Midterm

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Problem 1

1.1

***Proof*.**

$$
\begin{aligned}
&(\forall n) \quad A^{n+2}\boldsymbol{v} - A^{n+1}\boldsymbol{v} = A^{n+1}\boldsymbol{v} - A^{n}\boldsymbol{v} \\
\iff &A^2\boldsymbol{v} - A\boldsymbol{v} = A\boldsymbol{v} - \boldsymbol{v} \\
\iff &(A-I)^2\boldsymbol{v} = 0
\end{aligned}
$$

Since $(A-I)^2\boldsymbol{v} = 0$ for some non-zero $\boldsymbol{v}$, we know that $det((A-I)^2) = 0$, which means $det(A-I) = 0$. Thus A has eigenvalue 1 and v is in its generalized eigenspace. So $\boldsymbol{v}$ is a generalized eigenvector for A. □

1.2

***CounterExample*.**

$$
\begin{aligned}
&(\forall n) \quad A^{n+2}\boldsymbol{v} = A^{n+1}\boldsymbol{v} + A^{n}\boldsymbol{v} \\
\iff &A^2\boldsymbol{v} = A\boldsymbol{v} + \boldsymbol{v} \\
\iff &(A^2 - A - I)\boldsymbol{v} = 0
\end{aligned}
$$

Suppose

$$x_1 = \frac{1+\sqrt{5}}{2}, \ x_2 = \frac{1-\sqrt{5}}{2}$$

Then $x_1, \ x_2$ are distinct roots for equation $x^2 - x - 1 = 0$. Let

$$A = \begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix}$$

The only two generalized eigenspaces are $span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$, so pick $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly $\boldsymbol{v}$ is in neither of the two generalized eigenspaces. Also, we know that $A^2 - A - I = O$, which means the sequence $\{A^k\boldsymbol{v}\}$ here satisfies the required condition $A^{n+2}\boldsymbol{v} = A^{n+1}\boldsymbol{v} + A^{n}\boldsymbol{v}$, but $\boldsymbol{v}$ is NOT a generalized eigenvector for A. □

1.3

***Proof*.** Suppose M has a generalized eigenspace for eigenvalue λ. Let I be the identical mapping. We know for some none-zero $p(x) \in V$, $\exists t \in \mathbb{N}$ s.t.

$$(M - \lambda I)^t p(x) = 0$$

Which means

$$(x-\lambda)^t p(x) = 0$$

Since $(x-\lambda)^t$ is not a zero polynomial, then we must have $p(x) = \mathbf{0}$. Contradiction. So M has no generalized eigenvector.

□

1.4

Proof. Let $A = XJX^{-1}$, where J is the Jordan normal form of A. We know that all the Jordan blocks in J are nilpotent.

Suppose J has a Jordan block N which is larger then 5×5. We immdeiately know that $N^5 \neq O$. I.e. $J^5 \neq O$. This is contradictory to $A^5 = O$. So every Jordan block of J is not larger than 5×5.

Let $d = dim\ Ker(A) = dim\ Ker(J)$. So J have d Jordan blocks. Since the sum of the size of all Jordan blocks is the size of J, which is n, we know that $5d \geq n$, i.e. $d \geq \frac{n}{5}$. □

1.5

Proof. **For this subproblem, restrict the matrices to real matrices.** From singular value decomposition we know

$$A = U_r D V_r^T$$

where D is a diagonal matrix with the singular value of A (all positive real number), U_r and V_r are respectively a orthonormal basis for $Ran(A)$ and $Ran(A^T)$. From $AA^TAAA^T = O$, we have

$$U_r D V_r^T V_r D U_r^T U_r D V_r^T U_r D V_r^T V_r D U_r^T = O$$

Simplify, we get

$$U_r D^3 V_r^T U_r D^2 U_r^T = O$$

Multiply $U_r(D^{-1})^2 U_r^T$ from the left and $U_r D^{-1} V_r^T$ form the right, we get

$$A^2 = U_r D V_r^T U_r D V_r^T = O$$

Thus A and all its Jordan blocks are nilpotent. For all $l \times l$ nilpotent Jordan block N, $N^k = O \iff k \geq l$. So the size of any the Jordan block of A is at most 2, so $2 \times dim\ Ker(A) \geq n$, we get $dim\ Ker(A) \geq \frac{n}{2}$. □

Problem2

2.1

Let

$$P = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$$

Note that

$$PA \otimes BP^{-1} = P \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} P^{-1} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{bmatrix} = B \otimes A$$

□

2.2

Proof. First,

$$I \otimes e^B = \begin{bmatrix} e^B & \\ & e^B \end{bmatrix} = e^{\begin{bmatrix} B & \\ & B \end{bmatrix}} = e^{I \otimes B}$$

Then we know

$$\begin{aligned} e^{A\otimes I} &= e^{PI\otimes AP^{-1}} \\ &= Pe^{I\otimes A}P^{-1} \\ &= PI\otimes e^{A}P^{-1} \\ &= e^{A}\otimes I \end{aligned}$$

□

2.3

Proof. Note that $(A\otimes I)(I\otimes B) = (AI)\otimes(IB) = (IA)\otimes(BI) = (I\otimes B)(A\otimes I)$. So

$$e^{A\otimes I+I\otimes B} = e^{A\otimes I}e^{I\otimes B} \tag{*}$$

We have

$$\begin{aligned} e^{A}\otimes e^{B} &= (e^{A}I)(Ie^{B}) \\ &= (e^{A}\otimes I)(I\otimes e^{B}) \\ &= e^{A\otimes I}e^{I\otimes B} && \text{(From the previous subproblem)} \\ &= e^{A\otimes I+I\otimes B} && \text{(From (*))} \\ &= e^{A\oplus B} \end{aligned}$$

□

2.4

Calculate directly, we have

$$\begin{aligned} trace(A\oplus B) &= trace(A\otimes I)+trace(I\otimes B) \\ &= trace(\begin{bmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{bmatrix})+trace(\begin{bmatrix} B & \\ & B \end{bmatrix}) \\ &= 2trace(A)+2trace(B) \end{aligned}$$

□

2.5

Proof. **Case 1.** If A is diagonalizable, then let

$$A = P\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}P^{-1}$$

Note that A is invertible, so $\lambda_1,\ \lambda_2$ are non-zero. Let $\lambda_1 = e^{\mu_1},\ \lambda_2 = e^{\mu_2}$, where $\mu_1,\ \mu_2 \in \mathbb{C}$. Let $X = P\begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix}P^{-1}$. We have

$$\begin{aligned}
e^X &= e^{P\begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix}P^{-1}} \\
&= Pe^{\begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix}}P^{-1} \\
&= P\begin{bmatrix}e^{\mu_1} & \\ & e^{\mu_2}\end{bmatrix}P^{-1} \\
&= P\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}P^{-1} \\
&= A
\end{aligned}$$

Case 2. If A is not diagonalizable, then let

$$A = P\begin{bmatrix}\lambda & 1 \\ & \lambda\end{bmatrix}P^{-1}$$

Note that A is invertible, so $\lambda \neq 0$. Let $\lambda = e^{\mu}$, where $\mu \in \mathbb{C}$. Let $X = P\begin{bmatrix}\mu & \lambda^{-1} \\ & \mu\end{bmatrix}P^{-1}$. We have

$$X = P\begin{bmatrix}\lambda^{-1} & \\ & 1\end{bmatrix}\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}\begin{bmatrix}\lambda & \\ & 1\end{bmatrix}P^{-1}$$

So

$$\begin{aligned}
e^X &= P\begin{bmatrix}\lambda^{-1} & \\ & 1\end{bmatrix}e^{\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}}\begin{bmatrix}\lambda & \\ & 1\end{bmatrix}P^{-1} \\
&= P\begin{bmatrix}\lambda^{-1} & \\ & 1\end{bmatrix}\begin{bmatrix}e^{\mu} & e^{\mu} \\ & e^{\mu}\end{bmatrix}\begin{bmatrix}\lambda & \\ & 1\end{bmatrix}P^{-1} \\
&= P\begin{bmatrix}\lambda^{-1} & \\ & 1\end{bmatrix}\begin{bmatrix}\lambda & \lambda \\ & \lambda\end{bmatrix}\begin{bmatrix}\lambda & \\ & 1\end{bmatrix}P^{-1} \\
&= P\begin{bmatrix}\lambda & 1 \\ & \lambda\end{bmatrix}P^{-1} \\
&= A
\end{aligned}$$

$\square$

2.6

Consider $A \otimes B$, we have

$$A \otimes B = \begin{bmatrix}a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22}\end{bmatrix} := \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix}$$

where $\boldsymbol{v}_i \in \mathbb{C}^4$. When A is not invertible, we have λ_1, λ_2 s.t. $\lambda_1^2 + \lambda_2^2 \neq 0$ and

$$\lambda_1\begin{bmatrix}a_{11} \\ a_{21}\end{bmatrix} + \lambda_2\begin{bmatrix}a_{12} \\ a_{22}\end{bmatrix} = 0$$

So

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_3 = \begin{bmatrix} b_{11}(\lambda_1 a_{11} + \lambda_2 a_{12}) \\ b_{21}(\lambda_1 a_{11} + \lambda_2 a_{12}) \\ b_{11}(\lambda_1 a_{21} + \lambda_2 a_{22}) \\ b_{21}(\lambda_1 a_{21} + \lambda_2 a_{22}) \end{bmatrix} = 0$$

which means $\boldsymbol{v}_1, \boldsymbol{v}_3$ are linearly dependent, so $det(A \otimes B) = det^2(A)det^2(B) = 0$.
When B is not invertible, we have λ_1, λ_2 s.t. $\lambda_1^2 + \lambda_2^2 \neq 0$ and

$$\lambda_1 \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \lambda_2 \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = 0$$

So

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 = \begin{bmatrix} a_{11}(\lambda_1 b_{11} + \lambda_2 b_{12}) \\ a_{11}(\lambda_1 b_{21} + \lambda_2 b_{22}) \\ a_{21}(\lambda_1 b_{11} + \lambda_2 b_{12}) \\ a_{21}(\lambda_1 b_{21} + \lambda_2 a_{22}) \end{bmatrix} = 0$$

which means $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent, so $det(A \otimes B) = det^2(A)det^2(B) = 0$.

When both A, B are invertible, let $A = e^X,\ B = e^Y$. We get

$$\begin{aligned} det(A \otimes B) &= det(e^X \otimes e^Y) && \text{(subproblem2.5)} \\ &= det(e^{X \oplus Y}) && \text{(subproblem2.3)} \\ &= e^{trace(X \oplus Y)} \\ &= e^{2trace(X) + 2trace(Y)} && \text{(subproblem2.4)} \\ &= (e^{trace(X)} e^{trace(Y)})^2 \\ &= (det(e^X)det(e^Y))^2 \\ &= det^2(A)det^2(B) \end{aligned}$$

□

Problem3

3.1

Example. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

Then we have the Jordan decomposition

$$A = \begin{bmatrix} -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i & 1 & & \\ & i & & \\ & & -i & 1 \\ & & & -i \end{bmatrix} \begin{bmatrix} -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \end{bmatrix}^{-1}$$

□

3.2

Example. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

Calculate directly, we get

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \ BA = O$$

Clearly, the rank of AB and BA are different. So AB and BA are not similiar. □

3.3

Example. Let

$$A = \begin{bmatrix} 1 & i \\ & 1 \end{bmatrix}$$

Clearly A has non-real entry but only has real eigenvalue 1. □

3.4

Example. Let

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

By definition, $e^A = \begin{bmatrix} e^0 & \frac{d}{dx}e^x\big|_{x=0} & & \\ & e^0 & & \\ & & e^0 & \frac{d}{dx}e^x\big|_{x=0} \\ & & & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} = A + I.$ □

Let $A = XJX^{-1}$, where J is the Jordan normal form of A. We have $e^A = A + I \iff e^J = J + I$. Consider the single Jordan blocks of J.

When $J(\lambda)$ is 1×1, then we have $e^\lambda = \lambda + 1$.

When $J(\lambda)$ is 2×2, then we have $\begin{bmatrix} e^\lambda & e^\lambda \\ & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 1 \\ & \lambda + 1 \end{bmatrix}$, which yields

$$\begin{cases} e^\lambda = 1 \\ e^\lambda = \lambda + 1 \end{cases}$$

We know $\lambda = 0$. All 2×2 Jordan block of J is nilpotent.

When $J(\lambda)$ is 3×3, then we have $\begin{bmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2!} \\ & e^\lambda & e^\lambda \\ & & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 1 & \\ & \lambda + 1 & 1 \\ & & \lambda + 1 \end{bmatrix}$, which yields

$$\begin{cases} e^\lambda = 1 \\ e^\lambda = \lambda + 1 \\ \frac{e^\lambda}{2!} = 0 \end{cases}$$

We know it has no solution in $\mathbb{C}$, so J does not have a 3×3 single Jordan block. Similary there is no 4×4 single Jordan block.

So all the possible Jordan canonical forms of A are

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \mu_2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \mu_1 & \\ & & & \mu_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \mu_3 & \\ & & & \mu_4 \end{bmatrix}$$

Where μ_i are solutions to $e^\mu - \mu - 1 = 0$ in $\mathbb{C}$. (They can be distinct or some of them are equal.) □

Problem 4

4.1

Note that

$$M(1) = x,\ M(x) = x^2,\ M(x^2) = x^3,\ M(x^3) = -d - cx - bx^2 - ax^3$$

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}$$

We know that

$$M([1\ x\ x^2\ x^3]) = [1\ x\ x^2\ x^3]A$$

4.2

***Proof*.** Let's consider the case of $n \times n$ matrix A. Let V be the space of polynomials degree strictly less then n and $q(x)$ be a degree n polynomial.
For all polynomials $r(x),\ s(x),\ t(x)$. We have $r(x) = m(x)t(x) + (r(x)\ mod\ t(x))$, $s(x) = n(x)t(x) + (s(x)\ mod\ t(x))$ for some polynomial $m(x),\ n(x)$. So we have

$$(r(x) + s(x))\ mod\ t(x) = ((r(x)\ mod\ t(x)) + ((s(x)\ mod\ t(x))$$

$$(r(x) \cdot s(x))\ mod\ t(x) = ((r(x)\ mod\ t(x)) \cdot ((s(x)\ mod\ t(x))$$

We know $Mp(x) = xp(x)\ mod\ q(x)$, so for $\forall$ k

$$\begin{aligned} M^k p(x) &= M(M^{k-1}p)(x) \\ &= xM^{k-1}p(x)\ mod\ q(x) \\ &= (x\ mod\ q(x))(M^{k-1}p(x)\ mod\ q(x)) \\ &= (x\ mod\ q(x))((xM^{k-2}p(x)\ mod\ q(x))\ mod\ q(x)) \\ &= (x\ mod\ q(x))(xM^{k-2}p(x)\ mod\ q(x)) \\ &= x^2M^{k-2}p(x)\ mod\ q(x) \end{aligned}$$

Follow this, by induction, we know

$$M^k p(x) = x^k p(x)\ mod\ q(x)$$

Note that the polynomial of the operator M is the linear combination of M^k for some k. For $\forall$ polynomial $P(x)$

$$(P(M)p)(x) = P(x)p(x)\ mod\ q(x)$$

So we know $q(M)p(x) = q(x)p(x)\ mod\ q(x) = \mathbf{0}$. For $\forall$ polynomial $Q(x)$ s.t. $Q(M) = O$, we have

$$Q(x)p(x)\ mod\ q(x) = \mathbf{0}$$

So $Q(x)$ is a multiple of $q(x)$. Thus $q(x)$ is the minimal polynomial of operator M and matrix A. So the characteristic polynomial of A is a monic multiple of $q(x)$ with degree n. Note that $deg(q(x)) = n$. Thus, there can be no more factors for the characteristic polynomial, which means $q(x)$ is the characteristic polynomial.

□

Collaborator for **4.2**: Chen Siyuan.

4.3

We claim that A has a single Jordan block for each eigenvalue λ_i. If not, then the degree of factor $(x - \lambda_i)$ in the minimal polynomial will be strictly less then the degree of factor $(x - \lambda_i)$ in the characteristic polymonial, which is contradictory to the conclusion in the previous subproblem, that the characteristic polynomial and the minimal polynomial are the same.

Then we know that AB=BA implies $B = f(A)$ for some polynomial f. (Proven in the lecture note) □

4.4

From **subproblem 4.2** we know that p(x) is the minimal polynomial, which means the Jordan canonical form of A can be

$$\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$$

Or dependent on the choose of basis, there are other 23 possible forms, which correspond to other permutations of $\{1, 2, 3, 4\}$. □

4.5

From **subproblem 4.2** we know that p(x) is the minimal polynomial, which means the Jordan canonical form of A can be

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$$

Or dependent on the choose of basis, there are other 5 possible forms, which correspond to other permutations of $\{0, 1, 2\}$.

□

Problem 5

5.1

Proof. Note that $AA^T = I$.

$$\begin{aligned} & A + A^T - 2I \\ = & -(AA^T - A - A^T + I) \\ = & -(A - I)(A^T - I) \end{aligned}$$

□

5.2

***Proof*.**

$$\begin{aligned}
\lim_{t\to 0}\frac{-(f(t)-I)(f(t)^T-I)}{t} &= \lim_{t\to 0}\frac{(f(t)-f(0))+(f(t)^T-f(0)^T)}{t}\\
&= \lim_{t\to 0}\frac{(f(t)-f(0))}{t}+\lim_{t\to 0}\frac{(f(t)^T-f(0)^T)}{t}\\
&= f'(0)+f'(0)^T
\end{aligned}$$

□

5.3

***Proof*.**

$$\begin{aligned}
\lim_{t\to 0}\frac{-(f(t)-I)(f(t)^T-I)}{t} &= \lim_{t\to 0}-(f(t)-f(0))\frac{f(t)^T-f(0)^T}{t}\\
&= \lim_{t\to 0}-(f(t)-f(0))\cdot\lim_{t\to 0}\frac{f(t)^T-f(0)^T}{t}\\
&= O\cdot f'(0)^T\\
&= O
\end{aligned}$$

So $f'(0)+f'(0)^T = \lim_{t\to 0}\frac{-(f(t)-I)(f(t)^T-I)}{t} = O$, $f'(0)$ is skew-symmetric. □

5.4

***Proof*.** Note that $(f(t+h)f(t)^T)^T = f(t)f(t+h)^T$, we have

$$\begin{aligned}
f'(t)f(t)^T+f(t)f'(t)^T &= \lim_{h\to 0}\frac{f(t+h)-f(t)}{t}f(t)^T+f(t)\lim_{h\to 0}\frac{f(t+h)^T-f(t)^T}{t}\\
&= \lim_{h\to 0}\frac{f(t+h)f(t)^T+f(t)f(t+h)^T-2I}{t}\\
&= -\lim_{h\to 0}\frac{f(t+h)f(t)^T-f(t)f(t)^T}{t}(f(t)f(t+h)^T-I)\\
&= -f'(t)f(t)^T\lim_{h\to 0}(f(t)f(t+h)^T-I)\\
&= O
\end{aligned}$$

Note that $f(t)^{-1}=f(t)^T$, we have $f'(t)f(t)^{-1}+f(t)f'(t)^{-1}=O$. So $f'(t)f(t)^{-1}$ is skew-symmetric. □

5.5

***Proof*.** From $f'(t)f(t)^{-1}=A$, we know $f'(t)=Af(t)$. Let $f(t)=[\boldsymbol{v}_1(t)\ \cdots\ \boldsymbol{v}_n(t)]$. For $\forall i$, $\frac{d\boldsymbol{v}_i(t)}{dt}=A\boldsymbol{v}_i(t)$. So $\boldsymbol{v}_i(t)=e^{At}\boldsymbol{w}_i$ for some $\boldsymbol{w}_i\in\mathbb{C}^n$. Let $B=[\boldsymbol{w}_1\ \cdots\ \boldsymbol{w}_n]$, we have $f(t)=e^{At}B$.
Since A is skew-symmetric, we have $(e^{At})^{-1}=(e^{At})^T$. Thus $B=(e^{At})^Tf(t)$, $BB^T=(e^{At})^Tf(t)f(t)^T(e^{At})=I$. I.e. $B\in SO_n$. □