习题5.5.

3.(1)(6)(8)

(1) $\int \frac{1}{\sqrt{x}\cdot(\sqrt{x}+\sqrt[3]{x})}dx = \int \frac{dx}{x+x^{\frac{5}{6}}} = \int \frac{dx}{x^{\frac{5}{6}}(1+x^{\frac{1}{6}})}$

设 $x^{\frac{1}{12}} = \tan t$. 则 $t = \arctan x^{\frac{1}{12}}$ $x = (\tan t)^{12}$

$dx = 12(\tan t)^{11}\sec^2 t\, dt$ $x^{\frac{5}{6}}(1+x^{\frac{1}{6}}) = (\tan t)^{10}\cdot(1+\tan^2 t)$

$= (\tan t)^{10}\cdot\sec^2 t$

$\therefore \int \frac{12(\tan t)^{11}}{\tan^{10} t}dt = \int 12\tan t\, dt = 12\ln^{|\sec t|} + C$

$= 6\ln^{\sec^2 t} + C = 6\ln^{1+\tan^2 t} + C = 6\ln^{1+x^{\frac{1}{6}}} + C$

3.(6)

$\int x\sqrt{\frac{1+x}{1-x}}dx$ 设 $\sqrt{\frac{1+x}{1-x}} = t$ $\therefore x = \frac{t^2-1}{t^2+1}$ $dx = \frac{4t}{(t^2+1)^2}dt$

$\int \frac{t^2-1}{t^2+1}\cdot t\cdot \frac{4t}{(t^2+1)^2}dt = \int \frac{4t^2(t^2-1)}{(t^2+1)^3}dt$

设 $\frac{4t^2(t^2-1)}{(t^2+1)^3} = \frac{At+B}{t^2+1} + \frac{Ct+D}{(t^2+1)^2} + \frac{Et+F}{(t^2+1)^3}$

$4t^2(t^2-1) = (At+B)(t^2+1)^2 + (Ct+D)(t^2+1) + (Et+F)$

$At^5 + Bt^4 + (2A+C)t^3 + (2B+D)t^2 + (A+C+E)t + B+D+F$

解之有: $A=0$. $B=4$. $C=0$. $D=-12$. $E=0$, $F=8$

$\int\left(\frac{4}{t^2+1} - \frac{12}{(t^2+1)^2} + \frac{8}{(t^2+1)^3}\right)dt$

$4\int\frac{dt}{t^2+1}=4\arctan t+C$

$\int\frac{1}{(x^2+1)^2}dx=\frac{1}{2}\left[\frac{x}{(x^2+1)}+\int\frac{dx}{x^2+1}\right]$

$\int\frac{1}{(x^2+1)^3}dx=\frac{1}{4}\left[\frac{x}{(x^2+1)^2}+3\int\frac{1}{(x^2+1)^2}dx\right]$

$\therefore$ 原式 $=4\int\frac{dt}{t^2+1}-6\frac{t}{t^2+1}-6\int\frac{dt}{t^2+1}$

$+2\frac{t}{(t^2+1)^2}+3\frac{t}{t^2+1}+3\int\frac{dt}{t^2+1}$

$=\arctan t-\frac{3t}{t^2+1}+\frac{2t}{(t^2+1)^2}+C$

$=\arctan\left(\sqrt{\frac{1+x}{1-x}}\right)+\sqrt{\frac{1+x}{1-x}}\cdot\frac{x^2+x-2}{2}+C$

3.6另解：令 $x=\cos t$. $t\in[0,\pi]$

$\therefore\sqrt{\frac{1+\cos t}{1-\cos t}}=\frac{(1+\cos t)^2}{\sin^2 t}$

$\int\cos t\cdot\frac{1+\cos t}{\sin t}d\cos t$

$=-\int\cos t\cdot(1+\cos t)\,dt$

$=-\int\cos t\,dt-\int\cos^2 t\,dt$

$\int\cos^2 t\,dt=\int\frac{1+\cos 2t}{2}dt$

$=\frac{t}{2}+\frac{1}{4}\int\cos 2t\,d2t=\frac{t}{2}+\frac{\sin 2t}{4}+C$

$\therefore$ 原式 $=-\sin t-\frac{t+\sin t\cos t}{2}+C$

$=-\sqrt{1-x^2}-\frac{\arccos x+x\cdot\sqrt{1-x^2}}{2}+C$

3.(8) $\int\frac{1-x+x^2}{\sqrt{1+x-x^2}}dx$

$\sqrt{1+x-x^2}=\sqrt{\frac{5}{4}-(x^2-x+\frac{1}{4})}=\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}$

令 $x=\frac{\sqrt{5}\cos t+1}{2}$. $t\in[0,\pi]$.

故 $\sqrt{1+x-x^2}=\frac{\sqrt{5}}{2}\sin t$.

$1-x+x^2=(x-\frac{1}{2})^2+\frac{3}{4}=\frac{5}{4}\cos^2 t+\frac{3}{4}$

$dx=-\frac{\sqrt{5}}{2}\sin t\,dt$

$\int\frac{\frac{5}{2}\cos^2 t+\frac{3}{2}}{\sqrt{5}\sin t}\cdot-\frac{\sqrt{5}}{2}\sin t\,dt$

$=-\int\left(\frac{5}{4}\cos^2 t+\frac{3}{4}\right)dt$

$=-\int\left(\frac{5}{8}\cos 2t+\frac{11}{8}\right)dt$

$=-\frac{11}{8}t-\frac{5}{16}\sin 2t+C$

$t=\arccos\left(\frac{2x-1}{\sqrt{5}}\right)$

$\sin t=\frac{2}{\sqrt{5}}\sqrt{1+x-x^2}$

$\cos t=(2x-1)/\sqrt{5}$

$\therefore\sin 2t=\frac{4}{5}(2x-1)\cdot\sqrt{1+x-x^2}$

故：$-\frac{11}{8}\arccos\left(\frac{2x-1}{\sqrt{5}}\right)-\frac{1}{4}(2x-1)\sqrt{1+x-x^2}+C$

即 $\frac{11}{8}\arcsin x\left(\frac{2x-1}{\sqrt{5}}\right)-\frac{2x-1}{4}\sqrt{1+x-x^2}+C$

注意到 $\arcsin x+\arccos x=\frac{\pi}{2}$

$\therefore-\arccos x=\arcsin x-\frac{\pi}{2}$

另 $\arcsin(-x)=-\arcsin x$

$\arccos(-x)=\pi-\arccos x$

习题 5.6：

1.(1)

$$\int_0^{2\pi}|\sin x|\,dx$$
$$=\int_0^{\pi}\sin x\,dx-\int_{\pi}^{2\pi}\sin x\,dx$$
$$=-\cos x\Big|_0^{\pi}+\cos x\Big|_{\pi}^{2\pi}$$
$$=4$$

(5)

$$\int_1^2\frac{dx}{(x-3)(x+1)}=\frac{1}{4}\int_1^2\left(\frac{1}{x-3}-\frac{1}{x+1}\right)dx$$
$$=\frac{1}{4}\left(\ln|x-3|\Big|_1^2-\ln|x+1|\Big|_1^2\right)$$
$$=\frac{1}{4}\left(-\ln 2+\ln 2-\ln 3\right)$$
$$=-\frac{\ln 3}{4}$$

(12) $\int\frac{dx}{x^2\sqrt{x^2-1}}$ 令 $x=\sec t$；$t\in[\pi,\frac{3}{2}\pi]$

则 $dx=\sec t\cdot\tan t\cdot dt$

$\sqrt{x^2-1}=\tan t$. $x^2=\sec^2 t$

$$\therefore\int\frac{\sec t\cdot\tan t\cdot dt}{\tan t\cdot\sec^2 t}=\int\frac{1}{\sec t}dt$$
$$=\int\cos t\,dt=\sin t+C$$

而 $\sin^2 t+\frac{1}{\sec^2 t}=1$

$\therefore\sin^2 t=1-\frac{1}{x^2}$

$$\int\frac{dx}{x^2\sqrt{x^2-1}}=-\sqrt{1-\frac{1}{x^2}}$$
$$-\sqrt{1-\frac{1}{x^2}}\Big|_{-2}^{-\sqrt{2}}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}$$

注：被积 $f(x)$ 恒正与 $F(x)$ 正负性有何关系 !!!

2.(1)

$$\int(2x-1)\,d\cos x=(2x-1)\cdot\cos x$$
$$-\int\cos x\,d(2x-1)=(2x-1)\cdot\cos x$$
$$-2\int\cos x\,dx=(2x-1)\cdot\cos x-2\sin x+C$$
$$((2x-1)\cdot\cos x-2\sin x)\Big|_0^{\pi}$$
$$=1-2\pi+1=2-2\pi$$

(5) $\int e^{-x}\cdot\sin 2x\,dx=-\int\sin 2x\,de^{-x}$

$$=-\left(\sin 2x\cdot e^{-x}-\int e^{-x}\,d\sin 2x\right)$$
$$=-\left(\sin 2x\cdot e^{-x}-2\int e^{-x}\cdot\cos 2x\,dx\right)$$
$$=-\sin 2x\cdot e^{-x}+2\int e^{-x}\cdot\cos 2x\,dx$$
$$\int e^{x}\cos 2x\,dx=\int-\cos 2x\,de^{-x}$$
$$=-\left(\cos 2x\cdot e^{-x}-\int e^{-x}\,d\cos 2x\right)$$
$$=-\cos 2x\cdot e^{-x}-2\int e^{-x}\sin 2x\,dx$$
$$\therefore\int e^{-x}\sin 2x\,dx=-\frac{1}{5}e^{-x}(\sin 2x+2\cos 2x)$$

$\therefore \int_0^{\frac{\pi}{4}} e^{-x}\sin 2x\,dx$

$= -\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)\Big|_0^{\frac{\pi}{4}}$

$= -\frac{1}{5}\cdot\frac{1}{e^{\frac{\pi}{4}}} + \frac{2}{5}$

3.(1)

$\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$\int (1 - 2\cos 2x + \cos^2 2x)\,dx$

$= x - \int 2\cos 2x\,dx + \frac{1}{2}\int (1+\cos 4x)\,dx$

$= x - \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x + C$

$= \frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x + C$

$\therefore \int \sin^4 x\,dx = \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

$\left(\frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{3}{16}\pi$

(8) $\int_{-a}^{a} (1-x)\sqrt{a^2-x^2}\,dx$

设 $x = a\cos t.\ t\in[0,\pi]$.

$\int (1-x)\sqrt{a^2-x^2}\,dx$

$= -\int (1-a\cos t)\,a\sin t(a\sin t)\,dt$

$= -\int (a^2\sin^2 t - a^3\sin^2 t\cdot\cos t)\,dt$

$= -a^2\int \sin^2 t\,dt + a^3\int \sin^2 t\cdot\cos t\cdot dt$

$\int \sin^2 t\,dt = \frac{1}{2}t - \frac{1}{2}\int \cos 2t\,dt$

$= \frac{1}{2}t - \frac{1}{4}\sin 2t$

$\int \sin^2 t\cdot\cos t\,dt = \int \sin^2 t\,d\sin t$

$= \frac{1}{3}\sin^3 t$

$\therefore$ 原式 $= -\frac{1}{2}a^2 t + \frac{a^2}{4}\sin 2t + \frac{a^3}{3}\sin^3 t\Big|_{\pi}^{0}$

$= \frac{1}{2}a^2 t - \frac{a^2}{4}\sin 2t - \frac{a^3}{3}\sin^3 t\Big|_0^{\pi}$

$= \frac{1}{2}a^2\pi$

5. 令 $x = \pi - t$. 则 $\int_0^{\pi} x f(\sin x)\,dx$

$= \int_0^{\pi} (\pi - t) f(\sin t)\,dt$

$= \int_0^{\pi} \pi f(\sin t)\,dt - \int_0^{\pi} t f(\sin t)\,dt$

$\therefore \int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

8. 设 $g'(t) = e^{-t^2}$. 则 $f(x) = g(x)\Big|_1^{x^2}$

且 $\frac{df(x)}{dx} = \frac{df(x)}{dx^2}\cdot\frac{dx^2}{dx} = e^{-x^4}\cdot 2x$

$\int x f(x)\,dx = x^2 f(x) - \int x\cdot(f(x) + f'(x)\cdot x)\,dx$

$\therefore \int x f(x)\,dx = \frac{1}{2}x^2 f(x) - \frac{1}{2}\int x^2 f'(x)\,dx$

$= \frac{1}{2}x^2 f(x) - \int x^3\cdot e^{-x^4}\,dx$

$\int x^3\cdot e^{-x^4}\,dx = \frac{1}{4}\int e^{-x^4}\,dx^4 = \frac{-1}{4}e^{-x^4}$

$\therefore \int x f(x)\,dx = \frac{1}{2}x^2 f(x) + \frac{1}{4}e^{-x^4}$

注 $\int e^{-x}dx = -e^{-x}$　$de^{-x} = -e^{-x}dx$

$\int_0^1 x f(x)dx = \left(\frac{1}{2}x^2 f(x) + \frac{1}{4}e^{-x^4}\right)\Big|_0^1$

$= \left(\frac{1}{2}f(1) + \frac{1}{4e}\right) - \left(\frac{1}{4}\right)$

$f(1) = 0$

$\therefore$ 原式 $= \frac{1}{4e} - \frac{1}{4}$

14. 设 $g(x) = x \cdot f(x)$ 则 $g(1) = f(1)$

$2\int_0^{\frac{1}{2}} g(x)dx = g(1)$

由积分第一中值定理有

$\int_a^b f(x)\cdot 1\,dx = f(\eta)\cdot(b-a)$

$\eta \in [a,b]$

故：$2\int_0^{\frac{1}{2}} g(x)dx = g(\eta)\cdot(\frac{1}{2}-0)\cdot 2$

$= g(\eta)$. $\eta \in [0, \frac{1}{2}]$.

$\therefore g(\eta) = g(1)$. 而 $g(x) = x\cdot f(x)$

$f(x) \in C[0,1] \therefore g(x) \in C[0,1]$

$f(x)$ 在 $[0,1]$ 上可导 $\therefore g(x)$ 亦 $[0,1]$ 上可导

由罗尔定理. $g(x)$ 在 $[\eta,1]$ 上连

续. 在 $(\eta,1)$ 上可导. $g(\eta) = g(1)$

$\therefore \exists\, \xi \in (\eta,1)$ s.t. $f(\xi) = 0$

$\therefore \exists\, \xi \in (0,1)$, $f(\xi) + \xi f(\xi) = 0$

习题 5.7

7.(1) $y_1 = x^2$ 与 $y_2 = x^3$ 交于 $(1,1)$

$\int_0^1 \pi\cdot y_1^2 dx = \pi\int_0^1 x^4 dx = \frac{1}{5}\pi x^5\Big|_0^1$

$= \frac{1}{5}\pi$

$\int_0^1 \pi\cdot y_2^2 dx = \pi\cdot\int_0^1 x^6 dx = \frac{1}{7}\pi x^7\Big|_0^1$

$= \frac{1}{7}\pi$

$\therefore V = \int_0^1 \pi y_1^2 x - \int_0^1 \pi y_2^2 x$

$= \frac{2}{35}\pi$

(3) 取 $\frac{x^2}{9} + \frac{y^2}{4} = 1$ 上半部分则

$y = 2\sqrt{1-\frac{x^2}{9}}$

绕 x 轴旋转有：$V = \int_{-3}^3 \pi y^2 dx$

$= 4\pi\int_{-3}^3 (1-\frac{x^2}{9})dx$

$= 4\pi\cdot x\Big|_{-3}^3 - \frac{4\pi}{9}\cdot\frac{1}{3}x^3\Big|_{-3}^3$

$= 24\pi - \frac{4\pi}{9}\cdot\frac{1}{3}\cdot 54 = 16\pi$

取 y 轴右侧：$x = 3\sqrt{1-\frac{y^2}{4}}$

$V = \int_{-2}^2 \pi x^2 dy = 9\pi\int_{-2}^2 (1-\frac{y^2}{4})dy$

$= 9\pi\cdot y\Big|_{-2}^2 - \frac{9}{4}\pi\cdot\frac{1}{3}y^3\Big|_{-2}^2$

$= 36\pi - \frac{9}{4}\pi\cdot\frac{1}{3}\cdot 16 = 24\pi$

故绕 x 轴：16π

绕 y 轴：24π

(5) $y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$

$y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$

$V = \int_{-a}^{a} \pi y^2 dx$

$= \pi \int_{-a}^{a} (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3 dx$

$= \pi \int_{-a}^{a} (a^2 - 3a^{\frac{4}{3}} x^{\frac{2}{3}} + 3a^{\frac{2}{3}} x^{\frac{4}{3}} - x^2) dx$

$= \pi a^2 x \Big|_{-a}^{a} - 3a^{\frac{4}{3}} \cdot \pi \frac{3}{5} x^{\frac{5}{3}} \Big|_{-a}^{a}$

$+ 3a^{\frac{2}{3}} \pi \cdot \frac{3}{7} x^{\frac{7}{3}} \Big|_{-a}^{a} - \frac{\pi}{3} x^3 \Big|_{-a}^{a}$

$= 2a^3\pi - \frac{18}{5} a^3 \pi + \frac{18}{7} a^3 \pi - \frac{2}{3} a^3 \pi$

$= \frac{32}{105} \pi a^3$

PS: $\int_{-a}^{a} \pi y^2 dx$

$= \int_{\pi}^{0} \pi y^2 dx$

$= \int_{\pi}^{0} \pi (a \sin^3 t)^2 d(a\cos^3 t)$

3.(1) $l = \int \sqrt{1 + (\frac{dy}{dx})^2} dx$

$\frac{dy}{dx} = \sqrt{\cos x}$

$\therefore l = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \cos x} \, dx$

$= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2 \frac{x}{2}} \, dx$

$= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \frac{x}{2} \, dx$

$= \sqrt{2} \cdot 2 \cdot \sin \frac{x}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 4$

3.(2)

$l = \int_0^1 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$

$l = \int_0^1 \sqrt{(\frac{1}{1+t^2})^2 + (\frac{1}{\sqrt{1+t^2}} \cdot \frac{1}{2} \frac{2t}{\sqrt{1+t^2}})^2} dt$

$= \int_0^1 \sqrt{\frac{1+t^2}{(1+t^2)^2}} dt$

$= \int_0^1 \frac{dt}{\sqrt{1+t^2}}$ 令 $t = \tan m$

$dt = \sec^2 m \, dm$; $\int_0^{\frac{\pi}{4}} \sec m \, dm$

$\int \sec x \, dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{d\sin x}{1 - \sin^2 x}$

$= \frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + C$

$\therefore \int_0^{\frac{\pi}{4}} \sec m \, dm = \frac{1}{2} \ln \left| \frac{1+\sin m}{1-\sin m} \right| \Big|_0^{\frac{\pi}{4}}$

$= \frac{1}{2} \ln^{3+2\sqrt{2}} = \ln^{(1+\sqrt{2})}$

(5) $x = \rho \cos\theta \quad y = \rho \sin\theta$

$\int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$

$dx = d(a\theta \cos\theta) = a(\cos\theta - \sin\theta \cdot \theta) d\theta$

$dy = d(a\theta \sin\theta) = a(\sin\theta + \cos\theta \cdot \theta) d\theta$

$\therefore (\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = a^2(1 + \theta^2)$

$\int_0^{2\pi} a\sqrt{1 + \theta^2} \, d\theta$

$$a\int_0^{2\pi}\sqrt{1+\theta^2}\,d\theta$$
$$=a\left(\frac{\theta}{2}\sqrt{\theta^2+1}+\frac{1}{2}\ln\left|\theta+\sqrt{\theta^2+1}\right|\right)\Big|_0^{2\pi}$$
$$=a\pi\cdot\sqrt{4\pi^2+1}+\frac{1}{2}\ln(2\pi+\sqrt{4\pi^2+1})$$

2.(1)

$$x_1=y^2-2y \quad x_2=2y^2-8y+6$$
$x_1=x_2$ 则 $y_{1,2}=3\pm\sqrt{3}$
$$\int_{3-\sqrt{3}}^{3+\sqrt{3}}|x_2-x_1|\,dy$$
$$=-\int_{3-\sqrt{3}}^{3+\sqrt{3}}(y^2-6y+6)dy$$
$$=-\left(\frac{1}{3}y^3-3y^2+6y\right)\Big|_{3-\sqrt{3}}^{3+\sqrt{3}}$$
$$=\left(-\frac{1}{3}y^3+3y^2-6y\right)\Big|_{3-\sqrt{3}}^{3+\sqrt{3}}$$
$$=4\sqrt{3}$$

(4). $\int_0^a y\,dx$
$$=\int_{\frac{\pi}{2}}^{0}a\sin^3t\,d(a\cos^3t)$$
$$=3a^2\int_0^{\frac{\pi}{2}}\sin^4t\cdot\cos^2t\,dt$$
$$=3a^2\int_0^{\frac{\pi}{2}}(\sin^4t-\sin^6t)dt$$
$$\int(\sin^4t-\sin^6t)\,dt=\int\sin^4t\cdot\cos^2t\,dt$$
$$=\sin^5t\cdot\cos t-4\int\sin^4t\cdot\cos^2t\,dt+\int\sin^6dt$$
$$=\sin^5t\cdot\cos t-5\int\sin^4t\cdot\cos^2t\,dt+\int\sin^4t\,dt$$
$$=\frac{1}{6}\sin^5t\cdot\cos t+\frac{1}{6}\int\sin^4t\,dt$$

$$\int\sin^4t\,dt=\int(\sin^2t-\sin^2t\cdot\cos^2t)dt$$
$$=\int\frac{1-\cos 2t}{2}dt-\int\left(\frac{\sin 2t}{2}\right)^2dt$$
$$=\frac{1}{2}t-\frac{1}{2}\int\cos 2t\,dt-\frac{1}{8}\int(1-\cos 4t)\,dt$$
$$=\frac{3}{8}t-\frac{1}{4}\sin 2t+\frac{1}{32}\sin 4t$$
原式 $=\frac{1}{6}\sin^5t\cdot\cos t+\frac{1}{16}t-\frac{1}{24}\sin 2t+\frac{1}{192}\sin 4t$
$$\left(\frac{1}{6}\sin^5t\cos t+\frac{1}{16}t-\frac{1}{24}\sin 2t+\frac{\sin 4t}{192}\right)\Big|_0^{\frac{\pi}{2}}$$
$$=\frac{1}{32}\pi$$
$$S=12a^2\cdot\frac{\pi t}{32}=\frac{3}{8}a^2\pi$$

五章复习题

9. $\lim\limits_{x\to 0}\frac{f(t+x)-f(t)+f(t)-f(t-x)}{2x}$
$$=\frac{1}{2}f'(t)+\frac{1}{2}f'(t)=f'(t)$$
$$\lim_{x\to 0^+}\frac{1}{4x^2}\int_{-x}^{x}f(t)\cdot 2x\,dt$$
$$=\frac{1}{2x}\int_{-x}^{x}f(t)\cdot dt$$
$$=\frac{f(x)-f(-x)}{2x}=f'(0)$$

10. $S(a)=\int_0^a f(x)\,dx$

$=\frac{a^2}{2}+\frac{a}{2}\sin a+\frac{\pi}{2}\cos a$

$f(a)=S'(a)=a+\frac{\sin a}{2}+\frac{\cos a}{2}\cdot a-\frac{\pi}{2}\sin a$

$=a+\frac{\cos a}{2}\cdot a+\frac{1-\pi}{2}\sin a$

$f(\frac{\pi}{2})=\frac{1}{2}$

9.另解：记 $f(x)$ 原函数 $F(x)$

则：$\lim\limits_{x\to 0^+}\int_{-x}^{x}[f(t+x)-f(t-x)]\,dt$

$=\lim\limits_{x\to 0^+}\left(F(t+x)-F(t-x)\right)\Big|_{-x}^{x}$

$=\lim\limits_{x\to 0^+}\left(F(2x)+F(-2x)-2F(0)\right)$

$\lim\limits_{x\to 0^+}\frac{F(2x)+F(-2x)-2F(0)}{4x^2}$

$=\lim\limits_{x\to 0^+}\frac{2f(2x)-2f(-2x)}{8x}$

$=\lim\limits_{x\to 0^+}\frac{f(2x)-f(-2x)}{4x}=f'(0)$