Homework 4

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1 Exercise 1.4.1

1. Let

$$B = \begin{bmatrix} I & X \\ & I \end{bmatrix}, \quad A = \begin{bmatrix} M & C \\ & N \end{bmatrix}$$

Then

$$BAB^{-1} = \begin{bmatrix} M & XN + C - MX \\ & N \end{bmatrix} = \begin{bmatrix} M & \\ & N \end{bmatrix}$$

Thus

$$XN - MX = -C$$

Let $f : M_2(\mathbb{R}) \to M_2(\mathbb{R}), f(X) = XN - MX$. Then f is a linear map.
Pick basis for $M_2(\mathbb{R})$

$$T = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

Then the corresponding matrix of f under basis T is

$$F = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 5 & 3 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Let the coordinate of C under basis T be $\hat{c}$, the coordinate of X under basis T be $\hat{x}$.Then Solve the equation $F\hat{x} = -\hat{c}$.
Then we get

$$\hat{x} = \begin{bmatrix} -2 \\ \frac{31}{9} \\ -\frac{3}{2} \\ \frac{7}{6} \end{bmatrix}$$

So, let

$$B = \begin{bmatrix} 1 & 0 & -2 & \frac{31}{9} \\ & 1 & -\frac{3}{2} & \frac{7}{6} \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Then $BAB^{-1} = \begin{bmatrix} 1 & 2 & & \\ & 1 & & \\ & & 3 & 5 \\ & & & 4 \end{bmatrix}$. $\square$

2. Since$V_3 \cap V_4 = \{0\}$, only need to find basis for V_3 and V_4 respectively.

$$V_3 = Ker(A - 3I) = span(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix})$$

$$V_4 = Ker(A - 4I) = span(\begin{bmatrix} \frac{59}{9} \\ \frac{19}{3} \\ 5 \\ 1 \end{bmatrix})$$

Then, a basis for $V_3 + V_4$ is $\{\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{59}{5} \\ \frac{19}{3} \\ 5 \\ 1 \end{bmatrix}\}$.

Also, we can pick a basis directly from the previous calculation. I.e. the last 2 colum of B forms a basis of $V_3 + V_4$. So,

$$\{\begin{bmatrix} -2 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{31}{9} \\ \frac{7}{6} \\ 0 \\ 1 \end{bmatrix}\}$$

is a basis of $V_3 + V_4$. $\square$

2 Exercise 1.4.2

The minimal polynomial of A is not the square of the minimal polynomial of B.

Counter example Let

$$B = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Note that all the eigenvalue of B is 0, we know that the minimal polynomial of B is $p(x) = x^2$.
But

$$A^2 = \begin{bmatrix} B & I \\ & B \end{bmatrix} \begin{bmatrix} B & I \\ & B \end{bmatrix} = \begin{bmatrix} O & 2B \\ & O \end{bmatrix}, \quad A^3 = O$$

So the minimal polynomial of A is $q(x) = x^3 \neq p^2(x)$. $\square$

3 Exercise 1.4.3

1. Proof. Let

$$A = XUX^{-1}$$

where U is an upper triangluar matrix.
From$AB = BA$, we know that

$$UX^{-1}BX = X^{-1}BXU$$

Let $T = X^{-1}BX$.

The last row of UT is

$$\begin{bmatrix} u_{nn}t_{n1} & u_{nn}t_{n2} & u_{nn}t_{n3} & \cdots & u_{nn}t_{nn} \end{bmatrix}$$

Reduction Compare the entries of UT and TU.

The $(n, 1)$ entry of TU is

$$u_{11}t_{n1}$$

So $u_{11}t_{n1} - u_{nn}t_{n1} = 0$ is always true, but u_{11} is not always equal to u_{nn}.
So we conclude that

$$t_{n1} = 0 \tag{1}$$

The $(n, 2)$ entry of TU is

$$u_{12}t_{n1} + u_{22}t_{n2}$$

From (1), we know that $t_{n1} = 0$, so the $(n, 2)$ entry of TU is $u_{22}t_{n2}$.
So $u_{22}t_{n2} - u_{nn}t_{n2} = 0$ is always true, but u_{22} is not always equal to u_{nn}.
So we conclude that

$$t_{n2} = 0 \tag{2}$$

...

The (n,k) entry of TU is

$$u_{1k}t_{n1} + u_{2k}t_{n2} + \cdots + u_{kk}t_{nk}$$

From (1) to (k-1), we know that $t_{ni} = 0, \quad \forall i \in \{1, 2, \cdots, k-1\}$, so the (n,k) entry of TU is $u_{kk}t_{nk}$.
So $u_{kk}t_{nk} - u_{nn}t_{nk} = 0$ is always true, but u_{kk} is not always equal to u_{nn}.
So we conclude that

$$t_{nk} = 0 \tag{k}$$

...

Follow this iterating process, we conclude that for $\forall i \in \{1, 2, \cdots, n-1\}$, $t_{ni} = 0$. As for the (n,n) entry, we can finally get

$$u_{nn}t_{nn} = u_{nn}t_{nn}$$

which is always true.
So, we get

$$T = \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix}$$

Induction We try to apply this process to each sequential principal matrix of T.
We know

$$\begin{bmatrix} U_{n-1} & \alpha_1 \\ & u_{nn} \end{bmatrix} \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix} = \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix} \begin{bmatrix} U_{n-1} & \alpha_1 \\ & u_{nn} \end{bmatrix}$$

where U_{n-1} and T_{n-1} is respectively the (n-1) the sequential principal matrix of U and T.
According to the calculation of partitioned matrices, we get

$$U_{n-1}T_{n-1} = T_{n-1}U_{n-1}$$

From the **Reduction** process, we can similiarly conclude that

$$T_{n-1} = \begin{bmatrix} T_{n-2} & \beta_2 \\ & t_{(n-1)(n-1)} \end{bmatrix}$$

i.e.

$$T = \begin{bmatrix} T_{n-2} & \beta_2 & \beta_1 \\ & t_{(n-1)(n-1)} & \\ & & t_{nn} \end{bmatrix}$$

Also

$$\begin{bmatrix} U_{n-2} & \alpha_2 \\ & u_{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} T_{n-2} & \beta_2 \\ & t_{(n-1)(n-1)} \end{bmatrix} = \begin{bmatrix} T_{n-2} & \beta_2 \\ & t_{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} U_{n-2} & \alpha_2 \\ & u_{(n-1)(n-1)} \end{bmatrix}$$

which implies

$$U_{n-2}T_{n-2} = T_{n-2}U_{n-2}$$

Then, we repeat the **Reduction** process, until we get

$$T_2 = \begin{bmatrix} t_{11} & t_{12} \\ & t_{22} \end{bmatrix}$$

Thus,

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ & t_{22} & \cdots & t_{2n} \\ & & \ddots & \vdots \\ & & & t_{nn} \end{bmatrix}$$

T is upper triangluar, which means

$$B = XTX^{-1}$$

So A,B are simultaneously triangularized by X. $\square$

2. Counter example

Let

$$A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ & 0 \end{bmatrix}$$

We know that A and B commutes and A is already in Jordan Normal Form.

We try to find all the matrix X s.t. $B = X \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} X^{-1}$.

Note that

$$Ker(B) = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$Ker(B^2) - Ker(B) = span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

So $\forall m \in (Ker(B^2) - Ker(B))$

$$m = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

where $c \in \mathbb{C} - \{0\}$. So we get all the matrix X

$$X = c \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$$

But

$$AX = c \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \quad XA = c \begin{bmatrix} 2 & 2 \\ & 1 \end{bmatrix}$$

So $AX \neq XA$, which means $A \neq XAX^{-1} = X \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} X^{-1}$.

So A and B can not be simultaneously put into Jordan normal form. $\square$

4 Exercise 1.4.4

Lemma 0: Curiosity kills the cat.

Lemma 1: If $f \circ g$ is bijective, then f is surjective and g is injective. Proof. Let $f : B \to C,\ g : A \to B$.
If f is not surjective, then $\exists c \in C\ s.t.\ \forall b \in B,\ f(b) \neq c$. But $f \circ g$ is bijective, which means that

$$\forall c \in C, \exists a \in A\ s.t.\ c = f(g(a))$$

We get a contridiction.
If g is not injective, which means $\exists a_1 \neq a_2\ s.t.\ g(a_1) = g(a_2)$, so $f(g(a_1)) = f(g(a_2))$. But $f \circ g$ is bijective, which means

$$\forall a_1 \neq a_2,\ f(g(a_1)) \neq f(g(a_2))$$

We get a contridiction. $\square$

Lemma 2: Calculation rules of composite functions There's no need to proof, but for a clearer view, let's state them in advance.
If $\lim\limits_{x\to+\infty} g(x) = A,\ A \in \mathbb{R},\ and\ f\ is\ continuous,\ then \lim\limits_{x\to+\infty} f \circ g(x) = f(A)$.
If $\lim\limits_{x\to+\infty} g(x) = +\infty,\ and \lim\limits_{x\to+\infty} f(x)$ exists (in $\mathbb{R}$, or is infinity), then $\lim\limits_{x\to+\infty} f \circ g(x) = \lim\limits_{x\to+\infty} f(x)$.

From $f \circ f = -x$ and **Lemma 1**, we know that f is bijective. So $\forall y \in \mathbb{R},\ \exists! x \in \mathbb{R}\ s.t.\ y = f(x)$, which means f either strictly increases or decreases.
Without loss of generalization, let f strictly increases.
Further, suppose

$$\lim_{x\to+\infty} f(x) = +\infty$$

From **Lemma 2**, we know that

$$\lim_{x\to+\infty} f \circ f(x) = +\infty$$

But

$$\lim_{x\to+\infty} f \circ f(x) = \lim_{x\to+\infty} -x = -\infty$$

We get a contradiction, $\lim\limits_{x\to+\infty} f(x) \neq +\infty$.
Note that $f \in C(\mathbb{R})$, so when $x \to +\infty$, f must converge to a real number (or f will not increase strictly), i.e. $\exists c \in \mathbb{R}$ s.t.

$$\lim_{x\to+\infty} f(x) = c$$

From **Lemma 2** we know that

$$\lim_{x\to+\infty} f \circ f(x) = f(c)$$

And from the continuity of f, we know $f(c) \in \mathbb{R}$. But

$$\lim_{x \to +\infty} f \circ f(x) = \lim_{x \to +\infty} -x = -\infty$$

We get a contradiction.

As a result, there is no function that satisfy the all the following 3 properties:

1. $f : \mathbb{R} \to \mathbb{R}$
2. $f \in C(\mathbb{R})$
3. $f \circ f(x) + x = 0$

$\square$