homework2

曹茵雯 info ID: 2020010919 tel: 13051862801 collaborator: Chenyang Zhao

1.2.1

① incorrect. counter example:

for $V_1 = \text{span}\begin{pmatrix}1\\0\\0\end{pmatrix}$, $V_2 = \text{span}\begin{pmatrix}0\\1\\0\end{pmatrix}$, $V_3 = \text{span}\begin{pmatrix}0\\0\\1\end{pmatrix}$, $V_4 = \text{span}\begin{pmatrix}1\\1\\1\end{pmatrix}$

then any three of $V_1 \ldots V_4$ are linearly independent, but all of them are not.

② the example is the same as ①'s

③ Proof: V_1, V_2 are linearly independent $\Rightarrow \dim(V_1+V_2) = \dim V_1 + V_2$

also $\dim(V_3+V_4) = \dim V_3 + \dim V_4$,

$$\dim(V_1+V_2+V_3+V_4) = \dim(V_1+V_2) + \dim(V_3+V_4)$$
$$= \sum_{i=1}^{4} \dim V_i$$

$\therefore V_1 \cdots V_4$ are linearly independent.

1.2.2 for any A, $A = \underbrace{\frac{A+A^T}{2}}_{\substack{\text{symmetric}\\\text{matrix}\\ \parallel \\ S}} + \underbrace{\frac{A-A^T}{2}}_{\substack{\text{skew}\\\text{symmetric}\\\text{matrix}\\ \parallel \\ S'}}$

then $A = S+S' \xmapsto{T} S-S'$

let $V_1 = \{\text{all symmetric matrices in } V\}$, $V_2 = \{\text{all skew symmetric matrices in } V\}$.

$V = V_1 \oplus V_2$, V_1, V_2 are T-invariant subspaces.

since $\dim V_1 = \frac{n^2+n}{2}$, $\dim V_2 = \frac{n^2-n}{2}$, then the corresponding T is:

$$\begin{pmatrix} I_{\frac{n^2+n}{2}} & \\ & -I_{\frac{n^2-n}{2}} \end{pmatrix}$$

1.2.3.

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\begin{tikzcd}
\mathrm{Ran}(A^T) \arrow[r, "A_{11}"] \arrow[dr, "A_{21}"'] & \mathrm{Ran}(A) \\
\mathrm{Ker}(A) \arrow[r, "A_{22}"'] \arrow[ur, "A_{12}"] & \mathrm{Ker}(A^T)
\end{tikzcd}$$

A_{11} is injection. Proof. suppose $A_{11}x=0$ for some $x \in \mathrm{Ran}(A^T)$.

because $Ax = A_{11}x + A_{21}x$, and $Ax \in \mathrm{Ran}(A)$, $\mathrm{Ran}(A) \overset{\perp}{\oplus} \mathrm{Ker}(A^T) = \mathbb{R}^m$

$\therefore A_{21}x=0$, therefore $Ax = A_{11}x = 0$.

$\Rightarrow x \in \mathrm{Ker}(A) \cap \mathrm{Ran}(A^T) \Rightarrow x=0$. so A_{11} is injection.

Also A_{11} is surjection. Proof:

for $\forall v \in \mathrm{Ran}(A)$, $\exists x$, s.t. $v = Ax$.

see the proof for this at the end.

$\because \mathrm{Ran}(AA^T) = \mathrm{Ran}(A)$. $\therefore \exists y$, s.t. $v = Ax = AA^Ty$.

that is, for $\forall v \in \mathrm{Ran}(A)$, $\exists A^Ty \in \mathrm{Ran}(A^T)$, s.t. A^Ty is the preimage of v.

so A_{11} is surjection. So A_{11} is invertible. $\therefore$ rank $A_{11} = \dim \mathrm{Ran}(A^T) = r$.

since $\forall v \in \mathrm{Ker}(A)$, $A_{12}v = A_{22}v = 0 \Rightarrow$ rank of A_{12}, A_{22} are 0.

since for any $v \in \mathrm{Ran}(A^T)$, $Av \in \mathrm{Ran}(A)$. therefore $A_{21}v = 0$.

so rank $A_{21} = 0$.

$\Rightarrow$ rank $A_{11} = r$, rank $A_{12}, A_{21}, A_{22} = 0$.

1.2.4

① V is an A-invariant subspace.

suppose V has $\dim V = k$

let $\{b_1, b_2 \dots b_k\}$ be a basis of V.

T存在的前提是V是A-invariant subspace.

then $A(b_1 \dots b_k) = (b_1 \dots b_k)T$. where $T \in M_{k\times k}(\mathbb{C})$.

T是映射A在基$b_1 \dots b_k$下的表示矩阵

suppose $Tx = \lambda x$, $x = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \Rightarrow T\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$.

then $A(c_1b_1 + c_2b_2 + \dots + c_kb_k) = A(b_1 \dots b_k)\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = (b_1 \dots b_k)T\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$.

$$= (b_1 \cdots b_k)\lambda \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

$$= \lambda c_1 b_1 + \lambda c_2 b_2 + \cdots + \lambda c_k b_k = \lambda(c_1 b_1 + \cdots + c_k b_k)$$

let $w = c_1 b_1 + \cdots + c_k b_k$, then $Aw = \lambda w$.

$w \in V$, and w is A's eigenvector. $\square$.

② suppose $\exists x \neq 0, \lambda$, s.t. $Ax = \lambda x$.

then the eigenspace for A is $N(A-\lambda I)$.

for $\forall v \in N(A-\lambda I)$, $(A-\lambda I)v = 0$.

then $(A-\lambda I)Bv = ABv - \lambda Bv = BAv - \lambda Bv$

$$= B(A-\lambda I)v$$

$$= 0.$$

$\Rightarrow Bv \in N(A-\lambda I)$, $N(A-\lambda I)$ is B-invariant.

from ① we know that if $N(A-\lambda I)$ is an B-invariant subspace

then B has an eigenvector in $N(A-\lambda I)$.

$\therefore \exists w$, $(A-\lambda I)w = 0$ and also $Bw = \mu w$.

$\Rightarrow Aw = \lambda w$ and also $Bw = \mu w$ $\Rightarrow$ A, B has a common eigenvector w.

1.2.5.

$N_\infty(D) = \{ f(x) \mid f$ is a polynomial of $x \}$.

$N_\infty(D-I)$ is not spanned by e^x.

counter example: let $f = e^x(x+1)$.

then $(D-I)f = (x+2)e^x - (x+1)e^x = e^x$

$(D-I)^2 f = e^x - e^x = 0$

$\Rightarrow e^x(x+1) \in N_\infty(D-I)$. so $N_\infty(D-I)$ is not spanned by e^x.

Lemma. $\mathrm{Ran}(A^TA) = \mathrm{Ran}(A)$.

To prove $R(A^TA) = R(A)$.

we only need to prove $N(A^TA) = N(A)$.

$N(A^TA) \subseteq N(A)$ is trivial, since

for $\forall x$, $A^TAx = 0$.

$\Rightarrow x^TA^TAx = 0$.

$\Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0$. so $N(A) \subseteq N(A^TA)$. $\Rightarrow N(A) = N(A^TA)$.