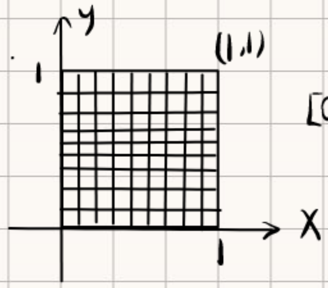
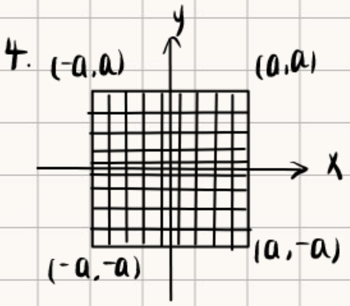
赵晨阳 软01 2020012363

3. 


$$\iint\limits_{[0,1]\times[0,1]} xy\,dxdy=\lim_{n\to+\infty} f\left(\frac{i}{n},\frac{j}{n}\right)\Delta x_i\Delta y_j=\lim_{n\to+\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ij}{n^2}\cdot\frac{1}{n^2}$$

$$=\lim_{n\to+\infty}\frac{1}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{n} ij=\lim_{n\to+\infty}\frac{1}{n^4}\cdot\frac{n^2+n}{2}\cdot\sum_{j=1}^{n} j=\lim_{n\to\infty}\frac{1}{n^4}\cdot\left(\frac{n^2+n}{2}\right)^2=\frac{1}{4}$$

4. 


将积分区域按 $x=a\cdot\frac{i}{n}$ $y=a\cdot\frac{j}{n}$ 划分为 $4n^2$ 个小正方形 $-n+1\le i,j\le n$ 并取小正方形右上角的值

$$\iint\limits_{[-a,a]\times[-a,a]}\sin(x+y)dxdy=\lim_{n\to+\infty}\sum_{i=-n+1}^{n}\sum_{j=-n+1}^{n} f\left(\frac{ai}{n},\frac{aj}{n}\right)\Delta x_i\Delta y_j$$

$$=\lim_{n\to+\infty}\sum_{i=-n+1}^{n}\sum_{j=-n+1}^{n}\sin\left(a\frac{i+j}{n}\right)\cdot\frac{1}{n^2}$$

而 $\left|\sum_{i=-n+1}^{n}\sum_{j=-n+1}^{n}\sin\left(a\frac{i+j}{n}\right)\right|=\left|\sum_{i=-n+1}^{n-1}\sum_{j=-n+1}^{n-1}\sin\left(a\frac{i+j}{n}\right)+\sum_{i=-n+1}^{n}\sin\left(a\frac{i+n}{n}\right)+\sum_{j=-n+1}^{n-1}\sin\left(a\frac{n+j}{n}\right)\right|$

考虑到 $\sin x$ 为奇函数 故 $\sum_{i=-n+1}^{n-1}\sum_{j=-n+1}^{n-1}\sin\left(a\frac{i+j}{n}\right)=0$

$\therefore$ 上式 $=\left|\sum_{i=-n+1}^{n}\sin\left(a\frac{i+n}{n}\right)+\sum_{j=-n+1}^{n-1}\sin\left(a\frac{n+j}{n}\right)\right|\le\left|\sum_{i=-n+1}^{n}1+\sum_{j=-n+1}^{n-1}1\right|\le 4n$

故 $\left|\iint\limits_{[-a,a]\times[-a,a]}\sin(x+y)dxdy\right|\le\frac{4n}{n^2}$ 而后者在 $n\to+\infty$ 时为 0

$\therefore\ \iint\limits_{[-a,a]\times[-a,a]}\sin(x+y)dxdy=0$

10. (1) $f(x,y)$ 在区域 $[-2,2]\times[-2,2]$ 上仅有有限个间断点:
$(1,-1)$ $(1,1)$ $(-1,1)$ $(-1,-1)$. 故间断点为零面积集.
而对于边界 $\forall\varepsilon>0$, 取4个闭矩形 $\{I_i\}$ 宽为 $\frac{\varepsilon}{25}$, 长为5
$\sum_{i=1}^{4}\sigma(I_i)=4\cdot\frac{\varepsilon}{25}\cdot5=\frac{4}{5}\varepsilon<\varepsilon$. 而 $\{I_i\}$ 又可将边界完全覆盖.
$f$ 的间断点与边界均为零面积集.
$\therefore f\in R([-2,2]\times[-2,2])$


(2) 间断为 $D=\{(x,y)\mid y=x^2, 0\le x\le1\}$.
如右图所示. 以 $\left(\frac{i-1}{n},\frac{(i-1)^2}{n^2}\right)$ $\left(\frac{i}{n},\frac{i^2}{n^2}\right)$
为左下与右上端点. $(1\le i\le n)$. $\sigma(I_i)=\frac{1}{n}\left(\frac{i^2}{n^2}-\frac{(i-1)^2}{n^2}\right)$. $n$ 个闭矩形 $\{I_n\}$
将D完全覆盖, $\sum_{i=1}^{n}\sigma(I_i)=\sum_{i=1}^{n}\left(\frac{i^2}{n^2}-\frac{(i-1)^2}{n^2}\right)\cdot\frac{1}{n}=\frac{1}{n}$
$\therefore\ \forall\varepsilon>0$, 取 $n>\varepsilon$, 则可用如上的 $n$ 个 $I_i$ 完全覆盖 D. 且面积和 $<\varepsilon$.
而边界为线段.

对于边界 $\forall \varepsilon>0$, 取4个闭矩形 $\{I_i\}$ 宽为 $\frac{\varepsilon}{10}$, 长为2

$\sum_{i=1}^{4}\sigma(I_i)=4\cdot\frac{\varepsilon}{10}\cdot 2=\frac{4}{5}\varepsilon<\varepsilon$. 而 $\{I_i\}$ 又可将边界完全覆盖

$\therefore f$ 的间断点与边界均为零面积集.

$\therefore f\in R([0,1]\times[0,1])$

4. 假定存在一内点 $P(x_0,y_0)$ st $f(x_0,y_0)=a>0$. 则由 $f(x,y)$ 连续可知, $\exists\,\varepsilon>0$, 使得

s.t. $\forall (x,y)\in B_\varepsilon(P,\varepsilon)\cap D$, $f(x,y)>0$.

即 $\exists\,\tilde{D}=B(P_0,\varepsilon)\cap D$, 则 $\forall (x,y)\in\tilde{D}$, $f(x,y)>0$

$\iint\limits_D f(x,y)\,dx\,dy=\iint\limits_{\tilde{D}} f(x,y)\,dx\,dy+\iint\limits_{D\setminus\tilde{D}} f(x,y)\,dx\,dy$ 而 $f(x,y)\geq 0$ 由保序性可知

$\iint\limits_D f(x,y)\,dx\,dy\geq\iint\limits_{\tilde{D}} f(x,y)\,dx\,dy$ 而 $f(x,y)$ 在 $\tilde{D}$ 内连续且不变号

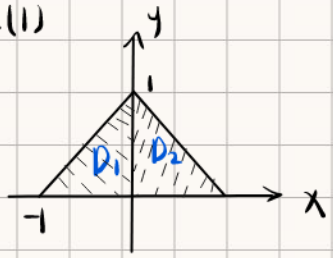
故 $\exists(\xi,\varphi)\in\tilde{D}$, st $\iint\limits_{\tilde{D}} f(x,y)\,dx\,dy=f(\xi,\varphi)\cdot\sigma(\tilde{D})>0$. 与题干矛盾.

故 $\forall$ 内点有 $f(x,y)\equiv 0$. 由连续性, 边界点同样 $f(x,y)=0$

故 $\forall (x,y)\in D$, $f(x,y)=0$

## 习题 3.3.

5.(1)

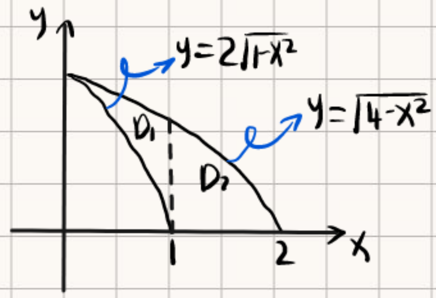



$\int_{-1}^{0}dx\int_0^{1+x}f(x,y)\,dy+\int_0^1 dx\int_0^{1-x}f(x,y)\,dy$

$=\iint\limits_{D_1+D_2}f(x,y)\,dx\,dy$

$=\int_0^1 dy\int_{y-1}^{1-y}f(x,y)\,dx$

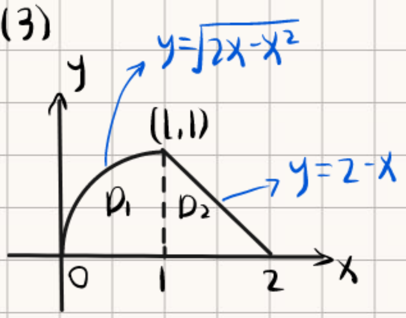
5.(2)




$\int_0^1 dx\int_{2\sqrt{1-x^2}}^{\sqrt{4-x^2}}f(x,y)\,dy+\int_1^2 dx\int_0^{\sqrt{4-x^2}}f(x,y)\,dy$

$=\iint\limits_{D_1+D_2}f(x,y)\,dx\,dy=\int_0^2 dy\int_{\sqrt{1-\frac{y^2}{4}}}^{\sqrt{4-y^2}}f(x,y)\,dx$

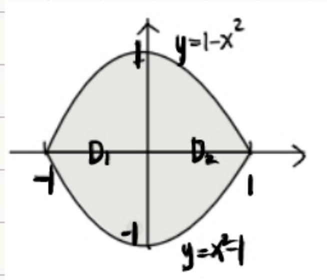
5.(3)




$\int_0^1 dx\int_0^{\sqrt{2x-x^2}}f(x,y)\,dy+\int_1^2\int_0^{2-x}f(x,y)\,dy$

$=\iint\limits_{D_1+D_2}f(x,y)\,dx\,dy=\int_0^1 dy\int_{1-\sqrt{1-y^2}}^{2-y}f(x,y)\,dx$

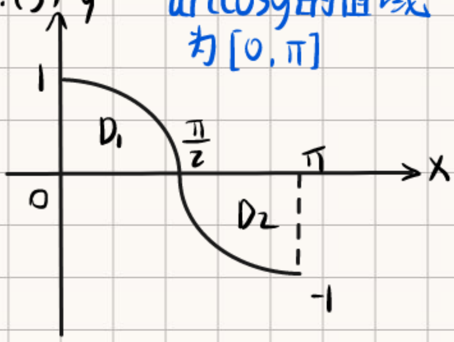
5.(4) 图实在不好画，我用纸质画了之后扫描图如下，望助教老师见谅




$$\int_{-1}^{1}dx\int_{x^2-1}^{1-x^2}f(x,y)dy=\iint_{D_1+D_2}f(x,y)dxdy$$

$$=\int_{-1}^{0}dy\int_{-\sqrt{y+1}}^{\sqrt{y+1}}f(x,y)dx+\int_{0}^{1}dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}}f(x,y)dx$$

5.(5)

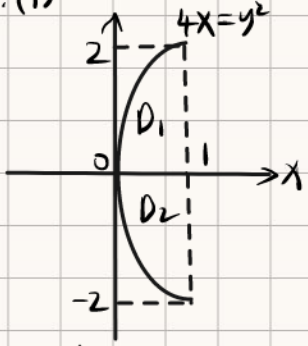



arccosy的值域为[0,π]

$$\iint_{D_1+D_2}f(x,y)dxdy$$

$$=\int_{-1}^{0}dy\int_{\arccos y}^{\pi}f(x,y)dx+\int_{0}^{1}dy\int_{0}^{\arccos y}f(x,y)dx$$

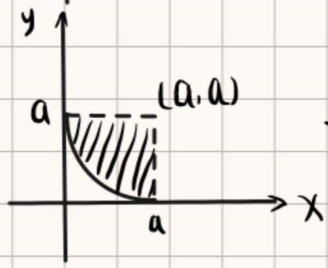
6.(1)




$$\iint_D f(x,y)dxdy=\int_0^1dx\int_{-2\sqrt{x}}^{2\sqrt{x}}xy^2dy$$

$$=\int_0^1 x\cdot\frac{1}{3}y^3\Big|_{-2\sqrt{x}}^{2\sqrt{x}}dx=\int_0^1\frac{16}{3}x^2\sqrt{x}\,dx=\frac{16}{3}\cdot\frac{2}{7}\cdot x^{\frac{7}{2}}\Big|_0^1=\frac{32}{21}$$

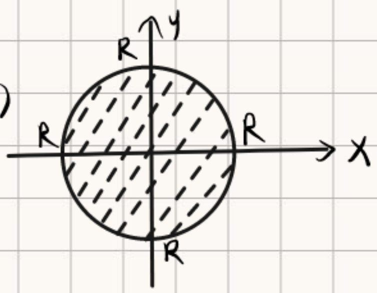
(2)




$$\iint_D\frac{1}{\sqrt{2a-x}}dxdy=\int_0^a dx\int_{a-\sqrt{a^2-(x-a)^2}}^{a}\frac{1}{\sqrt{2a-x}}dy=\int_0^a\frac{1}{\sqrt{2a-x}}y\Big|_{a-\sqrt{a^2-(a-x)^2}}^{a}dx$$

$$=\int_0^a\frac{\sqrt{a^2-(a-x)^2}}{\sqrt{2a-x}}dx=\int_0^a\sqrt{x}\,dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^a=\frac{2}{3}a^{\frac{3}{2}}$$

(3)

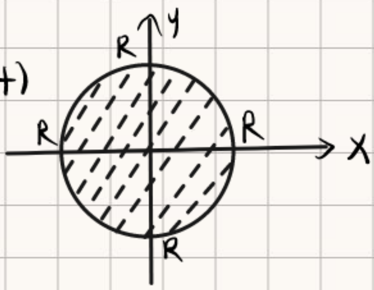


令 $x=r\cos\theta$ $y=r\sin\theta$ $E=\{(r,\theta)\,|\,0\le r\le R,\ 0\le\theta\le 2\pi\}$

$$\iint_D|xy|dxdy=\iint_E|r^2\sin\theta\cos\theta|\cdot r\,drd\theta=\frac{1}{2}\int_0^R r^3dr\int_0^{2\pi}|\sin2\theta|d\theta$$

$$=\frac{1}{2}\cdot\frac{R^4}{4}\cdot4\int_0^{\frac{\pi}{2}}\sin2\theta\,d\theta=\frac{R^4}{2}\cdot\frac{1}{2}(-\cos2\theta)\Big|_0^{\frac{\pi}{2}}=\frac{R^4}{2}$$

(4)

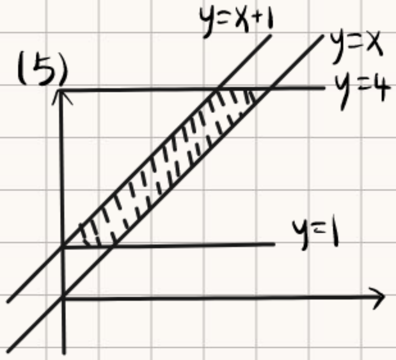


注意到积分区域关于 OY 轴对称且

$f(x,y)=x\cdot\cos xy$ 关于 x 为奇函数

$\therefore \iint_D x\cos(xy)dxdy=0$

(5)



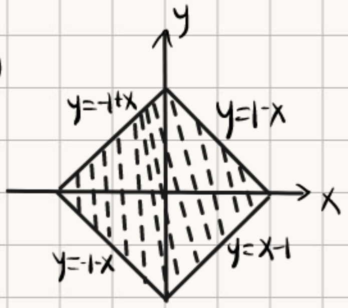

$$\iint_D (x^2+y^2)\,dx\,dy = \int_1^4 dy\int_{y-1}^{y}(x^2+y^2)\,dx = \int_1^4\left(\frac13x^3+xy^2\Big|_{y-1}^{y}\right)dy$$

$$= \int_1^4\left(\frac13y^3+y^3-\frac13(y-1)^3-(y-1)y^2\right)dy$$

$$= \int_1^4\left(y^2-y+\frac13+y^2\right)dy$$

$$= \int_1^4\left(2y^2-y+\frac13\right)dy = \left(\frac23y^3-\frac12y^2+\frac13y\right)\Big|_1^4 = \frac{71}{2}$$

(6)

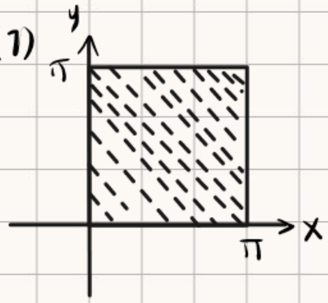



$$\iint_D e^{x+y}\,dxdy = \int_{-1}^0 dx\int_{-1-x}^{1+x}e^{x+y}\,dy + \int_0^1 dx\int_{x-1}^{1-x}e^{x+y}\,dy$$

$$= \int_{-1}^0\left(e^{x+y}\Big|_{-1-x}^{1+x}\right)dx + \int_0^1\left(e^{x+y}\Big|_{x-1}^{1-x}\right)dx = \int_{-1}^0\left(e^{2x+1}-e^{-1}\right)dx + \int_0^1\left(e-e^{2x-1}\right)dx$$

$$= \left(\frac12 e^{2x+1}-\frac1e x\right)\Big|_{-1}^0 + \left(ex-\frac12e^{2x-1}\right)\Big|_0^1 = e-\frac1e$$

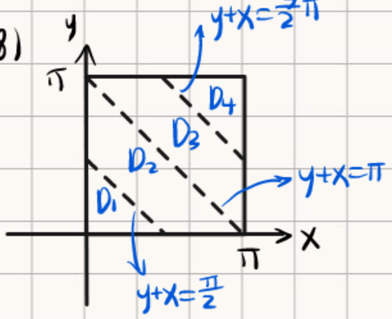
(7)




$$\iint_D \cos(x+y)\,dxdy = \int_0^\pi dx\int_0^\pi\cos(x+y)\,dy = \int_0^\pi \sin(x+y)\Big|_0^\pi dx$$

$$= \int_0^\pi -2\sin x\,dx = 2\cos x\Big|_0^\pi = -4$$

(8)




$$\iint_D |\cos(x+y)|\,dxdy$$

$$= \iint_{D_1}\cos(x+y)\,dxdy - \iint_{D_2}\cos(x+y)\,dxdy - \iint_{D_3}\cos(x+y)\,dxdy + \iint_{D_4}\cos(x+y)\,dxdy$$

$$= 2\iint_{D_1+D_2}\cos(x+y)\,dxdy - \iint_D\cos(x+y)\,dxdy$$

$$\iint_{D_1}\cos(x+y)\,dxdy = \int_0^{\frac\pi2}dx\int_0^{\frac\pi2-x}\cos(x+y)\,dy = \int_0^{\frac\pi2}\sin(x+y)\Big|_0^{\frac\pi2-x}dx = \int_0^{\frac\pi2}(1-\sin x)\,dx = (x+\cos x)\Big|_0^{\frac\pi2} = \frac\pi2-1$$

$$\iint_{D_4}\cos(x+y)\,dxdy = \int_{\frac\pi2}^{\pi}dx\int_{\frac32\pi-x}^{\pi}\cos(x+y)\,dy = \int_{\frac\pi2}^{\pi}\sin(x+y)\Big|_{\frac32\pi-x}^{\pi}dx = \int_{\frac\pi2}^{\pi}(-\sin x+1)\,dx = \cos x\Big|_{\frac\pi2}^{\pi}+x\Big|_{\frac\pi2}^{\pi}$$

$$= \frac\pi2-1$$

而由(7)知 $\iint_D\cos(x+y)\,dxdy=-4$. 故 $\iint_D|\cos(x+y)|\,dxdy = 2\cdot2\cdot\left(\frac\pi2-1\right)+4 = 2\pi$

$\sin^2\frac x2 = \frac{1-\cos t}{2}$ $\quad\cos t = 2\sin^2\frac x2+1$

(9) 设由参数方程确定函数 $y=f(x)$

$$\iint_D y^2\,dxdy = \int_0^{2\pi a}dx\int_0^{f(x)}y^2\,dy = \int_0^{2\pi a}\frac13 f^3(x)\,dx \xrightarrow{x=a(t-\sin t)} \frac13\int_0^{2\pi}a(1-\cos t)\,f(x(t))^3\,dt$$

$$= \frac13\int_0^{2\pi}a(1-\cos t)\cdot(a(1-\cos t))^3\,dt = \frac{a^4}{3}\int_0^{2\pi}(1-\cos t)^4\,dt = \frac{16}{3}a^4\int_0^{2\pi}\sin^8\frac t2\,dt \xrightarrow{u=\frac t2}$$

$= \frac{32}{3}a^4\int_0^\pi \sin^8u\,du$ $\quad h(u)=\sin^8(u)=\sin^8(\pi-u)$. 故 $h(u)$关于 $u=\frac\pi2$ 对称.

$\therefore \int_0^\pi h(u)\,du = 2\int_0^{\frac\pi2}h(u)\,du$. $\therefore \iint_D y^2\,dxdy = \frac{64}{3}a^4\int_0^{\frac\pi2}\sin^8u\,du$

(9)

$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ $\quad\therefore I_n=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d(\cos x)=-\sin^{n-1}x\cdot\cos x\Big|_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cdot\cos^2 x\,dx$

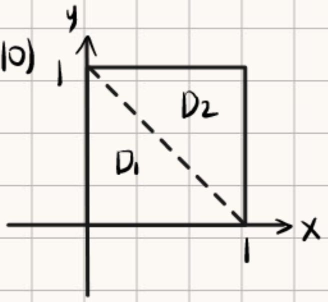
$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\,dx$ 即 $I_n=\frac{n-1}{n}I_{n-2}$.

$$I_n=\begin{cases}\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2} & (n\text{为偶})\\ \frac{(2k)!!}{(2k+1)!!} & (n\text{为奇数})\end{cases}$$

$8\cdot\frac{7\cdot5\cdot1}{6\cdot4\cdot2}$

故由递推公式: 原式 $=\frac{64}{3}a^4\cdot\frac{7\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2}=\frac{35}{12}a^4\pi$

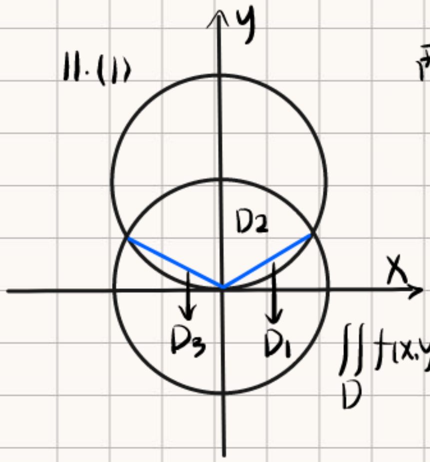
(10)




$$\iint_D[x+y]\,dxdy=\iint_{D_1}[x+y]\,dxdy+\iint_{D_2}[x+y]\,dxdy$$

$$=\iint_{D_2}[x+y]\,dxdy=\iint_{D_2}1\,dxdy=\sigma(D_2)=1\times1/2=\frac{1}{2}$$

11.(1)




两圆交点有 $\begin{cases}r=1\\ r^2\cos^2\theta+(r\sin\theta-1)^2=1\end{cases}\Rightarrow\begin{cases}r=1\\ \theta=\frac{\pi}{6}\end{cases}\quad\begin{cases}r=1\\ \theta=\frac{5}{6}\pi\end{cases}$

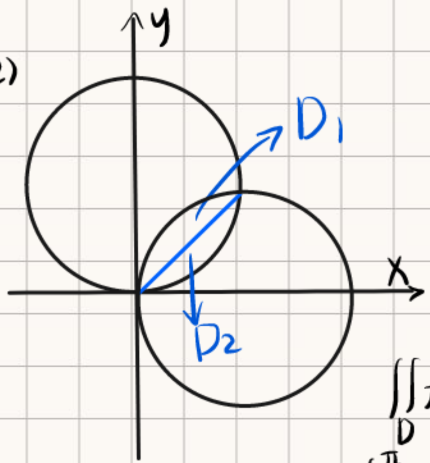
$D_1=\{(r,\theta)\mid 0\le r\le 2\sin\theta,\ 0\le\theta\le\frac{\pi}{6}\}$

$D_2=\{(r,\theta)\mid 0\le r\le 1,\ \frac{\pi}{6}\le\theta\le\frac{5}{6}\pi\}$

$D_3=\{(r,\theta)\mid 0\le r\le 2\sin\theta,\ \frac{5}{6}\pi\le\theta\le\pi\}$

$$\iint_D f(x,y)\,dxdy=\int_0^{\frac{\pi}{6}}d\theta\int_0^{2\sin\theta}f(r\cos\theta,r\sin\theta)\,r\,dr+\int_0^{\frac{\pi}{6}}d\theta\int_0^{2\sin\theta}f(r\cos\theta,r\sin\theta)\,dr$$
$$+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}d\theta\int_0^1 f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$$

(2)




两圆交点 $\begin{cases}r^2\cos^2\theta+(r\sin\theta-a)^2=a^2\\ (r\cos\theta-a)^2+r^2\sin\theta^2=a^2\end{cases}\quad\begin{cases}r=0\\ \theta=0\end{cases}\quad\begin{cases}r=\sqrt{2}a\\ \theta=\frac{\pi}{4}\end{cases}$

$D_1=\{(r,\theta)\mid 0\le r\le 2a\cos\theta,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{2}\}$

$D_2=\{(r,\theta)\mid 0\le r\le 2a\sin\theta,\ 0\le\theta\le\frac{\pi}{4}\}$

$$\iint_D f(x,y)\,dxdy$$
$$=\int_0^{\frac{\pi}{4}}d\theta\int_0^{2a\sin\theta}f(r\cos\theta,r\sin\theta)\,r\,dr+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{2a\cos\theta}f(r\cos\theta,r\sin\theta)\,r\,dr$$