# Project

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# problem 1

> *More generally, with suitable assumptions on the smoothness of $f$, the solution to the inhomogeneous system*
>
> $$\frac{dy}{dt} = Ay + f(t, y), \quad y(0) = c, \quad y \in \mathbb{C}^n, A \in \mathbb{C}^{n\times n} \tag{1}$$
>
> *satisfies*
>
> $$y(t) = e^{At}c + \int_0^t e^{A(t-s)} f(s, y) ds \tag{2}$$

*By the deriviation formula, we get this:*

$$\frac{d}{dt}(e^{-At}y) = -Ae^{-At}y + e^{-At}y'(t) = e^{-At}\left(y'(t) - Ay\right) \tag{3}$$

*And we let *

$$\begin{gathered}(y'(t) - Ay) = f(t, y) \\ e^0 y(0) = c\end{gathered} \tag{4}$$

*integrate it from 0 to t, thus we have:*

$$\begin{gathered}e^{-At}y - e^0 y(0) = \int_0^t e^{-As} f(s, y) ds \\ e^{-At}y - c = \int_0^t e^{-As} f(s, y) ds \\ y = e^{At}c + \int_0^t e^{A(t-s)} f(s, y) ds\end{gathered} \tag{5}$$

# Problem 2

> Trigonometric matrix functions, as well as matrix roots, arise in the solution of second order differential equations. For example, the problem
>
> $$\frac{d^2y}{dt^2} + Ay = 0, \qquad y(0) = y_0, \quad y'(0) = y_0' \tag{2.7}$$
>
> has solution
>
> $$y(t) = \cos(\sqrt{A}t)y_0 + \left(\sqrt{A}\right)^{-1} \sin(\sqrt{A}t)y_0', \tag{2.8}$$
>
> where $\sqrt{A}$ denotes any square root of $A$ (see Problems 2.2 and 4.1). The solution exists for all $A$. When $A$ is singular (and $\sqrt{A}$ possibly does not exist) this formula is interpreted by expanding $\cos(\sqrt{A}t)$ and $\left(\sqrt{A}\right)^{-1} \sin(\sqrt{A}t)$ as power series in $A$.

*First, any even function $f(x)$ only have even power of $x$ as it's expansion.*

*(If some power of $x$ are odd, then $f(x)$ could not be a even function)*

*And we know odd function time odd function would be an even function, even function plus an even function would still get an even function, hence $f(x) = cosx + x^{-1}sinx$ is a even function for $x$. So the expansion of $y(t) = cos(\sqrt{A}t)y_0 + (\sqrt{A})^{-1}sin(\sqrt{A}t)y_0'$ is even power of $\sqrt{A}$. Since $(\sqrt{A})^{2n} = A^n$, then the expansion of $y(t)$ is just the power series of ¥. Since the solution actually never depends on which $\sqrt{A}$ we choose, so $y(t) = cos(\sqrt{A}t)y_0 + (\sqrt{A})^{-1}sin(\sqrt{A}t)y_0'$ is a solution for any square root $\sqrt{A}$ of $A$.*

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# Problem 3

> **Theorem 2.1** (spectrum splitting via sign function). *Let $A \in \mathbb{R}^{n\times n}$ have no pure imaginary eigenvalues and define $W = (\operatorname{sign}(A) + I)/2$. Let*
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> $$Q^T W \Pi = \begin{array}{c c} & \begin{array}{cc} q & n-q \end{array} \\ \begin{array}{c} q \\ n-q \end{array} & \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \end{array}$$
>
> *be a rank-revealing QR factorization, where $\Pi$ is a permutation matrix and $q = \operatorname{rank}(W)$. Then*
>
> $$Q^T A Q = \begin{array}{c c} & \begin{array}{cc} q & n-q \end{array} \\ \begin{array}{c} q \\ n-q \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{array},$$
>
> *where the eigenvalues of $A_{11}$ lie in the open right half-plane and those of $A_{22}$ lie in the open left half-plane.*

*For the Jordan Canonical Form of $A$, we have $A = ZJZ^{-1}, Z = [Z_1, Z_2], J = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}$, where $p + q = n$, and $J_1$ is $p \times p$ with all eigenvalues lying in the left complex plane, and $J_2$ is $q \times q$ with all eigenvalues lying in the right complex plane. Hence:*

$$\begin{aligned} \operatorname{sign}(A) =& Z \begin{bmatrix} \operatorname{sign}(J_1) & \\ & \operatorname{sign}(J_2) \end{bmatrix} Z^{-1} \\ =& Z \begin{bmatrix} -I_p & \\ & I_q \end{bmatrix} Z^{-1} \end{aligned}$$

*By definition, Since $W = (\operatorname{sign}(A) + I)/2 = Z \begin{bmatrix} O & \\ & I_q \end{bmatrix} Z^{-1}$, $rank(W) = q$, and we know $Range(W)$ is the invariant subspace for these eigenvalues of $A$ which are in the open right half-plane.*

*Let $Q = [Q_1, Q_2]$, where $Q_1$ is $n \times q$ and $Q_2$ is $n \times p$. Hence:*

$$W\Pi = [Q_1 Q_2]] \begin{bmatrix} R_{11} & R_{12} \\ O & O \end{bmatrix} = Q_1 [R_{11} R_{12}] \tag{6}$$

*And since $\pi$ is a permutation matrix. we have: $\pi^{-1} = \pi^T$, so:*

$$W = Q_1 [R_{11} R_{12}] \Pi^T \tag{7}$$

*Because $\operatorname{rank}(W) = q$, and $Q_1$ has full column rank, then the column of $Q_1$ is a set of orthogonal basis of $\operatorname{range}(W)$. Then $AQ_1 = Q_1 X$, where $X$ is a $q \times q$ matrix whose eigenvalues are those of $A$ which are in the open right half-plane.*

*Note that columns of $Q$ span the who vector space, so $\mathrm{range}(Q_2)$ is the invariant subspace for these eigenvalues of $A$ which are in the open left half-plane. So similarly,we have $AQ_2 = Q_2 Y$ where $Y$ is a $p \times p$ matrix matrix whose eigenvalues are those of $A$ which are in the open left half-plane.*

*Since $Q$ is a matrix in $QR$ factorization, then $Q$ is orthogonal, hence $Q_1^T Q_1 = I, Q_2^T Q_2 = I$, then $Q_1^T AQ_1 = X, Q_2^T AQ_1 = Q_2^T Q_1 X = 0, Q_2^T AQ_2 = Y$*

*All in all, we have:*

$$\begin{aligned} Q^T AQ = \begin{bmatrix} Q_1^T & Q_2^T \end{bmatrix} A \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= \begin{bmatrix} Q_1^T AQ_1 & Q_1^T AQ_2 \\ Q_2^T AQ_1 & Q_2^T AQ_2 \end{bmatrix} \\ &= \begin{bmatrix} X & Q_1^T AQ_2 \\ Q_2^T Q_1 X & Y \end{bmatrix} \\ = \begin{bmatrix} X & Q_1^T AQ_2 \\ O & Y \end{bmatrix} &:= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{aligned}$$

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# Problem 4

> The geometric mean of positive scalars can be generalized to Hermitian positive definite matrices in various ways, which to a greater or lesser extent possess the properties one would like of a mean. Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. The *geometric mean* $A \# B$ is defined as the unique Hermitian positive definite solution to $XA^{-1}X = B$, or (cf. (2.19)–(2.21))
>
> $$X = B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2} = B(B^{-1}A)^{1/2} = (AB^{-1})^{1/2}B, \qquad (2.26)$$
>
> where the last equality can be seen using Corollary 1.34. The geometric mean has the properties (see Problem 2.5)
>
> $$A \# A = A, \qquad (2.27a)$$
> $$(A \# B)^{-1} = A^{-1} \# B^{-1}, \qquad (2.27b)$$
> $$A \# B = B \# A, \qquad (2.27c)$$
> $$A \# B \le \frac{1}{2}(A+B), \qquad (2.27d)$$

*If $A$ and $B$ commute , then* ***whatever*** *completely depend on $A$ would commute with* ***whatever*** *completely depend on $B$. Thus we have $A^{1/2}$ commute with $B^{1/2}$, $\log(A)$commute with $\log(B)$ and finally $\log(A^{1/2})$ commute with $\log(B^{1/2})$*

*By the definition above, $A\#B$ is the unique Hermitian positive definite solution to $XA^{-1}X = B$. Suppose the solution is$X = A^{1/2}B^{1/2}$ Then*

$$X^* = (B^*)^{1/2}(A^*)^{1/2} = B^{1/2}A^{1/2} = A^{1/2}B^{1/2} = X. \qquad (8)$$

*Wow $X$ is Hermitian.*😄

*From our lecture note, at least, a Hermitian positive definite matrix is always diagonalizable. On the other hand A and B commute, then they are simultaneously diagonalizable.*

*Hence $\exists P$, s.t., $A = P\Lambda_a P^{-1}. B = P\Lambda_b P^{-1}, \Lambda_a, \Lambda_b$ are both diagonal matrix with positive entry on the diagonal. So $X = P\sqrt{\Lambda_a}P^{-1}P\sqrt{\Lambda_b}P^{-1} = P\sqrt{\Lambda_a\Lambda_b}P^{-1}$ ,which is still positive definite.*

*All in all, we get:*

$$\begin{aligned} XA^{-1}X &= A^{1/2}B^{1/2}A^{-1}A^{1/2}B^{1/2} = A^{1/2}B^{1/2}A^{-1/2}B^{1/2} \\ &= A^{1/2}A^{-1/2}B^{1/2}B^{1/2} = B \end{aligned} \qquad (9)$$

*Surely $X = A^{1/2}B^{1/2}$ is the geometric mean $A\#B$ we are searching for.*

*We have seen that $\log(A^{1/2})$ commute with $\log(B^{1/2})$, and From our lecture note , we know that if $AB = BA$ , then $e^{A+B} = e^A e^B = e^B e^A$.*

*So in the end:*

$$\begin{aligned} E(A,B) &= e^{\frac{1}{2}\log(A)+\frac{1}{2}\log(B)} = e^{\frac{1}{2}\log(A)} e^{\frac{1}{2}\log(B)} \\ &= e^{\log(\sqrt{A})} e^{\log(\sqrt{B})} \\ &= A^{\frac{1}{2}} B^{\frac{1}{2}} \end{aligned} \tag{10}$$

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# Problem 5

$$A \# A = A, \tag{2.27a}$$
$$(A \# B)^{-1} = A^{-1} \# B^{-1}, \tag{2.27b}$$
$$A \# B = B \# A, \tag{2.27c}$$
$$A \# B \leq \frac{1}{2}(A+B), \tag{2.27d}$$

***2.27 a :***

*By the definition above, $A\#B$ is the unique Hermitian positive definite solution to $XA^{-1}X = B$. Since the solution is unique, and we know $AA^{-1}A = A$ , and $A$ itself is a Hermitian positive definite matrix. So A is the solution, i.e. $A\#A = A$*

***2.27 b :***

By definition, we have: $A\#B = \left(AB^{-1}\right)^{1/2}B$. On the other hand, the inverse of a Hermitian positive definite matrix is still a Hermitian positive definite matrix. So $A^{-1}\#B^{-1} = \left(A^{-1}B\right)^{1/2}B^{-1}$. All in all $(A\#B)^{-1} = \left(AB^{-1}\right)^{-1/2}B^{-1} = B^{1/2}A^{-1/2}B^{-1} = \left(A^{-1}B\right)^{1/2}B^{-1} = A^{-1}\#B^{-1}$.

***2.27 c :***

*Let $X_1 = A\#B$, which is the unique Hermitian positive definite matrix that satisfies $X_1A^{-1}X_1 = B$. And Let $X_2 = B\#A$ , which is the unique Hermitian positive definite matrix that satisfies $X_2B^{-1}X_2 = A$. Apply $X_1^{-1}$ on both side of $X_1A^{-1}X_1 = B$, then we have $A^{-1} = X_1^{-1}BX_1^{-1}$ .Take inverse of both side, $X_1B^{-1}X_1 = A$ . This is the same as $X_2 = B\#A$ which satisfies $X_2B^{-1}X_2 = A$. Since the solution is unique, so we have $X_1 = A\#B = X_2 = B\#A$*

***2.27 d :***

*From page 46, we know Here,X ≥ 0 denotes that the Hermitian matrix X is positive semidefinite. And by the original definition, we have:$A\#B = \left(B^{-1/2}AB^{-1/2}\right)^{1/2}$*

*We want to prove that $X := \frac{A+B}{2} - A\#B$ is positive semidefinite. Here we go.*

*Since we know that if B is semipositive then $B^{1/2}$ is also semipositive, and the result of two semipositive matrix's product is still semipositive.*

*We denote* $T$ *as* $B^{-1/2}AB^{-1/2}$*, and* $T^* = (B^*)^{-1/2}A^*(B^*)^{-1/2} = B^{-1/2}AB^{-1/2} = T$ *. From problems above, it's positive-definite, so it can be diagonalized into* $T = B\Lambda B^{-1}$ *for some positive diagonal matrix * $\Lambda$ . *Thus we have *$T^{1/2} = B\Lambda^{1/2}B^{-1}$, *let* $\Lambda' = I - \Lambda^{1/2}$*, okay:*

$$\begin{aligned} &\frac{1}{2}\left(B^{-1/2}AB^{-1/2}+I\right)-\left(B^{-1/2}AB^{-1/2}\right)^{1/2} \\ &=\frac{1}{2}(T+I)-T^{1/2} \\ &=\frac{1}{2}\left(I-T^{1/2}\right)^2 \\ &=\frac{1}{2}\left(B\Lambda' B^{-1}\right)^2 \\ &=\frac{1}{2}B(\Lambda')^2B^{-1} \end{aligned} \tag{11}$$

*And we see$(\Lambda')^2$ is semipositive. So $\frac{1}{2}\left(B^{-1/2}AB^{-1/2}+I\right)-\left(B^{-1/2}AB^{-1/2}\right)^{1/2}$ is semipositive. Apply $B^{1/2}$ on both side.$

$$\begin{aligned} X &:= \frac{A+B}{2} - A\#B \\ &= \frac{1}{2}(A+B) - B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{1/2} \\ &= B^{1/2}(\frac{1}{2}\left(B^{-1/2}AB^{-1/2}+I\right)-\left(B^{-1/2}AB^{-1/2}\right)^{1/2})B^{1/2} \end{aligned} \tag{12}$$

*So* $X$ *is positive semidefinite in the end.*

# Problem 6

**2.6.** (Bhatia [65, 2007, p. 111]) Show that for Hermitian positive definite $A, B \in \mathbb{C}^{2\times 2}$,

$$A \# B = \frac{\sqrt{\alpha\beta}}{\sqrt{\det(\alpha^{-1}A+\beta^{-1}B)}}(\alpha^{-1}A+\beta^{-1}B),$$

with $\alpha^2 = \det(A)$, $\beta^2 = \det(B)$.

*Because* $A, B$ *are both hermitian positive definite, they have real positive determinant.* ***WLOG****, assume* $\alpha, \beta > 0$. Let $T = \left(A^{-1}B\right)^{1/2}$ with two eigenvalue $\lambda_1, \lambda_2 \in \mathbb{C}$ .

*Thus we have* $\operatorname{trace}(T) = \lambda_1 + \lambda_2$, $\det(T) = \lambda_1\lambda_2 = \det\left(A^{-1/2}\right)\det\left(B^{1/2}\right) = \frac{\beta}{\alpha}$ .

*Furthermore, we see that the eigenvalue of $(\alpha^{-1}I + \beta^{-1}T^2)$ are $(\beta^{-1}\lambda_1^2 + \alpha^{-1})$ and $(\beta^{-1}\lambda_2^2 + \alpha^{-1})$. Hence:

$$\begin{aligned} &\det\left(\alpha^{-1}I+\beta^{-1}X^2\right) = (\beta^{-1}\lambda_1^2+\alpha^{-1})(\beta^{-1}\lambda_2^2+\alpha^{-1}) \\ &= 1 + \frac{\alpha}{\beta}\lambda_1^2 + \frac{\alpha}{\beta}\lambda_2^2 + \frac{\alpha^2}{\beta^2}\lambda_1^2\lambda_2^2 = \frac{\alpha}{\beta}\times\frac{\beta}{\alpha} + \frac{\alpha}{\beta}\lambda_1^2 + \frac{\alpha}{\beta}\lambda_2^2 + \frac{\alpha}{\beta}\lambda_1\lambda_2 \\ &= \frac{\alpha}{\beta}\lambda_1^2 + \frac{\alpha}{\beta}\lambda_2^2 + 2\frac{\alpha}{\beta}\lambda_1\lambda_2 = \frac{\alpha}{\beta}(\lambda_1+\lambda_2)^2 \end{aligned} \tag{13}$$

*All in all:*

$$\begin{aligned} \det\left(\alpha^{-1}A+\beta^{-1}B\right) &= \det(A)\det\left(\alpha^{-1}I+\beta^{-1}X^2\right) \\ &= \alpha^2(\beta^{-1}\lambda_1^2+\alpha^{-1})(\beta^{-1}\lambda_2^2+\alpha^{-1}) \\ &= \tfrac{\alpha}{\beta}(\lambda_1+\lambda_2)^2 \\ &= \tfrac{\alpha}{\beta}\operatorname{trace}(T)^2 \end{aligned} \tag{14}$$

*The product of two hermitian positive definite matrix is still hermitian positive definite, so* $T$ *is still hermitian positive definite,* $\operatorname{trace}(T)>0$*;So* $\operatorname{trace}(T)=\sqrt{\frac{\beta}{\alpha}\det(\alpha^{-1}A+\beta^{-1}B)}$

*From **Cayley-Hamiltion theorem**, we have:*

*When* $A$ *has characteristic polynomial* $p(x)=\det(A-\lambda I)=x^n+c_1x^{n-1}+\ldots+c_n$

*then* $p(A)=0$

*And the characteristic polynomial for A is* $(x-\lambda_1)(x-\lambda_2)=x^2-(\lambda_1+\lambda_2)+\lambda_1\lambda_2$

*Hence We know that* $T^2-\operatorname{trace}(T)T+\det(T)I=0$

Then $\operatorname{trace}(T)\left(A^{-1}B\right)^{1/2}=T^2+\det(T)I$ *Multiply* $A$ *from the left,we have*

$$\begin{aligned}
trace(T)A(A^{-1}B)^{1/2} &= B+det(T)A. \\
trace(T)(B\#A) &= B+det(T)A. \\
trace(T)(A\#B) &= B+det(T)A. \\
A\#B &= \frac{B+det(T)A}{trace(T)} \\
&= \frac{B+\frac{\beta}{\alpha}A}{\sqrt{\frac{\beta}{\alpha}det(\alpha^{-1}A+\beta^{-1}B)}} \\
&= \frac{\sqrt{\alpha\beta}}{\sqrt{det(\alpha^{-1}A+\beta^{-1}B)}}(a^{-1}A+\beta^{-1}B).
\end{aligned} \tag{15}$$

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# Problem 7

**2.7**. Consider the Riccati equation $XAX=B$, where $A$ and $B$ are Hermitian positive definite. Show that the Hermitian positive definite solution $X$ can be computed as $R^{-1}(RBR^*)^{1/2}R^{-*}$, where $A=R^*R$ is a Cholesky factorization.

> P161 the Cholesky factor of A: $A=R^*R$ (Cholesky factorization: R upper triangular)

*From **problem 5**, we define* $A>0$ *if* $A$ *is hermitian positive definite.*

*Since* $B>0$*, thus for* $\forall x\in\mathbb{C}^n$, $x^*RBR^*x=(R^*x)^*B(R^*x)>0$*. thus* $RBR^*>0$, $\sqrt{RBR^*}>0$.

*Similarly,for* $\forall x\in\mathbb{C}^n$, $x^*R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}x=(R^{-*}x)^*(RBR^*)^{\frac{1}{2}}(R^{-*}x)>0$*. So* $R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}>0$ *as well.*

*From the definition of* $A\#B$*, we know that the solution to* $X>0, XAX=B$ *is unique. Thus we only need to prove:* $R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}AR^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}=B$.

$$\begin{aligned}
&R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}AR^{-1}(RBR^*)^{\frac{1}{2}}R^{-*} \\
&= R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}R^*RR^{-1}(RBR^*)^{\frac{1}{2}}R^{-*} \\
&= R^{-1}(RBR^*)R^{-*} \\
&= (R^{-1}R)B(R^*R^{-*}) = B.
\end{aligned} \tag{16}$$

*So* $X=R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}$ *is the solution to* $XAX=B$ *that we are searching for.*

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**完结撒花，好耶！** 😄