Review

- 三重积分化累次积分

（先一后二）

$$\Omega:\begin{cases}(x,y)\in D_{xy},\\ z_1(x,y)\le z\le z_2(x,y),\end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint\limits_{D_{xy}}\mathrm{d}x\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z.$$

（先二后一）

$$\Omega:\begin{cases} c \le z \le d, \\ (x,y) \in \Omega_z, \end{cases}$$

$$\iiint_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_c^d dz \iint_{\Omega_z} f(x,y,z)\mathrm{d}x\mathrm{d}y.$$

- 投影法确定积分区域

- 三重积分的变量替换

$$u=u(x,y,z),v=v(x,y,z),w=w(x,y,z)$$

$$(x,y,z)\in\Omega\leftrightarrow(u,v,w)\in\Omega^*.$$

$$\iiint_{\Omega}f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\iiint_{\Omega^*}f\left(x(u,v,w),y(u,v,w),z(u,v,w)\right)$$

$$\cdot\left|\det\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

§5. 重积分的应用

- 曲面面积
- 质心
- 转动惯量
- 万有引力

原则：微元法

1. 曲面的面积

设曲面S的参数方程为

$$x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D,$$

简记为

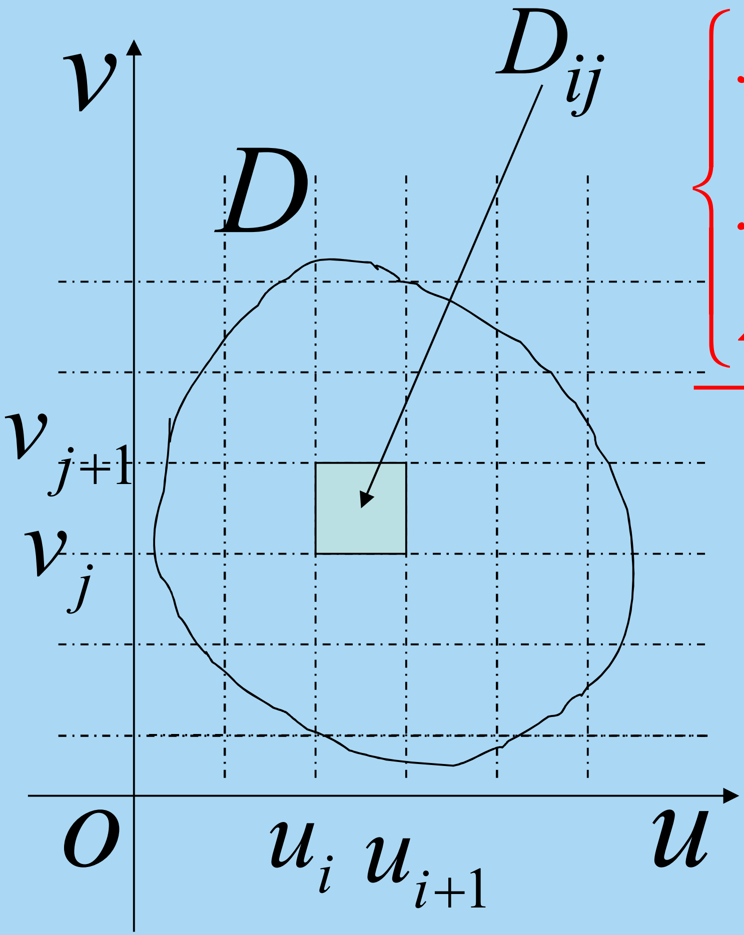
$$S : \mathrm{r} = \mathrm{r}(u,v), (u,v) \in D.$$

在ouv平面上,用平行于坐标轴的直线

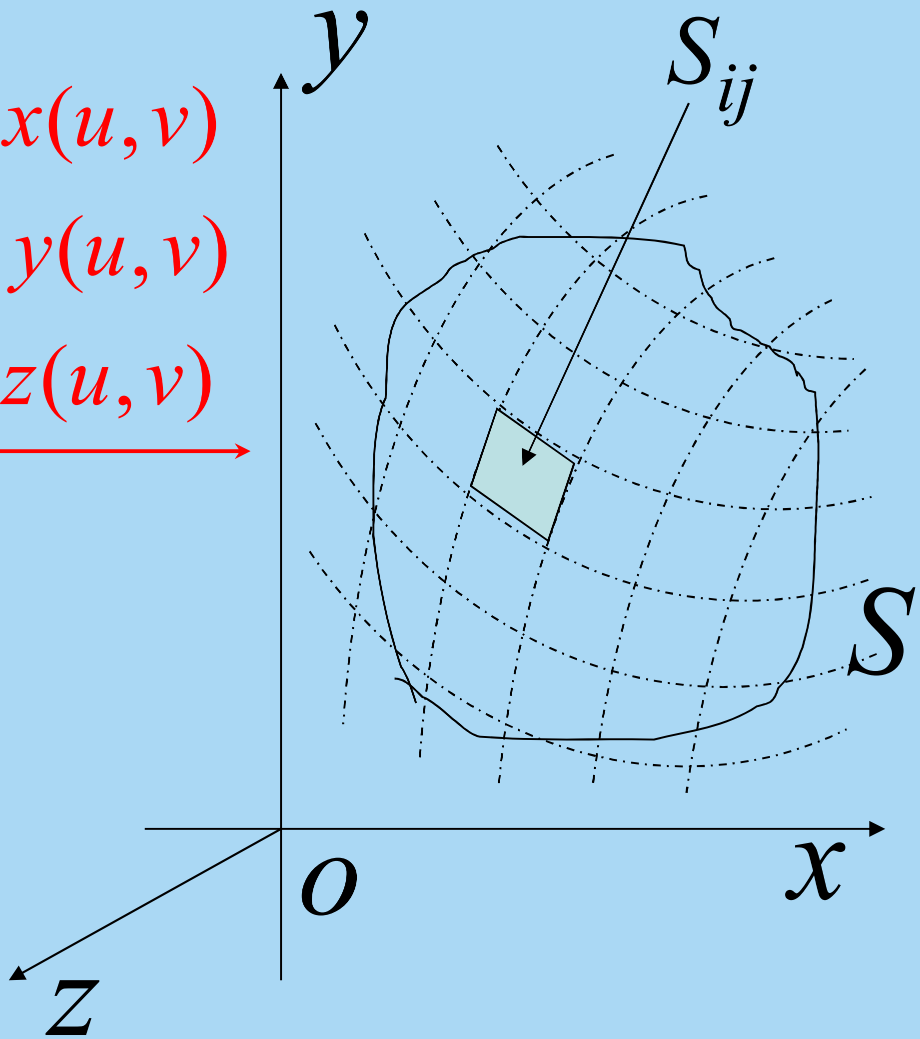
$$u = u_i (i = 1, 2, \cdots, n), v = v_j (j = 1, 2, \cdots, m)$$

将区域D分割成若干小矩形D_{ij}.

清華大學



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$



清華大學

D_{ij}的顶点为$(u_i,v_j),(u_{i+1},v_j),(u_i,v_{j+1}),(u_{i+1},v_{j+1})$.

对应地,空间曲边四边形S_{ij}的四个顶点为

$$P_{ij}(x(u_i,v_j),y(u_i,v_j),z(u_i,v_j)),$$

$$P_{i+1,j}(x(u_{i+1},v_j),y(u_{i+1},v_j),z(u_{i+1},v_j)),$$

$$P_{i,j+1}(x(u_i,v_{j+1}),y(u_i,v_{j+1}),z(u_i,v_{j+1})),$$

$$P_{i+1,j+1}(x(u_{i+1},v_{j+1}),y(u_{i+1},v_{j+1}),z(u_{i+1},v_{j+1})).$$

$$\overrightarrow{P_{ij}P_{i+1,j}} \approx (x'_u(u_i,v_j),y'_u(u_i,v_j),z'_u(u_i,v_j))\Delta u_i$$

$$= \mathrm{r}'_u(u_i,v_j)\Delta u_i$$

$$\overrightarrow{P_{ij}P_{i,j+1}} \approx \mathrm{r}'_v(u_i, v_j)\Delta v_j.$$

当分划很细时,空间曲面S_{ij}可近似地看成以线段$P_{ij}P_{i+1,j}, P_{ij}P_{i,j+1}$为邻边的平行四边形, 其面积

$$\Delta S_{ij} \approx \left\| \mathrm{r}'_u(u_i, v_j) \times \mathrm{r}'_v(u_i, v_j) \right\| \Delta u_i \Delta v_j$$

$$= \left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} \right\|_{(u_i, v_j)} \Delta u_i \Delta v_j$$

即 $\Delta S_{ij} \approx \sqrt{A^2+B^2+C^2}\,\Delta u_i \Delta v_j$，其中

$$A=\det\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{(u_i,v_j)},\quad B=\det\left.\frac{\partial(z,x)}{\partial(u,v)}\right|_{(u_i,v_j)},$$

$$C=\det\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_i,v_j)}.$$

•曲面 $S: x=x(u,v), y=y(u,v), z=z(u,v), (u,v)\in D$，的面积为

$$\iint_D \|\mathrm{r}'_u \times \mathrm{r}'_v\| \mathrm{d}u\mathrm{d}v = \iint_D \sqrt{A^2+B^2+C^2}\,\mathrm{d}u\mathrm{d}v.$$

•若曲面S的方程为$z=f(x,y),(x,y)\in D$,则

$$S: x=x, y=y, z=f(x,y),(x,y)\in D.$$

$$\mathrm{r}'_x\times\mathrm{r}'_y=\det\begin{pmatrix} i & j & k \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{pmatrix}=(-f'_x,-f'_y,1)$$

$$A=\det\begin{pmatrix} 0 & f'_x \\ 1 & f'_y \end{pmatrix}=-f'_x, B=-f'_y, C=1.$$

曲面S的面积为$\iint_D\sqrt{1+f_x'^2+f_y'^2}\,\mathrm{d}x\mathrm{d}y.$

清華大學

例: 求球面$S: x^2+y^2+z^2=R^2$的面积.

解: 球面S的参数方程为

$$x=R\sin\varphi\cos\theta, y=R\sin\varphi\sin\theta, z=R\cos\varphi,$$

$$(0\le\varphi\le\pi, 0\le\theta\le 2\pi).$$

则

$$\mathrm{r}'_\varphi\times\mathrm{r}'_\theta=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi\\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0\end{pmatrix}$$

$$=(R^2\sin^2\varphi\cos\theta, R^2\sin^2\varphi\sin\theta, R^2\sin\varphi\cos\varphi)$$

$$\left\|\mathrm{r}'_\varphi\times\mathrm{r}'_\theta\right\|=R^2\sin\varphi,$$

球面S的面积为

$$\iint\limits_{\substack{0\le\varphi\le\pi\\0\le\theta\le2\pi}} \left\|\mathrm{r}'_\varphi\times\mathrm{r}'_\theta\right\| \mathrm{d}\varphi\mathrm{d}\theta$$

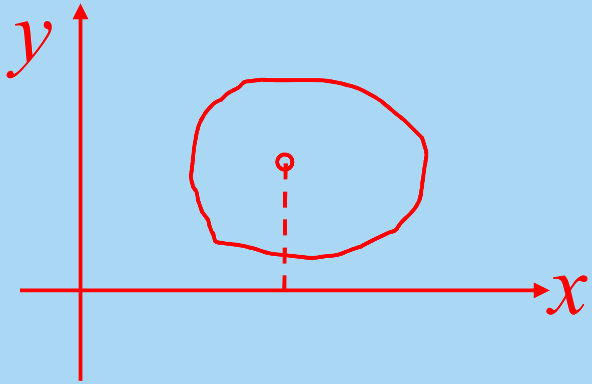
$$=\iint\limits_{\substack{0\le\varphi\le\pi\\0\le\theta\le2\pi}} R^2\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta$$

$$=R^2\int_0^\pi\sin\varphi\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta=4\pi R^2\ \square$$

2. 物体的质心

关于x轴的力矩微元为

$y\mu(x,y)dxdy$



- 平板D的质心$(\bar{x},\bar{y})$

平板密度$\mu(x,y)$

平板质量$\mathrm{M}=\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y$

平板关于x轴的静力矩为$\mathrm{M}\bar{y}=\iint_D y\mu(x,y)\mathrm{d}x\mathrm{d}y$

$$\text{故 } \bar{y}=\frac{\iint_D y\mu(x,y)\mathrm{d}x\mathrm{d}y}{\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y},\text{同理 } \bar{x}=\frac{\iint_D x\mu(x,y)\mathrm{d}x\mathrm{d}y}{\iint_D \mu(x,y)\mathrm{d}x\mathrm{d}y}.$$

• 空间物体Ω的质心$(\bar{x},\bar{y},\bar{z})$

密度$\mu(x,y,z)$，质量$\mathrm{M}=\iiint_{\Omega}\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Ω关于yz平面的静力矩为

$$\mathrm{M}\bar{x}=\iiint_{\Omega}x\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

故

$$\bar{x}=\frac{\iiint_{\Omega}x\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_{\Omega}\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z},$$

$$\bar{y}=\frac{\iiint_{\Omega}y\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_{\Omega}\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z},\bar{z}=\frac{\iiint_{\Omega}z\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\iiint_{\Omega}\mu(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}.$$

清华大学

3. 转动惯量

•位于(x,y,z)处质量为m的质点, 绕x,y,z轴的转动惯量分别为$m(y^2+z^2),m(z^2+x^2),m(x^2+y^2)$.

•$\Omega\subset\mathbb{R}^3$,密度$\rho(x,y,z)$,绕坐标轴的转动惯量为

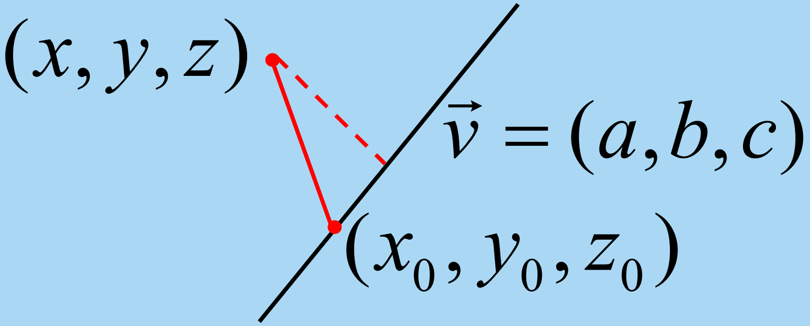
$$J_x=\iiint_\Omega(y^2+z^2)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$J_y=\iiint_\Omega(z^2+x^2)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$J_z=\iiint_\Omega(x^2+y^2)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Question. 直线l过点(x_0,y_0,z_0)沿方向(a,b,c), Ω绕l的转动惯量？

$$\iiint_\Omega d^2(x,y,z)\,\rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$



其中, $d(x,y,z)$

$$=\frac{1}{\sqrt{a^2+b^2+c^2}}\left\|\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ x-x_0 & y-y_0 & z-z_0\end{pmatrix}\right\|.$$

4. 万有引力

- 位于$P(x,y,z)$, $P_0(x_0,y_0,z_0)$的两质点,质量分别为m, m_0.记$r=\|PP_0\|$, $\overrightarrow{P_0P}$与x,y,z正半轴夹角为α,β,γ, m对m_0的万有引力的大小为$\frac{kmm_0}{r^2}$, 引力沿x,y,z轴的分量分别为

$$F_x=\frac{kmm_0}{r^2}\cos\alpha=\frac{kmm_0(x-x_0)}{r^3},$$

$$F_y=\frac{kmm_0}{r^2}\cos\beta=\frac{kmm_0(y-y_0)}{r^3},$$

$$F_z=\frac{kmm_0}{r^2}\cos\gamma=\frac{kmm_0(z-z_0)}{r^3}.$$

●密度为$\rho(x,y,z)$的物体Ω对$P_0(x_0,y_0,z_0)\notin\Omega$处质量为$m_0$的质点的万有引力:

$$F_x=\iiint_\Omega \frac{km_0(x-x_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_y=\iiint_\Omega \frac{km_0(y-y_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

$$F_z=\iiint_\Omega \frac{km_0(z-z_0)\rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\right)^3},$$

清華大學

例: 半径为R,质量为M的均匀球体$x^2+y^2+z^2\le R^2$对点$P(0,0,a)$ $(a>R)$处质量为 m 的质点的引力.

解: $$F_x=\iiint_\Omega \frac{kmx\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3}=0,\ F_y=0.$$

$$-F_z=\iiint_\Omega \frac{km(a-z)\rho\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(\sqrt{x^2+y^2+(a-z)^2}\right)^3},\quad \frac{4}{3}\pi R^3\rho=M.$$

在柱坐标系$x=r\cos\theta, y=r\sin\theta, z=z$下,

$$F_z=\int_{-R}^{R}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{R^2-z^2}}\frac{km(a-z)\rho r\mathrm{d}r}{(\sqrt{r^2+(a-z)^2})^3}$$

$$-F_z=\int_{-R}^{R}dz\int_0^{2\pi}d\theta\int_0^{\sqrt{R^2-z^2}}\frac{km(a-z)\rho r dr}{(\sqrt{r^2+(a-z)^2})^3}$$

$$=k\pi m\rho\int_{-R}^{R}(a-z)dz\int_0^{\sqrt{R^2-z^2}}\frac{dr^2}{(\sqrt{r^2+(a-z)^2})^3}$$

$$=2k\pi m\rho\int_{-R}^{R}\left(1-\frac{a-z}{\sqrt{R^2+a^2-2az}}\right)dz$$

$$=\frac{4k\pi m\rho R^3}{3a^2}=\frac{kMm}{a^2}.\square$$

(分部积分)

作业：习题3.5 No.1(单),9