第二次习题课：

方向导数,链式法则(高阶导),隐函数偏导

方向导数,链式法则

例1. 求函数 $f(x,y)=x^2-y^2$ 在 $P(1,1)$ 点沿与 x 轴成 $\dfrac{\pi}{3}$ 角方向的方向导数。

解：方向为 $\mathbf{l}=(\dfrac{1}{2},\dfrac{\sqrt{3}}{2})$。

$$\frac{\partial f}{\partial l}(P)=\left(\frac{\partial f}{\partial x}(P),\frac{\partial f}{\partial y}(P)\right)\cdot \mathbf{l}^0=(2,-2)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})=1-\sqrt{3}$$

例2. 求函数 $f(x,y)=1-\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$ 在 $P(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}})$ 点沿曲线 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 在该点的内法方向的方向导数。

解：曲线 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 在 $P(\dfrac{a}{\sqrt{2}},\dfrac{b}{\sqrt{2}})$ 点附近的显函数方程为 $y=\dfrac{b}{a}\sqrt{a^2-x^2}$，切线的斜率为 $y'(\dfrac{a}{\sqrt{2}})=-\dfrac{b}{a}$，所以内法方向的斜率为 $k=\dfrac{a}{b}$，$\mathbf{l}^0=(\dfrac{-b}{\sqrt{a^2+b^2}},\dfrac{-a}{\sqrt{a^2+b^2}})$。

$$\left(\frac{\partial f}{\partial x}(P),\frac{\partial f}{\partial y}(P)\right)=\left(-\frac{2x}{a^2},-\frac{2y}{b^2}\right)_P=\left(-\frac{\sqrt{2}}{a},-\frac{\sqrt{2}}{b}\right),$$

所以 $\dfrac{\partial f}{\partial l}(P)=\left(\dfrac{\partial f}{\partial x}(P),\dfrac{\partial f}{\partial y}(P)\right)\cdot \mathbf{l}^0=\dfrac{\sqrt{2(a^2+b^2)}}{ab}$。

例3. 设函数 $z=\arctan\dfrac{x-y}{x+y}$，求 $\dfrac{\partial z}{\partial x}$， $\dfrac{\partial z}{\partial y}$， dz， $\dfrac{\partial^2 z}{\partial x\partial y}$

解：记 $u=\dfrac{x-y}{x+y}$，则 $z=\arctan u$，由链式法则，

$$\frac{\partial z}{\partial x}=\frac{dz}{du}\frac{\partial u}{\partial x}=\frac{1}{1+u^2}\frac{\partial u}{\partial x}=\frac{1}{1+\left(\dfrac{x-y}{x+y}\right)^2}\frac{2y}{(x+y)^2};$$

$$\frac{\partial z}{\partial y}=\frac{dz}{du}\frac{\partial u}{\partial y}=\frac{1}{1+u^2}\frac{\partial u}{\partial y}=\frac{1}{1+\left(\dfrac{x-y}{x+y}\right)^2}\frac{-2x}{(x+y)^2},$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{ydx-xdy}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)=\frac{x^2-y^2}{(x^2+y^2)^2}\text{。}$$

例4. 若函数 $f(u)$ 有二阶导数，设函数 $z=\frac{1}{x}f(xy)+yf(x+y)$，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(\frac{1}{x}f(xy)\right)+\frac{\partial}{\partial y}\left(yf(x+y)\right)=\frac{1}{x}\frac{\partial}{\partial y}\left(f(xy)\right)+f(x+y)+y\frac{\partial}{\partial y}\left(f(x+y)\right)$

$\frac{\partial}{\partial y}\left(f(xy)\right)=\frac{d}{du}\left(f(u)\right)\frac{\partial u}{\partial y}=f'(u)\cdot x=xf'(xy)$，其中变量 $u=xy$。

$\frac{\partial}{\partial y}\left(f(x+y)\right)=\frac{\partial}{\partial y}\left(f(v)\right)\cdot\frac{\partial v}{\partial y}=f'(v)\cdot 1=f'(x+y)$，其中变量 $v=x+y$。

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(f'(xy)+f(x+y)+yf'(x+y)\right)$$

$$=yf''(xy)+f'(x+y)+yf''(x+y)$$

例5.　　已知 $y=(\frac{1}{x})^{-\frac{1}{x}}$，求 $\frac{dy}{dx}$.

解　考虑二元函数 $y=u^v$，$u=\frac{1}{x},v=-\frac{1}{x}$，应用推论得

$$\frac{dy}{dx}=\frac{\partial y}{\partial u}\frac{du}{dx}+\frac{\partial y}{\partial v}\frac{dv}{dx}.\,vu^{v-1}\frac{-1}{x^2}+(\ln u)u^v\frac{1}{x^2}=\left(\frac{1}{x}\right)^{2-\frac{1}{x}}(1-\ln x).$$

例6. 设 $f(x,y)$ 定义在 R^2 上, 若它对 x 连续,对 y 的偏导数在 R^2 上有界, 证明 $f(x,y)$ 连续.

【证明】$\forall(x_0,y_0)\in R$，

$$\begin{aligned}\left|f(x,y)-f(x_0,y_0)\right|&=\left|\left[f(x,y)-f(x,y_0)\right]+\left[f(x,y_0)-f(x_0,y_0)\right]\right|\\&\le\left|f(x,y)-f(x,y_0)\right|+\left|f(x,y_0)-f(x_0,y_0)\right|\end{aligned}$$

因为 $f(x,y)$ 对 x 连续，所以

$$\lim_{x\to x_0}\left[f(x,y_0)-f(x_0,y_0)\right]=0$$

又因为 $f(x,y)$ 对 y 的偏导数在 R^2 上有界，假设 $\left|\frac{\partial f}{\partial y}(x,y)\right|\le M$，

$$\left|f(x,y)-f(x,y_0)\right|=\left|\frac{\partial f}{\partial y}(x,\eta)(y-y_0)\right|\to 0,\quad y\to y_0$$

所以

$$\lim_{(x,y)\to(x_0,y_0)}\left(f(x,y)-f(x_0,y_0)\right)$$
$$=\lim_{(x,y)\to(x_0,y_0)}\left[f(x,y)-f(x,y_0)\right]+\lim_{(x,y)\to(x_0,y_0)}\left[f(x,y_0)-f(x_0,y_0)\right]=0$$

$f(x,y)$ 连续.

例7. 设 $g(x)=f\left(x,\varphi\left(x^2,x^2\right)\right)$，其中函数 f 于 φ 的二阶偏导数连续，求 $\dfrac{d^2g(x)}{dx^2}$

例8. 设 $z=f(xy,\dfrac{x}{y})$，f 二阶连续可微，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解 记 $u=xy, v=\dfrac{x}{y}$； $f_1'=\dfrac{\partial f}{\partial u}, f_2'=\dfrac{\partial f}{\partial v}$，

$$f_{11}''=\frac{\partial^2 f}{\partial u^2}, f_{22}''=\frac{\partial^2 f}{\partial v^2},\ f_{12}''=\frac{\partial^2 f}{\partial u\partial v}, f_{21}''=\frac{\partial^2 f}{\partial v\partial u}$$

则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=y\,f_1'+\frac{1}{y}\,f_2',$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=y\frac{\partial f_1'}{\partial x}+\frac{1}{y}\frac{\partial f_2'}{\partial x}$$

因为 $f_1'=\dfrac{\partial f}{\partial u}, f_2'=\dfrac{\partial f}{\partial v}$ 都是以 u,v 为中间变量，以 x,y 为自变量的函数，所以

$$\frac{\partial f_1'}{\partial x}=f_{11}''\frac{\partial u}{\partial x}+f_{12}''\frac{\partial v}{\partial x}=y\,f_{11}''+\frac{1}{y}\,f_{12}''$$
$$\frac{\partial f_2'}{\partial x}=f_{21}''\frac{\partial u}{\partial x}+f_{22}''\frac{\partial v}{\partial x}=y\,f_{21}''+\frac{1}{y}\,f_{22}''$$

将以上两式代入前式得: $\dfrac{\partial^2 z}{\partial x^2}=y^2 f''_{11}+2f''_{12}+\dfrac{1}{y^2}f''_{22}$.

例9. 设 $z=z(x,y)$ 二阶连续可微，并且满足方程

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=0$$

若令 $\begin{cases}u=x+\alpha y\\ v=x+\beta y\end{cases}$，试确定 α,β 为何值时能变原方程为 $\dfrac{\partial^2 z}{\partial u\partial v}=0$.

解 将 x,y 看成自变量，u,v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha^2\frac{\partial^2 z}{\partial u^2}+2\alpha\beta\frac{\partial^2 z}{\partial u\partial v}+\beta^2\frac{\partial^2 z}{\partial v^2}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha\frac{\partial^2 z}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 z}{\partial u\partial v}+\beta\frac{\partial^2 z}{\partial v^2}$$

$$=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

由此可得， $0=A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=$

$$=\left(A+2B\alpha+C\alpha^2\right)\frac{\partial^2 z}{\partial u^2}+2\left(A+B(\alpha+\beta)+C\alpha\beta\right)\frac{\partial^2 z}{\partial u\partial v}+\left(A+2B\beta+C\beta^2\right)\frac{\partial^2 z}{\partial v^2}=0$$

只要选取α,β使得 $\begin{cases}A+2B\alpha+C\alpha^2=0\\ A+2B\beta+C\beta^2=0\end{cases}$，可得 $\frac{\partial^2 z}{\partial u\partial v}=0$.

问题成为方程$A+2Bt+Ct^2=0$有两不同实根，即要求： $B^2-AC>0$.

令$\alpha=-B+\sqrt{B^2-AC}$，$\beta=-B-\sqrt{B^2-AC}$，即可。

此时，$\frac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\frac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=0\Rightarrow\frac{\partial z}{\partial v}=\varphi(v)\Rightarrow z=\int\varphi(v)dv+f(u)$.

$$z=f(u)+g(v)=f(x+\alpha y)+g(x+\beta y).$$

例10. 设$u(x,y)\in C^2$，又$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$，$u(x,2x)=x$，$u'_x(x,2x)=x^2$，求 $u''_{xx}(x,2x)$，

$u''_{xy}(x,2x)$ $u''_{yy}(x,2x)$

解: $$\frac{\partial u}{\partial x}(x,2x)=x^2,$$

两边对x求导,

$$\frac{\partial^2 u}{\partial x^2}(x,2x)+\frac{\partial^2 u}{\partial x\partial y}(x,2x)\cdot 2=2x. \qquad (1)$$

$$u(x,2x)=x,$$

两边对x求导,

$$\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1, \qquad \frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}.$$

两再边对x求导,

$$\frac{\partial^2 u}{\partial x\partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知 $$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

(1), (2), (3) 联立可解得:

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x, \quad \frac{\partial^2 u}{\partial x\partial y}(x,2x)=\frac{5}{3}x$$

隐函数的求导法

隐函数 若函数 $y=y(x)$，由方程 $F(x,y)=0$ 确定，求导函数？

按隐函数定义有恒等式：$F(x,y(x))\equiv 0 \Rightarrow \frac{d}{dx}F(x,y(x))=0$，

$$\Rightarrow F'_x(x,y(x))+F'_y(x,y(x))\cdot y'(x)=0 \Rightarrow y'(x)=-\frac{F'_x(x,y(x))}{F'_y(x,y(x))}。$$

由此可见：函数 $y=y(x)$ 可导有一个充分条件是，$F'_y(x,y)\neq 0$.

例11. 已知函数 $y=f(x)$ 由方程 $ax+by=f\left(x^2+y^2\right), a,b$ 是常数，$f'\left(x^2+y^2\right)$ 已知，求 $\frac{dy}{dx}$。

解：方法一。

方程 $ax+by=f\left(x^2+y^2\right)$ 两边对 x 求导，

$$a+b\frac{dy}{dx}=f'(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{2xf'(x^2+y^2)-a}{b-2yf'(x^2+y^2)}$$

方法二。

$F(x,y)=ax+by-f\left(x^2+y^2\right)$，

$$\frac{\partial F}{\partial x}=a-2xf'\left(x^2+y^2\right)$$

$$\frac{\partial F}{\partial y}=b-2yf'\left(x^2+y^2\right)$$

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\frac{2xf'(x^2+y^2)-a}{b-2yf'(x^2+y^2)}。$$

一般来说，若函数 $y=y(\vec{x})$，由方程 $F(\vec{x},y)=0$ 确定，求导之函数？

将 y 看作是 $x_1,\dots,x_n$ 的函数 $y=y(\vec{x})=y(x_1,\dots,x_n)$，对于方程

$$F(x_1,\dots,x_n,y(x_1,\dots,x_n))=0$$

两端分别关于 x_i 求偏导数得到，并解 $\frac{\partial f}{\partial x_i}$，可得到公式 ：$\frac{\partial y}{\partial x_i}=-\frac{F'_{x_i}(\vec{x},y)}{F'_y(\vec{x},y)}$

例12. 设 $F\in C^{(1)}$，证明：方程 $F\left(x+\frac{z}{y},y+\frac{z}{x}\right)=0$ 所确定的隐函数 $z=z(x,y)$ 满足

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy\text{。}$$

证明：记 $u=x+\frac{z}{y},v=y+\frac{z}{x}$。

方法一。$F\left(x+\frac{z(x,y)}{y},y+\frac{z(x,y)}{x}\right)\equiv 0,\quad \forall(x,y)$，

对 x 求偏导，$\frac{\partial}{\partial x}\left\{F\left(x+\frac{z(x,y)}{y},y+\frac{z(x,y)}{x}\right)\right\}\equiv 0,\quad \forall(x,y)$，

$\frac{\partial F}{\partial u}\left(1+\frac{\frac{\partial z}{\partial x}}{y}\right)+\frac{\partial F}{\partial v}\left(0+\frac{x\frac{\partial z}{\partial x}-z}{x^2}\right)=0$，解得 $\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial u}-\frac{z}{x^2}\frac{\partial F}{\partial v}}{\frac{1}{y}\frac{\partial F}{\partial u}+\frac{1}{x}\frac{\partial F}{\partial v}}$。

对 y 求偏导，$\frac{\partial}{\partial y}\left\{F\left(x+\frac{z(x,y)}{y},y+\frac{z(x,y)}{x}\right)\right\}\equiv 0,\quad \forall(x,y)$，

$\frac{\partial F}{\partial u}\left(0+\frac{y\frac{\partial z}{\partial y}-z}{y^2}\right)+\frac{\partial F}{\partial v}\left(1+\frac{\frac{\partial z}{\partial y}}{x}\right)=0$，解得 $\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial v}-\frac{z}{y^2}\frac{\partial F}{\partial u}}{\frac{1}{y}\frac{\partial F}{\partial u}+\frac{1}{x}\frac{\partial F}{\partial v}}$，

所以 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$。

方法二。记 $G(x,y,z)=F\left(x+\frac{z}{y},y+\frac{z}{x}\right)$，则方程 $G(x,y,z)=0$ 确定隐函数 $z=z(x,y)$。

$$\frac{\partial G}{\partial x}=\frac{\partial}{\partial x}\left\{F\left(x+\frac{z}{y},y+\frac{z}{x}\right)\right\}=\frac{\partial F}{\partial u}(1+0)+\frac{\partial F}{\partial v}(0-\frac{z}{x^2})\text{，}$$

$$\frac{\partial G}{\partial y}=\frac{\partial}{\partial y}\left\{F\left(x+\frac{z}{y},y+\frac{z}{x}\right)\right\}=\frac{\partial F}{\partial u}(0-\frac{z}{y^2})+\frac{\partial F}{\partial v}(1+0)\text{，}$$

$$\frac{\partial G}{\partial z}=\frac{\partial}{\partial z}\left\{F\left(x+\frac{z}{y},y+\frac{z}{x}\right)\right\}=\frac{\partial F}{\partial u}(0+\frac{1}{y})+\frac{\partial F}{\partial v}(0+\frac{1}{x})\text{，}$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial z}}=-\frac{\frac{\partial F}{\partial u}-\frac{z}{x^2}\frac{\partial F}{\partial v}}{\frac{1}{y}\frac{\partial F}{\partial u}+\frac{1}{x}\frac{\partial F}{\partial v}}\text{，}\qquad \frac{\partial z}{\partial y}=-\frac{\frac{\partial G}{\partial y}}{\frac{\partial G}{\partial z}}=-\frac{\frac{\partial F}{\partial v}-\frac{z}{y^2}\frac{\partial F}{\partial u}}{\frac{1}{y}\frac{\partial F}{\partial u}+\frac{1}{x}\frac{\partial F}{\partial v}}\text{。。。。。。}$$

例13. 设函数 $x=x(z),\ y=y(z)$ 由方程组 $\begin{cases}x^2+y^2+z^2-1=0\\x^2+2y^2-z^2-1=0\end{cases}$ 确定，求

$\dfrac{dx}{dz}, \dfrac{dy}{dz}$.

解：解法一。 $\begin{cases} x^2+y^2=-z^2+1 \\ x^2+2y^2=z^2+1 \end{cases} \Rightarrow \begin{cases} 2x\dfrac{dz}{dx}+2y\dfrac{dz}{dy}=-2z \\ 2x\dfrac{dz}{dx}+4y\dfrac{dz}{dy}=2z \end{cases}$ 解方程得：

$$\begin{bmatrix} \dfrac{dx}{dz} \\ \dfrac{dy}{dz} \end{bmatrix} = -\frac{1}{4xy}\begin{bmatrix} 4y & -2y \\ -2x & 2x \end{bmatrix}\begin{bmatrix} 2z \\ -2z \end{bmatrix} = -\frac{1}{4xy}\begin{bmatrix} 12yz \\ -8xz \end{bmatrix}$$

由此得到 $\dfrac{dx}{dz}=\dfrac{3z}{x}, \dfrac{dy}{dz}=-\dfrac{2z}{y}$.

解法二：记 $\begin{cases} F_1(x,y,z)=x^2+y^2+z^2-1 \\ F_2(x,y,z)=x^2+2y^2-z^2-1 \end{cases}$ ，则

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2x & 4y \end{pmatrix}, \qquad \begin{pmatrix} \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2z \\ -2z \end{pmatrix}$$

$$\begin{pmatrix} \dfrac{dx}{dz} \\ \dfrac{dy}{dz} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial z} \end{pmatrix} = -\begin{pmatrix} 2x & 2y \\ 2x & 4y \end{pmatrix}^{-1} \begin{pmatrix} 2z \\ -2z \end{pmatrix} = \text{略。}$$

例14. 已知函数 $z=z(x,y)$ 由参数方程: $\begin{cases} x=u\cos v \\ y=u\sin v \\ z=uv \end{cases}$ ，给定，试求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解 这个问题涉及到复合函数微分法与隐函数微分法. x, y 是自变量, u, v 是中间变量 (u, v 是 x, y 的函数), 先由 $z=uv$ 得到

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}$$

u, v 是由方程 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 的 x, y 的隐函数，在这两个等式两端分别关于 x, y 求偏导数，得

$$\begin{cases} 1=\cos v\dfrac{\partial u}{\partial x}-u\sin v\dfrac{\partial v}{\partial x} \\ 0=\sin v\dfrac{\partial u}{\partial x}+u\cos v\dfrac{\partial v}{\partial x} \end{cases}, \qquad \begin{cases} 0=\cos v\dfrac{\partial u}{\partial y}-u\sin v\dfrac{\partial v}{\partial y} \\ 1=\sin v\dfrac{\partial u}{\partial y}+u\cos v\dfrac{\partial v}{\partial y} \end{cases}$$

得到 $\dfrac{\partial u}{\partial x}=\cos v, \dfrac{\partial v}{\partial x}=\dfrac{-\sin u}{u}, \dfrac{\partial u}{\partial y}=\sin v, \dfrac{\partial v}{\partial x}=\dfrac{\cos v}{u}$

将这个结果代入前面的式子，得到

$$\frac{\partial z}{\partial x}=v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}=v\cos v-\sin v$$

与
$$\frac{\partial z}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}=v\sin v+\cos v$$

例 15. 函数$u=u(x,y)$由方程$\begin{cases}u=f(x,y,z,t)\\ g(y,z,t)=0\\ h(z,t)=0\end{cases}$ 确定，求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$

解： 函数关系分析: 5（变量）– 3（方程)=2(自变量)；

一函（u)，二自（ x， y ），二中（ z， t ）

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x},\qquad \frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial y}$$

$$\begin{pmatrix}\dfrac{\partial z}{\partial y}\\ \dfrac{\partial t}{\partial y}\end{pmatrix}=\left(\left|\frac{\partial(g,h)}{\partial(z,t)}\right|\right)^{-1}\begin{pmatrix}\dfrac{\partial h}{\partial t} & -\dfrac{\partial g}{\partial t}\\ -\dfrac{\partial h}{\partial z} & \dfrac{\partial g}{\partial z}\end{pmatrix}\begin{pmatrix}-\dfrac{\partial g}{\partial t}\\ 0\end{pmatrix},\quad \frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\left(\dfrac{\partial f}{\partial t}\dfrac{\partial h}{\partial z}-\dfrac{\partial f}{\partial z}\dfrac{\partial h}{\partial t}\right)\dfrac{\partial g}{\partial y}}{\dfrac{\partial g}{\partial z}\dfrac{\partial h}{\partial t}-\dfrac{\partial g}{\partial t}\dfrac{\partial h}{\partial z}}.$$