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Ex. 1-1-1

① For $A_{2k\times 2k}$, $A=\begin{bmatrix} J_2 & & \\ & J_2 & \\ & & \ddots \\ & & & J_2 \end{bmatrix}$ (k) s.t. $A^2=\begin{bmatrix} J_2^2 & & \\ & \ddots & \\ & & J_2^2 \end{bmatrix}=-I$

② $A^2=-I \Rightarrow$ the eigen of A^2 are all -1.

$\Rightarrow$ the eigenvalues of A a i and $-i$

but we only have real bers in A so i and $-i$ must come in pairs considering trace$(A)=$ $\cdots+\lambda_n$, which is impossible for $A_{n\times n}$ (n is odd number)

Ex. 1.1.2

① suppose $k=a+bi$

$B(kv)=B(a+bi)v=B(aI+bA)v=aBv+bBAv$

$kB(v)=(a+bi)Bv=(aI+bA)Bv=aBv+bABv$

$B(kv)=kB(v) \longleftarrow BA=AB$

② $\chi(kv)=i(a+bi)v=(-b+ai)v$

$k\chi(v)=(a+bi)i\vec{v}=(-b+a\vec{v})v$

so χ must be complex linear

$A_{m\times n}$ $B_{n\times s}$

$r(AB)\le r(A)+r(B)-n$

proof. $\begin{pmatrix} A & AB \\ I_{n\times n} & 0 \end{pmatrix}\begin{pmatrix} I & B \\ 0 & -I \end{pmatrix}=\begin{pmatrix} A & 0 \\ I & B \end{pmatrix}$ ← invertible

$r=n+r(AB)$ $\quad$ $r\ge r(A)+r(B)$

$\Rightarrow n-r(AB)\ge r(A)+r(B)$

③ $C^2=I$, $(C+I)(C-I)=0$

$r((C+I)(C-I))=0$

Inequality 1: $r(C+I)+r(C-I)\le n+r(C^2-I)$

$r(C+I)+r(C-I)\le n$

Inequality 2: $r(C+I)+r(C-I)$

$=r(C+I)+r(I-C)\ge r(2I)=n$

so we know $r(C+I)+r(I-C)=n$

$\dim_{\mathbb{R}}(N(C+I)+N(C-I))=n$

suppose $(C+I)\vec{v}=0$ has k solutions, the $(C-I)\vec{v}=0$ has $n-k$ solutions.

$\Rightarrow$ C is diagonalisable.

Considering $C\vec{x}=-\vec{x}$, multiply by A: $-ACx=Ax$

$CAx=Ax$

then $A\vec{x}$ is an eigenvector for eigenvalue 1.

Considering $C\vec{x}=\vec{x}$, multiply by A: $AC\vec{x}=A\vec{x}$

$CA\vec{x}=-A\vec{x}$

then $A\vec{x}$ is an eigenvector for -1

thus we know $C\vec{x}=\vec{x}$ and $C\vec{x}=-\vec{x}$ come in pairs.

their eigenspaces have the same dimension.

④ $A=\underbrace{\begin{pmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{pmatrix}}_{k}$ $\quad J_2=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$n=2k$

$CA=AC$

t come in pairs
or $A_{n\times n}$
n is odd number)

)

A)+r(B)
A)+r(B)

Ex 1.1.3

① for $\bar{x}\begin{pmatrix}a_1+b_1 i\\ \vdots\\ a_n+b_n i\end{pmatrix}$

if $k\in\mathbb{R}$, $C(kx)=\begin{pmatrix}k(a_1-b_1 i)\\ \vdots\\ k(a_n-b_n i)\end{pmatrix}$

$kC(x)=\begin{pmatrix}k(a_1-b_1 i)\\ \vdots\\ k(a_n-b_n i)\end{pmatrix}$ C is real linear

if $k\in\mathbb{C}$
$k=y+zi$

$C(kx)=C\begin{pmatrix}a_1y-b_1z+(b_1y+a_1z)i\\ \vdots\\ a_ny-b_nz+(b_ny+a_nz)i\end{pmatrix}=\begin{pmatrix}a_iy-b_iz-(b_iy+a_iz)i\\ \vdots\end{pmatrix}$

$kC(x)=(y+zi)\begin{pmatrix}a_1-b_1 i\\ \vdots\\ a_n-b_n i\end{pmatrix}=\begin{pmatrix}a_iy+b_iz-(b_iy-a_iz)i\end{pmatrix}$

apparently $kC(x)\neq C(kx)$

C is not complex linear

② $\mathbb{C}$-linear implies $\underbrace{\mathbb{R}\text{-linear}}_{\text{special occassion of } b=0}$

③ $\mathbb{C}^2$: $\mathbb{R}$-basis: $\begin{pmatrix}i\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$\mathbb{C}$-basis: $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$\mathbb{R}$-dimension: 4

$\mathbb{C}$-dimension: 2

④ $\mathbb{R}$-linearly indep-: $a_1\vec{v}_1+\dots+a_n\vec{v}_n=0$ implies $a_1=\dots=a_n=0$
(if $a_1,\dots a_n\in\mathbb{R}$)

$\mathbb{C}$-linearly indep-: (if $a_1,\dots a_n\in\mathbb{C}$)

$\mathbb{C}$-linearly independent implies $\mathbb{R}$-linearly independent.

⑤ Similarly, $\mathbb{C}$-spanning implies $\mathbb{R}$-spanning

Ex. 1.1.4 $P=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$ $F_4=\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix}$

① $P\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $P\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}i\\-1\\-i\\1\end{bmatrix}$

② $PF_4=\begin{pmatrix}1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\\1&1&1&1\end{pmatrix}=F_4\begin{pmatrix}1&&&\\&-i&&\\&&-1&\\&&&-i\end{pmatrix}$

$F_4^{-1}=\frac{1}{4}F_4$

$PF_4=F_4D \Rightarrow P=F_4DF_4^{-1}$

$\lambda_1=1,\ \vec{v}_1=(1,1,1,1)^T$

$\lambda_2=-1,\ \vec{v}_2=(-1,1,-1,1)^T$

$\lambda_3=-i,\ \vec{v}_3=(i\ -1\ -i\ 1)^T$

$\vec{v}_4=(-i\ -1\ i\ 1)^T$

③ $C\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}\sum_{i=0}^{3}c_i\\ \vdots\\ \sum c_i\end{bmatrix}$

$C\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}c_0-c_2+(c_1-c_3)i\\ c_3-c_1+(c_0-c_2)i\\ c_2-c_0+(c_3-c_1)i\\ c_1-c_3+(c_2-c_0)i\end{bmatrix}$

④ $C=c_1P+c_2P^2+c_3P^3+c_0P^4$