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习题 4.5 3,5,7

3.

(1) 对正方体分为6个面 $S_1, S_2, \cdots, S_6$.

在 S_1面: $x=0, |y|\le 1, |z|\le 1$, 而面的单位正法向量为 $(-1,0,0)$ $x=0, dx=0$. 故

$$\iint_{S_1} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{D_{yz}}-x\,dy\,dz=\iint_{\substack{|y|\le 1\\|z|\le 1}}0\cdot dy\,dz=0$$

同理: S_2面: $y=0, |x|\le 1, |z|\le 1$, 面的单位正法向量为 $(0,-1,0)$ $y=0$, $dy=0$. 故

$$\iint_{S_2} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{D_{xz}}-y\,dx\,dz=\iint_{\substack{|y|\le 1\\|z|\le 1}}0\cdot dy\,dz=0$$

S_3面: $z=0, |x|\le 1, |y|\le 1$, 面的单位正法向量为 $(0,0,-1)$ $z=0$, $dz=0$. 故

$$\iint_{S_3} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{D_{xy}}-z\,dx\,dy=\iint_{\substack{|x|\le 1\\|y|\le 1}}0\cdot dy\,dz=0$$

在 S_4面: $x=1, |y|\le 1, |z|\le 1$, 而面的单位正法向量为 $(1,0,0)$ $x=1, dx=0$. 故

$$\iint_{S_4} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{D_{yz}}x\,dy\,dz=\iint_{\substack{|y|\le 1\\|z|\le 1}}1\cdot dy\,dz=\sigma(D_{yz})=1$$

同理: S_5面: $y=1, |x|\le 1, |z|\le 1$, 面的单位正法向量为 $(0,1,0)$ $y=1$, $dy=0$. 故

$$\iint_{S_5} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\int_{D_{xz}}y\,dx\,dz=\iint_{\substack{|y|\le 1\\|z|\le 1}}1\cdot dy\,dz=\sigma(D_{xz})=1$$

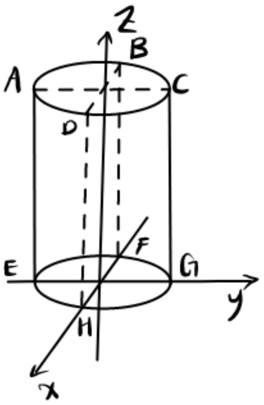
S_6面: $z=1, |x|\le 1, |y|\le 1$, 面的单位正法向量为 $(0,0,1)$ $z=1$, $dz=0$. 故

$$\iint_{S_6} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{D_{xy}}z\,dx\,dy=\iint_{\substack{|x|\le 1\\|y|\le 1}}1\cdot dy\,dz=\sigma(D_{xy})=1$$

$$\therefore \iint_{S^+} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=\iint_{\sum_{i=1}^{6}S_i} x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy=3$$

(2) 如果将柱面拆开,则分类太琐碎 试错.因为此题求单位正法向量太难,故直接利用对称性

$$\iint_{S^+} z^2\,dx\wedge dy=\pm\iint_{D_{xy}} z^2\,dx\,dy$$ 而 S^+在 xy平面上投影 D_{xy}为0. $\therefore \pm\iint_{D_{xy}} z^2\,dx\,dy=0$



对 $\iint_{S^+} x^2\,dy\wedge dz=\iint_{S_1} x^2\,dy\wedge dz+\iint_{S_2} x^2\,dy\wedge dz$. 其中 S_1为 ABCGFE 曲线外侧

S_2为ADCGHE 曲线外侧

S_1与S_2投影在 yz面上均为ACGE,但法向量大小相同方向相反,即:

$$\iint_{S^+} x^2\,dy\wedge dz=\iint_{D_{xy}}(1-y^2)\,dy\,dz-\iint_{D_{xy}}(1-y^2)\,dy\,dz=0$$ 函数关于 yoz面对称

对 $\iint_{S^+} y^2\,dz\wedge dx=\iint_{S_3} y^2\,dz\wedge dx+\iint_{S_4} y^2\,dz\wedge dx$. 其中 S_3为BCDHGF 曲线外侧

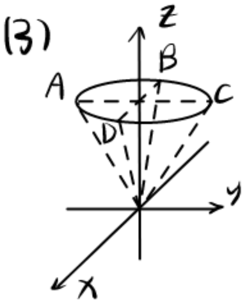
S_4为BADHEF 曲线外侧

S_3与S_4投影在 zx面上均为BDHF,但法向量大小相同方向相反,即:

$$\iint_{S^+} y^2\,dz\wedge dx=\iint_{D_{xy}}(1-x^2)\,dz\,dx-\iint_{D_{xy}}(1-x^2)\,dz\,dx=0$$

综上 $$\iint_{S^+} x^2\,dy\wedge dz+y^2\,dz\wedge dx+z^2\,dx\wedge dy=0$$

(3)



$\iint_{S^+}(y-z)dy\wedge dz=\iint_{S_{ADCO}}(y-z)dy\wedge dz+\iint_{S_{ABCO}}(y-z)dy\wedge dz$

ADCO面单位正法向量与(1,0,0)夹角为锐角, ABCO面单位正法向量与(1,0,0)夹角为钝角

$\iint_{S^+}(y-z)dy\wedge dz=\iint_{D_{ACO}}(y-z)dydz-\iint_{D_{ACO}}(y-z)dydz=0$

同理: $\iint_{S^+}(z-x)dz\wedge dx=\iint_{S_{BCDO}}(z-x)dz\wedge dx+\iint_{S_{BADO}}(z-x)dz\wedge dx$

$=\iint_{D_{DBO}}(z-x)dzdx-\iint_{D_{DBO}}(z-x)dzdx=0$

S^+单位正法向量与z轴正半轴为钝角

$\iint_{S^+}(x-y)dx\wedge dy=-\iint_{D_{xy}}(x-y)dxdy=\iint_{x^2+y^2\le h}(y-x)dxdy\xlongequal[\text{换元}]{\text{极坐标}}\int_0^{2\pi}d\theta\int_0^h(\rho\sin\theta-\rho\cos\theta)\rho d\rho$

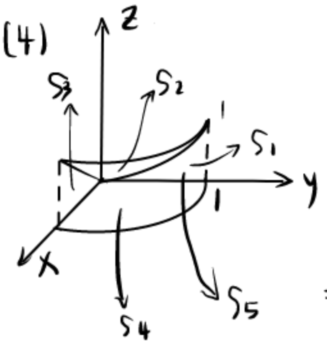
$=\int_0^{2\pi}d\theta\int_0^h\sqrt{2}\rho\sin(\theta-\frac{\pi}{4})\rho d\rho=\sqrt{2}\int_0^{2\pi}\sin(\theta-\frac{\pi}{4})\cdot\frac{1}{3}\rho^3\Big|_0^h d\theta=-\frac{\sqrt{2}}{3}h^3\cos(\theta-\frac{\pi}{4})\Big|_0^{2\pi}$

$=-\frac{\sqrt{2}}{3}h^3\cdot 0=0$

综上: $\iint_{S^+}(y-z)dy\wedge dz+\iint_{S^+}(x-y)dx\wedge dy+\iint_{S^+}(z-x)dz\wedge dx=0$

> remark: 轮换对称性
>
> $\iint_{x^2+y^2\le h}(y-x)dxdy=\iint_{x^2+y^2\le h}ydxdy-\iint_{x^2+y^2\le h}xdxdy=\iint_{x^2+y^2\le h}xdxdy-\iint_{x^2+y^2\le h}xdxdy=0$

(4)



S_1面法向量为(-1,0,0), x=0

$\iint_{S_1}y^2zdx\wedge dy+z^2xdy\wedge dz+x^2ydz\wedge dx=-\iint_{D_{yz}}z^2xdydz=-\iint_{D_{yz}}0dydz=0$

S_2面法向量与z轴正方向成锐角,与x,y轴正方向成钝角

$\iint_{S_2}y^2zdx\wedge dy+z^2xdy\wedge dz+x^2ydz\wedge dx=\iint_{D_{xy}}y^2zdxdy-\iint_{D_{yz}}z^2xdydz-\iint_{D_{zx}}x^2ydzdx$

$=\iint_{D_{xy}}(x^2+y^2)y^2dxdy-\iint_{D_{yz}}\sqrt{z-y^2}z^2dydz-\iint_{D_{zx}}x^2\sqrt{z-x^2}dzdx$

$=\int_0^{\frac{\pi}{2}}d\theta\int_0^1\rho^5\sin^2\theta d\rho-\int_0^1dz\int_0^{\sqrt{z}}\sqrt{z-y^2}z^2dy-\int_0^1dx\int_{x^2}^1x^2\sqrt{z-x^2}dz$

$\int_0^{\frac{\pi}{2}}d\theta\int_0^1\rho^5\sin^2\theta d\rho=\frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta=\frac{1}{6}\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}=\frac{\pi}{24}$

$\int_0^1dx\int_{x^2}^1x^2\sqrt{z-x^2}dz=\int_0^1x^2\cdot\frac{2}{3}(z-x^2)^{\frac{3}{2}}\Big|_{x^2}^1dx$

$=\frac{2}{3}\int_0^1(1-x^2)^{\frac{3}{2}}x^2dx\xlongequal{x=\sin t}\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos^4t\sin^2tdt$

$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos^4t-\cos^6tdt$

$=\frac{2}{3}\cdot\left(\frac{3!!}{4!!}-\frac{5!!}{6!!}\right)\cdot\frac{\pi}{2}=\frac{1}{48}\pi$

$\int_0^1dz\int_0^{\sqrt{z}}\sqrt{z-y^2}z^2dy\xlongequal{y=\sqrt{z}\sin t}$

$\int_0^1dz\int_0^{\frac{\pi}{2}}z^3\cos^2t\,dt$

$=\int_0^1z^3\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}dz=\frac{\pi}{16}$

$\therefore\iint_{S_2}y^2zdx\wedge dy+z^2xdy\wedge dz+x^2ydz\wedge dx=\frac{\pi}{24}-\frac{1}{48}\pi-\frac{1}{16}\pi=-\frac{1}{24}\pi$

S_3面法向量为$(0,-1,0)$，$y=0$

$$\iint_{S_3} y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx=-\iint_{D_{xz}}x^2y\,dxdz=-\iint_{D_{xz}}0\,dxdz=0$$

S_4面法向量为$(0,0,-1)$，$z=0$

$$\iint_{S_4} y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx=-\iint_{D_{xy}}y^2z\,dxdy=-\iint_{D_{xy}}0\,dxdy=0$$

S_5面法向量与z轴正方向成直角，与x,y轴正方向成锐角

$$\iint_{S_5} y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx=+\iint_{D_{yz}}z^2x\,dydz+\iint_{D_{zx}}x^2y\,dzdx$$

$$\iint_{D_{yz}}z^2x\,dydz=\iint_{D_{yz}}z^2\sqrt{1-y^2}\,dydz=\int_0^1dz\int_0^1z^2\sqrt{1-y^2}\,dy\ \underline{\underline{y=\sin t}}\ \int_0^1dz\int_0^{\frac{\pi}{2}}z^2\cos^2t\,dt$$

$$=\int_0^1z^2\frac{1!!}{2!!}\cdot\frac{\pi}{2}\,dz=\frac{1}{12}\pi$$

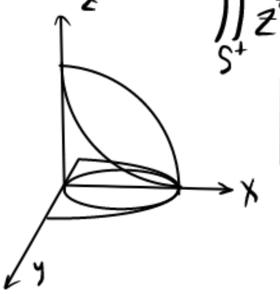
$$\iint_{D_{zx}}x^2y\,dzdx=\iint_{D_{zx}}x^2\sqrt{1-x^2}\,dzdx=\int_0^1dz\int_0^1x^2\sqrt{1-x^2}\,dx\ \underline{\underline{x=\sin t}}\ \int_0^{\frac{\pi}{2}}\sin^2t\cos^2t\,dt=\int_0^{\frac{\pi}{2}}(\sin^2t-\sin^4t)dt$$

$$=\left(\frac{1!!}{2!!}-\frac{3!!}{4!!}\right)\cdot\frac{\pi}{2}=\frac{1}{16}\pi$$

$$\therefore\iint_{S_5} y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx=\frac{1}{16}\pi+\frac{1}{12}\pi=\frac{7}{48}\pi$$

$$\therefore\iint_{S^+} y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx=\frac{7}{48}\pi-\frac{1}{24}\pi=\frac{5}{48}\pi$$

(5)



曲面法向量与z轴正方向成锐角 (注意此题极坐标代换与r范围)

$$\iint_{S^+}z^2\,dx\wedge dy=\iint_{D_{xy}}R^2-x^2-y^2\,dxdy \quad 令\begin{cases}x=\frac{R}{2}+r\cos\theta\\ y=r\sin\theta\end{cases}$$

$$\left|\frac{\nabla(x,y)}{\nabla(r,\theta)}\right|=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$ (平移不改变 det)

$$\iint_{D_{xy}}R^2-x^2-y^2\,dxdy=\int_0^{2\pi}d\theta\int_0^{\frac{R}{2}}\left(\frac{R^2}{2}-Rr\cancel{\sin\theta}\right)\cdot r\,dr$$

$x^2+y^2=Rx$仅在边界上成立

$$=\int_0^{2\pi}\left(\frac{R^2}{2}\cdot\frac12\cdot\frac{R^2}{4}-R\sin\theta\cdot\frac13\cdot\frac{R^3}{8}\right)d\theta$$

$$\cancel{=\frac{1}{48}\int_0^{2\pi}(3R^4-2R^4\sin\theta)d\theta=\frac18R^4}$$

$$\iint_{D_{xy}}R^2-x^2-y^2\,dxdy=\iint_{D_{xy}}R^2dxdy-\iint_{D_{xy}}x^2+y^2\,dxdy=R^2\cdot\sigma(xy)-\int_0^{2\pi}d\theta\int_0^{\frac{R}{2}}\left(\frac{R^2}{4}+Rr\cos\theta+r^2\right)r\,dr$$

$$后式=\int_0^{2\pi}\left(\frac{R^4}{16}+R\cos\theta\cdot\frac13\cdot\frac{R^3}{8}+\frac14\cdot\frac{R^4}{16}\right)d\theta=\frac54\cdot\frac{R^4}{16}\cdot2\pi=\frac{5}{32}R^4\pi$$

5. S^+为 $x^2+y^2+z^2=1$ 在一象限 背离原点一面. 单位正法向量与x, y, z轴均呈锐角

∴ $I=\iint_{S^+} xy\,dy\wedge dz + yz\,dz\wedge dx + zx\,dx\wedge dy$

$$\iint_{S^+} xy\,dy\wedge dz = \iint_{D_{yz}} xy\,dydz = \iint_{\substack{y^2+z^2\le 1\\ y\ge 0,\ z\ge 0}} \sqrt{1-y^2-z^2}\,y\,dydz = \int_0^{\frac{\pi}{2}}d\theta\int_0^1\sqrt{1-r^2}\,r^2\sin\theta\,dr$$

$$\overset{r=\sin t}{=\!=\!=} \int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin^2t\cos^2t\sin\theta\,dt = \int_0^{\frac{\pi}{2}}\sin\theta\cdot\left(\frac{1!!}{2!!}-\frac{3!!}{4!!}\right)\cdot\frac{\pi}{2}\,d\theta = -\frac{\pi}{16}\cdot\cos\theta\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{16}$$

又由曲面对称性: $\iint_{S^+} xy\,dy\wedge dz=\iint_{S^+} yz\,dz\wedge dx=\iint_{S^+} zx\,dx\wedge dy$

∴ $I=3\iint_{S^+} xy\,dy\wedge dz = \frac{3}{16}\pi$

5. 法二. $\vec{v}(xy, yz, zx)$. S_1^+上单位法向量 $\vec{n}=(x,y,z)=(\cos\alpha,\cos\beta,\cos\gamma)$

$I=\iint_{S^+}\vec{v}\cdot\vec{n}\,ds = \iint_{S^+} x^2y+y^2z+z^2x\,ds$ 而由面积性质 $dy\wedge dz=\cos\alpha\,ds = x\,ds$

$dz\wedge dx=\cos\beta\,ds=y\,ds$ $\quad dx\wedge dy=\cos\gamma\,ds = z\,ds$

∴ $I=\iint_{S^+}x^2dz\wedge dx + y^2dx\wedge dy+z^2dy\wedge dz = \iint_{D_{zx}}x^2dzdx+\iint_{D_{xy}}y^2dxdy+\iint_{D_{yz}}z^2dydz$

此处上的对应关系发生了转化

因 S^+与被积函数的对称性

$I=3\iint_{D_{zx}}x^2dzdx = 3\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta\int_0^1 r^3dr = \frac{3}{4}\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}=\frac{3\pi}{16}$

7. 原式 $=\iint_{S^+}y^2dy\wedge dz+z^2dz\wedge dx+(x^2+y^2)dx\wedge dy$

记 $S=r(u,v)$ $\quad r_u'=(\cos v,\sin v,0)$ $\quad r_v'=(-u\sin v,\ u\cos v,\ a)$

$\alpha = r_u'\times r_v' = (a\sin v,\ -a\cos v,\ u)$ 记为(A,B,C) 此向量与曲面正向呈锐角

原式 $=\iint_{S^+}(x^2+y^2,y^2,z^2)\cdot\frac{\alpha}{|\alpha|}ds$ $\quad ds=\sqrt{A^2+B^2+C^2}\,dudv$ ∴原式 $=\iint_{D_{uv}}(y^2,z^2,x^2+y^2)\cdot\alpha\,dudv$

$$=\iint_{D_{uv}}(u^2\sin^2v,\ a^2v^2,\ u^2)\cdot(a\sin v,\ -a\cos v,\ u)\,dudv$$

$$=\iint_{D_{uv}}(au^2\sin^3v-a^3v^2\cos v+u^3)\,du\,dv = \int_0^{2\pi}dv\int_0^1(au^2\sin^3v-a^3v^2\cos v+u^3)\,du$$

$$=\int_0^{2\pi}\left(\frac{1}{3}au^3\sin^3v\Big|_0^1-a^3v^2\cos v\cdot u\Big|_0^1+\frac{1}{4}u^4\Big|_0^1\right)dv=\int_0^{2\pi}\left(\frac{1}{3}a\sin^3v-a^3v^2\cos v+\frac{1}{4}\right)dv$$

$$\int_0^{2\pi}\frac{1}{3}a\sin^3v\,dv$$
$$=\int_0^{2\pi}\frac{1}{3}a\cdot(1-\cos^2v)\,d(-\cos v)$$
$$=-\frac{a}{3}\int_0^{2\pi}(1-\cos^2v)\,d\cos v$$
$$=-\frac{a}{3}\left(\cos v-\frac{1}{3}\cos^3v\right)\Big|_0^{2\pi}$$
$$=0$$

$$\int_0^{2\pi}a^3v^2\cos v\,dv$$
$$=a^3\int_0^{2\pi}v^2\,d\sin v$$
$$=a^3\left(v^2\sin v\Big|_0^{2\pi}-2\int_0^{2\pi}\sin v\cdot v\,dv\right)$$
$$=2a^3\int_0^{2\pi}v\,d\cos v$$
$$=2a^3\left(v\cos v\Big|_0^{2\pi}-\int_0^{2\pi}\cos v\,dv\right)$$
$$=2a^3\cdot 2\pi=4a^3\pi$$

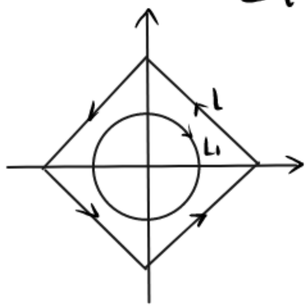
∴原式 $=0-4a^3\pi+\frac{\pi}{2}=\left(\frac{1}{2}-4a^3\right)\pi$

习题 4.6 No. 2(3)(4) 4(2) 8(2)(3) 9-11

2. 记 $\int_{L^+}\frac{(x+y)dx+(y-x)dy}{x^2+y^2} = \int_{L^+} Pdx+Qdy$. $\frac{\partial P}{\partial y} = \frac{(x^2+y^2)-2y(x+y)}{(x^2+y^2)^2}$ $\frac{\partial Q}{\partial x} = \frac{-(x^2+y^2)-2x(y-x)}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} = -\frac{x^2+y^2-2xy-2y^2+x^2+y^2+2xy-2x^2}{(x^2+y^2)^2}=0$

(3) 记D的有向边界为L^+, P与Q在L^+围成的区域内不连续可微.
取L_1为 $x^2+y^2=r^2$, r充分小的圆周. L_1为顺时针方向



记L与L_1间区域为M. 则 $\partial M = L+L_1$ 在M内, P,Q 连续可微, 在$M\cup\partial M$内, P,Q 连续, 故由格林公式: $\oint_L Pdx+Qdy+\oint_{L_1}Pdx+Qdy = \int_{\partial M}Pdx+Qdy$

$= \int_M\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \int_M 0\,dxdy = 0$

$\therefore \oint_L Pdx+Qdy = -\oint_{L_1}Pdx+Qdy = -\int_{L_1}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}$

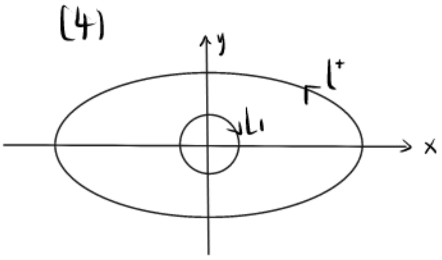
L_1参数方程: $x=r\cos\theta$ $y=r\sin\theta$ $(\theta\in[0,2\pi])$ 且θ增长方向与L_1正方向相反. $\sqrt{x_\theta'^2+y_\theta'^2}=r$

$\therefore -\oint_{L_1}(P,Q)d\tau = -\oint_{L_1}(P,Q)\cdot\tau\,dl = \int_0^{2\pi}(P,Q)\cdot(x_\theta', y_\theta')\,d\theta$

$= \int_0^{2\pi}\frac{-r^2(\sin\theta+\cos\theta)\sin\theta + r^2(\sin\theta-\cos\theta)\cos\theta}{r^2}\,d\theta$

$= \int_0^{2\pi} -\sin^2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta - \cos^2\theta\,d\theta = \int_0^{2\pi} -1\,d\theta = -2\pi$

(4)



记D的有向边界为L^+, P与Q在L^+围成的区域内不连续可微.
取L_1为 $x^2+y^2=r^2$, r充分小的圆周. L_1为顺时针方向
记L与L_1间区域为M. 则 $\partial M = L+L_1$
在M内, P,Q 连续可微, 在$M\cup\partial M$内, P,Q 连续
故由格林公式: $\oint_L Pdx+Qdy+\oint_{L_1}Pdx+Qdy = \int_{\partial M}Pdx+Qdy$

$= \int_M\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = \int_M 0\,dxdy = 0$

$\therefore \oint_L Pdx+Qdy = -\oint_{L_1}Pdx+Qdy = -\int_L\frac{(x+y)dx+(y-x)dy}{x^2+y^2}$

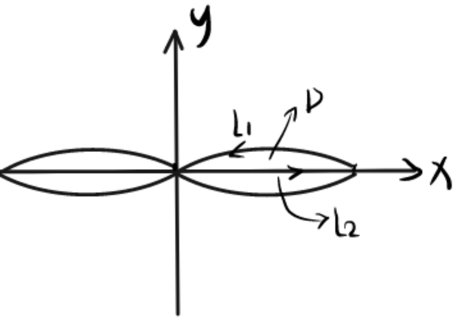
L_1参数方程: $x=r\cos\theta$ $y=r\sin\theta$ $(\theta\in[0,2\pi])$ 且θ增长方向与L_1正方向相反. $\sqrt{x_\theta'^2+y_\theta'^2}=r$

$\therefore -\oint_{L_1}(P,Q)d\tau = -\oint_{L_1}(P,Q)\cdot\tau\,dl = \int_0^{2\pi}(P,Q)\cdot(x_\theta', y_\theta')\,d\theta$

$= \int_0^{2\pi}\frac{-r^2(\sin\theta+\cos\theta)\sin\theta + r^2(\sin\theta-\cos\theta)\cos\theta}{r^2}\,d\theta$

$= \int_0^{2\pi} -\sin^2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta - \cos^2\theta\,d\theta = \int_0^{2\pi} -1\,d\theta = -2\pi$

4.(2) $x=r\cos\theta \quad y=r\sin\theta$ 代入有: $r^4=a^2r^2\cos2\theta \quad \therefore r^2=a^2\cos2\theta \quad \therefore r=a\sqrt{\cos2\theta}$



$x=a\sqrt{\cos2\theta}\cos\theta \quad y=a\sqrt{\cos2\theta}\sin\theta$. 在D部分: $\theta\in[0,\frac{\pi}{4}]$

由对称性 $\sigma(S)=4\sigma(D)$. 而∂D的边界由两段连续可微的曲线围成

$\sigma(S)=\iint_\Omega dxdy=\oint_{\partial\Omega}xdy$ 故 $\sigma(D)=\oint_{\partial D}xdy=\int_{L_1}xdy+\int_{L_2}xdy=\int_{L_1}xdy$

$$\int_{L_1}xdy=\int_0^{\frac{\pi}{4}}a\sqrt{\cos2\theta}\cos\theta\cdot a\left(\frac{1}{2}\cdot\frac{-\sin2\theta}{\sqrt{\cos2\theta}}\cdot2\cdot\sin\theta+\cos\theta\cdot\sqrt{\cos2\theta}\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}(-a^2\cos\theta\cdot\sin2\theta\cdot\sin\theta+a^2\cos^2\theta\cdot\cos2\theta)d\theta$$

$$=\int_0^{\frac{\pi}{4}}a^2\left(\frac{1+\cos2\theta}{2}\cdot\cos2\theta-\frac{1}{2}\sin^2 2\theta\right)d\theta=\int_0^{\frac{\pi}{4}}\left(a^2\cdot\frac{1}{2}\cos2\theta+a^2\cdot\frac{1}{2}(\cos^2 2\theta-\sin^2 2\theta)\right)d\theta$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}a^2\cos2\theta+a^2\cdot\frac{1}{2}\cos4\theta\right)d\theta=\frac{1}{2}a^2\int_0^{\frac{\pi}{4}}(\cos2\theta+\cos4\theta)d\theta=\frac{1}{2}a^2\cdot\left(\frac{1}{2}\sin2\theta\Big|_0^{\frac{\pi}{4}}+\frac{1}{4}\sin4\theta\Big|_0^{\frac{\pi}{4}}\right)=\frac{1}{4}a^2$$

$\therefore \sigma(S)=4\sigma(D)=a^2$

法二: $\left|\frac{\nabla(x,y)}{\nabla(\theta,r)}\right|=r$

$$\sigma(D)=\iint_{D_{xy}}1\,dxdy=\int_0^{\frac{\pi}{4}}d\theta\int_0^{a\sqrt{\cos2\theta}}r\,dr=\int_0^{\frac{\pi}{4}}\frac{1}{2}a^2\cos2\theta\,d\theta=\frac{1}{4}a^2\sin2\theta\Big|_0^{\frac{\pi}{4}}=\frac{1}{4}a^2$$

$\therefore S=4\sigma(D)=a^2$

8.(2) 乖乖: 注意题目改过与题目下的小字

$$\oint_{\partial D}v\frac{\partial u}{\partial n}dl=\oint_{\partial D}v\cdot\mathrm{grad}u\cdot\vec{n}\,dl=\oint_{\partial D}(u'_xv,u'_yv)\cdot\vec{n}\,dl=\oint_{\partial D}(-u'_yv,u'_xv)\cdot\vec{T}\,dl=\oint_{\partial D}-u'_yv\,dx+u'_xv\,dy$$

$$=\iint_D\frac{\partial(u'_xv)}{\partial x}+\frac{\partial(u'_yv)}{\partial y}dxdy=\iint_D(u''_{xx}v+u'_xv'_x+u''_{yy}v+u'_yv'_y)dxdy$$

$$=\iint_D v\cdot\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)\cdot\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)dxdy=\iint_D v\cdot\Delta u\,dxdy+\iint_D\nabla u\cdot\nabla v\,dxdy$$

(3) 原式 $=\oint_{\partial D}\left(v\cdot\frac{\partial u}{\partial n}-u\cdot\frac{\partial v}{\partial n}\right)dl=\oint_{\partial D}v\cdot\mathrm{grad}u\cdot\vec{n}\,dl-\oint_{\partial D}u\cdot\mathrm{grad}v\cdot\vec{n}\,dl$

$$=\iint_D[(u''_{xx}v+u'_xv'_x+u''_{yy}v+u'_yv'_y)-(v''_{xx}u+v'_xu'_x+v''_{yy}u+v'_yu'_y)]dxdy$$

$$=\iint_D\left(\left[\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right]v-\left[\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}\right]u\right)dxdy=\iint_D\begin{vmatrix}\Delta u&\Delta v\\u&v\end{vmatrix}dxdy$$

9. 切向量: $d\vec{T}=(dx,dy)$ $\vec{n}\cdot d\vec{T}=0$, 且 $\|\vec{n}\|=1$ 且$\vec{n}$为外法向量

$$\therefore \vec{n}=\left(\frac{dy}{\sqrt{dx^2+dy^2}},\frac{-dx}{\sqrt{dx^2+dy^2}}\right)=\left(\frac{dy}{\|d\vec{T}\|},\frac{-dx}{\|d\vec{T}\|}\right)=\left(\frac{dy}{dL},\frac{-dx}{dL}\right)$$

$$\therefore \cos\langle\vec{n},\vec{i}\rangle=\frac{\vec{n}\cdot\vec{i}}{\|\vec{n}\|\|\vec{i}\|}=\frac{dy}{dL}\quad \cos\langle\vec{n},\vec{j}\rangle=\frac{\vec{n}\cdot\vec{j}}{\|\vec{n}\|\|\vec{j}\|}=-\frac{dx}{dL}$$

$\therefore$ 原式 $=\oint_L x\,dy-y\,dx=\iint_M\left(\frac{\partial x}{\partial x}-\frac{\partial(-y)}{\partial y}\right)dxdy=2\sigma(M)$

其中$\sigma(M)$为L围成的封闭区域面积

10.

(1) 记原方程为 $Pdx+qdy=0$

$\frac{\partial q}{\partial x}-\frac{\partial P}{\partial y}=-1-(-1)=0$ ∴原方程为恰当方程. $\exists\, U(x,y)$ s.t. $Pdx+qdy=dU(x,y)$

$\frac{\partial u}{\partial x}=P=x^2-y$ ∴ $u=\int x^2-y\,dx+g(y)=\frac{1}{3}x^3-yx+g(y)$ $\frac{\partial u}{\partial y}=-x+g'(y)=q=-x-\sin^2 y$

∴ $g'(y)=-\sin^2 y$ ∴ $g(y)=-\int\sin^2 y\,dy=\int\frac{\cos 2y-1}{2}dy=\frac{1}{4}\sin 2y-\frac{1}{2}y$

∴原方程解为: $\frac{1}{3}x^3-yx+\frac{1}{4}\sin 2y-\frac{1}{2}y=C$ (C为一常数)

(2) 记原方程为 $Pdx+qdy=0$

$\frac{\partial q}{\partial x}-\frac{\partial P}{\partial y}=e^y-e^y=0$ ∴原方程为恰当方程. $\exists\, U(x,y)$ s.t. $Pdx+qdy=dU(x,y)$

$\frac{\partial u}{\partial x}=e^y$ ∴ $u=\int e^y dx=e^y x+g(y)$ $\frac{\partial u}{\partial y}=x\cdot e^y+g'(y)=q=xe^y-2y$ ∴ $g(y)'=-2y$

$g(y)=\int -2y\,dy=-y^2$

∴原方程解为: $e^y\cdot x-y^2=C$ (C为一常数)

(3) $\frac{xdy+ydx}{\sqrt{x^2+y^2}}=d(\sqrt{x^2+y^2})$ 记 $\frac{ydx-xdy}{x^2}=Pdx+qdy$ 则 $\frac{\partial q}{\partial x}-\frac{\partial P}{\partial y}=+\frac{1}{x^2}-\frac{1}{x^2}=0$

∴ $\exists\, U(x,y)$ s.t. $Pdx+qdy=dU(x,y)$ $\frac{\partial u}{\partial x}=P=\frac{y}{x^2}$ ∴ $u=\int\frac{y}{x^2}dx+g(y)=-\frac{y}{x}+g(y)$

$\frac{\partial u}{\partial y}=q=-\frac{1}{x}=-\frac{1}{x}+g'(y)$ ∴ $g(y)=0$ ∴ $g(y)=0$ ∴ $U=-\frac{y}{x}+C$ (C为一常数)

∴ $d(\sqrt{x^2+y^2})=du$ ∴方程之根为:

$\sqrt{x^2+y^2}=-\frac{y}{x}+C$ (C为一常数)

(4) 记原方程为 $Pdx+qdy=0$

$\frac{\partial q}{\partial x}-\frac{\partial P}{\partial y}=-\frac{1}{y^2}-(-\frac{1}{y^2})=0$ ∴原方程为恰当方程. $\exists\, U(x,y)$ s.t. $Pdx+qdy=dU(x,y)$

$\frac{\partial u}{\partial x}=\cos x+\frac{1}{y}$ ∴ $u=\int\cos x+\frac{1}{y}dx+g(y)=\sin x+\frac{x}{y}+g(y)$ $\frac{\partial u}{\partial y}=-\frac{x}{y^2}+g(y)'=\frac{1}{y}-\frac{x}{y^2}$

∴ $g(y)'=\frac{1}{y}$ ∴ $g(y)=\ln|y|$

∴解为 $\sin x+\frac{x}{y}+\ln|y|=C$. (C为一常数)

11.

(1) 记原方程为 $Pdx+qdy=0$ 则 $\frac{P'_y-q'_x}{-P}=\frac{\cos x-y\cos x-\cos x+x\sin x}{-(y\cos x-x\sin x)}=1$

则原方程存在仅与y有关的积分因子 M_y 且 $M_y=e^{\int 1 dy}=e^y$

∴原方程之根等价于 $e^y(y\cos x-x\sin x)dx+e^y(y\sin x+x\cos x)dy=0$ 记之为 $Mdx+Ndy=0$

则 $\frac{\partial N}{\partial y}-\frac{\partial M}{\partial x}=e^y(y\sin x+x\cos x)-\sin x\cdot e^y-(-e^y y\sin x-\sin x e^y-x\cos x e^y)=0$

$Mdx+Ndy=0$ 为恰当方程. $\exists\, U(x,y)$ s.t. $Mdx+Ndy=dU(x,y)$

$\frac{\partial u}{\partial x}=M=e^y y\cos x-e^y x\sin x$ ∴ $u=\int(e^y y\cos x-e^y x\sin x)dx+g(y)$

$=e^y y\sin x+e^y x\cos x-\int e^y\cos x\,dx+g(y)=e^y x\cos x+e^y y\sin x-e^y\sin x+g(y)$

$\frac{\partial u}{\partial y}=e^y x\cos x+e^y y\sin x+e^y\sin x-e^y\sin x+g(y)'=N=e^y(y\sin x+x\cos x)$

∴ $g(y)'=0$ $g(y)=0$

∴原方程之根为 $e^y x\cos x+e^y y\sin x-e^y\sin x=C$ (C为一常数)

(2) 讲义42页

$(xdx+ydy)+(ydx-xdy)=0$ $\qquad \frac{(xdx+ydy)}{x^2+y^2}+\frac{(ydx-xdy)}{x^2+y^2}=0$

$\therefore \frac{1}{2}d\ln(x^2+y^2)+d\arctan\frac{x}{y}=0$ ∴原方程通解为:

$\frac{1}{2}\ln(x^2+y^2)+\arctan\frac{x}{y}=C$ (C为一常数)

(3) 原方程等价于 $(3x+\frac{y}{x^2})dx+(2y-\frac{1}{x})dy=0$ 记为 $Pdx+Qdy=0$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=+\frac{1}{x^2}-\frac{1}{x^2}=0$ ∴此方程为恰当方程. $\exists\, U(x,y)$ s.t. $Pdx+Qdy=dU(x,y)$

$\frac{\partial u}{\partial x}=P=3x+\frac{y}{x^2}$ $\quad u=\int 3x+\frac{y}{x^2}dx+g(y)=\frac{3}{2}x^2-\frac{y}{x}+g(y)$ $\quad \frac{\partial u}{\partial y}=-\frac{1}{x}+g'(y)=2y-\frac{1}{x}$

$\therefore g'(y)=2y$ $\therefore g(y)=y^2$ ∴原方程之根为 $\frac{3}{2}x^2-\frac{y}{x}+y^2=C$ (C为一常数)

(4) $(x+y)(dx-dy)=dx+dy$ $\therefore (x+y-1)dx-(x+y+1)dy=0$ $\therefore (1-\frac{1}{x+y})dx+(1+\frac{1}{x+y})dy=0$

$dx-dy-\frac{dx+dy}{x+y}=0$ $\therefore d(x-y)-d\ln|x+y|=0$

∴原方程之根为 $x-y-\ln|x+y|=C$ (C为一常数)

(5) $(x^2-\sin^2y)dx+x\sin2y\,dy=0$ $\therefore (1-\frac{\sin^2y}{x^2})dx+\frac{\sin2y}{x}dy=0$ 记为 $Pdx+Qdy=0$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-\frac{\sin2y}{x^2}-(-\frac{2\sin y\cos y}{x^2})=0$ ∴此方程为恰当方程. $\exists\, U(x,y)$ s.t. $Pdx+Qdy=dU(x,y)$

$\frac{\partial u}{\partial x}=P=1-\frac{\sin^2y}{x^2}$ $\quad u=\int 1-\frac{\sin^2y}{x^2}dx+g(y)=x+\frac{\sin^2y}{x}+g(y)$ $\quad \frac{\partial u}{\partial y}=\frac{1}{x}\cdot\sin2y+g'(y)=\frac{\sin2y}{x}$

$\therefore g'(y)=0$ $\therefore g(y)=0$

∴原方程之根为 $x+\frac{\sin^2y}{x}=C$ (C为一常数)