2. suppose A has an eigenvector λ. $(\lambda \in \mathbb{C})$

so $\mathrm{Ker}(A-\lambda I)$

is an eigenspace of A.

for $\forall v \in \mathrm{Ker}(A-\lambda I)$

$(A-\lambda I)(Bv)$

$= ABv - \lambda Bv$

Conside $AB = BA$

$(A-\lambda I)(Bv)$

$= B(Av - \lambda v)$

$= B(A-\lambda I)v = 0$.

$\therefore Bv \in \mathrm{Ker}(A-\lambda I)$

so $\mathrm{Ker}(A-\lambda I)$ is B-invariant.

according to 1.

B has an eigenvector in $\mathrm{Ker}(A-\lambda I)$

and any vector in $\mathrm{Ker}(A-\lambda I)$

is the eigenvector of A.

so B, A has a common eigenvector. $\square$

Exercise 1.2.5.

$N_\infty(D)$

$= \{f \mid f$ is a polynomial$\}$.

conclusion: $N_\infty(D-I)$ is NOT spanned by e^x.

counter example:

let $f = (x+1)e^x$.

$(D-I)f = e^x$.

$(D-I)^2 f = 0$.

so $f \in N_\infty(D-I)$.

but f is not spanned by e^x. $\square$

under a basis $X_1, \dots X_{n^2}$.

in which $X_1, \dots, X_{\frac{n^2+n}{2}}$.

forms a basis for V_1

and the others form a basis for V_2.

~~Actually let $X_1 \sim X_n$ be~~

Exercise 1.2.3

① consider A_{11}:

for any $v \in R(A^T)$.

which is equivalent to consider

any $w \in \mathbb{R}^m$, (since: $v = A^T w$)

$$Av = AA^T w.$$

note that $R(AA^T) = R(A)$

so $\mathrm{Ran}(A_{11}) = R(AA^T) = R(A)$

$\Rightarrow \mathrm{rank}(A_{11}) = \mathrm{rank}(A)$

② consider A_{12}:

for any $v \in \mathrm{Ker}(A)$

$$Av = 0.$$

so $\dim \mathrm{Ran}(A_{12}) = 0$

$\Rightarrow \mathrm{rank}(A_{12}) = 0$

③ consider A_{21}:

from ①: we know that:

for any $v \in R(A^T)$

$A(v) \in R(AA^T) = R(A)$

note that $R(A) \cap \mathrm{Ker}(A^T) = \{0\}$

so $\mathrm{Ran}(A_{21}) = \{0\}$.

$\mathrm{rank}(A_{21}) = 0$.

④ consider A_{22}:

for any $v \in \mathrm{Ker}(A)$.

$$Av = 0$$

similiar to ②: $\mathrm{rank}(A_{22}) = 0$.

Exercise 1.2.4.

1. consider a basis for V: where $\dim V = k$.

$\{b_1, \dots, b_k\}$

Then:

$$A(\{b_1, \dots, b_k\}) = \{b_1, \dots, b_k\} T$$

and $T \in M_k(\mathbb{C})$.

pick $v \in \mathbb{R}^k$ s.t.

$$Tv = \lambda v. \quad (\lambda \in \mathbb{R}).$$

and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$

let $u = v_1 b_1 + v_2 b_2 + \dots + v_k b_k$.

Then: $A(u) = A\{b_1, \dots, b_k\} \cdot v$

$= \{b_1, \dots, b_k\} T v$

$= \lambda \{b_1, \dots, b_k\} v$

$= \lambda u.$

and $u \in V$.

so A has an eigenvector in V.

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Exercise 1.2.1.

1. counter example:

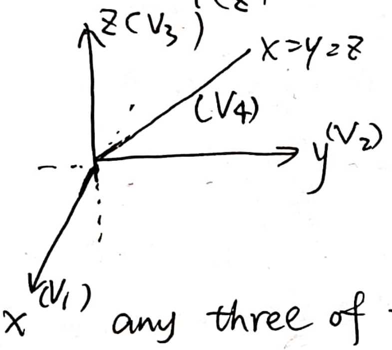
Consider subspaces in $\mathbb{R}^3$.

$V_1: \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x \in \mathbb{R}, y = z = 0 \right\}$.

$V_2: \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y \in \mathbb{R}, x = z = 0 \right\}$.

$V_3: \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z \in \mathbb{R}, y = x = 0 \right\}$

$V_4: \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = y = z ; x, y, z \in \mathbb{R} \right\}$



any three of them are linearly independent.

but: $\sum_{i=1}^{4} \dim V_i = 4$.

$\dim \sum_{i=1}^{4} V_i = 3$.

so they're not linearly independent.

2. counter example:

the same as the previous one.

3. Proof.

V_1, V_2 linearly independent

$\Rightarrow \dim(V_1+V_2) = \dim V_1 + \dim V_2$.

V_3, V_4 linearly independent

$\Rightarrow \dim(V_3+V_4) = \dim V_3 + \dim V_4$

V_1+V_2, V_3+V_4 linearly independent

$\Rightarrow \dim(V_1+V_2+V_3+V_4)$

$= \dim(V_1+V_2) + \dim(V_3+V_4)$

$= \dim V_1 + \dim V_2 + \dim V_3 + \dim V_4$

so, the four subspaces $V_1 \sim V_4$ are linearly independent.

Exercise 1.2.2.

let $V_1 = \{A : A \in V, A^T = A\}$.

$V_2 = \{A : A \in V, A^T = -A\}$

$\forall X \in V$.

note that: $X = \frac{X+X^T}{2} + \frac{X-X^T}{2}$.

$\frac{X+X^T}{2} \in V_1$; $\frac{X-X^T}{2} \in V_2$.

so $V_1 + V_2 = V$.

let $B \in V_1$ and $B \in V_2$

the $B^T = B = -B \Rightarrow B = 0$

so $V_1 \cap V_2 = \{0\}$.

$\therefore V = V_1 \oplus V_2$.

note that: $\dim V_1 = \frac{n^2+n}{2}$

$\dim V_2 = \frac{n^2-n}{2}$

so the block form of T is $\begin{bmatrix} I_{\frac{n^2+n}{2}} & \\ & I_{\frac{n^2-n}{2}} \end{bmatrix}$.

Ex. 1.2.1

① counter example: $(1,0,0)^T$ $(0\,1\,0)^T$ $(0\,0\,1)^T$ $(1,1,1)^T$

② same counter example as above.

③ $\dim(V_1+V_2+V_3+V_4) = \dim(V_1+V_2)+\dim(V_3+V_4)-\dim((V_1+V_2)\cap(V_3+V_4))$

$= \dim(V_1+V_2)+\dim(V_3+V_4)$

$= \dim V_1+\dim V_2+\dim V_3+\dim V_4-\dim(V_1\cap V_2)-\dim(V_3\cap V_4)$

$= \dim V_1+\dim V_2+\dim V_3+\dim V_4$

V_1, V_2, V_3, V_4 are independent.

Ex. 1.2.2

Any $n\times n$ Matrix can be written into the form of sysmmetric matrix + skew-sym matrix.

$V=\{A\}\oplus\{B\}$ where A statisfy $A=A^T$ and B statisfy $B^T=-B$

considering C. $C = A+B$

$C^T = A^T+B^T = A-B$

$\begin{cases} A=\frac{C+C^T}{2} \\ B=\frac{C-C^T}{2}\end{cases}$ note that A has $\frac{n^2+n}{2}$ decisive entry

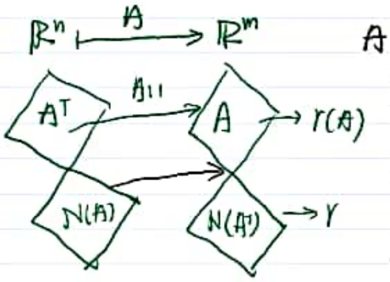
B has $\frac{n^2-n}{2}$ decisive entry

$\dim A=\frac{n^2+n}{2}$, $\dim B=\frac{n^2-n}{2}$

Ex. 1.2.3

	$\mathbb{R}^n \mapsto \mathbb{R}^m$	dim
A_{11} send	$Ran(A^T) \mapsto Ran(A)$	$r(A) \to r(A)$
A_{12} send	$Ker(A) \mapsto Ran(A)$	$n-r \mapsto r$
A_{21} send	$Ran(A^T) \mapsto Ker(A^T)$	$r \mapsto m-r$
A_{22} send	$Ker(A) \mapsto Ker(A^T)$	$n-r \to m-r$

rank is the dim of codomain!



A can only send vectors in $\mathbb{R}^n$ to $Ran(A)$,

so $r(A_{21}), r(A_{22})=0$

A send $\vec{v}\in N(A)$ to $\vec{0}$,

so $r(A_{12})=0$

A send $\vec{v}\in Ran(A^T)$ to $Ran(A)$

$r(A_{11})=r(A)$

Ex 1.2.4

① $\dim V=k$, $\vec{v}=c_1\vec{e_1}+\dots+c_k\vec{e_k}$ (c_i are not all 0)

$A\vec{v}=c_1A\vec{e_1}+\dots+c_kA\vec{e_k}\in V$

so $A\vec{v}$ can be written out with a linear combination like:

$d_1\vec{e_1}+\dots+d_k\vec{e_k}$. ($\vec{e_1},\dots,\vec{e_k}$ are independent)

$A\vec{e_i}=\frac{d_i}{c_i}\vec{e_i}$ $(i=1,\dots,k)$

so A has at least 1 eigenvector in V

②. suppose $A\vec{v}=\lambda\vec{v}$.

muptilpy $\vec{v}$ by B: $B\vec{v}$

$A(B\vec{v})=AB\vec{v}=BA\vec{v}=\lambda B\vec{v}$

$B\vec{v}$ is still in the eigenspace for A (λ)

$V=\{\vec{v}\mid(A-\lambda I)\vec{v}=0\}$ is B-invariant.

so B has an eigevector in V.

Ex. 1.2.5

$N_\infty(D)=\{c_0x^0+\dots+c_nx^n\}$

if $f(x)\in N_\infty(D-I)$ then must have

$\frac{d}{dx^k}f(x)=f(x)$

counter example: e^{-x} also satisfies: $(e^{-x})''=e^{-x}$

$k=2$.

false conclusion.

Collaborator: Ex 1.2.1③ & 1.2.2

MingDao Liu (student in the class)