Chenyang Zhao 赵晨阳 from the school of software
Info ID: 2020012363 Tel: 18015766633

Collaborations: Hanwen Cao, Mingdao Liu, Siyuan Chen

1.1: $A_t=\begin{bmatrix}1&1\\0&1+t\end{bmatrix}$ $t\in R$ and $t\neq 0$; Hence A_t can be diagnised:

$P(A_t)=\begin{vmatrix}1-\lambda & 1\\ & 1-\lambda+t\end{vmatrix}=(1-\lambda)(1+t-\lambda)$. $P(A_t)=0$ $\lambda_1=1$ $\lambda=t+1$

$(A-\lambda_1)x_1=0 \Rightarrow x_1=\begin{bmatrix}1\\0\end{bmatrix}$ $(A-\lambda_2)x_2=0 \Rightarrow x_2=\begin{bmatrix}1\\t\end{bmatrix}$ suppose $P=\begin{bmatrix}1&1\\&t\end{bmatrix}$

$P^{-1}=\frac{1}{\det P}\begin{bmatrix}t&-1\\&1\end{bmatrix}=\frac{1}{t}\begin{bmatrix}t&-1\\&1\end{bmatrix}=\begin{bmatrix}1&-\frac{1}{t}\\&\frac{1}{t}\end{bmatrix}$

$A_t=\begin{bmatrix}1&1\\&t\end{bmatrix}\begin{bmatrix}1&\\&1+t\end{bmatrix}\begin{bmatrix}1&-\frac{1}{t}\\&\frac{1}{t}\end{bmatrix}$ Then $f(A_t)=\begin{bmatrix}1&1\\&t\end{bmatrix}\begin{bmatrix}f(1)&\\&f(1+t)\end{bmatrix}\begin{bmatrix}1&-\frac{1}{t}\\&\frac{1}{t}\end{bmatrix}$

$\lim f(A_t)=\lim\limits_{t\to 0} f(A_t)=\lim\limits_{t\to 0}\left(\begin{bmatrix}1&1\\&t\end{bmatrix}\begin{bmatrix}1&\\&(1+t)^2\end{bmatrix}\begin{bmatrix}1&-\frac{1}{t}\\&\frac{1}{t}\end{bmatrix}\right)=\lim\limits_{t\to 0}\left(\begin{bmatrix}1&t+2\\&(t+1)^2\end{bmatrix}\right)=\begin{bmatrix}1&2\\&1\end{bmatrix}$

1.2: $A_t=\begin{bmatrix}1&1\\-t^2&1\end{bmatrix}$ $t\in R$ and $t\neq 0$ A_t can be diagnised when $t\neq 0$.

$P(A_t)=|A_t-\lambda I|=\begin{vmatrix}1-\lambda&1\\-t^2&1-\lambda\end{vmatrix}=(1-\lambda)^2+t^2$. $P(A_t)=0 \Rightarrow \lambda_1=ti+1$; $\lambda_2=-ti+1$

$(A-\lambda_1)x_1=0 \Rightarrow x_1=\begin{bmatrix}1\\ti\end{bmatrix}$ $(A-\lambda_2)x_2=0 \Rightarrow x_2=\begin{bmatrix}1\\-ti\end{bmatrix}$ suppose $P=\begin{bmatrix}1&1\\-ti&ti\end{bmatrix}$

$P^{-1}=\frac{1}{\det P}\begin{bmatrix}ti&-1\\ti&1\end{bmatrix}=\frac{1}{2ti}\begin{bmatrix}ti&-1\\ti&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{i}{2t}\\\frac{1}{2}&\frac{-i}{2t}\end{bmatrix}$

$A_t=\begin{bmatrix}1&1\\-ti&ti\end{bmatrix}\begin{bmatrix}1-ti&\\&1+ti\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{i}{2t}\\\frac{1}{2}&-\frac{i}{2t}\end{bmatrix}$ $f(1-ti)=(1-ti)\cdot|1-ti|=\sqrt{1+t^2}\cdot(1-ti)$
$f(1+ti)=(1+ti)\cdot|1+ti|=\sqrt{1+t^2}\cdot(1+ti)$

$\lim\limits_{t\to 0} f(A_t)=\lim\limits_{t\to 0}\left(\begin{bmatrix}1&1\\-ti&ti\end{bmatrix}\begin{bmatrix}f(1-ti)&\\&f(1+ti)\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{i}{2t}\\\frac{1}{2}&-\frac{i}{2t}\end{bmatrix}\right)$

$=\lim\limits_{t\to 0}\left(\begin{bmatrix}1&1\\-ti&ti\end{bmatrix}\begin{bmatrix}1-ti&\\&1+ti\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{i}{2t}\\\frac{1}{2}&-\frac{i}{2t}\end{bmatrix}\cdot\sqrt{1+t^2}\right)=\lim\limits_{t\to 0}\left(\sqrt{1+t^2}\begin{bmatrix}1&1\\-t^2&1\end{bmatrix}\right)=\begin{bmatrix}1&1\\&1\end{bmatrix}$

We define derivative $f(x)$ at x_0 as: $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ or $\lim\limits_{t\to 0}\frac{f(x_0+t)-f(x_0)}{t}$

However: Because a real matrix A may have imaginary eigenvalue
So $f(x)$ must also be well-defined in complex field.

2.1. $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots + (-1)^{m-1}\frac{1}{(2m-1)!}x^{2m-1} + o(x^{2m})$

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots + (-1)^{m}\frac{1}{(2m)!}x^{2m} + o(x^{2m+1})$

$$\sin x = \sum_{K=0}^{+\infty} \frac{(-1)^{K+1} \cdot x^{2K+1}}{(2K+1)!} \quad \sin(At) = \sum_{K=0}^{+\infty} \frac{(-1)^{K+1} \cdot (At)^{2K+1}}{(2K+1)!} \quad \frac{d\sin(At)}{dt} = \frac{d}{dt}\sum_{K=0}^{+\infty} \frac{(-1)^{K+1}(At)^{2K+1}}{(2K+1)!} = \sum_{K=0}^{+\infty} \frac{(2K+1)A^{2K+1} \cdot t^{2K} \cdot (-1)^{K+1}}{(2K+1)!}$$

$$\sum_{K=0}^{+\infty} \frac{A^{2K+1} \cdot t^{2K} (-1)^{K+1}}{(2K)!} = A \cdot \sum_{K=0}^{+\infty} \frac{(At)^{2K}(-1)^{K+1}}{(2K)!}$$ We also know that $\cos(At) = \sum_{K=0}^{+\infty} \frac{(At)^{2K} \cdot (-1)^{K+1}}{(2K)!}$

Hence: $\frac{d\sin(At)}{dt} = A \cdot \cos(At)$

Lemma: $\frac{d}{dt} f(At) = A f'(tA)$.

2.2. take $A_x = \begin{bmatrix} 2A & A \\ & (2+x)A \end{bmatrix}$ $\quad f(A) = \lim_{x \to 0} f(A_x)$

for A_x, $x \neq 0$.

$$A_x = \begin{bmatrix} I & \frac{1}{x}I \\ & I \end{bmatrix}\begin{bmatrix} 2A & \\ & (2+x)A \end{bmatrix}\begin{bmatrix} I & -\frac{1}{x}I \\ & I \end{bmatrix}$$

$$f(A_x) = \begin{bmatrix} I & \frac{1}{x}I \\ & I \end{bmatrix}\begin{bmatrix} f(2A) & \\ & f((2+x)A) \end{bmatrix}\begin{bmatrix} I & -\frac{1}{x}I \\ & I \end{bmatrix} = \begin{bmatrix} f(2A) & \frac{f((2+x)A) - f(2A)}{x} \\ & f((2+x)A) \end{bmatrix}$$

note that $\lim_{x \to 0} \frac{f((2+x)A) - f(2A)}{x} = \frac{d}{dx} f(Ax)\Big|_{x=2}$. from Lemma: We know $\frac{d}{dx} f(Ax)\Big|_{x=2} = A \cdot f'(Ax)\Big|_{x=2}$

$= A \cdot f'(2A)$

Hence: $f\left(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}\right) = \lim_{x \to 0} f(A_x) = \lim_{x \to 0}\left(\begin{bmatrix} f(2A) & \frac{f((2+x)A) - f(2A)}{x} \\ & f((2+x)A) \end{bmatrix}\right) = \begin{bmatrix} f(2A) & A \cdot f'(2A) \\ & f(2A) \end{bmatrix} \Rightarrow B = A \cdot f'(2A)$

3. Let $f(x) = x^2$. $A = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $f'(x) = 2x$. $A + Bt = \begin{bmatrix} 1+t & t \\ & 2+t \end{bmatrix}$ Then we diagnised

$P(A+Bt) = |A+Bt-\lambda I| = (1+t-\lambda)(2+t-\lambda)$ $\quad P(A+Bt) = 0 \Rightarrow \lambda_1 = 1+t \quad \lambda_2 = 2+t$

$(A+Bt-\lambda_1 I)X_1 = 0 \Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad (A+Bt-\lambda_2 I)X_2 = 0 \Rightarrow X_2 = \begin{bmatrix} t \\ -1-t \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & t \\ 0 & -1-t \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} -1-t & -t \\ & 1 \end{bmatrix} \cdot \frac{-1}{1+t} = \begin{bmatrix} 1 & \frac{t}{t+1} \\ & -\frac{1}{t+1} \end{bmatrix}$

$$A+Bt = \begin{bmatrix} 1 & t \\ 0 & -1-t \end{bmatrix}\begin{bmatrix} t+1 & \\ & t+2 \end{bmatrix}\begin{bmatrix} 1 & \frac{t}{t+1} \\ & -\frac{1}{t+1} \end{bmatrix} \quad f(A+Bt) = \begin{bmatrix} 1 & t \\ 0 & -1-t \end{bmatrix}\begin{bmatrix} f(t+1) & \\ & f(t+2) \end{bmatrix}\begin{bmatrix} 1 & \frac{t}{t+1} \\ & -\frac{1}{t+1} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & -1-t \end{bmatrix}\begin{bmatrix} (t+1)^2 & \\ & (t+2)^2 \end{bmatrix}\begin{bmatrix} 1 & \frac{t}{t+1} \\ & -\frac{1}{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)^2 & 2t^2+3t \\ & (2+t)^2 \end{bmatrix} = \begin{bmatrix} t^2+2t+1 & 2t^2+3t \\ & t^2+4t+4 \end{bmatrix}$$

$$\frac{d f(At+B)}{dt} = \frac{f(A(t+\Delta t)+B) - f(At+B)}{\Delta t} = \frac{\begin{bmatrix} 2\Delta t \cdot t + (\Delta t)^2 - 2\Delta t & 4\Delta t \cdot t + 3\Delta t \\ & 2\Delta t \cdot t + (\Delta t)^2 + 4\Delta t \end{bmatrix}}{\Delta t} = \begin{bmatrix} 2t+2 & 4t+3 \\ & 2t+4 \end{bmatrix}$$

$$\frac{d f(At+B)}{dt}\Big|_{t=0} = \begin{bmatrix} 2 & 3 \\ & 4 \end{bmatrix}$$

$A = I \cdot \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \cdot I$ $\quad f'(A) = I \cdot \begin{bmatrix} f'(1) & \\ & f'(2) \end{bmatrix} \cdot I = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix}$

$f'(A) \cdot B = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ & 4 \end{bmatrix} \neq \frac{d f(At+B)}{dt}\Big|_{t=0}$

Lemma: $\frac{d}{dt} f(At) = A' f'(tA)$.

To prove: consider a Jordan block $J(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}_{n\times n}$ $A = PJP^{-1}$ $tA = PtJP^{-1}$ find Jordan norm of $tJ(\lambda)$

Let Nilpotent $N = J(\lambda) - \lambda I_{n\times n}$. Hence $\ker(tJ(\lambda) - t\lambda I) = \ker(tN)$

$\ker((tJ(\lambda) - t\lambda I)^2) = \ker((tN)^2) \cdots \ker((tJ(\lambda) - t\lambda I)^{n-1}) = \ker((tN)^{n-1})$

pick basis $P = \mathrm{Diag}(t^{n-1}, t^{n-2}, \cdots, t, 1)$.

$$tJ(\lambda)\cdot P = \begin{bmatrix} t\lambda & t & & & & \\ & t\lambda & t & & & \\ & & t\lambda & t & & \\ & & & \ddots & \ddots & \\ & & & & t\lambda & t \\ & & & & & t\lambda \end{bmatrix} \begin{bmatrix} t^{n-1} & & & & & \\ & t^{n-2} & & & & \\ & & t^{n-3} & & & \\ & & & \ddots & & \\ & & & & t & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} t^n\lambda & t^{n-1} & & & & \\ & t^{n-1}\lambda & t^{n-2} & & & \\ & & t^{n-2}\lambda & t^{n-3} & & \\ & & & \ddots & \ddots & \\ & & & & t^2\lambda & t \\ & & & & & t\lambda \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} t^{1-n} & & & & & \\ & t^{2-n} & & & & \\ & & t^{3-n} & & & \\ & & & \ddots & & \\ & & & & t^{-1} & \\ & & & & & 1 \end{bmatrix} \quad P^{-1}\cdot tJ(\lambda)\cdot P = \begin{bmatrix} t^{1-n} & & & & & \\ & t^{2-n} & & & & \\ & & t^{3-n} & & & \\ & & & \ddots & & \\ & & & & t^{-1} & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} t^n\lambda & t^{n-1} & & & & \\ & t^{n-1}\lambda & t^{n-2} & & & \\ & & t^{n-2}\lambda & t^{n-3} & & \\ & & & \ddots & \ddots & \\ & & & & t^2\lambda & t \\ & & & & & t\lambda \end{bmatrix} = \begin{bmatrix} t\lambda & 1 & & & & \\ & t\lambda & 1 & & & \\ & & t\lambda & 1 & & \\ & & & \ddots & \ddots & \\ & & & & t\lambda & 1 \\ & & & & & t\lambda \end{bmatrix} = J(\lambda t)$$

ie. $tJ(\lambda) = P\cdot J(\lambda t)\cdot P^{-1}$ Then by definition:

$$f(tJ(\lambda)) = P\cdot \begin{bmatrix} \frac{f(t\lambda)}{0!} & \frac{f'(t\lambda)}{1!} & \cdots & \frac{f^{(n-1)}(t\lambda)}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \frac{f'(t\lambda)}{1!} \\ & & & \frac{f(t\lambda)}{0!} \end{bmatrix} \cdot P^{-1}$$

$$= \begin{bmatrix} t^{n-1} & & & & & \\ & t^{n-2} & & & & \\ & & t^{n-3} & & & \\ & & & \ddots & & \\ & & & & t & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \frac{f(t\lambda)}{0!} & \frac{f'(t\lambda)}{1!} & \cdots & \frac{f^{(n-1)}(t\lambda)}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \frac{f'(t\lambda)}{1!} \\ & & & \frac{f(t\lambda)}{0!} \end{bmatrix} \begin{bmatrix} t^{1-n} & & & & & \\ & t^{2-n} & & & & \\ & & t^{3-n} & & & \\ & & & \ddots & & \\ & & & & t^{-1} & \\ & & & & & 1 \end{bmatrix} = \begin{bmatrix} \frac{f(t\lambda)}{0!} & \frac{f'(t\lambda)t}{1!} & \cdots & \frac{f^{(n-1)}(t\lambda)t^{n-1}}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \frac{f'(t\lambda)t}{1!} \\ & & & \frac{f(t\lambda)}{0!} \end{bmatrix}$$

$$= \sum_{k=0}^{n-1} t^k \cdot \frac{f^{(k)}(t\lambda)}{k!} N^k = f(t\lambda)I + \sum_{k=1}^{n-1} t^k \cdot \frac{f^{(k)}(t\lambda)}{k!} N^k$$

$$\frac{df(tJ(\lambda))}{dt} = \lambda\cdot f'(t\lambda)\cdot I + \sum_{k=1}^{n-1} t^k \lambda \frac{f^{(k+1)}(t\lambda)}{k!} N^k + \sum_{k=1}^{n-1} k\cdot t^{k-1} \frac{f^{(k)}(t\lambda)}{k!} N^k$$

$$= \lambda \sum_{k=0}^{n-1} t^k \lambda \frac{f^{(k+1)}(t\lambda)}{k!} N^k + \sum_{k=0}^{n-2} t^k \lambda \frac{f^{(k+1)}(t\lambda)}{k!} N^{k+1}$$

On the other hand $\frac{df(tJ(\lambda))}{dt} = \dfrac{d \sum_{k=0}^{n-1} t^k \cdot \frac{f^{(k)}(t\lambda)}{k!} N^k}{dt} = \sum_{k=0}^{n-1} t^k \cdot \frac{f^{(k+1)}(\lambda t)}{k!} \cdot N^k$