Homework 5

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Exercise 1.5.1

1. First, diagonalize A_t, which can be

$$A_t = \begin{bmatrix} 1 & 1 \\ & t \end{bmatrix} \begin{bmatrix} 1 & \\ & 1+t \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ & \frac{1}{t} \end{bmatrix}$$

According to the definition,

$$\begin{aligned} f(A_t) &= \begin{bmatrix} 1 & 1 \\ & t \end{bmatrix} \begin{bmatrix} f(1) & \\ & f(1+t) \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ & \frac{1}{t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ & t \end{bmatrix} \begin{bmatrix} 1 & \\ & (1+t)^2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{t} \\ & \frac{1}{t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & t+2 \\ & (t+1)^2 \end{bmatrix} \end{aligned}$$

So

$$\lim_{t\to 0} f(A_t) = \begin{bmatrix} 1 & \lim_{t\to 0}(t+2) \\ & \lim_{t\to 0}(t+1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}$$

2. First, diagonalize A_t, which can be

$$A_t = \begin{bmatrix} 1 & 1 \\ -ti & ti \end{bmatrix} \begin{bmatrix} 1-ti & \\ & 1+ti \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{i}{2t} \\ \frac{1}{2} & -\frac{i}{2t} \end{bmatrix}$$

According to the definition,

$$\begin{aligned} f(A_t) &= \begin{bmatrix} 1 & 1 \\ -ti & ti \end{bmatrix} \begin{bmatrix} f(1-ti) & \\ & f(1+ti) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{i}{2t} \\ \frac{1}{2} & -\frac{i}{2t} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -ti & ti \end{bmatrix} \begin{bmatrix} |1-ti|(1-ti) & \\ & |1+ti|(1+ti) \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{i}{2t} \\ \frac{1}{2} & -\frac{i}{2t} \end{bmatrix} \\ &= \sqrt{1-t^2} \begin{bmatrix} 1 & 1 \\ -t^2 & 1 \end{bmatrix} \end{aligned}$$

So

$$\lim_{t\to 0} f(A_t) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

From the previous calculation, we know that using different matrix sequence $\{A_t\}$ that converge to J, the limit of $f(A_t)$ in the same limit process can be different. Thus, $f(J)$ is not well-defined.

Exercise 1.5.2

1. Using Taylor expansion, we know that

$$sin(x) = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{(2k+1)!}$$

So

$$sin(At) = \sum_{k=0}^{+\infty} \frac{(At)^{2k+1}}{(2k+1)!}$$

Take derivative

$$\begin{aligned} \frac{d}{dt}(sin(At)) &= \sum_{k=0}^{+\infty} \frac{A^{2k+1}x^{2k}}{(2k)!} \\ &= A\sum_{k=0}^{+\infty} \frac{(Ax)^{2k}}{(2k)!} \\ &= Acos(At) \end{aligned}$$

2. First, we try to block diagonalize $\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}$. Pick a sequence $A_x = \begin{bmatrix} 2A & A \\ & (2+x)A \end{bmatrix}$, so $f(\begin{bmatrix} 2A & A \\ & 2A \end{bmatrix}) = \lim_{x\to 0} f(A_x)$.

Diagnoalize A_x, we get

$$A_x = \begin{bmatrix} I & \frac{1}{x}I \\ & I \end{bmatrix} \begin{bmatrix} 2A & \\ & (2+x)A \end{bmatrix} \begin{bmatrix} I & -\frac{1}{x}I \\ & I \end{bmatrix}$$

So

$$\begin{aligned} f(A_x) &= \begin{bmatrix} I & \frac{1}{x}I \\ & I \end{bmatrix} \begin{bmatrix} f(2A) & \\ & f((2+x)A) \end{bmatrix} \begin{bmatrix} I & -\frac{1}{x}I \\ & I \end{bmatrix} \\ &= \begin{bmatrix} f(2A) & \frac{f((2+x)A)-f(2A)}{x} \\ & f((2+x)A) \end{bmatrix} \end{aligned}$$

Note that

$$\lim_{x\to 0} \frac{f((2+x)A) - f(2A)}{x} = \frac{d}{dx} f(xA)\Big|_{x=2}$$

To calculate the limit above, let's prove a lemma.

Lemma $\frac{d}{dt}(f(xA)) = Af'(tA)$.
Proof. First, prove this lemma for a Jordan block

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

Find the Jordan Normal From of $xJ(\lambda)$. Let N be an $n \times n$ nilpotent Jordan block. We have

$$\begin{gathered} Ker(xJ(\lambda) - x\lambda I) = Ker(xN) \\ Ker((xJ(\lambda) - x\lambda I)^2) = Ker((xN)^2) \\ \cdots \\ Ker((xJ(\lambda) - x\lambda I)^{n-1}) = Ker((xN)^{n-1}) \end{gathered}$$

So we pick basis D

$$D = diag(x^{n-1}, x^{n-2}, \cdots, x, 1)$$

And

$$xJ(\lambda) = DJ(x\lambda)D^{-1}$$

So

$$\begin{aligned} f(xJ(\lambda)) &= D \begin{bmatrix} \frac{f(x\lambda)}{0!} & \frac{f'(x\lambda)}{1!} & \cdots & \frac{f^{(n-1)}(x\lambda)}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & & \frac{f'(x\lambda)}{1!} \\ & & & \frac{f(x\lambda)}{0!} \end{bmatrix} D^{-1} \\ &= \begin{bmatrix} \frac{f(x\lambda)}{0!} & \frac{xf'(x\lambda)}{1!} & \cdots & \frac{x^{n-1}f^{(n-1)}(x\lambda)}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & & \frac{xf'(x\lambda)}{1!} \\ & & & \frac{f(x\lambda)}{0!} \end{bmatrix} \\ &= \sum_{k=0}^{n-1} x^k \frac{f^{(k)}(x\lambda)}{k!} N^k \qquad (*) \\ &= f(x\lambda)I + \sum_{k=1}^{n-1} x^k \frac{f^{(k)}(x\lambda)}{k!} N^k \end{aligned}$$

Take dirivative

$$
\begin{aligned}
\frac{d}{dx}(f(xJ(\lambda))) &= \lambda f'(x\lambda) + \lambda \sum_{k=1}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^k + \sum_{k=1}^{n-1} x^{k-1} \frac{f^{(k)}(x\lambda)}{(k-1)!} N^k \\
&= \lambda \sum_{k=0}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^k + \sum_{k=0}^{n-2} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^{k+1}
\end{aligned}
$$

From $(*)$, we know that

$$
f'(xJ(\lambda)) = \sum_{k=0}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^k
$$

And

$$
\begin{aligned}
Jf'(xJ(\lambda)) &= (\lambda I + N) f'(xJ(\lambda)) \\
&= \lambda \sum_{k=0}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^k + \sum_{k=0}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^{k+1}
\end{aligned}
$$

Note that $N^n = O$, we get

$$
\begin{aligned}
Jf'(xJ(\lambda)) &= \lambda \sum_{k=0}^{n-1} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^k + \sum_{k=0}^{n-2} x^k \frac{f^{(k+1)}(x\lambda)}{k!} N^{k+1} \\
&= \frac{d}{dx}(f(xJ(\lambda)))
\end{aligned}
$$

Then let's generalize this conclusion to any given $A \in M_n(\mathbb{C})$. Let

$$
A = X \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k) \end{bmatrix} X^{-1}
$$

where each $J_{n_i}(\lambda_i)$ is a Jordan block with only one eigenvalue λ_i. So

$$
\begin{aligned}
\frac{d}{dx}(f(xA)) &= X \begin{bmatrix} \frac{d}{dx}(f(xJ_{n_1}(\lambda_1))) & & \\ & \ddots & \\ & & \frac{d}{dx}(f(xJ_{n_k}(\lambda_k))) \end{bmatrix} X^{-1} \\
&= X \begin{bmatrix} J_{n_1}(\lambda_1) f'(J_{n_1}(\lambda_1)) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k) f'(J_{n_k}(\lambda_k)) \end{bmatrix} X^{-1} \\
&= X \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k) \end{bmatrix} X^{-1} X \begin{bmatrix} f'(J_{n_1}(\lambda_1)) & & \\ & \ddots & \\ & & f'(J_{n_k}(\lambda_k)) \end{bmatrix} X^{-1} \\
&= Af'(xA)
\end{aligned}
$$

$\square$

According to **Lemma**, we know that $\frac{d}{dx}f(xA)\Big|_{x=2} = Af'(2A)$. So

$$B = Af'(2A)$$

Collaborator for Exercise 1.5.2.2: Chen Siyuan, a student in this class.

3. Conter example. Let $f(x) = x^2$, $A = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$, then $f'(x) = 2x$. We have

$$\begin{aligned} f(A+tB) &= f(\begin{bmatrix} 1+t & t \\ & 2+t \end{bmatrix}) \\ &= \begin{bmatrix} (1+t)^2 & 2t^2+3t \\ & (2+t)^2 \end{bmatrix} \\ \frac{d}{dt}(f(A+tB)) &= \begin{bmatrix} 2+2t & 3+4t \\ & 4+2t \end{bmatrix} \end{aligned}$$

So

$$\frac{d}{dt}(f(A+tB))\Big|_{t=0} = \begin{bmatrix} 2 & 3 \\ & 4 \end{bmatrix}$$

But

$$f'(A)B = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ & 4 \end{bmatrix}$$

So for $\forall A,\ B,\ f$, $\frac{d}{dt}(f(A+tB))\big|_{t=0} = f'(A)B$ is NOT always true.