# Review

曲面的切平面与法线:

| 曲面方程 | 点 | 法向量 |
| --- | --- | --- |
| $\mathrm{r}=\mathrm{r}(u,v)$ | $\mathrm{r}_0=\mathrm{r}(u_0,v_0)$ | $\left(\mathrm{r}_u'\times\mathrm{r}_v'\right)\big\vert_{(u_0,v_0)}$ |
| $z=f(x,y)$ | $(x_0,y_0,z_0)$<br>$z_0=f(x_0,y_0)$ | $(-f_x',-f_y',1)^{\mathrm{T}}\big\vert_{(x_0,y_0)}$ |
| $F(x,y,z)=0$ | $\mathrm{r}_0=(x_0,y_0,z_0)$ | $\mathrm{grad}F(\mathrm{r}_0)$ |

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曲线的切向量：

| 曲线方程 | 点 | 切向量 |
|---|---|---|
| $\mathrm{r}=\mathrm{r}(t)$ | $\mathrm{r}_0=\mathrm{r}(t_0)$ | $\mathrm{r}'(t_0)=$ $\left(x'(t_0),y'(t_0),z'(t_0)\right)$ |
| $\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$ | $\mathrm{r}_0=$ $(x_0,y_0,z_0)$ | $\mathrm{grad}F(\mathrm{r}_0)\times\mathrm{grad}G(\mathrm{r}_0)$ |

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## §8. 多元函数的Taylor公式

对充分光滑的一元函数$f(x)$,有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$+\frac{1}{(n+1)!}f^{(n+1)}\big(x_0+\theta(x-x_0)\big)(x-x_0)^{n+1},$$

$\big(0<\theta<1\big)$, 带*Lagrange*余项的*n*阶*Taylor*公式

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$+o\big((x-x_0)^n\big)$, 带*Peano*余项的*n*阶*Taylor*公式

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$Taylor$公式的应用之一是近似计算.

对二元可微函数$g(x,y)$,利用全微分也可以进行近似计算.

$$g(x,y)=g(x_0,y_0)+g'_x(x_0,y_0)(x-x_0)$$
$$+g'_y(x_0,y_0)(y-y_0)$$
$$+o\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$
$$g(x,y)\approx g(x_0,y_0)+g'_x(x_0,y_0)(x-x_0)$$
$$+g'_y(x_0,y_0)(y-y_0)$$

Question: 如何进一步提高精度？

Lemma. 设 $\mathrm{A}=(a_{ij})_{n\times n}$ 为$n$阶实对称矩阵,则

$$\left|x^{\mathrm{T}}\mathrm{A}x\right| \le \left(n\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\right)\cdot\|x\|^{2}, \qquad \forall x\in\mathbb{R}^{n}.$$

Proof. 记 $\mathrm{A}=(\alpha_1,\cdots,\alpha_n), \forall x=(x_1,\cdots,x_n)^{\mathrm{T}}\in\mathbb{R}^{n}$,有

$$\begin{aligned}\|\mathrm{A}x\| &= \|x_1\alpha_1+\cdots+x_n\alpha_n\| \le |x_1|\|\alpha_1\|+\cdots+|x_n|\|\alpha_n\| \\ &\le \left(|x_1|+\cdots+|x_n|\right)\left(\|\alpha_1\|+\cdots+\|\alpha_n\|\right) \\ &\le n\|x\|\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|,\end{aligned}$$

于是 $\left|x^{\mathrm{T}}\mathrm{A}x\right| \le \|x\|\cdot\|\mathrm{A}x\| \le \left(n\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\right)\cdot\|x\|^{2}$. □

Def. 设$n$元函数$f$在$B(x_0,\delta)$中二阶连续可微，称实对称阵

$$H_f(\mathrm{x}_0)=\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}_{x_0}$$

为$f$在点$x_0$的Hessian矩阵.

**Thm.** 设$n$元函数$f$在$B(x_0,\delta)$中二阶连续可微, 则

$\forall x_0+\Delta x\in B(x_0,\delta),\exists\theta\in(0,1),s.t.$

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x$$

$$+\frac{1}{2}(\Delta x)^T H_f(x_0+\theta\Delta x)\Delta x$$

(称为带*Lagrange*余项的一阶*Taylor*公式), 且

$$f(x_0+\Delta x)=f(x_0)+J_f(x_0)\Delta x$$

$$+\frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x+o(\|\Delta x\|^2),\Delta x\to 0\text{时}$$

(称为带*Peano*余项的二阶*Taylor*公式).

Proof. 构造一元函数

$$g(t)=f(x_0+t\Delta x)=f(x_0^{(1)}+t\Delta x_1,\cdots,x_0^{(n)}+t\Delta x_n),$$

$$f\in C^2, \text{则} g'(t)=\sum_{i=1}^{n} f_i'(x_0+t\Delta x)\Delta x_i=J_f(x_0+t\Delta x)\Delta x$$

$$g''(t)=\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}''(x_0+t\Delta x)\Delta x_i\Delta x_j=(\Delta x)^T H_f(x_0+t\Delta x)\Delta x$$

$$g(t)=g(0)+g'(0)t+\frac{1}{2}g''(\theta t)t^2,\theta\in(0,1).$$

$$f(x_0+\Delta x)=g(1)=g(0)+g'(0)+\frac{1}{2}g''(\theta)$$

$$=f(x_0)+J_f(x_0)\Delta x+\frac{1}{2}(\Delta x)^T H_f(x_0+\theta\Delta x)\Delta x,\theta\in(0,1).$$

带*Lagrange*余项的一阶*Taylor*公式得证.

记 $P(\Delta x)=H_f(x_0+\theta\Delta x)-H_f(x_0)=(h_{ij})_{n\times n}$,

$$\alpha(\Delta x)=(\Delta x)^T H_f(x_0+\theta\Delta x)\Delta x-(\Delta x)^T H_f(x_0)\Delta x$$

$$=(\Delta x)^T P(\Delta x)\Delta x.$$

已知$f\in C^2$,则 $\lim\limits_{\Delta x\to 0} h_{ij}=0$. 由前面的引理得

$$|\alpha(\Delta x)|\le\left(n\sum_{i=1}^{n}\sum_{j=1}^{n}|h_{ij}|\right)\cdot\|\Delta x\|^2,$$

因此 $\alpha(\Delta x)=o(\|\Delta x\|^2)$, 当$\Delta x\to 0$时.

故带*Peano*余项的二阶*Taylor*公式得证.□

Thm. 设函数$f(x,y)$在区域$D$中$n+1$阶连续可微，$M_0(x_0,y_0)\in D, M(x,y)\in D$，且线段$\overline{M_0M}$完全包含在$D$中. 记

$$h=x-x_0, k=y-y_0,$$

记算子

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m \triangleq \sum_{i=0}^{m} C_m^i h^i k^{m-i}\frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则$f$在点$(x_0,y_0)$有

(1)带$Lagrange$余项的$n$阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$$

$$(0<\theta<1)$$

(2)带$Peano$余项的$n+1$阶$Taylor$公式

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1}f(x_0,y_0)$$

$$+o\left(\left(\sqrt{h^2+k^2}\right)^{n+1}\right).$$

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例：写出$f(x,y)=x^y$在点(1,1)的邻域内带*Peano*余项的3阶*Taylor*公式，并求$(1.1)^{1.02}$.

解：$f(x,y)=x^y, f'_x=yx^{y-1}, f'_y=x^y\ln x,$

$$f''_{xx}=y(y-1)x^{y-2}, f''_{yy}=x^y\ln^2 x, f''_{xy}=x^{y-1}+yx^{y-1}\ln x,$$

$$f'''_{xxx}=y(y-1)(y-2)x^{y-3},\quad f'''_{yyy}=x^y\ln^3 x$$

$$f'''_{xxy}=(2y-1)x^{y-2}+y(y-1)x^{y-2}\ln x,$$

$$f'''_{xyy}=yx^{y-1}\ln^2 x+2x^{y-1}\ln x.$$

令$x_0=y_0=1, h=x-1, k=y-1.$

$$f(1,1)=f'_x(1,1)=f''_{xy}(1,1)=f'''_{xxy}(1,1)=1,$$

$$f'_y(1,1)=f''_{xx}(1,1)=f''_{yy}(1,1)$$

$$=f'''_{xxx}(1,1)=f'''_{yyy}(1,1)=f'''_{xyy}(1,1)=0$$

$$f(x,y)=f(x_0,y_0)+(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(x_0,y_0)$$

$$+\frac{1}{2!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2 f(x_0,y_0)$$

$$+\frac{1}{3!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^3 f(x_0,y_0)+o((\sqrt{h^2+k^2})^3)$$

$$f(x,y)=1+(x-1)+(x-1)(y-1)$$

$$+\frac{1}{2}(x-1)^2(y-1)$$

$$+o\left(\left(\sqrt{(x-1)^2+(y-1)^2}\right)^3\right)$$

$$(1.1)^{1.02}\approx 1.1+0.1\times 0.02+\frac{1}{2}\times 0.01\times 0.02$$

$$=1.1021.\square$$

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Question. 二元函数在一点的Taylor多项式是否唯一？如何证明？

唯一！

例. $\cos(x^2+y^2)$在$(0,0)$的8阶带Peano余项的Taylor展开式.

解: $\cos t = 1-\dfrac{t^2}{2!}+\dfrac{t^4}{4!}+\cdots+(-1)^n\dfrac{t^{2n}}{(2n)!}+o(t^{2n}),\ t\to 0$时.

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\cdots+(-1)^n\frac{(x^2+y^2)^{2n}}{(2n)!}$$

$$+o((x^2+y^2)^{2n}),\ x^2+y^2\to 0\text{时}.$$

令$n=2$得

$$\cos(x^2+y^2)=1-\frac{(x^2+y^2)^2}{2!}+\frac{(x^2+y^2)^4}{4!}+o((x^2+y^2)^4),$$

$$x^2+y^2\to 0\text{时}.\ \square$$

例. $\ln(2+x+y+xy)$在$(0,0)$带Peano余项的2阶Taylor展开.

解: $x+y+xy\to 0$ 时,

$$\ln(2+x+y+xy)=\ln 2+\ln\left(1+\frac{x+y+xy}{2}\right)$$

$$=\ln 2+\frac{x+y+xy}{2}-\frac{1}{2}\left(\frac{x+y+xy}{2}\right)^2+o\left((x+y+xy)^2\right)$$

$x^2+y^2\to 0$时, 必有$x+y+xy\to 0$ 时, 因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2}=\frac{o((x+y+xy)^2)}{(x+y+xy)^2}\cdot\frac{(x+y+xy)^2}{x^2+y^2}\to 0,$$

$$\ln(2+x+y+xy)$$

$$=\ln 2+\frac{x+y}{2}-\frac{x^2+y^2-2xy}{8}+o(x^2+y^2).\square$$

作业：习题1.8 No.1,2(2)