习题1.4 No.1 2 8, 11, 12, 15

(1) $\frac{\partial z}{\partial x} = 2ayx + by^2$, $\frac{\partial z}{\partial y} = ax^2 + 2bxy$ (3) $\frac{\partial z}{\partial x} = \frac{1}{y} - \frac{y}{x^2}$, $\frac{\partial z}{\partial y} = \frac{1}{x} - \frac{x}{y^2}$

(5) $\frac{\partial z}{\partial x} = \frac{1+\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2-y^2}}}{x+\sqrt{x^2-y^2}} = \frac{1}{\sqrt{x^2-y^2}}$ $\frac{\partial z}{\partial y} = \frac{\frac{1}{2}\cdot\frac{-2y}{\sqrt{x^2-y^2}}}{x+\sqrt{x^2-y^2}} = \frac{-y}{x\sqrt{x^2-y^2}+x^2-y^2}$

(7) $\frac{\partial z}{\partial x} = -\sin(1+2^{xy})\cdot 2^{xy}\cdot \ln 2\cdot y$ $\frac{\partial z}{\partial y} = -\sin(1+2^{xy})\cdot 2^{xy}\cdot\ln 2\cdot x$

(9) 按定义: $\frac{\partial z}{\partial x} = \lim\limits_{t\to x}\frac{z(t,y)-z(x,y)}{t-x} = \lim\limits_{t\to x}\frac{\sqrt{|ty|}-\sqrt{|xy|}}{t-x} = \lim\limits_{t\to x}\sqrt{|y|}\,\frac{\sqrt{|t|}-\sqrt{|x|}}{t-x}$

注意到$t\to x$,故由x的正负性分类:

① $x>0$时$\Longrightarrow$上式$=\frac{\sqrt{|y|}}{2\sqrt{x}}$ ② $x<0$时,上式$=-\frac{\sqrt{|y|}}{2\sqrt{-x}}$ ③ $x=0$时,不存在

$\therefore \frac{\partial z}{\partial x} = \begin{cases} \sqrt{|y|}/2\sqrt{x} & (x>0\text{且}y\neq 0) \\ \text{不存在} & (x=0\text{且}y\neq 0) \\ 0 & (y=0) \\ -\sqrt{|y|}/2\sqrt{-x} & (x<0\text{且}y\neq 0) \end{cases}$ 同理 $\frac{\partial z}{\partial y} = \begin{cases} \sqrt{|x|}/2\sqrt{y} & (y>0\text{且}x\neq 0) \\ \text{不存在} & (y=0\text{且}x\neq 0) \\ -\sqrt{|x|}/2\sqrt{-y} & (y<0\text{且}x\neq 0) \\ 0 & (x=0) \end{cases}$

结果可想而知, $\sqrt{|x|}$不可能恒递增

No.2 考:

(1) 令$y=0$. $\frac{\partial f(x,0)}{\partial x} = \frac{|x|'}{2\sqrt{|x|}}$ 而$|x|$在0处无导数. 故$f(x,y)$在原点不可微

(3) $\frac{\partial f(x,0)}{\partial x} = \frac{\partial f(0,y)}{\partial y} = 0$ $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x^2\Delta y^2}{(\Delta x^2+\Delta y^2)^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}$

$= \frac{\Delta x^2\Delta y^2}{(\Delta x^2+\Delta y^2)^2}$ 令$\Delta x=\Delta y$,则原式$=\frac{1}{4}$. 不为0.

故$f(x,y)$在0处不可微

习题1.3.1.(7)的思考: 强硬消元处理分母: $\frac{x^3-y^3}{x+y}$

① 令$x+y=a$, $x-y=b$. 代入后为 $\frac{b}{a}\cdot\frac{3a^2+b^2}{4} = \frac{3}{4}ab+\frac{b^3}{4a}$. 取$b=k\sqrt[3]{a}$即有极限不存在

② 令$x+y=t$. $\frac{1}{t}((t-y)^3-y^3) = \frac{1}{t}(t^3-3t^2y+3ty^2-2y^3) = t^2-3ty+3y^2-2\frac{y^3}{t}$

即处理$\frac{y^3}{x+y}$. 令$y=k\sqrt[3]{x}$. 则 $\frac{k^3x}{x+k\sqrt[3]{x}} = \frac{k^3}{1+k\cdot x^{\frac{2}{3}}}$ 极限为k^3.

故原式极限不存在

习题1.4. 8, 11, 12, 15

8. 连续 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0) = 0$. 故$f$在原点连续. $\frac{\partial f(0,0)}{\partial x} = \lim\limits_{x\to 0}\frac{f(x,0)}{x} = 0$; 同理 $\frac{\partial f(0,0)}{\partial y} = 0$ 而方向导数 令$m=\frac{a}{\sqrt{a^2+b^2}}$ $n=\frac{b}{\sqrt{a^2+b^2}}$

令 $v=(m,n)$ $m\cdot n\neq 0$. 有 $g(t)=f(mt,nt)$ $(t>0)$

方向导数为 $\lim\limits_{t\to 0^+}\frac{g(t)-g(0)}{t} = \lim\limits_{t\to 0^+}\frac{\sqrt[3]{mn}\cdot t^{\frac{2}{3}}}{t} = \lim\limits_{t\to 0^+}\frac{\sqrt[3]{mn}}{t^{\frac{1}{3}}} \Longrightarrow \frac{M}{0}$型极限不存在

该极限不存在. 故f沿l方向的方向导数不存在

11. 1) $\frac{\partial z}{\partial x} = -\sin(x+y)$ $\frac{\partial z(P_0)}{\partial x} = -1$ $\frac{\partial z}{\partial y} = -\sin(x+y)$, $\frac{\partial z(P_0)}{\partial y} = -1$

$\frac{\partial z(P_0)}{\partial l} = -1\cdot\frac{3}{5}+(-1)\cdot\frac{-4}{5} = \frac{1}{5}$

3) $z = x_1x_1+x_1x_2+\cdots+x_1x_n+x_2x_1+\cdots+x_nx_n$ $\frac{\partial z}{\partial x_1} = 2x_1+2(x_2+\cdots+x_n)$

$\therefore \frac{\partial z(P_0)}{\partial x_1} = 2n = \frac{\partial z(P_0)}{\partial x_2} = \cdots = \frac{\partial z(P_0)}{\partial x_n}$

$$\frac{\partial Z(P_0)}{\partial l}=\frac{\partial Z(P_0)}{\partial x_1}\cdot\frac{V_1}{\|V\|}+\cdots+\frac{\partial Z(P_0)}{\partial x_n}\frac{V_n}{\|V\|}=-\frac{2n}{\sqrt{n}}-\frac{2n}{\sqrt{n}}-\cdots-\frac{2n}{\sqrt{n}}=-2n^{\frac{3}{2}}$$

12. 1) $\operatorname{grad}(u)=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$

3) $\operatorname{grad}(u)=\left(\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right)=(1,1,1,\cdots,1)$ n个1.

15. (1) $\frac{\partial u}{\partial y}=4\cos(x-\frac{y}{2})\cdot-\sin(x-\frac{y}{2})\cdot(-\frac{1}{2})=\sin(2x-y)$

$\frac{\partial^2 u}{\partial x\partial y}=2\cos(2x-y)$ $\frac{\partial^2 u}{\partial^2 y}=-\cos(2x-y)$

$\therefore 2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}=0$ 证毕

(3) $\frac{\partial u}{\partial x}=\cos y\cdot e^x$. $\frac{\partial u}{\partial y}=-e^x\sin y$ $\frac{\partial v}{\partial x}=\sin y\cdot e^x$ $\frac{\partial v}{\partial y}=e^x\cdot\cos y$

$\therefore \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial y}$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\cos y\cdot e^x-\cos y\cdot e^x=0$ $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=e^x\sin y-e^x\sin y=0$

证毕