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Exercise 1.1.1.

1. let $C=\begin{bmatrix}0&-1\\1&0\end{bmatrix} \Rightarrow C^2=-I_2$

Then let $A=\begin{bmatrix}C&&&\\&C&&\\&&\ddots&\\&&&C\end{bmatrix}_{2n\times 2n}$

where there's $\frac{n}{2}$ C in the diagonal

so $A^2=\begin{bmatrix}C^2&&&\\&C^2&&\\&&\ddots&\\&&&C^2\end{bmatrix}=-I_{2n}$.

2. Proof.

Suppose λ is the eigenvalue of A.

then $\lambda=i$ or $\lambda=-i$, let $P_A(x)$ be the eigenpolynomial of A.

① if $P_A(x)=(x-i)^3$.

from Hamilto-Caylay Theorem.

we know that:

$$P_A(A)=0$$

$$\Leftrightarrow A^3+3Ai^2-3A^2i-i^3I=0$$

$$\Leftrightarrow A^3-3A=i(3A^2-I)$$

if $A\in M_3(\mathbb{R})$, the $A^3-3A\in M_3(\mathbb{R})$

but the entry $i\cdot(3A^2-I)$ is either 0 or an imaginary number.

thus $i\cdot(3A^2-I)\notin M_3(\mathbb{R})$

(from $A^2=-I$, we know that $3A^2-I\neq 0$)

We get a contradiction.

so the assumption $A\in M_3(\mathbb{R})$ is incorrect.

② if $P_A(x)=(x+i)^3$

similarly, by Hamilton-Caylay Theorm,:

$$A^3-3A=i(I-3A)$$

suppose $A\in M_3(\mathbb{R})$

then $A^3-3A\in M_3(\mathbb{R})$

but $i(I-3A)\notin M_3(\mathbb{R})$.

We got a contridiction.

③ if $P_A(x)=(x+i)^2(x-i)$

similarly: $A^3+A=-i(I+A^2)$

if $A\in M_3(\mathbb{R})$, contradittion.

④ if $P_A(x)=(x-i)^2(x+i)$.

similarly, $A^3+A=i(I+A^2)$

if $A\in M_3(\mathbb{R})$. contradiction.

so, in any cases. $A\notin M_3(\mathbb{R})$.

Exercise 1.1.2

1. Proof. let $k = a+bi$, where $a, b \in \mathbb{R}$.

$B(kv) = k(Bv)$

$\Leftrightarrow B(av+biv) = aBv+b\,iBv$

$\Leftrightarrow aBv + bBAv = aBv + bABv$

$\Leftrightarrow b(BA-AB)v = 0$ for $\forall v \in \mathbb{R}^n, b \in \mathbb{R}$.

$\Leftrightarrow BA - AB = 0$

$\Leftrightarrow BA = AB$.

$\therefore$ B is complex linear if and only if $AB = BA$.

2. X doesn't have to be complex linear.

let $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. $B = i \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$.

Then $A \in M_n(\mathbb{R})$ and $A^2 = -I$

$B^2 = -I$.

But:

$AB = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$

$\Rightarrow AB \neq BA$

so B is NOT complex linear, B is a counter example.

3. $C^2 = I \Rightarrow \lambda_C = \pm 1$.

C has two eigenspaces: $\ker(C-I)$, $\ker(C+I)$.

$\dim N(C-I) = n - \dim C(C-I)$

$\dim N(C+I) = n - \dim C(C+I)$

note that: $(C-I)(C+I) = 0$

$\therefore r(C-I) + r(C+I) - n \leq r((C-I)(C+I)) = 0$

$\Rightarrow r(C-I) + r(C+I) \leq n$

~~but r(C-I)~~

$\Rightarrow \dim N(C-I) + \dim N(C+I)$

$\geq n+n-n = n$.

note that: $N(C-I) \cap N(C+I) = \{0\}$

$\therefore \dim(N(C-I)+N(C+I))$

$= \dim N(C-I) + \dim N(C+I) \geq n$

$\because N(C+I) + N(C-I) \subseteq \mathbb{R}^n$

$\therefore \dim(N(C-I) + N(C+I)) \leq n$

$\therefore \dim(N(C-I) + N(C+I)) = n$.

$\therefore$ C is diagonalizable.

$CA = -AC$

$\Rightarrow CA \cdot A = -ACA$

$\Rightarrow C = ACA$

$\Rightarrow tr(C) = tr(AC \cdot A) = tr(A \cdot AC)$

$= -tr(C)$

$\Rightarrow tr(C) = 0$

let multiplicity of 1 be n_1.

multiplicity of -1 be n_2.

then $tr(C) = \sum_{i=1}^{n} \lambda_i = n_1 \times 1 + n_2 \times (-1)$

$= n_1 - n_2 = 0$

$\Rightarrow n_1 = n_2$.

since C is ~~diagonalizable~~ diagonalizable

$\dim N(C-I) = n_1$

$\dim N(C+I) = n_2$.

$\therefore \dim N(C-I) = \dim N(C+I)$

which means C's eigenspaces for 1 and -1 has the same dimension.

4. let $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

pick: $C_1 = diag(-1, 1, -1, 1)$.

$C_2 = diag(1, -1, 1, -1)$,

Collaborators for Exercise 1.1.3.

Liu Sijia. (Question 1)

Kong Lingyu (Question 3).

both of them are students in this class.

Exercise 1.1.3

1. C is real linear.

Proof. let $v \in \mathbb{C}^n$, $v = a+bi$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$.

let $k \in \mathbb{R}$.

$C(kv) = C(ka + kbi)$

$= ka - kbi$

$= k(a-bi) = kC(v)$.

C is NOT complex linear.

Proof. let $v \in \mathbb{C}^n$, $v = a+bi$. $a, b \in \mathbb{R}^n$. let $k \in \mathbb{C}$. $k = c+di$; $c, d \in \mathbb{R}$.

$C(kv) = C((ac-bd) + (cb+da)i)$

$= (c \cdot a - d \cdot b) + (cib + dia)$.

$k \cdot C(v) = (a-bi)(c+di)$

$= (c \cdot a + d \cdot b) + (da - cb)i$

$\therefore$ for some a, b, c, d.

$C(kv) \neq kC(v)$

2. $\mathbb{C}$-linear implys $\mathbb{R}$-linear

for $\mathbb{R} \subseteq \mathbb{C}$, if $k = a+bi$, $\forall a, b \in \mathbb{R}$.

there is: $C(kv) = kC(v)$

then let $b = 0$.

the conclusion is still right.

3. $\mathbb{R}$-basis: $\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}i\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\i\end{bmatrix} \}$.

$\mathbb{C}$-basis: $\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \}$.

real dimension: 4

complex dimension: 2.

4. $\mathbb{C}$-linear independent implys $\mathbb{R}$-linear independent.

$\mathbb{C} \Rightarrow \mathbb{R}$:

if $\sum a_i v_i = 0 \wedge a_i \in \mathbb{C} \Rightarrow a_i = 0$

is right for $\mathbb{C}$,

it is also right for subset of $\mathbb{C}$, which can be $\mathbb{R}$.

$\mathbb{R} \not\Rightarrow \mathbb{C}$: $\begin{bmatrix}-1\\i\end{bmatrix}, \begin{bmatrix}i\\1\end{bmatrix}$

is $\mathbb{R}$-independent.

but is not $\mathbb{C}$-indepent (with coefficient $1, i$).

5. $\mathbb{R}$-spanning implys $\mathbb{C}$-spanning.

<1> if $\forall v \in \mathbb{C}^n$ can be decomposed into $v = \sum_{i=1}^{n} a_i \vec{t_i}$. $\vec{t_i} \in \mathbb{C}^n$.

where $a_i \in \mathbb{R}$.

since $\mathbb{R} \subseteq \mathbb{C}$, we can also find $b_i \in \mathbb{C}$ (e.g. $b_i = a_i$) s.t. $v = \sum_{i=1}^{n} b_i \vec{t_i}$.

<2> Conversly. let $\begin{bmatrix}i\\i\end{bmatrix} = i\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}$.

but for any $a, b \in \mathbb{R}$. $\begin{bmatrix}i\\i\end{bmatrix}$ can not be decomposed into $a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$.

Exercise 1.1.4

1. $P\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$,

$P\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}=\begin{pmatrix}i\\-1\\-i\\1\end{pmatrix}$

2. let $F_4=[\vec{f}_1\ \vec{f}_2\ \vec{f}_3\ \vec{f}_4]$

$\Rightarrow P\vec{f}_1=\vec{f}_1,\ P\vec{f}_2=i\vec{f}_2$

$P\vec{f}_3=-\vec{f}_3,\ P\vec{f}_4=-i\vec{f}_2$

$\Rightarrow D=\mathrm{diag}(1,i,-1,-i)$.

P has eigenvalues $1,i,-1,-i$,

eigenvectors $\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$.

3. $C\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\sum_{i=0}^{3}c_i\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$C\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{pmatrix}c_0+c_1i-c_2-c_3i\\c_3+c_0i-c_1-c_2i\\c_2+c_3i-c_0-c_1i\\c_1+c_2i-c_3-c_0i\end{pmatrix}=\sum_{j=0}^{3}c_jP^j f_2$

4. $C=\sum_{j=0}^{3}c_jP^j,\ P=F_4DF_4^{-1}$

$\Rightarrow C=F_4\left(\sum_{j=0}^{3}c_jD^j\right)\cdot F_4^{-1}$.

$\therefore C$ has eigenvalues:

$c_0+c_1+c_2+c_3,\ (c_0-c_2)+(c_1-c_3)i,$

$c_0-c_1+c_2-c_3,\ (c_0-c_2)+(c_3-c_1)i$

with respective eigenvectors:

$\begin{pmatrix}1\\1\\1\\1\end{pmatrix},\begin{pmatrix}1\\i\\-1\\i\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}$.