第七次习题课　二重积分

例.1 证明 $\iint\limits_{[0,1]^2}(xy)^{xy}dxdy=\int_0^1 t^t dt$（第三章的总复习题题 9， page 171）

证明：先将重积分化为累次积分。再对累次积分作适当变换。

$$\iint\limits_{[0,1]^2}(xy)^{xy}dxdy=\int_0^1 dx\boxed{\int_0^1(xy)^{xy}dy}=\int_0^1\frac{dx}{x}\int_0^1(xy)^{xy}d(xy)=\int_0^1\frac{dx}{x}\boxed{\int_0^x t^t dt}。$$

记 $f(x)=\int_0^x t^t dt$。则有 $\iint\limits_{[0,1]^2}(xy)^{xy}dxdy=\int_0^1 f(x)d(\ln x)$。作分部积分得

$$\int_0^1 f(x)d(\ln x)=f(x)\ln x\Big|_{x=0}^{x=1}-\int_0^1 x^x\ln xdx=-\int_0^1 x^x\ln xdx。$$

$\int_0^1 e^{x\ln x}d(x\ln x)=e^{x\ln x}\Big|_0^1=0$

注意到关系式 $[x\ln x]'=\ln x+1$，即 $\ln x=[x\ln x]'-1$，我们得到

①　②

$\int_0^1 f(x)d(\ln x)=\int_0^1 x^x dx-\int_0^1 x^x[x\ln x]'dx$。容易看出后一个积分为零。这是因为

$\int_0^1 x^x[x\ln x]'dx=\int_0^1 e^{x\ln x}d[x\ln x]=e^{x\ln x}\Big|_{x=0}^{x=1}=0$。故

$\iint\limits_{[0,1]^2}(xy)^{xy}dxdy=\int_0^1 t^t dt$。证毕。

例.2 利用二重积分理论，证明以下积分不等式。设 $f(x)$， $g(x)$ 于 $[a,b]$ 上连续，则

（1） $\left(\int_a^b f(x)dx\right)^2\le(b-a)\int_a^b f^2(x)dx$。

（2） $\left(\int_a^b f(x)g(x)dx\right)^2\le\int_a^b f^2(x)dx\int_a^b g^2(x)dx$.

$\left(\int_0^\infty e^{-x^2}dx\right)^2=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$

（3） $\iint\limits_{[a,b]^2}\frac{f(x)}{f(y)}dxdy\ge(b-a)^2$，这里补充假设 $f(x)>0$， $\forall x\in[a,b]$。

注：证明上述不等式的方法有许多。以下的证明方法表明，二重积分理论可以用于证明一些重要的不等式。

证明：

$xy\le|xy|\le\frac{x^2+y^2}{2}$

（1） $\left(\int_a^b f(x)dx\right)^2=\int_a^b f(x)dx\int_a^b f(y)dy=\iint\limits_{[a,b]^2}f(x)f(y)dxdy\le$

$$\le\frac{1}{2}\iint\limits_{[a,b]^2}[f^2(x)+f^2(y)]dxdy=\frac{1}{2}\iint\limits_{[a,b]^2}f^2(x)dxdy+\frac{1}{2}\iint\limits_{[a,b]^2}f^2(y)dxdy=$$

$(b-a)$

$$=\frac{1}{2}(b-a)\int_a^b f^2(x)dx+\frac{1}{2}\int_a^b f^2(y)dy=(b-a)\int_a^b f^2(x)dx.$$

（2）由不等式$[f(x)g(y)-f(y)g(x)]^2\geq 0$得

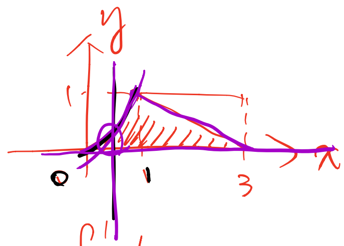
$$0\leq \iint\limits_{[a,b]^2}[f(x)g(y)-f(y)g(x)]^2dxdy=$$

$$=\iint\limits_{[a,b]^2}[f^2(x)g^2(y)+f^2(y)g^2(x)-2f(x)g(x)f(y)g(y)]dxdy=$$

$$=2\int_a^b f^2(x)dx\int_a^b g^2(x)dx-2\left(\int_a^b f(x)g(x)dx\right)^2$$ 。由此立刻得到不等式(ii).

（3）$$\iint\limits_{[a,b]^2}\frac{f(x)}{f(y)}dxdy=\int_a^b f(x)dx\int_a^b\frac{1}{f(x)}dx\geq\left(\int_a^b\sqrt{f(x)}\frac{1}{\sqrt{f(x)}}dx\right)^2=(b-a)^2,$$

上式的第二个不等式成立的根据是不等式(ii). 证毕。



例.3 改变累次积分顺序$\int_0^1dx\int_0^{x^2}f(x,y)dy+\int_1^3dx\int_0^{\frac{1}{2}(3-x)}f(x,y)dy$；

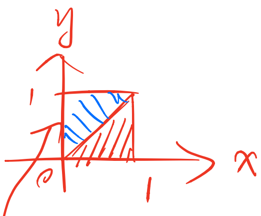
$\int_0^1 dy$

解：$$\int_0^1dx\int_0^{x^2}f(x,y)dy+\int_1^3dx\int_0^{\frac{1}{2}(3-x)}f(x,y)dy=\int_0^1dy\int_{\sqrt{y}}^{3-2y}f(x,y)dx$$

$$\begin{cases}y=x^2\\ y=\frac{1}{2}(3-x)\end{cases}\Rightarrow\begin{cases}x=\sqrt{y}\\ x=3-2y\end{cases}$$

例.4 设$f(x,y)$为连续函数,且$f(x,y)=f(y,x)$.证明:

$$\int_0^1dx\int_0^x f(x,y)dy=\int_0^1dx\int_0^x f(1-x,1-y)dy.$$



证：令$x=1-u,y=1-v$,则

$$\begin{cases}u=1-x\\ v=1-y\end{cases}$$

$$0\leq v\leq 1,0\leq u\leq v,|J|=1.$$

于是

$f(v,u)$

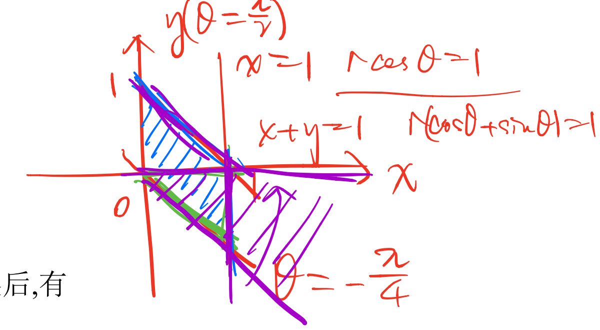
$$\int_0^1dx\int_0^x f(1-x,1-y)dy=\int_0^1dv\int_0^v f(u,v)du$$

$$=\int_0^1dv\int_0^v f(v,u)du=\int_0^1dx\int_0^x f(x,y)dy.$$

$$\int_0^1 du\int_u^1 f(u,v)dv$$

例.5 对积分$\iint\limits_D f(x,y)dxdy$， $D=\{(x,y)|0\leq x\leq 1,\ 0\leq x+y\leq 1\}$进行极坐标变换并写出

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



变换后不同顺序的累次积分

解：由 $D=\{(x,y)|0\le x\le 1,\ 0\le x+y\le 1\}$，用极坐标变换后,有

$$\iint_D f(x,y)dxdy=\int_{-\frac{\pi}{4}}^{0}d\theta\int_0^{\sec\theta}rf(r\cos\theta,r\sin\theta)dr+\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}}rf(r\cos\theta,r\sin\theta)dr$$

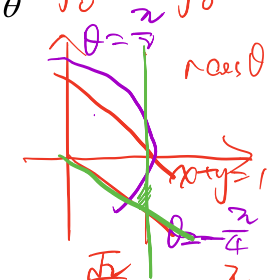
$$\int_{-\frac{\pi}{4}}^{0}d\theta\int_0^{\frac{1}{\cos\theta}}rf(r,\theta)dr$$

$$+\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{1}{\cos\theta+\sin\theta}}r\cdots$$

$$=\int_0^{\frac{\sqrt2}{2}}rdr\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}f(r\cos\theta,r\sin\theta)d\theta+\int_{\frac{\sqrt2}{2}}^{1}rdr\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}-\arccos\frac{1}{\sqrt2 r}}f(r\cos\theta,r\sin\theta)d\theta$$

$$+\int_{\frac{\sqrt2}{2}}^{1}rdr\int_{\frac{\pi}{4}+\arccos\frac{1}{\sqrt2 r}}^{\frac{\pi}{4}}f(r\cos\theta,r\sin\theta)d\theta+\int_1^{\sqrt2}rdr\int_{-\frac{\pi}{4}}^{-\arccos\frac{1}{r}}f(r\cos\theta,r\sin\theta)d\theta$$

$\frac{\pi}{2}$



例.6 计算二重积分：$\iint_D |xy|dxdy$,其中 D 为圆域: $x^2+y^2\le a^2$ 。

解：由对称性有

Jacobi 行列式

$$\iint_D|xy|dxdy=4\int_0^{\frac{\pi}{2}}d\theta\int_0^a r\sin\theta\cdot r\cos\theta\cdot rdr$$

$$=4\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin 2\theta d\theta\cdot\int_0^a r^3dr=2\cdot\frac{-\cos 2\theta}{2}\Big|_0^{\frac{\pi}{2}}\cdot\frac{r^4}{4}\Big|_0^a=\frac{a^4}{2}.$$

$$\int_0^{\frac{\sqrt2}{2}}dr\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

例.7 在下列积分中引入新变量 u,v 后，试将它化为累次积分：

$r(\cos\theta+\sin\theta)=1$

$$\iint_D f(x,y)dxdy,\text{其中}D=\{(x,y)\,|\,\sqrt{x}+\sqrt{y}\le\sqrt{a},x\ge 0,y\ge 0\},$$

若 $x=u\cos^4 v,\ y=u\sin^4 v.$　　$0\le v\le\frac{\pi}{2}$

$$+\int_{\frac{\sqrt2}{2}}^{1}dr\int_{-\frac{\pi}{4}}^{\arcsin\frac{1}{\sqrt2 r}-\frac{\pi}{4}}rf(r,\theta)d\theta$$

解：由 $x=u\cos^4 v,y=\sin^4 v$，得 $D'=\{(u,v)\,|\,0\le u\le a,0\le v\le\frac{\pi}{2}\}$,

$|J|=4u\sin^3 v\cos^3 v.$ 于是

$$+\int_{\frac{\sqrt2}{2}}^{1}dr\int_{\arcsin\frac{1}{\sqrt2 r}+\frac{\pi}{4}}^{\frac{\pi}{2}}rf(r,\theta)d\theta$$

$$\iint_D f(x,y)dxdy$$

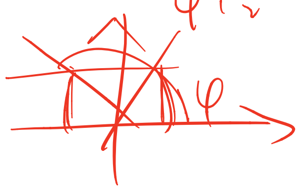
$$=\iint_{D'}f(u\sin^4 v,u\cos^3 v)4u\sin^3 v\cos^3 v\,dudv$$

$$=4\int_0^{\frac{\pi}{2}}dv\int_0^a u\sin^3 v\cos^3 v f(u\sin^3 v,u\cos^3 v)du$$

$$=4\int_0^a du\int_0^{\frac{\pi}{2}}u\sin^3 v\cos^3 v f(u\sin^3 v,u\cos^3 v)dv$$

$\sin(\theta+\frac{\pi}{4})=\frac{1}{\sqrt2 r}$

$\varphi+\frac{\pi}{2}$



例.8 试作适当变换,计算下列积分:

(1) $\iint_D (x+y)\sin(x-y)dxdy,D=\{(x,y)\,|\,0\le x+y\le\pi,0\le x-y\le\pi\}$;

$x+y=u$, $x-y=v$

$\theta=\arcsin\frac{1}{\sqrt2 r}-\frac{\pi}{4}$

$0\le u\le\pi$　$0\le v\le\pi$

$\theta=\arcsin\frac{1}{\sqrt2 r}+\frac{\pi}{4}$

$$\begin{cases} x=\frac{1}{2}(u+v) \\ y=\frac{1}{2}(u-v) \end{cases}\Rightarrow |J|$$

① 变量代换
② 代换后变量积分区间
③ Jacobi行列式

$\begin{cases} x = v-u \\ y = u \end{cases} \Rightarrow |J|$

(2) $\iint\limits_D e^{\frac{y}{x+y}}dxdy, D=\{(x,y)\mid x+y\le 1, x\ge 0, y\ge 0\}$.

解 (1)令 $x=\frac{1}{2}(u+v), y=\frac{1}{2}(u-v)$, 则 $D^{'}=\{(u,v)\mid 0\le u\le \pi, 0\le v\le \pi\}$,

$|J(u,v)|=\frac{1}{2}$.

于是 $\iint\limits_D (x+y)\sin(x-y)dxdy=\iint\limits_{D^{'}} u\sin v\cdot\frac{1}{2}dudv=\frac{1}{2}\int_0^{\pi}udu\int_0^{\pi}\sin vdv=\frac{1}{2}\pi^2$

(2) 令 $x=v-u, y=u$， 则 $D^{'}=\{(u,v)\mid \boxed{0\le u\le v, 0\le v\le 1}\}$, $|J(u,v)|=1$.

于是 $\iint\limits_D e^{\frac{y}{x+y}}dxdy=\iint\limits_{D^{'}} e^{\frac{u}{v}}dudv=\int_0^1 dv\int_0^v e^{\frac{u}{v}}du=\frac{1}{2}(e-1)$.

例.9 求由曲线所围的平面图形面积: $(\frac{x^2}{a^2}+\frac{y^2}{b^2})^2=x^2+y^2$。

$r^2=a^2\cos^2\theta+b^2\sin^2\theta$

解：令 $x=ar\cos\theta, y=br\sin\theta$,则 $|J|=abr$

$$D^{'}=\{(r,\theta): 0\le\theta\le 2\pi, 0\le r\le\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\}.$$

于是所求面积

Jacobi行列式

$$\Delta D=\iint\limits_D dxdy=\iint\limits_{D^{'}}\boxed{abr}drd\theta$$

$$=ab\int_0^{2\pi}d\theta\int_0^{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}rdr$$

$$=\frac{1}{2}ab\pi(a^2+b^2).$$

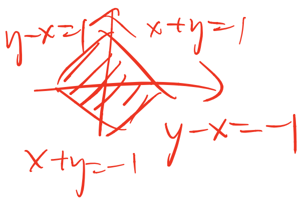
例.10 试作适当变换,把 $\iint\limits_D f(x+y)dxdy$,其中 $D=\{(x,y)\mid |x|+|y|\le 1\}$ 化为单重积分。

解：令 $x=\frac{1}{2}(u+v), y=\frac{1}{2}(u-v)$, 则

$u=x+y$
$v=x-y$ $\Rightarrow \begin{cases} x= \\ y= \end{cases} \Rightarrow$ Jacobi

$-1\le u\le 1, -1\le v\le 1; |J|=\frac{1}{2}$.

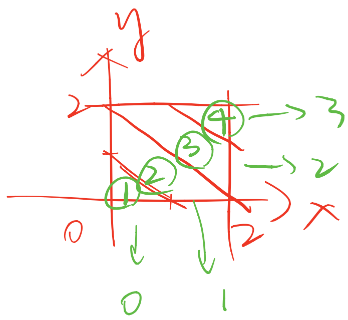
于是



$$\iint\limits_{|x|+|y|\le 1} f(x+y)dxdy=\frac{1}{2}\int_{-1}^1 f(u)du\int_{-1}^1 dv=\int_{-1}^1 f(u)du.$$

例.11 计算积分 $\iint\limits_{\substack{0\le x\le 2\\ 0\le y\le 2}}[x+y]d\sigma$; :

向下取整



解 （1）把 D 分成四个区域 D_1，D_2，D_3，D_4，f 分别在它上取值 0，1，2，3.于是

$$\iint_D [x+y]d\sigma = \iint_{D_1} 0\cdot d\sigma + \iint_{D_2} d\sigma + \iint_{D_3} 2d\sigma + \iint_{D_4} 3d\sigma$$

$$= 1\times\frac{3}{2} + 2\times\frac{3}{2} + 3\times\frac{1}{2} = 6.$$

$$\frac{1}{r}\cdot r \qquad = -\frac{\partial f}{\partial \theta}$$

例. 12 计算 $I = \iint_D \frac{1}{\sqrt{x^2+y^2}}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)d\sigma$，其中 $D = \{(x,y)|x^2+y^2 \le R^2\}$.

解：考虑极坐标系 $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$，$d\sigma = \rho d\rho d\theta$. $D = \{(x,y)|x^2+y^2 \le R^2\}$

$$\frac{1}{\sqrt{x^2+y^2}}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) = \frac{1}{\rho}\frac{\partial f}{\partial(x,y)}\begin{pmatrix} y \\ -x \end{pmatrix} =$$

$$I = \int_0^R dr \int_0^{2\pi} -\frac{\partial f}{\partial \theta} d\theta$$

$$= \frac{1}{\rho}\frac{\partial f}{\partial(x,y)}\begin{pmatrix} y \\ -x \end{pmatrix} = \frac{1}{\rho}\frac{\partial f}{\partial(\rho,\theta)}\cdot\frac{\partial(\rho,\theta)}{\partial(x,y)}\begin{pmatrix} y \\ -x \end{pmatrix} = -\frac{1}{\rho}\frac{\partial f}{\partial \theta}$$

$$-f\Big|_0^{2\pi}$$

因为：
$$\frac{\partial(\rho,\theta)}{\partial(x,y)}\begin{pmatrix} y \\ -x \end{pmatrix} = \left(\frac{\partial(x,y)}{\partial(\rho,\theta)}\right)^{-1}\begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} Cos\theta & -\rho Sin\theta \\ Sin\theta & \rho Cos\theta \end{pmatrix}^{-1}\begin{pmatrix} y \\ -x \end{pmatrix}$$

$$= \frac{1}{\rho}\begin{pmatrix} \rho Cos\theta & \rho Sin\theta \\ -Sin\theta & Cos\theta \end{pmatrix}\begin{pmatrix} y \\ -x \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} 0 \\ -\rho \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

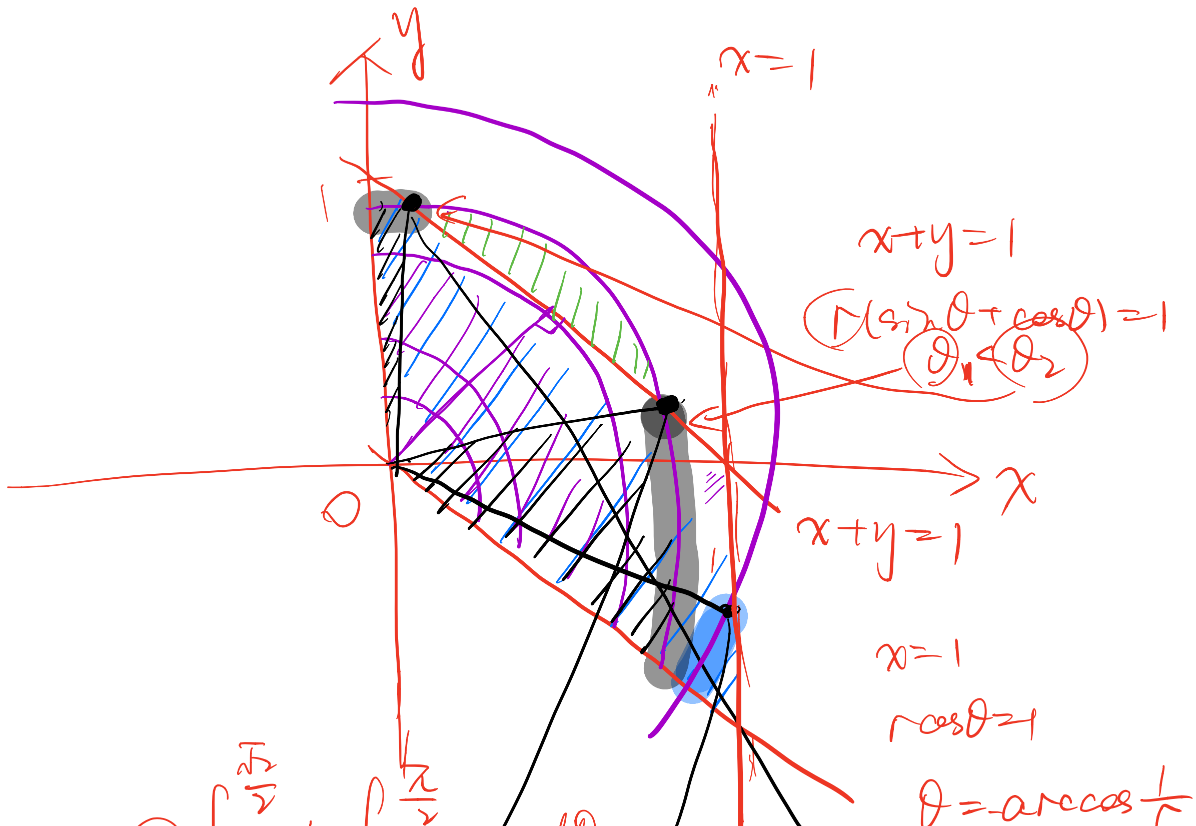
$$= 0$$

$$I = \iint_D \frac{1}{\sqrt{x^2+y^2}}\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)d\sigma = -\iint_{\rho\le R}\frac{1}{\rho}\frac{\partial f}{\partial \theta}\rho d\rho d\theta$$

$$= -\int_0^R d\rho\int_0^{2\pi}\frac{\partial f}{\partial \theta}d\theta = -\int_0^R \big(f(2\pi,\rho) - f(0,\rho)\big)d\rho = 0$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$
$$= \frac{\partial f}{\partial x}(-y) + \frac{\partial f}{\partial y}\cdot x$$



$$①\ \int_{0}^{\frac{\sqrt{2}}{2}} dr \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cdots d\theta$$

$$②\ \int_{\frac{\sqrt{2}}{2}}^{1} dr \int_{-\frac{\pi}{4}}^{\underline{\quad}} \cdots d\theta + \int_{\frac{\sqrt{2}}{2}}^{1} dr \int_{\underline{\quad}}^{\frac{\pi}{2}} \cdots d\theta$$

$$③\ \int_{1}^{\sqrt{2}} dr \int_{-\frac{\pi}{4}}^{\underline{\quad}} \cdots d\theta$$