习题 7.2

3.(1)(8)(14)

(1) $y'=(2-x+y)^2$

令 $p=2-x+y$. 则 $\frac{dp}{dx}=-1+\frac{dy}{dx}$

$\therefore \frac{dy}{dx}=\frac{dp}{dx}+1$

$\therefore p'+1=p^2$; $\frac{dp}{dx}=p^2-1$

$\frac{1}{p^2-1}dp=1\cdot dx$

$\frac{1}{2}\ln\left|\frac{p-1}{p+1}\right|=x+c$

$\frac{p-1}{p+1}=e^{2x}\cdot e^{2c}$

$\frac{1-x+y}{3-x+y}=e^{2x}\cdot e^{2c}$

$\therefore e^{2x}\cdot e^{c}(3-x+y)-(1-x+y)=0$

(8) 令 $x=\rho\cdot\cos\alpha$. $y=\rho\cdot\sin\alpha$. $\rho\geq 0$,

$dx=-\rho\sin\alpha\, d\alpha$ $dy=\rho\cos\alpha\, d\alpha$

$\rho^2\cdot\cos^2\alpha\, d\alpha+\rho^2\sin^2\alpha\, d\alpha=-\rho^2\sin\alpha\, d\alpha$

$\rho^2\cdot d\alpha=-\rho^2\sin\alpha\, d\alpha$

$\int d\alpha=\int d\cos\alpha$

$\alpha=\cos\alpha+c$

$\arctan\frac{y}{x}=\frac{x}{\sqrt{x^2+y^2}}+c$

Why not! $-\sin\alpha=1$

$\alpha=-\frac{\pi}{2}$?

为什么不能这么做

在以上的计算过程中,基于ρ为一个常数这显然是有问题的.这个作法会涉及偏微分

(8) 令 $\frac{y}{x}=t$. 则 $y=t\cdot x$ $\frac{dy}{dx}=\frac{dt}{dx}\cdot x+t$

$\therefore dy=dt\cdot x+t\cdot dx$

$\therefore x^2dt+tx\cdot dx-tx\cdot dx=x\cdot\sqrt{t^2+1}\,dx$

$\therefore x\,dt=\sqrt{t^2+1}\,dx$

$\therefore \frac{1}{x}dx=\frac{1}{\sqrt{t^2+1}}dt$

$\therefore \ln x+c=\ln(t+\sqrt{t^2+1})$

$\therefore x\cdot e^{c}=t+\sqrt{t^2+1}$

$\therefore x\cdot e^{c}=\frac{y}{x}+\sqrt{y^2/x^2+1}$

$\therefore \sqrt{x^2+y^2}+y-x^2\cdot e^{c}=0$

$\rightarrow x^2=\tilde{c}(y+\sqrt{x^2+y^2})$

(14) $y'+2xy=2x^3y^2$

$\frac{y'}{y^2}+\frac{2x}{y}=2x^3$ 令 $z=y^{-1}$. 则

$dz/dx=-\frac{1}{y^2}\cdot dy/dx$

$\therefore \frac{1}{y^2}\cdot\frac{dy}{dx}=-\frac{dz}{dx}$

$-\frac{dz}{dx}+2xz=2x^3$

$z'-2xz=2x^3$

积分因子为: $e^{\int-2x\,dx}=e^{-x^2}$

$\therefore e^{-x^2}\cdot z'-2x\cdot e^{-x^2}\cdot z=e^{-x^2}\cdot 2x^3$

$\frac{de^{-x^2}\cdot z}{dx}=e^{-x^2}\cdot 2x^3$

$\therefore e^{-x^2}\cdot z=2\int e^{-x^2}\cdot x^3dx$ 令 $x^2=t$.

则 $dt=2x\cdot dx$ $\therefore \int e^{-x^2}\cdot x^2\cdot 2x\,dx$

$=\int te^{-t}dt=-\int t\cdot de^{-t}=-(t\cdot e^{-t}-\int e^{-t}dt)$

$=-t\cdot e^{-t}-e^{-t}=-x^2\cdot e^{-x^2}-e^{-x^2}+c$

$\therefore z=-x^2-1+c\cdot e^{x^2}$

$\therefore \frac{1}{y}=-x^2-1+c\cdot e^{x^2}$ $\therefore y=\frac{1}{x^2+1+c\cdot e^{x^2}}$

习题 7.3 (2)(7)

(2) 令 $y'=p$. $y''=p'$ $x\cdot p'+p^2-p=0$

$p^2-p=-x\cdot\frac{dp}{dx}$

$-\frac{1}{x}\cdot dx=(p^2-p)^{-1}dp$

$\int\frac{1}{x}dx=\int\frac{1}{p(1-p)}dp$ $\frac{1}{p}+\frac{1}{1-p}$

$\ln x+c=\ln p-\ln(1-p)$

$x\cdot e^{c}=\frac{p}{1-p}$

$x\cdot e^{c}-px\cdot e^{c}=p$

$(1+x\cdot e^{c})p=x\cdot e^{c}$

$p=\frac{x\cdot e^{c}}{1+x\cdot e^{c}}=\frac{cx}{1+cx}$

$\therefore y=\int\frac{cx}{1+cx}dx=\int(1-\frac{1}{1+cx})dx$

$=x-\int\frac{1}{1+cx}dx=x-\frac{1}{c}\cdot\int\frac{c}{1+cx}dx$

$=x-\frac{1}{c_1}\ln(1+c_1x)+C_2$

$p(1)=\frac{c_1}{1+c_1}=\frac{1}{2}$ $\therefore c_1=1$

$y=x-\ln(1+x)+C_2$

$y(1)=1-\ln 2+C_2=1-\ln 2$

$\therefore C_2=0$

$\therefore y=x-\ln(1+x)$

(7) 令 $y'=t$. $x\cdot t'-t\cdot\ln t+t=0$

$t\cdot\ln t-t=x\cdot\frac{dt}{dx}$

$\therefore \frac{1}{x}dx=\frac{1}{t\ln t-t}dt$

$\therefore \ln x+c=\int\frac{d\ln t}{\ln t-1}=\ln|\ln t-1|$

$\therefore e^{c}\cdot x=\ln t-1$

$\therefore t=e^{e^{c}\cdot x+1}$

$y=\int e^{e^{c}\cdot x+1}dx=\frac{1}{e^{c_1}}\cdot e^{e^{c_1}\cdot x+1}+C_2$

$\therefore y=e^{-c_1}\cdot e^{e^{c_1}\cdot x+1}+C_2$

$=\frac{1}{c_1}e^{c_1\cdot x+1}+C_2$

七章复习题 1.(5) (7)题见后

(5) $\frac{y'}{y^2}-\frac{1}{1+x}\cdot\frac{1}{y}=1$ 令 $y^{-1}=p$, 则 $\frac{dp}{dx}=\frac{-1}{y^2}\frac{dy}{dx}$

$\therefore y'=-y^2\cdot p'$

$\therefore -p'-\frac{1}{1+x}\cdot p=1$ $\quad\frac{dp}{dx}+\frac{p}{1+x}=-1$

积分因子为 $e^{\int\frac{1}{1+x}dx}=1+x$

$\therefore (1+x)\cdot\frac{dp}{dx}+p=-(1+x)$

$\therefore \frac{d(1+x)p}{dx}=-(x+1)$

$(x+1)p=-\int(x+1)dx=-\frac{1}{2}x^2-x+C$

$\therefore y=\frac{-(2x+2)}{x^2+2x+C}$

习题7.4:

1. $k_1(y_1^{(n)}+a_{n-1}y_1^{(n-1)}+\cdots+a_0(x)\cdot y_1)=k_1f(x)$

$k_2(y_2^{(n)}+a_{n-1}y_2^{(n-1)}+\cdots+a_0(x)y_2)=k_2g(x)$

故 $(k_1y_1^{(n)}+k_2y_2^{(n)})+a_{n-1}(k_1y_1^{(n-1)}+k_2y_2^{(n-1)})+\cdots+a_n(x)(k_1y_1+k_2y_2)=k_1f(x)+k_2g(x)$

$\therefore y_3=k_1y_1+k_2y_2$ 为

$y^{(n)}+a_{n-1}(x)\cdot y^{(n-1)}+\cdots+a_n(x)y=k_1f(x)+k_2g(x)$

的根

2. $\frac{d\varphi(x)}{dx}+\varphi(x)=f(x)$ $\quad\frac{d\varphi(x+T)}{d(x+T)}+\varphi(x+T)=f(x+T)=f(x)$

$\therefore \varphi(x+T)$ 也为原方程的解

又 $y(0)=\varphi(0)=\varphi(T)$. 故由解的唯一性

$\varphi(x)=\varphi(x+T)$ 即 $\varphi(x)$ 以T为周期

3. $W=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

$a_1(x)W=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ a_1y_1^{(n-1)} & a_1y_2^{(n-1)} & \cdots & a_1y_n^{(n-1)} \end{vmatrix}$

$W'=\begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}+\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}+\cdots+\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$

$=0+0+\cdots+\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$

$W'+a_1(x)W=\begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n)}+a_1y_1^{(n-1)} & \cdots & y_n^{(n)}+a_1y_n^{(n-1)} \end{vmatrix}$

而 $y_i^{(n)}+a_1y_i^{(n-1)}+\cdots+a_ny_i=0$

$\therefore W'+a_1(x)W=0$

6. 方程的通解为 $y=C_1e^{x}+C_2\cdot e^{-x}$; $y'=C_1\cdot e^{x}-C_2\cdot e^{-x}$

$\begin{cases}y(0)=1\\y'(0)=0\end{cases}\Rightarrow\begin{cases}C_1+C_2=1\\C_1-C_2=0\end{cases}\Rightarrow C_1=C_2=\frac{1}{2}$

$\begin{cases}y(0)=0\\y'(0)=1\end{cases}\Rightarrow\begin{cases}C_1+C_2=0\\C_1-C_2=1\end{cases}\Rightarrow\begin{cases}C_1=\frac{1}{2}\\C_2=-\frac{1}{2}\end{cases}$

$\begin{cases}y(0)=a\\y'(0)=b\end{cases}\Rightarrow\begin{cases}C_1+C_2=a\\C_1-C_2=b\end{cases}\Rightarrow\begin{cases}C_1=\frac{a+b}{2}\\C_2=\frac{a-b}{2}\end{cases}$

8. $(x^3+x^2)-(x^2+x)=x^3-x$. $y_1=1$; $y_2=x$; $y_3=x^3-x$

y_1,y_2,y_3 线性无关

故通解为 $y=x^2+x+C_1+C_2x+C_3(x^3-x)$

$=a_1x+x^2+a_2x^3+a_3$

$W(x)=\begin{vmatrix} 1 & x & x^3-x \\ 0 & 1 & 3x^2-1 \\ 0 & 0 & 6x \end{vmatrix}$

关于定理7.4.3.(2)

① 注意在特定区间I上.

② 此题可以构造 $x\neq0$. 如 $xy'''-y''+2=0$

在写为标准形过程中x会作分母. $x\neq0$

5. 若$\exists x_0\in(a,b)$ $\varphi(x_0)$与$\varphi'(x_0)$均为0.则:
$y=\varphi(x)$与$y\equiv 0$均有: $\begin{cases} y''+a_1(x)y'+a_2(x)y=0 \\ y(x_0)=0;\ y'(x_0)=0 \end{cases}$

由解的唯一性 $\varphi(x)\equiv 0$
与题干矛盾
$\therefore \forall x_0\in(a,b)$ $\varphi(x_0)$与$\varphi'(x_0)$至多一个为0

思考题:
令 $m_1(x)=-\int_{x_0}^{x}\frac{\varphi_2(t)f(t)}{W(\varphi_1,\varphi_2)(t)}dt$ $m_2(x)=\int_{x_0}^{x}\frac{\varphi_1(t)f(t)}{W(\varphi_1,\varphi_2)(t)}dt$. 则:

$m_1'(x)=\frac{-\varphi_2(x)f(x)}{W(\varphi_1,\varphi_2)(x)}$ $m_2'(x)=\frac{\varphi_1(x)f(x)}{W(\varphi_1,\varphi_2)(x)}$ 从而:

$$\begin{cases} m_1'(x)\varphi_1(x)+m_2'(x)\varphi_2(x)=0 & \text{①} \\ m_1'(x)\varphi_1'(x)+m_2'(x)\varphi_2'(x)=f(x) & \text{②} \end{cases}$$

故令 $\psi(x)=m_1\varphi_1+m_2\varphi_2$, 结合①②有:
$\psi'(x)=m_1\varphi_1'+m_2\varphi_2'$ $\psi''(x)=m_1\varphi_1''+m_2\varphi_2''+f(x)$

而 $\begin{cases} \varphi_1''+p(x)\varphi_1'+q(x)\varphi_1=0 \\ \varphi_2''+p(x)\varphi_2'+q(x)\varphi_2=0 \end{cases}$

即 $\psi''(x)+p(x)\psi'(x)+q(x)\psi(x)=m_1(\varphi_1''+p(x)\varphi_1'+q(x)\varphi_1)$
$+m_2(\varphi_2''+p(x)\varphi_2'+q(x)\varphi_2)+f(x)=f(x)$

故$\psi(x)$为特解. 故
$\forall C_1、C_2\in R$. $\psi(x)+C_1\varphi_1+C_2\varphi_2$为通解

七章复习题 1.(7)
$y'=\frac{4x^3y}{x^4+y^2}$ 令 $\frac{x^2}{y}=u$ 则 $\frac{4u^2}{u+\frac{1}{u}}=2u-xu'$

$\therefore -\frac{dx}{x}=\frac{u^2+1}{2u^3-2u}du=(-\frac{1}{u}+\frac{1}{u+1}+\frac{1}{u-1})du$

$\therefore -\ln|x|=-\ln|u|+\ln|u+1|+\ln|u-1|+C$

$\therefore \frac{1}{x}=C_1\cdot\frac{u^2-1}{u}$

$\therefore C\cdot(x^4-y^2)=xy$