Chapter 1

Systems of Linear Equations

1.1 HW1 Complex Stuff

Excercise 1.1.1.

1. For $n = 2$, we have

$$A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and for each even $n > 2$, we have

$$A_n = \begin{bmatrix} A_2 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_2 \end{bmatrix},$$

such that $A_n^2 = -I$.

2. If $A^2 = -I$, then A^2 has the only eigenvalue -1 (with a mutiplicity of n). On the other hand, if n is odd and A is real, then the n-degree real-coeffcient eigenpolynomial of A does have at least one real root. Therefore A has at least one real eigenvalue, whose square is -1, which is impossible.

Excercise 1.1.2.

1. For any $k = a + b\,\mathrm{i} \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$,

$$\begin{aligned} B(k\mathbf{v}) &= B((a + b\,\mathrm{i})\mathbf{v}) = B(a\mathbf{v} + b\,\mathrm{i}\,\mathbf{v}) = B(a\mathbf{v} + bA\mathbf{v}) = aB\mathbf{v} + bBA\mathbf{v}, \\ kB\mathbf{v} &= (a + b\,\mathrm{i})(B\mathbf{v}) = a(B\mathbf{v}) + b\,\mathrm{i}\,(B\mathbf{v}) = aB\mathbf{v} + bAB\mathbf{v}. \end{aligned}$$

Therefore, $B(k\mathbf{v}) = kB\mathbf{v}$ if and only if $AB = BA$.

2. No. For example, let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

Then $A^2 = X^2 = -I$, but

$$\begin{aligned} AX &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \\ \neq XA &= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

3. Since $C^2 = I$, C have n complex eigenvalues whose square are 1. Hence the eigenvalues must be 1 or -1, and because they are all real, C is diagonalizable.
   Suppose the eigenspaces for 1 and -1 are F and G respectively. For each $\mathbf{v} \in F$, since $CA = -AC$, we have
   $$C(A\mathbf{v}) = (CA)\mathbf{v} = (-AC)\mathbf{v} = -A(C\mathbf{v}) = -A\mathbf{v},$$
   i.e. $A\mathbf{v}$ is in G. Note that A is invertible ($A^{-1} = -A$), so A gives a bijection between F and G, and therefore they have the same dimension.

4. An example is
   $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
   and
   $$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Excercise 1.1.3.

1. It is real linear but not complex linear, as for each element c in any vector, only if k is real do we have $\overline{kc} = k\bar{c}$.

2. $\mathbb{C}$-linear implies $\mathbb{R}$-linear, because if $L(k\mathbf{v}) = kL\mathbf{v}$ for all $k \in \mathbb{C}$ and $\mathbf{v} \in V$, then this is also true for any $k \in \mathbb{R}$, since a real number is as well a complex number.

3. An $\mathbb{R}$-basis for $\mathbb{C}^2$ is
   $$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathrm{i} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{i} \end{bmatrix}\}.$$
   It is 4-real-dimension. And a $\mathbb{C}$-basis for $\mathbb{C}^2$ is
   $$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}.$$
   It is 2-complex-dimension.

4. $\mathbb{C}$-linearly independent implies $\mathbb{R}$-linearly independent, because if a bunch of vectors $\{\mathbf{v}_i\}$ are not $\mathbb{R}$-linearly independent, then we have a set of real coeffcient $\{a_i\}$ all of which are not zero, such that $\sum a_i\mathbf{v}_i = \mathbf{0}$. And this is also true for complex $\{a_i\}$, since a real number is as well a complex number, and therefore $\{\mathbf{v}_i\}$ are not $\mathbb{C}$-linearly independent.

5. $\mathbb{R}$-spanning implies $\mathbb{C}$-spanning, because for a bunch of vectors $\{\mathbf{v}_i\}$, if any $\mathbf{v} \in V$ can be represented as $\mathbf{v} = \sum a_i\mathbf{v}_i$ where $\{a_i\}$ are all real, then this is as well true for complex $\{a_i\}$, since a real number is as well a complex number.

Excercise 1.1.4.

1.
   $$P\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad P\begin{bmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{bmatrix} = \begin{bmatrix} \mathrm{i} \\ -1 \\ -\mathrm{i} \\ 1 \end{bmatrix}.$$

2.

$$
\begin{aligned}
PF_4 &= P \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix} \\
&= F_4 D.
\end{aligned}
$$

Therefore, the four eigenvectors and eigenvalues of P are the columns of F_4 and the diagonal entries of D respectively.

3.

$$
C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_3 + c_0 + c_1 + c_2 \\ c_2 + c_3 + c_0 + c_1 \\ c_1 + c_2 + c_3 + c_0 \end{bmatrix},
$$

$$
C \begin{bmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{bmatrix} = \begin{bmatrix} c_0 + c_1\,\mathrm{i} + c_2(-1) + c_3(-\mathrm{i}) \\ c_3 + c_0\,\mathrm{i} + c_1(-1) + c_2(-\mathrm{i}) \\ c_2 + c_3\,\mathrm{i} + c_0(-1) + c_1(-\mathrm{i}) \\ c_1 + c_2\,\mathrm{i} + c_3(-1) + c_0(-\mathrm{i}) \end{bmatrix} = \begin{bmatrix} c_0 - c_2 + (c_1 - c_3)\,\mathrm{i} \\ c_3 - c_1 + (c_0 - c_2)\,\mathrm{i} \\ c_2 - c_0 + (c_3 - c_1)\,\mathrm{i} \\ c_1 - c_3 + (c_2 - c_0)\,\mathrm{i} \end{bmatrix}.
$$

4. Obviously $C = c_0 P^0 + c_1 P^1 + c_2 P^2 + c_3 P^3 = \sum_n c_n P^n$. For the i-th column of F (also a eigenvector of P, with eigenvalue λ_i) $\mathbf{F}_i$, we have

$$
P^n \mathbf{F}_i = \lambda_i^n \mathbf{F}_i.
$$

Note that $\mathbf{F}_i = (\lambda_i^0, \lambda_i^1, \lambda_i^2, \lambda_i^3)$, therefore

$$
C\mathbf{F}_i = \sum_n c_n P^n \mathbf{F}_i = \sum_n c_n \lambda_i^n \mathbf{F}_i = (\mathbf{c} \cdot \mathbf{F}_i)\mathbf{F}_i,
$$

where $\mathbf{c} = (c_0, c_1, c_2, c_3)$. Therefore the eigenvectors and the eigenvalues are $\mathbf{F}_i$ and $(\mathbf{c} \cdot \mathbf{F}_i)$ respectively.