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1. $\forall\ v_1, v_2, v \in H_A,\ w_1, w_2, w \in H_B$

① symmetric: $(v_1 \otimes w_1, v_2 \otimes w_2) = (v_1, v_2)(w_1, w_2)$ Since H_A and H_B are inner product space, they are all symmetric

Hence $(v_1, v_2)(w_1, w_2) = (v_2, v_1)(w_2, w_1) = (v_2 \otimes w_2, v_1 \otimes w_1)$

$\therefore (v_1 \otimes w_1, v_2 \otimes w_2) = (v_2 \otimes w_2, v_1 \otimes w_1)$ it's symmetric.

② Postive defined: $(v \otimes w, v \otimes w) = (v, v)(w, w)$ Since H_A and H_B are inner product space, they are all positive-defined Hence $(v \otimes w, v \otimes w) = (v, v)(w, w) \geqslant 0$, and if and only if $v = 0$ or $w = 0$ will let $(v \otimes w, v \otimes w) = 0$. But when $v = 0$ or $w = 0$, then $v \otimes w = 0$ (zero map)

Hence it's positively-defined

2. Suppose $a e_1 \otimes e_1 + b e_2 \otimes e_2 = v \otimes w$ for some $v \in H_A$ and $w \in H_B$.

we have $v = [e_1\ e_2]\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = e_1 v^1 + e_2 v^2$ $w = [e_1\ e_2]\begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = e_1 w^1 + e_2 w^2$

$v \otimes w = (e_1 v^1 + e_2 v^2) \otimes (e_1 w^1 + e_2 w^2) = v^1 w^1 e_1 \otimes e_1 + v^1 w^2 e_1 \otimes e_2 + v^2 w^1 e_2 \otimes e_1 + v^2 w^2 e_2 \otimes e_2$

So we have $v^1 w^1 = a \neq 0$ $v^1 w^2 = 0$ $v^2 w^1 = 0$ $v^2 w^2 = b \neq 0$

So $v^1 v^2 w^1 w^2 = (v^1 w^1)(v^2 w^2) = ab \neq 0$ but $v^1 v^2 w^1 w^2 = (v^1 w^2)(v^2 w^1) = 0 \cdot 0 = 0$

Contradict!

Hence $a e_1 \otimes e_1 + b e_2 \otimes e_2$ can not have rank 1.

It has alredy be expressed by two different rank 1 tensor

Hence it has rank 2.

3. $(w, L \otimes I_B(w)) = (v \otimes w, L \otimes I_B(v \otimes w)) = (v \otimes w, Lv \otimes I_B w) = (v, Lv)(w, I_B w)$

$((v^1)^2 - (v^2)^2)((w^1)^2 + (w^2)^2)$

$(w, I_A \otimes L(w)) = (v \otimes w, I_A \otimes L(v \otimes w)) = (v \otimes w, I_A v \otimes L w) = (v, I_A v)(w, Lw)$

$= ((v^1)^2 + (v^2)^2)((w^1)^2 - (w^2)^2)$

Let $A = ((v^1)^2 - (v^2)^2)((w^1)^2 + (w^2)^2)$ $B = ((v^1)^2 + (v^2)^2)((w^1)^2 - (w^2)^2)$

There is real solution for Any $A, B \in \mathbb{R}$.

Hence $(w, L \otimes I_B(w))\ (w, I_A \otimes L(w))$ Can be any pair of real numbers.

Proof: $\forall x, y \in \mathbb{R}$ let $R = \sqrt{x^2 + y^2}$. we only need to find one solution.

Let $\|v\| = (v^1)^2 + (v^2)^2 = R^2$ $v^1 = R\cos\theta$, $v^2 = R\sin\theta$

$\|w\| = (w^1)^2 + (w^2)^2 = R^2$ $w^1 = R\cos\beta$, $w^2 = R\sin\beta$

Then Let: $\begin{cases} x = (w, I_A \otimes L(w)) = R^4 \cos 2\theta \\ y = (w, L \otimes I_B(w)) = R^4 \cos 2\beta \end{cases}$

1° If $R = 0$, then $x = y = 0$, $v^1 = v^2 = w^1 = w^2 = 1$ is a solution

2° if $R \neq 0$, then $\cos 2\theta = \frac{x}{R^4}$ $\cos 2\beta = \frac{y}{R^4}$

a solution is: $\beta = \frac{1}{2}\arccos\frac{y}{R^4}$ $\theta = \frac{1}{2}\arccos\frac{x}{R^4}$ the we have v and w.

In the end, we find any $x, y \in \mathbb{R}$ have at least one solution.

4. from 3. $L(e_1)=\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}=e_1$ $L(e_2)=\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=-\begin{bmatrix}0\\1\end{bmatrix}=-e_2$

$(e_1\otimes e_1, e_2\otimes e_2)=(e_1,e_2)(e_1,e_2)=0\cdot 0=0$ (e_1, e_2 are orthogonal to each other)

Since $(-,-)$ is symmetric, the $(e_2\otimes e_2, e_1\otimes e_1)=(e_1\otimes e_1, e_2\otimes e_2)=0$

$(e_1\otimes e_1, e_1\otimes e_1)=(e_1,e_1)(e_1,e_1)=1$ $(e_2\otimes e_2, e_2\otimes e_2)=(e_2,e_2)(e_2,e_2)=1\cdot 1=1$

$L\otimes I_B(w)=L\otimes I_B(ae_1\otimes e_1+be_2\otimes e_2)=L\otimes I_B(a\,e_1\otimes e_1)+L\otimes I_B(b\,e_2\otimes e_2)$

$=a\,L(e_1)\otimes I_B(e_1)+b\,L(e_2)\otimes I_B(e_2)=a\,e_1\otimes e_1-b\,e_2\otimes e_2$

Similarly $I_A\otimes L(w)=I_A\otimes L(ae_1\otimes e_1+be_2\otimes e_2)=a\,e_1\otimes e_1-be_2\otimes e_2=L\otimes I_B(w)$

Hence $(w, L\otimes I_B(w))=(w, I_A\otimes L(w))=(ae_1\otimes e_1+be_2\otimes e_2, ae_1\otimes e_1-be_2\otimes e_2)$

$=a^2(e_1\otimes e_1, e_1\otimes e_1)-b^2(e_2\otimes e_2, e_2\otimes e_2)+ab(e_2\otimes e_2, e_1\otimes e_1)-ab(e_1\otimes e_1, e_2\otimes e_2)$

$=a^2-b^2$

1.9.2: B is a $(1,1)$ tensor. Let $B=(B^i_j)$ Hence $V_C=(V_C)^i$, $V_B=(V_B)^j$ $(V_C)^i=(B^i_j)(V_B^j)$

let old base $B=[V_1, V_2, \cdots, V_n]$ New basis $C=[w_1, w_2, \cdots, w_n]$

We have $V=[V_1,\cdots,V_n]V_B=[w_1\cdots w_n]B\,V_B$ Hence: $[V_1\ V_2\cdots V_n]=[w_1\cdots w_n]B$

Change of basis is invertible, hence $[w_1\ w_2\cdots w_n]=[V_1, V_2\cdots V_n]B^{-1}$

let $B^{-1}=[b_1, b_2, \cdots, b_n]$ Hence $w_i=[V_1\cdots V_n]b_i$ for $i=1,2,\cdots,n$

$\lambda_C=[\lambda(w_1), \lambda(w_2)\cdots, \lambda(w_n)]=(\lambda([V_1,\cdots,V_n]b_1), \cdots\cdots, \lambda([V_1\cdots V_n]b_n))$

$=((\lambda[V_1,\cdots,V_n])b_1, \cdots\cdots, (\lambda[V_1\cdots V_n])b_n)=\lambda_B[b_1\ b_2\cdots b_n]=\lambda_B B^{-1}$

2. L is a tensor $(\mathbb{R}^n)\otimes(\mathbb{R}^n)\otimes(\mathbb{R}^n)^*\otimes(\mathbb{R}^n)^*$ i.e. a $(2,2)$-tensor

It is also: $L=(\mathbb{R}^n\otimes(\mathbb{R}^n)^*)\otimes(\mathbb{R}^n\otimes(\mathbb{R}^n)^*)$ i.e. a matrix tensor another matrix

$L(V_B\otimes W_B)=L(V_B, W_B)=V_C\otimes W_C=BV_B\otimes BW_B=B(V_B)\otimes B(W_B)=(B\otimes B)(V_B, W_B)$

$=(B\otimes B)(V_B\otimes W_B)$ Hence: $L=B\otimes B$

3. $\forall v\in V$, We have $v_C=Bv_B$; $\forall \lambda\in V^*$, we have $\lambda_C=\lambda_B B^{-1}$

Above all: we have

$(V_1)_B\otimes(V_2)_B\otimes\cdots\otimes(V_a)_B\otimes(\lambda^1)_B\otimes\cdots\otimes(\lambda^b)_B$

$\xrightarrow[\text{of Basis}]{\text{change}} B(V_1)_B\otimes B(V_2)_B\otimes\cdots\otimes B(V_a)_B\otimes(\lambda^1)_B B^{-1}\otimes\cdots\otimes(\lambda^b)_B B^{-1}$

Yany Sir told us if we take grad as a row vector, we only need to verify $\nabla f_{new} = \nabla f_{old} \cdot B^{-1}$

1.9.3 By definition: Grad $f = [\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}] = [2x\ \ 2y\ \ 2z]$

Suppose $V_{old} = [x\ y\ z]^T$ $V_{new} = B(V_{old}) = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y+z \\ z \end{bmatrix}$

$f_{new}(\begin{bmatrix} x+y \\ y+z \\ z \end{bmatrix}) = f(V_{old}) = x^2+y^2+z^2 = (x+y-y-z+z)^2+(y+z-z)^2+z^2$

Let $a=x+y$ $b=y+z$ $c=z$

$f_{new}(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = (a-b+c)^2+(b-c)^2+c^2$ $\frac{\partial f}{\partial a} = 2(a-b+c)$

$\frac{\partial f}{\partial b} = -2(a-b+c)+2(b-c) = -2a+2b-2c+2b-2c = -2a+4b-4c$

$\frac{\partial f}{\partial c} = 2(a-b+c)-2(b-c)+2c = 2a-2b+2c-2b+2c+2c = 2a-4b+6c$

Hence grad$(f_{new}) = [2a-2b+2c, -2a+4b-4c, 2a-4b+6c]$

$= [2(x+y)-2(y+z)+2z, -2(x+y)+4(y+z)-4z, 2(x+y)-4(y+z)+6z] = [2x, -2x+2y, 2x-2y+2z]$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$(B^{-1})(\nabla f(x,y,z)) = [2x\ 2y\ 2z]\begin{bmatrix} 1 & -1 & 1 \\ & 1 & -1 \\ & & 1 \end{bmatrix} = [2x\ \ -2x+2y\ \ 2x-2y+2z]$

Hence $\nabla f_{new}(a,b,c) = (B^{-1})^T(\nabla f(x,y,z))$

Grad f behaves like a dual vector

Hence it should be a row vector