Chenyang Zhao 赵晨阳 from the school of software
Info ID: 2020012363 Tel: 18015766633

Collaborations: Hanwen Cao, Mingdao Liu, Sijia Liu
↳ My girlfriend ^^!

1.1.1: From Example 1.1.1 we notice that $C=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is a real matrix which satisfies $A^2=-I$. Hence $A=\begin{bmatrix}C&&\\&\ddots&\\&&C\end{bmatrix}_{2k\times 2k}$ will satisfy $A^2_{2n\times 2n}=-I$

Therefore, a sonlution is:

$$A=\begin{bmatrix}C&&\\&\ddots&\\&&C\end{bmatrix}_{n\times n}$$

(n is an even number and $C=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, we have $\frac{n}{2}$ C in the diagonal)

2) $A^2=-I$. Hence the eigenvalue of A^2 is only -1.
If $AX=\lambda X$, then $A^2X=A\lambda X=\lambda^2 X$. So A's eigenvalues are $\pm i$. But for the trace of A, we know trace $A=\sum\lambda$ (λ is the eigenvalue). If n is odd, trace A must be imaginary number. So A can not be a real matrix.

On the other hand, if $A_{3\times3}$ is real and $A^2=-I$, then A only has $\pm i$ for its eigenvalue. The imaginary eigenvalues for a real matrix appear in pair. How could odd number appears in pair.
So generally speaking, if n is odd, there is no real solution.

1.1.2:

① B is complex linear $\Rightarrow AB=BA$.
$B(iv)=B(Av)=BAv$. $iB(v)=A\cdot B\cdot v$. $\Rightarrow BAv=ABv$.
$(AB-BA)v=0$. for $\forall v\in R^n$. $\Rightarrow \dim(N(AB-BA))=n$
$\Rightarrow \text{rank}(AB-BA)=0 \Rightarrow AB=BA$.

② $AB=BA \Rightarrow$ B is complex linear
suppose $k=a+bi$. Then $kB(v)=(aI+bA)Bv=aBv+bABv$
$=aBv+bBAv$. $B(kv)=B(aIv+bAv)=aBv+bBAv$
$\therefore kB(v)=B(kv) \Rightarrow$ B is complex linear

③ $X^2=-I$. $A^2=-I$. But AX do not have to satisfy $AX=XA$.
for example, $A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $X=\begin{bmatrix}i&0\\0&-i\end{bmatrix}$ $AX=\begin{bmatrix}0&i\\i&0\end{bmatrix}$ $XA=\begin{bmatrix}0&-i\\-i&0\end{bmatrix}$
$AX\neq XA$. So X is not complex linear

④ $C^2 = I \Rightarrow \lambda(C) = \pm 1$. Hence C has two eigenspace. $\ker(C-I)$ and $\ker(C+I)$. $\dim N(C-I) = n - \text{rank}(C-I)$

$\dim(C+I) = n - \text{rank}(C+I)$

$\dim N(C-I) + \dim N(C+I) = 2n - \text{ran}(C+I) - \text{rank}(C-I)$

$\text{rank}(C+I) + \text{rank}(C-I) \geq \text{rank}(C+I-C+I) = n$. $\longrightarrow$ proof is attached in 4th page

而 $C^2 = I \quad \therefore (C-I)(C+I) = 0 \quad \therefore C(C+I) \subseteq N(C-I)$

$\therefore \dim C(C+I) \leq \dim N(C-I) \quad \therefore \text{rank}(C+I) \leq n - \text{rank}(C-I)$

$\therefore \quad n \leq \text{rank}(C+I) + \text{rank}(C-I) \leq n \quad \therefore \text{rank}(C+I) + \text{rank}(C-I) = n$

$\therefore \dim N(C-I) + \dim N(C+I) = n$

Thereby, we know C is diagonlizable, and only has eigenvalues 1 and -1. Then, $CA = -AC \Rightarrow CA^2 = -ACA \quad \therefore C = ACA$

$\therefore \text{trace}(C) = \text{trace}(ACA) = \text{trace}(AC \cdot A) = \text{trace}(A \cdot AC)$

$= -\text{trace}(C) \Rightarrow \text{trace}(C) = 0$ $\longrightarrow$ It is because $tr(A \cdot B) = tr(B \cdot A)$

However, $\text{trace}(C) = \sum \text{eigenvalue} \Rightarrow$ We have the same number of -1 and 1 in eigenvalues. $\Rightarrow$ eigenspace for 1 and -1 have the same dimension.

⑤ $A = \begin{bmatrix} J_2 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_2 \end{bmatrix}_{2n \times 2n}$ $J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $C_1 = \begin{bmatrix} 1 & & & & & & \\ & -1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & -1 \end{bmatrix}_{2n \times 2n}$ $C_2 = \begin{bmatrix} -1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & -1 & \\ & & & & & & 1 \end{bmatrix}_{2n \times 2n}$

1.1.3: ① C is $\mathbb{R}$-linear but not $\mathbb{C}$-linear. because we know that $\overline{w_1} \cdot \overline{w_2} = \overline{w_1 w_2}$ but may not equal to $w_1 \cdot \overline{w_2}$

So if w_1 is real, $w \cdot C\,v = w \cdot \bar{v}$ but if w is complex, $w \cdot C\, v$ may not equal to $\overline{wv} = \bar{w} \cdot \bar{v}$. Hence, C is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

② we know that $\mathbb{R} \subsetneq \mathbb{C}$. So if there is a complex-linear map T which satisfy $(a+bi) \cdot T \cdot v = T((a+bi) \cdot v)$, it must satisfy $a \cdot T \cdot v = T \cdot (av)$. So $\mathbb{C}$-linear implies $\mathbb{R}$-linear.

③ When we talk about $\mathbb{R}$-bases, we assume that the coefficients are real while the bases can be complex vector.

So, $\mathbb{R}$-bases are $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} i \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ i \end{bmatrix}$. with 4 dimension.

Therefore, $\mathbb{C}$-bases are $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with 2 dimension.

④ C-linearly independent implies R-linearly indepedent.
Because we know if $(a_1+b_1 i)V_1+\cdots+(a_n+b_n i)V_n=0$
$\Rightarrow a_1=b_1=a_2=b_2=\cdots=b_n=0$, then $a_1V_1+a_2V_2+\cdots+a_nV_n=0$
must implies $a_1=\cdots=a_n=0$. So C-independent $\Rightarrow$ R-independent

But, R-independent $\not\Rightarrow$ C-independent. for example:
$\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\i\end{bmatrix}$ is R-independent, but $1\cdot\begin{bmatrix}0\\1\end{bmatrix}+i\cdot\begin{bmatrix}0\\i\end{bmatrix}=0$.

⑤ R-spanning implies C-spanning. (It is the same reason as before, $R\subsetneqq C$)
But C-spanning does not imply R-spanning.
R-spanning $\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)$ is $a\begin{bmatrix}0\\1\end{bmatrix}+b\begin{bmatrix}1\\0\end{bmatrix}$ But C-spanning $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$
is $(a+bi)\begin{bmatrix}0\\1\end{bmatrix}+(c+di)\begin{bmatrix}1\\0\end{bmatrix}$

1.1.4

① $P\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $P\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}i\\-1\\-i\\1\end{bmatrix}$ $F_4=\begin{bmatrix}1&1&1&1\\1&\omega&\omega^2&\omega^3\\1&\omega^2&\omega^4&\omega^6\\1&\omega^3&\omega^6&\omega^9\end{bmatrix}$ $\omega=\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})$

② $F_4=\begin{bmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{bmatrix}$ we take $D=\begin{bmatrix}1&&&\\&i&&\\&&-1&\\&&&-i\end{bmatrix}$ then

$PF_4=\begin{bmatrix}1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\\1&1&1&1\end{bmatrix}=F_4D=\begin{bmatrix}1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\\1&1&1&1\end{bmatrix}$

$P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$ $\therefore P=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$ $P-\lambda I=\begin{bmatrix}-\lambda&1&0&0\\0&-\lambda&1&0\\0&0&-\lambda&1\\1&0&0&-\lambda\end{bmatrix}$

$\det(P-\lambda I)=0\Rightarrow\lambda^4=1$. $\therefore$ eigenvalues are $1,-1,i,-i$
and the corresponding eigenvectors are

$\lambda=1\Rightarrow x=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $\lambda=-1\Rightarrow x=\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$ $\lambda=i\Rightarrow x=\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}$ $\lambda=-i\Rightarrow x=\begin{bmatrix}1\\-i\\-1\\i\end{bmatrix}$

③ $C\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}C_0+C_1+C_2+C_3\\C_0+C_1+C_2+C_3\\C_0+C_1+C_2+C_3\\C_0+C_1+C_2+C_3\end{bmatrix}$ $C\cdot\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}=\begin{bmatrix}C_0+C_1i-C_2-C_3i\\C_3+C_0i-C_1-C_2i\\C_2+C_3i-C_0-C_1i\\C_1+C_2i-C_3-C_0i\end{bmatrix}$

④ $P^0=I$ $P=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$ $P^2=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$ $P^3=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$ $P^4=I$

$\therefore C = C_1 \cdot P + C_0 \cdot I + C_2 \cdot P^2 + C_3 \cdot P^3 = \sum_{j=0}^{3} C_j \cdot P^j$

$\therefore$ eigenvalues $\Rightarrow$ eigenvectors

$\lambda_1 = C_2 + C_1 + C_0 + C_3 \Rightarrow X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = C_2 - C_1 + C_0 - C_3 \Rightarrow X = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$\lambda_3 = C_2 - C_1 i - C_0 + C_3 i \Rightarrow X = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$ $\quad \lambda_4 = C_2 + C_1 i - C_0 - C_3 i \Rightarrow X = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$

proof : $\text{rank}(A) + \text{rank}(B) \geq \text{rank}(A+B)$

① $r(AB) \leq \min\{r(A), r(B)\}$ (It is obvious that $r(A) \leq r(AB)$ and $r(B) \leq r(AB)$)

② $r([A\ B]) \leq r(A) + r(B)$

$\because$ $[I \quad I]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = [A \quad B]$ and $r(AB) \leq \min\{r(A), r(B)\}$

$r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = r(A) + r(B) \Rightarrow r([A\ B]) \leq r(A) + r(B)$

③ $[A \quad B]\begin{bmatrix} I \\ I \end{bmatrix} = A + B$. so $r(A+B) \leq \min\{r([A\ B]), r(\begin{bmatrix} I \\ I \end{bmatrix})\}$

$\therefore r(A+B) \leq r([A\ B]) \leq r(A) + r(B)$

Hence: $r(C-I) + r(C+I) \geq r(C - I - C - I) = n$