Chenyang Zhao ID:2020012363

collaborators: Hanwen Cao, Mingdao Liu

Pray for India.

Problem One

1.1

Let $g = c$ for some constant real number c. Since we know that dual vector v is a a linear map, hence, by product rule, for any $p \in R^3$:

$$v(g^2) = v(g \cdot g) = g(p) \cdot v(g) + g(p)v(g) = 2c \cdot v(g) \tag{1}$$

On the other hand, we have:

$$v(g^2) = v(g \cdot g) = v(c \cdot g) = c \cdot v(g) \text{ (becuase v is linear)} \tag{2}$$

So:

$$c \cdot v(g) = 2c \cdot v(g) \tag{3}$$

On one hand, if $c = 0$, then $v(g) = v(0) = v(g^2) = g(p) \cdot v(g) + g(p)v(g) = 0$.

On the other hand, if $c \neq 0$, then from (4) we have $c \cdot v(g) = 0$, hence $v(g) = 0$

In conclusion, $v(g) = 0$ for all constant function g.

1.2

By definition of x,y,z we have $x(p) = p_1$, $y(p) = p_2$, $z(p) = p_3$.

by product rule and since v is linear,hence:

$$\begin{aligned} v((x-p_1)f) &= (x-p_1)(p) \cdot v(f) + f(p) \cdot v(x-p_1) = x(p) \cdot v(f) \\ &= x(p) \cdot v(f) - p_1(p) \cdot v(f) + f(p) \cdot v(x) - f(p) \cdot v(p_1) \end{aligned} \tag{4}$$

and from 1.1 and $x(p) = p_1, p_1(p) = p_1$ we know that:

$$(4) = p_1 \cdot v(f) - p_1 \cdot v(f) + f(p) \cdot v(x) - f(p) \cdot 0 = f(p) \cdot v(x) \tag{5}$$

Same thing with x, we have:

$y(p) = p_2, p_2(p) = p_2$, v is linear:

$$\begin{aligned} v((y-p_2)f) &= (y-p_2)(p) \cdot v(f) + f(p) \cdot v(y-p_2) = y(p) \cdot v(f) \\ &= y(p) \cdot v(f) - p_2(p) \cdot v(f) + f(p) \cdot v(y) - f(p) \cdot v(p_2) \\ &= p_2 \cdot v(f) - p_2 \cdot v(f) + f(p) \cdot v(y) - f(p) \cdot 0 = f(p) \cdot v(y) \end{aligned} \tag{6}$$

$z(p) = p_3, p_3(p) = p_3$, v is linear:

$$\begin{aligned} v((z-p_3)f) &= (z-p_3)(p) \cdot v(f) + f(p) \cdot v(z-p_3) = z(p) \cdot v(f) \\ &= z(p) \cdot v(f) - p_3(p) \cdot v(f) + f(p) \cdot v(z) - f(p) \cdot v(p_3) \\ &= p_3 \cdot v(f) - p_3 \cdot v(f) + f(p) \cdot v(z) - f(p) \cdot 0 = f(p) \cdot v(z) \end{aligned} \tag{7}$$

1.3

Suppose a < 1, b < 1, c < 1, then a = b = c = 0, it is a contradiction. So at least one number in {a, b, c} ≥ 1. Without loss of generalization, assume a≥ 1. let d=a-1

let $f=(x-p_1)^d\cdot(y-p_2)^b\cdot(z-p_3)^c$*, and for* d,b,c*, if* $d=b=c=0$*,hence* $a+b+c=d+1=1$*, which is a contradiction to a + b + c > 1, so there must be at least one number in {d, b, c} ≥ 1. Without loss of generalization, assume d≥ 1.*

Then we know:

$$\begin{aligned} f(p) &= [(x-p_1)^d\cdot(y-p_2)^b\cdot(z-p_3)^c](p) \\ &= (x-p_1)(p)\cdot[(x-p_1)^{(d-1)}\cdot(y-p_2)^b\cdot(z-p_3)^c](p) \\ &= (p_1-p_1)\cdot[(x-p_1)^{(d-1)}\cdot(y-p_2)^b\cdot(z-p_3)^c](p)=0 \end{aligned} \tag{8}$$

and we know that
$(x-p_1)^a\cdot(y-p_2)^b\cdot(z-p_3)^c=(x-p_1)\cdot(x-p_1)^d\cdot(y-p_2)^b\cdot(z-p_3)^c=(x-p_1)\cdot f$

hence:

$$\begin{aligned} v((x-p_1)^a\cdot(y-p_2)^b\cdot(z-p_3)^c) &= v((x-p_1)\cdot f) \\ &= f(p)\cdot v(x)=0\cdot v(x)=0 \end{aligned} \tag{9}$$

1.4

Using Taylor Expansion at $\boldsymbol{p}$ *for* f*, we have:*

$$f(x,y,z)=f\left(p_1,p_2,p_3\right)+\frac{\partial f}{\partial x}(\boldsymbol{p})\left(x-p_1\right)+\frac{\partial f}{\partial y}(\boldsymbol{p})\left(y-p_2\right)+\frac{\partial f}{\partial z}(\boldsymbol{p})(z-p_3)+\triangle\left(x,y,z\right) \tag{10}$$

where $\triangle\,(x,y,z)$ *is the remainder term.*

By the definition of Taylor Expansion, we know that $\triangle\,(x,y,z)$ *is a linear combination of terms with form* $(x-p_1)^a\cdot(y-p_2)^b\cdot(z-p_3)^c$ *where* $a+b+c\geq 2$*, since terms with form* $(x-p_1)^a\cdot(y-p_2)^b\cdot(z-p_3)^c$ *with* $0\leq a+b+c\leq 1$ *have all be included in* $f\left(p_1,p_2,p_3\right)+\frac{\partial f}{\partial x}(\boldsymbol{p})\left(x-p_1\right)+\frac{\partial f}{\partial y}(\boldsymbol{p})\left(y-p_2\right)+\frac{\partial f}{\partial z}(\boldsymbol{p})(z-p_3)$*.*

So from 1.3 we know $v(\triangle\,(x,y,z))=0$*. and* $f(p_1,p_2,p_3)$ *is a constant number, so* $v(f(p_1,p_2,p_3))=0$*, and we know* v *is linear, hence*

$$\begin{aligned} v(f) &= v(f\left(p_1,p_2,p_3\right)+\frac{\partial f}{\partial x}(\boldsymbol{p})\left(x-p_1\right)+\frac{\partial f}{\partial y}(\boldsymbol{p})\left(y-p_2\right)+\frac{\partial f}{\partial z}(\boldsymbol{p})(z-p_3)+\triangle\left(x,y,z\right) \\ &= v(f\left(p_1,p_2,p_3\right))+v(\frac{\partial f}{\partial x}(\boldsymbol{p})\left(x-p_1\right))+v(\frac{\partial f}{\partial y}(\boldsymbol{p})\left(y-p_2\right))+v(\frac{\partial f}{\partial z}(\boldsymbol{p})(z-p_3))+v(\triangle\left(x,y,z\right)) \\ &= v((x-p_1)\cdot\frac{\partial f}{\partial x}(\boldsymbol{p}))+v((y-p_2)\cdot\frac{\partial f}{\partial y}(\boldsymbol{p}))+v((z-p_3)\cdot\frac{\partial f}{\partial z}(\boldsymbol{p})) \\ &= \frac{\partial f}{\partial x}(\boldsymbol{p})v(x)+\frac{\partial f}{\partial y}(\boldsymbol{p})v(y)+\frac{\partial f}{\partial z}(\boldsymbol{p})v(z)-v(p_1)\cdot\frac{\partial f}{\partial x}(\boldsymbol{p})-v(p_2)\cdot\frac{\partial f}{\partial y}(\boldsymbol{p})-v(p_3)\cdot\frac{\partial f}{\partial z}(\boldsymbol{p}) \\ &= \frac{\partial f}{\partial x}(\boldsymbol{p})v(x)+\frac{\partial f}{\partial y}(\boldsymbol{p})v(y)+\frac{\partial f}{\partial z}(\boldsymbol{p})v(z) \end{aligned} \tag{11}$$

1.5

By definition of direction derivative, we know that for $\forall f$ *and given point* $\boldsymbol{p} \in \mathbb{R}^3$

$$\begin{aligned}
\nabla_v f &= \lim_{t \to 0^+} \frac{f(\boldsymbol{p} + t\boldsymbol{v}) - f(\boldsymbol{p})}{t} \\
&= \lim_{t \to 0^+} \frac{f(\boldsymbol{p}) + \frac{\partial f}{\partial x}(\boldsymbol{p})v(x)t + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y)t + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z)t + o(\|\boldsymbol{v}\|t) - f(\boldsymbol{p})}{t} \\
&= \frac{\partial f}{\partial x}(\boldsymbol{p})v(x) + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y) + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z) \\
&= v(f)
\end{aligned}$$

$Problem \quad two$

2.1 *Target——to show that* $Xp(fg) = f(p)Xp(g) + g(p)Xp(f)$

$$\begin{aligned}
Xp(f \cdot g) = (X(f \cdot g))(p) = (fX(g) + gX(f))(p) = f(X(g))(p) + g(X(f))(p) \\
= f(p)(X(g))(p) + g(p)(X(f))(p) = f(p)Xp(g) + g(p)Xp(f)
\end{aligned} \tag{12}$$

2.2

If we input any $p \in \mathbb{R}^3$*, hence* $df(X)(p) = df_p(X_p)$*,* df_p *is a covector. From 1.5 we know that deriviation at p is canonical isomoprphism to tangent vector at p, and from 2.1 we see that* X_p *is a deriviation at p, it is exactly a tangent vector, let the corresponding tangent vector be* $\boldsymbol{w}$ *, hence.*

$$\boldsymbol{w} = \begin{bmatrix} X_{\boldsymbol{p}}(X) \\ X_{\boldsymbol{p}}(Y) \\ X_{\boldsymbol{p}}(Z) \end{bmatrix} \text{,where X,Y,Z are the coordinate functions.} \tag{13}$$

$$\begin{aligned}
\text{so } df(X)|_p = df_p(\boldsymbol{w}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\boldsymbol{p}) & \frac{\partial f}{\partial y}(\boldsymbol{p}) & \frac{\partial f}{\partial z}(\boldsymbol{p}) \end{bmatrix} \begin{bmatrix} X_{\boldsymbol{p}}(X) \\ X_{\boldsymbol{p}}(Y) \\ X_{\boldsymbol{p}}(Z) \end{bmatrix} \\
= \frac{\partial f}{\partial x}(\boldsymbol{p})X_{\boldsymbol{p}}(X) + \frac{\partial f}{\partial y}(\boldsymbol{p})X_{\boldsymbol{p}}(Y) + \frac{\partial f}{\partial z}(\boldsymbol{p})X_{\boldsymbol{p}}(Z)
\end{aligned} \tag{14}$$

Also from 1.4, we have:

$$(X(f))(\boldsymbol{p}) = X_p(f) = \frac{\partial f}{\partial x}(\boldsymbol{p})X_{\boldsymbol{p}}(X) + \frac{\partial f}{\partial y}(\boldsymbol{p})X_{\boldsymbol{p}}(Y) + \frac{\partial f}{\partial z}(\boldsymbol{p})X_{\boldsymbol{p}}(Z) \tag{15}$$

So $df(X)|_{\boldsymbol{p}}$=$X(f)(\boldsymbol{p})$*, ie.* $df(X) = X(f)$.

2.3

$$
\begin{aligned}
(X \circ Y - Y \circ X)(f \cdot g) &= X(Y(f \cdot g)) - Y(X(f \cdot g)) \\
&= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\
&= f \cdot X(Y(g)) + Y(g) \cdot X(f) + g \cdot X(Y(f)) + Y(f) \cdot X(g) \\
&- (X(g) \cdot Y(f) + f \cdot Y(X(g)) + g \cdot Y(X(f)) + X(f) \cdot Y(g)) \\
&= f \cdot X(Y(g)) - f \cdot Y(X(g)) + g \cdot X(Y(f)) - g \cdot Y(X(f)) \\
&= f \cdot (X \circ Y - Y \circ X)(g) + g \cdot (X \circ Y - Y \circ X)(f)
\end{aligned} \tag{16}
$$

Hence by definition, $X \circ Y - Y \circ X$ is always a vector field.

2.4

Since A and B are skew-symmetric , then $A^T = -A$, $B^T = -B$. Hence

$$
\begin{aligned}
(AB - BA)^T &= (AB)^T - (BA)^T \\
&= \left(B^T A^T\right) - \left(A^T B^T\right) \\
&= (-B)(-A) - (-A)(-B) \\
&= BA - AB \\
&= -(AB - BA)
\end{aligned}
$$

So we know $(AB - BA)$ is skew-symmetric.
