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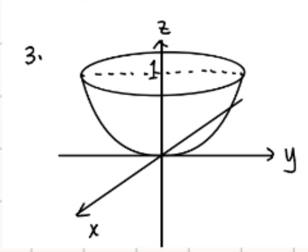


3. $z=f(x,y)=(x^2+y^2)/2 \quad f'(x)=x \quad f'y=y$

$M=\iint_S z\,ds=\iint_{D_{xy}}(\frac12x^2+\frac12y^2)\sqrt{1+f_x'^2+f_y'^2}\,dxdy=\iint_{D_{xy}}(\frac12x^2+\frac12y^2)\sqrt{1+x^2+y^2}\,dxdy$

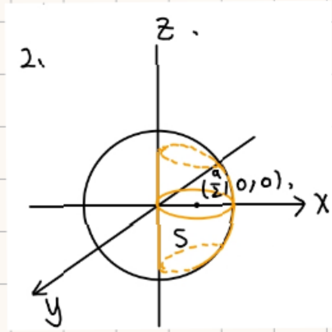
$D_{xy}=\{0\le x^2+y^2\le 2\}$ 换用极坐标

$M=\int_0^{2\pi}d\theta\int_0^{\sqrt2}\frac12\rho^3\sqrt{1+\rho^2}\,d\rho=\pi\cdot\int_0^{\sqrt2}\rho^3\sqrt{1+\rho^2}\,d\rho$

$\int_0^{\sqrt2}\rho^3\sqrt{1+\rho^2}\,d\rho \xlongequal[t\in(0,\arctan\sqrt2)]{\rho=\tan t} \int_0^{\arctan\sqrt2}\tan^3t\cdot\sec t\cdot\sec^2t\,dt=\int_0^{\arctan\sqrt2}(\sec^2t-1)\sec^2t\,d\sec t$

$=(-\frac13\sec^3t+\frac15\sec^5t)\Big|_0^{\arctan\sqrt2}=(\frac15\cdot9\sqrt3-\sqrt3)-(\frac15-\frac13)=\frac45\sqrt3+\frac2{15}$

$\therefore M=\iint_{D_{xy}}(\frac12x^2+\frac12y^2)\sqrt{1+x^2+y^2}\,dxdy=\frac{12\sqrt3+2}{15}\pi$



2. 此题视为曲面积分则：柱体底面圆心 $(\frac a2,0)$

S: $x=\frac a2+\frac a2\cos\theta \quad y=\frac a2\sin\theta \quad z=t.$ 又 $\because 0\le x^2+y^2+z^2\le a^2 \quad \therefore 0\le z^2\le a^2-\frac{a^2}{2}-\frac{a^2}{2}\cos\theta$

$\therefore 0\le z^2\le\frac{a^2}{2}-\frac{a^2}{2}\cos\theta \quad \therefore t\in[-\frac{a}{\sqrt2}\sqrt{1-\cos\theta},\frac{a}{\sqrt2}\sqrt{1-\cos\theta}]$

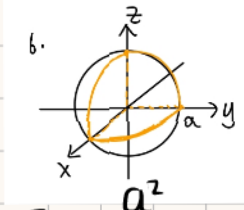
记 $S=r(\theta,t)$. 则 $r_\theta'=(-\frac a2\sin\theta,\frac a2\cos\theta,0) \quad r_t'=(0,0,1)$

$r_\theta'\times r_t'=(\frac a2\cos\theta,\frac a2\sin\theta,0) \quad \|r_\theta'\times r_t'\|=\frac a2$

$\sigma(S)=\int_0^{2\pi}d\theta\int_{-\frac{a}{\sqrt2}\sqrt{1-\cos\theta}}^{\frac{a}{\sqrt2}\sqrt{1-\cos\theta}}\frac a2\,dt=\int_0^{2\pi}\frac{\sqrt2}{2}a^2\sqrt{1-\cos\theta}\,d\theta=\int_0^{2\pi}\frac{\sqrt2}{2}a^2\cdot\sqrt{2\sin^2\frac\theta2}\,d\theta=\int_0^{2\pi}a^2\sin\frac\theta2\,d\theta$

令 $=\theta/2$ 则 $d\theta=2d$ 原式 $=2\int_0^{\pi}a^2\sin u\,du=2a^2(-\cos u)\Big|_0^{\pi}=4a^2$

6. 在第一卦限内. $M=\frac{1}{8}4\pi a^2=\frac{1}{2}\pi a^2$ $M_{yz}=\iint\limits_{S} x\,ds$.



曲面方程为 $z=f(x,y)=\sqrt{a^2-x^2-y^2}$

$$1+f_x'^2+f_y'^2=1+\left(\frac{1}{2}\cdot\frac{-2x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{1}{2}\cdot\frac{-2y}{\sqrt{a^2-x^2-y^2}}\right)^2=1+\frac{x^2+y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

∴ $\iint\limits_{S} x\,ds=\iint\limits_{D_{xy}} x\sqrt{1+f_x'^2+f_y'^2}\,dydx$ $D_{xy}=\{(x,y)\mid x\ge0,y\ge0,x^2+y^2\le a^2\}$

极坐标换元. 则 $\theta\in[0,\frac{\pi}{2}]$ $\rho\in[0,a]$ $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$

原式 $=\int_0^{\frac{\pi}{2}}d\theta\int_0^a\frac{a\rho^2\cos\theta}{\sqrt{a^2-\rho^2}}d\rho$ $\int_0^a\frac{\rho^2}{\sqrt{a^2-\rho^2}}d\rho$ 令 $\rho=a\sin t$. $t\in[0,\frac{\pi}{2}]$. 则 $d\rho=a\cos t$

$$\int_0^a\frac{\rho^2}{\sqrt{a^2-\rho^2}}d\rho=\int_0^{\frac{\pi}{2}}\frac{a^2\sin^2t}{a\cos t}a\cos t\,dt=a^2\int_0^{\frac{\pi}{2}}\sin^2t\,dt=a^2\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}=\frac{a^2}{4}\pi$$

$$\int_0^{\frac{\pi}{2}}d\theta\int_0^a\frac{a\rho^2\cos\theta}{\sqrt{a^2-\rho^2}}d\rho=\frac{a^3}{4}\pi\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta=\frac{a^3}{4}\pi$$

∴ $\bar{x}=\frac{M_{yz}}{M}=\frac{a}{2}$. 由对称性 $\bar{y}=\bar{z}=\bar{x}=\frac{1}{2}a$ 重心坐标 $(\frac{a}{2},\frac{a}{2},\frac{a}{2})$

上半球面 由对称性,质心在z轴上 $M=\frac{1}{2}\cdot4\pi a^2=2\pi a^2$

$M_{xy}=\iint\limits_{S} z\,ds$ 曲面方程为 $z=f(x,y)=\sqrt{a^2-x^2-y^2}$ 同上方计算

$$1+f_x'^2+f_y'^2=1+\left(\frac{1}{2}\cdot\frac{-2x}{\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{1}{2}\cdot\frac{-2y}{\sqrt{a^2-x^2-y^2}}\right)^2=1+\frac{x^2+y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$\iint\limits_{S} z\,ds=\iint\limits_{D_{xy}}\sqrt{a^2-x^2-y^2}\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy=a\iint\limits_{D_{xy}}dxdy=a\sigma(xy)=\pi a^2\cdot a=\pi a^3$$

∴ $\bar{z}=\frac{M_{xy}}{M}=\frac{a}{2}$

∴ 质心为 $(0,0,\frac{a}{2})$

10. 切平面法向量为 $\vec{n}=(\frac{2}{a^2}x,\frac{2}{b^2}y,\frac{2}{c^2}z)$ 记点 $P(x,y,z)$

$$L(x,y,z)=\left|\frac{\vec{OP}\cdot\vec{n}}{|\vec{n}|}\right|=\left|\frac{\frac{x^2}{a^2}\cdot2+\frac{y^2}{b^2}\cdot2+\frac{z^2}{c^2}\cdot2}{\sqrt{\frac{4}{a^2}\cdot\frac{x^2}{a^2}+\frac{4}{b^2}\cdot\frac{y^2}{b^2}+\frac{4}{c^2}\cdot\frac{z^2}{c^2}}}\right|=\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$$

记S上半部分为 S_1. $\iint\limits_{S_1}L(x,y,z)ds=\iint\limits_{D_{xy}}\frac{\sqrt{1+z_x'^2+z_y'^2}}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}dxdy$ $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ $D_{xy}=\{(x,y)\mid\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1\}$

$z_x'=\frac{c}{2}\cdot\frac{-\frac{2}{a^2}x}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}=-\frac{c}{a^2}\frac{x}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$ ∴ $1+z_x'^2+z_y'^2=1+\frac{c^2}{a^4}\cdot\frac{x^2}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}+\frac{c^2}{b^4}\cdot\frac{y^2}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$=\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{c^2}{a^4}x^2+\frac{c^2}{b^4}y^2}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}=\frac{\frac{c^2}{c^4}z^2+\frac{c^2}{b^4}y^2+\frac{c^2}{a^4}x^2}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$\iint\limits_{S_1}L(x,y,z)ds=\iint\limits_{D_{xy}}\frac{c}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dxdy$ $D_{xy}=\{(x,y)\mid\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1\}$. 换用极坐标. $x=ar\cos\theta$ $y=br\sin\theta$

原式 $=\int_0^1dr\int_0^{2\pi}c\cdot\frac{1}{\sqrt{1-r^2}}abr\,d\theta=2\pi abc\int_0^1\frac{r}{\sqrt{1-r^2}}dr=2\pi abc\cdot(-\sqrt{1-r^2})\Big|_0^1=2\pi abc$

由对称性 xy平面下方与xy平面上方积分结果相同

∴ $\iint\limits_{S}L(x,y,z)ds=2\iint\limits_{S_1}L(x,y,z)ds=4\pi abc$