Policy

This is a takehome midterm. You take it home, do it for a week, and then hand it back. It is due Apr 26th. (So you have a total of 2 weeks.) I encourage collaborations on this midterm. However, you must obey the following rule:

1. You MUST each hand in your own work individially in your own words.

2. You MUST understand everything you wrote. (Say you copied your friend's WRONG answer without thinking, and that will most likely be in violation of this rule.)

3. You need to write down the names of your collaborator, if any.

4. Failure to comply rule 2 and rule 3 will be treated as plagiarism.

5. Collaboration with people not in this class (such as a math grad student) is not forbidden but not recommended. If you choose to, then write down their names as well.

We have a total of five problems, and a total of 55 points. Full credit is 50 points, and if you get more than 50 points, then your score is simply 50 points. (So you have some room for mistakes.)

The Midterm

Problem 1. *Prove or find counter example to the statements below.*

1. *(2pt) Suppose for the sequence $\boldsymbol{v}, A\boldsymbol{v}, A^2\boldsymbol{v}, \ldots$, the differences between consecutive terms are always constant. Also suppose that $\boldsymbol{v} \neq \boldsymbol{0}$. Then $\boldsymbol{v}$ is a generalized eignevector for A.*

2. *(2pt) Suppose for the sequence $\boldsymbol{v}, A\boldsymbol{v}, A^2\boldsymbol{v}, \ldots$, each term is the sum of previous two terms. (Similar to the famous Fibonacci sequence.) Also suppose that $\boldsymbol{v} \neq \boldsymbol{0}$. Then $\boldsymbol{v}$ is a generalized eignevector for A.*

3. *(2pt) Let V be the space of all polynomials (so it is an infinite dimensional space). Let $M : V \to V$ be the map that sends $p(x)$ to $xp(x)$. Then M has NO generalized eigenvectors at all. (We use the convension that $\boldsymbol{0}$ is NOT a generalized eigenvector.)*

4. *(2pt) For a $n \times n$ matrix A, suppose $A^5 = 0$, then $\dim \operatorname{Ker}(A)$ is at least $\frac{n}{5}$.*

5. *(2pt) For a $n \times n$ matrix A, suppose $AA^{\mathrm{T}}AAA^{\mathrm{T}} = 0$, then $\dim \operatorname{Ker}(A)$ is at least $\frac{n}{2}$.*

Problem 2. *Consider the Kronecker product $A \otimes B$ on 2×2 matrices A, B. We define the Kronecker sum to be $A \oplus B = A \otimes I + I \otimes B$.*

1. *(2pt) Find a matrix P such that $P(A \otimes B)P^{-1} = B \otimes A$.*

2. *(2pt) Show that $\mathrm{e}^{A \otimes I} = \mathrm{e}^A \otimes I$ and $\mathrm{e}^{I \otimes B} = I \otimes \mathrm{e}^B$.*

3. *(2pt) Show that $\mathrm{e}^A \otimes \mathrm{e}^B = \mathrm{e}^{A \oplus B}$. (Hint: $\mathrm{e}^{X+Y} = \mathrm{e}^X \mathrm{e}^Y$ when $XY = YX$.)*

4. *(2pt) Find $\operatorname{trace}(A \oplus B)$ in terms of $\operatorname{trace}(A), \operatorname{trace}(B)$.*

5. *(2pt) If the 2×2 matrix A is invertible, show that there is a matrix X such that $\mathrm{e}^X = A$. (You may use without proof that any non-zero complex number z is e^μ for some complex number μ.) (Hint: Verify for each Jordan block.)*

6. *(2pt) Find $\det(A \otimes B)$ in terms of $\det(A), \det(B)$. (Hint: Combine last two subproblems.)*

Problem 3. *Construct matrices that are examples of the following, and then show that they indeed satisfy the desired property.*

1. *(3pt) A real matrix which has NO 1×1 Jordan block, and no real eigenvalues.*

2. *(3pt) Two matrices A, B such that AB and BA are NOT similar. (Different Jordan form.)*

3. *(3pt) A non-real matrix with real eigenvalues.*

4. *(3pt) A 4×4 matrix A such that $\mathrm{e}^A = A + I$. For this sub-problem, find all possible Jordan canonical forms that satisfy this.*

Problem 4 (Campanion matrices for polynomials)**.** *Let V be the space of polynomials of degree at most 3. For each $p(x) \in V$, let Mp be the remainder if we divide $xp(x)$ by $q(x) = x^4 + ax^3 + bx^2 + cx + d$ for constants a, b, c, d.*

For example, if $a = b = c = d = 1$ and $p(x) = x^3 + 2x + 3$, then $xp(x) = x^4 + 2x^2 + 3x =$ (multiple of $q(x)$) $- x^3 + x^2 + 2x - 1$, hence $(Mp)(x) = -x^3 + x^2 + 2x - 1$.

1. *(2pt) Write out the matrix A for the linear map $M : V \to V$ under the basis $1, x, x^2, x^3$. (The answer will be in terms of a, b, c, d.)*

2. *(2pt) Show that $q(x)$ is the minimal polynomial of A and the characteristic polynomial of A. (Hint: if you are tired of brute force computation, there is a fast way. Think about the meaning of $q(A)$ in terms of polynomial manipulations.)*

3. *(2pt) If $AB = BA$ for some matrix B, prove that $B = f(A)$ for some function f.*

4. *(2pt) Suppose $q(x) = x(x-1)(x-2)(x-3)$, find the Jordan canonical form of A.*

5. *(2pt) Suppose $q(x) = x^2(x-1)(x-2)$, find the Jordan canonical form of A.*

Problem 5 (Straight curve in the same direction)**.** *Suppose $f : \mathbb{R} \to SO_n$ is an <mark>infinitely differentiable</mark> function from <mark>$\mathbb{R}$ to the set of $n \times n$ real orthogonal matrices with determinant 1</mark>. Here differentiable means $\lim_{dt \to 0} \frac{f(t+dt) - f(t)}{dt}$ <mark>exists at all t</mark>. (The <mark>numerator</mark> is a matrix, while the <mark>denominator</mark> is a number. So the result of this limit is also a matrix.) For simplicity, let us <mark>assume that $f(0) = I$ the identity matrix.</mark>*

1. *(2pt) Show that $A + A^{\mathrm{T}} - 2I = -(A - I)(A^{\mathrm{T}} - I)$ for any orthogonal matrix A.*

2. *(2pt) Show that $\lim_{t \to 0} \frac{-(f(t) - I)(f(t)^{\mathrm{T}} - I)}{t} = f'(0) + f'(0)^{\mathrm{T}}$.*

3. *(3pt) Show that $f'(0)$ must be skew symmetric. (Hint: What if we replace the t in the last subproblem by t^2?)*

4. *(2pt) Show that $f'(t)f(t)^{-1}$ must be skew symmetric. (Hint: Note that $f(t)^{-1} = f(t)^{\mathrm{T}}$.)*

5. *(2pt) Suppose we have $f'(t)f(t)^{-1} = A$ for a constant skew-symmetric matrix A. (So it is a straight curve in the same A-direction.) Show that $f(t) = e^{At}B$ for some $B \in SO_n$. (Hint: consider the differential equation for each column of $f(t)$.)*