重积分习题课题目解答

第 1 部分 课堂内容回顾

1. 重积分的概念及其性质

(1) $\mathbb{R}^n$ **中的坐标平行体上的积分:** $\mathbb{R}^n$ 中的区间或者坐标平行体及其体积, 分割, 步长, 带点分割, Riemann 和, 重积分, Riemann 可积.

(2) **有界集上的函数的 Riemann 积分:** 零延拓成坐标平行体上的函数, 再研究其积分. 有界集 Ω 上所有 Riemann 可积函数的全体记作 $\mathscr{R}(\Omega)$,

(3) **二重积分的几何意义:** 立体的体积.

(4) **Jordan 可测集:** 定义, 典型的 Jordan 可测集.

(5) **典型的 Riemann 可积函数:** 如果有界闭集 $\Omega \subset \mathbb{R}^n$ 为 Jordan 可测集, 则我们有 $\mathscr{C}(\Omega) \subset \mathscr{R}(\Omega)$.

(6) **Jordan 可测集上重积分的性质:** 有界性, 线性, 区域可加性, (严格) 保号性, (严格) 保序性, 绝对值不等式, 积分的上、下界, 积分中值定理及其应用, 变量替换.

2. 重积分的计算

(1) 直角坐标系下二重积分的累次积分法,

(2) 极坐标坐标系下二重积分的累次积分法,

(3) 直角坐标系下三重积分的累次积分法,

(4) 柱坐标系下三重积分的累次积分法,

(5) 球坐标系下三重积分的累次积分法,

(6) 一般坐标变换: 目的在于转化成累次积分,

(7) **对称性在重积分计算当中的应用.**

3. 重积分应用: 质心、重心、形心, 曲面面积.

第 2 部分 习题课题目

§1. 二重积分

1. 设A是实二阶对称矩阵，我们假设A是正定矩阵，试确定有界闭区域$\Omega \subset \mathbb{R}^2$，使得二重积分

$$I = \int_{\Omega} (1 - x^t Ax) dx$$

的值取得最大值，其中$x = (x_1, x_2)^t$是二维列向量，$dx = dx_1 dx_2$.

证明: 我们记I_Ω是集合Ω的示性函数，即I_Ω取值为$0, 1$，且$I_\Omega(x) = 1$当且仅当$x \in \Omega$. 此时，我们得到

$$I = \int_{\mathbb{R}^2} I_\Omega(x) \cdot (1 - x^t Ax) dx,$$

通过划分$f(x) = (1 - x^t Ax) \cdot I_\Omega(x)$的值域，我们得到$\mathbb{R}^2$中的集合$V_+ = \{y : f(y) > 0\}$, $V_0 = \{y : f(y) = 0\}$ 与$V_- = \{y : f(y) < 0\}$，我们得到

$$I = \int_{V_+} f(x) dx + \int_{V_-} f(x) dx,$$

注意到上面的积分等式右端的第一个积分非负而第二个积分非正，再注意到二次型$x^t Ax$是x的连续函数，从而为使得积分值I最大，我们知开区域Ω°包含集合

$$\{y : 1 - y^t Ay > 0\},$$

且与集合

$$\{y : 1 - y^t Ay < 0\}$$

的交非空. 最后由配方法得到，$\{y : 1 - y^t Ay = 0\}$是一个椭圆或者圆，由教材*P122*例*3.1.1*知其为二维零面积集*(*或者*Jordan*可测集*)*，从而我们所求的有界闭区域为

$$\Omega = \{y \in \mathbb{R}^2 : 1 - y^t Ay \geq 0\}.$$

2. 改变下述累次积分的积分次序*:*

(1) $\displaystyle\int_0^1 \Big(\int_0^{x^2} f(x, y)\, \mathrm{d}y \Big) \mathrm{d}x + \int_1^3 \Big(\int_0^{\frac{1}{2}(3-x)} f(x, y)\, \mathrm{d}y \Big) \mathrm{d}x;$

(2) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta) r\, \mathrm{d}r \Big)\, \mathrm{d}\theta.$

解: (1) 由题设可知积分区域中的点 (x, y) 满足

$$0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2,\ \text{或}\ 1 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant \frac{1}{2}(3 - x),$$

而这等价于说 $0 \leqslant y \leqslant 1$, $\sqrt{y} \leqslant x \leqslant 3-2y$. 于是

$$\int_0^1 \Big(\int_0^{x^2} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x + \int_1^3 \Big(\int_0^{\frac{1}{2}(3-x)} f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x = \int_0^1 \Big(\int_{\sqrt{y}}^{3-2y} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y.$$

(2) 由题设立刻可知

$$\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\Big)\,\mathrm{d}\theta \\
= &\iint\limits_{\substack{-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2},\\ 0\leqslant r\leqslant 2\cos\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\mathrm{d}\theta \\
= &\iint\limits_{\substack{0\leqslant r\leqslant 2,\\ -\arccos\frac{r}{2}\leqslant\theta\leqslant\arccos\frac{r}{2}}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\mathrm{d}\theta \\
= &\int_0^2 \Big(\int_{-\arccos\frac{r}{2}}^{\arccos\frac{r}{2}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}\theta\Big)\,\mathrm{d}r.
\end{aligned}$$

3. 假设 $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, 而 $f \in \mathscr{C}[-1,1]$, 求证:

$$\iint\limits_{x^2+y^2\leqslant 1} f(ax+by)\,\mathrm{d}x\mathrm{d}y = 2\int_{-1}^1 \sqrt{1-u^2} f(\sqrt{a^2+b^2}u)\,\mathrm{d}u.$$

证明: $\forall (x,y) \in \mathbb{R}^2$, 作变换

$$u = \frac{ax+by}{\sqrt{a^2+b^2}},\ v = \frac{-bx+ay}{\sqrt{a^2+b^2}}.$$

上述线性变换为正交变换, 因此该变换及其逆连续可导且

$$u^2+v^2 = x^2+y^2.$$

另外, 我们还有

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{vmatrix} = 1.$$

于是由变量变换公式立刻可得

$$\begin{aligned}
\iint\limits_{x^2+y^2\leqslant 1} f(ax+by)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{u^2+v^2\leqslant 1} f(\sqrt{a^2+b^2}u)\,\mathrm{d}u\mathrm{d}v \\
&= \int_{-1}^1 \Big(\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(\sqrt{a^2+b^2}u)\,\mathrm{d}v\Big)\mathrm{d}u \\
&= 2\int_{-1}^1 \sqrt{1-u^2} f(\sqrt{a^2+b^2}u)\,\mathrm{d}u.
\end{aligned}$$

4. 计算 $I = \iint\limits_D \frac{1}{\sqrt{x^2+y^2}}\Big(y\frac{\partial f}{\partial x}(x,y) - x\frac{\partial f}{\partial y}(x,y)\Big)\,\mathrm{d}x\mathrm{d}y$, 其中

$$D = \Big\{(x,y) \mid x^2+y^2 \leqslant R^2\Big\},\ R > 0.$$

解: 作极坐标变换 $\begin{cases} x=\rho\cos\varphi, \\ y=\rho\sin\varphi. \end{cases}$ 在此变换下, 积分区域 D 变为

$$D'=\Big\{(\rho,\varphi)\mid 0\leqslant\rho\leqslant R,\ 0\leqslant\varphi\leqslant 2\pi\Big\}.$$

$\forall(\rho,\varphi)\in D'$, 定义 $F(\rho,\varphi)=f(\rho\cos\varphi,\rho\sin\varphi)$, 则我们有

$$\begin{aligned}\frac{\partial(F)}{\partial(\rho,\varphi)}(\rho,\varphi) &= \frac{\partial(f)}{\partial(x,y)}(\rho\cos\varphi,\rho\sin\varphi)\frac{\partial(x,y)}{\partial(\rho,\varphi)} \\ &= \frac{\partial(f)}{\partial(x,y)}(\rho\cos\varphi,\rho\sin\varphi)\begin{pmatrix}\cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi\end{pmatrix},\end{aligned}$$

由此立刻可得

$$\begin{aligned}\frac{\partial(f)}{\partial(x,y)}(\rho\cos\varphi,\rho\sin\varphi) &= \frac{\partial(F)}{\partial(\rho,\varphi)}(\rho,\varphi)\begin{pmatrix}\cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi\end{pmatrix}^{-1} \\ &= \frac{\partial(F)}{\partial(\rho,\varphi)}(\rho,\varphi)\begin{pmatrix}\cos\varphi & \sin\varphi \\ -\frac{1}{\rho}\sin\varphi & \frac{1}{\rho}\cos\varphi\end{pmatrix},\end{aligned}$$

于是我们有

$$\begin{aligned}I &= \iint\limits_D \frac{1}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial(x,y)}(x,y)\begin{pmatrix} y \\ -x\end{pmatrix}\mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{D'} \frac{1}{\rho}\frac{\partial(f)}{\partial(x,y)}(\rho\cos\varphi,\rho\sin\varphi)\begin{pmatrix}\rho\sin\varphi \\ -\rho\cos\varphi\end{pmatrix}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= \iint\limits_{D'} \frac{\partial(F)}{\partial(\rho,\varphi)}(\rho,\varphi)\begin{pmatrix}\cos\varphi & \sin\varphi \\ -\frac{1}{\rho}\sin\varphi & \frac{1}{\rho}\cos\varphi\end{pmatrix}\begin{pmatrix}\rho\sin\varphi \\ -\rho\cos\varphi\end{pmatrix}\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= \iint\limits_{D'} \frac{\partial(F)}{\partial(\rho,\varphi)}(\rho,\varphi)\begin{pmatrix}0 \\ -1\end{pmatrix}\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= -\iint\limits_{D'} \frac{\partial F}{\partial\varphi}(\rho,\varphi)\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= -\int_0^R\Big(\int_0^{2\pi}\frac{\partial F}{\partial\varphi}(\rho,\varphi)\,\mathrm{d}\varphi\Big)\mathrm{d}\rho \\ &= -\int_0^R\Big(F(\rho,2\pi)-F(\rho,0)\Big)\,\mathrm{d}\rho \\ &= -\int_0^R\Big(f(\rho,0)-f(\rho,0)\Big)\,\mathrm{d}\rho=0.\end{aligned}$$

5. 对二重积分 $\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y$ 作极坐标变换并且给出极坐标系下不同积分次序的累次积分, 其中 $D=\{(x,y)\mid 0\leqslant x\leqslant 1,\ 0\leqslant x+y\leqslant 1\}$.

解: 在极坐标下, 积分区域 D 变为

$$D'=\Big\{(\rho,\varphi)\mid 0\leqslant \rho\cos\varphi\leqslant 1,\ 0\leqslant \rho(\cos\varphi+\sin\varphi)=\sqrt{2}\rho\cos\Big(\frac{\pi}{4}-\varphi\Big)\leqslant 1\Big\},$$

则 $(\rho,\varphi)\in D'$ 当且仅当

$$-\frac{\pi}{4}\leqslant\varphi\leqslant 0,\ 0\leqslant\rho\leqslant\frac{1}{\cos\varphi},\ \text{或}\ 0\leqslant\varphi\leqslant\frac{\pi}{2},\ 0\leqslant\rho\leqslant\frac{1}{\cos\varphi+\sin\varphi}.$$

由此我们可立刻导出

$$\begin{aligned}\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{D'} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\rho\mathrm{d}\varphi\\ &= \int_{-\frac{\pi}{4}}^{0}\Big(\int_0^{\frac{1}{\cos\varphi}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\rho\Big)\mathrm{d}\varphi\\ &\quad+\int_0^{\frac{\pi}{2}}\Big(\int_0^{\frac{1}{\cos\varphi+\sin\varphi}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\Big)\rho\mathrm{d}\varphi.\end{aligned}$$

另外, 我们也有 $(\rho,\varphi)\in D'$ 当且仅当下面五种情况之一出现:

(1) $0\leqslant\rho\leqslant 1,\ -\frac{\pi}{4}\leqslant\varphi\leqslant 0;$ (2) $1\leqslant\rho\leqslant\sqrt{2},\ -\frac{\pi}{4}\leqslant\varphi\leqslant-\arccos\frac{1}{\rho};$
(3) $0\leqslant\rho\leqslant\frac{\sqrt{2}}{2},\ 0\leqslant\varphi\leqslant\frac{\pi}{2};$ (4) $\frac{\sqrt{2}}{2}\leqslant\rho\leqslant 1,\ 0\leqslant\varphi\leqslant\frac{\pi}{4}-\arccos\frac{1}{\sqrt{2}\rho};$
(5) $\frac{\sqrt{2}}{2}\leqslant\rho\leqslant 1,\ \frac{\pi}{4}+\arccos\frac{1}{\sqrt{2}\rho}\leqslant\varphi\leqslant\frac{\pi}{2}.$

而这又等价于说下面四种情况之一出现:

(1) $0\leqslant\rho\leqslant\frac{\sqrt{2}}{2},\ -\frac{\pi}{4}\leqslant\varphi\leqslant\frac{\pi}{2};$
(2) $\frac{\sqrt{2}}{2}\leqslant\rho\leqslant 1,\ -\frac{\pi}{4}\leqslant\varphi\leqslant\frac{\pi}{4}-\arccos\frac{1}{\sqrt{2}\rho};$
(3) $1\leqslant\rho\leqslant\sqrt{2},\ -\frac{\pi}{4}\leqslant\varphi\leqslant-\arccos\frac{1}{\rho};$
(4) $\frac{\sqrt{2}}{2}\leqslant\rho\leqslant 1,\ \frac{\pi}{4}+\arccos\frac{1}{\sqrt{2}\rho}\leqslant\varphi\leqslant\frac{\pi}{2}.$

由此我们立刻可得

$$\begin{aligned}\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{D'} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\rho\mathrm{d}\varphi\\ &= \int_0^{\frac{\sqrt{2}}{2}}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\varphi\Big)\mathrm{d}\rho\\ &\quad+\int_{\frac{\sqrt{2}}{2}}^{1}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}-\arccos\frac{1}{\sqrt{2}\rho}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\varphi\Big)\mathrm{d}\rho\\ &\quad+\int_1^{\sqrt{2}}\Big(\int_{-\frac{\pi}{4}}^{-\arccos\frac{1}{\rho}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\varphi\Big)\mathrm{d}\rho\\ &\quad+\int_{\frac{\sqrt{2}}{2}}^{1}\Big(\int_{\frac{\pi}{4}+\arccos\frac{1}{\sqrt{2}\rho}}^{\frac{\pi}{2}} f(\rho\cos\varphi,\rho\sin\varphi)\rho\,\mathrm{d}\varphi\Big)\mathrm{d}\rho.\end{aligned}$$

6. 将 $\iint\limits_D f(x+y)\,\mathrm{d}x\mathrm{d}y$ 化成单重积分, 其中 $D=\Big\{(x,y)\mid |x|+|y|\leqslant 1\Big\}$.

解: 令 $u=x+y$, $v=x-y$. 在此变换下 D 变为

$$D'=\Big\{(u,v)\mid -1\leqslant u\leqslant 1,\ -1\leqslant v\leqslant 1\Big\},$$

并且我们还有 $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2$, 故 $\frac{D(x,y)}{D(u,v)}=-\frac{1}{2}$. 于是我们有

$$\iint\limits_D f(x+y)\,\mathrm{d}x\mathrm{d}y=\int_{-1}^{1}\Big(\int_{-1}^{1}f(u)\cdot\frac{1}{2}\,\mathrm{d}u\Big)\mathrm{d}v=\int_{-1}^{1}f(u)\,\mathrm{d}u.$$

7. 计算下列二重积分:

(1) $\iint\limits_D(x+y)\sin(x-y)\,\mathrm{d}x\mathrm{d}y$, $D=\Big\{(x,y)\mid 0\leqslant x+y\leqslant\pi,\ 0\leqslant x-y\leqslant\pi\Big\}$;

(2) $\iint\limits_D e^{\frac{y}{x+y}}\,\mathrm{d}x\mathrm{d}y$, $D=\Big\{(x,y)\mid x+y\leqslant 1,\ x\geqslant 0,\ y\geqslant 0\Big\}$;

(3) $\iint\limits_{\substack{0\leqslant x\leqslant 2\\ 0\leqslant y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y$, 其中 $[x+y]$ 表示 $x+y$ 的整数部分;

(4) $\iint\limits_D\Big|\frac{x+y}{\sqrt{2}}-x^2-y^2\Big|\mathrm{d}x\mathrm{d}y$, 其中 $D=\Big\{(x,y)\mid x^2+y^2\leqslant 1\Big\}$;

(5) $\iint\limits_D(x-y)\,\mathrm{d}x\mathrm{d}y$, 其中 $D=\Big\{(x,y)\mid (x-1)^2+(y-1)^2\leqslant 2,\ y\geqslant x\Big\}$;

(6) $\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y$, 其中 $D=\Big\{(x,y)\mid |x|+|y|\leqslant 2\Big\}$ 且 $\forall(x,y)\in D$,

$$f(x,y)=\begin{cases}1, & \text{若 } |x|+|y|\leqslant 1,\\ 2, & \text{若 } 1<|x|+|y|\leqslant 2.\end{cases}$$

(7) $\iint\limits_D\frac{x^2}{y}\sin(xy)\,\mathrm{d}x\mathrm{d}y$, 其中

$$D=\left\{(x,y):0<a\leq\frac{x^2}{y}\leq b\quad 0<p\leq\frac{y^2}{x}\leq q\right\},$$

此处p,q为常数.

解: (1) $\forall x,y\in\mathbb{R}$, 定义 $u=x+y$, $v=x-y$, 在此变换下, 积分区域 D 变为

$$D'=\{(u,v)\mid 0\leqslant u\leqslant\pi,\ 0\leqslant v\leqslant\pi\},$$

并且我们还有 $\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2$, 故 $\frac{D(x,y)}{D(u,v)}=-\frac{1}{2}$, 于是我们有

$$\iint\limits_D(x+y)\sin(x-y)\,\mathrm{d}x\mathrm{d}y=\int_0^{\pi}\Big(\int_0^{\pi}u\sin v\cdot\frac{1}{2}\,\mathrm{d}u\Big)\mathrm{d}v=\frac{\pi^2}{2}.$$

(2) 令 $u=x+y$, $v=y$, 则 D 变为 $D'=\Big\{(u,v)\mid 0\leqslant v\leqslant u\leqslant 1\Big\}$, 且

$$\frac{D(u,v)}{D(x,y)}=\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}=1,$$

故 $\frac{D(x,y)}{D(u,v)}=1$, 于是我们有

$$\iint\limits_D e^{\frac{y}{x+y}}\,\mathrm{d}x\mathrm{d}y=\int_0^1\Big(\int_0^u e^{\frac{v}{u}}\,\mathrm{d}v\Big)\mathrm{d}u=\int_0^1 u(e-1)\,\mathrm{d}u=\frac{1}{2}(e-1).$$

(3) 由题设可知

$$\begin{aligned}\iint\limits_{\substack{0\leqslant x\leqslant 2\\0\leqslant y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{\substack{0\leqslant x+y<1\\0\leqslant x,y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y+\iint\limits_{\substack{1\leqslant x+y<2\\0\leqslant x,y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y\\ &\quad+\iint\limits_{\substack{2\leqslant x+y<3\\0\leqslant x,y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y+\iint\limits_{\substack{3\leqslant x+y\leqslant 4\\0\leqslant x,y\leqslant 2}}[x+y]\,\mathrm{d}x\mathrm{d}y\\ &= 1\times\frac{3}{2}+2\times\frac{3}{2}+3\times\frac{1}{2}=6.\end{aligned}$$

(4) 借助极坐标系立刻可得

$$\begin{aligned}&\iint\limits_D\Big|\frac{x+y}{\sqrt{2}}-x^2-y^2\Big|\,\mathrm{d}x\mathrm{d}y=\iint\limits_{\substack{0\leqslant\rho\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\Big|\frac{\rho(\cos\varphi+\sin\varphi)}{\sqrt{2}}-\rho^2\Big|\rho\,\mathrm{d}\rho\mathrm{d}\varphi\\ =\;&\iint\limits_{\substack{0\leqslant\rho\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\Big|\sin\Big(\varphi+\frac{\pi}{4}\Big)-\rho\Big|\rho^2\,\mathrm{d}\rho\mathrm{d}\varphi\\ =\;&\int_0^{2\pi}\Big(\int_0^1\Big|\sin\Big(\varphi+\frac{\pi}{4}\Big)-\rho\Big|\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\ \overset{\theta=\varphi+\frac{\pi}{4}}{=}\;&\int_{\frac{\pi}{4}}^{\frac{9}{4}\pi}\Big(\int_0^1|\sin\theta-\rho|\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta=\int_0^{2\pi}\Big(\int_0^1|\sin\theta-\rho|\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta\\ =\;&\int_0^{\pi}\Big(\int_0^{\sin\theta}(\sin\theta-\rho)\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta+\int_0^{\pi}\Big(\int_{\sin\theta}^1(\rho-\sin\theta)\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta\\ &+\int_{\pi}^{2\pi}\Big(\int_0^1(\rho-\sin\theta)\rho^2\,\mathrm{d}\rho\Big)\,\mathrm{d}\theta\\ =\;&\frac{1}{12}\int_0^{\pi}\sin^4\theta\,\mathrm{d}\theta+\int_0^{\pi}\Big(\frac{1}{4}-\frac{1}{3}\sin\theta+\frac{1}{12}\sin^4\theta\Big)\,\mathrm{d}\theta\\ &+\int_{\pi}^{2\pi}\Big(\frac{1}{4}-\frac{1}{3}\sin\theta\Big)\,\mathrm{d}\theta\end{aligned}$$

$$\begin{aligned}=\;&\frac{1}{6}\int_0^{\pi}\sin^4\theta\,\mathrm{d}\theta+\int_0^{\pi}\Big(\frac{1}{4}-\frac{1}{3}\sin\theta\Big)\,\mathrm{d}\theta+\int_0^{\pi}\Big(\frac{1}{4}+\frac{1}{3}\sin\theta\Big)\,\mathrm{d}\theta\\ =\;&\frac{\pi}{2}+\frac{1}{3}\int_0^{\frac{\pi}{2}}\sin^4\theta\,\mathrm{d}\theta=\frac{\pi}{2}+\frac{1}{6}\int_0^{\frac{\pi}{2}}(\cos^4\theta+\sin^4\theta)\,\mathrm{d}\theta\\ =\;&\frac{\pi}{2}+\frac{1}{6}\int_0^{\frac{\pi}{2}}(1-2\cos^2\theta\sin^2\theta)\,\mathrm{d}\theta=\frac{\pi}{2}+\frac{1}{6}\int_0^{\frac{\pi}{2}}\Big(1-\frac{1}{2}\sin^2(2\theta)\Big)\,\mathrm{d}\theta\\ =\;&\frac{7\pi}{12}-\frac{1}{12}\int_0^{\frac{\pi}{2}}\sin^2(2\theta)\,\mathrm{d}\theta=\frac{7\pi}{12}-\frac{1}{24}\int_0^{\frac{\pi}{2}}\Big(\cos^2(2\theta)+\sin^2(2\theta)\Big)\,\mathrm{d}\theta\\ =\;&\frac{7\pi}{12}-\frac{\pi}{48}=\frac{9}{16}\pi.\end{aligned}$$

(5) 考虑变换 $x=1+\rho\cos\varphi$, $y=1+\rho\sin\varphi$, 该变换连续可导且 $\frac{D(x,y)}{D(\rho,\varphi)}=\rho$. 在此变换下, 积分区域 D 变为 $D_1=\{(\rho,\varphi)\mid \frac{\pi}{4}\leqslant\varphi\leqslant\frac{5}{4}\pi, 0\leqslant\rho\leqslant\sqrt{2}\}$, 则

$$\begin{aligned}\iint\limits_D (x-y)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{D_1}(\rho\cos\varphi-\rho\sin\varphi)\rho\,\mathrm{d}\rho\mathrm{d}\varphi\\ &= \Big(\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(\cos\varphi-\sin\varphi)\,\mathrm{d}\varphi\Big)\Big(\int_0^{\sqrt{2}}\rho^2\,\mathrm{d}\rho\Big)\\ &= \Big((\sin\varphi+\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\Big)\Big(\frac{1}{3}\rho^3\Big|_0^{\sqrt{2}}\Big)=-\frac{8}{3}.\end{aligned}$$

(6) 由题设以及重积分区域可加性立刻可得

$$\begin{aligned}\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y &= \iint\limits_{|x|+|y|\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y+\iint\limits_{1<|x|+|y|\leqslant 2} 2\,\mathrm{d}x\mathrm{d}y\\ &= \iint\limits_{|x|+|y|\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y+\iint\limits_{|x|+|y|\leqslant 2} 2\,\mathrm{d}x\mathrm{d}y-\iint\limits_{|x|+|y|\leqslant 1} 2\,\mathrm{d}x\mathrm{d}y\\ &= 2\iint\limits_{|x|+|y|\leqslant 2} 1\,\mathrm{d}x\mathrm{d}y-\iint\limits_{|x|+|y|\leqslant 1} 1\,\mathrm{d}x\mathrm{d}y\\ &= 2\times(2\sqrt{2})^2-(\sqrt{2})^2=14.\end{aligned}$$

(7) 我们作坐标变换如下:

$$X=\frac{x^2}{y}\quad Y=\frac{y^2}{x},$$

注意到$\frac{D(X,Y)}{D(x,y)}=3$, 我们知道此变换将区域D变为区域$D_1=\{(X,Y):0<a\leq X\leq b\quad 0<p\leq Y\leq q\}$, 且注意到$\frac{x^2}{y}\sin(xy)dxdy=\frac{1}{3}X\sin(XY)dXdY$, 我们得到原积分等于

$$\begin{aligned}\frac{1}{3}\int_{D_1}X\sin(XY)dXdY &=\frac{1}{3}\int_a^b dX\int_p^q X\sin(XY)dY\\ &=\frac{1}{3}\int_a^b(-\cos(XY))\big|_p^q dX\\ &=\frac{1}{3}\int_a^b(\cos(pX)-\cos(qX))dX\\ &=\frac{\sin(pb)-\sin(pa)}{3p}-\frac{\sin(qb)-\sin(qa)}{3q}.\end{aligned}$$

8. 设 $D=\{(x,y)\mid x^2+y^2\leqslant 1\}$, 而 $f\in\mathscr{C}^{(2)}(D)$ 在 ∂D 上恒为零, 求证:

$$\iint\limits_D f(x,y)\Big(\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)\Big)\,\mathrm{d}x\mathrm{d}y\leqslant 0.$$

证明: 由于 f 在 ∂D 上恒为零, 从而 $\forall x,y\in[-1,1]$, 均有

$f(x,\sqrt{1-x^2})=f(x,-\sqrt{1-x^2})=0,\ \ f(\sqrt{1-y^2},y)=f(-\sqrt{1-y^2},y)=0,$

由此立刻可得

$$
\begin{aligned}
&\iint\limits_D f(x,y)\Big(\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)\Big)\,\mathrm{d}x\mathrm{d}y\\
=&\int_{-1}^{1}\Big(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\frac{\partial^2 f}{\partial x^2}(x,y)\,\mathrm{d}x\Big)\mathrm{d}y\\
&+\int_{-1}^{1}\Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\frac{\partial^2 f}{\partial y^2}(x,y)\,\mathrm{d}y\Big)\mathrm{d}x\\
=&\int_{-1}^{1}\left(f(x,y)\frac{\partial f}{\partial x}(x,y)\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}-\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\Big(\frac{\partial f}{\partial x}(x,y)\Big)^2\mathrm{d}x\right)\mathrm{d}y\\
&+\int_{-1}^{1}\left(f(x,y)\frac{\partial f}{\partial y}(x,y)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}-\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\Big(\frac{\partial f}{\partial y}(x,y)\Big)^2\mathrm{d}y\right)\mathrm{d}x\\
=&-\int_{-1}^{1}\Big(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\Big(\frac{\partial f}{\partial x}(x,y)\Big)^2\mathrm{d}x\Big)\mathrm{d}y-\int_{-1}^{1}\Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\Big(\frac{\partial f}{\partial y}(x,y)\Big)^2\mathrm{d}y\Big)\mathrm{d}x\\
\leqslant&\quad 0,
\end{aligned}
$$

因此所证结论成立.

注: 将来利用 Green 公式可以得到更为简单的证明.

9. 利用二重积分理论, 证明以下积分不等式. 设$f(x),g(x)$是$[a,b]$上的连续函数.

(1) 求证:

$$
\left(\int_a^b f(x)dx\right)^2\le (b-a)\int_a^b (f(x))^2dx
$$

(2) 如果对于任意的$x\in[a,b]$, 我们有$f(x)>0$, 求证:

$$
\int_a^b f(x)dx\cdot\int_a^b\frac{1}{f(x)}dx\ge (b-a)^2.
$$

证明: *(1)* 我们直接证明如下: 此处我们记$f^2(x)=(f(x))^2$,

$$
\begin{aligned}
\left(\int_a^b f(x)dx\right)^2 &=\int\int_{[a,b]^2} f(x)f(y)dxdy\\
&\le\int\int_{[a,b]^2}\frac12\left(f^2(x)+f^2(y)\right)dxdy\\
&=\frac12\int\int_{[a,b]^2} f^2(x)dxdy+\frac12\int\int_{[a,b]^2} f^2(y)dxdy\\
&=\frac12\int_a^b dy\int_a^b f^2(x)dx+\frac12\int_a^b dx\int_a^b f^2(y)dy\\
&=(b-a)\int_a^b f^2(x)dx.
\end{aligned}
$$

(2) 首先, 我们注意到

$$\int_a^b f(x)dx \cdot \int_a^b \frac{dy}{f(y)} = \int\int_{[a,b]^2} \frac{f(x)}{f(y)} dxdy$$

然后我们交换x, y得到

$$\int_a^b f(y)dy \cdot \int_a^b \frac{dx}{f(x)} = \int\int_{[a,b]^2} \frac{f(y)}{f(x)} dxdy$$

从而我们有

$$\begin{aligned}\int_a^b f(x)dx \cdot \int_a^b \frac{dx}{f(x)} =& \frac{1}{2}\int\int_{[a,b]^2} \left(\frac{f(y)}{f(x)} + \frac{f(x)}{f(y)}\right) dxdy \\ \geq& \int\int_{[a,b]^2} 1 dxdy = (b-a)^2.\end{aligned}$$

§2. 三重积分

10. 设$f(u)$是$[0,1]$上的连续函数, 求证*:*

$$\int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_n) dx_n = \frac{1}{(n-1)!}\int_0^1 (1-x)^{n-1} f(x) dx.$$

证明: 此题当然有很多解法, 我们的解法是将积分写成示性函数乘以原来函数的多重积分, 然后通过解不等式换成可以积分的区域积出积分. 我们设

$$\Omega = \{x = (x_1, x_2, \cdots, x_n) : 0 \leq x_n \leq x_{n-1} \leq x_{n-2} \leq \cdots \leq x_2 \leq x_1 \leq 1\}.$$

则此时我们知道待证等式左端的积分等于

$$\int_{\mathbb{R}^n} I_\Omega(x_1, \cdots, x_n) f(x_n) dx_1 \cdots dx_n.$$

此时注意到Ω等于

$$\{x = (x_1, x_2, \cdots, x_n) : 0 \leq x_n \leq 1, x_n \leq x_{n-1} \leq 1, \cdots, x_2 \leq x_1 \leq 1\}$$

故而此时我们得到

$$\begin{aligned}&\int_{\mathbb{R}^n} I_\Omega(x_1, \cdots, x_n) f(x_n) dx_1 \cdots dx_n \\ =& \int_0^1 dx_n \int_{x_n}^1 dx_{n-1} \cdots \int_{x_3}^1 dx_2 \int_{x_2}^1 f(x_n) dx_1 \\ =& \int_0^1 f(x_n) dx_n \int_{x_n}^1 dx_{n-1} \cdots \int_{x_3}^1 dx_2 \int_{x_2}^1 dx_1\end{aligned} \tag{$*$}$$

我们计算里面的$n-1$重积分得到

$$
\begin{aligned}
&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_3}^1 dx_2\int_{x_2}^1 dx_1\\
=&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_3}^1 (1-x_2)dx_2\\
=&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_4}^1 \left(-\frac{(1-x_2)^2}{2!}\right)\Big|_{x_3}^1 dx_3\\
=&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_4}^1 \frac{(1-x_3)^2}{2!}dx_3\\
=&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_5}^1 \left(-\frac{(1-x_3)^3}{3!}\right)\Big|_{x_4}^1 dx_4\\
=&\int_{x_n}^1 dx_{n-1}\cdots\int_{x_5}^1 \frac{(1-x_4)^3}{3!}dx_4\\
=&\cdots\\
=&\int_{x_n}^1 \frac{(1-x_{n-1})^{n-2}}{(n-2)!}dx_{n-1}\\
=&\frac{(1-x_n)^{n-1}}{(n-1)!}
\end{aligned}
$$

回代到式子$(*)$得到结论成立.

11. 记 Ω 为曲面 $x^2+y^2=az$, $z=2a-\sqrt{x^2+y^2}$ $(a>0)$ 所围立体. 分别在直角坐标系、柱坐标系、球坐标系下将 $\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ 化成累次积分.

解: 由于 $\Omega=\left\{(x,y,z)\mid \frac{1}{a}(x^2+y^2)\leqslant z\leqslant 2a-\sqrt{x^2+y^2}, x^2+y^2\leqslant a^2\right\}$, 故

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_{-a}^{a}\Big(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\Big(\int_{\frac{1}{a}(x^2+y^2)}^{2a-\sqrt{x^2+y^2}} f(x,y,z)\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x.$$

在柱坐标系下积分区域 Ω 变为

$$\Omega_1=\left\{(\rho,\varphi,z)\mid \frac{1}{a}\rho^2\leqslant z\leqslant 2a-\rho,\ 0\leqslant\rho\leqslant a,\ 0\leqslant\varphi\leqslant 2\pi\right\},$$

由此我们立刻可得

$$
\begin{aligned}
\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z&=\iiint\limits_{\Omega_1} f(\rho\cos\varphi,\rho\sin\varphi,z)\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z\\
=\quad&\int_0^{2\pi}\Big(\int_0^a\Big(\int_{\frac{1}{a}\rho^2}^{2a-\rho} f(\rho\cos\varphi,\rho\sin\varphi,z)\rho\,\mathrm{d}z\Big)\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi.
\end{aligned}
$$

设在球坐标系下积分区域 Ω 变为 Ω_2. 则 $(r,\theta,\varphi)\in\Omega_2$ 当且仅当

$$\frac{1}{a}r^2\sin^2\theta\leqslant r\cos\theta\leqslant 2a-r\sin\theta,\ 0\leqslant r\sin\theta\leqslant a,\ 0\leqslant\varphi\leqslant 2\pi,\ 0\leqslant\theta\leqslant\pi,$$

而这又等价于说

$$0 \leqslant \theta \leqslant \frac{\pi}{2},\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant r \leqslant \min\Big(\frac{a\cos\theta}{\sin^2\theta}, \frac{\sqrt{2}a}{\sin(\theta+\frac{\pi}{4})}\Big).$$

注意到 $\frac{a\cos\theta}{\sin^2\theta} \leqslant \frac{\sqrt{2}a}{\sin(\theta+\frac{\pi}{4})}$ 当且仅当 $\cos^2\theta + \cos\theta\sin\theta \leqslant 2\sin^2\theta$, 也就是说

$$(\sin\theta - \cos\theta)(2\sin\theta + \cos\theta) \geqslant 0,$$

这又等价于说 $\sin\theta \geqslant \cos\theta$, 也即 $\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$, 于是我们有

$$\begin{aligned}
&\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
= &\iiint\limits_{\Omega_2} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2 \sin\theta\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta \\
= &\int_0^{2\pi}\Big(\int_0^{\frac{\pi}{4}}\Big(\int_0^{\frac{\sqrt{2}a}{\sin(\theta+\frac{\pi}{4})}} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\sin\theta\,\mathrm{d}r\Big)\,\mathrm{d}\theta\Big)\,\mathrm{d}\varphi \\
&+\int_0^{2\pi}\Big(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big(\int_0^{\frac{a\cos\theta}{\sin^2\theta}} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) r^2\sin\theta\,\mathrm{d}r\Big)\,\mathrm{d}\theta\Big)\,\mathrm{d}\varphi.
\end{aligned}$$

12. 交换积分 $\int_0^1\Big(\int_0^{1-x}\Big(\int_0^{x+y} f(x,y,z)\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x$:

(1) 先对 y 积, 再对 x 积, 最后再对 z 积;

(2) 先对 x 积, 再对 z 积, 最后再对 y 积.

解: (1) 由题设立刻可得

$$\begin{aligned}
&\int_0^1\Big(\int_0^{1-x}\Big(\int_0^{x+y} f(x,y,z)\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x = \iiint\limits_{\substack{0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1-x\\ 0\leqslant z\leqslant x+y}} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
= &\iiint\limits_{\substack{0\leqslant z\leqslant 1, 0\leqslant x\leqslant 1\\ \max(0,z-x)\leqslant y\leqslant 1-x}} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1\Big(\int_0^z\Big(\int_{z-x}^{1-x} f(x,y,z)\,\mathrm{d}y\Big)\,\mathrm{d}x\Big)\,\mathrm{d}z \\
&+\int_0^1\Big(\int_z^1\Big(\int_0^{1-x} f(x,y,z)\,\mathrm{d}y\Big)\,\mathrm{d}x\Big)\,\mathrm{d}z.
\end{aligned}$$

(2) 由题设立刻可得

$$\begin{aligned}
&\int_0^1\Big(\int_0^{1-x}\Big(\int_0^{x+y} f(x,y,z)\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x = \iiint\limits_{\substack{0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1-x\\ 0\leqslant z\leqslant x+y}} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
= &\iiint\limits_{\substack{0\leqslant y\leqslant 1, 0\leqslant z\leqslant 1\\ \max(0,z-y)\leqslant x\leqslant 1-y}} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1\Big(\int_y^1\Big(\int_{z-y}^{1-y} f(x,y,z)\,\mathrm{d}x\Big)\,\mathrm{d}z\Big)\,\mathrm{d}y \\
&+\int_0^1\Big(\int_0^y\Big(\int_0^{1-y} f(x,y,z)\,\mathrm{d}x\Big)\,\mathrm{d}z\Big)\,\mathrm{d}y.
\end{aligned}$$

13. 求下列立体的体积:

(1) 曲面 $(x^2+y^2)^2+z^4=z$ 围成的立体;

(2) 曲面 $z=1-\sqrt{x^2+y^2}$, $z=x$, $x=0$ 围成的立体.

解: (1) 将曲面所围立体记作 Ω, 则 $\Omega=\left\{(x,y,z)\mid(x^2+y^2)^2+z^4\leqslant z\right\}$, 它在柱坐标系下变为 $\Omega'=\left\{(\rho,\varphi,z)\mid 0\leqslant z\leqslant 1,\ 0\leqslant\varphi\leqslant 2\pi,\ 0\leqslant\rho\leqslant(z-z^4)^{\frac14}\right\}$, 于是所求体积为

$$
\begin{aligned}
|\Omega| &= \iiint\limits_{\Omega}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\Omega'}\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z=\int_0^{2\pi}\Big(\int_0^1\Big(\int_0^{(z-z^4)^{\frac14}}\rho\,\mathrm{d}\varphi\Big)\mathrm{d}\rho\Big)\,\mathrm{d}z\\
&= \pi\int_0^1\rho^2\Big|_0^{(z-z^4)^{\frac14}}\mathrm{d}z=\pi\int_0^1(z-z^4)^{\frac12}\,\mathrm{d}z=\pi\int_0^1(1-z^3)^{\frac12}\cdot\sqrt{z}\,\mathrm{d}z\\
&\overset{u=z^{\frac32}}{=} \frac{2\pi}{3}\int_0^1(1-u^2)^{\frac12}\,\mathrm{d}u\overset{u=\sin\theta}{=}\frac{2\pi}{3}\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)^{\frac12}\,\mathrm{d}(\sin\theta)\\
&= \frac{2\pi}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta=\frac{\pi}{3}\int_0^{\frac{\pi}{2}}\Big(\cos^2\theta+\sin^2\theta\Big)\,\mathrm{d}\theta=\frac{\pi^2}{6}.
\end{aligned}
$$

(2) 设所围成的立体为 Ω, 则由二重积分的意义可知其体积为

$$
\begin{aligned}
|\Omega| &= \iint\limits_{0\leqslant x\leqslant 1-\sqrt{x^2+y^2}}(1-\sqrt{x^2+y^2}-x)\,\mathrm{d}x\mathrm{d}y\\
&\overset{\substack{x=\rho\cos\varphi\\ y=\rho\sin\varphi}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\int_0^{\frac{1}{1+\cos\varphi}}\Big(1-\rho(1+\cos\varphi)\Big)\rho\,\mathrm{d}\rho\Big)\,\mathrm{d}\varphi\\
&= \frac16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{(1+\cos\varphi)^2}=\frac13\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\varphi}{(1+\cos\varphi)^2}\\
&= \frac16\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\frac{\varphi}{2}}{\cos^4\frac12\varphi}=\frac16\int_0^{\frac{\pi}{2}}\Big(1+\tan^2\frac{\varphi}{2}\Big)\,\mathrm{d}\Big(\tan\frac{\varphi}{2}\Big)\\
&= \frac16\Big(\tan\frac{\varphi}{2}+\frac13\tan^3\frac{\varphi}{2}\Big)\Big|_0^{\frac{\pi}{2}}\\
&= \frac29.
\end{aligned}
$$

14. 计算下列积分:

(1) $\iiint_\Omega xyzdxdydz$, 其中, 区域Ω为下列不等式确定

$$
\begin{cases}
0<a\le\sqrt{xy}\le b\\
0<\alpha\le\frac{y}{x}\le\beta\\
0<m\le\frac{x^2+y^2}{z}\le n\\
x>0\\
y>0\\
z>0
\end{cases}
$$

(2) $\displaystyle\iiint\limits_{\substack{0\leqslant x\leqslant a,0\leqslant y\leqslant b,\\ 0\leqslant z\leqslant c}} (x+2y+3z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z;$

(3) $\displaystyle\iiint\limits_{\Omega}(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, 其中 Ω 是由球面 $x^2+y^2+z^2=R^2$ 和锥面 $z=\sqrt{x^2+y^2}$ 所围成的区域;

(4) $\displaystyle\iiint\limits_{\Omega}(x+y+z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, $\Omega=\left\{(x,y,z)\mid\sqrt{x^2+y^2}\leqslant z\leqslant\sqrt{1-y^2-x^2}\right\}$.

解: (1) 我们令$X=\sqrt{xy}$, $Y=\frac{y}{x}$, $Z=z$, 则

$$\frac{D(X,Y,Z)}{D(x,y,z)}=\frac{\sqrt{y}}{x\sqrt{x}}=\frac{Y}{X}.$$

此时区域Ω被变换成区域D_1

$$\begin{cases}0<a\le X\le b\\ 0<\alpha\le Y\le\beta\\ \frac{\frac{X^2}{Y}+X^2Y}{n}\le Z\le\frac{\frac{X^2}{Y}+X^2Y}{m}.\end{cases}$$

此时原来的积分化为

$$\begin{aligned}&\int_a^b dX\int_\alpha^\beta dY\int_{\frac{1}{n}\left(\frac{X^2}{Y}+X^2Y\right)}^{\frac{1}{m}\left(\frac{X^2}{Y}+X^2Y\right)}X^3Z\frac{1}{Y}dZ\\=&\int_a^b X^3dX\int_\alpha^\beta\frac{1}{Y}dY\int_{\frac{1}{n}\left(\frac{X^2}{Y}+X^2Y\right)}^{\frac{1}{m}\left(\frac{X^2}{Y}+X^2Y\right)}ZdZ\\=&\int_a^b X^3dX\int_\alpha^\beta\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)\left(\frac{X^2}{Y}+X^2Y\right)^2\frac{1}{Y}dY\end{aligned}$$

由于

$$\int_\alpha^\beta\left(\frac{X^2}{Y}+X^2Y\right)^2\frac{1}{Y}dY=X^4\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}+\frac{\beta^2-\alpha^2}{2}+2\log\frac{\beta}{\alpha}\right).$$

此时我们得到上面的积分等于

$$\begin{aligned}&\int_a^b X^3dX\int_\alpha^\beta\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)\left(\frac{X^2}{Y}+X^2Y\right)^2\frac{1}{Y}dY\\=&\frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{n^2}\right)\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}+\frac{\beta^2-\alpha^2}{2}+2\log\frac{\beta}{\alpha}\right)\int_a^b X^7dX\\=&\frac{1}{32}(b^8-a^8)\left(\beta^2-\alpha^2+4\log\frac{\beta}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\left(\frac{1}{m^2}-\frac{1}{n^2}\right)\end{aligned}$$

(2) 由题设立刻可得

$$\begin{aligned}
&\iiint\limits_{\substack{0\leqslant x\leqslant a,0\leqslant y\leqslant b,\\ 0\leqslant z\leqslant c}}(x+2y+3z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^c\Big(\int_0^b\Big(\int_0^a x\,\mathrm{d}x\Big)\,\mathrm{d}y\Big)\mathrm{d}z\\
&+\int_0^c\Big(\int_0^a\Big(\int_0^b 2y\,\mathrm{d}y\Big)\,\mathrm{d}x\Big)\mathrm{d}z+\int_0^a\Big(\int_0^b\Big(\int_0^c 3z\,\mathrm{d}z\Big)\,\mathrm{d}y\Big)\mathrm{d}x\\
=\ &\frac{1}{2}a^2bc+ab^2c+\frac{3}{2}abc^2=\frac{1}{2}abc(a+2b+3c).
\end{aligned}$$

(3) 由题设知 $\Omega=\Big\{(x,y,z)\mid\sqrt{x^2+y^2}\leqslant z\leqslant\sqrt{R^2-x^2-y^2}\Big\}$, 并且它在球坐标系下变为 $\Omega'=\Big\{(\rho,\theta,\varphi)\mid 0\leqslant\rho\leqslant R,\ 0\leqslant\theta\leqslant\frac{\pi}{4},\ 0\leqslant\varphi\leqslant 2\pi\Big\}$, 故

$$\begin{aligned}
\iiint\limits_{\Omega}(x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^R\Big(\int_0^{2\pi}\Big(\int_0^{\frac{\pi}{4}}\rho^2\cdot\rho^2\sin\theta\,d\theta\Big)\mathrm{d}\varphi\Big)\mathrm{d}\rho\\
&= 2\pi\Big(\frac{\rho^5}{5}\Big)\Big|_0^R(-\cos\theta)\Big|_0^{\frac{\pi}{4}}=2\pi\cdot\frac{R^5}{5}\cdot\Big(1-\frac{\sqrt{2}}{2}\Big)\\
&= \frac{\pi R^5}{5}(2-\sqrt{2}).
\end{aligned}$$

(4) 借助对称性以及柱坐标系, 我们有

$$\begin{aligned}
I &= \iiint\limits_{\Omega}z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\limits_{\substack{\rho\leqslant z\leqslant\sqrt{1-\rho^2}\\ 0\leqslant\rho\leqslant\frac{\sqrt{2}}{2},\ 0\leqslant\varphi\leqslant 2\pi}}z\rho\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z\\
&= \int_0^{2\pi}\Big(\int_0^{\frac{\sqrt{2}}{2}}\Big(\int_\rho^{\sqrt{1-\rho^2}}z\rho\,\mathrm{d}z\Big)\mathrm{d}\rho\Big)\mathrm{d}\varphi\\
&= 2\pi\int_0^{\frac{\sqrt{2}}{2}}\Big(\frac{1}{2}z^2\rho\Big|_\rho^{\sqrt{1-\rho^2}}\Big)\mathrm{d}\rho\\
&= \pi\int_0^{\frac{\sqrt{2}}{2}}(1-2\rho^2)\rho\,\mathrm{d}\rho\\
&= \pi\Big(\frac{1}{2}\rho^2-\frac{1}{2}\rho^4\Big)\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{\pi}{8}.
\end{aligned}$$

15. 设曲面 S 的球坐标方程为 $r=a(1+\cos\theta)$, 求该曲面在直角坐标系下的形心坐标.

解: 曲面 S 的参数方程为

$$\begin{cases}
x=r\sin\theta\cos\varphi=a(1+\cos\theta)\sin\theta\cos\varphi,\\
y=r\sin\theta\sin\varphi=a(1+\cos\theta)\sin\theta\sin\varphi, \quad (0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi).\\
z=r\cos\theta=a(1+\cos\theta)\cos\theta,
\end{cases}$$

由此可知

$$\frac{\partial(x,y,z)}{\partial(\theta,\varphi)}=\begin{pmatrix}
a(\cos\theta+\cos 2\theta)\cos\varphi & -a(1+\cos\theta)\sin\theta\sin\varphi\\
a(\cos\theta+\cos 2\theta)\sin\varphi & a(1+\cos\theta)\sin\theta\cos\varphi\\
-a(1+2\cos\theta)\sin\theta & 0
\end{pmatrix}.$$

于是我们有

$$\begin{aligned}
E &= a^2(\cos\theta+\cos 2\theta)^2\cos^2\varphi + a^2(\cos\theta+\cos 2\theta)^2\sin^2\varphi + a^2(1+2\cos\theta)^2\sin^2\theta \\
&= 2a^2(1+\cos\theta), \\
G &= a^2(1+\cos\theta)^2\sin^2\theta\sin^2\varphi + a^2(1+\cos\theta)^2\sin\theta^2\cos^2\varphi = a^2(1+\cos\theta)^2\sin^2\theta, \\
F &= 0.
\end{aligned}$$

从而曲面微元为 $\mathrm{d}\sigma = \sqrt{EG}\,\mathrm{d}\varphi\mathrm{d}\theta = \sqrt{2}a^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\varphi\mathrm{d}\theta$. 故

$$\begin{aligned}
|S| &= \int_0^{2\pi}\Big(\int_0^{\pi}\sqrt{2}a^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\theta\Big)\mathrm{d}\varphi \\
&= 2\sqrt{2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\theta \\
&= -2\sqrt{2}\pi a^2\cdot\frac{2}{5}(1+\cos\theta)^{\frac{5}{2}}\Big|_0^{\pi} = \frac{32}{5}\pi a^2.
\end{aligned}$$

设曲面 S 的形心为 $(\bar{x},\bar{y},\bar{z})$. 则

$$\begin{aligned}
\bar{x} &= \frac{1}{|S|}\iint\limits_S x\,\mathrm{d}\sigma \\
&= \frac{1}{|S|}\int_0^{2\pi}\Big(\int_0^{\pi}a(1+\cos\theta)\sin\theta\cos\varphi\cdot\sqrt{2}a^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\theta\Big)\mathrm{d}\varphi \\
&= \frac{1}{|S|}\Big(\int_0^{2\pi}\cos\varphi\,\mathrm{d}\varphi\Big)\Big(\int_0^{\pi}\sqrt{2}a^3(1+\cos\theta)^{\frac{5}{2}}\sin^2\theta\,\mathrm{d}\theta\Big) = 0, \\
\bar{y} &= \frac{1}{|S|}\iint\limits_S y\,\mathrm{d}\sigma \\
&= \frac{1}{|S|}\int_0^{2\pi}\Big(\int_0^{\pi}a(1+\cos\theta)\sin\theta\sin\varphi\cdot\sqrt{2}a^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\theta\Big)\mathrm{d}\varphi \\
&= \frac{1}{|S|}\Big(\int_0^{2\pi}\sin\varphi\,\mathrm{d}\varphi\Big)\Big(\int_0^{\pi}\sqrt{2}a^3(1+\cos\theta)^{\frac{5}{2}}\sin^2\theta\,\mathrm{d}\theta\Big) = 0, \\
\bar{z} &= \frac{1}{|S|}\iint\limits_S z\,\mathrm{d}\sigma \\
&= \frac{1}{|S|}\int_0^{2\pi}\Big(\int_0^{\pi}a(1+\cos\theta)\cos\theta\cdot\sqrt{2}a^2(1+\cos\theta)^{\frac{3}{2}}\sin\theta\,\mathrm{d}\theta\Big)\mathrm{d}\varphi \\
&= \frac{2\sqrt{2}\pi a^3}{|S|}\int_0^{\pi}(1+\cos\theta)^{\frac{5}{2}}\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= -\frac{2\sqrt{2}\pi a^3}{|S|}\int_0^{\pi}(1+\cos\theta)^{\frac{5}{2}}\cos\theta\,\mathrm{d}(\cos\theta) \\
&\overset{t=\cos\theta}{=} \frac{2\sqrt{2}\pi a^3}{|S|}\int_{-1}^{1}(1+t)^{\frac{5}{2}}t\,\mathrm{d}t \\
&= \frac{2\sqrt{2}\pi a^3}{|S|}\Big(\frac{2}{7}t(1+t)^{\frac{7}{2}} - \frac{2}{7}\cdot\frac{2}{9}(1+t)^{\frac{9}{2}}\Big)\Big|_{-1}^{1} \\
&= \frac{320}{63}\frac{\pi a^3}{|S|} = \frac{50}{63}a.
\end{aligned}$$

16. 设A是3阶实对称矩阵，且A正定，$\sum_{i,j=1}^{3} a_{ij}x_ix_j = 1$表示$\mathbb{R}^3$中的一个椭球面. 求证：该椭球面所围的立体$V$的体积为

$$|V| = \frac{4\pi}{3\sqrt{\det A}}.$$

证明： 由于A对称正定，我们知道存在三阶可逆实矩阵P，使得$A = P^tP$，故在线性变换

$$y = Px$$

下，V变为$\mathbb{R}^3$中的单位球U. 此原因是

$$1 = \sum_{i,j=1}^{3} a_{ij}x_ix_j = x^tAx = x^tP^tPx = y^ty = y_1^2 + y_2^2 + y_3^2.$$

故而由重积分的换元公式及$dy = |\mathrm{det}P|dx$，我们得到

$$|V| = \iiint\limits_V dx_1dx_2dx_3 = \iiint\limits_U |\mathrm{det}P|^{-1}dy_1dy_2dy_3 = |\mathrm{det}P|^{-1}\cdot|U| = \frac{4\pi}{3}|\mathrm{det}P|^{-1}.$$

又由$A = P^tP$，我们得到

$$\det(P)^2 = \det A$$

从而$\det(P)^{-1} = \frac{1}{\sqrt{\det A}}$，此时我们得到

$$|V| = \frac{4\pi}{2\sqrt{\det A}}.$$

17. 设

$$V = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}, \quad h = \sqrt{a^2 + b^2 + c^2} > 0,$$

$f(u)$在区间$[-h, h]$上连续，求证：

$$\iiint\limits_V f(ax + by + cz)dxdydz = \pi\int_{-1}^{1}(1 - t^2)f(ht)dt.$$

证明： 作变量替换

$$\begin{cases} u = \frac{1}{h}(ax + by + cz) \\ v = a_2x + b_2y + c_2z \\ w = a_3x + b_3y + c_3z \end{cases}$$

其中系数矩阵为正交矩阵，则由$|u| \leq 1$得到

$$\iiint\limits_V f(ax + by + cz)dxdydz = \int_{-1}^{1}du\iint\limits_{D_u} f(hu)dvdw,$$

其中，

$$D_u = \{(v,w) : v^2 + w^2 \leq 1 - u^2\}.$$

从而我们得到

$$\iint\limits_V f(ax+by+cz)dxdydz = \pi\int_{-1}^{1}(1-u^2)f(hu)du.$$

得证.