LiuMingdao 2020011156 Week4 HW3

Exercise 1.3.1

1. let an R-basis of $\mathbb{C}^2$ be $T = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}i\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\i\end{bmatrix}\}$.

Then $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}a+bi\\c+di\end{bmatrix} = T\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$.

$A(\begin{bmatrix}x\\y\end{bmatrix}) = T\begin{bmatrix}1&0&-1&0\\0&-1&0&0\\0&1&-1&0\\0&1&0&-1\end{bmatrix}\hat{x}$,

where $\hat{x}$ is the coordinate of $\begin{bmatrix}x\\y\end{bmatrix}$ under basis T.

let $B = \begin{bmatrix}1&0&-1&0\\0&-1&0&0\\0&1&-1&0\\0&1&0&-1\end{bmatrix}$.

$\det(B-\lambda I) = (\lambda+1)^3(\lambda-1)$

$Ker(B-I) = span(\begin{bmatrix}1\\0\\0\\0\end{bmatrix})$.

$Ker(B+I) = span(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\0\end{pmatrix})$

$Ker((B+I)^2) = span(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}1\\4\\8\end{bmatrix})$.

note that $\begin{bmatrix}1\\4\\0\\0\end{bmatrix} \in Ker((B+2)^2) - Ker(B+I)$

calcutate: $(B+I)\begin{bmatrix}1\\4\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\4\\4\end{bmatrix}$.

note that $\begin{bmatrix}2\\0\\4\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, linearly independent.

let $X = \begin{bmatrix}2&1&0&1\\0&4&0&0\\4&0&0&0\\4&0&1&0\end{bmatrix}$.

Then $B = X\begin{bmatrix}-1&1&&\\&-1&&\\&&-1&\\&&&1\end{bmatrix}X^{-1}$

let $H = TX = \{\begin{bmatrix}2\\4+4i\end{bmatrix}, \begin{bmatrix}1+4i\\0\end{bmatrix}, \begin{bmatrix}0\\i\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$

so: under Basis $H = \{\begin{bmatrix}2\\4+4i\end{bmatrix}, \begin{bmatrix}1+4i\\0\end{bmatrix}, \begin{bmatrix}0\\i\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$.

A has Jordan normal Form $\begin{bmatrix}-1&1&&\\&-1&&\\&&-1&\\&&&1\end{bmatrix}$. □

2. Pick a basis $T = \{x^4, x^3, x^2, x, 1\}$.

Then let $p = T\hat{x}$, $\hat{x}$ is the coordinate of p under basis T.

Then: $A(T\hat{x}) = A(p) = A(TB\hat{x})$

where $B = \begin{bmatrix}0&0&0&0&0\\4&0&0&0&0\\0&3&0&1&0\\0&0&2&0&0\\0&0&0&1&1\end{bmatrix}$

$\det(B-\lambda I) = -(\lambda-1)(\lambda+\sqrt{2})(\lambda-\sqrt{2})\lambda^2$

$Ker(B) = span\begin{pmatrix}0\\1\\0\\-3\\3\end{pmatrix}$

$Ker(B^2) = span(\begin{bmatrix}1\\-4\\-6\\12\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-6\\0\\12\end{bmatrix})$

note that: $\begin{bmatrix}1\\0\\-6\\0\\12\end{bmatrix} \in Ker(B^2) - Ker(B)$

Then: $B\begin{bmatrix}1\\0\\-6\\0\\12\end{bmatrix} = \begin{bmatrix}0\\4\\0\\-12\\12\end{bmatrix} := \vec{x_1}$.

let $\begin{bmatrix}1\\0\\-6\\0\\12\end{bmatrix} := \vec{x_2}$

$Ker(B-I) = span[\begin{smallmatrix}0\\0\\0\\0\\1\end{smallmatrix}] := \vec{x_3}$

$Ker(B-\sqrt{2}I) = span(\begin{bmatrix}0\\0\\\sqrt{2}-1\\2-\sqrt{2}\\\sqrt{2}\end{bmatrix}) := \vec{x_4}$

$Ker(B+\sqrt{2}I) = span(\begin{bmatrix}0\\0\\-\sqrt{2}-1\\2+\sqrt{2}\\-\sqrt{2}\end{bmatrix}) := \vec{x_5}$.

let $X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \vec{x}_4 & \vec{x}_5 \end{bmatrix}$.

let $H = TX$.

Then: $B = X \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \sqrt{2} & \\ & & & & -\sqrt{2} \end{bmatrix} X^{-1}$

so under basis $H =$

$\{x^3-12x+12,\ x^4-6x^2+12,\ 1,\ (\sqrt{2}-1)x^2+(2-\sqrt{2})x+\sqrt{2},$

$(-\sqrt{2}-1)x^2+(2+\sqrt{2})x-\sqrt{2}\}$

A has Jordan Normal Form

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \sqrt{2} & \\ & & & & -\sqrt{2} \end{bmatrix}.$$

3. $\det(A-\lambda I) = (\lambda^2 - a_1a_4)(\lambda^2 - a_2a_3)$

3. $\boxed{\text{Situation 0}}$ $a_1 \sim a_4$ are all non-zero.

the eigonvalues of A are $\pm\sqrt{a_1a_4}, \pm\sqrt{a_2a_3}$.

though $a_1a_4 = a_2a_3$ may be true.

but that doesn't matter.

under basis $\left\{ \begin{bmatrix} 0 \\ \sqrt{a_2} \\ \sqrt{a_3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{a_2} \\ -\sqrt{a_3} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ \sqrt{a_4} \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ -\sqrt{a_4} \end{bmatrix} \right\}$

Jordan Normal Form is $\begin{bmatrix} \sqrt{a_2a_3} & & & \\ & -\sqrt{a_2a_3} & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4} \end{bmatrix}$

$\boxed{\text{Situation 1}}$

$\boxed{1.1}$ $a_1 = 0$, a_2, a_3, a_4 non-zero.

~~pick base~~ the eigenvalue $0, \pm\sqrt{a_2a_3}$.

algebraird multiplicity of 0 is 2

while $\dim Ker(A) = 1$.

$Ker(A^2) = span\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in Ker(A^2) - Ker(A)$

$\Rightarrow A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{pmatrix}$.

so under basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{a_2} \\ \sqrt{a_3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{a_2} \\ \sqrt{a_3} \\ 0 \end{bmatrix} \right\}$

Jordan Normal Form is $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}$.

$\boxed{1.2}$ similarly, $a_2 = 0$; a_1, a_3, a_4 non-zero

basis: $\left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ \sqrt{a_4} \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ -\sqrt{a_4} \end{bmatrix} \right\}$.

Jordan Normal Form $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4} \end{bmatrix}$

$\boxed{1.3}$ $a_3 = 0$, a_1, a_2, a_4 non-zero.

basis: $\left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ \sqrt{a_4} \end{bmatrix}, \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ -\sqrt{a_4} \end{bmatrix} \right\}$.

Jordan Normal Form.

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4} \end{bmatrix}$$

$\boxed{2.6}$ $a_2=a_4=0$.

basis $\left\{\begin{bmatrix}a_1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\a_3\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$

Jordan Normal Form: Ditto

Situation 3

$\boxed{3.1}$ $a_1=a_2=a_3=0$, $a_4\neq0$. all eigenvalues $=0$.

$A=\begin{bmatrix} & 0\,0\,0 \\ a_4 & \end{bmatrix}$. $A^2=0$.

under basis $\left\{\begin{bmatrix}0\\0\\0\\a_4\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$

Jordan Normal Form: $\begin{bmatrix}0&1&&\\&0&&\\&&0&\\&&&0\end{bmatrix}$

$\boxed{3.2}$ $a_1=a_2=a_4=0$, $a_3\neq0$.

basis: $\left\{\begin{bmatrix}0\\0\\a_3\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$

Jordan Normal Form: Ditto.

$\boxed{3.3}$ $a_1=a_3=a_4=0$, $a_2\neq0$

basis: $\left\{\begin{bmatrix}0\\a_2\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

Jordan Normal Form: Ditto.

$\boxed{3.4}$ $a_2=a_3=a_4=0$; $a_1\neq0$.

basis: $\left\{\begin{bmatrix}a_1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$.

Jordan Normal Form: Ditto.

Situation 4

$a_1=a_2=a_3=a_4=0$

$A=0$.

basis: $\{\vec{e_1},\vec{e_2},\vec{e_3},\vec{e_4}\}$.

Jordan Normal Form: $\begin{bmatrix}0&&&\\&0&&\\&&0&\\&&&0\end{bmatrix}$

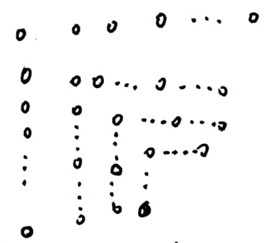
□

Exercise 1.3.2.

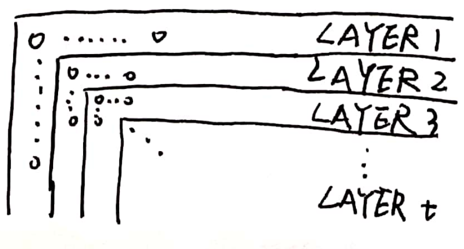
1. The two diagrams are transposition of the other. (When zero content is omitted).

2. Proof.

Suppose n has a self-conjugate partition, ϕ.



because of its symetric structure, it can always be divided into different layers as ~~foo~~ follow:

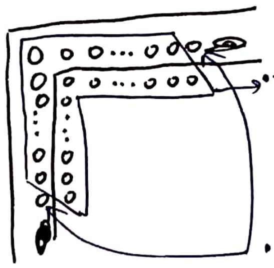


note the following facts:

① the number of dots in each layer is odd:

~~be~~ First, omit the dot on the main ~~diagonal diagnal~~ diagonal, then the symetric parts have the same number of dots, ~~then~~ then add the dot on the diagonal. We got an odd number.

② ~~the inn~~ each inner layer's number of dots must be ~~at least~~ smaller than its ~~adjacent~~ adjacent outer layer by at least 2.



$\boxed{\mathrm{I}}$

From ①, ②, we conclude that all the number of dots in ~~each~~ each layer are odd numbers and are distinct,

And from outer layer to inner layer the number decrease strictly.

So every self~~conjug~~-conjugate diagram of dots induces ~~as~~ a distinct odd partition.

$\boxed{\mathrm{II}}$

Suppose n is partition to several distinct odd numbers $\alpha_1, \dots, \alpha_k$. (from larger to small)

we place dots, s.t. number of ~~dan~~ dot's in Layer-i $= \alpha_i$.

So every distinct odd partition induces a self-conjugat diagram of dots.

From $\boxed{\mathrm{I}}$, $\boxed{\mathrm{II}}$, we get a bijection.

the number of self-conjugate partition = the number of distinct odd partition. □

3.

$\boxed{1.4}$ $a_4=0$. a_2, a_1, a_3 non-zero.

basis: $\left\{ \begin{bmatrix} a_1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\\sqrt{a_2}\\\sqrt{a_3}\\0 \end{bmatrix}, \begin{bmatrix} 0\\\sqrt{a_2}\\-\sqrt{a_3}\\0 \end{bmatrix} \right\}$

Jordan Normal Form:

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}.$$

$\boxed{\text{Situation 2}}$

$\boxed{2.1}$ $a_1=a_4=0$, a_2, a_3 non-zero ~~non-zero~~

basis: $\left\{ \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\\sqrt{a_2}\\\sqrt{a_3}\\0 \end{bmatrix}, \begin{bmatrix} 0\\\sqrt{a_2}\\-\sqrt{a_3}\\0 \end{bmatrix} \right\}$

Jordan Normal Form:

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}$$

$\boxed{2.2}$ $a_2=a_3=0$, a_1, a_4 non-zero.

basis: $\left\{ \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} \sqrt{a_1}\\0\\0\\\sqrt{a_4} \end{bmatrix}, \begin{bmatrix} \sqrt{a_1}\\0\\0\\-\sqrt{a_4} \end{bmatrix} \right\}$

Jordan Normal Form

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4} \end{bmatrix}$$

$\boxed{2.3}$ $a_1=a_2=0$ all eigenvalues are zero.
a_3, a_4 non-zero

$A=\begin{bmatrix} & & & 0\\ & & 0 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$, $A^2=\begin{bmatrix} & & & 0\\ & & 0 & \\ & 0 & & \\ 0 & & & \end{bmatrix}=O$.

$\mathrm{Ker}(A)=\mathrm{span}\left\{ \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix} \right\}$.

$\mathrm{Ker}(A^2)$ = domain of A.

select: $\begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} \in \mathrm{Ker}(A^2)-\mathrm{Ker}(A)$

$A\begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}=\begin{bmatrix} 0\\0\\0\\a_4 \end{bmatrix}$, $A\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}=\begin{bmatrix} 0\\0\\a_3\\0 \end{bmatrix}$.

So, under basis

$\left\{ \begin{bmatrix} 0\\0\\0\\a_4 \end{bmatrix}, \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\a_3\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} \right\}$

Jordan Normal Form $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1\\ & & & 0 \end{bmatrix}$.

$\boxed{2.4}$ $a_1=a_3=0$, a_2, a_4 non-zero.

~~basis:~~

basis: $\left\{ \begin{bmatrix} 0\\0\\0\\a_4 \end{bmatrix}, \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\a_2\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix} \right\}$

Jordan Normal Form: Ditto.

$\boxed{2.5}$ $a_4=a_3=0$, a_2, a_1 non-zero:

basis: $\left\{ \begin{bmatrix} a_1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\a_2\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix} \right\}$.

Jordan Normal Form: Ditto.