx$

$= \lim\limits_{N\to+\infty} \frac{-e^{-x}(\cos x + \sin x)}{2} \Big|_0^N$

$= 0 - \frac{-1\cdot(1+0)}{2} = \frac{1}{2}$

$\int e^{-x} \sin x \, dx$

$= e^{-x} \cdot (-\cos x) - \int \cos x \, e^{-x} dx$

$-\cos x \cdot e^{-x} - (e^{-x} \cdot \sin x + \int \sin x \cdot e^{-x} dx)$

$= \frac{-e^{-x}(\cos x + \sin x)}{2}$

⑦ $\int_0^1 \frac{1}{e^{\frac{1}{x}} x^2} dx$ 瑕点为0.

$= \lim\limits_{\varepsilon\to 0^+} \int_\varepsilon^1 e^{-\frac{1}{x}} \cdot \frac{1}{x^2} dx$ 令 $u = -\frac{1}{x}$

$\int e^{-\frac{1}{x}} \frac{1}{x^2} dx = \int e^u du = e^u + C = e^{-\frac{1}{x}} + C$

$\lim\limits_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{e^{\frac{1}{x}} \cdot x^2} dx = \lim\limits_{\varepsilon\to 0^+} e^{-\frac{1}{x}} \Big|_\varepsilon^1 = \frac{1}{e} - 0 = \frac{1}{e}$

⑧ $\int_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}}$

令 $\ln x = t$. 则 $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

$= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C$

$\therefore$ 原式 $= \lim\limits_{N\to e^-} \int_1^N \frac{dx}{x \cdot \sqrt{1-(\ln x)^2}}$

$= \lim\limits_{N\to e^-} \arcsin(\ln x) \Big|_1^N$

$= \arcsin 1 = \frac{\pi}{2}$

3.(1)(6)

① $\int_1^{+\infty} \frac{1}{x^2\sqrt{1+x^2}} dx$

令 $x = \tan t$. 则 $dx = \frac{1}{\cos^2 t} dt$

$\int \frac{\sec^2 t \, dt}{\tan^2 t \cdot \sec t} = \int \frac{\sec t \, dt}{\tan^2 t}$

$= \int \frac{\cos t}{\sin^2 t} dt = \int \frac{d\sin t}{\sin^2 t}$

$= -\frac{1}{\sin t} = -\frac{\sqrt{x^2+1}}{x}$

原式 $= \lim\limits_{N\to+\infty} (-\frac{\sqrt{x^2+1}}{x}) \Big|_1^N$

$= -1 - (-\frac{\sqrt{2}}{1})$

$= \sqrt{2} - 1$

⑥ 令 $\sqrt{1-x} = t$. 则 $x = 1-t^2$; $dx = -2t\,dt$

$\int \frac{-2t\,dt}{(3-t^2)\,t} = 2\int \frac{dt}{t^2-3}$

$= -2 \cdot (\frac{1}{2\sqrt{3}} \cdot \ln(\frac{\sqrt{3}+t}{\sqrt{3}-t})) + C$

$= -\frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+t}{\sqrt{3}-t} + C$

$= -\frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+\sqrt{1-x}}{\sqrt{3}-\sqrt{1-x}} + C$

$\lim\limits_{n\to 1^-} -\frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+\sqrt{1-x}}{\sqrt{3}-\sqrt{1-x}} \Big|_0^n$

$= -\frac{1}{\sqrt{3}} (\ln^1 - \ln^{\frac{\sqrt{3}+1}{\sqrt{3}-1}}) = \frac{\sqrt{3}}{3} \cdot \ln^{2+\sqrt{3}}$

4.(1)(3)

① $\int_{-2}^{2}\frac{2x}{x^2-4}dx$ 为瑕积分. 瑕点为 $-2, 0, 2$; 拆分为 $\int_{-2}^{-1}\frac{2x}{x^2-4}dx + \int_{-1}^{0}\frac{2x}{x^2-4}dx + \int_{0}^{1}\frac{2x}{x^2-4}dx + \int_{1}^{2}\frac{2x}{x^2-4}dx$

$\int\frac{2x}{x^2-4}dx$. 令 $t=x^2$ 则 $2xdx=dt$

$\therefore \int\frac{dt}{t-4} = \ln|t-4| = \ln|x^2-4|$

$\int_{-2}^{-1}\frac{2x}{x^2-4}dx = \lim\limits_{n\to-2^+}\ln|x^2-4|\Big|_n^{-1}$

$=+\infty$

故 $\int_{-2}^{-1}\frac{2x}{x^2-4}dx$ 不收敛

故 $\int_{-2}^{2}\frac{2x}{x^2-4}dx$ 不收敛

③ $\int_1^{+\infty}\frac{dx}{x\sqrt{x^2-1}}$ 为无穷限积分与瑕积分的混合

$\int\frac{dx}{x\sqrt{x^2-1}}$. 令 $x=\sec t$. 则

$dx=\sec t\cdot\tan t\cdot dt$

$\int\frac{\sec t\cdot\tan t\, dt}{\sec t\cdot\tan t}=t$

$\therefore \int\frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}x$

原式有两个瑕点., 拆为:

$\operatorname{arcsec}x\Big|_1^2 + \operatorname{arcsec}x\Big|_2^{+\infty}$

$=\frac{\pi}{3}-0+\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{2}$

故收敛

习题6.2

4.(2)(6)(10)

② $x\to+\infty$, $\arctan\frac{1}{x}\sim\frac{1}{x}$

$\int_1^{+\infty}\frac{\arctan\frac{1}{x}}{x^2}dx$ 与 $\int_1^{+\infty}\frac{1}{x^3}dx$ 同敛散

而 $\int_1^{+\infty}\frac{dx}{x^3} = \lim\limits_{n\to+\infty}-\frac{1}{2}x^{-2}\Big|_1^n$ 收敛

$\therefore \int_1^{+\infty}\frac{\arctan\frac{1}{x}}{x^2}dx$ 收敛

10. $\int_0^1\frac{\sqrt{x}}{e^{\sin x}-1}dx$ 有一个瑕点 0

$x\to0$ 时, $e^{\sin x}-1\sim\sin x\sim x$

$\therefore \int_0^1\frac{\sqrt{x}}{e^{\sin x}-1}dx$ 与 $\int_0^1\frac{dx}{\sqrt{x}}$ 同敛散

而 $\int_0^1\frac{dx}{\sqrt{x}} = \lim\limits_{n\to0^+}2\sqrt{x}\Big|_n^1$ 收敛

$\therefore \int_0^1\frac{\sqrt{x}}{e^{\sin x}-1}dx$ 收敛

⑥ $\int_0^1\frac{1}{\sqrt{x-x^3}}dx$ 有两个瑕点 0 与 1

$=\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x-x^3}}dx + \int_{\frac{1}{2}}^1\frac{1}{\sqrt{x-x^3}}dx$

对前者 $x\to0$ 时 $\sqrt{x-x^3}\sim\sqrt{x}$

故 $\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x-x^3}}dx$ 与 $\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x}}dx$ 同敛散

$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x}}dx = \lim\limits_{n\to0^+}2\sqrt{x}\Big|_n^{\frac{1}{2}}$ 收敛

故 $\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x-x^3}}dx$ 收敛

$\int_{\frac{1}{2}}^1\frac{1}{\sqrt{x-x^3}}dx$. $\frac{1}{\sqrt{x-x^3}}$ 在 $[\frac{1}{2},1)$ 上恒正

令 $x=t+1$. 则 $dx=dt$

$\int_{-\frac{1}{2}}^0\frac{dt}{\sqrt{(t+1)-(t+1)^3}} = \int_{-\frac{1}{2}}^0\frac{dt}{\sqrt{-t^3-3t^2-2t}}$

$x\to0$ 时, $-t^3-3t^2-2t\sim-2t$

$\therefore$ 原式与 $\int_{-\frac{1}{2}}^0\frac{dt}{\sqrt{-2t}}$ 同敛散

由上方讨论有该式收敛

$\therefore \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{x-x^3}}$ 与 $\int_{\frac{1}{2}}^1\frac{dx}{\sqrt{x-x^3}}$ 均收敛

$\therefore \int_0^1 f(x)dx$ 收敛. $(f(x)=(x-x^3)^{-\frac{1}{2}})$

5. (2)(7)(15)

② $\int_0^{+\infty}\frac{\ln(1+x^2)}{x^p}dx$ 有两个瑕点, 0与$+\infty$

$=\int_0^1 \frac{\ln(1+x^2)}{x^p}dx+\int_1^{+\infty}\frac{\ln(x^2+1)}{x^p}dx$

$x\to0^+$时, $\ln(1+x^2)\sim x^2$

$\int_0^1\frac{\ln(1+x)^2}{x^p}dx$ 与 $\int_0^1\frac{1}{x^{p-2}}dx$ 同敛散.

$p\in[3,+\infty)$时, 发散

$p\in(-\infty,3)$时, 收敛

当$p>1$时, 取 $q=\frac{p+1}{2}$ $\lim_{x\to+\infty}x^q\cdot\frac{\ln(x^2+1)}{x^p}=\lim_{x\to+\infty}\frac{\ln(x^2+1)}{x^{p-q}}$

$=\lim_{x\to+\infty}\frac{2x^2}{(p-q)\cdot x^{p-q}\cdot(1+x^2)}=0$ 且$q>1$.

故 $\int_1^{+\infty}\frac{\ln(1+x^2)}{x^p}dx$ 收敛

$p\le1$时, $\frac{\ln(1+x^2)}{x^p}\geqslant\frac{\ln2}{x^p}$ $(x\geqslant1)$

$\int_1^{+\infty}\frac{\ln2}{x^p}dx$ 发散, 则 $\int_1^{+\infty}\frac{\ln(1+x^2)}{x^p}dx$

也发散. 综上:

$p\in(1,3)\Longleftrightarrow\int_0^{+\infty}\frac{\ln(1+x^2)}{x^p}dx$ 收敛

⑦ $\int_0^{\frac{\pi}{2}}|\ln\sin x|^p dx$ 一个瑕点, $x=0$

在 $x\to0^+$时, $x^\alpha\cdot\ln\sin x=0$

对$\forall\alpha>0$成立.

而 $x=\frac{\pi}{2}$时, $x^\alpha\cdot\ln\sin x=0$. 且该函数连续. 故记 $f(x)_{max}=C$.

有 $x\in(0,\frac{\pi}{2})$时, $\ln\sin x\le\frac{C}{x^\alpha}$

$(0<\alpha)$.

$\int_0^{\frac{\pi}{2}}|\ln\sin x|^pdx\le\int_0^{\frac{\pi}{2}}\frac{C^p}{x^{\alpha p}}dx$

$\forall p>0$, $\exists\alpha>0$使$\alpha\cdot p<1$. 故

$\int_0^{\frac{\pi}{2}}\frac{C^p}{x^{\alpha p}}dx$ 一定收敛

故$\forall p>1$, $\int_0^{\frac{\pi}{2}}|\ln\sin x|^pdx$ 收敛

换言之:

$\lim_{x\to0^+}x^{\frac12}\cdot|\ln\sin x|^p=\lim_{x\to0^+}\left|\frac{\ln\sin x}{x^{-\frac{1}{2p}}}\right|^p$

$=\lim_{x\to0^+}\left|2p\cdot x^{\frac{1}{2p}}\cdot\cos x\cdot\frac{x}{\sin x}\right|^p=0$

而 $\frac12<1$, 故原式收敛

⑮ $\int_0^1 x^{p-1}\cdot(1-x)^{q-1}\ln x\,dx$ 有两个瑕点

$=\int_0^{\frac12}x^{p-1}\cdot(1-x)^{q-1}\cdot\ln x\,dx+\int_{\frac12}^1 x^{p-1}(1-x)^{q-1}\ln x\,dx$

$\lim_{x\to1^-}-(1-x)^{-q}\cdot x^{p-1}(1-x)^{q-1}\cdot\ln x=\lim_{x\to1^-}\frac{-\ln x}{1-x}=\lim_{x\to1^-}\frac{\frac{1}{x}}{+1}=+1$

故 $q>-1\Longleftrightarrow\int_{\frac12}^1(1-x)^{q-1}\ln x\,dx$ 收敛

而 $-x^{p-1}\cdot(1-x)^{q-1}\cdot\ln x\geqslant x^{p-1}\cdot(1-x)^{q-1}\ln2\geqslant0$ $(x\in(0,\frac12])$

$p\le0$时 $\int_0^{\frac12}x^{p-1}(1-x)^{q-1}\ln2\,dx$ 发散

故 $\int_0^{\frac12}-x^{p-1}(1-x)^{q-1}\ln x\,dx$ 发散

$p>0$时, $\lim_{x\to0^+}x^{1-\frac{p}{2}}\cdot(-x^{p-1}\cdot(1-x)^{q-1}\ln x)=\lim_{x\to0^+}\frac{-\ln x}{x^{-\frac{p}{2}}}$

$=\lim_{x\to0^+}\frac{2}{p}\cdot x^{\frac{p}{2}}=0$ 故 $\int_0^{\frac12}x^{p-1}(1-x)^{q-1}\ln x\,dx$ 收敛

故: $p>0, q>-1\Longleftrightarrow\int_0^1 x^{p-1}(1-x)^{q-1}\ln x\,dx$ 收敛

习题 5.7

4.(2)(3) $k=\dfrac{|x'(t)y''(t)-x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^{\frac{3}{2}}}$

②

$x'(t)=a-a\cos t \quad x''(t)=a\sin t$

$y'(t)=a\sin t \quad y''(t)=a\cos t$

$k=\dfrac{|a(1-\cos t)a\cos t-a^2\sin t^2|}{(a^2(1-\cos t)^2+a^2\sin^2 t)^{\frac{3}{2}}}$

$=\dfrac{1-\cos t}{a\,(2-2\cos t)^{\frac{3}{2}}}=\dfrac{1}{2a\sqrt{2-2\cos t}}$

③ $y=e^x$. $f(x)=e^x$ $f'(x)=e^x$

$k=\dfrac{|f''(x)|}{(1+f'(x)^2)^{\frac{3}{2}}}$

$=\dfrac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$

5. ① $x=\rho\cos\theta=a(\cos\theta+\cos^2\theta)$

$y=\rho\sin\theta=a(\sin\theta+\sin\theta\cdot\cos\theta)$

$x'=-a(\sin\theta+\sin2\theta)$

$y'=a(\cos\theta+\cos2\theta)$

$x''=-a(\cos\theta+2\cos2\theta)$

$y''=-a(\sin\theta+2\sin2\theta)$

$|x'y''-x''y'|$

$=a^2|\sin^2\theta+3\sin\theta\sin2\theta+2\sin^22\theta+$

$\cos^2\theta+3\cos\theta\cdot\cos2\theta+2\cos^22\theta|$

$=a^2\cdot(3+3\cos\theta)$

$(x'^2+y'^2)^{\frac{3}{2}}$

$=a^3(2+2\cos\theta)^{\frac{3}{2}}$

$k=\dfrac{3}{2a\sqrt{2+2\cos\theta}}$

6. $f(x)=\frac{1}{x}$ $f'(x)=-\frac{1}{x^2}$

$k=\dfrac{\frac{1}{x^2}}{(1+\frac{1}{x^2})^{\frac{3}{2}}}=\dfrac{1}{x^2\cdot(1+\frac{1}{x^2})\cdot\sqrt{1+\frac{1}{x^2}}}$

$=\dfrac{1}{(x^2+1)\sqrt{1+\frac{1}{x^2}}}=\dfrac{1}{\sqrt{\frac{(x^2+1)^3}{x^2}}}$

令 $t=x^2$

$g(t)=\dfrac{(t+1)^3}{t}$ $\quad g'(t)=\dfrac{3(t+1)^2\cdot t-(t+1)^3}{t^2}$

$=\dfrac{2t^3+3t^2-1}{t^2}=\dfrac{(t+1)^2(t-\frac{1}{2})}{t^2}$

$\therefore g(t)\min=g(\frac{1}{2})=\frac{27}{4}$

$k\max=\dfrac{2\sqrt{3}}{9}$

设曲率圆心为 (a,b)

$(1,0)$ 处 $k=\frac{1}{2\sqrt{2}}$ $r=2\sqrt{2}$

$(a-1)^2+b^2=8$ 且 $\dfrac{b-0}{a-1}\cdot1=-1$

$\therefore a=3$ 或 -1

而 $a<1$ 对 $\ln x$ 凹侧

$\therefore a=-1,\ b=2$

$(x+1)^2+(y-2)^2=8$

8. (1)(2)(3)

① $S = 2\pi \int_a^b |y(t)| \sqrt{x'(t)^2+y'(t)^2}\, dt$

$x=t.\ \ y=\sqrt{t}.\ \ x'(t)=1\ \ \ y'(t)=\frac{1}{2\sqrt{t}}$

$S=2\pi\int_0^2 \sqrt{t}\cdot\sqrt{1+\frac{1}{4t}}\,dt$

$\int\sqrt{t+\frac14}\,dt=\frac23\left(t+\frac14\right)^{\frac32}+C$

$\therefore S=\frac{4\pi}{3}\left(t+\frac14\right)^{\frac32}\Big|_0^2$

$=\frac{13}{3}\pi$

③ $x=a\sin t.\ \ y=b+a\cos t\ \ \ t\in[0,2\pi]$

$S=2\pi\int_0^\pi |y|\cdot\sqrt{x'^2+y'^2}\,dt$

$x'=a\cos t\ \ \ y'=-a\sin t$

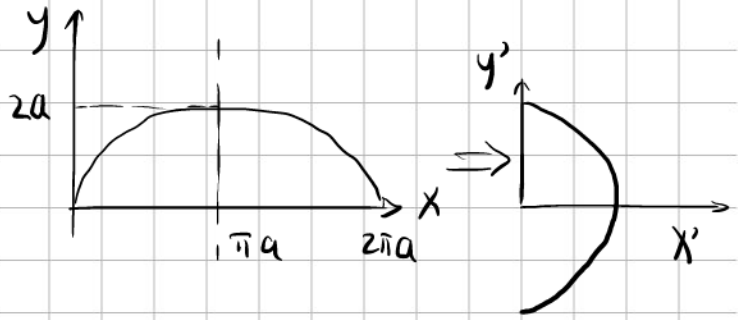
$S_1=2\pi\int_0^\pi(b+a\cos t)\,a\,dt$

$=2\pi\int_0^\pi ab\,dt+2\pi\int_0^\pi a^2\cos t\,dt$

$=2\pi abt\Big|_0^\pi+2\pi a^2\sin t\Big|_0^\pi$

$=2\pi^2ab$

$S=2S_1=4ab\pi^2$

② 

$y'=a(t-\sin t)-a\pi$

$x'=a(1-\cos t)$

$\int_\pi^{2\pi}2\pi\cdot(at-a\sin t-a\pi)\sqrt{a^2(1-\cos t)^2+\sin^2 t}\,dt$

$=2\pi a^2\cdot\int_\pi^{2\pi}(t-\sin t-\pi)\sqrt{2-2\cos t}\,dt$

$=4\pi a^2\cdot\int_\pi^{2\pi}(t-\sin t-\pi)\cdot\sin\frac t2\,dt$

$\int_\pi^{2\pi}t\sin\frac t2\,dt-\int_\pi^{2\pi}\sin t\cdot\sin\frac t2\,dt-\pi\int_\pi^{2\pi}\sin\frac t2\,dt$

$\int t\sin\frac t2\,dt=-2t\cos\frac t2+2\int\cos\frac t2\,dt$

$=-2t\cos\frac t2+4\sin\frac t2+C$

$\int\sin t\cdot\sin\frac t2\,dt=\int2\sin^2\frac t2\cos\frac t2\,dt$

$=\int4\sin^2\frac t2\,d\sin\frac t2=\frac43\sin^3\frac t2+C$

$\int\sin\frac t2\,dt=-2\cos\frac t2+C$

原式$=4\pi a^2\left(-2t\cos\frac t2+4\sin\frac t2-\frac43\sin^3\frac t2+2\pi\cdot\cos\frac t2\right)\Big|_\pi^{2\pi}=4\pi a^2\left(2\pi-\frac83\right)$

第五章复习题:

20. $\lim_{n\to\infty}\int_0^\pi f(x)|\sin nx|\,dx=\lim_{n\to\infty}\left(\int_0^{\frac\pi n}f(x)|\sin nx|\,dx+\int_{\frac\pi n}^{\frac2n\pi}f(x)|\sin nx|\,dx+\cdots+\int_{\frac{n-1}{n}\pi}^{\pi}f(x)|\sin nx|\,dx\right)$

$=\lim_{n\to\infty}\left[f(\xi_1)\int_0^{\frac\pi n}|\sin nx|\,dx+\cdots+f(\xi_n)\int_{\frac{n-1}{n}\pi}^{\pi}|\sin nx|\,dx\right]$

其中 $\xi_i\in\left[\frac{i-1}{n}\pi,\frac in\pi\right]$ 而 $\int_{\frac{i-1}{n}\pi}^{\frac in\pi}|\sin nx|\,dx=\frac2n$

$\lim_{n\to\infty}\sum_{i=1}^n f(\xi_i)\frac2n=\frac2\pi\lim_{n\to\infty}\sum_{i=1}^n f(\xi_i)\frac\pi n=\frac2\pi\int_0^\pi f(x)\,dx$