微积分A(2)期中复习

经73班　　罗承扬

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0/ 学期初的建议

- 重视作业, 一定认真完成作业, 切实理解方法

标准. 不会做的题,听完讲解后,自己能够独立做出来.

- 建议多预习、自学, 赶在大课进度前面

本学期所学内容:

- 多元函数微分学(比较容易)
- 含参数积分(难、抽象)

期中

- 多重积分和曲线曲面积分(理论简单但难于计算)
- 常数项级数(中等), 函数项级数(难), 幂级数(容易)

期末

1/ 多元连续函数、偏导数、全微分

1. 多元函数在一点处的极限

Def. $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, x_0\in\mathbb{R}^n, \mathrm{A}\in\mathbb{R}^m$, f在x_0的某个去心邻域$B_0(x_0,r)$中有定义. 若$\forall\varepsilon>0,\exists\delta\in(0,r)$, s.t.

$$\|f(x)-\mathrm{A}\|<\varepsilon,\quad \forall x\in B_0(x_0,\delta),$$

则称$x\to x_0$时, $f(x)$以A为极限, 记作$\lim\limits_{x\to x_0} f(x)=\mathrm{A}$.

Remark. $\lim\limits_{x\to x_0} f(x)=\mathrm{A}$, 则:

不论动点x沿什么路径趋于定点x_0, 都有$f(x)\to\mathrm{A}$.

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$不存在？

例. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x+y}$是否存在？

解：$\lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{x}{x+y} = \lim\limits_{x\to 0} \dfrac{x}{x} = \lim\limits_{x\to 0} 1 = 1$

$$\lim\limits_{\substack{y\to 0\\ x=0}} \frac{x}{x+y} = \lim\limits_{y\to 0} 0 = 0.$$

故 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x+y}$不存在. □

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$不存在？

例. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$是否存在？

解：$\lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{xy}{x^2+y^2} = \lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{0}{x^2} = 0$

$$\lim_{\substack{x\to 0\\ y=x}} \frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y=x}} \frac{x^2}{2x^2} = \lim_{\substack{x\to 0\\ y=x}} \frac{1}{2} = \frac{1}{2}$$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$不存在！

Question. 如何证明 $\lim\limits_{x \to x_0} f(x)$不存在？

例. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x+y}$是否存在？

解：$\lim\limits_{\substack{x\to 0 \\ y=0}} \dfrac{xy}{x+y} = 0,$

$$\lim_{\substack{x\to 0 \\ y=x^2-x}} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3 - x^2}{x^2} = -1.$$

故 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x+y}$不存在. □

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$不存在？

练. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x+y}$是否存在？

解：$\lim\limits_{\substack{x\to 0\\ y=0}} \dfrac{x^2}{x+y} = \lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$

$\lim\limits_{\substack{y\to 0\\ y=x^2-x}} \dfrac{x^2}{x+y} = \lim\limits_{y\to 0} \dfrac{x^2}{x^2} = \lim\limits_{y\to 0} 1 = 1$

故 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x+y}$不存在. □

Question. 如何证明 $\lim\limits_{x\to x_0} f(x)$ 不存在？

练. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x+y}$ 是否存在？

$y = x^3 - x$!

2. 多元函数极限的性质:四则运算、夹挤原理、复合极限定理

Thm. $f,g:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, x_0\in\mathbb{R}^n$,若 $\lim\limits_{x\to x_0} f(x)$与 $\lim\limits_{x\to x_0} g(x)$

都存在, 则

(1) $\lim\limits_{x\to x_0}\big(f(x)\pm g(x)\big)=\lim\limits_{x\to x_0} f(x)\pm\lim\limits_{x\to x_0} g(x)$;

(2) $m=1$时, $\lim\limits_{x\to x_0} f(x)g(x)=\lim\limits_{x\to x_0} f(x)\cdot\lim\limits_{x\to x_0} g(x)$;

(3) $m=1$且 $\lim\limits_{x\to x_0} g(x)\neq 0$时, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$.

2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

例. $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^l, g:f(\Omega)\subset\mathbb{R}^l\to\mathbb{R}^m$,若

$$\lim_{x\to x_0} f(x)=\mathrm{A},\ \lim_{y\to \mathrm{A}} g(y)=\mathrm{B},$$

且$\exists B(x_0,\delta)\subset\Omega, s.t.\forall x\in B(x_0,\delta)$, 有$f(x)\neq \mathrm{A}$,则

$$\lim_{x\to x_0}(g\circ f)(x)=\mathrm{B}.$$

Thm. (夹挤原理) $f,g,h:B_0(x_0,\delta)\subset\mathbb{R}^n\to\mathbb{R}$,若

$$f(x)\le g(x)\le h(x),\forall x\in B_0(x_0,\delta),$$

$$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} h(x)=\mathrm{A},$$

则

$$\lim_{x\to x_0} g(x)=\mathrm{A}.$$

思路. 均值不等式是常用技巧：$|xy|\le\dfrac{x^2+y^2}{2}$

1/ 多元连续函数、偏导数、全微分

2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

例(夹挤).

(1) $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^3+y^3}{x^2+y^2}=$__________;(2) $\lim\limits_{(x,y)\to(0,0)} x\sin\dfrac{1}{y}+y\cos\dfrac{1}{x}=$__________;

解.(1) $\dfrac{x^3+y^3}{x^2+y^2}=\dfrac{(x+y)(x^2-xy+y^2)}{x^2+y^2}$ $\because |x^2-xy+y^2|\le x^2+y^2+|xy|\le\dfrac{3}{2}(x^2+y^2)$

$$\therefore \left|\frac{(x+y)(x^2-xy+y^2)}{x^2+y^2}\right|\le\frac{3}{2}|x+y| \quad \therefore \lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}=0$$

$$(2)\left|x\sin\frac{1}{y}+y\cos\frac{1}{x}\right|\le\left|x\sin\frac{1}{y}\right|+\left|y\cos\frac{1}{x}\right|\le|x|+|y|$$

$$\therefore \lim_{(x,y)\to(0,0)} x\sin\frac{1}{y}+y\cos\frac{1}{x}=0$$

2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

例(复合极限定理允许了结合一元函数的一些极限).

$$\lim_{(x,y)\to(0,0)}\frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)}=\underline{\qquad\qquad}$$

解. 视$x^2+y^2=r$

$$\lim_{(x,y)\to(0,0)}\frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)}=\lim_{r\to0+}\frac{\sqrt[3]{1+r}-1}{\sin r}=\lim_{r\to0+}\frac{\frac{1}{3}r}{\sin r}=\frac{1}{3}$$

2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

练习 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy-\sin(xy)}{xy-xy\cos(xy)} =$ ________

解. 视 $xy=r$

$$\lim_{(x,y)\to(0,0)} \frac{xy-\sin(xy)}{xy-xy\cos(xy)} = \lim_{r\to 0} \frac{r-\sin(r)}{r-r\cos(r)} = \lim_{r\to 0} \frac{r-\sin r}{r\left(1-\cos r\right)}$$

$$= \lim_{r\to 0} \frac{r-\sin r}{\frac{1}{2}r^3} = 2\lim_{r\to 0} \frac{r-\sin r}{r^3} = 2\lim_{r\to 0} \frac{1-\cos r}{3r^2} = 1/3$$

3. 累次极限和二重极限

Def.(累次极限) $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y) \triangleq \lim\limits_{y\to y_0}\left(\lim\limits_{x\to x_0} f(x,y)\right)$

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) \triangleq \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right)$$

Remark. 任意固定$y\neq y_0$, 若 $\lim\limits_{x\to x_0} f(x,y)$存在, 记为

$$g(y)=\lim_{x\to x_0} f(x,y).$$

若 $\lim\limits_{y\to y_0} g(y)=A$, 则 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)=\lim\limits_{y\to y_0} g(y)=A.$

3. 累次极限和二重极限

Remark. 求算 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$时候,先计算 $\lim\limits_{x\to x_0} f(x,y)$，此时把$y$看做常数,

显然这次极限计算后x被消掉,之后再令$y\to y_0$.

例. 求算 $\lim\limits_{y\to y_0}\lim\limits_{x\to x_0} f(x,y)$时候,先计算 $\lim\limits_{x\to x_0} f(x,y)$，此时把$y$看做常数,

显然这次极限计算后x被消掉,之后再令$y\to y_0$.

1 / 多元连续函数、偏导数、全微分

例. (2020春) $D=\{(x,y)\,|\,x+y\neq 0\}$, $f(x,y)=\dfrac{x-y}{x+y}$

问: $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$, $\lim_{(x,y)\to(0,0)} f(x,y)$是否存在

解: $\lim_{x\to 0}\lim_{y\to 0}\dfrac{x-y}{x+y}=\lim_{x\to 0}\dfrac{x}{x}=\lim_{x\to 0}1=1$

$\lim_{y\to 0}\lim_{x\to 0}\dfrac{x-y}{x+y}=\lim_{y\to 0}\dfrac{-y}{y}=\lim_{y\to 0}-1=-1$

$\lim_{(x,y)\to(0,0)} f(x,y)$不存在

选择路径$y=2x$, $f(x,2x)=\dfrac{x-2x}{x+2x}=-\dfrac{1}{3}$

选择路径$y=3x$, $f(x,3x)=\dfrac{x-3x}{x+3x}=-\dfrac{1}{2}$

$\therefore$ 极限不存在

在(x_0,y_0)连续 $\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$

4. 向量值函数的连续

Def. 设$f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$, $x_0\in\Omega$, 若 $\lim\limits_{x\to x_0} f(x)=f(x_0)$, 也即

$\forall\varepsilon>0,\exists\delta>0, s.t.$

$$\|f(x)-f(x_0)\|<\varepsilon,\quad \forall x\in\Omega\bigcap B(x_0,\delta),$$

则称f在点x_0处连续, 称f的不连续点为间断点.

Def. 设$f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$, 若$f$在$\Omega$上点点连续, 则称$f$在$\Omega$上连续, 记作$f\in C(\Omega)$.

Remark. $f=(f_1,f_1,\ldots,f_m):\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$, 则

$$f\text{在点}x_0\text{连续}\Leftrightarrow f_i\text{在点}x_0\text{连续}, i=1,2,\cdots m.$$

例:讨论 $f(x,y)=\begin{cases}\dfrac{x^2y^2}{\left(x^2+y^2\right)^{3/2}} & (x,y)\neq(0,0)\\ 0 & \text{其它情形}\end{cases}$ 的连续性.

解:只需研究 $\lim_{(x,y)\to(0,0)}\dfrac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}$ 是否存在.

$$0\le\frac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}=\frac{(xy)^2}{\left(x^2+y^2\right)^{3/2}}\le\frac{\left(\dfrac{x^2+y^2}{2}\right)^2}{\left(x^2+y^2\right)^{3/2}}=\frac{1}{4}\sqrt{x^2+y^2}$$

$$\therefore \lim_{(x,y)\to(0,0)}\frac{x^2y^2}{\left(x^2+y^2\right)^{3/2}}=0$$

例:讨论 $f(x,y)=\begin{cases}1 & y=x^2, x>0 \\ 0 & \text{其它情形}\end{cases}$ 的连续性.

解:f 在开区域$\{(x,y)\,|\,x\neq\sqrt{y}\}$中为初等函数,故处处连续. 而$f$在曲线$x=\sqrt{y}$上每一点都不连续.

事实上,任取(x_0,y_0), $x_0=\sqrt{y_0}$, 当点列$\{P_k(x_k,y_k)\}$沿曲线$x=\sqrt{y}$趋于(x_0,y_0)时,$f(x_k,y_k)\to 1$; 当点列$\{P_k\}$沿直线$x=x_0$趋于(x_0,y_0)时,$f(x_k,y_k)\to 0$.□

Thm.(最值定理) 设$\Omega\subset\mathbb{R}^n$为有界闭集, $f\in C(\Omega)$, 则f在Ω上存在最大值M和最小值m, 即$\exists\xi,\eta\in\Omega, s.t.\forall x\in\Omega$, 都有

$$m=f(\xi)\le f(x)\le f(\eta)=M.$$

Thm.(介值定理) 设$\Omega\subset\mathbb{R}^n$为连通区域, $f\in C(\Omega), x_1,x_2\in\Omega$, $f(x_1)=\lambda\le\mu=f(x_2)$, 则$\forall\sigma\in[\lambda,\mu], \exists x\in\Omega, s.t. f(x)=\sigma$.

例(P24−T8) : $\lim_{x^2+y^2\to+\infty} f(x,y)=+\infty \Rightarrow f(x,y)$有最小值

证明 : $\lim_{x^2+y^2\to+\infty} f(x,y)=+\infty \Rightarrow$

$\forall M>0, \exists R>0, s.t. \forall (x,y)$,满足$x^2+y^2\ge R^2$, $f(x,y)\ge M$

$x^2+y^2\le R^2$是有界闭集, 故$f(x,y)$在$x^2+y^2\le R^2$有最小值

取$M=f(0,0)$,

$\exists R>0, s.t. \forall (x,y)$,满足$x^2+y^2\ge R^2$, $f(x,y)\ge f(0,0)$

$f(x,y)$在$x^2+y^2\le R^2$有最小值

$$f(x_0,y_0)=\min_{x^2+y^2\le R^2} f(x,y) \le f(0,0)\le f(x,y),\ \ \forall x^2+y^2\ge R^2$$

5. 偏导数

Def. $u=f(\mathrm{x})=f(x_1,x_2,\cdots,x_n)$在$\mathrm{x}_0=(x_0^{(1)},x_0^{(2)},\cdots,x_0^{(n)})\in\mathbb{R}^n$的某个邻域中有定义, 若极限

$$\lim_{\Delta x_i\to 0}\frac{\Delta_{x_i}u}{\Delta x_i}=\lim_{\Delta x_i\to 0}\frac{f(x_0^{(1)},\cdots,x_0^{(i-1)},x_0^{(i)}+\Delta x_i,x_0^{(i+1)},\cdots,x_0^{(n)})-f(\mathrm{x}_0)}{\Delta x_i}$$

存在, 则称之为$f(\mathrm{x})$在x_0关于x_i的偏导数, 记作$\dfrac{\partial f}{\partial x_i}(\mathrm{x}_0)$, $\dfrac{\partial u}{\partial x_i}(\mathrm{x}_0)$, $\left.\dfrac{\partial f}{\partial x_i}\right|_{\mathrm{x}_0}$, $\left.\dfrac{\partial u}{\partial x_i}\right|_{\mathrm{x}_0}$, $u'_{x_i}(\mathrm{x}_0)$或$f'_{x_i}(\mathrm{x}_0)$.

5. 偏导数

$$f'_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Remark: 1) 对某个变量求偏导数时, 视其余变量为常数, 按一元函数求导法则和公式去求.

2)求分段函数的偏导函数时, 用定义求分界点处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

3)求某一点的偏导数时, 可以先带入其他变量的值,使之完全退化为一元函数,再求导

例. $f(x,y)=x^2e^y+(x-1)\arctan\dfrac{y}{x}$, 求$f_x'(1,0)$.

解法一: $f(x,0)=x^2$, 所以$f_x'(1,0)=2$.

解法二:

$$f_x'(x,y)=2xe^y+\arctan\frac{y}{x}+(x-1)\cdot\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$=2xe^y+\arctan\frac{y}{x}+\frac{y(1-x)}{x^2+y^2}.$$

所以$f_x'(1,0)=2$.□

Remark: 求具体点处的偏导数时, 第一种方法较好.

5. 偏导数

4)偏导数仅仅说明了沿着坐标轴方向,函数是光滑的, 因此和连续性互不蕴含

例: $f(x,y)=\begin{cases}1 & y=x^2,x>0\\ 0 & \text{其它情形}\end{cases}$ 在(0, 0)处不连续, 俩偏导数都为0

偏导数的局限性:只看坐标轴方向, 不全面

——引出方向导数、可微两个概念

$(x+1)\sin y+\sin x$

5. 偏导数

例. $z=f(x,y)$偏导数存在, $\dfrac{\partial z}{\partial x}=\sin y+\cos x,\ f(0,y)=\sin y$, 求$f(x,y)=$________.

$\dfrac{\partial z}{\partial x}$的得出: 视$y$为常数, 对$x$求导 $\therefore\ f(x,y)=\int \sin y+\cos x dx$

$\therefore\ f(x,y)=x\sin y+\sin x+g(y)$

$\because\ g(y)=f(0,y)\ =\sin y$

$\therefore\ f(x,y)=(x+1)\sin y+\sin x$

6. 可微

$$f(x_0+\Delta x)-f(x_0)=\frac{\partial f}{\partial x_1}(x_0)\Delta x_1+\frac{\partial f}{\partial x_2}(x_0)\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}(x_0)\Delta x_n+o(\|\Delta x\|)$$

$$\Leftrightarrow \lim_{\Delta x\to O}\frac{f(x_0+\Delta x)-f(x_0)-\left(\frac{\partial f}{\partial x_1}(x_0)\Delta x_1+\frac{\partial f}{\partial x_2}(x_0)\Delta x_2+\cdots+\frac{\partial f}{\partial x_n}(x_0)\Delta x_n\right)}{\|\Delta x\|}=0$$

二元函数特殊情况

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-f_x'(x_0,y_0)(x-x_0)-f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

可微一定连续, 偏导数也一定存在.

7. 总结（二元函数版本的连续可偏导可微）

连续：$f(x_0, y_0) = \lim\limits_{(x,y)\to(0,0)} f(x, y)$

可偏导：$f_x'(x_0, y_0) = \lim\limits_{\Delta x\to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$，

$f_y'(x_0, y_0) = \lim\limits_{\Delta y\to 0} \dfrac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$.

可微 $\Leftrightarrow$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_x'(x_0,y_0)(x-x_0) - f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

例. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点的可微性

解： .Step1. 计算偏导数 $f(x,0)=\begin{cases}x^2\sin\dfrac{1}{x^2}, & x\neq 0\\ 0, & x=0\end{cases}$,

$$f_x'(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}x\sin\frac{1}{x^2}=0;$$

同理$f_y'(0,0)=0.$

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

本题中 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)}{\sqrt{x^2+y^2}}=0$是否成立?

例. $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$ 在原点的可微性

解: .

$$\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)\sin\dfrac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}=0 \qquad \therefore 可微$$

Hint. 分段函数分析可微性:

(1)用定义计算偏导数;(不是用求导法则)

(2)用定义验证可微:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-f(x_0,y_0)-f_x'(x_0,y_0)(x-x_0)-f_y'(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

例. P42-2(4)

$f(x,y)=|x-y|\varphi(x,y)$, $\varphi(x,y)$在(0,0)的邻域内连续, $\varphi(0,0)=0$

问: $f(x,y)=|x-y|\varphi(x,y)$是否可微

解. P42-2(4)

Step1. 计算偏导数

$$\left|\frac{|x|\varphi(x,0)}{x}\right|=|\varphi(x,0)|$$

$$x\to 0, |\varphi(x,0)|\to|\varphi(0,0)|=0$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{|x|\varphi(x,0)}{x}=0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)}=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{x\to 0}\frac{|y|\varphi(0,y)}{y}=0$$

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

例. P42-2(4)

$f(x,y)=|x-y|\varphi(x,y)$, $\varphi(x,y)$在$(0,0)$的邻域内连续, $\varphi(0,0)=0$

问: $f(x,y)=|x-y|\varphi(x,y)$是否可微

解. P42-2(4)

Step2. 考察 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=0$是否成立

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-f_x'(0,0)x-f_y'(0,0)y}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$=\lim_{(x,y)\to(0,0)}\frac{|x-y|\varphi(x,y)}{\sqrt{x^2+y^2}}=0$$

$$\left|\frac{|x-y|\varphi(x,y)}{\sqrt{x^2+y^2}}\right|\le|\varphi(x,y)|\frac{|x|+|y|}{\sqrt{x^2+y^2}}$$

$$\le 2|\varphi(x,y)|$$

$2|\varphi(x,y)|\to 0$, 当$x,y\to(0,0)$

Remark: 函数的连续性、可微性、偏导数存在性与偏导数连续性之间的蕴含关系图.

$$
\begin{tikzcd}
\text{连续} & \text{可微} \arrow[l] \arrow[d] \\
\text{偏导函数连续} \arrow[u] \arrow[ur] \arrow[r] & \text{偏导数存在}
\end{tikzcd}
$$

Def. f 在 $\mathrm{x}_0\in\mathbb{R}^n$ 的邻域中有定义, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$ 为非零向量, l 为过 x_0 沿 $\vec{v}$ 方向的射线, 若 t 的函数

$$g(t)=f(\mathrm{x}_0+\frac{\vec{v}}{\|\vec{v}\|}t)=f(\mathrm{x}_0^{(1)}+\frac{v_1}{\|\vec{v}\|}t,\cdots,\mathrm{x}_0^{(n)}+\frac{v_n}{\|\vec{v}\|}t)$$

在 $t=0$ 存在右导数, 即极限

$$\lim_{\substack{\mathrm{x}\to\mathrm{x}_0\\ \mathrm{x}\in l}}\frac{f(\mathrm{x})-f(\mathrm{x}_0)}{\|\mathrm{x}-\mathrm{x}_0\|}=\lim_{t\to0^+}\frac{g(t)-g(0)}{t}$$

存在, 则称该极限为 $f(\mathrm{x})$ 在 x_0 沿方向 $\vec{v}$ 的方向导数, 记作

$$\frac{\partial f(\mathrm{x}_0)}{\partial\vec{v}},\left.\frac{\partial f}{\partial\vec{v}}\right|_{\mathrm{x}_0}\text{ 或 }f'_{\vec{v}}(\mathrm{x}_0).$$

Remark. $\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$是函数$f(\mathrm{x})$在点$\mathrm{x}_0$沿方向$\vec{v}$的变化率.

Remark. $\dfrac{\partial f(\mathrm{x}_0)}{\partial x_i}$为$f$在$\mathrm{x}_0$沿$e_i=(0,\cdots 0,\overset{\text{第}i\text{个分量}}{\overset{\downarrow}{1}},0,\cdots 0)$的方向导数.

Thm. 设f在$\mathrm{x}_0\in\mathbb{R}^n$可微, $\vec{v}=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$为非零向量,

则方向导数$\dfrac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}$存在, 且

$$\frac{\partial f(\mathrm{x}_0)}{\partial \vec{v}}=\frac{\partial f(\mathrm{x}_0)}{\partial x_1}\frac{v_1}{\|\vec{v}\|}+\frac{\partial f(\mathrm{x}_0)}{\partial x_2}\frac{v_2}{\|\vec{v}\|}+\cdots+\frac{\partial f(\mathrm{x}_0)}{\partial x_n}\frac{v_n}{\|\vec{v}\|}.$$

例. (1)计算$f(x,y)=\sin(x+2y)$在$(0,0)$处,沿着$I=(1,1)$方向的方向导数;
(2)求出方向导数最大的方向(单位化为单位向量)

解. (1) $\frac{\partial f}{\partial x}(x,y)=\cos(x+2y), \frac{\partial f}{\partial y}(x,y)=2\cos(x+2y) \therefore \frac{\partial f}{\partial x}(0,0)=1, \frac{\partial f}{\partial y}(0,0)=2$

$$\therefore \frac{\partial f}{\partial I}(0,0)=1\times\frac{1}{\sqrt{2}}+2\times\frac{1}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

(2)求出方向导数最大的方向

设这一方向为$I=(\cos\theta,\sin\theta)$

$\therefore \frac{\partial f}{\partial I}(0,0)=1\times\cos\theta+2\times\sin\theta$,由柯西-施瓦茨不等式,$\frac{\partial f}{\partial I}(0,0)\le\sqrt{5}$,

当$(\cos\theta,\sin\theta)=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})$

1 / 多元连续函数、偏导数、全微分

例.（2020春模拟）$f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

(1) $f(x,y)$在$(0,0)$处的连续性？;(2) $f(x,y)$在$(0,0)$处两个一阶偏导数的存在性?;

(3) $f(x,y)$在$(0,0)$处是否可微？

解:(1) $|x^3+y^3|=|x+y||x^2-xy+y^2|\leq|x^2+|xy|+y^2||x+y|\leq\dfrac{3}{2}|x^2+y^2||x+y|$

$\therefore\left|\dfrac{x^3+y^3}{x^2+y^2}\right|\leq\dfrac{3}{2}|x+y|\quad\therefore\lim_{(x,y)\to(0,0)}\dfrac{x^3+y^3}{x^2+y^2}=0=f(0,0)$ ∴连续

(2) $\quad f'_x(0,0)=\lim_{x\to0}\dfrac{f(x,0)-f(0,0)}{x}=\lim_{x\to0}\dfrac{x-0}{x}=1\qquad f'_y(0,0)=1$

(3)不可微. $\quad f(x,y)-f(0,0)-xf'(0,0)-yf'(0,0)=\dfrac{x^3+y^3}{x^2+y^2}-x-y=-\dfrac{xy(x+y)}{x^2+y^2}$

考虑极限$\lim_{(x,y)\to(0,0)}\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}$是否存在,并且是否为0

1 / 多元连续函数、偏导数、全微分

例. (2020春模拟) $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2},(x,y)\neq(0,0)\\ 0,(x,y)=(0,0)\end{cases}$

(1) $f(x,y)$在$(0,0)$处的连续性？ ;(2) $f(x,y)$在$(0,0)$处两个一阶偏导数的存在性?;

(3) $f(x,y)$在$(0,0)$处是否可微？

(3)不可微. 考虑极限 $\lim_{(x,y)\to(0,0)}\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}$ 是否存在,并且是否为0

取$y=x$, $\dfrac{-\dfrac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}=\dfrac{-\dfrac{2x^3}{2x^2}}{\sqrt{2x^2}}=\dfrac{-x}{\sqrt{2}\,|x|}$ $\qquad \lim_{x\to0+}\dfrac{-x}{\sqrt{2}\,|x|}=-\dfrac{1}{\sqrt{2}}$

2)求分段函数的偏导函数时, 用定义求**分界点**处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

2/ 链锁法则和隐函数定理

•Chain Rule

$u=g(x):\Omega\subset\mathbb{R}^n\to\mathbb{R}^m,\ y=f(u):g(\Omega)\subset\mathbb{R}^m\to\mathbb{R}^k,$

$g(x)$在$x_0\in\Omega$可微, $f(u)$在$u_0=g(x_0)$可微, 则

$$J(f\circ g)\big|_{x_0}=J(f)\big|_{u_0}\cdot J(g)\big|_{x_0},$$

即$$\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0}=\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(u_1,u_2,\cdots,u_m)}\right|_{u_0}\cdot\left.\frac{\partial(u_1,u_2,\cdots,u_m)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0},$$

简记为$$\left.\frac{\partial y}{\partial x}\right|_{x_0}=\left.\frac{\partial y}{\partial u}\right|_{u_0}\cdot\left.\frac{\partial u}{\partial x}\right|_{x_0}.$$

$k=1$时, $$\frac{\partial y}{\partial x_i}=\frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i}+\frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i}+\cdots+\frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i},\ i=1,2,\cdots,n.$$

2 / 链锁法则和隐函数定理

例. $z = f(xy, x^2 + y^2)$, 计算z'_x, z'_y

解. $z'_x = f'_1(xy, x^2 + y^2)y + f'_2(xy, x^2 + y^2)2x \quad z'_y = f'_1(xy, x^2 + y^2)x + f'_2(xy, x^2 + y^2)2y$

例. $u = u(x, y, z)$, u在全空间可微, u满足

$$u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z, \text{其中} k > 0$$

证明: $ku(x, y, z) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

证. $\because u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z.$ 等式两边对t求导

$\because xu'_1(tx, ty, tz) + yu'_2(tx, ty, tz) + zu'_3(tx, ty, tz) = kt^{k-1}u(x, y, z)$

取$t = 1$, 得$ku(x, y, z) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

2 链锁法则和隐函数定理

例. (2020期末) $f \in C^{(2)}(\mathbb{R}), z = f(x^2 + xy + y^2)$, 计算$z'_y, z''_{xy}$在(1,1)的值.

解. $z'_y = f'(x^2 + xy + y^2)(2y + x) \quad \therefore z'_y(1,1) = 3f'(3)$

$\because z'_y(x,1) = f'(x^2 + x + 1)(2 + x)$

$\therefore z''_{yx}(x,1) = (z'_y(x,1))' = f''(x^2 + x + 1)(2 + x)^2 + f'(x^2 + x + 1)$

$\therefore z''_{yx}(1,1) = 9f''(3) + f'(3)$

2 / 链锁法则和隐函数定理

例. $u=u(x,y,z)$,u在全空间可微, u满足

$$ku(x,y,z)=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z},\forall t,x,y,z,\text{其中}k>0$$

证明:$u(tx,ty,tz)=t^k u(x,y,z)$

证. 构造辅助函数$F(t)=u(tx,ty,tz)-t^k u(x,y,z)$

$$F'(t)=xu_1'(tx,ty,tz)+yu_2'(tx,ty,tz)+zu_3'(tx,ty,tz)-kt^{k-1}u(x,y,z)$$

$$=\frac{1}{t}(txu_1'(tx,ty,tz)+tyu_2'(tx,ty,tz)+tzu_3'(tx,ty,tz)-kt^k u(x,y,z))$$

$$=\frac{1}{t}(ku(tx,ty,tz)-kt^k u(x,y,z))=\frac{k}{t}(u(tx,ty,tz)-t^k u(x,y,z))=\frac{k}{t}F(t)$$

$$\therefore F'(t)=\frac{k}{t}F(t)\Rightarrow F(t)=Ct^k\ \because F(1)=u(x,y,z)-u(x,y,z)=0\qquad\therefore C=0\ \therefore F(t)=0$$

Remark: $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m,(x,y)\mapsto F(x,y)$,若$\dfrac{\partial F}{\partial y}$可逆,

则$F(x,y)=0$确定隐"函数"$y=y(x)$,求$\dfrac{\partial y}{\partial x}$有两种方法:

• 套用定理:$\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

这里求Jaccobi矩阵时 x,y 相互独立!

• 将$F(x,y)=0$中y视为$y=y(x)$,利用复合映射的链式法则,方程组 $F(x,y(x))=0$两边对x求Jaccobi矩阵.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. φ可微, $x^2+z^2=y\varphi\left(\dfrac{z}{y}\right)$确定隐函数$z=z(x,y)$.求$z'_x, z'_y$.

解: 视$x^2+z^2=y\varphi(z/y)$中$z=z(x,y)$为隐函数. 两边分别对x,y求偏导, 有

$$2x+2zz'_x=y\varphi'(z/y)\cdot\frac{1}{y}z'_x,$$

$$2zz'_y=\varphi(z/y)+y\varphi'(z/y)\cdot\frac{1}{y^2}(yz'_y-z).$$

求解得

$$z'_x=\frac{2x}{\varphi'(z/y)-2z},\quad z'_y=\frac{y\varphi(z/y)-\varphi'(z/y)}{2yz-y\varphi'(z/y)}.\square$$

例. $u=f\left(x-ut,y-ut,z-ut\right),g(x,y,z)=0$,求$u'_x,u'_y$.

分析: 五个变量x,y,z,t,u,两个方程, 确定两个隐函数 $z=z(x,y,t)=z(x,y),u=u(x,y,t)$.

解法一: 视$u=f\left(x-ut,y-ut,z-ut\right)$中$z=z(x,y)$为隐函数, 两边分别对$x,y$求偏导, 有

$$u'_x=(1-tu'_x)f'_1+(-tu'_x)f'_2+(z'_x-tu'_x)f'_3,$$

$$u'_y=(-tu'_y)f'_1+(1-tu'_y)f'_2+(z'_y-tu'_y)f'_3.$$

其中 f'_1,f'_2,f'_3 在$(x-ut,y-ut,z-ut)$处取值.

视$g(x,y,z)=0$中$z=z(x,y)$, 两边对x,y求偏导,有

$$\begin{cases} g'_x + g'_z z'_x = 0, \\ g'_y + g'_z z'_y = 0, \end{cases} \Rightarrow \begin{cases} z'_x = -g'_x / g'_z\,, \\ z'_y = -g'_y / g'_z\,. \end{cases}$$

代入前两式,求解得

$$u'_x = \frac{f'_1 + f'_3 z'_x}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_1 g'_z - f'_3 g'_x}{[1 + t(f'_1 + f'_2 + f'_3)]g'_z}$$

$$u'_y = \frac{f'_2 + f'_3 z'_y}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_2 g'_z - f'_3 g'_y}{[1 + t(f'_1 + f'_2 + f'_3)]g'_z}.$$

例. (2020真题)$u(t)\in C^2(\mathbb{R}), z=u(\sqrt{x^2+y^2})$满足：

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=x^2+y^2(x^2+y^2>0)$$

证明：$u(t)$满足$u''+\frac{1}{t}u'=t^2$

证. $\because z=u(\sqrt{x^2+y^2})\ \therefore z'_x=\frac{x}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2}), z'_y=\frac{y}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2})$

$$\therefore z''_{xx}=\frac{y^2}{(x^2+y^2)^{3/2}}u'(\sqrt{x^2+y^2})+\frac{x^2}{(x^2+y^2)}u''(\sqrt{x^2+y^2})$$

$$z''_{yy}=\frac{x^2}{(x^2+y^2)^{3/2}}u'(\sqrt{x^2+y^2})++\frac{y^2}{(x^2+y^2)}u''(\sqrt{x^2+y^2})$$

$\therefore z''_{xx}+z''_{yy}=\frac{1}{\sqrt{x^2+y^2}}u'(\sqrt{x^2+y^2})+u''(\sqrt{x^2+y^2})=x^2+y^2$ 令$t=\sqrt{x^2+y^2}$即可.

例. (2020模拟)二阶连续可微函数$z=z(x,y)$满足:$x^3+y^3+z^3=x+y+z$

计算$\dfrac{\partial^2 z}{\partial x\partial y}$

解. $\because x^3+y^3+z^3(x,y)=x+y+z(x,y)$

$\therefore$ 对x偏导,$3x^2+3z^2\dfrac{\partial z}{\partial x}=1+\dfrac{\partial z}{\partial x}$ $\quad\therefore \dfrac{\partial z}{\partial x}=\dfrac{1-3x^2}{3z^2-1},\dfrac{\partial z}{\partial y}=\dfrac{1-3y^2}{3z^2-1}$

再对y偏导,$6z\dfrac{\partial z}{\partial y}\dfrac{\partial z}{\partial x}+3z^2\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial x\partial y}$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{6z}{1-3z^2}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}=\frac{6z}{(1-3z^2)^3}(1-3x^2)(1-3y^2)$$

2 / 链锁法则和隐函数定理

例. (2020期末) $y=y(x), z=z(x)$由方程组$\begin{cases} x^3+y^3-z^3=10 \\ x+y+z=0 \end{cases}$在$(1,1,-2)$处确定隐函数

求$y=y(x), z=z(x)$在$x=1$处的导数

解. 在方程组$\begin{cases} x^3+y^3-z^3=10 \\ x+y+z=0 \end{cases}$两边对$x$求导 得$\begin{cases} 3x^2+3y^2y'-3z^2z'=0 \\ 1+y'+z'=0 \end{cases}$

按照$x=1, y=1, z=-2$带入得$\begin{cases} 3+3y'-12z'=0 \\ 1+y'+z'=0 \end{cases}$ $\therefore \begin{cases} y'(1)=-1 \\ z'(1)=0 \end{cases}$

例. (2020期中-类似)$f \in C^2(\mathbb{R}^2)$, $f > 0$, $f''_{xy} f = f'_x f'_y$,

求证:存在一元函数$u(x), v(y)$, s.t. $f(x,y) = u(x)v(y)$

分析:

$$\ln f(x,y) = \ln u(x) + \ln v(y) \Leftrightarrow \frac{\partial \ln f(x,y)}{\partial x} = \frac{u'(x)}{u(x)} \Leftrightarrow \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = 0$$

证明:

$$\frac{\partial \ln f(x,y)}{\partial x} = \frac{1}{f}\frac{\partial f}{\partial x},$$

$$\frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = \frac{\partial(\frac{f'_x}{f})}{\partial y} = \frac{f''_{xy} f - f'_y f'_x}{f^2} = 0$$

例. $f\in C^1(\mathbb{R}^2), a\dfrac{\partial f}{\partial x}+b\dfrac{\partial f}{\partial y}=0$, 选取合适的变量替换 $\begin{cases}u=x+py\\ v=x+qy\end{cases}$, p,q为常数,

将原方程化为 $\dfrac{\partial f}{\partial u}=0$, 从而解为 $f=g(x+qy)$

解. $\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial u}+\dfrac{\partial f}{\partial v}$ $\qquad$ $\dfrac{\partial f}{\partial y}=p\dfrac{\partial f}{\partial u}+q\dfrac{\partial f}{\partial v}$

$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=(a+bp)\frac{\partial f}{\partial u}+(a+bq)\frac{\partial f}{\partial v}$$

$$\therefore a+bp=1, a+bq=0\Rightarrow p=\frac{1-a}{b}, q=-\frac{a}{b} \quad \therefore f=g(x-\frac{a}{b}y)$$

例. (2020模拟)$f \in C^2(\mathbb{R}^2)$, 满足(1) $f_x' = f_y'$, (2) $f(x,0) > 0$;

证明: $f(x,y) > 0$.

分析. $f_x' - f_y' = 0 \Rightarrow$

$$u = f(x+y, x-y), u_y' = f_x'(x+y, x-y) - f_y'(x+y, x-y) = 0$$

进一步$f_x' - f_y' = 0 \Rightarrow u = f(x+y, -y), u_y' = f_x'(x+y, -y) - f_y'(x+y, -y) = 0$

$$\therefore u = f(x+y, -y) = f(x,0) > 0$$

$$\therefore f(x,y) = f(x+y,0) > 0$$

Thm. 设n元函数f在$B(x_0,\delta)$中二阶连续可微, 则

$\forall x_0+\Delta x\in B(x_0,\delta), \exists\theta\in(0,1), s.t.$

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x+\frac{1}{2}(\Delta x)^T H_f(x_0+\theta\Delta x)\Delta x$$

(称为带*Lagrange*余项的一阶*Taylor*公式), 且

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x+\frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x+o(\|\Delta x\|^2), \Delta x\to 0\text{时}$$

(称为带*Peano*余项的二阶*Taylor*公式).

Thm. 设函数$f(x,y)$在区域D中$n+1$阶连续可微，$M_0(x_0,y_0)\in D, M(x,y)\in D$，且线段$\overline{M_0M}$完全包含在$D$中. 记

$$h=x-x_0, k=y-y_0,$$

记算子

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m \triangleq \sum_{i=0}^{m} C_m^i h^i k^{m-i}\frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则f在点(x_0,y_0)有

(1)带$Lagrange$余项的n阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$$

$$(0<\theta<1)$$

(2)带$Peano$余项的$n+1$阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(x_0,y_0)$$

$$+o\left(\left(\sqrt{h^2+k^2}\right)^{n+1}\right).$$

Note.一般来说, 我们不用如此复杂的公式, 而是设法化为一元函数的泰勒公式

例. $\cos(x^2+y^2)$在$(0,0)$的8阶带Peano余项的Taylor展开式.

解: $\cos t = 1-\dfrac{t^2}{2!}+\dfrac{t^4}{4!}+\cdots+(-1)^n\dfrac{t^{2n}}{(2n)!}+o(t^{2n}),\ t\to 0$时.

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\cdots+(-1)^n\frac{(x^2+y^2)^{2n}}{(2n)!}$$

$$+o((x^2+y^2)^{2n}),\ x^2+y^2\to 0\text{时}.$$

令$n=2$得

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\frac{(x^2+y^2)^4}{4!}+o((x^2+y^2)^4),$$

$$x^2+y^2\to 0\text{时}.\square$$

例：求$f(x,y)=x^y$在点(1,1)的邻域内带Peano余项的3阶Taylor公式

$$f(x,y)=x^y=(x-1+1)^y \qquad \rho=\sqrt{(x-1)^2+(y-1)^2}$$

解：$$(x-1+1)^y=1+y(x-1)+\frac{y(y-1)}{2!}(x-1)^2+\frac{y(y-1)(y-2)}{3!}(x-1)^3+o(\rho^3)$$

$$y(x-1)=(y-1)(x-1)+(x-1)$$

$$\frac{y(y-1)}{2!}(x-1)^2=\frac{1}{2}(x-1)^2(y-1)+\frac{1}{2}(x-1)^2(y-1)^2$$

$$\frac{1}{6}y(y-1)(y-2)(x-1)^3=\frac{1}{6}(y-1)(y^2-2y)(x-1)^3=\frac{1}{6}(y-1)^3(x-1)^3-\frac{1}{6}(y-1)(x-1)^3$$

$$\therefore \text{原式}=1+(x-1)+(y-1)(x-1)+\frac{(y-1)}{2!}(x-1)^2+o(\rho^3)$$

例. $\ln(2+x+y+xy)$在$(0,0)$带Peano余项的2阶Taylor展开.

解：$x+y+xy\to 0$ 时，

$$\ln(2+x+y+xy)=\ln 2+\ln\left(1+\frac{x+y+xy}{2}\right)$$

$$=\ln 2+\frac{x+y+xy}{2}-\frac{1}{2}\left(\frac{x+y+xy}{2}\right)^2+o\left((x+y+xy)^2\right)$$

$x^2+y^2\to 0$时，必有$x+y+xy\to 0$ 时，因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2}=\frac{o((x+y+xy)^2)}{(x+y+xy)^2}\cdot\frac{(x+y+xy)^2}{x^2+y^2}\to 0,$$

$$\ln(2+x+y+xy)$$

$$=\ln 2+\frac{x+y}{2}-\frac{x^2+y^2-2xy}{8}+o(x^2+y^2).\square$$

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$,求$z=z(x,y)$在(0,0)带Peano余项的2阶Taylor展开.

解: 计算$\frac{\partial z}{\partial x}(0,0),\frac{\partial z}{\partial y}(0,0),\frac{\partial^2 z}{\partial x^2}(0,0),\frac{\partial^2 z}{\partial y^2}(0,0),\frac{\partial^2 z}{\partial y\partial x}(0,0)$

在$\sin(x+y)+ze^z-ye^x=0$两边同时对x求偏导, 得

$$\cos(x+y)+\frac{\partial z}{\partial x}e^z+\frac{\partial z}{\partial x}ze^z-ye^x=0$$

在$\sin(x+y)+ze^z-ye^x=0$两边同时对y求偏导, 得

$$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$$

$x=0, y=0$时, $z=0$

$$\therefore \frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=0$$

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$,求$z=z(x,y)$在$(0,0)$带Peano余项的2阶Taylor展开.

解： 计算$\frac{\partial z}{\partial x}(0,0),\frac{\partial z}{\partial y}(0,0),\frac{\partial^2 z}{\partial x^2}(0,0),\frac{\partial^2 z}{\partial y^2}(0,0),\frac{\partial^2 z}{\partial y\partial x}(0,0)$

在$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$两边再对$x$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial y\partial x}e^z+\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}e^z+\frac{\partial^2 z}{\partial y\partial x}ze^z+\frac{\partial z}{\partial y}\frac{\partial ze^z}{\partial x}-e^x=0$$

取$x=y=z=0$ $\quad\therefore\frac{\partial^2 z}{\partial y\partial x}=1$

在$\cos(x+y)+\frac{\partial z}{\partial y}e^z+\frac{\partial z}{\partial y}ze^z-e^x=0$两边再对$y$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial y^2}e^z+(\frac{\partial z}{\partial y})^2e^z+\frac{\partial^2 z}{\partial y^2}ze^z+\frac{\partial z}{\partial y}\frac{\partial ze^z}{\partial y}=0 \qquad \therefore\frac{\partial^2 z}{\partial y^2}=0$$

例. $\sin(x+y)+ze^z-ye^x=0$确定了隐函数$z=z(x,y)$,求$z=z(x,y)$在(0,0)带Peano余项的2阶Taylor展开.

解： 计算$\frac{\partial z}{\partial x}(0,0),\frac{\partial z}{\partial y}(0,0),\frac{\partial^2 z}{\partial x^2}(0,0),\frac{\partial^2 z}{\partial y^2}(0,0),\frac{\partial^2 z}{\partial y\partial x}(0,0)$

在$\cos(x+y)+\frac{\partial z}{\partial x}e^z+\frac{\partial z}{\partial x}ze^z-ye^x=0$两边再对$x$求偏导, 得

$$-\sin(x+y)+\frac{\partial^2 z}{\partial x^2}e^z+(\frac{\partial z}{\partial x})^2e^z+\frac{\partial^2 z}{\partial x^2}ze^z+(\frac{\partial z}{\partial x})^2e^z+(\frac{\partial z}{\partial x})^2ze^z-ye^x=0$$

$$\therefore \frac{\partial^2 z}{\partial x^2}=-2$$

$$\therefore z(x,y)=-x+\frac{1}{2!}(2xy-2x^2)+o(x^2+y^2)=-x+xy-x^2+o(x^2+y^2)$$

例. (2020真题) f 二阶连续可微, 求证: $\lim_{h\to 0^+}\dfrac{f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)}{h^2}=f''_{xx}(0,0)$

解: $\because f$ 二阶连续可微

$$\therefore f(x,y)=f(0,0)+xf'_x(0,0)+yf'_y(0,0)+\frac{1}{2}x^2f''_{xx}(0,0)+\frac{1}{2}y^2f''_{yy}(0,0)$$

$$+xyf''_{xy}(0,0)+o(x^2+y^2)$$

$$f(2h,e^{-\frac{1}{2h}})=f(0,0)+2hf'_x(0,0)+e^{-\frac{1}{2h}}f'_y(0,0)+2h^2f''_{xx}(0,0)+\frac{1}{2}e^{-\frac{1}{h}}f''_{yy}(0,0)+2he^{-\frac{1}{2h}}f''_{xy}(0,0)+o(h^2)$$

$$=f(0,0)+2hf'_x(0,0)+2h^2f''_{xx}(0,0)+o(h^2)$$

$$f(h,e^{-\frac{1}{h}})=f(0,0)+hf'_x(0,0)+e^{-\frac{1}{h}}f'_y(0,0)+\frac{1}{2}h^2f''_{xx}(0,0)+\frac{1}{2}e^{-\frac{2}{h}}f''_{yy}(0,0)+he^{-\frac{1}{h}}f''_{xy}(0,0)+o(h^2)$$

$$=f(0,0)+hf'_x(0,0)+\frac{1}{2}h^2f''_{xx}(0,0)+o(h^2)$$

$$\therefore f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)=h^2f''_{xx}(0,0)+o(h^2)$$

3/ 多元泰勒公式和极值原理

Thm. n元函数f在x_0的某个邻域中可微, x_0为f的极值点,则x_0为f的驻点, 即$\mathrm{grad}f(x_0)=0$.

Thm. n元函数f在x_0的邻域中二阶连续可微,

$\mathrm{grad}f(x_0)=0$,

(1)若$H_f(x_0)$正定,则$f(x_0)$严格极小.

(2)若$H_f(x_0)$负定,则$f(x_0)$严格极大.

(3)若$H_f(x_0)$不定,则$f(x_0)$不是极值.

例: 求$z=x^4+y^4-2x^2+4xy-2y^2$的极值.

解: $z'_x=4x^3-4x+4y,\ \ z'_y=4y^3+4x-4y.$

得驻点$(\sqrt{2},-\sqrt{2}),(-\sqrt{2},\sqrt{2}),(0,0).$

$$z''_{xx}=12x^2-4,\quad z''_{xy}=4,\quad z''_{yy}=12y^2-4.$$

(1)在$(\sqrt{2},-\sqrt{2})$,

$$A=C=20, B=4,\ AC-B^2>0,$$

取得极小值.

(2)同理$z(x,y)$在$(-\sqrt{2},\sqrt{2})$取得极小值.

(3)在$(0,0)$,

$$A=C=-4, B=4,\ \ AC-B^2=0,$$

判别法失效. 注意到

$$z(x,x)=2x^4>0, \text{当} x\neq 0 \text{时}.$$

$$z(x,0)=x^4-2x^2$$
$$=x^2(x^2-2)<0, \text{当} 0<x^2<2 \text{时}.$$

故$(0,0)$不是极值点.□

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

解: 在$2x^2+2y^2+z^2+8xz-z+8=0$两边分别对x,y求偏导数.

$$4x+2z\frac{\partial z}{\partial x}+8z+8x\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}=0(1)$$

$$4y+2z\frac{\partial z}{\partial y}+8x\frac{\partial z}{\partial y}-\frac{\partial z}{\partial y}=0(2)$$

先计算驻点,即$\frac{\partial z}{\partial x}=0,\frac{\partial z}{\partial y}=0$ $\quad\therefore 4x+8z=0,4y=0$

结合$2x^2+2y^2+z^2+8xz-z+8=0.$ $\quad\therefore -7z^2-z+8=0,\therefore z=1$或$-\frac{8}{7}$

$\therefore$两个驻点为$(-2,0)$和$(16/7,0)$

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

$\therefore$ 两个驻点为$(-2,0)$和$(16/7,0)$ 下面计算$(-2,0)$和$(16/7,0)$处的海塞矩阵

对$4x+2z\frac{\partial z}{\partial x}+8z+8x\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}=0$两边对$x,y$求导:

$$4+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}+8\frac{\partial z}{\partial x}+8\frac{\partial z}{\partial x}+8x\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x^2}=0$$

$$2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+2z\frac{\partial^2 z}{\partial x\partial y}+8\frac{\partial z}{\partial y}+8x\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial x\partial y}=0$$

对$4y+2z\frac{\partial z}{\partial y}+8x\frac{\partial z}{\partial y}-\frac{\partial z}{\partial y}=0$两边同时对$y$求导 得$4+2\left(\frac{\partial z}{\partial y}\right)^2+2z\frac{\partial^2 z}{\partial y^2}+8x\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial y^2}=0$

3 多元泰勒公式和极值原理

例: $2x^2+2y^2+z^2+8xz-z+8=0$确定隐函数$z=z(x,y)$.求$z(x,y)$的极值.

$\therefore$两个驻点为$(-2,0)$和$(16/7,0)$ 下面计算$(-2,0)$和$(16/7,0)$处的海塞矩阵

在$(-2,0)$处, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\ z=1$

$\frac{\partial^2 z}{\partial x^2}=\frac{4}{15},\frac{\partial^2 z}{\partial x\partial y}=0,\frac{\partial^2 z}{\partial y^2}=\frac{4}{15}$

极小

在$(16/7,0)$处, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0,\ z=1$

$\frac{\partial^2 z}{\partial x^2}=-\frac{4}{15},\frac{\partial^2 z}{\partial x\partial y}=0,\frac{\partial^2 z}{\partial y^2}=-\frac{4}{15}$

极大

例:f连续, $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-xy}{(x^2+y^2)^2}=1$. $f(0,0)$是否极值？

解: $\lim\limits_{(x,y)\to(0,0)}\left(f(x,y)-xy\right)=0,\ f(0,0)=0.$

存在$\varepsilon>0$, 当$x^2+y^2<\varepsilon$时,

$$\frac{3}{2}(x^2+y^2)^2>f(x,y)-xy>\frac{1}{2}(x^2+y^2)^2.$$

于是对充分大的n, $f\left(\frac{1}{n},\frac{1}{n}\right)>\frac{1}{n^2}+\frac{2}{n^4}>0,$

$$f\left(\frac{1}{n},-\frac{1}{n}\right)<-\frac{1}{n^2}+\frac{6}{n^4}=-\frac{1}{n^2}(1-\frac{6}{n^2})<0.$$

故$f(0,0)$不是极值. □

3/ 多元泰勒公式和极值原理

Note: 无条件极值问题求解步骤:

(1)计算驻点,即偏导数为0的点;

(2)计算驻点处的*Hessen*矩阵,正定极小, 负定极大, 不定不是极值点

(3)如果(2)失效, 要考虑其他方法.

3/ 多元泰勒公式和极值原理

Ex. (巧妙运用极值原理 $-P94T5$)

(1) $f(x,y)$在$x^2+y^2\le 1$上连续, 在$x^2+y^2<1$内可导,

满足方程$\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=kf(x,y)(k>0)$, 若在$x^2+y^2=1$上$f(x,y)=0$,

求证f在$x^2+y^2\le 1$内部恒为0.

证明　反证, 如果f在$x^2+y^2\le 1$内部不恒为0

即存在(x_0,y_0), $s.t.\ f(x_0,y_0)\ne 0$

1° $f(x_0,y_0)>0$, 则$f(x,y)$在$x^2+y^2\le 1$上有大于0的最大值

如果最大值在(x_1,y_1)处取, 有$f(x_1,y_1)>0$, 并且${x_1}^2+{y_1}^2<1$

$\therefore f(x_1,y_1)$为极大值, $\therefore \dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=0$　$\because \dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=kf(x,y)$, $\therefore f(x_1,y_1)=0$, 矛盾!

2° $f(x_0,y_0)<0$, 取$-f$带入上面证明即可.□

•条件极值与Lagrange乘子法

$$\max(\min)\ f(\mathrm{x}) = f(x_1,\cdots,x_n)$$

$$s.t.\quad g_i(\mathrm{x}) = g_i(x_1,\cdots,x_n) = 0,\quad i = 1,\cdots,m.$$

其中rank $\dfrac{\partial(g_1,\cdots,g_m)}{\partial(x_1,\cdots,x_n)} = m$ (正则性条件).

结论: x_0是条件极值问题的最大(小)值点, 则$\exists\lambda_0$,s.t. (x_0,λ_0)是

$$L(\mathrm{x},\lambda) = L(x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_m)$$

$$= f(x_1,\cdots,x_n) + \sum_{i=1}^{m}\lambda_i g_i(x_1,\cdots,x_n)$$

的驻点.

3 / 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球$x^2+y^2+\frac{z^2}{4}=1$上找一点, 位于$x>0, y>0, z>0$.

使得切平面与三个坐标轴的交点到原点距离的平方和最小

解. 设该点坐标为(a,b,c), 法向量为$(2a,2b,\frac{c}{2})$

切平面为$2a(x-a)+2b(y-b)+\frac{c}{2}(z-c)=0$ 即$ax+by+\frac{c}{4}z=1$

解得三个交点坐标为$(\frac{1}{a},0,0),(0,\frac{1}{b},0),(0,0,\frac{4}{c})$

求解如下条件极值问题

$$\min:\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}$$

$$s.t. a^2+b^2+\frac{c^2}{4}=1;\quad a>0,b>0,c>0$$

3/ 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球$x^2+y^2+\frac{z^2}{4}=1$上找一点, 位于$x>0, y>0, z>0$.

使得切平面与三个坐标轴的交点到原点距离的平方和最小

$$L(a,b,c,\lambda)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{16}{c^2}+\lambda(a^2+b^2+\frac{c^2}{4}-1)$$

$$L'_a(a,b,c,\lambda)=-\frac{2}{a^3}+2\lambda a=0$$

$$L'_b(a,b,c,\lambda)=-\frac{2}{b^3}+2\lambda b=0$$

$$L'_b(a,b,c,\lambda)=-\frac{32}{c^3}+\frac{\lambda c}{2}=0$$

$$\therefore \lambda=\frac{1}{a^4}=\frac{1}{b^4}=\frac{64}{c^4} \qquad \therefore a=b=\frac{c}{2\sqrt{2}}$$

结合$a^2+b^2+\frac{c^2}{4}=1$

$$\therefore a=1/2, b=1/2, c=\sqrt{2}$$

3 多元泰勒公式和极值原理

例. (2020春期末) 求函数$u=\sin x\sin y\sin z$在条件$x+y+z=\dfrac{\pi}{2}, x>0, y>0, z>0$下的极值,并说明是极大值还是极小值.

解: (化为无条件极值) 考虑$v(x,y)=\sin x\sin y\sin(\dfrac{\pi}{2}-x-y)=\sin x\sin y\cos(x+y)$

在$\{(x,y): x>0, y>0, \dfrac{\pi}{2}-x-y>0\}$上的极值

$$\frac{\partial v(x,y)}{\partial x}=\cos x\sin y\cos(x+y)-\sin x\sin y\sin(x+y)=\sin y(\cos x\cos(x+y)-\sin x\sin(x+y))=$$

$\sin y\cos(2x+y)=0\qquad \because y>0, \therefore 2x+y=\dfrac{\pi}{2}$

$$\frac{\partial v(x,y)}{\partial y}=\sin x\cos(x+2y)=0\Rightarrow\ x+2y=\frac{\pi}{2}\quad \therefore x=y=z=\frac{\pi}{6}$$ 是唯一驻点. $v(\dfrac{\pi}{6},\dfrac{\pi}{6})=\left(\dfrac{1}{2}\right)^3=\dfrac{1}{8}$

进一步考虑$D=\{(x,y): x\ge 0, y\ge 0, \dfrac{\pi}{2}-x-y\ge 0\}$ v在D上有最大值和最小值

D的边界上, $v(x,y)=0$.可知上面所求为最大值.

3/ 多元泰勒公式和极值原理

例. (2020春期末) 求函数$u=\sin x\sin y\sin z$在条件$x+y+z=\dfrac{\pi}{2}, x>0, y>0, z>0$下的极值,并说明是极大值还是极小值.

解: (化为无条件极值) 考虑$v(x,y)=\sin x\sin y\sin(\dfrac{\pi}{2}-x-y)=\sin x\sin y\cos(x+y)$

在$\{(x,y): x>0, y>0, \dfrac{\pi}{2}-x-y>0\}$上的极值

$\therefore x=y=z=\dfrac{\pi}{6}$ 是唯一驻点. $\because v'_x=\sin y\cos(2x+y), v'_y=\sin x\cos(x+2y)$

$$\because \begin{cases} v''_{xx}=-2\sin y\sin(2x+y) \\ v''_{xy}=\cos y\cos(x+2y)-\sin y\sin(2x+y)=\cos(2x+2y) \\ v''_{yy}=-2\sin x\sin(x+2y) \end{cases} \quad \because \begin{cases} v''_{xx}=-1 \\ v''_{xy}=\cos(\dfrac{2\pi}{3})=-\dfrac{1}{2} \\ v''_{yy}=-1 \end{cases}$$ 海塞矩阵负定.

4 含参数积分

- 含参数定积分：$\int_{\alpha}^{\beta} g(t,x)dx$
- 含参数广义积分：$\int_{\alpha}^{+\infty} g(t,x)dx$
- 无论是含参数定积分还是含参数的广义积分，本质上都是关于参数t的函数
  - 对于一个函数来讲，主要研究其连续性、可导性、可积性
  - 连续性：$\lim_{t\to t_0}\int_{\alpha}^{\beta} g(t,x)dx=\int_{\alpha}^{\beta}\lim_{t\to t_0} g(t,x)dx=\int_{\alpha}^{\beta} g(t_0,x)dx$
  - 可导性：$f'(t)=\dfrac{d}{dt}\int_{\alpha}^{\beta} g(t,x)dx=\int_{\alpha}^{\beta} g_t'(t,x)dx.$
  - 可积性：$\int_{a}^{b}\left(\int_{\alpha}^{\beta} g(t,x)dx\right)dt=\int_{\alpha}^{\beta}\left(\int_{a}^{b} g(t,x)dt\right)dx$
- 对于含参数定积分，一般只要求被积函数$g(t,x)$及g_t'的连续性即可.
- 对于含参数广义积分，除了含参数定积分的条件外，还需要更强的条件.

4 / 含参数积分

Thm1.(连续性) 设二元函数$g(t,x)$在$D=[a,b]\times[\alpha,\beta]$上连续,则

$$f(t)=\int_{\alpha}^{\beta} g(t,x)dx \text{在}[a,b]\text{上一致连续.}$$

$$\text{也即}\ \lim_{t\to t_0}\int_{\alpha}^{\beta} g(t,x)dx=\int_{\alpha}^{\beta}\lim_{t\to t_0} g(t,x)dx.$$

Thm2.(在积分号下求导)设$D=[a,b]\times[\alpha,\beta]$,且$g(t,x),g_t'(t,x)\in C(D)$,则

$$f(t)=\int_{\alpha}^{\beta} g(t,x)dx \text{在}[a,b]\text{上连续可导,且}$$

$$f'(t)=\frac{d}{dt}\int_{\alpha}^{\beta} g(t,x)dx=\int_{\alpha}^{\beta} g_t'(t,x)dx.$$

例. 求$a,b,s.t.\int_1^3 (ax+b-x^2)^2 dx$取最小值

解. $I(a,b)=\int_1^3 (ax+b-x^2)^2 dx$

$$\frac{\partial I(a,b)}{\partial a}=\int_1^3 2(ax+b-x^2)xdx=2\int_1^3 ax^2+bx-x^3dx=2(\frac{26}{3}a+4b-20)=0$$

$$\frac{\partial I(a,b)}{\partial b}=\int_1^3 2(ax+b-x^2)dx \quad =2\int_1^3 ax+b-x^2dx \quad =2(4a+2b-\frac{26}{3})=0$$

解方程$\begin{cases}\frac{26}{3}a+4b=20\\ 4a+2b=\frac{26}{3}\end{cases}\Rightarrow\begin{cases}a=4\\ b=-\frac{11}{3}\end{cases}$

例. $F(x)=\int_0^{2\pi}e^{x\cos\theta}\cos(x\sin\theta)d\theta$, 证明$F(x)\equiv 2\pi$.

Proof. 令$f(x,\theta)=e^{x\cos\theta}\cos(x\sin\theta)$, 则$\forall r>0$, $f(x,\theta)$, $f_x'(x,\theta)$在$[-r,r]\times[0,2\pi]$上连续. 因此

$$F'(x)=\int_0^{2\pi}f_x'(x,\theta)d\theta=\int_0^{2\pi}e^{x\cos\theta}\cos\theta\cos(x\sin\theta)d\theta$$

$$-\int_0^{2\pi}e^{x\cos\theta}\sin(x\sin\theta)\sin\theta d\theta\triangleq I-J.$$

$$I=\int_0^{2\pi}\frac{1}{x}e^{x\cos\theta}\cos(x\sin\theta)d\sin\theta=\int_0^{2\pi}\frac{1}{x}e^{x\cos\theta}d\sin(x\sin\theta)$$

$$=\frac{1}{x}e^{x\cos\theta}\sin(x\sin\theta)\Big|_{\theta=0}^{2\pi}-\frac{1}{x}\int_0^{2\pi}\sin(x\sin\theta)de^{x\cos\theta}=J,\forall x\neq 0.$$

于是, $F'(x)\equiv 0,\forall|x|\leq r$.由$F(0)=2\pi$及r的任意性, $F(x)\equiv 2\pi$.□

Thm3. 设$g(t,x), g_t'(t,x)\in C([a,b]\times[c,d]), \alpha(t), \beta(t)$在$[a,b]$上可导，且

$$c\le \alpha(t), \beta(t)\le d, \quad \forall t\in[a,b],$$

则

$$f(t)=\int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

在区间$[a,b]$上可导，且

$$f'(t)=\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

$$=\int_{\alpha(t)}^{\beta(t)} g_t'(t,x)dx+g(t,\beta(t))\beta'(t)-g(t,\alpha(t))\alpha'(t).$$

例. $f(x)=\int_x^{x^2} e^{-x^2u^2}du,\ f'(x)=$ ________

解. $f'(x)=2xe^{-x^6}-e^{-x^4}+\int_x^{x^2} e^{-x^2u^2}\dfrac{d(-x^2u^2)}{dx}du$

$$=2xe^{-x^6}-e^{-x^4}-2\int_x^{x^2} e^{-x^2u^2}xu^2du$$

要点. 上限替代被积变量*上限的导数-下限替代被积变量*下限的导数+积分号下求导的部分

例. $f(y)=\int_y^{y^2}\dfrac{\sin(xy)}{x}dx,\ f'(y)=$ $\underline{\dfrac{3\sin(y^3)}{y}-\dfrac{2\sin(y^2)}{y}}$

解. $f(y)=\int_y^{y^2}\dfrac{\sin(xy)}{x}dx,\ f'(y)=\dfrac{\sin(y^3)}{y^2}2y-\dfrac{\sin(y^2)}{y}+\int_y^{y^2}\cos(xy)dx$

$$=\frac{2\sin(y^3)}{y}-\frac{\sin(y^2)}{y}+\frac{\sin(y^3)-\sin(y^2)}{y}$$

4. 含参积分的可积性

Thm4. (累次积分交换次序的充分条件)

设$g(t,x)$在$(t,x)\in D=[a,b]\times[\alpha,\beta]$上连续, 则$\int_\alpha^\beta g(t,x)dx$在$t\in[a,b]$上可积, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积, 且

$$\int_a^b\left(\int_\alpha^\beta g(t,x)dx\right)dt=\int_\alpha^\beta\left(\int_a^b g(t,x)dt\right)dx,$$

简记为 $$\int_a^b dt\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta dx\int_a^b g(t,x)dt.$$

Proof. 由$g(t,x)$的连续性及Thm1, $\int_\alpha^\beta g(t,x)dx$在$t\in[a,b]$上连续, 从而可积. 同理, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积.

例. 计算$\int_0^1 \frac{x^b - x^a}{\ln x}\sin(\ln\frac{1}{x})dx(a,b>0)$

要点. 交换积分次序, 会让难算的积分变得好算.

如果给你的是定积分,需要先变出两重积分号!

解. $\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy$

$$原式 = \int_0^1 \int_a^b x^y \sin(\ln\frac{1}{x})dydx(a,b>0) \quad = \int_a^b (\int_0^1 x^y \sin(\ln\frac{1}{x})dx)dy$$

$$= \int_a^b \frac{1}{(y+1)^2+1}dy = \arctan(b+1) - \arctan(a+1)$$

$x^y \sin(\ln\frac{1}{x})$在$\{(x,y): a \le y \le b, 0 \le x \le 1\}$上连续

注:$\lim_{(x,y)\to(0,y_0)} x^y \sin(\ln\frac{1}{x}) = 0 \quad \because |x^y \sin(\ln\frac{1}{x})| \le x^y, 0 < a \le y_0 \le b$

$$\int_0^1 x^y \sin(\ln\frac{1}{x})dx \overset{\ln\frac{1}{x}=t, x=e^{-t}}{=} \int_{+\infty}^0 e^{-ty}\sin(t)d(e^{-t}) = \int_0^{+\infty} e^{-t(y+1)}\sin(t)dt = \frac{1}{(y+1)^2+1}$$

4/ 含参数广义积分

Note.对于含参数广义积分而言，需要更强的条件以满足以上定理

Def. $\forall t \in \Omega \subset \mathbb{R}, \int_a^{+\infty} f(t,x)dx$ 收敛. 若$\forall \varepsilon > 0, \exists \mathrm{M}(\varepsilon), s.t.$

$$\left|\int_a^{\mathrm{A}} f(t,x)dx - \int_a^{+\infty} f(t,x)dx\right| < \varepsilon, \quad \forall \mathrm{A} > \mathrm{M}, \forall t \in \Omega,$$

则称含参广义积分$\int_a^{+\infty} f(t,x)dx$ 关于$t \in \Omega$一致收敛.

一致性体现在，一旦ε被指定，

则$\forall t \in \Omega$，$\exists$同一个$M, s.t. \left|\int_a^{\mathrm{A}} f(t,x)dx - \int_a^{+\infty} f(t,x)dx\right| < \varepsilon, \quad \forall \mathrm{A} > \mathrm{M}$

4/ 含参数广义积分

Thm.(Weirstrass判别法) $\forall t\in\Omega\subset\mathbb{R}$, $\int_a^{+\infty} f(t,x)dx$收敛,

若存在$[a,+\infty)$上的广义可积函数$g(x)$, $s.t.$

$$|f(t,x)|\le g(x),\quad \forall (t,x)\in\Omega\times[a,+\infty),$$

则$\int_a^{+\infty} f(t,x)dx$ 在$t\in\Omega$上一致收敛.

问:如何证明不一致收敛?

Thm.(Cauchy收敛原理)

$$\int_a^{+\infty} f(t,x)dx \text{ 关于} t\in\Omega \text{一致收敛} \Leftrightarrow\ \forall\varepsilon>0, \exists \mathrm{M}(\varepsilon), s.t.$$

$$\left|\int_{\mathrm{A}}^{\mathrm{A}'} f(t,x)dx\right|<\varepsilon,\quad \forall \mathrm{A},\mathrm{A}'>\mathrm{M}, \forall t\in\Omega.$$

问：如何证明不一致收敛？

Remark.(Cauchy收敛原理逆否)

$$\int_a^{+\infty} f(t,x)dx \text{ 关于} t\in\Omega \text{不一致收敛} \Leftrightarrow \exists \varepsilon_0>0, \forall \mathrm{M}, s.t.$$

$$\left|\int_{\mathrm{A}}^{\mathrm{A}'} f(t_0,x)dx\right| > \varepsilon_0, \quad \exists \mathrm{A},\mathrm{A}'>\mathrm{M}, \exists t_0\in\Omega.$$

例. (1)设$c>0, \int_0^{+\infty} e^{-xy}dx$ 在$y\in[c,+\infty)$上是否一致收敛?

(2)$\int_0^{+\infty} e^{-xy}dx$ 在$y\in(0,+\infty)$上是否一致收敛?

解: (1) $c>0$, 则$\int_0^{+\infty} e^{-cx}dx=-\frac{1}{c}e^{-cx}\Big|_{x=0}^{+\infty}=\frac{1}{c}$收敛, 且

$$e^{-xy}\le e^{-cx},\quad \forall(x,y)\in[0,+\infty)\times[c,+\infty).$$

故$\int_0^{+\infty} e^{-xy}dx$ 在$y\in[c,+\infty)$上一致收敛(Weirstrass).

(2) $\exists\varepsilon_0=e^{-1}-e^{-2}, \forall \mathrm{M}>0, \exists \mathrm{A}=\mathrm{M}+1, \mathrm{A}'=2\mathrm{A}, y_0=\frac{1}{\mathrm{A}}, s.t.$

$$\left|\int_{\mathrm{A}}^{\mathrm{A}'} e^{-xy_0}dx\right|=-\frac{1}{y_0}e^{-xy_0}\Big|_{x=\mathrm{A}}^{\mathrm{A}'}=\frac{1}{y_0}(e^{-\mathrm{A}y_0}-e^{-\mathrm{A}'y_0})=\mathrm{A}\varepsilon_0>\varepsilon_0,$$

故$\int_0^{+\infty} e^{-xy}dx$ 在$y\in[0,+\infty)$上不一致收敛(Cauchy).□

4/ 含参数广义积分

Thm1. $f(t,x)\in C\left([\alpha,\beta]\times[a,+\infty)\right), I(t)=\int_a^{+\infty} f(t,x)dx$ 关于$t\in[\alpha,\beta]$一致收敛,则$I(t)\in C[\alpha,\beta]$.

Thm1(逆否). $f(t,x)\in C\left([\alpha,\beta]\times[a,+\infty)\right), I(t)\notin C[\alpha,\beta]$.则$I(t)=\int_a^{+\infty} f(t,x)dx$ 关于$t\in[\alpha,\beta]$不一致收敛,则

例. 证明$\int_0^{+\infty}\frac{\sin tx}{x}dx$在$t\in[0,+\infty)$上不一致收敛.

解: $I(t)=\int_0^{+\infty}\frac{\sin tx}{x}dx=\begin{cases}\int_0^{+\infty}\frac{\sin x}{x}dx, & t>0\\ 0, & t=0\end{cases}.$

若$\int_0^{+\infty}\frac{\sin tx}{x}dx$在$t\in[0,+\infty)$上一致收敛, 则$I(t)$ $\in C[0,+\infty)$, 矛盾.□

Remark. 证明含参积分不一致收敛的方法:

定义、Cauchy准则、含参积分不连续.

Thm2. 设(1) $f(t,x),\ f_t'(t,x)\in C\big([\alpha,\beta]\times[a,+\infty)\big)$;

(2)$\forall t\in[\alpha,\beta], I(t)=\int_a^{+\infty} f(t,x)dx$ 收敛;

(3)$\int_a^{+\infty} f_t'(t,x)dx$ 关于$t\in[\alpha,\beta]$一致收敛;

则$I(t)\in C^1[\alpha,\beta]$, 且

$$I'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{+\infty} f(t,x)dx = \int_a^{+\infty} f_t'(t,x)dx.$$

注意. 是$\int_a^{+\infty} f_t'(t,x)dx$一致收敛! 不是$\int_a^{+\infty} f(t,x)dx$一致收敛

例: $I=\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}dx\quad (b>a>0).$

思想1:通过积分号下求导,把难算的积分转化为好算的.

解:引入参数$t\in[a,b]$,令$I(t)=\int_0^{+\infty}\frac{e^{-tx}-e^{-bx}}{x}dx.$

则 $I(b)=0$, 欲求$I(a)$,可以先求$I'(t)$,再积分. 因为

• $f(t,x)=(e^{-tx}-e^{-bx})/x,\ f_t'(t,x)\in C\big([a,b]\times[0,+\infty)\big),$

• $\int_0^{+\infty}f(t,x)dx$对任意$t\in[a,b]$收敛,

• $\int_0^{+\infty}f_t'(t,x)dx=-\int_0^{+\infty}e^{-tx}dx$对$t\in[a,b]$一致收敛,

所以 $I'(t)=\int_0^{+\infty}f_t'(t,x)dx=\int_0^{+\infty}-e^{-tx}dx=\left.\frac{e^{-tx}}{t}\right|_{x=0}^{+\infty}=-1/t,$

$I(t)=-\ln t+C.$ 又 $I(b)=0, C=\ln b, I(t)=\ln b-\ln t.$

所求积分为 $I=I(a)=\ln b-\ln a.\square$

例(2020样题): $I=\int_0^{+\infty}\frac{\arctan bx-\arctan ax}{x}dx \quad (b>a>0).$

解:引入参数$t\in[a,b]$,令$I(t)=\int_0^{+\infty}\frac{\arctan tx-\arctan ax}{x}dx.$

$$\because \left|\frac{1}{1+t^2x^2}\right|\le\frac{1}{1+a^2x^2},\forall t\in[a,b]$$

$$\therefore I'(t)=\int_0^{+\infty}f_t'(t,x)dx=\int_0^{+\infty}\frac{1}{1+t^2x^2}dx=\frac{\pi}{2t}$$

$$\therefore I(a)=0, I(b)=I(a)+\int_a^b I'(t)dt=\int_a^b I'(t)dt=\frac{\pi}{2}\ln(b/a)$$

4/ 含参数广义积分

例. 计算积分$\int_0^{+\infty} e^{-ax^2}\cos bx dx, a>0$

解. 视b为参数,定义$I(b)=\int_0^{+\infty} e^{-ax^2}\cos bx dx$,

思想2:通过积分号下求导,

虽然仍旧不好算,但是构造了ODE

$\because |xe^{-ax^2}\sin bx|\le xe^{-ax^2}, \forall a>0, \int_0^{+\infty} xe^{-ax^2}dx$存在 $\therefore I'(b)=-\int_0^{+\infty} xe^{-ax^2}\sin bx dx$

$$\therefore I'(b)=-\int_0^{+\infty} xe^{-ax^2}\sin bx dx=-\frac{1}{2}\int_0^{+\infty} e^{-ax^2}\sin bx dx^2=\frac{1}{2a}\int_0^{+\infty}\sin bx d(e^{-ax^2})$$

$$=\frac{1}{2a}(\sin bx e^{-ax^2}\Big|_0^{+\infty}-b\int_0^{+\infty}\cos bx e^{-ax^2}dx)=-\frac{b}{2a}\int_0^{+\infty}\cos bx e^{-ax^2}dx=-\frac{b}{2a}I(b)$$

即$I'(b)=-\frac{b}{2a}I(b)$ 结合初值$I(0)=\frac{1}{2}\sqrt{\pi/a}$,解出$I(b)=\frac{1}{2}\sqrt{\pi/a}e^{-b^2/4a}$

例(2020真题) $I=\int_0^{+\infty} e^{-x^2-\frac{t^2}{x^2}}dx \quad (t>0)$.

(1)计算$I'(t)$.(不必算出结果,表示为广义积分即可);

(2)求$I(t)$满足的ODE.

解: $I'(t)=\int_0^{+\infty} -\frac{2t}{x^2}e^{-x^2-\frac{t^2}{x^2}}dx \quad \forall t>0,\exists[a,b]\subset(0,+\infty), s.t. t\in[a,b]$

$\therefore \left|\frac{2t}{x^2}e^{-x^2-\frac{t^2}{x^2}}\right| \le \frac{2b}{x^2}e^{-x^2-\frac{a^2}{x^2}}, a,b>0$

在0附近, $\lim_{x\to0+}\frac{\frac{2b}{x^2}e^{-x^2-\frac{a^2}{x^2}}}{\frac{2b}{x^2}e^{-\frac{a^2}{x^2}}}=1, \therefore \int_0^1 \frac{2b}{x^2}e^{-\frac{a^2}{x^2}}dx$和$\int_0^1 \frac{2b}{x^2}e^{-x^2-\frac{a^2}{x^2}}dx$敛散性相同

例(2020真题) $I=\int_0^{+\infty} e^{-x^2-\frac{t^2}{x^2}}dx \quad (t>0)$.

(1)计算$I'(t)$.(不必算出结果,表示为广义积分即可);

(2)求$I(t)$满足的ODE.

解:$I'(t)=\int_0^{+\infty} -\frac{2t}{x^2}e^{-x^2-\frac{t^2}{x^2}}dx$

在0附近,$\lim_{x\to0+}\dfrac{\frac{2b}{x^2}e^{-\frac{a^2}{x^2}}}{\frac{1}{\sqrt{x}}}=\lim_{x\to0+}\frac{2b}{x^{1.5}}e^{-\frac{a^2}{x^2}}=0$ $\therefore$ 原积分收敛,由$Weierstrass$判敛法.

(2)

$$I=\int_0^{+\infty} e^{-x^2-\frac{t^2}{x^2}}dx=$$

例. 利用$\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$计算$\int_0^{+\infty}\frac{\sin^2 x}{x^2}dx$.

解: 令$I(t)\triangleq\int_0^{+\infty}\frac{\sin^2 tx}{x^2}dx$, 则$I(0)=0$, 欲求$I(1)$.

$\forall t\in[0,+\infty)$, $x=0$是关于x的一元函数$f(t,x)=\frac{\sin^2 tx}{x^2}$的可去间断点.

$$\frac{\sin^2 tx}{x^2}\le\frac{1}{x^2},\ \int_1^{+\infty}\frac{1}{x^2}dx\text{收敛},$$

由Weirstrass判别法, $\int_0^{+\infty}f(t,x)dx$关于$t\in[0,1]$一致收敛.

故$I(t)\in C[0,1]$.

$$\text{已知}\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2},\text{收敛},\text{则}\forall\varepsilon>0,\exists \mathrm{M}>0,s.t.$$

$$\left|\int_{\mathrm{A}}^{\mathrm{B}}\frac{\sin x}{x}dx\right|\le\varepsilon>0,\quad\forall \mathrm{A},\mathrm{B}>\mathrm{M}.$$

对任意取定的$t_0\in(0,1]$,有

$$\left|\int_a^b f_t'(t,x)dx\right|=\left|\int_a^b\frac{2\sin tx\cos tx}{x}dx\right|=\left|\int_a^b\frac{\sin 2tx}{x}dx\right|$$

$$=\left|\int_{2at}^{2bt}\frac{\sin x}{x}dx\right|<\varepsilon,\quad\forall a,b>\frac{\mathrm{M}}{2t_0},\forall t>t_0.$$

因此$\int_0^{+\infty} f_t'(t,x)dx$ 关于$t\ge t_0>0$一致收敛(Cauchy), 故

$$I'(t)=\int_0^{+\infty} f_t'(t,x)dx \quad = \int_0^{+\infty} \frac{2\sin tx\cos tx}{x}dx$$

$$= \int_0^{+\infty} \frac{\sin 2tx}{x}dx \quad = \int_0^{+\infty} \frac{\sin x}{x}dx \quad = \frac{\pi}{2}, \forall t \in [t_0,1].$$

由t_0的任意性, 有

$$I'(t)=\frac{\pi}{2}, \quad \forall t \in (0,1].$$

又$I(t)\in C[0,1], I(0)=0$, 所以

$$I(t)=\frac{\pi}{2}t, \forall t \in [0,1], \quad \int_0^{+\infty} \frac{\sin^2 x}{x^2}dx = I(1) = \frac{\pi}{2}. \square$$

Question. $\int_0^{+\infty} \frac{\sin^4 x}{x^2}dx = ? \int_0^{+\infty} \frac{\sin^4 x}{x^4}dx = ? \quad \frac{\pi}{4}, \frac{\pi}{3}.$