赵晨阳 软01 2020012363

习题4.2：

3.(1) $t=0$时，(x,y,z)点在$(0,0,0)$ $t=1$时，(x,y,z)点在$(3,3,2)$

$x'(t)=3$，$y'(t)=6t$ $z'(t)=6t^2$

$I=\int_L dL=\int_0^1\sqrt{9+36t^2+36t^4}\,dt=3\int_0^1(2t^2+1)dt=3\cdot\left(\frac{2}{3}t^3+t\right)\Big|_0^1=3\cdot\frac{5}{3}=5$

3.(2) $x'=-e^{-t}(\cos t+\sin t)$ $y'=-e^{-t}\cdot\sin t+e^{-t}\cos t$ $z'=-e^{-t}$

$dL=\sqrt{(x')^2+(y')^2+(z')^2}\,dt=e^{-t}\sqrt{1+2\sin^2t+2\cos^2t}\,dt=\sqrt{3}e^{-t}dt$

$l=\sqrt{3}\int_0^{+\infty}e^{-t}dt=-\sqrt{3}(e^{-t})\Big|_0^{+\infty}=\sqrt{3}$

4. 参数方程$\begin{cases}x=x\\y=\ln x\end{cases}$ $dx=dx$ $dy=\frac{1}{x}dx$ $dL=\sqrt{(x')^2+(y')^2}\,dx=\sqrt{1+\frac{1}{x^2}}\,dx$

$M=\int_L\rho\,dl=\int_{\sqrt{3}}^{\sqrt{15}}x^2\sqrt{1+\frac{1}{x^2}}\,dx=\int_{\sqrt{3}}^{\sqrt{15}}x\sqrt{x^2+1}\,dx=\frac{1}{2}\int_{\sqrt{3}}^{\sqrt{15}}\sqrt{x^2+1}\,dx^2=\frac{1}{2}\int_3^{15}\sqrt{t+1}\,dt=\frac{1}{2}\cdot\frac{2}{3}\cdot(t+1)^{\frac{3}{2}}\Big|_3^{15}$

$=\frac{1}{3}\times\left(16^{\frac{3}{2}}-4^{\frac{3}{2}}\right)=\frac{56}{3}$

5. 记 $L=\begin{cases}x^2+y^2=a^2\\z=0\end{cases}$ $\sigma(s)=\oint_L a+\frac{x^2}{a}dL$，$L$参数为$\begin{cases}x=a\cos\theta\\y=a\sin\theta\\z=0\end{cases}$ $(\theta\in[0,2\pi])$ $dL'=\sqrt{x'^2+y'^2+z'^2}\,d\theta$ $=a\,d\theta$

且参数θ与曲线L增长方向一致

$\sigma(s)=a^2\int_0^{2\pi}(1+\cos^2\theta)\,d\theta=a^2\int_0^{2\pi}1+\frac{1+\cos2\theta}{2}d\theta=3a^2\pi+\frac{a^2}{4}\cdot\sin2\theta\Big|_0^{2\pi}=3a^2\pi$

6. 设线密度为ρ，则 $\bar{x}=\frac{\int_L x\rho\,dL}{\int_L\rho\,dL}=\frac{\int_L x\,dL}{\int_L dL}$ 同理 $\bar{y}=\frac{\int_L y\,dL}{\int_L dL}$

$x'(t)=a(1-\cos t)$ $y'(t)=a\sin t$. $dl=\sqrt{(x')^2+(y')^2}\,dt=\sqrt{2}a\sqrt{1-\cos t}\,dt$

$\int_L x\,dL=\sqrt{2}a^2\int_0^{\pi}(t-\sin t)\sqrt{1-\cos t}\,dt=\sqrt{2}a^2\int_0^{\pi}(t-\sin t)\sqrt{2}\cdot\sin\frac{t}{2}\,dt$

$=2a^2\int_0^{\pi}(t-\sin t)\sin\frac{t}{2}dt$

$\int_0^{\pi}t\sin\frac{t}{2}dt=-2t\cos\frac{t}{2}\Big|_0^{\pi}+\int_0^{\pi}2\cos\frac{t}{2}dt=4\int_0^{\pi}\cos\frac{t}{2}d\frac{t}{2}=4\sin\frac{t}{2}\Big|_0^{\pi}=4$

$\int_0^{\pi}\sin t\cdot\sin\frac{t}{2}dt=2\int_0^{\pi}\sin^2\frac{t}{2}\cdot\cos\frac{t}{2}dt=4\int_0^{\pi}\sin^2\frac{t}{2}\,d\sin\frac{t}{2}=4\int_0^1x^2dx=\frac{4}{3}$

$\therefore\int_L x\,dL=2a^2\left(4-\frac{4}{3}\right)=\frac{16}{3}a^2$

$\int_L dl=\sqrt{2}a\int_0^{\pi}\sqrt{1-\cos t}\,dt=4a\int_0^{\pi}\sin\frac{t}{2}d\frac{t}{2}=-4a\cos\frac{t}{2}\Big|_0^{\pi}=4a$

$\therefore\bar{x}=\frac{\frac{16}{3}a^2}{4a}=\frac{4}{3}a$

同理：$\int_L y\,dL=\sqrt{2}a^2\int_0^{\pi}(1-\cos t)\sqrt{1-\cos t}\,dt=8a^2\int_0^{\pi}\sin^3\frac{t}{2}d\frac{t}{2}=8a^2\int_0^{\frac{\pi}{2}}\sin^3y\,dy$

$=8a^2\cdot\frac{2!!}{3!!}=\frac{16}{3}a^2$

$\bar{y}=\frac{\int_L y\,dL}{\int_L dL}=\frac{4}{3}a$ $\therefore$质心为$\left(\frac{4}{3}a,\frac{4}{3}a\right)$

7. $x'=-a\sin t$ $y'=a\cos t$ $z'=\frac{b}{2\pi}$ $dl=\sqrt{a^2+\frac{b^2}{4\pi^2}}\,dt$ $J_x=\int_L(y^2+z^2)dL=\sqrt{a^2+\frac{b^2}{4\pi^2}}\int_0^{2\pi}(a^2\sin^2 t+\frac{b^2t^2}{4\pi^2})dt$

$\int_0^{2\pi}\sin^2 t\,dt=4\cdot\int_0^{\frac{\pi}{2}}\sin^2 t\,dt=4\cdot\frac{1}{2!!}\cdot\frac{\pi}{2}=\pi$

$\int_0^{2\pi}t^2dt=\frac{t^3}{3}\Big|_0^{2\pi}=\frac{8}{3}\pi^3$ $\therefore J_x=\sqrt{a^2+\frac{b^2}{4\pi^2}}(\pi a^2+\frac{2}{3}\pi b^2)=\sqrt{4\pi^2a^2+b^2}\,(\frac{a^2}{2}+\frac{b^2}{3})$

习题4.4

2.(1) x的增长方向与L^+同向.

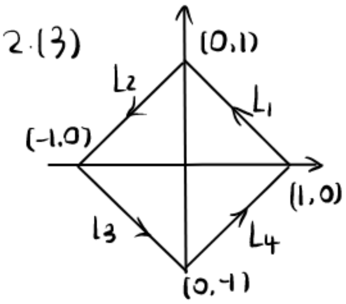
$\int_{L^+}(x^2-y^2)dx=\int_0^2(x^2-x^4)dx=(\frac{x^3}{3}-\frac{x^5}{5})\Big|_0^2=\frac{8}{3}-\frac{32}{5}=-\frac{56}{15}$

2.(2) 参考: $\begin{cases}x=a\cos t\\ y=a\sin t\end{cases}$ $t\in[0,2\pi]$ 且t的增长方向与L^+同向.

原式$=\int_0^{2\pi}\frac{-a^2(\cos t+\sin t)\cdot\sin t+a^2(\sin t-\cos t)\cos t}{a^2}dt=\int_0^{2\pi}-\sin t\cos t-\sin^2 t+\sin t\cos t-\cos^2 t\,dt$

$=-2\pi$

2.(3)



$L_1\begin{cases}y=1-x\\ x=x\end{cases}$ x的增长方向与L_1反向. $x\in[0,1]$

$\int_{L_1}\frac{dx+dy}{|x|+|y|}=\int_1^0 dx+d(1-x)=\int_1^0dx-\int_1^0dx=0$

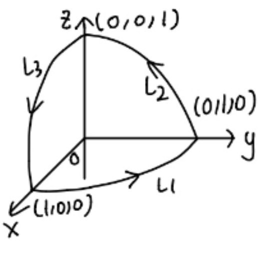
$L_2\begin{cases}y=1+x\\ x=x\end{cases}$ $x\in[-1,0]$ 参数增加方向与曲线相反

$\int_{L_2^+}\frac{dx+dy}{|x|+|y|}=\int_0^{-1}dx+d(1-x)=2\int_0^{-1}dx=-2$

$L_3\begin{cases}y=-1-x\\ x=x\end{cases}$, $x\in[-1,0]$ x的增长方向与L_1同向. $\int_{L_3}\frac{dx+dy}{|x|+|y|}=\int_0^{-1}dx-dx=0$

$L_4\begin{cases}y=x-1\\ x=x\end{cases}$, $x\in[0,1]$ x的增长方向与L_1同向. $\int_{L_4}\frac{dx+dy}{|x|+|y|}=2\int_0^1dx=2$

$\therefore \oint_{L^+}\frac{dx+dy}{|x|+|y|}=2-2=0$



(4) $L_1\begin{cases}x=\cos t\\ y=\sin t\\ z=0\end{cases}$ $t\in[0,\frac{\pi}{2}]$ t的增长方向与L_1同向.

$\int_{L_1^+}(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz=\int_0^{\frac{\pi}{2}}\sin^2t(-\sin t)-\cos^3t\,dt$

$=-\int_0^{\frac{\pi}{2}}(\sin^3t+\cos^3t)dt=-(\frac{2!!}{3!!}+\frac{2!!}{3!!})=-\frac{4}{3}$

$L_2\begin{cases}x=0\\ y=\cos t\\ z=\sin t\end{cases}$ $t\in[0,\frac{\pi}{2}]$ t的增长方向与L_2同向.

$\int_{L_2^+}(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz=\int_0^{\frac{\pi}{2}}\sin^2t(-\sin t)-\cos^3t\,dt$

$=-\int_0^{\frac{\pi}{2}}(\sin^3t+\cos^3t)dt=-(\frac{2!!}{3!!}+\frac{2!!}{3!!})=-\frac{4}{3}$

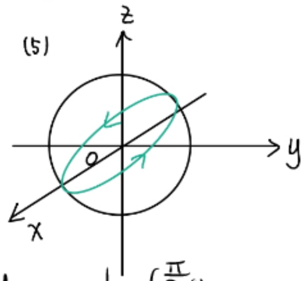
$L_3\begin{cases}x=\sin t\\ y=0\\ z=\cos t\end{cases}$ $t\in[0,\frac{\pi}{2}]$ t的增长方向与L_3同向.

$\int_{L_3^+}(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz=\int_0^{\frac{\pi}{2}}\sin^2t(-\sin t)-\cos^3t\,dt$

$=-\int_0^{\frac{\pi}{2}}(\sin^3t+\cos^3t)dt=-(\frac{2!!}{3!!}+\frac{2!!}{3!!})=-\frac{4}{3}$

$\therefore \oint_L (y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz = -\frac{4}{3}\times 3 = -4$

(5) 代入边界条件 $z=y$ 到 $x^2+y^2+z^2=1$ 有 $x^2+2y^2=1$



令 $\begin{cases} x=\cos t \\ y=\sin t/\sqrt{2} \\ z=\sin t/\sqrt{2} \end{cases}$ $t\in[0,2\pi]$ t的增长方向与L同向.

$$\int_{L^+} xyz\,dz = \frac{1}{2}\int_0^{2\pi}\cos t\cdot\sin^2 t\cdot\frac{\cos t}{\sqrt{2}}dt = \frac{1}{2\sqrt{2}}\cdot\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

$$= \frac{1}{16\sqrt{2}}\int_0^{2\pi}\sin^2 2t\,d2t = \frac{1}{16\sqrt{2}}\int_0^{4\pi}\sin^2 y\,dy = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^2 y\,dy = \frac{1}{4\sqrt{2}}\int_0^{\frac{\pi}{2}}(1-\cos 2y)dy$$

$$= \frac{\sqrt{2}}{16}\pi - \frac{1}{8\sqrt{2}}\sin 2y\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{16}\pi$$

4.(1) $|\vec{F}| = \sqrt{x^2+y^2}$ 方向为 $(-x,-y)$ 即 $\vec{F} = \begin{pmatrix}-x\\-y\end{pmatrix}$

记弧为 L_1 $\begin{cases} x=a\cos t \\ y=b\sin t \end{cases}$ $t\in[0,\frac{\pi}{2}]$ t增加方向与L同向

$$\int_{L_1^+}\vec{F}\,d\vec{r} = \int_0^{\frac{\pi}{2}}(-x,-y)\cdot(x'(t),y'(t))dt = -\int_0^{\frac{\pi}{2}} a^2\cos t(-\sin t)+b^2\sin t\cos t\,dt$$

$$= (a^2-b^2)\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt = \frac{a^2-b^2}{4}\int_0^{\frac{\pi}{2}}\sin 2t\,d2t = \frac{a^2-b^2}{4}\int_0^{\pi}\sin y\,dy = \frac{a^2-b^2}{4}(-\cos y)\Big|_0^{\pi} = \frac{a^2-b^2}{2}$$

(2) 记一周弧长为L. L: $\begin{cases} x=a\cos t \\ y=b\sin t \end{cases}$ $t\in[0,2\pi]$ t的增长方向与L同向.

$$\int_{L^+}\vec{F}\,d\vec{r} = \int_0^{2\pi}(-x(t),-y(t))\cdot(x'(t),y'(t))dt = (a^2-b^2)\int_0^{2\pi}\sin t\cos t\,dt = \frac{a^2-b^2}{4}\int_0^{2\pi}\sin^2 2t\,d2t$$

$$= \frac{a^2-b^2}{4}\int_0^{4\pi}\sin y\,dy = \frac{a^2-b^2}{4}(-\cos y)\Big|_0^{4\pi} = 0 \quad \therefore W=0$$