Problem Set 6.5

1. (i) ac (而 $ac > b^2 > 0$)

(ii) $a+c$ (而 $a>0$ $c > \frac{b^2}{a} > 0$)

11. 1阶顺序主子式 $= 2$ 记作 C_1

2阶：$2\times5-2\times2=6$ 记作 C_2

3阶：$2\times(5\times8-3\times3)-2\times(2\times8-0)=30$ 记作 C_3

$$S=\begin{pmatrix}2&2&0\\2&5&3\\0&3&8\end{pmatrix}\xrightarrow{(2)-(1)}\begin{pmatrix}2&2&0\\0&3&3\\0&3&8\end{pmatrix}$$

$$\xrightarrow{(3)-(2)}\begin{pmatrix}2&2&0\\0&3&3\\0&0&5\end{pmatrix}$$

$\therefore$ $C_1 = 2 = \text{pivot }1$

$\frac{C_2}{C_1}=\frac{6}{2}=3=\text{pivot }2$

$\frac{C_3}{C_2}=\frac{30}{6}=5=\text{pivot }3$

14. 填空 ① S^{-1} 的特征值是 S 特征值的倒数，而 S 正定，故其特征值均大于0，故 S^{-1} 特征值均大于0 即 S^{-1} 正定；

② $\begin{cases} c>0 \\ ac-b^2>0 \end{cases}$

15. 证明：法一：$\because$ S、T 均正定

$\therefore$ $\forall x \neq 0$ $x^TSx>0$ $x^TTx>0$

$\therefore$ $x^T(S+T)x = x^TSx + x^TTx > 0$

$\therefore$ $S+T$ 正定

法二：$\because$ S、T 均正定

$\therefore$ $\exists$ 可逆矩阵 A、B, s.t. $S=A^TA$ $T=B^TB$

$\therefore$ $S+T=A^TA+B^TB=\begin{pmatrix}A^T & B^T\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$ 正定

19. $\because$ 所有对称矩阵均能相似正交对角化

$\therefore$ S 有 n 个线性无关的特征向量

不同特征值对应的特征向量相互正交；

相同特征值对应的特征向量，将之正交化；

从而这 n 个特征向量两两正交

即 $x_i^Tx_j=0$ ($i\neq j$)

$\therefore$ $x=c_1x_1+c_2x_2+\cdots+c_nx_n$

$x^TSx=\sum_{i=1}^{n}c_i^2\lambda_i x_i^T x_i>0$

21. (1) $S=\begin{pmatrix}s&-4&-4\\-4&s&-4\\-4&-4&s\end{pmatrix}$

若 S 正定，则有 $\det(s)=s>0$

$\det\begin{pmatrix}s&-4\\-4&s\end{pmatrix}=s^2-16>0$

$\det(S)=s\cdot(s^2-16)$

$+4\cdot(-4s-16)-4(16+4s)>0$

即 $(s-8)(s+4)^2>0$

$\therefore$ $s>8$

(2) $T=\begin{pmatrix}t&3&0\\3&t&4\\0&4&t\end{pmatrix}$

若 T 正定，则有 $\det(t)=t>0$

$\det\begin{pmatrix}t&3\\3&t\end{pmatrix}=t^2-9>0$

$\det(T)=t(t^2-16)-3(3t)>0$

$\therefore$ $t>5$

26.

(1) $S=\begin{pmatrix}9&0&0\\0&1&2\\0&2&8\end{pmatrix} \xrightarrow{(3)-2(2)} \begin{pmatrix}9&0&0\\0&1&2\\0&0&4\end{pmatrix}$

$\therefore S = LDL^T = \begin{pmatrix}1&&\\&1&\\&2&1\end{pmatrix}\begin{pmatrix}9&&\\&1&\\&&4\end{pmatrix}\begin{pmatrix}1&&\\&1&2\\&&1\end{pmatrix}$

$C=\sqrt{D}L^T = \begin{pmatrix}3&&\\&1&\\&&2\end{pmatrix}\begin{pmatrix}1&&\\&1&2\\&&1\end{pmatrix}$

(2) $S=\begin{pmatrix}1&1&1\\1&2&2\\1&2&7\end{pmatrix} \xrightarrow[(3)-(1)]{(2)-(1)} \begin{pmatrix}1&1&1\\0&1&1\\0&1&6\end{pmatrix} \xrightarrow{(3)-(2)} \begin{pmatrix}1&1&1\\0&1&1\\0&0&5\end{pmatrix}$

$\therefore S = LDL^T = \begin{pmatrix}1&&\\1&1&\\1&1&1\end{pmatrix}\begin{pmatrix}1&&\\&1&\\&&5\end{pmatrix}\begin{pmatrix}1&1&1\\&1&1\\&&1\end{pmatrix}$

$\therefore C = \sqrt{D}L^T = \begin{pmatrix}1&&\\&1&\\&&\sqrt{5}\end{pmatrix}\begin{pmatrix}1&1&1\\&1&1\\&&1\end{pmatrix}$

Problem Set 7.1

1.(1) $A=\begin{pmatrix}1&2&3&4\\2&4&6&8\\3&6&9&12\\4&8&12&16\end{pmatrix}$ $r=1$

(2) $B=\begin{pmatrix}2&3&4&5\\3&4&5&6\\4&5&6&7\\5&6&7&8\end{pmatrix} = \begin{pmatrix}2&3&4&5\\2&3&4&5\\2&3&4&5\\2&3&4&5\end{pmatrix} + \begin{pmatrix}0&0&0&0\\1&1&1&1\\2&2&2&2\\3&3&3&3\end{pmatrix}$ $r=2$

$A=\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1&2&3&4\end{pmatrix}$

$B=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}2&3&4&5\end{pmatrix} + \begin{pmatrix}0\\1\\2\\3\end{pmatrix}\begin{pmatrix}1&1&1&1\end{pmatrix}$

4. $B=\begin{pmatrix}1&2&2\\1&3&3\end{pmatrix}$

$BB^T=\begin{pmatrix}9&13\\13&19\end{pmatrix}$ $tr(BB^T)=28$

7.

$det(BB^T)=19\times 9-13\times 3=2$

19

$B^TB=\begin{pmatrix}2&5&5\\5&13&13\\5&13&13\end{pmatrix}$ $tr(B^TB)=28$

$det(B^TB)=2\times 0-5\times 0+5\times 0=0$

B不可压缩.

$r(B)=2$ 若将B进行SVD，则其形式为 $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$=\sigma_1 \begin{pmatrix}-\\-\end{pmatrix}\begin{pmatrix}-&-&-\end{pmatrix} + \sigma_2 \begin{pmatrix}-\\-\end{pmatrix}\begin{pmatrix}-&-&-\end{pmatrix}$

共10个数字，比B本身6个元素还多. 故B不可压缩

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Problem Set 7-2

1. (1) $A=\begin{pmatrix}0&4\\0&0\end{pmatrix}$

设特征值为 λ $(-\lambda)^2=0$

$\therefore \lambda=0$

$A^TA=\begin{pmatrix}0&0\\0&16\end{pmatrix}$ $\sigma_2=0$ $v_1=\begin{pmatrix}1\\0\end{pmatrix}$ $v_2=\begin{pmatrix}0\\1\end{pmatrix}$

$AA^T=\begin{pmatrix}16&0\\0&0\end{pmatrix}$ $\sigma_1=4$ $u_1=\begin{pmatrix}0\\1\end{pmatrix}$ $u_2=\begin{pmatrix}1\\0\end{pmatrix}$

$A=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\\&4\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(2) $A=\begin{pmatrix}0&4\\1&0\end{pmatrix}$

设特征值为 λ $(-\lambda)^2-4=0$

$\therefore \lambda=\pm 2$

$AA^T=\begin{pmatrix}16&0\\0&1\end{pmatrix}$ $u_1=\begin{pmatrix}0\\1\end{pmatrix}$ $u_2=\begin{pmatrix}1\\0\end{pmatrix}$

$\sigma_1=1$ $\sigma_2=4$

$A^TA=\begin{pmatrix}1&0\\0&16\end{pmatrix}$ $v_1=\begin{pmatrix}1\\0\end{pmatrix}$ $v_2=\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore A=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&\\&4\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

7. Av

15. $\because A_0$ 奇异且 $\sigma_1>\sigma_2>0$

$\therefore$ 只需将 A 中的 σ_2 转化为 0 即可

$A=(u_1\sigma_1\ \ u_2\sigma_2)(v_1^T\ \ v_2^T)$ 设 A 为 $m\times n$ 矩阵

$A_0=(u_1\sigma_1\ \ u_2\sigma_2)(v_1^T\ \ v_2^T)(\underbrace{1\ 1\cdots 1}_{n个}\ \underbrace{0\ 0\cdots 0}_{n个})$

$=(u_1\sigma_1\ \ u_2\sigma_2)(v_1^T\ \ 0)$

$=\sigma_1 u_1 v_1^T$

22. 因为当 $A\in M_{m\times n}$ 时，

U 是 $m\times m$ 的正交矩阵

(AA^T 的一组标准正交 $\boxed{向量}\boxed{特征}$)

V 是 $n\times n$ 的正交矩阵

两正交矩阵之积仍为正交矩阵

$\therefore Q_1U$、$Q_2V\in$ Orthogonal Matrix.

25. (1) 当 x 对应的特征向量在 S 所有特征向量中最小时，

$\frac{x^TSx}{x^Tx}=\lambda_i$ 取最小值

(2)

Problem Set 7-3

1. $A=\begin{pmatrix}2&1&0&-1&-2\\-1&1&0&1&-1\end{pmatrix}$ $\begin{pmatrix}2&-1\\1&1\\0&0\\-1&1\\-2&-1\end{pmatrix}$

$S=\frac{AA^T}{4}=\begin{pmatrix}\frac{5}{2}&0\\0&1\end{pmatrix}$

$\lambda_1=\frac{5}{2}$ $\lambda_2=1$

$\sigma_1=\frac{\sqrt{10}}{2}>\sigma_2=1$

$u_1=\begin{pmatrix}1\\0\end{pmatrix}$

$\therefore$ 直线 $y=0$ (即 x 轴) 最接近样本中的5点.

Problem Set 7-4

2. $A = \begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$ $\quad \begin{matrix}1 & 3\\ 2 & 6\end{matrix}$

(a) $AA^T = \begin{pmatrix}5 & 15\\ 15 & 45\end{pmatrix}$

$A^TA = \begin{pmatrix}10 & 20\\ 20 & 40\end{pmatrix}$

$\lambda_1 = 0 \quad \lambda_2 = 50$

$u_1 = \begin{pmatrix}-3\\ 1\end{pmatrix} \quad u_2 = \begin{pmatrix}\frac{1}{3}\\ 1\end{pmatrix}$

$v_1 = \begin{pmatrix}-2\\ 1\end{pmatrix} \quad v_2 = \begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}$

$\therefore$ $\cancel{C(A)}$ $N(A^T)$: $\frac{1}{\sqrt{10}}\begin{pmatrix}-3\\ 1\end{pmatrix}$ $\frac{3}{\sqrt{10}}\begin{pmatrix}\frac{1}{3}\\ 1\end{pmatrix}$ $C(A)$

$\cancel{C(A^T)}$ $N(A)$: $\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\ 1\end{pmatrix}$ $\frac{2}{\sqrt{5}}\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}$ $C(A^T)$

~~$N(A)$: $\begin{pmatrix}-2\\ 1\end{pmatrix}$ $N(A^T)$~~

(b) k. $\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$?

3. $Q = UV^T$

$$= \begin{pmatrix}-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}\begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{7\sqrt{2}}{10} & -\frac{\sqrt{2}}{10}\\ \frac{\sqrt{2}}{10} & \frac{7\sqrt{2}}{10}\end{pmatrix}$$

$$S = V\Sigma V^T = \begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}0 & \\ & 5\sqrt{2}\end{pmatrix}\begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{2} & 2\sqrt{2}\\ 2\sqrt{2} & 4\sqrt{2}\end{pmatrix}$$

$QS = \begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix} = A$

$S^2 = A^TA = \begin{pmatrix}10 & 20\\ 20 & 40\end{pmatrix}$ 填空：S的主元为$\sqrt{2}$和0，≥0但不严格大于0

4. $A^+ = V\Sigma^+U^T$

$$= \begin{pmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{pmatrix}\begin{pmatrix}0 & \\ & \frac{1}{5\sqrt{2}}\end{pmatrix}\begin{pmatrix}-\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{50} & \frac{3}{50}\\ \frac{1}{25} & \frac{3}{25}\end{pmatrix}$$

$$A^+A = \begin{pmatrix}\frac{1}{50} & \frac{3}{50}\\ \frac{1}{25} & \frac{3}{25}\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{4}{5}\end{pmatrix} \begin{matrix}1\\ 2\end{matrix}$$

$$AA^+ = \begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}\begin{pmatrix}\frac{1}{50} & \frac{3}{50}\\ \frac{1}{25} & \frac{3}{25}\end{pmatrix} = \begin{pmatrix}\frac{1}{10} & \frac{3}{10}\\ \frac{3}{10} & \frac{9}{10}\end{pmatrix}$$

$A^+A\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}1\\ 2\end{pmatrix} \quad A^+A\begin{pmatrix}3\\ 6\end{pmatrix} = \begin{pmatrix}3\\ 6\end{pmatrix}$

$A^+A\begin{pmatrix}1\\ 3\end{pmatrix} = \begin{pmatrix}\frac{7}{5}\\ \frac{14}{5}\end{pmatrix} \quad A^+A\begin{pmatrix}2\\ 6\end{pmatrix} = \begin{pmatrix}\frac{14}{5}\\ \frac{28}{5}\end{pmatrix}$

$AA^+\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}\frac{7}{10}\\ \frac{21}{10}\end{pmatrix} \quad AA^+\begin{pmatrix}3\\ 1\end{pmatrix} = \begin{pmatrix}\frac{7}{5}\\ \frac{21}{5}\end{pmatrix}$

$AA^+\begin{pmatrix}1\\ 3\end{pmatrix} = \begin{pmatrix}1\\ 3\end{pmatrix} \quad AA^+\begin{pmatrix}2\\ 6\end{pmatrix} = \begin{pmatrix}2\\ 6\end{pmatrix}$

即 AA^+ 将向量投影到列空间上

A^+A 将向量投影到行空间上

A		A^+
row	$\longrightarrow$	column
column	$\longrightarrow$	row
nullspace	$\longrightarrow$	left nullspace
left nullspace	$\longrightarrow$	nullspace

7. (1) $A=\begin{pmatrix}3&3\\-1&1\end{pmatrix}$

$AA^T=\begin{pmatrix}18&0\\0&2\end{pmatrix}$

$A^TA=\begin{pmatrix}10&8\\8&10\end{pmatrix}$

$\lambda_1=18 \quad \lambda_2=2$

$\sigma_1=3\sqrt{2} \quad \sigma_2=\sqrt{2}$

$u_1=\begin{pmatrix}1\\0\end{pmatrix} \quad u_2=\begin{pmatrix}0\\1\end{pmatrix}$

$v_1=\begin{pmatrix}1\\1\end{pmatrix} \quad v_2=\begin{pmatrix}-1\\1\end{pmatrix}$

$A=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}3\sqrt{2}&\\&\sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$

(2) $A=U\Sigma V^T$

$=(u_1\ u_2 \cdots u_r \cdots u_m)\left(\begin{array}{ccc|c}\sigma_1&&&\\&\ddots&&\\&&\sigma_r&\\\hline&&&0\end{array}\right)\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\\\vdots\\v_n\end{pmatrix}$

$=\sum_{i=1}^{r}\sigma_i u_i v_i^T$

上式中0为$(m-r)(n-r)$矩阵

即任意秩r矩阵均可分解为r个秩1矩阵之和.

11. $A=(3\ 4\ 0)$

$AA^T=(25)$

$A^TA=\begin{pmatrix}9&12&0\\12&16&0\\0&0&0\end{pmatrix}$

$\lambda=25$或0 $\quad \sigma=5$或0

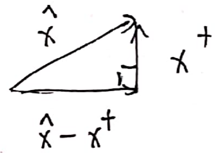
$u=\begin{pmatrix}1\\0\\0\end{pmatrix} \quad v_1=\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\\0\end{pmatrix} \quad v_2=\begin{pmatrix}-\frac{4}{5}\\\frac{3}{5}\\0\end{pmatrix} \quad v_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\therefore A=\begin{pmatrix}1\\0\\0\end{pmatrix}(5\ 0\ 0)\begin{pmatrix}\frac{3}{5}&\frac{4}{5}&0\\-\frac{4}{5}&\frac{3}{5}&0\\0&0&1\end{pmatrix}$

$A^+=\begin{pmatrix}\frac{3}{5}&-\frac{4}{5}&0\\\frac{4}{5}&\frac{3}{5}&0\\0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{5}&0&0\end{pmatrix}^T(1\ 0\ 0)$

$=\begin{pmatrix}\frac{3}{25}\\\frac{4}{25}\\0\end{pmatrix}=\begin{pmatrix}\frac{3}{5}&-\frac{4}{5}&0\\\frac{4}{5}&\frac{3}{5}&0\\0&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{5}\\0\\0\end{pmatrix}(1\ 0\ 0)$

16.



$\Rightarrow$ 在A^TA的Nullspace里

与x^+垂直

由勾股定理知 $\|\hat{x}\|^2=\|\hat{x}-x^+\|^2+\|x^+\|^2$

证毕