1.4 Applications of Functions of Matrices

Excercise 1.4.1.

1. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$, and assume $B = \begin{bmatrix} I & X \\ O & I \end{bmatrix}$, where all blocks are 2×2 matrices. Therefore we have

$$\begin{aligned} BAB^{-1} &= \begin{bmatrix} I & X \\ O & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix} \begin{bmatrix} I & -X \\ O & I \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} - A_{11}X + XA_{22} \\ O & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}, \end{aligned}$$

i.e. $A_{11}X - XA_{22} = A_{12}$. Note that A_{11} and A_{22} have no common eigenvalue, so we have a unique X satisfying the equation. Suppose $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{aligned} A_{11}X - XA_{22} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -2a + 2c & -5a - 3b + 2d \\ -2c & -5c - 3d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \end{aligned}$$

Solve it and we get $(a, b, c, d) = \left(-2, \frac{31}{9}, -\frac{3}{2}, \frac{7}{6}\right)$. Thus

$$B = \begin{bmatrix} 1 & 0 & -2 & \frac{31}{9} \\ 0 & 1 & -\frac{3}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. $V_3 + V_4$ corresponds to the block A_{22} in $BAB^{-1} = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$. Therefore a basis of $V_3 + V_4$ is $\{\mathbf{e}_3, \mathbf{e}_4\}$ under the basis B^{-1}, i.e. $\{B^{-1}\mathbf{e}_3, B^{-1}\mathbf{e}_4\} = \left\{\left(2, \frac{3}{2}, 1, 0\right), \left(-\frac{31}{9}, -\frac{7}{6}, 0, 1\right)\right\}$.

Excercise 1.4.2. No. A counter example is

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where the minimal polynomial of B is $p_B(x) = x^2$, but

$$A^3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^3 = O,$$

i.e. $p_A(x) = x^3 \neq (p_B(x))^2$.

Note that $A = \begin{bmatrix} B & O \\ O & B \end{bmatrix} \begin{bmatrix} I & B^{-1} \\ O & I \end{bmatrix} = \begin{bmatrix} I & B^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} B & O \\ O & B \end{bmatrix}$, therefore for any positive integer n,

$$\begin{aligned} A^n &= \left(\begin{bmatrix} B & O \\ O & B \end{bmatrix} \begin{bmatrix} I & B^{-1} \\ O & I \end{bmatrix} \right)^n \\ &= \begin{bmatrix} B & O \\ O & B \end{bmatrix}^n \begin{bmatrix} I & B^{-1} \\ O & I \end{bmatrix}^n \\ &= \begin{bmatrix} B^n & nB^{n-1} \\ O & B^n \end{bmatrix}. \end{aligned}$$

Hence $p(A) = \begin{bmatrix} p(B) & p'(B) \\ O & p(B) \end{bmatrix}$, and $p(A) = O \Leftrightarrow p(B) = p'(B) = O$. Since $p_A(B) = p'_A(B) = O$, they are both a multiple of p_B. Suppose $p_B(x) = \sum (x - \lambda_i)^{k_i}$, then we have $p_A(x) = \sum (x - \lambda_i)^{k_i + 1}$.

Excercise 1.4.3.

1. Suppose A and B are both $n \times n$ matrices. For any $\mathbf{v}$ in the generalized eigenspace of A with the eigenvalue λ, we have

   $$(A - \lambda I)B = AB - \lambda B = BA - \lambda B = B(A - \lambda I).$$

   Applying this repeatedly, we have

   $$(A - \lambda I)^n B = B(A - \lambda I)^n,$$

   and therefore

   $$(A - \lambda I)^n B\mathbf{v} = B(A - \lambda I)^n \mathbf{v} = \mathbf{0},$$

   i.e. $B\mathbf{v}$ is in the same generalized eigenspace of A, too. Hence any generalized eigenspace of A is an invariant space of B, and vice versa. In other words, they have the same generalized eigenspaces, so they can be simultaneously triangularized.

2. No. A counter example is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, where A is already in its Jordan canonical form (i.e. $C = I$) but B is not.

Excercise 1.4.4. There is no such f. If $f(f(x)) + x = 0$, then $(f \circ f)(x) = f(f(x)) = -x$ is a bijection between $\mathbb{R}$ and $\mathbb{R}$, and thus f is injective and f is surjective, producing f a bijection. Therefore since f is continuous, it is strictly monotone. On the other hand, since $f(-x) = f(f(f(x))) = -f(x)$, f is odd. Combining the conclusion above, $f(x)$ is either sign-preserving or sign-inverting, and thus $f(f(x))$ is always sign-preserving, contradictory to $f(f(x)) = -x$.