清华大学数学作业纸

（科目微积分）

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习题4.4

4.(2) $y=e^{x\ln(1+\frac{1}{x})}$　$y'=e^{x\ln(1+\frac{1}{x})}\cdot(\ln(1+\frac{1}{x})+\frac{x}{1+\frac{1}{x}}\cdot\frac{-1}{x^2})$

$y'=(1+\frac{1}{x})^x\cdot(\ln(1+\frac{1}{x})-\frac{1}{x+1})$ ~~取$h(x)=\ln(1+\frac{1}{x})-\frac{1}{x+1}$~~

~~则$h'(x)=$~~ 取$t=1+\frac{1}{x}\in(2,+\infty)$. $h(t)=\ln t+\frac{1}{t}-1$

则 $\ln(1+\frac{1}{x})-\frac{1}{x+1}=h(t)$　$h'(t)=\frac{t-1}{t^2}>0$

又$h(2)>0$. $\therefore h(t)$恒正

$\therefore y$单调递增　无极值

(4) $y'=6x^2-6x-12=6(x^2-x-2)$

$=6(x+1)(x-2)$. 故y在$(-\infty,-1)$上↑

$(-1,2)$↓, $(2,+\infty)$↑

y连续且可导, $\therefore$极大值点为$y(-1)=8$

极小值点为$y(2)=-19$

习题4.5

1.(1) $y'=6x-3x^2$　$y''=6-6x$. 则y在$(-\infty,1)$内

y''恒为正, $(1,+\infty)$内, y''恒为负

$\therefore$上凸区间$(1,+\infty)$ 下凸区间:$(-\infty,1)$

拐点: $(1,2)$

1.(4) $y=\frac{x}{e^x}$　$y'=\frac{e^x-e^x\cdot x}{(e^x)^2}=\frac{1-x}{\cancel{(e^x)^2}}$　$\frac{1-x}{e^x}$

$y''(x)=\cancel{\frac{-1\cdot e^{2x}-e^{2x}\cdot 2(1-x)}{(e^x)^4}=\frac{2x-3}{e^{2x}}}=\frac{-e^x-e^x(1-x)}{e^{2x}}=\frac{x-2}{e^x}$

$\therefore$上凸区间: $(-\infty,\frac{3}{2})$ 下凸区间$(\frac{3}{2},+\infty)$

拐点: ~~$(0,0)$~~ $(2,\frac{2}{e^2})$

5(1) 令$f(x)=x^p$ $(x>0,p\geqslant1)$. 则可知$f'(x)=px^{p-1}$

$f''(x)=p(p-1)x^{p-2}>0$. 则$f(x)$为下凸函数

由定理4.5.1可知:

$f(\frac{1}{n}x_1+\frac{1}{n}x_2+\cdots+\frac{1}{n}x_n)\leq\frac{1}{n}f(x_1)+\frac{1}{n}f(x_2)+\cdots+\frac{1}{n}f(x_n)$

即$(\frac{x_1+x_2+\cdots+x_n}{n})^p\leq\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}$

7. $f(x)$下凸. 则: ~~①$f(x)$不可导时.~~

①$f(a)=f(b)$时, $\forall x\in[a,b]$ $\exists$唯一λ

$(\lambda\in[0,1])$ s.t. $x=\lambda a+(1-\lambda)b$

由下凸函数定义: $f(x)=f(\lambda a+(1-\lambda)b)$

$\leq\lambda f(a)+(1-\lambda)f(b)=\max\{f(a),f(b)\}$

② $f(a)\neq f(b)$时, 不妨$f(b)>f(a)$

则$f(x)=f(\lambda a+(1-\lambda)b)\leq\lambda f(a)+(1-\lambda)f(b)$

$\leq f(b)=\max\{f(a),f(b)\}$

综上: $f(x)$ $x\in[a,b]$时恒$\leq\max\{f(a),f(b)\}$

$\therefore f(x)_{max}=\max\{f(a),f(b)\}$

8. $f(x)$可导(微分)且$f(x)$下凸, 则

$f'(x)$单调递增且$f'(x_0)=0$. 可知(a,x_0)上

$f'(x_0)<0$. (x_0,b)上$f'(x_0)>0$. 且函数无不

可导点. 则$f(x)$在(a,x_0)↓ (x_0,b)↑

$\therefore f(x)_{min}=f(x_0)$

习题4.6

1.(1) $y=\sqrt{\frac{x^3}{x-1}}$. 可知$x\in(-\infty,0)\cup(1,+\infty)$

$\lim\limits_{x\to\infty}\frac{y}{x}=\frac{\sqrt{\frac{x^3}{x-1}}}{x}$　1° $x\in(-\infty,0)$时, 原式

$\lim\limits_{x\to-\infty}\frac{y}{x}=\frac{\sqrt{\frac{x^3}{x-1}}}{-(-x)}=-\sqrt{\frac{x}{x-1}}=-1$. 2° $\lim\limits_{x\to+\infty}\frac{y}{x}=\sqrt{\frac{x}{x-1}}=1$

$\lim\limits_{x\to-\infty}(\sqrt{\frac{x^3}{x-1}}-(-x))=\lim\limits_{x\to-\infty}\sqrt{\frac{x^3}{x-1}}+x=\lim\limits_{x\to-\infty}x-x\sqrt{\frac{x}{x-1}}$

$=\lim\limits_{x\to-\infty}x(1-\sqrt{\frac{x}{x-1}})=\lim\limits_{x\to-\infty}\frac{x\cdot\frac{-1}{x-1}}{1+\sqrt{\frac{x}{x-1}}}=-\frac{1}{2}$

$\lim\limits_{x\to+\infty}(\sqrt{\frac{x^3}{x-1}}-x)=\lim\limits_{x\to+\infty}x(\sqrt{\frac{x}{x-1}}-1)=\lim\limits_{x\to+\infty}\frac{\frac{x}{x-1}}{1+\sqrt{\frac{x}{x-1}}}=\frac{1}{2}$

$\therefore$渐近线为 $l_1: y=x+\frac{1}{2}$

$l_2: y=-x-\frac{1}{2}$

问题是直接漏了水平与竖直渐近线

$\lim\limits_{x\to1^+}\sqrt{\frac{x^3}{x-1}}=+\infty$　$x=1$为竖直渐近线

无水平渐近线

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（科目微积分7）

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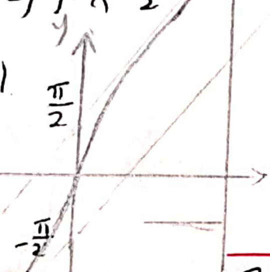
2.(4) $y=x+\arctan x$ ①定义域：R

②有；奇函数 ③ $f'(x)=1+\frac{1}{1+x^2}>0$. 恒↑.无极值点； ④ $f''(x)=-\frac{1}{(1+x^2)^2}\cdot 2x$. 可知

$f''(x)$在$x<0$时为正,$f(x)$下凸；$x>0$时为负，$f(x)$上凸. 拐点$(0,0)$ ⑤无水平与竖直渐近线，斜渐线为$y=x+\frac{\pi}{2}$与$y=x-\frac{\pi}{2}$

$$\lim_{x\to\infty}\frac{y}{x}=\lim_{x\to\infty}\left(1+\frac{\arctan x}{x}\right)=1$$

$$\lim_{x\to+\infty} y-x=\frac{\pi}{2} \qquad \lim_{x\to-\infty} y-x=-\frac{\pi}{2}$$



习题4.2 ▲导求错了

2.(2) $\lim_{x\to0}\frac{e^x-e^{\sin x}}{x-\sin x}$

$$=\lim_{x\to0}\frac{e^x-e^{\sin x}\cdot\cos x}{1-\cos x}=\lim_{x\to0}\frac{e^x-(e^{\sin x}\cdot\cos^2x-\sin x\cdot e^{\sin x})}{\cancel{1+\sin x}\ \sin x}$$

$$\cancel{=0}=\lim_{x\to0}\left(\frac{e^x-e^{\sin x}\cdot\cos^2x}{\sin x}+e^{\sin x}\right)=\lim_{x\to0}\frac{e^x-e^{\sin x}}{x}+1\ (泰勒)$$

方式 $=\lim_{x\to0}\frac{1+x+o(x)-(1+\sin x+o(\sin x))}{x}=\lim_{x\to0}\frac{x-x+o(x)+o(\sin x)}{x}=0$

3. $\lim_{h\to0}\frac{\cancel{(f(a+h)-f(a))}+(f(a-h)-f(a))}{h^2}=1$

$$=\lim_{h\to0}\left(\cancel{\frac{f(a+h)-f(a)}{h^2}}-\frac{f(a)-f(a-h)}{h^2}\right)$$

$$=\cancel{\lim_{h\to0}}\frac{f'(a)}{\cancel{h}}$$

a为恒量

$$\lim_{h\to0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=\lim_{h\to0}\frac{\cancel{f(a+h)-2f(a)+f(a-h)}}{2h}$$

$$=\lim_{h\to0}\frac{f(a+h)-f(a)+f(a)-f(a-h)}{2h}$$

$$=\frac12\lim_{h\to0}\frac{f(a+h)-f(a)}{h}+\frac12\lim_{h\to0}\frac{f(a)-f(a-h)}{h}$$

$$=f''(a)$$

4. $\lim_{x\to0}\frac{f(\sin x)-1}{\ln f(x)}=\lim_{x\to0}\frac{f'(\sin x)\cdot\cos x}{\frac{1}{f(x)}\cdot f'(x)}=1$

2(10) $\lim_{x\to\infty}\frac{\ln(x+1)}{x^2}=\lim_{x\to0}\frac{1}{2x(x+1)}=0$

(13) $\lim_{x\to1}\frac{\tan\frac{\pi}{2}x}{\frac{1}{x-1}}=\lim_{x\to1}\frac{\frac{1}{\cos^2\frac{\pi}{2}x}\cdot\frac{\pi}{2}}{-\frac{1}{(x-1)^2}}=\lim_{x\to1}-\frac{\frac{\pi}{2}\cdot(x-1)^2}{\cos^2\frac{\pi}{2}x}$

$(\cot x)'=-\frac{1}{\sin^2x}$ $\lim\frac{x-1}{\cot\frac{\pi}{2}x}=\lim\frac{1}{-\frac{1}{\sin^2\frac{\pi}{2}x}\cdot\frac{\pi}{2}}=-\frac{2}{\pi}$

$$=\lim_{x\to1}-\frac{\pi}{2}\cdot\frac{2(x-1)}{2\cos\frac{\pi}{2}x\cdot-\sin\frac{\pi}{2}x\cdot\frac{\pi}{2}}$$

$$=\lim_{x\to1}\frac{x-1}{\cos\frac{\pi}{2}x\cdot\sin\frac{\pi}{2}x}=\lim_{x\to1}2\cdot\frac{x-1}{\sin\pi x}$$

$$=\lim_{x\to1}2\cdot\frac{1}{\cos\pi x\cdot\pi}=-\frac{2}{\pi}$$

(17) $\lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{x}-1\right)^{\tan x}$

$$=\lim_{x\to\frac{\pi}{2}}e^{\tan x\cdot\ln\left(\frac{\pi}{x}-1\right)}$$

$$=e^{\lim_{x\to\frac{\pi}{2}}\frac{\ln(\frac{\pi}{x}-1)}{\cot x}}=e^{\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\frac{\pi}{x}-1}\cdot\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{\sin^2x}}}$$

$$=e^{\frac{-\frac{4}{\pi}}{-1}}=e^{\frac{4}{\pi}}$$

对tanx的处理很经典

习题4.3.

3(2) $\sin^{(n)}x=\sin\left(x+\frac{n\pi}{2}\right)\ (n=0,1,\cdots)$

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}\cdot(x-x_0)^k+o((x-x_0)^n)$$

$$\frac{\sqrt2}{2}+\left(+\frac{\sqrt2}{2}\right)\left(x-\frac{\pi}{4}\right)-\frac{\sqrt2}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt2}{12}\left(x-\frac{\pi}{4}\right)^3$$

$+o((x-x_0)^n)$ —— 皮亚诺余项

$f(x)$可4阶导.故

$$f(x)=\frac{\sqrt2}{2}+\frac{\sqrt2}{2}\left(x-\frac{\pi}{4}\right)-\frac{\sqrt2}{4}\left(x-\frac{\pi}{4}\right)^2-\frac{\sqrt2}{12}\left(x-\frac{\pi}{4}\right)^3+\frac{\sin(\xi)}{24}\left(x-\frac{\pi}{4}\right)^4$$

3(4) $\tan'x=\frac{1}{\cos^2x}$ $\tan''(x)=\frac{2\sin x}{\cos^3x}$

$$\tan'''x=\frac{2\cos^4x-3\cos^2x\cdot(-\sin x)\cdot\sin x}{\cos^6x}$$

$$=\frac{2\cos^4x+3\cos^2x\cdot\sin^2x}{\cos^6x}=\frac{2\cos^2x+3\sin^2x}{\cos^4x}$$

$$f(x)=1+2\cdot\left(x-\frac{\pi}{4}\right)^1+2\cdot\left(x-\frac{\pi}{4}\right)^2+o\left(\left(x-\frac{\pi}{4}\right)^2\right)$$

$$f(x)=1+2\left(x-\frac{\pi}{4}\right)^1+2\cdot\left(x-\frac{\pi}{4}\right)^2+\frac{2\cos^2\xi+3\sin^2\xi}{6\cos^4\xi}\left(x-\frac{\pi}{4}\right)^3$$

9. $f(x)=f(\frac12)+f'(\frac12)(x-\frac12)+\frac{f''(\frac12)}{2}(x-\frac12)^2+\frac{f'''(\xi)}{6}(x-\frac12)^3$

$f(0)=f(\frac12)+\frac{f''(\frac12)}{8}-\frac{f'''(\xi_1)}{48}$ $\xi_1\in(0,\frac12)$

$f(1)=f(\frac12)+\frac{f''(\frac12)}{8}+\frac{f'''(\xi_2)}{48}$ $\xi_2\in(\frac12,1)$

$\therefore f'''(\xi_1)+f'''(\xi_2)=24$ 由达布定理(书例4.1.6)

$\exists\xi_3\in(\xi_1,\xi_2)$且$f'''(\xi_3)=\frac{f'''(\xi_1)+f'''(\xi_2)}{2}=12$

10. $f(x)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})\cdot(x-\frac{a+b}{2})+\frac{f''(\xi)}{2}(x-\frac{a+b}{2})^2$

$f(a)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(\frac{a-b}{2})+\frac{f''(\xi_1)}{8}(a-b)^2$

$f(b)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(\frac{b-a}{2})+\frac{f''(\xi_2)}{8}(a-b)^2$

$\therefore f(a)+f(b)-2f(\frac{a+b}{2})=\frac{f''(\xi_1)+f''(\xi_2)}{8}(a-b)^2$

由达布定理. $\exists\ \xi_3\in(\xi_1,\xi_2)$ s.t. $f''(\xi_3)=\frac{f''(\xi_1)+f''(\xi_2)}{2}$

$\therefore f(a)+f(b)-2f(\frac{a+b}{2})=\frac{f''(\xi_3)}{4}(a-b)^2$

$f(x)=f(a)+f'(a)(x-a)+\frac{f''(\xi)}{2}(x-a)^2$

$f(x)=f(b)+f'(b)(x-b)+\frac{f''(\xi)}{2}(x-b)^2$

$f(x)=f(\frac{a+b}{2})+f'(\frac{a+b}{2})(x-\frac{a+b}{2})+\frac{f''(\xi)}{2}(x-\frac{a+b}{2})^2$

$\therefore f(b)=f(a)+\frac{f''(\xi_1)}{2}(b-a)^2$

$f(a)=f(b)+\frac{f''(\xi_2)}{2}(b-a)^2$

$\therefore\ f''(\xi_1)+f''(\xi_2)=0$

$f(\frac{a+b}{2})=f(a)+\frac{f''(\xi_3)}{2}(\frac{a-b}{2})^2\ (\xi_3\in(a,\frac{a+b}{2}))$

$f(\frac{a+b}{2})=f(b)+\frac{f''(\xi_4)}{2}(\frac{a-b}{2})^2\ (\xi_4\in(\frac{a+b}{2},b))$

$\therefore\ f(b)-f(a)+\frac{(a-b)^2}{8}(f''(\xi_4)-f''(\xi_3))=0$

$\therefore\ \frac{f''(\xi_4)-f''(\xi_3)}{2}=\frac{f(a)-f(b)}{(b-a)^2}\cdot 4$

$\therefore\ \left|\frac{4}{(b-a)^2}(f(b)-f(a))\right|\le\frac{|f''(\xi_4)|+|f''(\xi_3)|}{2}$

$\le\max\{|f''(\xi_4)|,|f''(\xi_3)|\}$

$\therefore\ \exists\ \xi\in(a,b)$ s.t. $|f''(\xi)|\geqslant\frac{4}{(b-a)^2}|f(b)-f(a)|$