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习题1.7 1.(5)(6) 2. 3. 5.6 勿忘逆映射

1. 7: (5)(6)

(5): $r=r(u,v)$ 法向量为 $r_u' \times r_v' |_{(u_0,v_0)}$

$r_u'=(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u})=(\cos v,\sin v,0)$ $r_v'=(-u\sin v,u\cos v,a)$

$r_u'\times r_v'=(a\sin v,-a\cos v,u)$ 故法线为

$(u_0\cos v_0,u_0\sin v_0,av_0)+t(a\sin v_0,-a\cos v_0,u_0)$ $(t\in R)$

切平面为: $(x-u_0\cos v_0,y-u_0\sin v_0,z-av_0)\cdot(a\sin v_0,-a\cos v_0,u_0)$

$a\sin v_0 x-au_0\sin v_0\cos v_0-a\cos v_0 y+au_0\sin v_0\cos u_0+u_0z-av_0u_0=0$

$a\sin v_0 x-a\cos v_0 y+u_0z-av_0u_0=0$ 即为法平面

(6) $r_u'=(1,2u,3u^2)$ $r_v'=(1,2v,3v^2)$ 代入(1,2)有点(3,5,9) $r_u'=(1,2,3)$

$r_v'=(1,4,12)$ 法线向量为 $r_v'\times r_u'=(12,-9,2)$

故 法平面为 $(x-3,y-5,z-9)\cdot(12,-9,2)=0$

$12x-36-9y+45+2z-18=0$ $12x-9y+2z-9=0$

法线为 $l: (x,y,z)=(3,5,9)+(12,-9,2)t$ $(t\in R)$

2. 设 $P(x_0,y_0,z_0)$, $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ $\text{Grad}F(x_0,y_0,z_0)=(\frac{2x_0}{a^2},\frac{2y_0}{b^2},\frac{2z_0}{c^2})=M$

设 i,j,k 为 x,y,z 正方向单位向量

$\because \frac{i\cdot M}{|i||M|}=\frac{j\cdot M}{|j||M|}=\frac{k\cdot M}{|k||M|}\Rightarrow \frac{2x_0}{a^2}=\frac{2y_0}{b^2}=\frac{2z_0}{c^2}$ 设之为 K,则

$x_0=a^2K/2$ $y_0=b^2K/2$ $z_0=c^2\cdot K/2$ 代入椭球方程

$\frac{K^2}{4}\cdot(a^2+b^2+c^2)=1$ $\therefore K=\pm 2\cdot(a^2+b^2+c^2)^{-\frac{1}{2}}$

$\therefore$ 点 $P_1(+a^2\cdot(a^2+b^2+c^2)^{-\frac{1}{2}},+b^2\cdot(a^2+b^2+c^2)^{-\frac{1}{2}},+c^2(a^2+b^2+c^2)^{-\frac{1}{2}})$

点 $P_2(-a^2\cdot(a^2+b^2+c^2)^{-\frac{1}{2}},-b^2\cdot(a^2+b^2+c^2)^{-\frac{1}{2}},-c^2(a^2+b^2+c^2)^{-\frac{1}{2}})$

3. 记 $f(x,y,z)=x^2+2y^2+3z^2-21$ $\text{Grad}f=(2x,4y,6z)$ 取点 (x_0,y_0,z_0). $f(x_0,y_0,z_0)=0$

且 $\text{Grad}f(x_0,y_0,z_0)\perp$ 平面 $x+4y+6z=0$ 取三点 $(0,0,0)$ $(4,-1,0)$ $(6,0,-1)$

平面上两向量 $\alpha=(4,-1,0)$; $\beta=(6,0,-1)$

$$\begin{cases}\text{Grad}f(x_0,y_0,z_0)\cdot\alpha=0\\ \text{Grad}f(x_0,y_0,z_0)\cdot\beta=0\\ f(x_0,y_0,z_0)=0\end{cases}\Rightarrow\begin{cases}8x_0-4y_0=0\\ 12x_0-6z_0=0\\ x_0^2+2y_0^2+3z^2-21=0\end{cases}\Rightarrow\begin{cases}x_0=1,-1\\ y_0=2,-2\\ z_0=2,-2\end{cases}$$

对应切平面为: $(x-1,y-2,z-2)\cdot(2,8,12)=0$ $(x+1,y+2,z+2)(-2,-8,-12)=0$

即 $x+4y+6z-21=0$ 与 $x+4y+6z+21=0$

5. $f(x,y,z)=x^2+y^2+z^2-6$ $\quad \mathrm{Grad} f(p_0)=(2,-4,2)$ 切向量为 $(2,-4,2)\times(1,1,1)$

$g(x,y,z)=x+y+z$ $\quad \mathrm{Grad}\, g(p_0)=(1,1,1)$ 记 $\alpha=(-1,0,1)$

切线为

$$l:\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}+t\begin{bmatrix}1\\0\\-1\end{bmatrix}\ (t\in\mathbb{R})$$

法平面：$(x-1,y+2,z-1)\cdot(-1,0,1)=0$

即 $x=z$.（可以设想 $x=z$ 确实是个平面）

6.

切向量为 $\alpha=(x'(t),y'(t),z'(t))=(-a\sin t,a\cos t,b)$ 取 z 方向单位向量 $i=(0,0,1)$

$\cos\langle\alpha,i\rangle=\dfrac{\alpha\cdot(0,0,1)}{|\alpha|\cdot|1|}=\dfrac{b}{\sqrt{a^2+b^2}}$ 故切线与 z 轴所成角的余弦恒定

$\cos x$ 在 $[0,\pi]$ 内单调，切线与 z 轴成定角

习题 1.6.9:

(1) $J(f)=\begin{bmatrix}2x&-2y\\2y&2x\end{bmatrix}$ $\therefore |J(f)|=4x^2+4y^2$ $\quad J(f^{-1})=J(f)^{-1}=\dfrac{1}{2x^2+2y^2}\begin{bmatrix}x&y\\-y&x\end{bmatrix}$

$|J(f^{-1})|=\dfrac{1}{4x^2+4y^2}$

(2) $\begin{cases}u=e^x\cdot\cos y\\v=e^y\cdot\sin y\end{cases}$ $\quad J(f)=\begin{bmatrix}e^x\cos y&-e^x\sin y\\0&e^y(\sin y+\cos y)\end{bmatrix}$ $\quad |J(f)|=e^x\cdot e^y(\sin y\cos y+\cos^2 y)$

注：另一题干版本解答在末页

$\therefore \det(J(f^{-1}))=\dfrac{1}{e^x\cdot e^y\cdot(\sin y\cos y+\cos^2 y)}$

$J(f^{-1})=\dfrac{1}{e^x\cdot e^y\cdot(\sin y\cos y+\cos^2 y)}\begin{bmatrix}e^y(\sin y+\cos y)&e^x\sin y\\0&e^x\cos y\end{bmatrix}$

(3) $J(f)=\begin{bmatrix}3x^2&-3y^2\\y^2&2xy\end{bmatrix}$ $\quad \det=6x^3y+3y^4$

$\det(J(f^{-1}))=\dfrac{1}{6x^3y+3y^4}$ $\quad J(f^{-1})=\dfrac{1}{6x^3y+3y^4}\begin{bmatrix}2xy&3y^2\\-y^2&3x^2\end{bmatrix}$

(4) $J(f)=\begin{bmatrix}\mathrm{ch}x&\mathrm{sh}y\\-\mathrm{sh}x&\mathrm{ch}y\end{bmatrix}$ $\quad \det(J(f))=\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y=\mathrm{ch}(x+y)$

$\therefore \det(J(f^{-1}))=\dfrac{1}{\mathrm{ch}(x+y)}$ $\quad J(f^{-1})=\dfrac{1}{\mathrm{ch}(x+y)}\begin{bmatrix}\mathrm{ch}y&-\mathrm{sh}y\\\mathrm{sh}x&\mathrm{ch}x\end{bmatrix}$

(5) $J(f)=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $\quad \det(J(f))=ad-bc$

$\det(J(f^{-1}))=\dfrac{1}{ad-bc}$ $\quad J(f^{-1})=\dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

(6) $J(f)=\begin{bmatrix}3x^2&-1\\1&3y^2\end{bmatrix}$ $\quad |J(f)|=9x^2y^2+1$

$\det(J(f^{-1}))=\dfrac{1}{9x^2y^2+1}$ $\quad J(f^{-1})=\dfrac{1}{9x^2y^2+1}\begin{bmatrix}3y^2&1\\-1&3x^2\end{bmatrix}$

习题 1.8 1. 2.(2)

$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$

Larange: $+\frac{1}{6}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h,y_0+\theta h)\quad(0<h<1)$

Peano: $O\left((x_0^2+y_0^2)^{\frac{3}{2}}\right)$

1.(1) $\frac{\partial z}{\partial x}=-2x\cdot\sin(x^2+y^2)$ $\frac{\partial z}{\partial y}=-2y\cdot\sin(x^2+y^2)$ $\frac{\partial^2 z}{\partial x^2}=-2\sin(x^2+y^2)-4x^2\cos(x^2+y^2)$

$\frac{\partial^2 z}{\partial y^2}=-2\sin(x^2+y^2)-4y^2\cos(x^2+y^2)$ $\frac{\partial^2 z}{\partial x\partial y}=-4xy\cdot\cos(x^2+y^2)$

$\frac{\partial^3 z}{\partial x^3}=+8x^3\cdot\sin(x^2+y^2)-12x\cdot\cos(x^2+y^2)$ $\frac{\partial^3 z}{\partial y^3}=+8y^3\cdot\sin(x^2+y^2)-12y\cdot\cos(x^2+y^2)$

$\frac{\partial^3 z}{\partial x^2\partial y}=8x^2y\cdot\sin(x^2+y^2)-4y\cdot\cos(x^2+y^2)$ $\frac{\partial^3 z}{\partial y^2\partial x}=8xy^2\cdot\sin(x^2+y^2)-4x\cdot\cos(x^2+y^2)$

注意到代入(0,0)后，以上各导数为0

∴ 2阶Peano: $z=1+O(x^2+y^2)$

$$\begin{aligned}T(x,y)=&y^3\left(+8\theta^3y^3\sin(\theta^2x^2+\theta^2y^2)-12\theta y\cos(\theta^2x^2+\theta^2y^2)\right)\\&+3xy^2\left(8\theta^3xy^2\sin^2(\theta^2x^2+\theta^2y^2)-4\theta x\cdot\cos(\theta^2x^2+\theta^2y^2)\right)\\&+3x^2y\left(8\theta^3x^2y\sin^2(\theta^2x^2+\theta^2y^2)-4\theta y\cdot\cos(\theta^2x^2+\theta^2y^2)\right)\\&+x^3\left(+8\theta^3x^3\sin(\theta^2y^2+\theta^2x^2)-12\theta x\cos(\theta^2x^2+\theta^2y^2)\right)\\=&-12\theta(x^2+y^2)^2\cos(\theta^2x^2+\theta^2y^2)+8\theta^3(x^2+y^2)^3\cdot\sin(\theta^2x^2+\theta^2y^2)\end{aligned}$$

2阶Larange: $z=1+\frac{1}{6}\cdot T(x,y)\quad(\theta\in(0,1))$

1.2: $z_x'=e^{x^2-y^2}\cdot(2x)$ $z_y'=e^{x^2-y^2}(-2y)$ $z_{xx}''=(4x^2+2)\cdot e^{x^2-y^2}$ $z_{xy}''=-4xy\cdot e^{x^2-y^2}$

$z_{yy}''=e^{x^2-y^2}\cdot(4y^2-2)$ $z_{xxx}'''=e^{x^2-y^2}\cdot(8x^3+12x)$ $z_{xxy}'''=e^{x^2-y^2}\cdot(-8x^2y-4y)$

$z_{xyy}'''=e^{x^2-y^2}\cdot(8xy^2-4x)$ $z_{yyy}'''=e^{x^2-y^2}\cdot(-8y^3+12y)$

代入(0,0)后，仅 $z_{xx}''=2$，$z_{yy}''=-2$ 其余为0

故: $z=1+0+\frac{1}{2}\cdot(x,y)\begin{bmatrix}2&0\\0&-2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+O(x^2+y^2)$

Peano: $=1+x^2-y^2+O(x^2+y^2)$

Lagrange: $=1+x^2-y^2+\frac{1}{6}T(x,y)$

$$\begin{aligned}T(x,y)=&y^3\cdot e^{\theta^2x^2-\theta^2y^2}\cdot(-8\theta^3y^3+12\theta y)+3xy^2\cdot e^{\theta^2x^2-\theta^2y^2}\cdot(8\theta^3xy^2-4\theta x)\\&+3x^2y\cdot e^{\theta^2x^2-\theta^2y^2}\cdot(-8\theta^3x^2y-4\theta y)+x^3\cdot e^{\theta^2x^2-\theta^2y^2}\cdot(8\theta^3x^3+12\theta x)\\=&(8\theta^3x^6+12\theta x^4-24\theta^3x^4y^2+24\theta^3x^2y^4-24\theta x^2y^2+12\theta y^4-8\theta^3y^6)\cdot e^{\theta^2x^2-\theta^2y^2}\end{aligned}$$

1.3: $u=\ln(1+x+y+z)$ 设 $t=x+y+z$. $u=f(t)=\ln^{1+t}$.

而 $\ln^{t+1}=t-\frac{1}{2}t^2+o(t^2)$

已求出了所有二阶项,则余项必为高阶小量 $o(x^2+y^2+z^2)$

故 Peano余项: $f(x,y,z)=x+y+z-\frac{1}{2}(x+y+z)^2+o(x^2+y^2+z^2)$

lagrange: $\ln^{t+1}=t-\frac{1}{2}t^2+\frac{1}{6}\cdot\frac{2}{(\theta t+1)^3}\cdot t^3$

代入有: $f(x,y,z)=x+y+z-\frac{1}{2}(x+y+z)^2+\frac{1}{3}\cdot\frac{1}{(\theta(x+y+z)+1)^3}\cdot(x+y+z)^3$

为什么与(1)不同,此处可以如此换元.

本质上:是因为 $\ln(1+x+y+z)$ 的三个偏导数全同!

严格来做:

$$u'_x=\frac{1}{1+x+y+z}=u'_y=u'_z \qquad u''_{xx}=-\frac{1}{(1+x+y+z)^2}=u''_{xy}=u''_{yy}=u''_{xz}=u''_{zz}=u''_{yz}$$

$$u'''_{xxx}=\frac{2}{(1+x+y+z)^3}=u_{xxy}=\cdots=u_{zzz}$$

由多元函数求导法则: f的泰勒展开的n次项为 $\frac{1}{n!}\left(\frac{\partial}{\partial x_1}\Delta x_1+\frac{\partial}{\partial x_2}\Delta x_2+\cdots+\frac{\partial}{\partial x_m}\Delta x_m\right)^n f(x_1,\cdots,x_m)$

对 $f(x,y,z)$ 的3次项为 $\frac{1}{3!}\left(\frac{\partial}{\partial x}\cdot x+\frac{\partial}{\partial y}\cdot y+\frac{\partial}{\partial z}\cdot z\right)^3 f(x,y,z)$

即 $\frac{\partial^3 f}{\partial x^3}\cdot(x+y+z)^3$ 故为 $\frac{1}{3}\cdot\frac{1}{(1+\theta x+\theta y+\theta z)^3}\cdot(x+y+z)^3$

2.(2)

$$z=\frac{\cos x}{\cos y} \qquad f'_x=-\frac{\sin x}{\cos y} \qquad f'_y=\frac{\cos x\cdot\sin y}{\cos^2 y} \qquad f''_{xx}=-\frac{\cos x}{\cos y} \qquad f''_{yy}=\frac{\cos x(\sin^2 y+1)}{\cos^3 y}$$

$f''_{xy}=-\frac{\sin x\cdot\sin y}{\cos^2 y}$ 代入 $(0,0)$

有:泰勒多项式为

$$z=f(x_0,y_0)+(hf'_{x_0}+kf'_{y_0})+\frac{1}{2!}(h^2f''_{x_0x_0}+k^2f''_{y_0y_0}+2hkf''_{x_0y_0})$$

$$=1+\frac{1}{2}(y^2-x^2)$$

1.6.9.(2): 如果题目为 $\begin{cases}u=e^x\cos y\\ v=e^x\sin y\end{cases}$

则 $J(f)=\begin{bmatrix}\cos y\cdot e^x & -\sin y\cdot e^x\\ \sin y\cdot e^x & \cos y\cdot e^x\end{bmatrix}$ $\qquad |J(f)|=e^{2x}$

$\therefore |J(f^{-1})|=e^{-2x}$ $\qquad J(f^{-1})=\begin{bmatrix}\frac{\cos y}{e^x} & \frac{-\sin y}{e^x}\\ \frac{-\sin y}{e^x} & \frac{\cos y}{e^x}\end{bmatrix}$