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1.4.1.1: It is a Sylvestor equation:

$A=\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ we block it into $\begin{bmatrix} A_{1\,2\times2} & C_{2\times2} \\ & A_{2\,2\times2} \end{bmatrix}$ $A_1=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_{2\times2}$ $C=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$ $A_2=\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}_{2\times2}$

where A_1's eigenvalue is 1 and A_2's eigenvalues are 3 and 4. They have no common eigenvalues.

We want to perform $B\begin{bmatrix} A_1 & C \\ & A_2 \end{bmatrix}B^{-1}$ so that the resulting matrix is block diagonal. Note that since B here is invertible, it must corresponds to some row/column operations that must happen in pairs. Suppose $B=\begin{bmatrix} I_{2\times2} & X_{2\times2} \\ & I_{2\times2} \end{bmatrix}$ ie, it is a block operation. $B^{-1}=\begin{bmatrix} I_{2\times2} & -X_{2\times2} \\ & I_{2\times2} \end{bmatrix}$

$$\left(\begin{bmatrix} I_{2\times2} & X_{2\times2} \\ & I_{2\times2} \end{bmatrix}\cdot\begin{bmatrix} I_{2\times2} & -X_{2\times2} \\ & I_{2\times2} \end{bmatrix}=\begin{bmatrix} I_{2\times2} & -X_{2\times2}\cdot I_{2\times2}+X_{2\times2}\cdot I_{2\times2} \\ & I_{2\times2} \end{bmatrix}=\begin{bmatrix} I_{2\times2} & \\ & I_{2\times2} \end{bmatrix}\right)$$

Then $BAB^{-1}=\begin{bmatrix} I_{2\times2} & X_{2\times2} \\ & I_{2\times2} \end{bmatrix}\begin{bmatrix} A_{1\,2\times2} & C_{2\times2} \\ & A_{2\,2\times2} \end{bmatrix}\begin{bmatrix} I_{2\times2} & -X_{2\times2} \\ & I_{2\times2} \end{bmatrix}=\begin{bmatrix} A_{1\,2\times2} & -A_{1\,2\times2}X_{2\times2}+X_{2\times2}A_{2\,2\times2}+C_{2\times2} \\ & A_{2\,2\times2} \end{bmatrix}$

ie $A_1X-XA_2=C$. Let $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}-\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 5 \\ & 4 \end{bmatrix}=\begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}-\begin{bmatrix} 3a & 5a+4b \\ 3c & 5c+4d \end{bmatrix}=\begin{bmatrix} -2a+2c & 5a+3b-2d \\ -2c & -5c-3d \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$-2a+c=1$ $-5a-3b+2d=2$ $-2c=3$ $-5c-3d=4$

$\Rightarrow (a,b,c,d)=(-2,\frac{31}{9},-\frac{3}{2},\frac{7}{6})$ Thus: $B=\begin{bmatrix} 1 & 0 & -2 & \frac{31}{9} \\ 0 & 1 & -\frac{3}{2} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1.4.1.2: $\lambda=3$, Hence $(A-3I)X=0 \Rightarrow \begin{bmatrix} -2 & 2 & 1 & 4 \\ & -2 & 3 & 4 \\ & & 0 & 5 \\ & & & 1 \end{bmatrix}x=0 \Rightarrow x=\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix} \Rightarrow V_3=\mathrm{span}\left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}\right)$

$\lambda=4$, Hence $(A-4I)X=0 \Rightarrow \begin{bmatrix} -3 & 2 & 1 & 2 \\ & -3 & 3 & 4 \\ & & -1 & 5 \\ & & & 0 \end{bmatrix}X=0 \Rightarrow x=\begin{bmatrix} 59 \\ 57 \\ 45 \\ 9 \end{bmatrix} \Rightarrow V_4=\mathrm{span}\left(\begin{bmatrix} 59 \\ 57 \\ 45 \\ 9 \end{bmatrix}\right)$

Charateristic polynomial：特征多项式

1.4.2. : counter example: Let $B=\begin{bmatrix}0&1\\&0\end{bmatrix}$. $B^2=0$.

If any $f(x)$ that satisfies $\deg P(x)=1$ or 0 and $P(B)=0$

$B^0=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $B^1=\begin{bmatrix}0&1\\&0\end{bmatrix}$ $f(x)=ax+b$. $(a,b\in \mathbb{C})$ Then $f(A)=\begin{bmatrix}a&b\\0&a\end{bmatrix}$ if $f(A)=0$, $a=b=0$.

Hence: B's minimal polynomial is $m_B(x)=X^2$

Hence: $A=\begin{bmatrix}0&1&1&0\\&0&0&1\\&&0&1\\&&&0\end{bmatrix}$ $A^3=0$ So Let $G(x)=X^3$. $G(A)=0$. Hence

A's minimal polynomial $m_A(x)$ satisfies $m_A(x)\mid G(x)$. But $[m_B(x)]^2=X^4$

$[m_B(x)]^2\nmid G(x)$ So $m_A(x)\neq[m_B(x)]^2$ proof is below

Proof: $A\in M_n(F)$, $\forall f(x)\in F[x]$. $f(A)=0 \Longleftrightarrow m_A(x)\mid f(x)$ ($m_A(x)$ is A's minimal polynomial)

$\Leftarrow$: if $f(x)=g(x)\cdot m_A(x)$ Then $f(A)=g(A)\cdot m_A(A)=0$

$\Rightarrow$: if $m_A(x)\nmid f(x)$. Then $f(x)=m_A(x)\cdot g(x)+q(x)$. where $\deg q(x)<\deg m_A(x)$

$f(A)=0\Rightarrow m_A(A)\cdot g(A)+q(A)=0$ $\therefore q(A)=0$. But $\deg q(x)<\deg m_A(x)$

So $m_A(x)$ is not A's minimal polynomial, It's a contradiction)

Hence $f(A)=0\Longleftrightarrow m_A(x)\mid f(x)$

1.4.3.

① from homework 1.2.4. We know that

If V is an A-invariant subspace, show that A has an eigenvector in V .

And AB = BA for two complex square matrices A, B. Then A, B has a common eigenvector.

We have the proof already

② we use mathematicial induction to prove if any two $(n-1)\times(n-1)$ matrix A and B, AB=BA then A, B can be simultaneously triangularised. Then for any two $n\times n$ matrix C and D.

If $CD=DC$.

Then from ① we know C, D have a common eigenvector. Let it be X_1. Then we choose another n-1 vector, $X_2, X_3 \cdots X_n$, which together with X_1 can is a basis of F^n.

Let $P=[X_1\ X_2 \cdots X_n]$ The $A[X_1\ X_2\cdots X_n]=[X_1\cdots X_n]\begin{bmatrix}\lambda_1&\cdots\\0&\cdots\\\vdots&\\0&\cdots\end{bmatrix}$ $\therefore P^{-1}AP=\begin{bmatrix}\lambda_1&M\\0&A_1\end{bmatrix}$ M is $1\times(n-1)$

A_1 is $(n-1)\times(n-1)$ 0 is $(n-1)\times 1$. And $P^{-1}BP=\begin{bmatrix}\lambda_1&N\\0&B_1\end{bmatrix}$ N is $1\times(n-1)$ B_1 is $(n-1)\times(n-1)$ 0 is $(n-1)\times 1$.

We know that $A=P\begin{bmatrix}\lambda_1&M\\&A_1\end{bmatrix}P^{-1}$ $B=P\begin{bmatrix}\lambda_1&N\\&B_1\end{bmatrix}P^{-1}$ $AB=P\begin{bmatrix}\lambda_1^2&\lambda_1N+MB_1\\0&A_1B_1\end{bmatrix}P^{-1}$

$BA=P\begin{bmatrix}\lambda_1^2&\lambda_1M+NA_1\\0&B_1A_1\end{bmatrix}P^{-1}$ $AB=BA$. Hence: $AB-BA=P\begin{bmatrix}0&\lambda_1N+MB_1-\lambda_1M-NA_1\\0&A_1B_1-B_1A_1\end{bmatrix}P^{-1}=0$

Since P is invertible, then $\begin{bmatrix}0&\lambda_1N+MB_1-\lambda_1M-NA_1\\0&A_1B_1-B_1A_1\end{bmatrix}=0$

$\therefore A_1B_1=B_1A_1$. And we know any two $(n-1)\times(n-1)$ matrix A and B, AB=BA then A, B can be simultaneously triangularised. So A_1, B_1 can be simultaneously triangularised. Choose the corresponding invertible $(n-1)\times(n-1)$ matrix Q_1.

$Q_1^{-1}AQ=\hat{A}_1$, which is upper triangulized $Q_1^{-1}BQ=\hat{B}_1$, which is upper triangulized

Let $Q=\begin{bmatrix}1&0\\0&Q_1\end{bmatrix}$ then: $Q^{-1}=\begin{bmatrix}1&\\&Q^{-1}\end{bmatrix}$ $\left(Q\cdot Q^{-1}=\begin{bmatrix}1&\\&Q_1\end{bmatrix}\begin{bmatrix}1&\\&Q^{-1}\end{bmatrix}=\begin{bmatrix}1&\\&I\end{bmatrix}\right)$

$Q^{-1}\cdot P^{-1}\cdot A\cdot P\cdot Q=\begin{bmatrix}1&\\&Q_1^{-1}\end{bmatrix}\begin{bmatrix}\lambda_1&M\\&A_1\end{bmatrix}\begin{bmatrix}1&0\\0&Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&M\\0&Q^{-1}A_1\end{bmatrix}\begin{bmatrix}1&\\&Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&MQ_1\\0&Q^{-1}A_1Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&MQ_1\\&\hat{A}_1\end{bmatrix}$

$Q^{-1}\cdot P^{-1}\cdot B\cdot P\cdot Q=\begin{bmatrix}1&\\&Q_1^{-1}\end{bmatrix}\begin{bmatrix}\lambda_1&N\\&B_1\end{bmatrix}\begin{bmatrix}1&0\\0&Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&N\\0&Q^{-1}B_1\end{bmatrix}\begin{bmatrix}1&\\&Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&NQ_1\\0&Q^{-1}B_1Q_1\end{bmatrix}=\begin{bmatrix}\lambda_1&MQ_1\\&\hat{B}_1\end{bmatrix}$

(1) 设 A 是 $m\times m$ 可逆矩阵，B 是 $m\times n$ 矩阵，C 是 $n\times m$ 矩阵，D 是 $n\times n$ 矩阵，$D-CA^{-1}B$ 是 $n\times n$ 可逆矩阵，则有

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}.$$

当 $A=I$ 时有

$$\begin{bmatrix} I & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I+B(D-CB)^{-1}C & -B(D-CB)^{-1} \\ -(D-CB)^{-1}C & (D-CB)^{-1} \end{bmatrix};$$

当 $B=O$ 时有

$$\begin{bmatrix} A & O \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & O \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix};$$

当 $C=O$ 时有

$$\begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix};$$

当 $D=O$ 时有

$$\begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}-A^{-1}B(CA^{-1}B)^{-1}CA^{-1} & A^{-1}B(CA^{-1}B)^{-1} \\ (CA^{-1}B)^{-1}CA^{-1} & -(CA^{-1}B)^{-1} \end{bmatrix}.$$

Hence $Q^{-1}P^{-1}APQ$ and $Q^{-1}P^{-1}BPQ$ are both upper triangulized, then $A \cdot B$ can be simultaneously triangularised.

1.4.3.2 Counter example. $A=\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $B=\begin{bmatrix} 0 & 2 \\ & 0 \end{bmatrix}$ $AB=\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ & 0 \end{bmatrix}=\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$BA=\begin{bmatrix} 0 & 2 \\ & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}=\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ $AB=BA$. A is already in jordan norm

All eigenvanlue of A is 1 and All eigenvanlue of B is 0

$(A-I)X_1=0$

$\Rightarrow X_1=\begin{bmatrix} a \\ 0 \end{bmatrix}$ $a\in F$ and $a\neq 0$ $(A-I)X_2=X_1 \Rightarrow X_2=\begin{bmatrix} 0 \\ a \end{bmatrix}$. so $A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}$

For B: $BX_1=0 \Rightarrow X_1=\begin{bmatrix} 2b \\ 0 \end{bmatrix}$ $BX_2=X_1 \Rightarrow X_2=\begin{bmatrix} 0 \\ b \end{bmatrix}$ So $B=\begin{bmatrix} 2b & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2b} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$

But $\begin{bmatrix} a & \\ & a \end{bmatrix}\cdot B\begin{bmatrix} \frac{1}{a} & \\ & \frac{1}{a} \end{bmatrix}=a\cdot\frac{1}{a}\cdot I\cdot B\cdot I=B$. is not in jordan form

$\begin{bmatrix} 2b & 0 \\ 0 & b \end{bmatrix}A\begin{bmatrix} \frac{1}{2b} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is also not in jordan form

So they can not always be simultaneously put into Jordan normal form

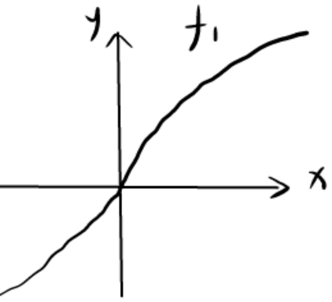
1.4.4. There is no $f(x)$ that satisfy $f(f(x))+x=0$.

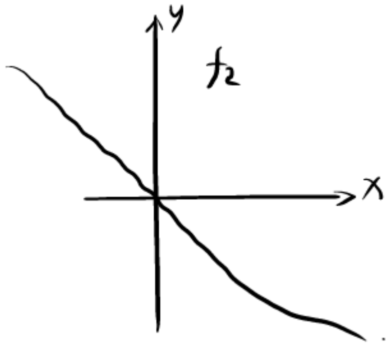
① From my discrete math class. I get that for $g: A\to B$ $f: B\to C$ if $f\circ g$ is bijective, the g is injective and f is surjective.

too prove it:

1. $f\circ g$ is surjective, then for every $z\in C$. $\exists x\in A$ that $f\circ g(x)=z$. then $\exists y\in B$. $g(x)=y$ and $f(y)=z$. So for every $z\in C$. $\exists y\in B$. $f(y)=z$. Then z is surjective
2. $f\circ g$ is injective, then for any $y\in Ran(g)$ if $\exists x_1, x_2\in A$. and $g(x_1)=g(x_2)=y$. For this $y\in B$. since $Ran(g)\subseteq B$. $\exists z\in C$. that $f(y)=z$. So $f(g(x_1))=f(g(x_2))=z$. However $f\circ g$ is injective. then $x_1=x_2$. So $g(x_1)=g(x_2)\Rightarrow x_1=x_2$. g is injective

② $f\circ f(x)=-x$. $f\circ f$ is bijective. So f is injective and surjective. then f is bijective. Thereby $f(x)$ should be seriously decreasing or increasing. And $f(-x)=f(f(f(x)))=-f(x)$. Hence $f(x)$ is odd. And $f(x)$ is continuous. $f(x)$ is gonna be like f_1 or f_2.





③ if f is like f_2. where $f(x)$ is sign-invertible.
for $x<0$. $f(x)>0$. $f(f(x))<0$. but $-x>0$
then $f(f(x)) \neq -x$
if f is like f_1 where $f(x)$ is sign-perserving
for $x<0$. $f(x)<0$ $f(f(x))<0$ but $-x>0$
then $f(f(x)) \neq -x$
So there gonna be no f that exist

Proof Attached $AB=BA$ then A and B can be simutaneously diagnised

A is diagnisable. so exist An invertible Matrix X_1. $X_1^{-1} A X_1 = \hat{\Lambda}_1$ And we take corresponding permutation matrix P so that: $P^{-1}\hat{\Lambda}_1 P = \Lambda_1 = \operatorname{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_r I_{n_r})$ and $\lambda_1, \lambda_2, \cdots \lambda_r$ are distinct eigenvalue. The size of each I_{n_i} is $k_i \times k_i$, and k_i is the geometric multiplicity of λ_i.

The $P^{-1}X_1^{-1}AX_1P = \Lambda_1$. Let $\hat{B}$ be $P^{-1}X_1^{-1}BX_1P$. Since $AB=BA$. We have $\Lambda_1\hat{B}$
$= P^{-1}X_1^{-1}AX_1P \cdot P^{-1}X_1^{-1}BX_1P = P^{-1}X_1^{-1}A \cdot BX_1P = P^{-1}X_1^{-1}BAX_1P = P^{-1}X_1^{-1}BX_1P \cdot P^{-1}X_1^{-1} \cdot AX_1P = \hat{B}\Lambda_1$

Let $\hat{B} = \begin{bmatrix} \hat{B}_{11} & \cdots & \hat{B}_{1r} \\ \vdots & & \vdots \\ \hat{B}_{r1} & \cdots & \hat{B}_{rr} \end{bmatrix}$ The size of $\hat{B}_{mi}$ is $k_i \times k_i$, and k_i is the geometric multiplicity of λ_i.

$$\Lambda_1\hat{B} = \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_r I_{n_r} \end{bmatrix} \begin{bmatrix} \hat{B}_{11} & \cdots & \hat{B}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{B}_{r1} & \cdots & \hat{B}_{rr} \end{bmatrix} = \begin{bmatrix} \lambda_1\hat{B}_{11} & \lambda_1\hat{B}_{12} \cdots & \lambda_1\hat{B}_{1r} \\ \lambda_2\hat{B}_{12} & \lambda_2\hat{B}_{22} \cdots & \lambda_2\hat{B}_{2r} \\ \vdots & \vdots & \vdots \\ \lambda_r\hat{B}_{r1} & \lambda_r\hat{B}_{r2} \cdots & \lambda_r\hat{B}_{rr} \end{bmatrix}$$

$$\hat{B}\Lambda_1 = \begin{bmatrix} \hat{B}_{11} & \cdots & \hat{B}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{B}_{r1} & \cdots & \hat{B}_{rr} \end{bmatrix} \begin{bmatrix} \lambda_1 I_{n_1} & & & \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ & & & \lambda_r I_{n_r} \end{bmatrix} = \begin{bmatrix} \lambda_1\hat{B}_{11} & \lambda_2\hat{B}_{12} \cdots & \lambda_r\hat{B}_{1r} \\ \lambda_1\hat{B}_{12} & \lambda_2\hat{B}_{22} \cdots & \lambda_r\hat{B}_{2r} \\ \vdots & \vdots & \vdots \\ \lambda_1\hat{B}_{r1} & \lambda_2\hat{B}_{r2} \cdots & \lambda_r\hat{B}_{rr} \end{bmatrix}$$

So $\lambda_i I_{n_i}\hat{B}_{ij} = \hat{B}_{ij}\lambda_j I_{n_j}$. $(\lambda_i - \lambda_j)\hat{B}_{ij} = 0$ for $i \neq j$. $\lambda_i \neq \lambda_j$. Then $\hat{B}_{ij} = 0$

i.e. $\hat{B} = \begin{bmatrix} \hat{B}_{11} & & \\ & \hat{B}_{22} & \\ & & \ddots \\ & & & \hat{B}_{rr} \end{bmatrix}$ $\hat{B}$ is also block diagnised.

Since B is block diagnisable, $\hat{B}$ is also diagnisable, then $\hat{B}_{ii}$ is diagnisable let $X_{ii}^{-1}\hat{B}_{ii}X_{ii} = \Lambda_{ii}$. where X_{ii} is invertible and Λ_{ii} is diagnised. the size of X_{ii} and Λ_{ii} are $k_i \times k_i$, and k_i is the geometric multiplicity of λ_i.

Let $X_2 = \begin{bmatrix} X_{11} & & \\ & X_{22} & \\ & & \ddots \\ & & & X_{rr} \end{bmatrix}$ So X_2 is invertible. $X_2^{-1} = \begin{bmatrix} X_{11}^{-1} & & \\ & X_{22}^{-1} & \\ & & \ddots \\ & & & X_{rr}^{-1} \end{bmatrix}$

$X_2^{-1}\hat{B}X_2 = \begin{bmatrix} \Lambda_{11} & & \\ & \Lambda_{22} & \\ & & \ddots \\ & & & \Lambda_{rr} \end{bmatrix}$ Let it be Λ_2

And $X_2^{-1} \cdot \Lambda_1 \cdot X_2 = \begin{bmatrix} X_{11}^{-1}\lambda_1 I_{n_1}X_{11} & & \\ & X_{22}^{-1}\lambda_2 I_{n_2}X_{22} & \\ & & \ddots \\ & & & X_{rr}^{-1}\lambda_r I_{n_r}X_{rr} \end{bmatrix} = \Lambda_1$

Hence $X_2^{-1}P^{-1}X_1^{-1}AX_1PX_2 = \Lambda_1$ $X_2^{-1}P^{-1}X_1^{-1}BX_1PX_2 = \Lambda_2$

A, B can be simultaneously triangularized.