Chenyang Zhao 赵晨阳 from the school of software
Info ID: 2020012363 Tel: 18015766633

Collaborations: Hanwen Cao, Mingdao Liu, Sijia Liu
↳ My girlfriend ♡!

1.1. a very famous example: consider 4 subspace in R^3:

$V_1=\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ $V_2=\text{span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ $V_3=\text{span}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ $V_4=\text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

It's a very obvious counter example.

1.2: Take V_1, V_2, V_3, V_4 in answer 1.1 as the same counter example.
V_1 (the x-axis) and V_2 (the y axis) are linear-independent, so as $V_1 V_3 V_4$ and V_2, V_3, V_4. But the four subspace are not linear-independent.

1.3: $V_1 \cap V_2 = V_3 \cap V_4 = (V_1+V_2)\cap(V_3+V_4)=0$

$\dim(V_1+V_2+V_3+V_4) = \dim(V_1+V_2) + \dim(V_3+V_4) + \dim((V_1+V_2)\cap(V_3+V_4))$
$= \dim(V_1+V_2)+\dim(V_3+V_4) = \dim V_1+\dim V_2+\dim V_3+\dim V_4 + \dim(V_1\cap V_2)+\dim(V_3\cap V_4)$
$= \dim V_1+\dim V_2+\dim V_3+\dim V_4$

Hence: V_1, V_2, V_3, V_4 are linear independent.

2: Target: find $V_1\oplus V_2\oplus\cdots\oplus V_n=V$. $T(V_1)\subseteq V_1$ $T(V_2)\subseteq V_2$ ...
We know that any Matrix $X_{n\times n}$ can be devided into a symmetrical matrix A and a skew-symetrical matrix A.

$A=\frac{A+A^T}{2}$ $B=\frac{A-A^T}{2}$ A satisfies $A^T=A$. B satisfies $B^T=-B$.

$\therefore V=V_1\oplus V_2$ $V_1=\{X_{n\times n}\,|\,X^T=X\}$ $V_2=\{X_{n\times n}\,|\,X^T=-X\}$

And if $X\in(V_1\cap V_2)\Rightarrow X^T=X=-X\Rightarrow X=0$ $\therefore V_1\cap V_2=0$
and we know that $\dim V_1=\frac{n^2+n}{2}$ (To confirm a symetric matrix we need to know all the upper triangular element, including diagnal)
and we know that $\dim V_2=\frac{n^2-n}{2}$ (To confirm an anti-symetric matrix we need to know all the upper triangular element, except diagnal)

Hence the block form is $\begin{bmatrix} I_{\frac{n^2+n}{2}} & \\ & -I_{\frac{n^2-n}{2}}\end{bmatrix}$

3. $A_{m\times n} X_n = Y_m$

$$\begin{tikzcd}
V_r \quad \operatorname{Ran}(A^T) \arrow[r, "T_1"] \arrow[dr, "T_2"'] & \operatorname{Ran}(A) \quad W_r \\
V_{n-r} \quad \operatorname{Ker}(A) \arrow[r, "T_4"'] \arrow[ur, "T_3"] & \operatorname{Ker}(A^T) \quad W_{m-r}
\end{tikzcd}$$

We block $A: R^n \to R^m$ into four smaller linear transformation T_1, T_2, T_3, T_4

$T_1: R(A^T) \to R(A)$ Choose the bases of $R(A^T)$. $V_1, V_2 \cdots V_r$ bases of $R(A)$: $W_1 \cdots W_r$

$T_1([V_1\ V_2 \cdots V_r]) = [W_1\ W_2 \cdots W_r]\ A_{11\,r\times r}$ Chose Any vector $X \in R^r$. Hence we have

$T_1([V_1 \cdots V_n] \cdot X) = [W_1 W_2 \cdots W_r] \cdot A_{11} \cdot X \Rightarrow T_1(V_r \cdot X) = W_r \cdot A_{11} X$

And $T_2(V_r X) = W_{m-r} \cdot A_{21} \cdot X$ $\quad T_3(V_{n-r} X) = W_r \cdot A_{12} \cdot X$ $\quad T_4(V_{n-r} \cdot X) = W_{m-r} \cdot A_{22} X$

target ①: T_1 is bijection.

1) T_1 is injection. That is to prove if $T_1(A^T X) = 0 \Rightarrow A^T X = 0$

$A \cdot A^T X = T_1(A^T X) + T_2(A^T X)$. if $T_1(A^T X) = 0$

$\Rightarrow A \cdot A^T X = T_2(A^T X) \in N(A^T)$. But $A \cdot A^T X \in \operatorname{Ran}(A)$

$\operatorname{Ran}(A) \cap N(A^T) = 0 \quad \therefore A \cdot A^T X = 0$

$\therefore A^T X \in N(A)$ and $A^T X \in \operatorname{Ran}(A^T)$ $\quad N(A) \cap \operatorname{Ran}(A^T) = 0$

$\therefore A^T X = 0$

So T_1 is injection

2) T_1 is surjection.

$A \cdot A^T X \in \operatorname{Ran}(A) \quad \therefore T_2(A^T X) \in N(A^T) \quad T_1(A^T X) \in R(A)$ proof is attached

Hence $T_2(A^T X)$ must be 0

For any $V \in \operatorname{Ran}(A)$ $\quad V = A \cdot X$. $\operatorname{Ran}(A) = \operatorname{Ran}(A \cdot A^T)$ $\therefore V = AX = A \cdot A^T y$

every v from $\operatorname{Ran}(A)$ we have a vector $A^T y \in \operatorname{Ran}(A^T)$ as its preimage

So T_1 is bijection. So A_{11} is invertible $\operatorname{Rank}(A_{11}) = \operatorname{Rank}(A)$

② We already know that $T_2(A^T X)$ must be 0 for any $X \in R^m$.

So $T_2(V_r \cdot X) = W_{m-r} \cdot A_{21} \cdot X = 0$ (for All X)

W_{m-r} is column full rank. So A_{21} must be 0

Hence $\operatorname{rank}(A_{21}) = 0$

Hint: T的定义域内所有向量均为0 则T的表示必为0

③ for T_3 and T_4. any $V_1 \in N(A)$ we have:

$A \cdot V_1 = T_3(V_1) + T_4(V_1) = 0 \quad T_3(V_1) \in \operatorname{Ran}(A) \quad T_4(V_1) \in N(A^T)$

Hence $T_3(V_1) = T_4(V_1) = 0$ for $\forall\, V_1 \in N(A)$

$T_3(V_{n-r} X) = W_r \cdot A_{12} \cdot X$ always be 0

$\therefore A_{12} = 0$. Similarly we have $T_4(V_{n-r} X) = W_{m-r} \cdot A_{22} X = 0$

$\therefore A_{12} = A_{22} = 0$

$\therefore \operatorname{rank}(A_{12}) = \operatorname{rank}(A_{22}) = 0$

proof: $ran(A) = ran(A \cdot A^T)$

which is to say: $N(A^T) = N(A \cdot A^T)$

if $A^T x = 0$, it is obvious $A \cdot A^T x = 0$

if $A \cdot A^T X = 0 \Rightarrow X^T A \cdot A^T X = 0 \Rightarrow (A^T X)^T \cdot AX = 0 \Rightarrow AX = 0$

So $ran(A) = ran(A \cdot A^T)$

1.2.4: Take $[V_1\ V_2 \cdots V_r]$ as the bases of V.

$A[V_r] = [V_r] \cdot T$ So $A \cdot [V_r] \cdot X = [V_r] \cdot T \cdot X$

Chose A vector x which satisfies $TX = \lambda X$

And $M = [V_r] \cdot X$.

Then $A \cdot M = A[V_r]X = [V_r]TX = \lambda[V_r]X = \lambda M$

Hence A has an eigenvector in V.

1.2.4.2: Suppose A have an eigenvalue λ. Hence $|A - \lambda I| = 0$

$ker(A - \lambda I)$ is an eigenspace of A.

$\forall v \in ker(A - \lambda I)$ $(A - \lambda I)v = 0$

$(A - \lambda I)(Bv) = ABv - \lambda Bv = BAv - B\lambda v = B(Av - \lambda v) = 0$

$\therefore Bv \in ker(A - \lambda I)$

$\therefore ker(A - \lambda I)$ is B-invariant

According to 1.2.4.1

$(A - \lambda I)$ is B invarriant.

Then B has an eigenvactor in $ker(A - \lambda I)$, we suppose it's X.

And $X \in ker(A - \lambda I)$ so X is Also an eigenvector for A

Hence A and B have common eigenvector

1.2.5: $N_\infty(D) = \{f \mid f \text{ is a polynomial}\}$

Because we know if $f = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} \cdots + a_0 \cdot x^0$

Then: $D^n \cdot f = 0$.

$N_\infty(D - I)$ isn't spanned by e^x.

(by spanning, we mean $f(x) = K \cdot e^x$ and K is a const number)

Take $G(x) = (x+1) \cdot e^x$

So $(D - I)^2 G(x)$

$= (D^2 - 2D + I) G(x)$

$= e^x(x+3) - 2 \cdot e^x(x+2) + e^x(x+1)$

$= 0$

But $G(x)$ is not spanned by e^x.