1.3 HW3 Jordan Canonical Form

Excercise 1.3.1.

1. Choose basis $\{\mathbf{e}_i\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathrm{i} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{i} \end{bmatrix} \right\}$, and then

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Therefore $\det(A - \lambda I) = (1-\lambda)(-1-\lambda)^3$, i.e. A has eigenvalue $\lambda_1 = 1$ and $\lambda_2 = -1$ with algebraic mutiplicity 1 and 3 respectively. For λ_1 we have

$$\mathrm{Ker}(A - \lambda_1 I) = \mathrm{Ker} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} = \mathrm{Span} \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

And for λ_2,

$$\mathrm{Ker}(A - \lambda_2 I) = \mathrm{Ker} \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathrm{Span} \left\{ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 4 \end{bmatrix} \right\},$$

$$\mathrm{Ker}(A - \lambda_2 I)^2 = \mathrm{Ker} \begin{bmatrix} 4 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{Ker}(A - \lambda_2 I) \oplus \mathrm{Span} \left\{ \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Note that $\mathbf{v}_3 = (A - \lambda_2 I)\mathbf{v}_4$, therefore the Jordan canonical form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

under the new basis $\{\mathbf{v}_i\}$, or equally $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{i} \end{bmatrix}, \begin{bmatrix} 2 \\ 4+4\mathrm{i} \end{bmatrix}, \begin{bmatrix} 1+4\mathrm{i} \\ 0 \end{bmatrix} \right\}$.

2. Choose basis $\{\mathbf{e}_i\} = \{1, x, x^2, x^3, x^4\}$, and then

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $\det(A - \lambda I) = \lambda^2(1-\lambda)(\sqrt{2}-\lambda)(-\sqrt{2}-\lambda)$, i.e. A has eigenvalue $\lambda_1 = 0$ with algebraic mutiplicity 2 and eigenvalues $\lambda_{2,3,4} = 1, \pm\sqrt{2}$ with algebraic mutiplicity 1 respectively. For λ_1 we have

$$\mathrm{Ker}(A - \lambda_1 I) = \mathrm{Ker} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{Span} \left\{ \mathbf{v}_1 = \begin{bmatrix} 12 \\ -12 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\},$$

$$\mathrm{Ker}(A-\lambda_1 I)^2 = \mathrm{Ker}\begin{bmatrix}1&1&2&0&0\\0&2&0&6&0\\0&0&2&0&12\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} = \mathrm{Ker}(A-\lambda_1 I)\oplus \mathrm{Span}\left\{\mathbf{v}_2=\begin{bmatrix}12\\0\\-6\\0\\1\end{bmatrix}\right\}.$$

And for $\lambda_{2,3,4}$,

$$\mathrm{Ker}(A-\lambda_2 I) = \mathrm{Ker}\begin{bmatrix}0&1&0&0&0\\0&-1&2&0&0\\0&1&-1&3&0\\0&0&0&-1&4\\0&0&0&0&-1\end{bmatrix} = \mathrm{Span}\left\{\mathbf{v}_3=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right\},$$

$$\mathrm{Ker}(A-\lambda_3 I) = \mathrm{Ker}\begin{bmatrix}1-\sqrt{2}&1&0&0&0\\0&-\sqrt{2}&2&0&0\\0&1&-\sqrt{2}&3&0\\0&0&0&-\sqrt{2}&4\\0&0&0&0&-\sqrt{2}\end{bmatrix} = \mathrm{Span}\left\{\mathbf{v}_4=\begin{bmatrix}2+\sqrt{2}\\\sqrt{2}\\1\\0\\0\end{bmatrix}\right\},$$

$$\mathrm{Ker}(A-\lambda_4 I) = \mathrm{Ker}\begin{bmatrix}1+\sqrt{2}&1&0&0&0\\0&\sqrt{2}&2&0&0\\0&1&\sqrt{2}&3&0\\0&0&0&\sqrt{2}&4\\0&0&0&0&\sqrt{2}\end{bmatrix} = \mathrm{Span}\left\{\mathbf{v}_5=\begin{bmatrix}2-\sqrt{2}\\-\sqrt{2}\\1\\0\\0\end{bmatrix}\right\}.$$

Note that $\mathbf{v}_1=(A-\lambda_1 I)\mathbf{v}_2$, therefore the Jordan canonical form of A is

$$\begin{bmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&\sqrt{2}&0\\0&0&0&0&-\sqrt{2}\end{bmatrix}$$

under the new basis $\{\mathbf{v}_i\}$, or equally $\{12-12x+4x^3, 12-6x^2+x^4, 1, 2+\sqrt{2}+\sqrt{2}x+x^2, 2-\sqrt{2}-\sqrt{2}x+x^2\}$.

3. $\det(A-\lambda I)=(\lambda^2-a_1a_4)(\lambda^2-a_2a_3)$, i.e. (in terms of complex field) A has eigenvalues $\lambda_{1,2}=\pm\sqrt{a_1a_4}$, and $\lambda_{3,4}=\pm\sqrt{a_2a_3}$, each with algebraic mutiplicity 1.

   Consider the invariant subspace Span $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$, where the submap A_1 is represented as

$$\begin{bmatrix}0&a_1\\a_4&0\end{bmatrix}.$$

   There are three different cases depending on the values of a_1 and a_4.

   1) If both are non-zero, then we have the Jordan form of A_1 as $\begin{bmatrix}\sqrt{a_1a_4}&0\\0&-\sqrt{a_1a_4}\end{bmatrix}$, under the basis $\left\{\begin{bmatrix}\sqrt{a_1}\\0\\0\\\sqrt{a_4}\end{bmatrix},\begin{bmatrix}\sqrt{a_1}\\0\\0\\-\sqrt{a_4}\end{bmatrix}\right\}$. That's actually two Jordan blocks.

   2) If both are zero, then we have the Jordan form of A_1 as $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, under the basis $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

      That's actually two Jordan blocks, too.

3) If one of them say a_4 is 0 and the other is not, then we have the Jordan form of A_1 as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

under the basis $\left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The submap A_2 in subspace Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is similar. Hence we get the final answer by directly combining them.

Excercise 1.3.2.

1. One is the transpose of the other, because for any a_i, J_{a_i} contributes one to each of $\{\dim \operatorname{Ker}(A^j) - \dim \operatorname{Ker}(A^{j-1}) \,|\, 1 \leq j \leq a_i\}$, like this (suppose $\{a_i\}$ is in the order from large to small):

$$\begin{array}{r c} \text{sum} & \begin{array}{cccc} a_1 & a_2 & \cdots & a_k \end{array} \\ \begin{array}{r} \dim \operatorname{Ker}(A) \\ \dim \operatorname{Ker}(A^2) - \dim \operatorname{Ker}(A) \\ \vdots \\ \dim \operatorname{Ker}(A^{a_1}) - \dim \operatorname{Ker}(A^{a_1 - 1}) \end{array} & \begin{bmatrix} \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \\ \vdots & \vdots & & \\ \bullet & & & \end{bmatrix} \end{array}.$$

2. A partition of n is self-conjugate if and only if the dot diagram is symmetric, so the number of dots either in the first column or in the first row is $b_1 = 2a_1 - 1$ in total, which is odd. Now get rid of the first column and the first row and we get a new symmetric diagram. Do it repeatedly and we have the second odd number $b_2 = 2(a_2 - 1) - 1 < b_1$, the third $b_3 = 2(a_3 - 2) - 1 < b_2$ ... until no dots left. By this means each self-conjugate partition $\{a_i\}$ gives a partition into distinct odd numbers $\{b_i\}$, and vice versa, so the numbers of the two partitions are equal.

3. It *almost surely* corresponds to the partition $4 = 4$. Just look at A^3, whose $(1, 4)$ entry is a non-trivial and complicated polynomial of all (i, j) entries of A where $i < j$, and thus it is *almost never* zero. On the other hand, A^3 is not a zero matrix if and only if the Jordan canonical form of A corresponds to the partition $4 = 4$.