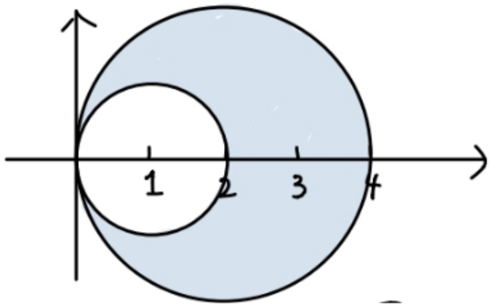
赵晨阳 软01 2020012363



12.(1) 令 $x=r\cos\theta, y=r\sin\theta$. 则积分区域 $2r\cos\theta\le r^2\le 4r\cos\theta$

即 $2\cos\theta\le r\le 4\cos\theta$. $E=\{(r,\theta)\mid -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, 2\cos\theta\le r\le 4\cos\theta\}$

$\iint_D (x^2+y^2)dxdy=\iint_E r^3 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\int_{2\cos\theta}^{4\cos\theta} r^3 dr$

$=\frac14 r^4\Big|_{2\cos\theta}^{4\cos\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta = 60\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta = 120\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta$

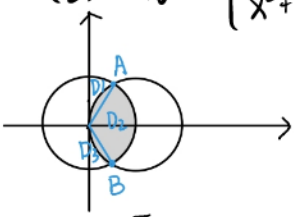
$=120\int_0^{\frac{\pi}{2}}\cos^4(\frac{\pi}{2}-\theta)d\theta = 120\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta = 120\cdot\frac{3!!}{4!!}\frac{\pi}{2}=\frac{45}{2}\pi$

$I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx$ ∴ $I_n=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\, d(\cos x) = -\sin^{n-1}x\cdot\cos x\Big|_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cdot\cos^2 x dx$

$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}xdx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x dx$. 即 $I_n=\frac{n-1}{n}I_{n-2}$.

$$I_n=\begin{cases}\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2} & (n为偶)\\ \frac{(2k)!!}{(2k+1)!!} & (n为奇数)\end{cases}$$

(2) 联立 $\begin{cases}x^2+y^2=1\\x^2+y^2=2x\end{cases}$ 有 $A(\frac12,\frac{\sqrt3}{2})$ $B(\frac12,-\frac{\sqrt3}{2})$ 将D沿左圆分割并换用极坐标



$E_1=\{(r,\theta)\mid\frac{\pi}{3}\le\theta\le\frac{\pi}{2}, 0\le r\le 2\cos\theta\}$ $E_2=\{(r,\theta)\mid -\frac{\pi}{3}\le\theta\le\frac{\pi}{3},0\le r\le1\}$

$E_3=\{(r,\theta)\mid-\frac{\pi}{2}\le\theta\le-\frac{\pi}{3},0\le r\le2\cos\theta\}$

$\iint_D(x^2+y^2)^{\frac32}dxdy=\iint_{E_1+E_2+E_3}r^4drd\theta$ 由于 r^4 为关于θ的偶函数，而 E_1 与 E_3 关于 $\theta=0$ 对称

∴ $\iint_{E_1}r^4drd\theta=\iint_{E_3}r^4drd\theta = 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^4dr=\frac{32}{5}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^5\theta d\theta$

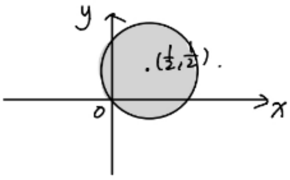
而 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^5\theta d\theta=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1-\sin^2\theta)^2 d\sin\theta=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1-2\sin^2\theta+\sin^4\theta)d\sin\theta=(\sin\theta-\frac23\sin^3\theta+\frac15\sin^5\theta)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$=\frac{8}{15}-\frac{49}{160}\sqrt3$ ∴ $\iint_{E_1}r^4drd\theta=\frac{32}{5}(\frac{8}{15}-\frac{49}{160}\sqrt3)=\frac{256}{75}-\frac{49}{25}\sqrt3$

$\iint_{E_2}r^4drd\theta=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}d\theta\int_0^1r^4dr=\frac15\cdot\frac23\pi=\frac{2}{15}\pi$

∴ $\iint_D(x^2+y^2)^{\frac32}dxdy=\frac{2}{15}\pi+\frac{512}{75}-\frac{98}{25}\sqrt3$

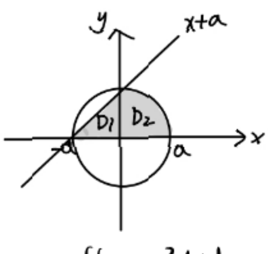
(3) $\iint_D x^2+y^2dxdy$ 令 $x=\frac12+r\cos\theta$ $y=\frac12+r\sin\theta$ $E=\{(r,\theta)\mid0\le\theta\le2\pi,0\le r\le\frac{\sqrt2}{2}\}$



$\iint_D(x+y)dxdy=\iint_E(1+r\cos\theta+r\sin\theta)rdrd\theta=\int_0^{\frac{\sqrt2}{2}}dr\int_0^{2\pi}(r+r^2\cos\theta+r^2\sin\theta)d\theta$

$=\int_0^{\frac{\sqrt2}{2}}(r\theta+r^2\sin\theta-r^2\cos\theta)\Big|_0^{2\pi}dr=\int_0^{\frac{\sqrt2}{2}}2\pi rdr=\pi r^2\Big|_0^{\frac{\sqrt2}{2}}=\frac12\pi$

(4) $D_1=\{(x,y)\mid-a\le x\le0,0\le y\le x+a\}$ $D_2=\{(x,y)\mid0\le x\le a,0\le y\le\sqrt{a^2-x^2}\}$



$\iint_{D_1}(y-x)^2dxdy=\int_{-a}^0dx\int_0^{x+a}(y-x)^2dy=\int_{-a}^0\frac13(y-x)^3\Big|_0^{x+a}dx$

$=\int_{-a}^0(\frac13a^3+\frac13x^3)dx=\frac13a^4+\frac1{12}x^4\Big|_{-a}^0=\frac14a^4$

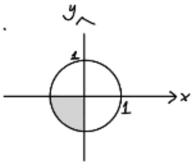
在 D_2 上极坐标换元. $E_2=\{(r,\theta)\mid\theta\in[0,\frac{\pi}{2}],0\le r\le a\}$

$\iint_{D_2}(y-x)^2dxdy=\iint_{E_2}r^3(\cos^2\theta+\sin^2\theta-2\cos\theta\cdot\sin\theta)d\theta dr=\int_0^a r^3dr\int_0^{\frac{\pi}{2}}1-2\sin\theta d\theta$

$=\frac14r^4\Big|_0^a\cdot(\frac{\pi}{2}+\frac12\cos2\theta\Big|_0^{\frac{\pi}{2}})=\frac14a^4(\frac{\pi}{2}-1)$

∴ $\iint_D(y-x)^2dxdy=\frac{a^4}{4}+\frac{a^4}{4}(\frac{\pi}{2}-1)=\frac{a^4}{4}$

(5) 极坐标换元 $E=\{(r,\theta)\mid \pi\le\theta\le\frac{3}{2}\pi,\ 0\le r\le 1\}$

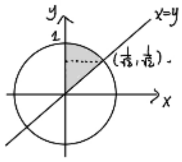


$\iint_D \arctan\frac{y}{x}dxdy=\iint_E \arctan(\tan\theta)\,r\,drd\theta$ 注意到 $\arctan(\tan\theta)$ 的值应将 θ 化归到其值域 故此题 $\arctan(\tan\theta)=\theta-\pi$

原式 $=\iint_E(\theta-\pi)r\,drd\theta=\int_0^1 dr\int_\pi^{\frac{3}{2}\pi} r(\theta-\pi)d\theta=\int_0^1\left(\frac{1}{2}r\theta^2-r\pi\theta\right)\Big|_\pi^{\frac{3}{2}\pi}dr=\int_0^1\left(\frac{5}{8}\pi^2 r-\frac{r}{2}\pi^2\right)dr$

$=\frac{\pi^2}{8}\int_0^1 r\,dr=\frac{\pi^2}{16}$

(6)



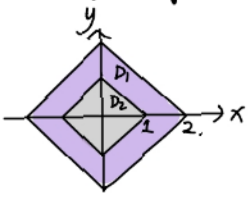
$\int_0^{\frac{\sqrt2}{2}}dy\int_0^y e^{-x^2-y^2}dx+\int_{\frac{\sqrt2}{2}}^1 dy\int_0^{\sqrt{1-y^2}}e^{-x^2-y^2}dx=\iint_D e^{-x^2-y^2}dxdy$

极坐标换元 $E=\{(r,\theta)\mid 0\le r\le1,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{2}\}$

$\iint_E e^{-x^2-y^2}dxdy=\iint_E e^{-r^2}r\,drd\theta=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^1 e^{-r^2}dr^2=\frac{1}{2}\cdot\frac{\pi}{4}\cdot\left(-e^{-r^2}\right)\Big|_0^1=\frac{\pi}{8}\left(-\frac{1}{e}+1\right)$

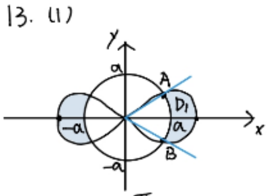
$=\frac{\pi}{8}\left(1-\frac{1}{e}\right)$

(7) 如下图分割 $f(x,y)=\begin{cases}1, & |x|+|y|\le1\\ 2, & |x|+|y|\le2\end{cases}$



$\iint_D f(x,y)dxdy=\sigma(D_1)+2\sigma(D_2)=(\sqrt2)^2+2\left((2\sqrt2)^2-(\sqrt2)^2\right)=14$

13. (1)



极坐标换元后：双纽线为 $r^2=2a^2\cos2\theta$

与 $r=a$ 联立有：$A(a,\frac{\pi}{6})$ $B(a,-\frac{\pi}{6})$

由 D_1 与左侧对称，故 $S=2\sigma(D_1)$

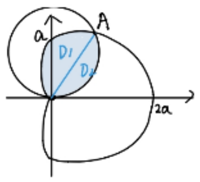
$D_1=\{(r,\theta)\mid a\le r\le a\sqrt{2\cos2\theta},\ -\frac{\pi}{6}\le\theta\le\frac{\pi}{6}\}$

$\sigma(D_1)=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta\int_a^{a\sqrt{2\cos2\theta}}r\,dr=\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(a^2\cdot2\cos2\theta-a^2)d\theta=\frac{1}{2}a^2\sin2\theta\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}-\frac{1}{2}\cdot a^2\cdot\frac{\pi}{3}$

$=\frac{\sqrt3}{2}a^2-\frac{\pi}{6}a^2=\left(\frac{\sqrt3}{2}-\frac{\pi}{6}\right)a^2$

$\therefore\ \sigma(D)=2\sigma(D_1)=\left(\sqrt3-\frac{\pi}{3}\right)a^2$

(2)



极坐标换元，圆化为 $r^2=\sqrt3 a r\sin\theta$ 与心脏线联立

有 $A(\frac{3}{2}a,\frac{\pi}{3})$

$D_1=\{(r,\theta)\mid \theta\in[\frac{\pi}{3},\pi],\ r\in[0,a(1+\cos\theta)]\}$

$D_2=\{(r,\theta)\mid \theta\in[0,\frac{\pi}{3}],\ r\in[0,\sqrt3 a\sin\theta]\}$

$\sigma(D_1)=\iint_{D_1}r\,drd\theta=\int_{\frac{\pi}{3}}^{\pi}d\theta\int_0^{a(1+\cos\theta)}r\,dr=\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}a^2(1+2\cos\theta+\cos^2\theta)d\theta$

$=\frac{1}{2}a^2\cdot\frac{2}{3}\pi+a^2\sin\theta\Big|_{\frac{\pi}{3}}^{\pi}+\frac{a^2}{4}\int_{\frac{\pi}{3}}^{\pi}(1+\cos2\theta)d\theta=\frac{a^2}{3}\pi-\frac{\sqrt3}{2}a^2+\frac{a^2}{6}\pi+\frac{a^2}{8}\sin2\theta\Big|_{\frac{\pi}{3}}^{\pi}$

$=\frac{a^2}{2}\pi-\frac{9}{16}\sqrt3 a^2$

$\sigma(D_2)=\int_0^{\frac{\pi}{3}}d\theta\int_0^{\sqrt3 a\sin\theta}r\,dr=\frac{3a^2}{2}\int_0^{\frac{\pi}{3}}\sin^2\theta\,d\theta=\frac{3a^2}{4}\int_0^{\frac{\pi}{3}}(1-\cos2\theta)d\theta$

$=\frac{3}{4}a^2\cdot\frac{\pi}{3}-\frac{3}{8}a^2\int_0^{\frac{\pi}{3}}\cos2\theta\,d2\theta=\frac{a^2}{4}\pi-\frac{3\sqrt3}{16}a^2$

$\sigma(D)=\sigma(D_1)+\sigma(D_2)=\frac{a^2}{2}\pi-\frac{9}{16}\sqrt3a^2+\frac{a^2}{4}\pi-\frac{3\sqrt3}{16}a^2=\frac{3}{4}a^2\pi-\frac{3}{4}\sqrt3a^2$

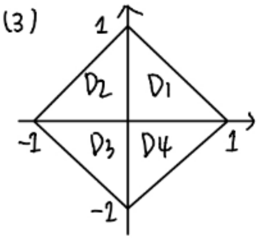
14. 令$\begin{cases}u=y/x\\v=xy\end{cases}$ 则 $D=\{(u,v)\mid u\in[1,3],v\in[2,4]\}$ $\frac{\nabla(u,v)}{\nabla(x,y)}=\begin{vmatrix}-\frac{y}{x^2} & \frac{1}{x}\\ y & x\end{vmatrix}=-\frac{2y}{x}=-2u$

$\iint\limits_D x^2y^2dxdy=\frac{1}{2}\int_2^4dv\int_1^3\frac{v^2}{u}du=\frac{1}{2}\int_2^4\ln3\cdot v^2dv=\frac{1}{2}\ln3\cdot\frac{1}{3}v^3\Big|_2^4=\frac{28}{3}\ln3$

(2) 令$\begin{cases}u=x^2-y^2\\v=xy\end{cases}$, $D=\{(u,v)\mid u\in[1,2],v\in[1,2]\}$ $\frac{\nabla(u,v)}{\nabla(x,y)}=2x^2+2y^2$

$\iint\limits_D(x^2+y^2)dxdy=\frac{1}{2}\iint\limits_D dudv=\frac{1}{2}\cdot1\cdot1=\frac{1}{2}$

(3)

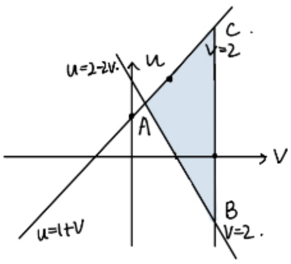


$\begin{cases}u=x+y\\v=x-y\end{cases}$ $D_1=\{(u,v)\mid u\in[-1,1],v\in[-1,1]\}$ $\frac{\nabla(u,v)}{\nabla(x,y)}=-2$

$\iint\limits_D(x^2+y^2)dxdy=\frac{1}{4}\iint\limits_D(v^2+u^2)dvdu=\frac{1}{4}\int_{-1}^1du\int_{-1}^1u^2+v^2dv$

$=\frac{1}{4}\int_{-1}^1(\frac{v^3}{3}+u^2v)\Big|_{-1}^1du=\frac{1}{4}\int_{-1}^1(2u^2+\frac{2}{3})du=\frac{1}{3}+\frac{1}{6}u^3\Big|_{-1}^1=\frac{2}{3}$

(4)



令$\begin{cases}u=y^2-x\\v=y\end{cases}$ 则边界曲线即为$\begin{cases}v=2\\u-v=1\\u+2v=2\end{cases}$

解得$A(\frac{1}{3},\frac{4}{3})$ $B(2,-2)$ $C(2,3)$

$D=\{(u,v)\mid v\in[\frac{1}{3},2],\ u\in[2-2v,v+1]\}$ $\frac{\nabla(u,v)}{\nabla(x,y)}=-1$

$\therefore\iint\limits_D(x-y^2)dxdy=-\iint\limits_D udvdu=-\int_{\frac{1}{3}}^2dv\int_{2-2v}^{v+1}udu=-\frac{1}{2}\int_{\frac{1}{3}}^2(v+1)^2-(2v-2)^2dv$

$=-\frac{1}{2}\int_{\frac{1}{3}}^2 3v^2-10v+3\,dv=\frac{1}{2}(v^3-5v^2+3v)\Big|_{\frac{1}{3}}^2=-\frac{175}{54}$

17. 积分区域 D $x^2+y^2\le R^2$ 关于 $y=x$ 对称

由轮换对称性：$I=\iint\limits_D\frac{af(x)+bf(y)}{f(x)+f(y)}dxdy=\iint\limits_D\frac{af(y)+bf(x)}{f(x)+f(y)}dxdy=\frac{1}{2}\iint\limits_D\frac{(a+b)(f(x)+f(y))}{f(x)+f(y)}dxdy$

而$f(x)+f(y)>0$，故

$I=\iint\limits_D\frac{a+b}{2}dxdy=\frac{a+b}{2}\sigma(D)=\frac{a+b}{2}\pi R^2$

18. 令$g(t,x)=\int_{t^2}^{x^2}f(t,s)ds$ 则 $F(x)=\int_0^x g(t,x)dt$. 由含参积分求导法则有：

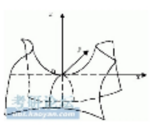
$F'(x)=\int_0^x g'_x(t,x)dt+g(x,x)=\int_0^x g'_x(t,x)dx$ 同理：$g'_x(t,x)=\int_{t^2}^{x^2}f'_x(t,s)ds+2x\cdot f(t,x^2)-0$

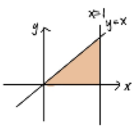
$=2x\cdot f(t,x^2)$

综上：$F'(x)=\int_0^x 2x\cdot f(t,x^2)dt$

习题3.4

5.(1)





Ω在XOY面内投影$D=\{(x,y)\mid0\le x\le1,0\le y\le x\}$

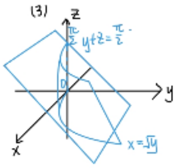
$\Omega=\{(x,y,z)\mid(x,y)\in D,\ 0\le z\le xy\}$

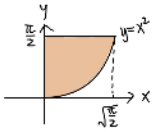
$\iiint\limits_\Omega xy^2z^3dxdydz=\iint\limits_D dxdy\int_0^{xy}xy^2z^3dz$

$=\frac{1}{4}\iint\limits_D x^5y^6dxdy=\frac{1}{4}\int_0^1dx\int_0^x x^5y^6dy=\frac{1}{4}\cdot\frac{1}{7}\int_0^1x^{12}dx$

$=\frac{1}{28}\cdot\frac{1}{13}\cdot x^{13}\Big|_0^1=\frac{1}{364}$

(3)





在XOY面上投影为 $D=\{(x,y)\mid y\in[0,\frac{\pi}{2}],x\in[0,\sqrt{y}]\}$

$\Omega=\{(x,y,z)\mid z\in[0,\frac{\pi}{2}-y],(x,y)\in D\}$

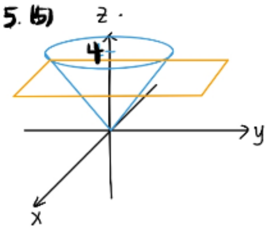
$\iiint_\Omega x\cos(y+z)\,dxdydz=\iint_D dxdy\int_0^{\frac{\pi}{2}-y}\cos(y+z)x\,dz=\iint_D x\sin(y+z)\Big|_0^{\frac{\pi}{2}-y}dxdy=\iint_D x-x\sin(y)\,dxdy$

$=\int_0^{\frac{\pi}{2}}dy\int_0^{\sqrt{y}}x-x\sin y\,dx=\int_0^{\frac{\pi}{2}}(\frac{x^2}{2}-\frac{x^2}{2}\sin y)\Big|_0^{\sqrt{y}}dy=\int_0^{\frac{\pi}{2}}(\frac{y}{2}-\frac{y}{2}\sin y)dy=\frac{1}{2}\int_0^{\frac{\pi}{2}}y\,dy-\frac{1}{2}\int_0^{\frac{\pi}{2}}y\sin y\,dy$

$\int_0^{\frac{\pi}{2}}y\sin y\,dy=-y\cos y\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos y\,dy=1$

∴原式 $=\frac{1}{2}\cdot\frac{1}{2}\cdot y^2\Big|_0^{\frac{\pi}{2}}-\frac{1}{2}=\frac{\pi^2}{16}-\frac{1}{2}$

5.(5)



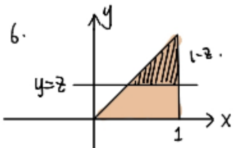
$D_z=\{(x,y)\mid x^2+y^2\le z^2\}$ $\Omega=\{(x,y,z)\mid 0\le z\le 4,(x,y)\in D_z\}$

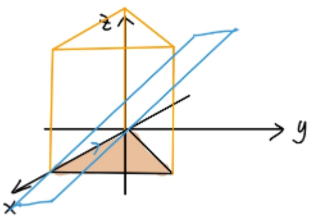
$\iiint_\Omega\frac{\sin z}{z}dxdydz=\int_0^4dz\iint_{D_z}\frac{\sin z}{z}dxdy=\int_0^4\frac{\sin z}{z}\sigma(D_z)dz=\int_0^4\pi z^2\cdot\frac{\sin z}{z}dz$

$=\pi\int_0^4\sin z\cdot z\,dz=-\pi(z\cdot\cos z\Big|_0^4-\int_0^4\cos z\,dz)=-\pi(4\cos4-\sin4)$

$=\pi\sin4-4\pi\cos4$

6.

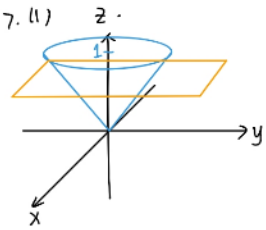




$D_z=\{(x,y)\mid y\in[z,1],x\in[y,1]\}$ $\Omega=\{(x,y,z)\mid z\in[0,1],(x,y)\in D_z\}$

$I=\iiint_\Omega\frac{\cos z}{(1-z)^2}dz=\int_0^1\frac{\cos z}{(1-z)^2}dz\iint_{D_z}dxdy=\int_0^1\frac{\cos z}{(1-z)^2}\cdot\frac{1}{2}(1-z)^2dz=\frac{1}{2}\int_0^1\cos z\,dz=\frac{1}{2}\sin1$

7.(1)



令 $\begin{cases}x=r\cos\theta\\y=r\sin\theta\\z=z\end{cases}$ $D_z=\{(r,\theta)\mid 0\le\theta\le2\pi,0\le r\le z\}$ $\Omega=\{(r,\theta,z)\mid z\in[0,1],(r,\theta)\in D_z\}$

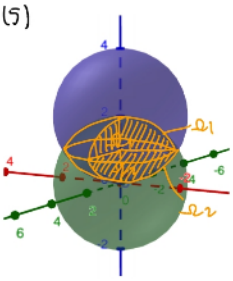
$\iiint_\Omega\sqrt{x^2+y^2}\,dxdydz=\iiint_\Omega r^2drd\theta dz=\int_0^1dz\int_0^{2\pi}d\theta\int_0^z r^2dr=2\pi\cdot\int_0^1z^3dz=\frac{\pi}{6}$

(3) $D=\{(x,y)\mid x\in[0,1],y\in[0,1-x]\}$ $\Omega=\{(x,y,z)\mid z\in[0,x^2+y^2],(x,y)\in D\}$

$\iiint_\Omega\frac{z}{x^2+y^2}dxdydz=\iint_D\frac{dxdy}{x^2+y^2}\int_0^{x^2+y^2}z\,dz=\frac{1}{2}\int_0^1dx\int_0^{1-x}\frac{1}{x^2+y^2}(x^2+y^2)^2dy$

$=\frac{1}{2}\int_0^1dx\int_0^{1-x}x^2+y^2dy=\frac{1}{2}\int_0^1x^2(1-x)+\frac{1}{3}(1-x)^3dx=\int_0^1x^2-\frac{2}{3}x^3-\frac{x}{2}+\frac{1}{6}dx=(\frac{x^3}{3}-\frac{2}{3}\cdot\frac{x^4}{4}-\frac{x^2}{4}+\frac{x}{6})\Big|_0^1=\frac{1}{12}$

(5)



令 $\begin{cases}x=r\cos\theta\\y=r\sin\theta\\z=z\end{cases}$ $D_{z_1}=\{(r,\theta)\mid 0\le\theta\le\frac{\pi}{2},0\le r\le\sqrt{4-z^2}\}$ $D_{z_2}=\{(r,\theta)\mid 0\le\theta\le\frac{\pi}{2},r\in[0,\sqrt{4z-z^2}]\}$

$\Omega_1=\{(r,\theta,z)\mid 1\le z\le2,(r,\theta)\in D_{z_1}\}$ $\Omega_2=\{(r,\theta,z)\mid 0\le z\le1,(r,\theta)\in D_{z_2}\}$

$\iiint_\Omega xyz\,dxdydz=\int_0^1z\,dz\iint_{D_{z_2}}r^3\sin\theta\cos\theta\,drd\theta+\int_1^2z\,dz\iint_{D_{z_1}}r^3\sin\theta\cos\theta\,drd\theta$

$\int_0^1z\,dz\iint_{D_{z_2}}r^3\sin\theta\cos\theta\,drd\theta=\frac{1}{4}\int_0^1z\,dz\int_0^{\sqrt{4z-z^2}}dr\int_0^{\frac{\pi}{2}}r^3\sin2\theta\,d2\theta$

$=\frac{1}{4}\int_0^1z\,dz\int_0^{\sqrt{4z-z^2}}2r^3dr=\frac{1}{8}\int_0^1z(4z-z^2)^2dz=\frac{1}{8}\int_0^1 16z^3-8z^4+z^5dz=\frac{1}{8}\cdot16\cdot\frac{1}{4}-\frac{1}{8}\cdot8\cdot\frac{1}{5}+\frac{1}{8}\cdot\frac{1}{6}$

$=\frac{1}{2}-\frac{1}{5}+\frac{1}{48}$

$$\int_1^2 zdz\iint r^3\sin\theta\cos\theta\, drd\theta=\int_1^2 zdz\int_0^{\sqrt{4-z^2}}dr\int_0^{\frac{\pi}{2}}r^3\sin\theta\cos\theta\, d\theta=\frac{1}{2}\int_1^2 zdz\int_0^{\sqrt{4-z^2}}r^3dr$$

D_{z2} $$=\frac{1}{8}\int_1^2 z\cdot(4-z^2)^2dz=\frac{1}{8}\int_0^1(16-8z^2+z^4)zdz=\frac{1}{8}\left(8z^2-2z^4+\frac{z^6}{6}\right)\Big|_1^2$$

$$=\frac{4}{3}-\frac{3}{4}-\frac{1}{48}$$

$$\iiint_\Omega f(x,y,z)dxdydz=\iiint_{\Omega_1}f(x,y,z)dxdydz+\iiint_{\Omega_2}f(x,y,z)dxdydz=\frac{1}{2}-\frac{1}{5}+\frac{1}{48}+\frac{4}{3}-\frac{3}{4}-\frac{1}{48}=\frac{53}{60}$$

8.(1) 令 $\begin{cases}x=ar\sin\varphi\cos\theta\\ y=br\sin\varphi\sin\theta\\ z=cr\cos\varphi\end{cases}$ $\Omega=\{(r,\varphi,\theta)\mid 0\le r\le 1,\ \varphi\in[0,\pi],\ \theta\in[0,2\pi]\}$

$$\iiint_\Omega f(x,y,z)dxdzdy=\iiint_\Omega\sqrt{1-r^2}\,abcr^2\sin\varphi\, drd\varphi d\theta=2\pi abc\int_0^1 r^2\sqrt{1-r^2}dr\int_0^\pi\sin\varphi d\varphi=4\pi abc\int_0^1 r^2\sqrt{1-r^2}dr$$

$$\int_0^1 r^2\sqrt{1-r^2}dr\xrightarrow{令 r=\sin t.}\int_0^{\frac{\pi}{2}}\sin^2t\cos^2t\,dt=\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt=\frac{1}{8}\int_0^{\frac{\pi}{2}}1-\cos4t\,dt$$

$$=\frac{1}{8}\cdot\frac{\pi}{2}-\frac{1}{8}\cdot\int_0^{\frac{\pi}{2}}\cos4t\,dt=\frac{\pi}{16}$$

$$\therefore\iiint_\Omega f(x,y,z)dxdydz=4\pi abc\cdot\frac{\pi}{16}=\frac{\pi^2}{4}abc$$