Review

- 偏导数的定义与几何意义

- 方向导数的定义与几何意义

$$g(t) = f(x_0 + \frac{\vec{v}}{\|\vec{v}\|}t) = f(x_0^{(1)} + \frac{v_1}{\|\vec{v}\|}t, \cdots, x_0^{(n)} + \frac{v_n}{\|\vec{v}\|}t)$$

$$\frac{\partial f(x_0)}{\partial \vec{v}} = \lim_{t \to 0^+} \frac{f(x_0 + \vec{v}t) - f(x_0)}{\|\vec{v}\|t} = \lim_{t \to 0^+} \frac{g(t) - g(0)}{t} = g'_+(0)$$

$$\frac{\partial f(x_0, y_0)}{\partial \vec{v}} \xrightarrow{f在(x_0, y_0)可微时} \mathrm{grad}f(x_0, y_0) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

- 梯度$\mathrm{grad}f(x_0)$的定义与意义

- n元函数可微的定义与判别

f 则 x_0 可微 $\Leftrightarrow \exists$ 常数 $a_1, a_2, \cdots, a_n \in \mathbb{R}, s.t.$

$$\lim_{\mathrm{x} \to \mathrm{x}_0} \frac{f(\mathrm{x}) - f(\mathrm{x}_0) - a_1 \Delta x_1 - a_2 \Delta x_2 - \cdots - a_n \Delta x_n}{\|\mathrm{x} - \mathrm{x}_0\|} = 0.$$

$\Leftrightarrow f(\mathrm{x}_0 + \Delta \mathrm{x}) - f(\mathrm{x}_0)$

$$= \frac{\partial f}{\partial x_1}(\mathrm{x}_0)\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathrm{x}_0)\Delta x_n + o(\|\Delta \mathrm{x}\|), \Delta \mathrm{x} \to 0 \text{ 时.}$$

$$\Leftrightarrow f(\mathrm{x}_0 + \Delta \mathrm{x}) - f(\mathrm{x}_0) = \sum_{i=1}^{n} f'_{x_i}(\mathrm{x}_0)\Delta x_i + \sum_{i=1}^{n} \varepsilon_i \Delta x_i,$$

$$\lim_{\Delta \mathrm{x} \to 0} \varepsilon_i(\Delta \mathrm{x}) = 0, \quad i = 1, 2, \cdots, n.$$

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●多元函数连续、可微、偏导数存在、
偏导函数连续之间的关系

●f''_{xy}, f''_{yx} 都在(x_0, y_0)连续

$$\Rightarrow f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$$

●f''_{xy}, f''_{yx} 也记为 f''_{12}, f''_{21}

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§5. 向量值函数的微分

1. 线性映射

Def. 称向量值函数$f:\mathbb{R}^n \to \mathbb{R}^m$为线性的, 若对任意$\alpha,\beta$ $\in\mathbb{R}, x, y \in \mathbb{R}^n$,都有$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$.

Thm. $\mathbb{R}^n \to \mathbb{R}^m$的线性映射与$\mathrm{M}_{mn}$中矩阵一一对应.

Proof. 记$\mathbb{R}^n \to \mathbb{R}^m$的线性映射构成的集合为$\mathrm{Y}$,定义

$$\varphi:\mathrm{M}_{mn} \to \mathrm{Y}$$

$$\mathrm{A} \mapsto \varphi(\mathrm{A})$$

使得$\varphi(\mathrm{A})x = \mathrm{A}x, \forall x \in \mathbb{R}^n$.下证$\varphi$为一一映射.

任给 $f \in Y$, 记$e_1, e_2, \cdots, e_n$为$\mathbb{R}^n$中自然基,则

$$\mathrm{A} \triangleq (f(e_1), f(e_2), \cdots, f(e_n)) \in M_{mn}.$$

对任意$x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$,有

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) = \mathrm{A}x.$$

因此φ为满射.

任给 $f \in Y$,若$\exists \mathrm{A}, \mathrm{B} \in \mathrm{M}_{m,n}, s.t. \varphi(\mathrm{A}) = \varphi(\mathrm{B}) = f$,则

$$f(x) = \mathrm{A}x = \mathrm{B}x, \forall x \in \mathbb{R}^n, 从而\mathrm{A} = \mathrm{B}. 因此\varphi为单射. \square$$

Thm. $A \in M_{mn}$, 则存在$C \geq 0, s.t.$

$$\|Ax\|_m \leq C\|x\|_n, \forall x \in \mathbb{R}^n.$$

使得此不等式成立的最小的C记为$\|A\|$.

Proof. $y = \|Ax\|_m$ 连续, $C = \max\{\|Ax\|_m : x \in \mathbb{R}^n, \|x\|_n = 1\}$存在,

$$\|Ax\|_m = \left\|A\left(\|x\|_n \frac{x}{\|x\|_n}\right)\right\|_m = \left\|A\frac{x}{\|x\|_n}\right\|_m \|x\|_n \leq C\|x\|_n, \forall x \in \mathbb{R}^n. \square$$

2.向量值函数的微分

Def. 设 $f: \Omega\left(\subset \mathbb{R}^n\right) \to \mathbb{R}^m$, $x_0 \in \Omega$. 若存在 $m \times n$ 矩阵 A, $s.t.$

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = A\Delta x + \alpha,$$

其中, $\alpha = \alpha(\Delta x) = \left(\alpha_1(\Delta x), \cdots, \alpha_m(\Delta x)\right)^{\mathrm{T}}$, 且当 $\Delta x \to 0$ 时,

$\alpha = o(\Delta x)$, 即

$$\lim_{\Delta x \to 0} \frac{\|\alpha\|_m}{\|\Delta x\|_n} = 0,$$

则称 f 在点 x_0 可微, 称 $A\Delta x$ 为 f 在点 x_0 的微分, 记为

$$\mathrm{d}f(x_0) = A\Delta x = Adx,$$

称 A 为 f 在 x_0 的 Jacobi 矩阵, 记作 $A = J(f)\big|_{x_0} = J_f(x_0)$.

Thm.$f = (f_1, f_2, \cdots, f_m)^{\mathrm{T}} : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^m$ 在 x_0 可微

$\Leftrightarrow n$ 元函数 $f_i : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$ 在 x_0 可微, $i = 1, 2, \cdots, m.$

Proof.(必要性) 设 f 在点 x_0 可微, 则

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = \mathrm{A}\Delta x + \alpha, \lim_{\Delta x \to 0} \frac{\|\alpha\|_m}{\|\Delta x\|_n} = 0,$$

比较第 i 个分量 $(1 \le \mathrm{i} \le m),$ 得

$$f_i(x_0 + \Delta x) - f_i(x_0) = \sum_{j=1}^{n} a_{ij}\Delta x_j + \alpha_i(\Delta x), \lim_{\Delta x \to 0} \frac{\alpha_i(\Delta x)}{\|\Delta x\|_n} = 0,$$

即 f_i 在 x_0 可微, $i = 1, 2, \cdots, m.$

(充分性) 设 f_i 在 x_0 可微, $i = 1, 2, \cdots, m$, 则

$$\begin{pmatrix} \Delta f_1(x_0) \\ \Delta f_2(x_0) \\ \vdots \\ \Delta f_m(x_0) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}\Bigg|_{x_0} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{pmatrix} + \begin{pmatrix} \alpha_1(\Delta x) \\ \alpha_2(\Delta x) \\ \vdots \\ \alpha_m(\Delta x) \end{pmatrix},$$

且 $\alpha_i(\Delta x) = o(\|\Delta x\|_n), \Delta x \to 0$ 时, $1 \le i \le m$. 故 f 在 x_0 可微. □

Remark: $f = \left(f_1, f_2, \cdots, f_m \right)^{\mathrm{T}} : \Omega \left(\subset \mathbb{R}^n \right) \rightarrow \mathbb{R}^m$ 在 x_0 可微,

则 f 在 x_0 的 $Jacobi$ 矩阵为

$$\mathrm{J}f(x_0) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\[2mm] \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\[2mm] \vdots & \vdots & \vdots & \vdots \\[2mm] \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix} \Bigg|_{x_0} \triangleq \dfrac{\partial(f_1, f_2, \cdots, f_m)}{\partial(x_1, x_2, \cdots, x_n)} \Bigg|_{x_0},$$

也记为 $\dfrac{\partial f}{\partial x}(x_0)$.

3.复合映射的微分

Thm. $u = g(x): \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $y = f(u): g(\Omega) \subset \mathbb{R}^m \to \mathbb{R}^k$, $x_0 \in \Omega$, $u_0 = g(x_0)$, 若$g(x)$在x_0可微, $f(u)$在u_0可微,则复合映射$y = (f \circ g)(x) = f\big(g(x)\big)$在$x_0$可微, 且

$$J(f \circ g)\big|_{x_0} = J(f)\big|_{u_0} \cdot J(g)\big|_{x_0},$$

即 $\dfrac{\partial\big(y_1, y_2, \cdots, y_k\big)}{\partial\big(x_1, x_2, \cdots, x_n\big)}\bigg|_{x_0} = \dfrac{\partial\big(y_1, y_2, \cdots, y_k\big)}{\partial\big(u_1, u_2, \cdots, u_m\big)}\bigg|_{u_0} \cdot \dfrac{\partial\big(u_1, u_2, \cdots, u_m\big)}{\partial\big(x_1, x_2, \cdots, x_n\big)}\bigg|_{x_0}$,

简记为 $\dfrac{\partial y}{\partial x}\bigg|_{x_0} = \dfrac{\partial y}{\partial u}\bigg|_{u_0} \cdot \dfrac{\partial u}{\partial x}\bigg|_{x_0}$

Question.可微能否替换成偏导存在？ 否！

Proof. $u = g(x)$在x_0可微,$y = f(u)$在u_0可微,记

$$A = \frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\bigg|_{u_0}, B = \frac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\bigg|_{x_0},$$

则

$$\Delta u = g(x_0 + \Delta x) - g(x_0) = B\Delta x + o(\Delta x),$$

$$\Delta y = f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u),$$

于是

$$\Delta(f \circ g)(x_0) = f(g(x_0 + \Delta x)) - f(g(x_0))$$

$$= f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u)$$

$$= A(B\Delta x + o(\Delta x)) + o(\Delta u) = AB\Delta x + Ao(\Delta x) + o(\Delta u),$$

往证: $Ao(\Delta x) + o(\Delta u) = o(\Delta x), \Delta x \to 0.$

$$\frac{\|\mathrm{A}o(\Delta x)\|}{\|\Delta x\|} \leq \frac{\|\mathrm{A}\|\|o(\Delta x)\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0\text{时}.$$

$$\frac{\|\Delta u\|}{\|\Delta x\|} = \frac{\|\mathrm{B}\Delta x + o(\Delta x)\|}{\|\Delta x\|} \leq \frac{\|\mathrm{B}\Delta x\|}{\|\Delta x\|} + \frac{\|o(\Delta x)\|}{\|\Delta x\|}$$

$$\leq \|\mathrm{B}\| + \frac{\|o(\Delta x)\|}{\|\Delta x\|} \rightarrow \|\mathrm{B}\|, \text{当} \Delta x \rightarrow 0\text{时}.$$

$$\frac{\|o(\Delta u)\|}{\|\Delta x\|} = \frac{\|o(\Delta u)\|}{\|\Delta u\|} \cdot \frac{\|\Delta u\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0\text{时}.$$

$$\frac{\|\mathrm{A}o(\Delta x) + o(\Delta u)\|}{\|\Delta x\|} \leq \frac{\|\mathrm{A}o(\Delta x)\|}{\|\Delta x\|} + \frac{\|o(\Delta u)\|}{\|\Delta x\|} \rightarrow 0, \text{当} \Delta x \rightarrow 0\text{时}. \square$$

Remark. $k = 1$时$, y = f(u_1, u_2, \cdots, u_m), u_i = g_i(x_1, x_2, \cdots, x_n),$

$1 \le i \le m,$均可微$,$则$\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} \dfrac{\partial u}{\partial x},$

$$\left(\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n} \right) = \left(\frac{\partial y}{\partial u_1}, \cdots, \frac{\partial y}{\partial u_m} \right) \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial u_m}{\partial x_1} & \cdots & \dfrac{\partial u_m}{\partial x_n} \end{pmatrix},$$

因此 $y = y(x_1, x_2, \cdots, x_n)$的偏导数满足链式法则

$$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1} \frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2} \frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial y}{\partial u_m} \frac{\partial u_m}{\partial x_i}, i = 1, 2, \cdots, n.$$

例. 球坐标变换 $\begin{cases} x = r\sin\varphi\cos\theta, \\ y = r\sin\varphi\sin\theta, \\ z = r\cos\varphi, \end{cases}$

是 $\Omega = \{(r,\varphi,\theta) \mid 0 \le r < \infty, 0 \le \varphi \le \pi, 0 \le \theta < 2\pi\}$ 到 $\mathbb{R}^3$ 的连续可微映射.

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$$

$$\frac{D(x,y,z)}{D(r,\varphi,\theta)} \triangleq \det\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = r^2\sin\varphi. \square$$

例：$\begin{cases} y_1 = u_1 u_2 - u_1 u_3, \\ y_2 = u_1 u_3 - u_2^2, \end{cases}$ $\begin{cases} u_1 = x_1 \cos x_2 + (x_1 + x_2)^2 \\ u_2 = x_1 \sin x_2 + x_1 x_2 \\ u_3 = x_1^2 - x_1 x_2 + x_2^2 \end{cases}$

求 $\left. \dfrac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|_{(1,0)}$.

解：$(x_1, x_2) = (1, 0)$时,$(u_1, u_2, u_3) = (2, 0, 1)$,

$$\left. \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|_{(1,0)} = \left. \frac{\partial(y_1, y_2)}{\partial(u_1, u_2, u_3)} \right|_{(2,0,1)} \cdot \left. \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2)} \right|_{(1,0)}$$

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$$= \begin{pmatrix} u_2 - u_3 & u_1 & -u_1 \\ u_3 & -2u_2 & u_1 \end{pmatrix} \Bigg|_{(2,0,1)}$$

$$\cdot \begin{pmatrix} \cos x_2 + 2(x_1 + x_2) & -x_1 \sin x_2 + 2(x_1 + x_2) \\ \sin x_2 + x_2 & x_1 \cos x_2 + x_1 \\ 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix} \Bigg|_{(1,0)}$$

$$= \begin{pmatrix} -1 & 2 & -2 \\ 1 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 0 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ 7 & 0 \end{pmatrix}. \square$$

例. $z = 2yf(\dfrac{x^2}{y}, 3y)$, f 二阶连续可微. 求 z''_{xy}.

解: f 二阶连续可微, 则 $z''_{xy} = z''_{yx}$. 后者求解相对容易.

设 $u = x^2/y$, $v = 3y$, $z = 2yf(u,v)$. 则

$$z'_x = 2y\left[f'_u \cdot (2x/y) + f'_v \cdot 0 \right] = 4xf'_u(u,v)$$

$$z''_{xy} = z''_{yx} = \frac{\partial}{\partial y}\left(4xf'_u\right) = 4x\left[f''_{uu} \cdot \frac{-x^2}{y^2} + f''_{vu} \cdot 3 \right]$$

$$= -\frac{4x^3}{y^2} f''_{uu} + 12xf''_{vu}. \qquad \square$$

例. $u(x,y)$ 二阶连续可微, 且 $u''_{xx} - u''_{yy} = 0.$ 令 $\xi = x - y,$

$\eta = x + y.$ 试证 : $u''_{\xi\eta} = 0.$

解: $x = (\xi + \eta)/2, \ y = (\eta - \xi)/2,$

$u'_\eta = u'_x x'_\eta + u'_y y'_\eta = (u'_x + u'_y)/2$

$u''_{\xi\eta} = \dfrac{1}{2}\dfrac{\partial}{\partial \xi}\left(u'_x + u'_y\right)$

Remark. 此处不能用 u'_1, u''_{12} 这样的记号.

$= \dfrac{1}{2}(u''_{xx}x'_\xi + u''_{yx}y'_\xi + u''_{xy}x'_\xi + u''_{yy}y'_\xi)$

$= \dfrac{1}{4}(u''_{xx} - u''_{yx} + u''_{xy} - u''_{yy}) = \dfrac{1}{4}(u''_{xx} - u''_{yy}) = 0.\ \square$

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例. $f(x,y)$可微,$f(x,x^2)=1$,$f'_x(x,x^2)=x$,求$f'_y(x,x^2)$,$(x\neq 0)$.

分析: $f'_x(x,x^2)=\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(x,x^2)}$.

解: 将$f(x,x^2)=1$两边同时对x求导, 有

$$f'_x(x,x^2)\cdot 1+f'_y(x,x^2)\cdot 2x=0,$$

$$x+2xf'_y(x,x^2)=0,$$

故$f'_y(x,x^2)=-\dfrac{1}{2}$, $\forall x\neq 0$. □

Question. $f'_x(x,x^2)$与$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x,x^2)$有何区别?

例. $y = (1/x)^{-(1/x)}$, 求 $y'(x)$.

解: 令 $u = 1/x, v = -1/x, y = u(x)^{v(x)}$.

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} \cdot \frac{du}{dx} + \frac{\partial y}{\partial v} \cdot \frac{dv}{dx}$$

$$= vu^{v-1}(-1/x^2) + u^v(\ln u)(1/x^2)$$

$$= u^{v+2}(1 + \ln u)$$

$$= (1/x)^{2-\frac{1}{x}}(1 - \ln x). \square$$

作业 ： 习题**1.5 No. 4,5,7,9.**