1.9 HW9 Tensor Calculations

Excercise 1.9.1.

1. 1) Symmetry:

$$\begin{aligned}(\phi^{ij}\mathbf{e}_i\mathbf{e}_j, \psi^{kl}\mathbf{e}_k\mathbf{e}_l) &= \phi^{ij}\psi^{kl}(\mathbf{e}_i\mathbf{e}_j, \mathbf{e}_k\mathbf{e}_l) && \text{(Bilinearity)}\\ &= \phi^{ij}\psi^{kl}(\mathbf{e}_i, \mathbf{e}_k)(\mathbf{e}_j, \mathbf{e}_l)\\ &= \psi^{kl}\phi^{ij}(\mathbf{e}_k, \mathbf{e}_i)(\mathbf{e}_l, \mathbf{e}_j) && \text{(Symmetry of inner product on } H)\\ &= \psi^{kl}\phi^{ij}(\mathbf{e}_k\mathbf{e}_l, \mathbf{e}_i\mathbf{e}_j)\\ &= (\psi^{kl}\mathbf{e}_k\mathbf{e}_l, \phi^{ij}\mathbf{e}_i\mathbf{e}_j). && \text{(Bilinearity)}\end{aligned}$$

2) Positive definiteness:

$$\begin{aligned}(\phi^{ij}\mathbf{e}_i\mathbf{e}_j, \phi^{kl}\mathbf{e}_k\mathbf{e}_l) &= \phi^{ij}\phi^{kl}(\mathbf{e}_i\mathbf{e}_j, \mathbf{e}_k\mathbf{e}_l) && \text{(Bilinearity)}\\ &= \phi^{ij}\phi^{kl}(\mathbf{e}_i, \mathbf{e}_k)(\mathbf{e}_j, \mathbf{e}_l)\\ &= \phi^{ij}\phi^{kl}g_{ik}g_{jl}\\ &= \phi^{ij}\phi_{ij}\\ &\begin{cases}= 0, & \phi = 0,\\ > 0, & \phi \neq 0.\end{cases}\end{aligned}$$

2. Given the expression, it has rank at most two. If the rank is 1, there exists $\mathbf{v} \in H_A$ and $\mathbf{w} \in H_B$ such that

$$a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2 = \mathbf{v} \otimes \mathbf{w} = v^i w^j \mathbf{e}_i\mathbf{e}_j,$$

and therefore $ab = (v^1w^1)(v^2w^2) = (v^1w^2)(v^2w^1) = 0$, contradictory to that a, b are non-zero. Hence the rank is not 1 but 2.

3. Suppose $\omega = \mathbf{v} \otimes \mathbf{w}$, then

$$\begin{aligned}A &= (\omega, (I_A \otimes L)(\omega)) = (\mathbf{v} \otimes \mathbf{w}, \mathbf{v} \otimes L\mathbf{w}) = (\mathbf{v}, \mathbf{v})(\mathbf{w}, L\mathbf{w}) = (v_1^2 + v_2^2)(w_1^2 - w_2^2),\\ B &= (\omega, (L \otimes I_B)(\omega)) = (\mathbf{v} \otimes \mathbf{w}, L\mathbf{v} \otimes \mathbf{w}) = (\mathbf{v}, L\mathbf{v})(\mathbf{w}, \mathbf{w}) = (v_1^2 - v_2^2)(w_1^2 + w_2^2).\end{aligned}$$

There is solution to the above equations for whatever $A, B \in \mathbb{R}$.

4.

$$\begin{aligned}A = (\omega, (I_A \otimes L)(\omega)) &= (a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2, (I_A \otimes L)(a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2))\\ &= (a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2, a\,\mathbf{e}_1 \otimes \mathbf{e}_1 - b\,\mathbf{e}_2 \otimes \mathbf{e}_2)\\ &= a^2 - b^2,\\ B = (\omega, (L \otimes I_B)(\omega)) &= (a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2, (L \otimes I_B)(a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2))\\ &= (a\,\mathbf{e}_1 \otimes \mathbf{e}_1 + b\,\mathbf{e}_2 \otimes \mathbf{e}_2, a\,\mathbf{e}_1 \otimes \mathbf{e}_1 - b\,\mathbf{e}_2 \otimes \mathbf{e}_2)\\ &= a^2 - b^2.\end{aligned}$$

We always have $A = B$.

Excercise 1.9.2.

1.

$${\alpha_{\mathcal{C}}}_i = g_{ik}{\alpha_{\mathcal{C}}}^k = g_{ik}B^k_l{\alpha_{\mathcal{B}}}^l = g_{ik}B^k_l g^{lj}{\alpha_{\mathcal{B}}}_j = B_i^j{\alpha_{\mathcal{B}}}_j.$$

2.

$$(\mathbf{v}_{\mathcal{C}} \otimes \mathbf{w}_{\mathcal{C}})^{ij} = {\mathbf{v}_{\mathcal{C}}}^i {\mathbf{w}_{\mathcal{C}}}^j = B^i_k {\mathbf{v}_{\mathcal{B}}}^k B^j_l {\mathbf{w}_{\mathcal{B}}}^l = B^i_k B^j_l (\mathbf{v}_{\mathcal{C}} \otimes \mathbf{w}_{\mathcal{C}})^{kl},$$

i.e. $L = B \otimes B$.

3.

$$\begin{aligned} &L\Big((\mathbf{v}_1)_{\mathcal{B}}{}^{j_1} \cdots (\mathbf{v}_a)_{\mathcal{B}}{}^{j_a} (\alpha^1)_{\mathcal{B}_{l_1}} \cdots (\alpha^b)_{\mathcal{B}_{l_b}} \Big)^{i_1 \cdots i_a}_{k_1 \cdots k_b} \\ =&\Big(B^{i_1}_{j_1} \cdots B^{i_a}_{j_a} B^{l_1}_{k_1} \cdots B^{l_b}_{k_b} \Big) (\mathbf{v}_1)_{\mathcal{B}}{}^{j_1} \cdots (\mathbf{v}_a)_{\mathcal{B}}{}^{j_a} (\alpha^1)_{\mathcal{B}_{l_1}} \cdots (\alpha^b)_{\mathcal{B}_{l_b}}. \end{aligned}$$

Excercise 1.9.3. Under the old basis,

$$\nabla f(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}.$$

Take the change of basis and

$$f_{new}(\mathbf{v}_{new}) = f_{new}(B\mathbf{v}_{old}) = f(\mathbf{v}_{old})$$

i.e. $f_{new} = f \circ B^{-1}$, where $B^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{bmatrix}$. Hence for $(a,b,c) = B(x,y,z)$,

$$f_{new}(a,b,c) = f(B^{-1}(a,b,c)) = f(a-b+c, b-c, c) = a^2 + 2b^2 + 3c^2 - 2ab + 2ac - 4bc,$$

and

$$\begin{aligned} \nabla f_{new}(a,b,c) &= \begin{bmatrix} 2a - 2b + 2c \\ -2a + 4b - 4c \\ 2a - 4b + 6c \end{bmatrix} \\ &= \begin{bmatrix} 2x \\ -2x + 2y \\ 2x - 2y + 2c \end{bmatrix} \\ &= (B^{-1})^{\mathrm{T}} \nabla f(x,y,z). \end{aligned}$$