

# 微积分A(2)期中复习

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# 目录

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# 0 / 学期初的建议

- 重视作业, 一定认真完成作业, 切实理解方法  
标准. 不会做的题, 听完讲解后, 自己能够独立做出来.

- 建议多预习、自学, 赶在大课进度前面

## 本学期所学内容:

- 多元函数微分学(比较容易)
  - 含参数积分(难、抽象)
  - 多重积分和曲线曲面积分(理论简单但难于计算)
  - 常数项级数(中等), 函数项级数(难), 幂级数(容易)
- 期中
- 期末

# 1 / 多元连续函数、偏导数、全微分

## 1. 多元函数在一点处的极限

**Def.**  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, x_0 \in \mathbb{R}^n, A \in \mathbb{R}^m, f$  在  $x_0$  的某个去心邻域  $B_0(x_0, r)$  中有定义. 若  $\forall \varepsilon > 0, \exists \delta \in (0, r), s.t.$

$$\|f(x) - A\| < \varepsilon, \quad \forall x \in B_0(x_0, \delta),$$

则称  $x \rightarrow x_0$  时,  $f(x)$  以  $A$  为极限, 记作  $\lim_{x \rightarrow x_0} f(x) = A$ .

**Remark.**  $\lim_{x \rightarrow x_0} f(x) = A$ , 则:

不论动点  $x$  沿什么路径趋于定点  $x_0$ , 都有  $f(x) \rightarrow A$ .

**Question.** 如何证明  $\lim_{x \rightarrow x_0} f(x)$  不存在?

**例.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x+y}$  是否存在?

**解:**  $\lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{x}{x+y} = \lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1 = 1$

$\lim_{\substack{y \rightarrow 0 \\ x=0}} \frac{x}{x+y} = \lim_{y \rightarrow 0} 0 = 0.$

故  $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x+y}$  不存在.  $\square$

**Question.** 如何证明  $\lim_{x \rightarrow x_0} f(x)$  不存在?

**例.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$  是否存在?

**解:**  $\lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{xy}{x^2 + y^2} = \lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{0}{x^2} = 0$

$$\lim_{\substack{x \rightarrow 0 \\ y=x}} \frac{xy}{x^2 + y^2} = \lim_{\substack{x \rightarrow 0 \\ y=x}} \frac{x^2}{2x^2} = \lim_{\substack{x \rightarrow 0 \\ y=x}} \frac{1}{2} = \frac{1}{2}$$

$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$  不存在!



**Question.** 如何证明  $\lim_{x \rightarrow x_0} f(x)$  不存在?

**例.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x+y}$  是否存在?

**解:**  $\lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{xy}{x+y} = 0,$

$$\lim_{\substack{x \rightarrow 0 \\ y=x^2-x}} \frac{xy}{x+y} = \lim_{x \rightarrow 0} \frac{x^3 - x^2}{x^2} = -1.$$

故  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x+y}$  不存在.  $\square$

**Question.** 如何证明  $\lim_{x \rightarrow x_0} f(x)$  不存在?

**练.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x+y}$  是否存在?

**解:**  $\lim_{\substack{x \rightarrow 0 \\ y=0}} \frac{x^2}{x+y} = \lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$

$$\lim_{\substack{y \rightarrow 0 \\ y=x^2-x}} \frac{x^2}{x+y} = \lim_{y \rightarrow 0} \frac{x^2}{x^2} = \lim_{y \rightarrow 0} 1 = 1$$

故  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x+y}$  不存在.  $\square$



**Question.** 如何证明  $\lim_{x \rightarrow x_0} f(x)$  不存在?

**练.**  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x + y}$  是否存在?

$$y = x^3 - x!$$

# 1 / 多元连续函数、偏导数、全微分

## 2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

**Thm.**  $f, g : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, x_0 \in \mathbb{R}^n$ , 若  $\lim_{x \rightarrow x_0} f(x)$  与  $\lim_{x \rightarrow x_0} g(x)$

都存在, 则

$$(1) \lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x);$$

$$(2) m = 1 \text{ 时, } \lim_{x \rightarrow x_0} f(x)g(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x);$$

$$(3) m = 1 \text{ 且 } \lim_{x \rightarrow x_0} g(x) \neq 0 \text{ 时, } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}.$$

# 1 / 多元连续函数、偏导数、全微分

## 2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

例.  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^l, g : f(\Omega) \subset \mathbb{R}^l \rightarrow \mathbb{R}^m$ , 若

$$\lim_{x \rightarrow x_0} f(x) = A, \lim_{y \rightarrow A} g(y) = B,$$

且  $\exists B(x_0, \delta) \subset \Omega, s.t. \forall x \in B(x_0, \delta), \text{有 } f(x) \neq A$ , 则

$$\lim_{x \rightarrow x_0} (g \circ f)(x) = B.$$

Thm. (夹挤原理)  $f, g, h : B_0(x_0, \delta) \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , 若

$$f(x) \leq g(x) \leq h(x), \forall x \in B_0(x_0, \delta),$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = A,$$

则

$$\lim_{x \rightarrow x_0} g(x) = A.$$

思路. 均值不等式是常用技巧:  $|xy| \leq \frac{x^2 + y^2}{2}$

# 1 / 多元连续函数、偏导数、全微分

2. 多元函数极限的性质: 四则运算、夹挤原理、复合极限定理  
例(夹挤).

$$(1) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \underline{\hspace{2cm}}; (2) \lim_{(x,y) \rightarrow (0,0)} x \sin \frac{1}{y} + y \cos \frac{1}{x} = \underline{\hspace{2cm}};$$

解. (1)  $\frac{x^3 + y^3}{x^2 + y^2} = \frac{(x+y)(x^2 - xy + y^2)}{x^2 + y^2} \quad \because |x^2 - xy + y^2| \leq x^2 + y^2 + |xy| \leq \frac{3}{2}(x^2 + y^2)$

$$\therefore \left| \frac{(x+y)(x^2 - xy + y^2)}{x^2 + y^2} \right| \leq \frac{3}{2}|x+y| \quad \therefore \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0$$

$$(2) \left| x \sin \frac{1}{y} + y \cos \frac{1}{x} \right| \leq \left| x \sin \frac{1}{y} \right| + \left| y \cos \frac{1}{x} \right| \leq |x| + |y|$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} x \sin \frac{1}{y} + y \cos \frac{1}{x} = 0$$

# 1 / 多元连续函数、偏导数、全微分

2. 多元函数极限的性质: 四则运算、夹挤原理、复合极限定理  
例 (复合极限定理允许了结合一元函数的一些极限).

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)} = \underline{\hspace{2cm}}$$

解. 视  $x^2 + y^2 = r$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt[3]{1+x^2+y^2}-1}{\sin(x^2+y^2)} = \lim_{r \rightarrow 0^+} \frac{\sqrt[3]{1+r}-1}{\sin r} = \lim_{r \rightarrow 0^+} \frac{\frac{1}{3}r}{\sin r} = \frac{1}{3}$$

# 1 / 多元连续函数、偏导数、全微分

## 2. 多元函数极限的性质：四则运算、夹挤原理、复合极限定理

练习  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy - \sin(xy)}{xy - xy \cos(xy)} = \underline{\hspace{2cm}}$

解. 视  $xy = r$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy - \sin(xy)}{xy - xy \cos(xy)} = \lim_{r \rightarrow 0} \frac{r - \sin(r)}{r - r \cos(r)} = \lim_{r \rightarrow 0} \frac{r - \sin r}{r(1 - \cos r)}$$

$$= \lim_{r \rightarrow 0} \frac{r - \sin r}{\frac{1}{2} r^3} = 2 \lim_{r \rightarrow 0} \frac{r - \sin r}{r^3} = 2 \lim_{r \rightarrow 0} \frac{1 - \cos r}{3r^2} = 1/3$$



# 1 / 多元连续函数、偏导数、全微分

## 3. 累次极限和二重极限

**Def.**(累次极限)  $\lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y) \triangleq \lim_{y \rightarrow y_0} \left( \lim_{x \rightarrow x_0} f(x, y) \right)$

$$\lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} f(x, y) \triangleq \lim_{x \rightarrow x_0} \left( \lim_{y \rightarrow y_0} f(x, y) \right)$$

**Remark.** 任意固定  $y \neq y_0$ , 若  $\lim_{x \rightarrow x_0} f(x, y)$  存在, 记为

$$g(y) = \lim_{x \rightarrow x_0} f(x, y).$$

若  $\lim_{y \rightarrow y_0} g(y) = A$ , 则  $\lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y) = \lim_{y \rightarrow y_0} g(y) = A$ .

# 1 / 多元连续函数、偏导数、全微分

## 3. 累次极限和二重极限

**Remark.** 求算  $\lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y)$  时候, 先计算  $\lim_{x \rightarrow x_0} f(x, y)$ , 此时把  $y$  看做常数, 显然这次极限计算后  $x$  被消掉, 之后再令  $y \rightarrow y_0$ .

**例.** 求算  $\lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y)$  时候, 先计算  $\lim_{x \rightarrow x_0} f(x, y)$ , 此时把  $y$  看做常数, 显然这次极限计算后  $x$  被消掉, 之后再令  $y \rightarrow y_0$ .

# 1 / 多元连续函数、偏导数、全微分

例. (2020春)  $D = \{(x, y) \mid x + y \neq 0\}$ ,  $f(x, y) = \frac{x - y}{x + y}$

问:  $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$ ,  $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y)$ ,  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$  是否存在

$$\text{解: } \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x - y}{x + y} = \lim_{x \rightarrow 0} \frac{x}{x} = \lim_{x \rightarrow 0} 1 = 1$$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x - y}{x + y} = \lim_{y \rightarrow 0} \frac{-y}{y} = \lim_{y \rightarrow 0} -1 = -1$$

$\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$  不存在

$$\text{选择路径 } y = 2x, \quad f(x, 2x) = \frac{x - 2x}{x + 2x} = -\frac{1}{3}$$

$$\text{选择路径 } y = 3x, \quad f(x, 3x) = \frac{x - 3x}{x + 3x} = -\frac{1}{2}$$

$\therefore$  极限不存在

在 $(x_0, y_0)$ 连续  $\Leftrightarrow \lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$

#### 4. 向量值函数的连续

**Def.** 设 $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, x_0 \in \Omega$ , 若  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ , 也即

$\forall \varepsilon > 0, \exists \delta > 0, s.t.$

$$\|f(x) - f(x_0)\| < \varepsilon, \quad \forall x \in \Omega \cap B(x_0, \delta),$$

则称 $f$ 在点 $x_0$ 处连续, 称 $f$ 的不连续点为间断点.

**Def.** 设 $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , 若 $f$ 在 $\Omega$ 上点点连续, 则称 $f$ 在 $\Omega$ 上连续, 记作 $f \in C(\Omega)$ .

**Remark.**  $f = (f_1, f_1, \dots, f_m) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , 则

$f$ 在点 $x_0$ 连续  $\Leftrightarrow f_i$ 在点 $x_0$ 连续,  $i = 1, 2, \dots, m$ .

例: 讨论  $f(x, y) = \begin{cases} \frac{x^2 y^2}{(x^2 + y^2)^{3/2}} & (x, y) \neq (0, 0) \\ 0 & \text{其它情形} \end{cases}$  的连续性.

解: 只需要研究  $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y^2}{(x^2 + y^2)^{3/2}}$  是否存在.

$$0 \leq \frac{x^2 y^2}{(x^2 + y^2)^{3/2}} = \frac{(xy)^2}{(x^2 + y^2)^{3/2}} \leq \frac{\left(\frac{x^2 + y^2}{2}\right)^2}{(x^2 + y^2)^{3/2}} = \frac{1}{4} \sqrt{x^2 + y^2}$$

$$\therefore \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y^2}{(x^2 + y^2)^{3/2}} = 0$$



例: 讨论  $f(x, y) = \begin{cases} 1 & y = x^2, x > 0 \\ 0 & \text{其它情形} \end{cases}$  的连续性.

解:  $f$  在开区域  $\{(x, y) \mid x \neq \sqrt{y}\}$  中为初等函数, 故处处连续. 而  $f$  在曲线  $x = \sqrt{y}$  上每一点都不连续. 事实上, 任取  $(x_0, y_0)$ ,  $x_0 = \sqrt{y_0}$ , 当点列  $\{P_k(x_k, y_k)\}$  沿曲线  $x = \sqrt{y}$  趋于  $(x_0, y_0)$  时,  $f(x_k, y_k) \rightarrow 1$ ; 当点列  $\{P_k\}$  沿直线  $x = x_0$  趋于  $(x_0, y_0)$  时,  $f(x_k, y_k) \rightarrow 0$ .  $\square$



**Thm.(最值定理)** 设 $\Omega \subset \mathbb{R}^n$ 为有界闭集,  $f \in C(\Omega)$ , 则 $f$ 在 $\Omega$ 上存在最大值 $M$ 和最小值 $m$ , 即 $\exists \xi, \eta \in \Omega, s.t. \forall x \in \Omega$ , 都有

$$m = f(\xi) \leq f(x) \leq f(\eta) = M.$$

**Thm.(介值定理)** 设 $\Omega \subset \mathbb{R}^n$ 为连通区域,  $f \in C(\Omega)$ ,  $x_1, x_2 \in \Omega$ ,  $f(x_1) = \lambda \leq \mu = f(x_2)$ , 则 $\forall \sigma \in [\lambda, \mu], \exists x \in \Omega, s.t. f(x) = \sigma$ .

例 (P24-T8) :  $\lim_{x^2+y^2 \rightarrow +\infty} f(x, y) = +\infty \Rightarrow f(x, y)$  有最小值

证明 :  $\lim_{x^2+y^2 \rightarrow +\infty} f(x, y) = +\infty \Rightarrow$

$\forall M > 0, \exists R > 0, s.t. \forall (x, y),$  满足  $x^2 + y^2 \geq R^2, f(x, y) \geq M$

$x^2 + y^2 \leq R^2$  是有界闭集, 故  $f(x, y)$  在  $x^2 + y^2 \leq R^2$  有最小值

取  $M = f(0, 0),$

$\exists R > 0, s.t. \forall (x, y),$  满足  $x^2 + y^2 \geq R^2, f(x, y) \geq f(0, 0)$

$f(x, y)$  在  $x^2 + y^2 \leq R^2$  有最小值

$f(x_0, y_0) = \min_{x^2+y^2 \leq R^2} f(x, y) \leq f(0, 0) \leq f(x, y), \quad \forall x^2 + y^2 \geq R^2$

## 5. 偏导数

**Def.**  $u = f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$  在  $\mathbf{x}_0 = (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(n)}) \in \mathbb{R}^n$  的某个邻域中有定义, 若极限

$$\lim_{\Delta x_i \rightarrow 0} \frac{\Delta_{x_i} u}{\Delta x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_0^{(1)}, \dots, x_0^{(i-1)}, \mathbf{x}_0^{(i)} + \Delta x_i, x_0^{(i+1)}, \dots, x_0^{(n)}) - f(\mathbf{x}_0)}{\Delta x_i}$$

存在, 则称之为  $f(\mathbf{x})$  在  $\mathbf{x}_0$  关于  $x_i$  的偏导数, 记作  $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ ,

$$\frac{\partial u}{\partial x_i}(\mathbf{x}_0), \left. \frac{\partial f}{\partial x_i} \right|_{\mathbf{x}_0}, \left. \frac{\partial u}{\partial x_i} \right|_{\mathbf{x}_0}, u'_{x_i}(\mathbf{x}_0) \text{ 或 } f'_{x_i}(\mathbf{x}_0).$$

## 5. 偏导数

$$f'_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

**Remark:** 1) 对某个变量求偏导数时, 视其余变量为常数, 按一元函数求导法则和公式去求.

2) 求分段函数的偏导函数时, 用定义求分界点处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

3) 求某一点的偏导数时, 可以先带入其他变量的值, 使之完全退化为一元函数, 再求导

例.  $f(x, y) = x^2 e^y + (x-1) \arctan \frac{y}{x}$ , 求  $f'_x(1, 0)$ .

解法一:  $f(x, 0) = x^2$ , 所以  $f'_x(1, 0) = 2$ .

解法二:

$$f'_x(x, y) = 2xe^y + \arctan \frac{y}{x} + (x-1) \cdot \frac{\frac{-y}{x^2}}{1 + \left(\frac{y}{x}\right)^2}$$
$$= 2xe^y + \arctan \frac{y}{x} + \frac{y(1-x)}{x^2 + y^2}.$$

所以  $f'_x(1, 0) = 2$ .  $\square$

**Remark:** 求具体点处的偏导数时, 第一种方法较好.



## 5. 偏导数

4)偏导数仅仅说明了沿着坐标轴方向,函数是光滑的,因此和连续性互不蕴含

例:  $f(x, y) = \begin{cases} 1 & y = x^2, x > 0 \\ 0 & \text{其它情形} \end{cases}$  在  $(0, 0)$  处不连续, 俩偏导数都为0

偏导数的局限性: 只看坐标轴方向, 不全面

——引出方向导数、可微两个概念



## 5. 偏导数

$$(x+1)\sin y + \sin x$$

例.  $z = f(x, y)$  偏导数存在,  $\frac{\partial z}{\partial x} = \sin y + \cos x$ ,  $f(0, y) = \sin y$ , 求  $f(x, y) = \underline{\hspace{2cm}}$ .

$\frac{\partial z}{\partial x}$  的得出: 视  $y$  为常数, 对  $x$  求导  $\therefore f(x, y) = \int \sin y + \cos x dx$

$$\therefore f(x, y) = x \sin y + \sin x + g(y)$$

$$\because g(y) = f(0, y) = \sin y$$

$$\therefore f(x, y) = (x+1)\sin y + \sin x$$

## 6. 可微

$$f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \frac{\partial f}{\partial x_1}(\mathbf{x}_0)\Delta x_1 + \frac{\partial f}{\partial x_2}(\mathbf{x}_0)\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\Delta x_n + o(\|\Delta \mathbf{x}\|)$$

$$\Leftrightarrow \lim_{\Delta \mathbf{x} \rightarrow 0} \frac{f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) - \left( \frac{\partial f}{\partial x_1}(\mathbf{x}_0)\Delta x_1 + \frac{\partial f}{\partial x_2}(\mathbf{x}_0)\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\Delta x_n \right)}{\|\Delta \mathbf{x}\|} = 0$$

二元函数特殊情况

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f'_x(x_0,y_0)(x-x_0) - f'_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

可微一定连续, 偏导数也一定存在.

## 7. 总结(二元函数版本的连续可偏导可微)

连续:  $f(x_0, y_0) = \lim_{(x,y) \rightarrow (0,0)} f(x, y)$

可偏导:  $f'_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$

$$f'_y(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

可微  $\Leftrightarrow$

$$\lim_{(x,y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - f'_x(x_0, y_0)(x - x_0) - f'_y(x_0, y_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

例.  $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$  在原点的

可微性

解: .Step1. 计算偏导数  $f(x, 0) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$

$$f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} x \sin \frac{1}{x^2} = 0;$$

同理  $f'_y(0, 0) = 0$ .

Step2. 考察  $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - f'_x(0, 0)x - f'_y(0, 0)y}{\sqrt{x^2 + y^2}} = 0$  是否成立

本题中  $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = 0$  是否成立?

例.  $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$  在原点的

可微性

解: .

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2) \sin \frac{1}{x^2 + y^2}}{\sqrt{x^2 + y^2}} = 0 \quad \therefore \text{可微}$$

Hint. 分段函数分析可微性:

(1) 用定义计算偏导数;(不是用求导法则)

(2) 用定义验证可微:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - f'_x(x_0, y_0)(x - x_0) - f'_y(x_0, y_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$



例. P42-2(4)

$f(x, y) = |x - y| \varphi(x, y)$ ,  $\varphi(x, y)$  在  $(0, 0)$  的邻域内连续,  $\varphi(0, 0) = 0$

问:  $f(x, y) = |x - y| \varphi(x, y)$  是否可微

解. P42-2(4)

Step1. 计算偏导数

$$\left| \frac{|x| \varphi(x, 0)}{x} \right| = |\varphi(x, 0)|$$

$$x \rightarrow 0, |\varphi(x, 0)| \rightarrow |\varphi(0, 0)| = 0$$

$$\frac{\partial f}{\partial x}_{(0,0)} = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{|x| \varphi(x, 0)}{x} = 0$$

$$\frac{\partial f}{\partial y}_{(0,0)} = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{|y| \varphi(0, y)}{y} = 0$$

Step2. 考察  $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0) - f'_x(0, 0)x - f'_y(0, 0)y}{\sqrt{x^2 + y^2}} = 0$  是否成立



例. P42-2(4)

$f(x, y) = |x - y| \varphi(x, y)$ ,  $\varphi(x, y)$  在  $(0, 0)$  的邻域内连续,  $\varphi(0, 0) = 0$

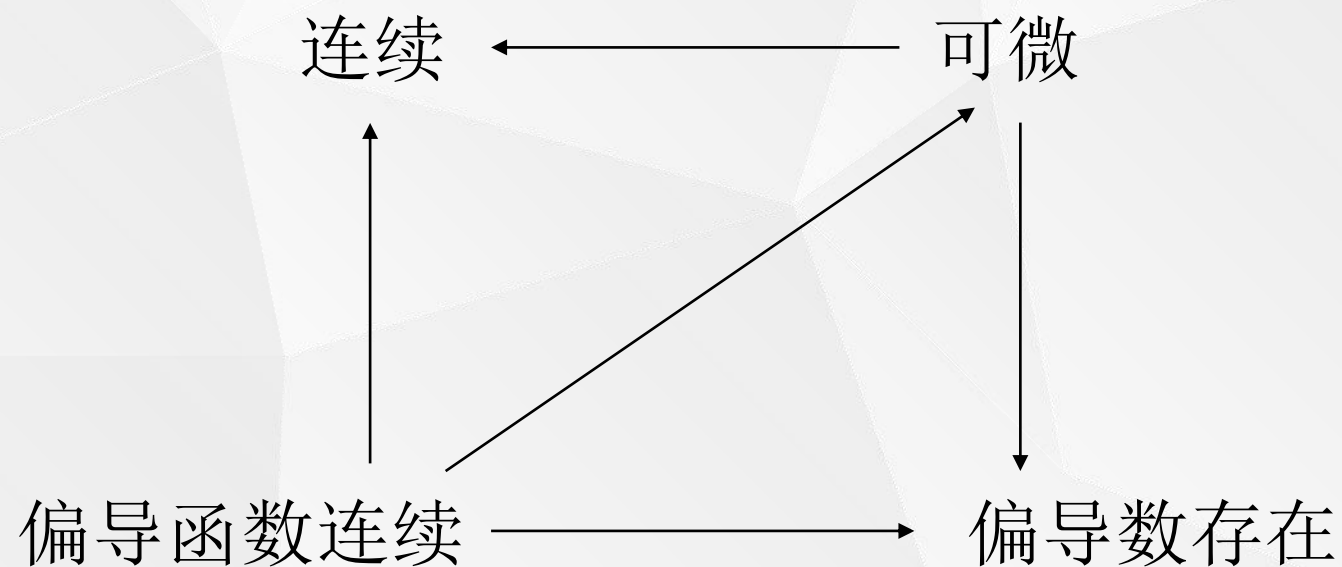
问:  $f(x, y) = |x - y| \varphi(x, y)$  是否可微

解. P42-2(4)

Step2. 考察  $\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - f'_x(0, 0)x - f'_y(0, 0)y}{\sqrt{x^2 + y^2}} = 0$  是否成立

$$\begin{aligned} \lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - f'_x(0, 0)x - f'_y(0, 0)y}{\sqrt{x^2 + y^2}} &= \lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} \\ &= \lim_{(x, y) \rightarrow (0, 0)} \frac{|x - y| \varphi(x, y)}{\sqrt{x^2 + y^2}} = 0 \end{aligned}$$
$$\left| \frac{|x - y| \varphi(x, y)}{\sqrt{x^2 + y^2}} \right| \leq |\varphi(x, y)| \frac{|x| + |y|}{\sqrt{x^2 + y^2}}$$
$$2|\varphi(x, y)| \rightarrow 0, \text{ 当 } x, y \rightarrow (0, 0) \leq 2|\varphi(x, y)|$$

**Remark:** 函数的连续性、可微性、偏导数存在性与偏导数连续性之间的蕴含关系图.



例:  $f(A) = A^2, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $f$  在  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  处的微分  $df =$  \_\_\_\_\_

$$f(A) = A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & cb + d^2 \end{pmatrix},$$

$$Jf(A) = \begin{pmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} df = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} da \\ db \\ dc \\ dd \end{pmatrix}$$

**Def.**  $f$  在  $\mathbf{x}_0 \in \mathbb{R}^n$  的邻域中有定义,  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$  为非零向量,  $l$  为过  $\mathbf{x}_0$  沿  $\vec{v}$  方向的射线, 若  $t$  的函数

$$g(t) = f\left(\mathbf{x}_0 + \frac{\vec{v}}{\|\vec{v}\|} t\right) = f\left(\mathbf{x}_0^{(1)} + \frac{v_1}{\|\vec{v}\|} t, \dots, \mathbf{x}_0^{(n)} + \frac{v_n}{\|\vec{v}\|} t\right)$$

在  $t = 0$  存在右导数, 即极限

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \mathbf{x} \in l}} \frac{f(\mathbf{x}) - f(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \lim_{t \rightarrow 0^+} \frac{g(t) - g(0)}{t}$$

存在, 则称该极限为  $f(\mathbf{x})$  在  $\mathbf{x}_0$  沿方向  $\vec{v}$  的方向导数, 记作

$$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}}, \left. \frac{\partial f}{\partial \vec{v}} \right|_{\mathbf{x}_0} \text{ 或 } f'_{\vec{v}}(\mathbf{x}_0).$$

**Remark.**  $\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}}$  是函数  $f(\mathbf{x})$  在点  $\mathbf{x}_0$  沿方向  $\vec{v}$  的变化率.

**Remark.**  $\frac{\partial f(\mathbf{x}_0)}{\partial x_i}$  为  $f$  在  $\mathbf{x}_0$  沿  $e_i = (0, \dots, 0, \overset{\substack{\text{第 } i \text{ 个分量} \\ \downarrow}}{1}, 0, \dots, 0)$  的方向导数.

**Thm.** 设  $f$  在  $\mathbf{x}_0 \in \mathbb{R}^n$  **可微**,  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$  为非零向量,  
则方向导数  $\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}}$  存在, 且

$$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}} = \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \frac{v_1}{\|\vec{v}\|} + \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \frac{v_2}{\|\vec{v}\|} + \dots + \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \frac{v_n}{\|\vec{v}\|}.$$



例. (1) 计算  $f(x, y) = \sin(x + 2y)$  在  $(0, 0)$  处, 沿着  $I = (1, 1)$  方向的方向导数;  
(2) 求出方向导数最大的方向 (单位化为单位向量)

解. (1)  $\frac{\partial f}{\partial x}(x, y) = \cos(x + 2y), \frac{\partial f}{\partial y}(x, y) = 2 \cos(x + 2y) \therefore \frac{\partial f}{\partial x}(0, 0) = 1, \frac{\partial f}{\partial y}(0, 0) = 2$

$$\therefore \frac{\partial f}{\partial I}(0, 0) = 1 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

(2) 求出方向导数最大的方向

设这一方向为  $I = (\cos \theta, \sin \theta)$

$$\therefore \frac{\partial f}{\partial I}(0, 0) = 1 \times \cos \theta + 2 \times \sin \theta, \text{由柯西-施瓦茨不等式, } \frac{\partial f}{\partial I}(0, 0) \leq \sqrt{5},$$

$$\text{当} (\cos \theta, \sin \theta) = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

# 1 / 多元连续函数、偏导数、全微分

例. (2020春模拟)  $f(x, y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(1)  $f(x, y)$  在  $(0, 0)$  处的连续性? ; (2)  $f(x, y)$  在  $(0, 0)$  处两个一阶偏导数的存在性?;

(3)  $f(x, y)$  在  $(0, 0)$  处是否可微?

解: (1)  $|x^3 + y^3| = |x + y| |x^2 - xy + y^2| \leq |x^2| + |xy| + |y^2| \leq |x + y| (|x|^2 + |x||y| + |y|^2) \leq \frac{3}{2} |x + y| (x^2 + y^2)$

$\therefore \left| \frac{x^3 + y^3}{x^2 + y^2} \right| \leq \frac{3}{2} |x + y| \therefore \lim_{(x, y) \rightarrow (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = 0 = f(0, 0) \therefore$  连续

(2)  $f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x - 0}{x^3 + 0} = 1 \quad f'_y(0, 0) = 1$

(3) 不可微.  $f(x, y) - f(0, 0) - xf'_x(0, 0) - yf'_y(0, 0) = \frac{x^3 + y^3}{x^2 + y^2} - x - y = -\frac{xy(x + y)}{x^2 + y^2}$

考虑极限  $\lim_{(x, y) \rightarrow (0, 0)} \frac{-\frac{xy(x + y)}{x^2 + y^2}}{\sqrt{x^2 + y^2}}$  是否存在, 并且是否为 0

# 1 / 多元连续函数、偏导数、全微分

例. (2020春模拟)  $f(x, y) = \begin{cases} \frac{x^3 + y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(1)  $f(x, y)$  在  $(0, 0)$  处的连续性? ; (2)  $f(x, y)$  在  $(0, 0)$  处两个一阶偏导数的存在性?;

(3)  $f(x, y)$  在  $(0, 0)$  处是否可微?  $-\frac{xy(x+y)}{x^2+y^2}$

(3) 不可微. 考虑极限  $\lim_{(x,y) \rightarrow (0,0)} \frac{-\frac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}}$  是否存在, 并且是否为 0

$$\text{取 } y = x, \frac{-\frac{xy(x+y)}{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{-\frac{2x^3}{2x^2}}{\sqrt{2x^2}} = \frac{-x}{\sqrt{2}|x|} \quad \lim_{x \rightarrow 0+} \frac{-x}{\sqrt{2}|x|} = -\frac{1}{\sqrt{2}}$$

2) 求分段函数的偏导函数时, 用定义求\*\*分界点\*\*处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

## 2 / 链锁法则和隐函数定理

### • Chain Rule

$$u = g(x) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, y = f(u) : g(\Omega) \subset \mathbb{R}^m \rightarrow \mathbb{R}^k,$$

$g(x)$  在  $x_0 \in \Omega$  可微,  $f(u)$  在  $u_0 = g(x_0)$  可微, 则

$$J(f \circ g)|_{x_0} = J(f)|_{u_0} \cdot J(g)|_{x_0},$$

$$\text{即 } \frac{\partial(y_1, y_2, \dots, y_k)}{\partial(x_1, x_2, \dots, x_n)} \Big|_{x_0} = \frac{\partial(y_1, y_2, \dots, y_k)}{\partial(u_1, u_2, \dots, u_m)} \Big|_{u_0} \cdot \frac{\partial(u_1, u_2, \dots, u_m)}{\partial(x_1, x_2, \dots, x_n)} \Big|_{x_0},$$

$$\text{简记为 } \frac{\partial y}{\partial x} \Big|_{x_0} = \frac{\partial y}{\partial u} \Big|_{u_0} \cdot \frac{\partial u}{\partial x} \Big|_{x_0}.$$

$$k=1 \text{ 时, } \frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1} \frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2} \frac{\partial u_2}{\partial x_i} + \dots + \frac{\partial y}{\partial u_m} \frac{\partial u_m}{\partial x_i}, i = 1, 2, \dots, n.$$

## 2 / 链锁法则和隐函数定理

例.  $z = f(xy, x^2 + y^2)$ , 计算  $z'_x, z'_y$

解.  $z'_x = f'_1(xy, x^2 + y^2)y + f'_2(xy, x^2 + y^2)2x$     $z'_y = f'_1(xy, x^2 + y^2)x + f'_2(xy, x^2 + y^2)2y$

例.  $u = u(x, y, z)$ ,  $u$  在全空间可微,  $u$  满足

$$u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z, \text{ 其中 } k > 0$$

证明:  $ku(x, y, z) = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$

证.  $\because u(tx, ty, tz) = t^k u(x, y, z), \forall t, x, y, z$ . 等式两边对  $t$  求导

$$\because xu'_1(tx, ty, tz) + yu'_2(tx, ty, tz) + zu'_3(tx, ty, tz) = kt^{k-1}u(x, y, z)$$

取  $t = 1$ , 得  $ku(x, y, z) = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$



## 2 / 链锁法则和隐函数定理

例. (2020期末)  $f \in C^{(2)}(\mathbb{R})$ ,  $z = f(x^2 + xy + y^2)$ , 计算  $z'_y, z''_{xy}$  在  $(1,1)$  的值.

解.  $z'_y = f'(x^2 + xy + y^2)(2y + x) \quad \therefore z'_y(1,1) = 3f'(3)$

$$\therefore z'_y(x,1) = f'(x^2 + x + 1)(2 + x)$$

$$\therefore z''_{yx}(x,1) = (z'_y(x,1))' = f''(x^2 + x + 1)(2 + x)^2 + f'(x^2 + x + 1)$$

$$\therefore z''_{yx}(1,1) = 9f''(3) + f'(3)$$

## 2 / 链锁法则和隐函数定理

例.  $u = u(x, y, z)$ ,  $u$  在全空间可微,  $u$  满足

$$ku(x, y, z) = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}, \forall t, x, y, z, \text{ 其中 } k > 0$$

证明:  $u(tx, ty, tz) = t^k u(x, y, z)$

证. 构造辅助函数  $F(t) = u(tx, ty, tz) - t^k u(x, y, z)$

$$\begin{aligned} F'(t) &= xu'_1(tx, ty, tz) + yu'_2(tx, ty, tz) + zu'_3(tx, ty, tz) - kt^{k-1}u(x, y, z) \\ &= \frac{1}{t} (txu'_1(tx, ty, tz) + tyu'_2(tx, ty, tz) + tzu'_3(tx, ty, tz) - kt^k u(x, y, z)) \\ &= \frac{1}{t} (ku(tx, ty, tz) - kt^k u(x, y, z)) = \frac{k}{t} (u(tx, ty, tz) - t^k u(x, y, z)) = \frac{k}{t} F(t) \end{aligned}$$

$$\therefore F'(t) = \frac{k}{t} F(t) \Rightarrow F(t) = Ct^k \because F(1) = u(x, y, z) - u(x, y, z) = 0 \quad \therefore C = 0 \therefore F(t) = 0$$

**Remark:**  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m, (x, y) \mapsto F(x, y)$ , 若  $\frac{\partial F}{\partial y}$  可逆,

则  $F(x, y) = 0$  确定隐“函数”  $y = y(x)$ , 求  $\frac{\partial y}{\partial x}$  有两种方法:

- 套用定理:  $\frac{\partial y}{\partial x} = - \left( \frac{\partial F}{\partial y} \right)^{-1} \frac{\partial F}{\partial x}.$

这里求Jaccobi矩阵时  $x, y$  相互独立!

- 将  $F(x, y) = 0$  中  $y$  视为  $y = y(x)$ , 利用复合映射的链式法则, 方程组  $F(x, y(x)) = 0$  两边对  $x$  求Jaccobi矩阵.

**Remark:** 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

**Remark:**  $m$ 个方程确定 $m$ 个隐函数, 将某 $m$ 个变量看成函数, 其它变量相互独立.

例.  $\varphi$ 可微,  $x^2 + z^2 = y\varphi\left(\frac{z}{y}\right)$  确定隐函数  $z = z(x, y)$ . 求  $z'_x, z'_y$ .

解: 视  $x^2 + z^2 = y\varphi(z/y)$  中  $z = z(x, y)$  为隐函数. 两边分别对  $x, y$  求偏导, 有

$$2x + 2zz'_x = y\varphi'(z/y) \cdot \frac{1}{y} z'_x,$$

$$2zz'_y = \varphi(z/y) + y\varphi'(z/y) \cdot \frac{1}{y^2} (yz'_y - z).$$

求解得

$$z'_x = \frac{2x}{\varphi'(z/y) - 2z}, \quad z'_y = \frac{y\varphi(z/y) - \varphi'(z/y)}{2yz - y\varphi'(z/y)}. \quad \square$$



例.  $u = f(x - ut, y - ut, z - ut)$ ,  $g(x, y, z) = 0$ , 求  $u'_x, u'_y$ .

分析: 五个变量  $x, y, z, t, u$ , 两个方程, 确定两个隐函数  $z = z(x, y, t) = z(x, y)$ ,  $u = u(x, y, t)$ .

解法一: 视  $u = f(x - ut, y - ut, z - ut)$  中  $z = z(x, y)$  为隐函数, 两边分别对  $x, y$  求偏导, 有

$$u'_x = (1 - tu'_x)f'_1 + (-tu'_x)f'_2 + (z'_x - tu'_x)f'_3,$$

$$u'_y = (-tu'_y)f'_1 + (1 - tu'_y)f'_2 + (z'_y - tu'_y)f'_3.$$

其中  $f'_1, f'_2, f'_3$  在  $(x - ut, y - ut, z - ut)$  处取值.

视  $g(x, y, z) = 0$  中  $z = z(x, y)$ , 两边对  $x, y$  求偏导, 有

$$\begin{cases} g'_x + g'_z z'_x = 0, \\ g'_y + g'_z z'_y = 0, \end{cases} \Rightarrow \begin{cases} z'_x = -g'_x / g'_z, \\ z'_y = -g'_y / g'_z. \end{cases}$$

代入前两式,求解得

$$u'_x = \frac{f'_1 + f'_3 z'_x}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_1 g'_z - f'_3 g'_x}{[1 + t(f'_1 + f'_2 + f'_3)] g'_z}$$

$$u'_y = \frac{f'_2 + f'_3 z'_y}{1 + t(f'_1 + f'_2 + f'_3)} = \frac{f'_2 g'_z - f'_3 g'_y}{[1 + t(f'_1 + f'_2 + f'_3)] g'_z}.$$

例. (2020真题)  $u(t) \in C^2(\mathbb{R})$ ,  $z = u(\sqrt{x^2 + y^2})$  满足:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y^2 \quad (x^2 + y^2 > 0)$$

证明:  $u(t)$  满足  $u'' + \frac{1}{t}u' = t^2$

证.  $\because z = u(\sqrt{x^2 + y^2}) \therefore z'_x = \frac{x}{\sqrt{x^2 + y^2}} u'(\sqrt{x^2 + y^2}), z'_y = \frac{y}{\sqrt{x^2 + y^2}} u'(\sqrt{x^2 + y^2})$

$$\therefore z''_{xx} = \frac{y^2}{(x^2 + y^2)^{3/2}} u'(\sqrt{x^2 + y^2}) + \frac{x^2}{(x^2 + y^2)} u''(\sqrt{x^2 + y^2})$$

$$z''_{yy} = \frac{x^2}{(x^2 + y^2)^{3/2}} u'(\sqrt{x^2 + y^2}) + \frac{y^2}{(x^2 + y^2)} u''(\sqrt{x^2 + y^2})$$

$$\therefore z''_{xx} + z''_{yy} = \frac{1}{\sqrt{x^2 + y^2}} u'(\sqrt{x^2 + y^2}) + u''(\sqrt{x^2 + y^2}) = x^2 + y^2 \quad \text{令 } t = \sqrt{x^2 + y^2} \text{ 即可.}$$

例. (2020模拟) 二阶连续可微函数  $z = z(x, y)$  满足:  $x^3 + y^3 + z^3 = x + y + z$

计算  $\frac{\partial^2 z}{\partial x \partial y}$

解.  $\because x^3 + y^3 + z^3(x, y) = x + y + z(x, y)$

$$\therefore \text{对 } x \text{ 偏导, } 3x^2 + 3z^2 \frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x} \qquad \therefore \frac{\partial z}{\partial x} = \frac{1-3x^2}{3z^2-1}, \frac{\partial z}{\partial y} = \frac{1-3y^2}{3z^2-1}$$

$$\text{再对 } y \text{ 偏导, } 6z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + 3z^2 \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{6z}{1-3z^2} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} = \frac{6z}{(1-3z^2)^3} (1-3x^2)(1-3y^2)$$

## 2 / 链锁法则和隐函数定理

例. (2020期末)  $y = y(x), z = z(x)$  由方程组  $\begin{cases} x^3 + y^3 - z^3 = 10 \\ x + y + z = 0 \end{cases}$  在  $(1, 1, -2)$  处确定隐函数

求  $y = y(x), z = z(x)$  在  $x = 1$  处的导数

解. 在方程组  $\begin{cases} x^3 + y^3 - z^3 = 10 \\ x + y + z = 0 \end{cases}$  两边对  $x$  求导 得  $\begin{cases} 3x^2 + 3y^2 y' - 3z^2 z' = 0 \\ 1 + y' + z' = 0 \end{cases}$

按照  $x = 1, y = 1, z = -2$  带入得  $\begin{cases} 3 + 3y' - 12z' = 0 \\ 1 + y' + z' = 0 \end{cases} \therefore \begin{cases} y'(1) = -1 \\ z'(1) = 0 \end{cases}$



例.  $f \in C^2(\mathbb{R}^2)$ ,  $f''_{xx} + f''_{yy} = 0$ ,

$$\text{令 } p = \frac{x}{x^2 + y^2}, q = \frac{y}{x^2 + y^2}$$

求证:  $u(x, y) = f\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$  也满足  $u''_{xx} + u''_{yy} = 0$ .

分析: 直接思路是强算, 但是可以预见到这种计算过于复杂

证明:  $\because u(x, y) = f(p(x, y), q(x, y)). \therefore u'_x = f'_1 p'_x + f'_2 q'_x, u'_y = f'_1 p'_y + f'_2 q'_y$

$$\begin{aligned} \therefore u''_{xx} &= \left( f''_{11} p'_x + f''_{12} q'_x \right) p'_x + f'_1 p''_{xx} + \left( f''_{21} p'_x + f''_{22} q'_x \right) q'_x + f'_2 q''_{xx} \\ &= f''_{11} p'^2_x + 2f''_{12} q'_x p'_x + f''_{22} q'^2_x + f'_1 p''_{xx} + f'_2 q''_{xx} \end{aligned}$$

$$\therefore u''_{yy} = f''_{11} p'^2_y + 2f''_{12} q'_y p'_y + f''_{22} q'^2_y + f'_1 p''_{yy} + f'_2 q''_{yy}$$

$$\begin{aligned} \therefore u''_{xx} + u''_{yy} &= f''_{11} (p'^2_y + p'^2_x) + 2f''_{12} (q'_x p'_x + q'_y p'_y) + f''_{22} (q'^2_y + q'^2_x) \\ &\quad + f'_1 (p''_{yy} + p''_{xx}) + f'_2 (q''_{yy} + q''_{xx}) \end{aligned}$$

$$\text{令 } p = \frac{x}{x^2 + y^2}, q = \frac{y}{x^2 + y^2} \quad \text{则 } p'_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, q'_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\text{则 } p'_y = \frac{-2xy}{(x^2 + y^2)^2}, q'_x = \frac{-2xy}{(x^2 + y^2)^2} \quad \therefore p'_y = q'_x, \quad p'_x = -q'_y$$

$$\therefore p''_{yy} = q''_{xy}, \quad p''_{yx} = q''_{xx}, \quad p''_{xy} = -q''_{yy}, \quad p''_{xx} = -q''_{xy}$$

$$\begin{aligned} \therefore u''_{xx} + u''_{yy} &= f''_{11} (p'^2_y + p'^2_x) + 2f''_{12} (q'_x p'_x + q'_y p'_y) + f''_{22} (q'^2_y + q'^2_x) \\ &+ f'_1 (p''_{yy} + p''_{xx}) + f'_2 (q''_{yy} + q''_{xx}) \end{aligned}$$

$$= f''_{11} (q'^2_x + q'^2_y) + f''_{22} (q'^2_y + q'^2_x) = (f''_{11} + f''_{22}) (q'^2_x + q'^2_y) = 0. \square$$

例. (2020期中-类似)  $f \in C^2(\mathbb{R}^2)$ ,  $f > 0$ ,  $f''_{xy}f = f'_xf'_y$ ,

求证: 存在一元函数  $u(x), v(y)$ , s.t.  $f(x, y) = u(x)v(y)$

分析:

$$\ln f(x, y) = \ln u(x) + \ln v(y) \Leftrightarrow \frac{\partial \ln f(x, y)}{\partial x} = \frac{u'(x)}{u(x)} \Leftrightarrow \frac{\partial^2 \ln f(x, y)}{\partial x \partial y} = 0$$

证明:

$$\begin{aligned} \frac{\partial \ln f(x, y)}{\partial x} &= \frac{1}{f} \frac{\partial f}{\partial x} \\ \frac{\partial^2 \ln f(x, y)}{\partial x \partial y} &= \frac{\partial \left( \frac{f'_x}{f} \right)}{\partial y} = \frac{f''_{xy}f - f'_y f'_x}{f^2} = 0 \end{aligned}$$

例.  $f \in C^1(\mathbb{R}^2)$ ,  $a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = 0$ , 选取合适的变量替换  $\begin{cases} u = x + py \\ v = x + qy \end{cases}$ ,  $p, q$  为常数,

将原方程化为  $\frac{\partial f}{\partial u} = 0$ , 从而解为  $f = g(x + qy)$

解.  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \quad \frac{\partial f}{\partial y} = p \frac{\partial f}{\partial u} + q \frac{\partial f}{\partial v}$

$$a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = (a + bp) \frac{\partial f}{\partial u} + (a + bq) \frac{\partial f}{\partial v}$$

$$\therefore a + bp = 1, a + bq = 0 \Rightarrow p = \frac{1-a}{b}, q = -\frac{a}{b} \quad \therefore f = g\left(x - \frac{a}{b}y\right)$$

例. (2020模拟)  $f \in C^2(\mathbb{R}^2)$ , 满足 (1)  $f'_x = f'_y$ , (2)  $f(x, 0) > 0$ ;

证明:  $f(x, y) > 0$ .

分析.  $f'_x - f'_y = 0 \Rightarrow$

$$u = f(x+y, x-y), u'_y = f'_x(x+y, x-y) - f'_y(x+y, x-y) = 0$$

$$\text{进一步 } f'_x - f'_y = 0 \Rightarrow u = f(x+y, -y), u'_y = f'_x(x+y, -y) - f'_y(x+y, -y) = 0$$

$$\therefore u = f(x+y, -y) = f(x, 0) > 0$$

$$\therefore f(x, y) = f(x+y, 0) > 0$$



### 3 / 曲线的切线、曲面的切平面

#### 1. 空间曲线的表示

(1) 参数方程表示: 
$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

切线的方向向量. 
$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left( \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right) = (x'(t), y'(t), z'(t)).$$

切线方程.

$$\begin{cases} x = x_0 + x'(t_0)t \\ y = y_0 + y'(t_0)t \\ z = z_0 + z'(t_0)t \end{cases}$$

法平面方程.

$$x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0$$

### 3 / 曲线的切线、曲面的切平面

#### 1. 空间曲线的表示

例. 求曲线  $\begin{cases} x = e^t \\ y = t \\ z = t^2 \end{cases}$  在  $(1, 0, 0)$  处的切线方程

$\because t = 0$ , 方向向量为  $(e^t, 1, 2t)$ ,  $\therefore$  带入数字得到  $(1, 1, 0)$

$\therefore$  切线方程为  $\begin{cases} x = 1 + t \\ y = t \\ z = 0 \end{cases}$

# 3 / 曲线的切线、曲面的切平面

## 2. 空间曲面的表示

(1) 一般方程表示:  $F(x, y, z) = 0$

曲面的法向量.  $\nabla F = (F'_x, F'_y, F'_z)$ .

切平面方程.

$$F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + F'_z(x_0, y_0, z_0)(z - z_0) = 0$$

法线方程. 
$$\begin{cases} x = x_0 + F'_x(x_0, y_0, z_0)t \\ y = y_0 + F'_y(x_0, y_0, z_0)t \\ z = z_0 + F'_z(x_0, y_0, z_0)t \end{cases}$$

### 3 / 曲线的切线、曲面的切平面

#### 2. 空间曲面的表示

例. 求曲面  $3x^2 + 2y^2 - 2z - 1 = 0$  在  $(1, 1, 2)$  处的法向量和切平面方程

第一步. 计算梯度向量为  $(6x, 4y, -2)$

代入数字, 得到  $(6, 4, -2)$  ← 法向量

$6(x-1) + 4(y-1) - 2(z-2) = 0$  ← 切平面



6. 设曲面  $z = x^2 - y^2$  在点  $(1, 0, 1)$  的切平面方程为  $z = f(x, y)$ , 则  $f(2, 1) = \underline{\hspace{2cm}}$ .

### 3 / 曲线的切线、曲面的切平面

#### 1. 空间曲线的表示

(2) 两个平面的交线: 
$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

切线的方向向量. 
$$\nabla F \times \nabla G = \begin{vmatrix} i & j & k \\ F'_x(x_0, y_0, z_0) & F'_y(x_0, y_0, z_0) & F'_z(x_0, y_0, z_0) \\ G'_x(x_0, y_0, z_0) & G'_y(x_0, y_0, z_0) & G'_z(x_0, y_0, z_0) \end{vmatrix}$$

切线方程和法平面方程怎么写?



### 3 / 曲线的切线、曲面的切平面

#### 1. 空间曲线的表示


4. 曲线  $\begin{cases} x^2 + y^2 = 2, \\ x^2 + z^2 = 2 \end{cases}$  在点  $(1,1,1)$  的法平面方程是

(A)  $x - y - z = -1$

(B)  $y - x - z = -1$

(C)  $z - x - y = -1$

(D)  $x - 1 = 1 - y = 1 - z$


$$\nabla F \times \nabla G = \begin{vmatrix} i & j & k \\ 2x & 2y & 0 \\ 2x & 0 & 2z \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix} = (4, -4, 4) \quad 4(x-1) - 4(y-1) + 4(z-1) = 0$$
$$\therefore x - y + z = 1$$

## 2. 参数方程下曲面的切平面

设曲面 $S$ 的参数方程为 $\mathbf{r} = \mathbf{r}(u, v)$ , 即

$$S: \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

•  $S$ 在 $r_0$ 的法向量:  $\vec{n} = (\mathbf{r}'_u \times \mathbf{r}'_v)|_{(u_0, v_0)}$ .

其中 $r_0 = \mathbf{r}(u_0, v_0)$

例: 求球面  $\begin{cases} x = a \sin \varphi \cos \theta \\ y = a \sin \varphi \sin \theta \\ z = a \cos \varphi \end{cases} \begin{pmatrix} 0 \leq \varphi \leq \pi \\ 0 \leq \theta < 2\pi \end{pmatrix}$  在  $\varphi = \pi/6$ ,

$\theta = \pi/3$  的切平面和法向量.

解:  $\mathbf{r}'_{\theta} = (-a \sin \varphi \sin \theta, a \sin \varphi \cos \theta, 0)$

$\mathbf{r}'_{\varphi} = (a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, -a \sin \varphi).$

当  $\varphi = \pi/6, \theta = \pi/3$  时,

$$(x, y, z) = (a/4, \sqrt{3}a/4, \sqrt{3}a/2),$$

$$\mathbf{r}'_{\varphi} = (\sqrt{3}a/4, 3a/4, -a/2),$$

$$\mathbf{r}'_{\theta} = (-\sqrt{3}a/4, a/4, 0).$$

$$\vec{n} // (\mathbf{r}'_{\varphi} \times \mathbf{r}'_{\theta}) = \det \begin{pmatrix} i & j & k \\ \sqrt{3}a/4 & 3a/4 & -a/2 \\ -\sqrt{3}a/4 & a/4 & 0 \end{pmatrix}$$

$$\vec{n} // (1/8, \sqrt{3}/8, \sqrt{3}/4).$$

切平面方程为

$$(x - a/4, y - \sqrt{3}a/4, z - \sqrt{3}a/2) \cdot \vec{n} = 0,$$

即

$$x + \sqrt{3}y + 2\sqrt{3}z - 4a = 0. \square$$

### 3 / 多元泰勒公式和极值原理

**Thm.** 设 $n$ 元函数 $f$ 在 $B(x_0, \delta)$ 中二阶连续可微, 则

$\forall x_0 + \Delta x \in B(x_0, \delta), \exists \theta \in (0, 1), s.t.$

$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0)\Delta x$$

$$+ \frac{1}{2}(\Delta x)^T H_f(x_0 + \theta\Delta x)\Delta x$$

(称为带 $Lagrange$ 余项的一阶 $Taylor$ 公式), 且

$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0)\Delta x$$

$$+ \frac{1}{2}(\Delta x)^T H_f(x_0)\Delta x + o(\|\Delta x\|^2), \Delta x \rightarrow 0 \text{ 时}$$

(称为带 $Peano$ 余项的二阶 $Taylor$ 公式).



**Thm.** 设函数 $f(x, y)$ 在区域 $D$ 中 $n + 1$ 阶连续可微,  
 $M_0(x_0, y_0) \in D, M(x, y) \in D$ , 且线段 $\overline{M_0M}$ 完全包  
含在 $D$ 中. 记

$$h = x - x_0, k = y - y_0,$$

记算子

$$\left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m \triangleq \sum_{i=0}^m C_m^i h^i k^{m-i} \frac{\partial^m}{\partial x^i \partial y^{m-i}},$$

则 $f$ 在点 $(x_0, y_0)$ 有

(1)带*Lagrange*余项的*n*阶*Taylor*公式

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) \\ & + \cdots + \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0, y_0) \\ & + \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k) \\ & (0 < \theta < 1) \end{aligned}$$

(2)带 $Peano$ 余项的 $n+1$ 阶 $Taylor$ 公式

$$\begin{aligned} f(x, y) = & f(x_0, y_0) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) \\ & + \cdots + \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0, y_0) \\ & + o \left( \left( \sqrt{h^2 + k^2} \right)^{n+1} \right). \end{aligned}$$

### 3 / 多元泰勒公式和极值原理

**Note.**一般来说, 我们不用如此复杂的公式, 而是设法化为一元函数的泰勒公式

**例.**  $\cos(x^2 + y^2)$  在  $(0,0)$  的 8 阶带 Peano 余项的 Taylor 展开式.

**解:**  $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + o(t^{2n}), t \rightarrow 0$  时.

$$\cos(x^2 + y^2) = 1 - \frac{(x^2 + y^2)^2}{2!} + \cdots + (-1)^n \frac{(x^2 + y^2)^{2n}}{(2n)!}$$

令  $n = 2$  得  $+o((x^2 + y^2)^{2n}), x^2 + y^2 \rightarrow 0$  时.

$$\cos(x^2 + y^2) = 1 - \frac{(x^2 + y^2)^2}{2!} + \frac{(x^2 + y^2)^4}{4!} + o((x^2 + y^2)^4),$$

$x^2 + y^2 \rightarrow 0$  时.  $\square$

### 3 / 多元泰勒公式和极值原理

例：求 $f(x, y) = x^y$ 在点 $(1, 1)$ 的邻域内带 $Peano$ 余项的3阶 $Taylor$ 公式

$$f(x, y) = x^y = (x-1+1)^y \qquad \rho = \sqrt{(x-1)^2 + (y-1)^2}$$

解： $(x-1+1)^y = 1 + y(x-1) + \frac{y(y-1)}{2!}(x-1)^2 + \frac{y(y-1)(y-2)}{3!}(x-1)^3 + o(\rho^3)$

$$y(x-1) = (y-1)(x-1) + (x-1)$$

$$\frac{y(y-1)}{2!}(x-1)^2 = \frac{1}{2}(x-1)^2(y-1) + \frac{1}{2}(x-1)^2(y-1)^2$$

$$\frac{1}{6}y(y-1)(y-2)(x-1)^3 = \frac{1}{6}(y-1)(y^2-2y)(x-1)^3 = \frac{1}{6}(y-1)^3(x-1)^3 - \frac{1}{6}(y-1)(x-1)^3$$

$$\therefore \text{原式} = 1 + (x-1) + (y-1)(x-1) + \frac{(y-1)}{2!}(x-1)^2 + o(\rho^3)$$



例.  $\ln(2+x+y+xy)$  在  $(0,0)$  带Peano余项的2阶Taylor展开.

解:  $x+y+xy \rightarrow 0$  时,

$$\begin{aligned}\ln(2+x+y+xy) &= \ln 2 + \ln\left(1 + \frac{x+y+xy}{2}\right) \\ &= \ln 2 + \frac{x+y+xy}{2} - \frac{1}{2} \left( \frac{x+y+xy}{2} \right)^2 + o\left((x+y+xy)^2\right)\end{aligned}$$

$x^2+y^2 \rightarrow 0$  时, 必有  $x+y+xy \rightarrow 0$  时, 因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2} = \frac{o((x+y+xy)^2)}{(x+y+xy)^2} \cdot \frac{(x+y+xy)^2}{x^2+y^2} \rightarrow 0,$$

$$\begin{aligned}\ln(2+x+y+xy) &= \ln 2 + \frac{x+y}{2} - \frac{x^2+y^2-2xy}{8} + o(x^2+y^2). \square\end{aligned}$$

例.  $\sin(x+y) + ze^z - ye^x = 0$  确定了隐函数  $z = z(x, y)$ , 求  $z = z(x, y)$  在  $(0, 0)$  带Peano余项的2阶Taylor展开.

解: 计算  $\frac{\partial z}{\partial x}(0, 0), \frac{\partial z}{\partial y}(0, 0), \frac{\partial^2 z}{\partial x^2}(0, 0), \frac{\partial^2 z}{\partial y^2}(0, 0), \frac{\partial^2 z}{\partial y \partial x}(0, 0)$

在  $\sin(x+y) + ze^z - ye^x = 0$  两边同时对  $x$  求偏导, 得

$$\cos(x+y) + \frac{\partial z}{\partial x} e^z + \frac{\partial z}{\partial x} ze^z - ye^x = 0$$

在  $\sin(x+y) + ze^z - ye^x = 0$  两边同时对  $y$  求偏导, 得

$$\cos(x+y) + \frac{\partial z}{\partial y} e^z + \frac{\partial z}{\partial y} ze^z - e^x = 0$$

$x = 0, y = 0$  时,  $z = 0$

$$\therefore \frac{\partial z}{\partial x} = -1 \quad \frac{\partial z}{\partial y} = 0$$

例.  $\sin(x+y) + ze^z - ye^x = 0$  确定了隐函数  $z = z(x, y)$ , 求  $z = z(x, y)$  在  $(0, 0)$  带Peano余项的2阶Taylor展开.

解: 计算  $\frac{\partial z}{\partial x}(0, 0), \frac{\partial z}{\partial y}(0, 0), \frac{\partial^2 z}{\partial x^2}(0, 0), \frac{\partial^2 z}{\partial y^2}(0, 0), \frac{\partial^2 z}{\partial y \partial x}(0, 0)$

在  $\cos(x+y) + \frac{\partial z}{\partial y}e^z + \frac{\partial z}{\partial y}ze^z - e^x = 0$  两边再对  $x$  求偏导, 得

$$-\sin(x+y) + \frac{\partial^2 z}{\partial y \partial x}e^z + \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}e^z + \frac{\partial^2 z}{\partial y \partial x}ze^z + \frac{\partial z}{\partial y} \frac{\partial ze^z}{\partial x} - e^x = 0$$

$$\text{取 } x = y = z = 0 \quad \therefore \frac{\partial^2 z}{\partial y \partial x} = 1$$

在  $\cos(x+y) + \frac{\partial z}{\partial y}e^z + \frac{\partial z}{\partial y}ze^z - e^x = 0$  两边再对  $y$  求偏导, 得

$$-\sin(x+y) + \frac{\partial^2 z}{\partial y^2}e^z + \left(\frac{\partial z}{\partial y}\right)^2 e^z + \frac{\partial^2 z}{\partial y^2}ze^z + \frac{\partial z}{\partial y} \frac{\partial ze^z}{\partial y} = 0 \quad \therefore \frac{\partial^2 z}{\partial y^2} = 0$$

例.  $\sin(x+y) + ze^z - ye^x = 0$  确定了隐函数  $z = z(x, y)$ , 求  $z = z(x, y)$  在  $(0, 0)$  带Peano余项的2阶Taylor展开.

解: 计算  $\frac{\partial z}{\partial x}(0, 0), \frac{\partial z}{\partial y}(0, 0), \frac{\partial^2 z}{\partial x^2}(0, 0), \frac{\partial^2 z}{\partial y^2}(0, 0), \frac{\partial^2 z}{\partial y \partial x}(0, 0)$

在  $\cos(x+y) + \frac{\partial z}{\partial x}e^z + \frac{\partial z}{\partial x}ze^z - ye^x = 0$  两边再对  $x$  求偏导, 得

$$-\sin(x+y) + \frac{\partial^2 z}{\partial x^2}e^z + \left(\frac{\partial z}{\partial x}\right)^2 e^z + \frac{\partial^2 z}{\partial x^2}ze^z + \left(\frac{\partial z}{\partial x}\right)^2 e^z + \left(\frac{\partial z}{\partial x}\right)^2 ze^z - ye^x = 0$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = -2$$

$$\therefore z(x, y) = -x + \frac{1}{2!}(2xy - 2x^2) + o(x^2 + y^2) = -x + xy - x^2 + o(x^2 + y^2)$$



例. (2020真题)  $f$  二阶连续可微, 求证:  $\lim_{h \rightarrow 0+} \frac{f(2h, e^{-\frac{1}{2h}}) - 2f(h, e^{-\frac{1}{h}}) + f(0, 0)}{h^2} = f''_{xx}(0, 0)$

解:

$$\because f \text{ 二阶连续可微} \therefore f(x, y) = f(0, 0) + xf'_x(0, 0) + yf'_y(0, 0) + \frac{1}{2}x^2 f''_{xx}(0, 0) + \frac{1}{2}y^2 f''_{yy}(0, 0) \\ + xyf''_{xy}(0, 0) + o(x^2 + y^2)$$

$$f(2h, e^{-\frac{1}{2h}}) = f(0, 0) + 2hf'_x(0, 0) + e^{-\frac{1}{2h}} f'_y(0, 0) + 2h^2 f''_{xx}(0, 0) + \frac{1}{2}e^{-\frac{1}{h}} f''_{yy}(0, 0) + 2he^{-\frac{1}{2h}} f''_{xy}(0, 0) + o(h^2) \\ = f(0, 0) + 2hf'_x(0, 0) + 2h^2 f''_{xx}(0, 0) + o(h^2)$$

$$f(h, e^{-\frac{1}{h}}) = f(0, 0) + hf'_x(0, 0) + e^{-\frac{1}{h}} f'_y(0, 0) + \frac{1}{2}h^2 f''_{xx}(0, 0) + \frac{1}{2}e^{-\frac{2}{h}} f''_{yy}(0, 0) + he^{-\frac{1}{h}} f''_{xy}(0, 0) + o(h^2) \\ = f(0, 0) + hf'_x(0, 0) + \frac{1}{2}h^2 f''_{xx}(0, 0) + o(h^2)$$

$$\therefore f(2h, e^{-\frac{1}{2h}}) - 2f(h, e^{-\frac{1}{h}}) + f(0, 0) = h^2 f''_{xx}(0, 0) + o(h^2)$$



### 3 / 多元泰勒公式和极值原理

**Thm.**  $n$ 元函数 $f$ 在 $x_0$ 的某个邻域中可微,  $x_0$ 为 $f$ 的极值点, 则 $x_0$ 为 $f$ 的驻点, 即 $\text{grad}f(x_0) = 0$ .

**Thm.**  $n$ 元函数 $f$ 在 $x_0$ 的邻域中二阶连续可微,  
 $\text{grad}f(x_0) = 0$ ,

- (1) 若 $H_f(x_0)$ 正定, 则 $f(x_0)$ 严格极小.
- (2) 若 $H_f(x_0)$ 负定, 则 $f(x_0)$ 严格极大.
- (3) 若 $H_f(x_0)$ 不定, 则 $f(x_0)$ 不是极值.

### 3 / 多元泰勒公式和极值原理

**Thm.**  $n$ 元函数 $f$ 在 $\mathbf{x}_0$ 的邻域中二阶连续可微,  
 $\mathbf{x}_0$ 为极值点,则

(1)  $f(\mathbf{x}_0)$ 极小, 则 $H_f(\mathbf{x}_0)$ 半正定

(2)  $f(\mathbf{x}_0)$ 极大, 则 $H_f(\mathbf{x}_0)$ 半负定

例: 求 $z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ 的极值.

解:  $z'_x = 4x^3 - 4x + 4y$ ,  $z'_y = 4y^3 + 4x - 4y$ .

得驻点 $(\sqrt{2}, -\sqrt{2})$ ,  $(-\sqrt{2}, \sqrt{2})$ ,  $(0, 0)$ .

$$z''_{xx} = 12x^2 - 4, \quad z''_{xy} = 4, \quad z''_{yy} = 12y^2 - 4.$$

(1) 在 $(\sqrt{2}, -\sqrt{2})$ ,

$$A = C = 20, B = 4, AC - B^2 > 0,$$

取得极小值.

(2) 同理 $z(x, y)$ 在 $(-\sqrt{2}, \sqrt{2})$ 取得极小值.

(3)在(0,0),

$$A = C = -4, B = 4, AC - B^2 = 0,$$

判别法失效. 注意到

$$z(x, x) = 2x^4 > 0, \text{当} x \neq 0 \text{时.}$$

$$\begin{aligned} z(x, 0) &= x^4 - 2x^2 \\ &= x^2(x^2 - 2) < 0, \text{当} 0 < x^2 < 2 \text{时.} \end{aligned}$$

故(0,0)不是极值点.□

### 3 / 多元泰勒公式和极值原理

例:  $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$  确定隐函数  $z = z(x, y)$ . 求  $z(x, y)$  的极值.

解: 在  $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$  两边分别对  $x, y$  求偏导数.

$$4x + 2z \frac{\partial z}{\partial x} + 8z + 8x \frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 0(1)$$

$$4y + 2z \frac{\partial z}{\partial y} + 8x \frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 0(2)$$

先计算驻点, 即  $\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0 \quad \therefore 4x + 8z = 0, 4y = 0$

结合  $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0. \quad \therefore -7z^2 - z + 8 = 0, \therefore z = 1 \text{ 或 } -\frac{8}{7}$

$\therefore$  两个驻点为  $(-2, 0)$  和  $(16/7, 0)$



### 3 / 多元泰勒公式和极值原理

例:  $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$  确定隐函数  $z = z(x, y)$ . 求  $z(x, y)$  的极值.

$\therefore$  两个驻点为  $(-2, 0)$  和  $(16/7, 0)$  下面计算  $(-2, 0)$  和  $(16/7, 0)$  处的海塞矩阵

对  $4x + 2z \frac{\partial z}{\partial x} + 8z + 8x \frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 0$  两边对  $x, y$  求导:

$$4 + 2\left(\frac{\partial z}{\partial x}\right)^2 + 2z \frac{\partial^2 z}{\partial x^2} + 8 \frac{\partial z}{\partial x} + 8 \frac{\partial z}{\partial x} + 8x \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x^2} = 0$$

$$2 \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + 2z \frac{\partial^2 z}{\partial x \partial y} + 8 \frac{\partial z}{\partial y} + 8x \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\text{对 } 4y + 2z \frac{\partial z}{\partial y} + 8x \frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 0 \text{ 两边同时对 } y \text{ 求导 得 } 4 + 2\left(\frac{\partial z}{\partial y}\right)^2 + 2z \frac{\partial^2 z}{\partial y^2} + 8x \frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

### 3 / 多元泰勒公式和极值原理

例:  $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$  确定隐函数  $z = z(x, y)$ . 求  $z(x, y)$  的极值.

$\therefore$  两个驻点为  $(-2, 0)$  和  $(16/7, 0)$  下面计算  $(-2, 0)$  和  $(16/7, 0)$  处的海塞矩阵

$$\text{在 } (-2, 0) \text{ 处, } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0, z = 1$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{4}{15}, \frac{\partial^2 z}{\partial x \partial y} = 0, \frac{\partial^2 z}{\partial y^2} = \frac{4}{15}$$

极小

$$\text{在 } (16/7, 0) \text{ 处, } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0, z = 1$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{4}{15}, \frac{\partial^2 z}{\partial x \partial y} = 0, \frac{\partial^2 z}{\partial y^2} = -\frac{4}{15}$$

极大

例:  $f$  连续,  $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - xy}{(x^2 + y^2)^2} = 1$ .  $f(0,0)$  是否极值?

解:  $\lim_{(x,y) \rightarrow (0,0)} (f(x,y) - xy) = 0, f(0,0) = 0$ .

存在  $\varepsilon > 0$ , 当  $x^2 + y^2 < \varepsilon$  时,

$$\frac{3}{2}(x^2 + y^2)^2 > f(x,y) - xy > \frac{1}{2}(x^2 + y^2)^2.$$

于是对充分大的  $n$ ,  $f\left(\frac{1}{n}, \frac{1}{n}\right) > \frac{1}{n^2} + \frac{2}{n^4} > 0$ ,

$$f\left(\frac{1}{n}, -\frac{1}{n}\right) < -\frac{1}{n^2} + \frac{6}{n^4} = -\frac{1}{n^2} \left(1 - \frac{6}{n^2}\right) < 0.$$

故  $f(0,0)$  不是极值.  $\square$

### 3 / 多元泰勒公式和极值原理

**Note:** 无条件极值问题求解步骤:

- (1) 计算驻点, 即偏导数为0的点;
- (2) 计算驻点处的 *Hessen* 矩阵, 正定极小, 负定极大, 不定不是极值点
- (3) 如果(2)失效, 要考虑其他方法.

### 3 / 多元泰勒公式和极值原理

Ex. (巧妙运用极值原理 – P94T5)

(1)  $f(x, y)$  在  $x^2 + y^2 \leq 1$  上连续, 在  $x^2 + y^2 < 1$  内可导,

满足方程  $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = kf(x, y) (k > 0)$ , 若在  $x^2 + y^2 = 1$  上  $f(x, y) = 0$ ,

求证  $f$  在  $x^2 + y^2 \leq 1$  内部恒为 0.

证明 反证, 如果  $f$  在  $x^2 + y^2 \leq 1$  内部不恒为 0

即存在  $(x_0, y_0)$ , s.t.  $f(x_0, y_0) \neq 0$

1°  $f(x_0, y_0) > 0$ , 则  $f(x, y)$  在  $x^2 + y^2 \leq 1$  上有大于 0 的最大值

如果最大值在  $(x_1, y_1)$  处取, 有  $f(x_1, y_1) > 0$ , 并且  $x_1^2 + y_1^2 < 1$

$\therefore f(x_1, y_1)$  为极大值,  $\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$   $\because \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = kf(x, y), \therefore f(x_1, y_1) = 0$ , 矛盾!

2°  $f(x_0, y_0) < 0$ , 取  $-f$  带入上面证明即可.  $\square$



## • 条件极值与Lagrange乘子法

$$\max(\min) \ f(\mathbf{x}) = f(x_1, \cdots, x_n)$$

$$s.t. \quad g_i(\mathbf{x}) = g_i(x_1, \cdots, x_n) = 0, \quad i = 1, \cdots, m.$$

其中  $\text{rank} \frac{\partial(g_1, \cdots, g_m)}{\partial(x_1, \cdots, x_n)} = m$  (正则性条件).

结论:  $\mathbf{x}_0$  是条件极值问题的最大(小)值点, 则  $\exists \lambda_0, s.t. (\mathbf{x}_0, \lambda_0)$  是

$$L(\mathbf{x}, \lambda) = L(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m)$$

$$= f(x_1, \cdots, x_n) + \sum_{i=1}^m \lambda_i g_i(x_1, \cdots, x_n)$$

的驻点.

### 3 / 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球  $x^2 + y^2 + \frac{z^2}{4} = 1$  上找一点, 位于  $x > 0, y > 0, z > 0$ .

使得切平面与三个坐标轴的交点到原点距离的平方和最小

解. 设该点坐标为  $(a, b, c)$ , 法向量为  $(2a, 2b, \frac{c}{2})$

切平面为  $2a(x-a) + 2b(y-b) + \frac{c}{2}(z-c) = 0$  即  $ax + by + \frac{c}{4}z = 1$

解得三个交点坐标为  $(\frac{1}{a}, 0, 0), (0, \frac{1}{b}, 0), (0, 0, \frac{4}{c})$

求解如下条件极值问题

$$\begin{aligned} \min : & \frac{1}{a^2} + \frac{1}{b^2} + \frac{16}{c^2} & \frac{1}{a^2} + \frac{1}{b^2} + \frac{16}{c^2} &= \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{16}{c^2} \right) \left( a^2 + b^2 + \frac{c^2}{4} \right) \\ s.t. & a^2 + b^2 + \frac{c^2}{4} = 1; \quad a > 0, b > 0, c > 0 & & \geq \left( \frac{1}{a}a + \frac{1}{b}b + \frac{4}{c} \frac{c}{2} \right)^2 = 4^2 \end{aligned}$$

### 3 / 多元泰勒公式和极值原理

例. (2020春模拟) 在椭球  $x^2 + y^2 + \frac{z^2}{4} = 1$  上找一点, 位于  $x > 0, y > 0, z > 0$ .

使得切平面与三个坐标轴的交点到原点距离的平方和最小

$$L(a, b, c, \lambda) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{16}{c^2} + \lambda(a^2 + b^2 + \frac{c^2}{4} - 1)$$

$$L'_a(a, b, c, \lambda) = -\frac{2}{a^3} + 2\lambda a = 0$$

$$L'_b(a, b, c, \lambda) = -\frac{2}{b^3} + 2\lambda b = 0$$

$$L'_c(a, b, c, \lambda) = -\frac{32}{c^3} + \frac{\lambda c}{2} = 0$$

$$\therefore \lambda = \frac{1}{a^4} = \frac{1}{b^4} = \frac{64}{c^4} \quad \therefore a = b = \frac{c}{2\sqrt{2}}$$

$$\text{结合 } a^2 + b^2 + \frac{c^2}{4} = 1$$

$$\therefore a = 1/2, b = 1/2, c = \sqrt{2}$$

### 3 / 多元泰勒公式和极值原理

例. (2020春期末) 求函数  $u = \sin x \sin y \sin z$  在条件  $x + y + z = \frac{\pi}{2}, x > 0, y > 0, z > 0$

下的极值, 并说明是极大值还是极小值.

解: (化为无条件极值) 考虑  $v(x, y) = \sin x \sin y \sin(\frac{\pi}{2} - x - y) = \sin x \sin y \cos(x + y)$

在  $\{(x, y) : x > 0, y > 0, \frac{\pi}{2} - x - y > 0\}$  上的极值

$$\begin{aligned} \frac{\partial v(x, y)}{\partial x} &= \cos x \sin y \cos(x + y) - \sin x \sin y \sin(x + y) = \sin y (\cos x \cos(x + y) - \sin x \sin(x + y)) = \\ &\sin y \cos(2x + y) = 0 \quad \because y > 0, \therefore 2x + y = \frac{\pi}{2} \end{aligned}$$

$$\frac{\partial v(x, y)}{\partial y} = \sin x \cos(x + 2y) = 0 \Rightarrow x + 2y = \frac{\pi}{2} \quad \therefore x = y = z = \frac{\pi}{6} \text{ 是唯一驻点. } v(\frac{\pi}{6}, \frac{\pi}{6}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

进一步考虑  $D = \{(x, y) : x \geq 0, y \geq 0, \frac{\pi}{2} - x - y \geq 0\}$   $v$  在  $D$  上有最大值和最小值

$D$  的边界上,  $v(x, y) = 0$ . 可知上面所求为最大值.

### 3 / 多元泰勒公式和极值原理

例. (2020春期末) 求函数  $u = \sin x \sin y \sin z$  在条件  $x + y + z = \frac{\pi}{2}, x > 0, y > 0, z > 0$  下的极值, 并说明是极大值还是极小值.

解: (化为无条件极值) 考虑  $v(x, y) = \sin x \sin y \sin(\frac{\pi}{2} - x - y) = \sin x \sin y \cos(x + y)$

在  $\{(x, y) : x > 0, y > 0, \frac{\pi}{2} - x - y > 0\}$  上的极值

$\therefore x = y = z = \frac{\pi}{6}$  是唯一驻点.  $\because v'_x = \sin y \cos(2x + y), v'_y = \sin x \cos(x + 2y)$

$$\because \begin{cases} v''_{xx} = -2 \sin y \sin(2x + y) \\ v''_{xy} = \cos y \cos(x + 2y) - \sin y \sin(2x + y) = \cos(2x + 2y) \\ v''_{yy} = -2 \sin x \sin(x + 2y) \end{cases} \quad \because \begin{cases} v''_{xx} = -1 \\ v''_{xy} = \cos(\frac{2\pi}{3}) = -\frac{1}{2} \\ v''_{yy} = -1 \end{cases} \quad \text{海塞矩阵负定.}$$



## 4 / 含参数积分

- 含参数定积分:  $\int_{\alpha}^{\beta} g(t, x) dx$
- 含参数广义积分:  $\int_{\alpha}^{+\infty} g(t, x) dx$
- 无论是含参数定积分还是含参数的广义积分, 本质上都是关于参数  $t$  的函数
  - 对于一个函数来讲, 主要研究其连续性、可导性、可积性
  - 连续性:  $\lim_{t \rightarrow t_0} \int_{\alpha}^{\beta} g(t, x) dx = \int_{\alpha}^{\beta} \lim_{t \rightarrow t_0} g(t, x) dx = \int_{\alpha}^{\beta} g(t_0, x) dx$
  - 可导性:  $f'(t) = \frac{d}{dt} \int_{\alpha}^{\beta} g(t, x) dx = \int_{\alpha}^{\beta} g'_t(t, x) dx.$
  - 可积性:  $\int_a^b \left( \int_{\alpha}^{\beta} g(t, x) dx \right) dt = \int_{\alpha}^{\beta} \left( \int_a^b g(t, x) dt \right) dx$
- 对于含参数定积分, 一般只要求被积函数  $g(t, x)$  及  $g'_t$  的连续性即可.
- 对于含参数广义积分, 除了含参数定积分的条件外, 还需要更强的条件.

## 4 / 含参数积分

**Thm1.(连续性)** 设二元函数 $g(t, x)$ 在 $D = [a, b] \times [\alpha, \beta]$ 上连续, 则

$$f(t) = \int_{\alpha}^{\beta} g(t, x) dx \text{ 在 } [a, b] \text{ 上一致连续.}$$

$$\text{也即 } \lim_{t \rightarrow t_0} \int_{\alpha}^{\beta} g(t, x) dx = \int_{\alpha}^{\beta} \lim_{t \rightarrow t_0} g(t, x) dx.$$

**Thm2.(在积分号下求导)** 设 $D = [a, b] \times [\alpha, \beta]$ , 且 $g(t, x), g'_t(t, x) \in C(D)$ , 则

$$f(t) = \int_{\alpha}^{\beta} g(t, x) dx \text{ 在 } [a, b] \text{ 上连续可导, 且}$$

$$f'(t) = \frac{d}{dt} \int_{\alpha}^{\beta} g(t, x) dx = \int_{\alpha}^{\beta} g'_t(t, x) dx.$$

例. 求 $a, b, s.t. \int_1^3 (ax + b - x^2)^2 dx$ 取最小值

解.  $I(a, b) = \int_1^3 (ax + b - x^2)^2 dx$

$$\frac{\partial I(a, b)}{\partial a} = \int_1^3 2(ax + b - x^2)x dx = 2 \int_1^3 ax^2 + bx - x^3 dx = 2\left(\frac{26}{3}a + 4b - 20\right) = 0$$

$$\frac{\partial I(a, b)}{\partial b} = \int_1^3 2(ax + b - x^2) dx = 2 \int_1^3 ax + b - x^2 dx = 2\left(4a + 2b - \frac{26}{3}\right) = 0$$

$$\text{解方程} \begin{cases} \frac{26}{3}a + 4b = 20 \\ 4a + 2b = \frac{26}{3} \end{cases} \Rightarrow \begin{cases} a = 4 \\ b = -\frac{11}{3} \end{cases}$$

例. 计算  $\int_0^{\pi/2} \frac{\arctan(a \tan x)}{\tan x} dx$

解. 记  $I(a) = \int_0^{\pi/2} \frac{\arctan(a \tan x)}{\tan x} dx$

$$\begin{aligned} I'(a) &= \int_0^{\pi/2} \left( \frac{\arctan(a \tan x)}{\tan x} \right)'_a dx = \int_0^{\pi/2} \frac{1}{1+a^2 \tan^2 x} dx \stackrel{y=\tan x}{=} \int_0^{+\infty} \frac{1}{(1+y^2)(1+a^2 y^2)} dy \\ &= \frac{1}{1-a^2} \left( \int_0^{+\infty} \frac{1}{(1+y^2)} dy - \int_0^{+\infty} \frac{a^2}{(1+a^2 y^2)} dy \right) = \frac{1}{1-a^2} \left( \frac{\pi}{2} - a \int_0^{+\infty} \frac{1}{(1+a^2 y^2)} d(ay) \right) \\ &= \frac{1}{1-a^2} \left( \frac{\pi}{2} - \frac{\pi}{2} a \operatorname{sgn}(a) \right) \end{aligned}$$

$$a > 0 \text{ 时: } = \frac{1}{1-a^2} \left( \frac{\pi}{2} - \frac{\pi}{2} a \right) = \frac{\pi}{2} \frac{1}{1+a}$$

$$I(a) = \frac{\pi}{2} \ln(1+a) + C, \quad I(0) = C = 0$$

$$\therefore I(a) = \frac{\pi}{2} \ln(1+a) \quad a < 0 \text{ 时: 奇函数!}$$

课后习题. 计算(1)  $\int_0^{\pi/2} \ln \frac{1+a \cos x}{1-a \cos x} \frac{dx}{\cos x}$  ( $|a| < 1$ );

(2)  $\int_0^{\pi/2} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx$ ;



答案. (1)  $\arcsin a$   
(2)  $\pi \ln \frac{a+b}{2}$



## 4 / 含参数积分

**Thm3.** 设  $g(t, x), g'_t(t, x) \in C([a, b] \times [c, d]), \alpha(t), \beta(t)$  在  $[a, b]$  上可导, 且

$$c \leq \alpha(t), \beta(t) \leq d, \quad \forall t \in [a, b],$$

则

$$f(t) = \int_{\alpha(t)}^{\beta(t)} g(t, x) dx$$

在区间  $[a, b]$  上可导, 且

$$\begin{aligned} f'(t) &= \frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} g(t, x) dx \\ &= \int_{\alpha(t)}^{\beta(t)} g'_t(t, x) dx + g(t, \beta(t))\beta'(t) - g(t, \alpha(t))\alpha'(t). \end{aligned}$$

例.  $f(x) = \int_x^{x^2} e^{-x^2 u^2} du, f'(x) = \underline{\hspace{2cm}}$

解. 
$$f'(x) = 2xe^{-x^6} - e^{-x^4} + \int_x^{x^2} e^{-x^2 u^2} \frac{d(-x^2 u^2)}{dx} du$$

$$= 2xe^{-x^6} - e^{-x^4} - 2 \int_x^{x^2} e^{-x^2 u^2} xu^2 du$$

要点. 上限替代被积变量\*上限的导数-下限替代被积变量\*下限的导数  
+积分号下求导的部分

例.  $f(y) = \int_y^{y^2} \frac{\sin(xy)}{x} dx, f'(y) = \frac{\frac{3 \sin(y^3)}{y}}{y} - \frac{\frac{2 \sin(y^2)}{y}}{y}$

解. 
$$f(y) = \int_y^{y^2} \frac{\sin(xy)}{x} dx, f'(y) = \frac{\sin(y^3)}{y^2} 2y - \frac{\sin(y^2)}{y} + \int_y^{y^2} \cos(xy) dx$$

$$= \frac{2 \sin(y^3)}{y} - \frac{\sin(y^2)}{y} + \frac{\sin(y^3) - \sin(y^2)}{y}$$

## 4. 含参积分的可积性

Thm4. (累次积分交换次序的充分条件)

设 $g(t, x)$ 在 $(t, x) \in D = [a, b] \times [\alpha, \beta]$ 上连续, 则 $\int_{\alpha}^{\beta} g(t, x) dx$ 在 $t \in [a, b]$ 上可积,  $\int_a^b g(t, x) dt$ 在 $x \in [\alpha, \beta]$ 上可积, 且

$$\int_a^b \left( \int_{\alpha}^{\beta} g(t, x) dx \right) dt = \int_{\alpha}^{\beta} \left( \int_a^b g(t, x) dt \right) dx,$$

简记为  $\int_a^b dt \int_{\alpha}^{\beta} g(t, x) dx = \int_{\alpha}^{\beta} dx \int_a^b g(t, x) dt.$

Proof. 由 $g(t, x)$ 的连续性 & Thm1,  $\int_{\alpha}^{\beta} g(t, x) dx$  在  $t \in [a, b]$  上连续, 从而可积. 同理,  $\int_a^b g(t, x) dt$  在  $x \in [\alpha, \beta]$  上可积.

例. 计算  $\int_0^1 \frac{x^b - x^a}{\ln x} \sin(\ln \frac{1}{x}) dx (a, b > 0)$

解.  $\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy$

要点. 交换积分次序, 会让难算的积分变得好算.  
如果给你的是定积分, 需要先变出两重积分号!

$$\begin{aligned} \text{原式} &= \int_0^1 \int_a^b x^y \sin(\ln \frac{1}{x}) dy dx (a, b > 0) = \int_a^b (\int_0^1 x^y \sin(\ln \frac{1}{x}) dx) dy \\ &= \int_a^b \frac{1}{(y+1)^2 + 1} dy = \arctan(b+1) - \arctan(a+1) \end{aligned}$$

$x^y \sin(\ln \frac{1}{x})$  在  $\{(x, y) : a \leq y \leq b, 0 \leq x \leq 1\}$  上连续

注:  $\lim_{(x,y) \rightarrow (0,y_0)} x^y \sin(\ln \frac{1}{x}) = 0 \quad \because |x^y \sin(\ln \frac{1}{x})| \leq x^y, 0 < a \leq y_0 \leq b$

$$\int_0^1 x^y \sin(\ln \frac{1}{x}) dx \stackrel{\ln \frac{1}{x} = t, x = e^{-t}}{=} \int_{+\infty}^0 e^{-ty} \sin(t) d(e^{-t}) = \int_0^{+\infty} e^{-t(y+1)} \sin(t) dt = \frac{1}{(y+1)^2 + 1}$$



## 4 / 含参数广义积分

**Note.** 对于含参数广义积分而言, 需要更强的条件以满足以上定理

**Def.**  $\forall t \in \Omega \subset \mathbb{R}, \int_a^{+\infty} f(t, x) dx$  收敛. 若  $\forall \varepsilon > 0, \exists M(\varepsilon), s.t.$

$$\left| \int_a^A f(t, x) dx - \int_a^{+\infty} f(t, x) dx \right| < \varepsilon, \quad \forall A > M, \forall t \in \Omega,$$

则称含参广义积分  $\int_a^{+\infty} f(t, x) dx$  关于  $t \in \Omega$  一致收敛.

一致性体现在, 一旦  $\varepsilon$  被指定,

则  $\forall t \in \Omega, \exists$  同一个  $M, s.t. \left| \int_a^A f(t, x) dx - \int_a^{+\infty} f(t, x) dx \right| < \varepsilon, \quad \forall A > M$



## 4 / 含参数广义积分

Thm.(Weistrass判别法)  $\forall t \in \Omega \subset \mathbb{R}, \int_a^{+\infty} f(t, x)dx$  收敛,

若存在  $[a, +\infty)$  上的广义可积函数  $g(x), s.t.$

$$|f(t, x)| \leq g(x), \quad \forall (t, x) \in \Omega \times [a, +\infty),$$

则  $\int_a^{+\infty} f(t, x)dx$  在  $t \in \Omega$  上一致收敛.

问: 如何证明不一致收敛?

Thm.(Cauchy收敛原理)

$$\int_a^{+\infty} f(t, x)dx \text{ 关于 } t \in \Omega \text{ 一致收敛} \Leftrightarrow \forall \varepsilon > 0, \exists M(\varepsilon), s.t.$$

$$\left| \int_A^{A'} f(t, x)dx \right| < \varepsilon, \quad \forall A, A' > M, \forall t \in \Omega.$$

## 4 / 含参数广义积分

问:如何证明不一致收敛?

Remark.(Cauchy收敛原理逆否)

$\int_a^{+\infty} f(t, x)dx$  关于  $t \in \Omega$  不一致收敛  $\Leftrightarrow \exists \varepsilon_0 > 0, \forall M, s.t.$

$$\left| \int_A^{A'} f(t_0, x)dx \right| > \varepsilon_0, \quad \exists A, A' > M, \exists t_0 \in \Omega.$$

例. (1) 设  $c > 0$ ,  $\int_0^{+\infty} e^{-xy} dx$  在  $y \in [c, +\infty)$  上是否一致收敛?

(2)  $\int_0^{+\infty} e^{-xy} dx$  在  $y \in (0, +\infty)$  上是否一致收敛?

解: (1)  $c > 0$ , 则  $\int_0^{+\infty} e^{-cx} dx = -\frac{1}{c} e^{-cx} \Big|_{x=0}^{+\infty} = \frac{1}{c}$  收敛, 且

$$e^{-xy} \leq e^{-cx}, \quad \forall (x, y) \in [0, +\infty) \times [c, +\infty).$$

故  $\int_0^{+\infty} e^{-xy} dx$  在  $y \in [c, +\infty)$  上一致收敛(Weirstrass).

(2)  $\exists \varepsilon_0 = e^{-1} - e^{-2}, \forall M > 0, \exists A = M + 1, A' = 2A, y_0 = \frac{1}{A}, s.t.$

$$\left| \int_A^{A'} e^{-xy_0} dx \right| = -\frac{1}{y_0} e^{-xy_0} \Big|_{x=A}^{A'} = \frac{1}{y_0} (e^{-Ay_0} - e^{-A'y_0}) = A\varepsilon_0 > \varepsilon_0,$$

故  $\int_0^{+\infty} e^{-xy} dx$  在  $y \in [0, +\infty)$  上不一致收敛(Cauchy).  $\square$

## 4 / 含参数广义积分

**Thm1.**  $f(t, x) \in C([\alpha, \beta] \times [a, +\infty))$ ,  $I(t) = \int_a^{+\infty} f(t, x) dx$  关于  $t \in [\alpha, \beta]$  一致收敛, 则  $I(t) \in C[\alpha, \beta]$ .

**Thm1(逆否).**  $f(t, x) \in C([\alpha, \beta] \times [a, +\infty))$ ,  $I(t) \notin C[\alpha, \beta]$ . 则  $I(t) = \int_a^{+\infty} f(t, x) dx$  关于  $t \in [\alpha, \beta]$  **不**一致收敛, 则

例. 证明  $\int_0^{+\infty} \frac{\sin tx}{x} dx$  在  $t \in [0, +\infty)$  上不一致收敛.

解: 
$$I(t) = \int_0^{+\infty} \frac{\sin tx}{x} dx = \begin{cases} \int_0^{+\infty} \frac{\sin x}{x} dx, & t > 0 \\ 0, & t = 0 \end{cases}.$$

若  $\int_0^{+\infty} \frac{\sin tx}{x} dx$  在  $t \in [0, +\infty)$  上一致收敛, 则  $I(t) \in C[0, +\infty)$ , 矛盾.  $\square$

Remark. 证明含参积分不一致收敛的方法:  
定义、Cauchy准则、含参积分不连续.



Thm2. 设(1)  $f(t, x), f'_t(t, x) \in C([\alpha, \beta] \times [a, +\infty))$ ;  
(2)  $\forall t \in [\alpha, \beta], I(t) = \int_a^{+\infty} f(t, x) dx$  收敛;  
(3)  $\int_a^{+\infty} f'_t(t, x) dx$  关于  $t \in [\alpha, \beta]$  一致收敛;

则  $I(t) \in C^1[\alpha, \beta]$ , 且

$$I'(t) = \frac{d}{dt} \int_a^{+\infty} f(t, x) dx = \int_a^{+\infty} f'_t(t, x) dx.$$

注意. 是  $\int_a^{+\infty} f'_t(t, x) dx$  一致收敛! 不是  $\int_a^{+\infty} f(t, x) dx$  一致收敛

例 (2020样题):  $I = \int_0^{+\infty} \frac{\arctan bx - \arctan ax}{x} dx \quad (b > a > 0).$

解: 引入参数  $t \in [a, b]$ , 令  $I(t) = \int_0^{+\infty} \frac{\arctan tx - \arctan ax}{x} dx.$

$$\because \left| \frac{1}{1+t^2x^2} \right| \leq \frac{1}{1+a^2x^2}, \forall t \in [a, b]$$

$$\therefore I'(t) = \int_0^{+\infty} f'_t(t, x) dx = \int_0^{+\infty} \frac{1}{1+t^2x^2} dx = \frac{\pi}{2t}$$

$$\therefore I(a) = 0, I(b) = I(a) + \int_a^b I'(t) dt = \int_a^b I'(t) dt = \frac{\pi}{2} \ln(b/a)$$

例.(2019) 计算  $\int_0^{+\infty} \frac{1-e^{-ax}}{xe^x} dx, a \geq 0$

解: 令  $I(a) = \int_0^{+\infty} \frac{1-e^{-ax}}{xe^x} dx$ , 考虑积分号下求导.

$$\left( \frac{1-e^{-ax}}{xe^x} \right)'_a = \frac{xe^{-ax}}{xe^x} = e^{-(a+1)x} \quad \int_0^{\infty} e^{-(a+1)x} dx \text{ 对 } a \geq 1 \text{ 一致收敛 (Weierstrass)}$$

$$I'(a) = \int_0^{\infty} e^{-(a+1)x} dx = \frac{1}{1+a}, \therefore I(a) = \ln(1+a) + C, \quad I(0) = 0 \Rightarrow C = 0,$$

$$\therefore I(a) = \ln(1+a)$$

例. 计算积分  $\int_0^{+\infty} e^{-ax^2} \cos bxdx, a > 0$

思想2: 通过积分号下求导,  
虽然仍旧不好算, 但是构造了 *ODE*

解. 视  $b$  为参数, 定义  $I(b) = \int_0^{+\infty} e^{-ax^2} \cos bxdx,$

$\because |xe^{-ax^2} \sin bx| \leq xe^{-ax^2}, \forall a > 0, \int_0^{+\infty} xe^{-ax^2} dx$  存在  $\therefore I'(b) = -\int_0^{+\infty} xe^{-ax^2} \sin bxdx$

$$\therefore I'(b) = -\int_0^{+\infty} xe^{-ax^2} \sin bxdx = -\frac{1}{2} \int_0^{+\infty} e^{-ax^2} \sin bxdx^2 = \frac{1}{2a} \int_0^{+\infty} \sin bxd(e^{-ax^2})$$

$$= \frac{1}{2a} (\sin bxe^{-ax^2} \Big|_0^{+\infty} - b \int_0^{+\infty} \cos bxe^{-ax^2} dx) = -\frac{b}{2a} \int_0^{+\infty} \cos bxe^{-ax^2} dx = -\frac{b}{2a} I(b)$$

$$\text{即 } I'(b) = -\frac{b}{2a} I(b) \text{ 结合初值 } I(0) = \frac{1}{2} \sqrt{\pi/a}, \text{ 解出 } I(b) = \frac{1}{2} \sqrt{\pi/a} e^{-b^2/4a}$$

Ex.  $I = \int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}} dx$ . (瑕积分)

引入参变量  $t$

解: 令  $I(t) = \int_0^1 \frac{\arctan(tx)}{x\sqrt{1-x^2}} dx$ , 则  $I(0) = 0$ , 求  $I(1)$ .

$$I'(t) \xrightarrow{\text{Abel}} \int_0^1 \frac{1}{(1+t^2x^2)\sqrt{1-x^2}} dx \xrightarrow{x=\sin\theta} \int_0^{\pi/2} \frac{d\theta}{1+t^2\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{\csc^2\theta d\theta}{\csc^2\theta + t^2} = \int_0^{\pi/2} \frac{-d\cot\theta}{1+t^2+\cot^2\theta}$$

思想3: 引入参变量, 化定积分为含参积分

$$= \frac{-1}{\sqrt{1+t^2}} \arctan \frac{\cot\theta}{\sqrt{1+t^2}} \Big|_{\theta=0}^{\pi/2} = \frac{\pi}{2\sqrt{1+t^2}}.$$

$$I(1) = \int_0^1 \frac{\pi}{2\sqrt{1+t^2}} dt = \frac{\pi}{2} \ln(t + \sqrt{1+t^2}) \Big|_{t=0}^1 = \frac{\pi}{2} \ln(1 + \sqrt{2}). \square$$



例. 计算积分  $\int_0^{+\infty} \frac{\arctan ax \arctan bx}{x^2} dx, a > 0, b > 0$

解.  $I(a, b) = \int_0^{+\infty} \frac{\arctan ax \arctan bx}{x^2} dx$

$$I'_a(a, b) = \int_0^{+\infty} \frac{x \arctan bx}{(1 + a^2 x^2) x^2} dx = \int_0^{+\infty} \frac{\arctan bx}{(1 + a^2 x^2) x} dx$$

$$I''_{ab}(a, b) = \int_0^{+\infty} \frac{1}{(1 + a^2 x^2)(1 + b^2 x^2)} dx = \frac{1}{b^2 - a^2} \left( \int_0^{+\infty} \frac{b^2}{(1 + b^2 x^2)} dx - \int_0^{+\infty} \frac{a^2}{(1 + a^2 x^2)} dx \right)$$

$$= \frac{1}{b^2 - a^2} \left( b \int_0^{+\infty} \frac{1}{(1 + b^2 x^2)} d(bx) - a \int_0^{+\infty} \frac{1}{(1 + a^2 x^2)} d(ax) \right) = \frac{1}{b + a} \frac{\pi}{2}$$

例. 计算积分  $\int_0^{+\infty} \frac{\arctan ax \arctan bx}{x^2} dx, a > 0, b > 0$

$$I'_a(a, b) = \frac{\pi}{2} \ln(a+b) + C(a) \quad 0 = I'_a(a, 0) = \frac{\pi}{2} \ln(a) + C(a) \Rightarrow C(a) = -\frac{\pi}{2} \ln(a)$$

$$\therefore I'_a(a, b) = \frac{\pi}{2} (\ln(a+b) - \ln(a)) \quad \therefore 0 = I(0, b) = \frac{\pi}{2} (b \ln(b) - b) + C(b)$$

$$\therefore I(a, b) = \frac{\pi}{2} ((a+b) \ln(a+b) - (a+b) - a \ln(a) + a) + C(b)$$

$$\therefore 0 = I(0, b) = \frac{\pi}{2} (b \ln b - b) + C(b) \quad \therefore C(b) = \frac{\pi}{2} (b - b \ln b)$$

$$\therefore I(a, b) = \frac{\pi}{2} ((a+b) \ln(a+b) - (a+b) - a \ln(a) + a) + \frac{\pi}{2} (b - b \ln b)$$

$$\therefore I(a, b) = \frac{\pi}{2} ((a+b) \ln(a+b) - a \ln(a) - b \ln b)$$

**Thm3.**  $f(x, y) \in C([a, +\infty) \times [\alpha, \beta])$ ,  $I(y) = \int_a^{+\infty} f(x, y) dx$

关于  $y \in [\alpha, \beta]$  一致收敛, 则  $I(y)$  在  $[\alpha, \beta]$  上可积, 且

$$\int_{\alpha}^{\beta} I(y) dy = \int_a^{+\infty} \left( \int_{\alpha}^{\beta} f(x, y) dy \right) dx,$$

即 
$$\int_{\alpha}^{\beta} \left( \int_a^{+\infty} f(x, y) dx \right) dy = \int_a^{+\infty} \left( \int_{\alpha}^{\beta} f(x, y) dy \right) dx,$$

也记为 
$$\int_{\alpha}^{\beta} dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} dx \int_{\alpha}^{\beta} f(x, y) dy.$$

**Thm4.**  $f(x, y) \in C([a, +\infty) \times [\alpha, +\infty])$ , 且满足

(1)  $\forall \beta > \alpha, \int_a^{+\infty} f(x, y)dx$  在  $y \in [\alpha, \beta]$  上一致收敛;

$\forall b > a, \int_\alpha^{+\infty} f(x, y)dy$  在  $x \in [a, b]$  上一致收敛;

(2)  $\int_\alpha^{+\infty} dy \int_a^{+\infty} |f(x, y)|dx$  与  $\int_a^{+\infty} dx \int_\alpha^{+\infty} |f(x, y)|dy$  中至少有一个存在;

则  $I(y) = \int_a^{+\infty} f(x, y)dx$  在  $[\alpha, +\infty]$  上可积, 且

$$\int_\alpha^{+\infty} dy \int_a^{+\infty} f(x, y)dx = \int_a^{+\infty} dx \int_\alpha^{+\infty} f(x, y)dy.$$

例  $I = \int_0^{+\infty} \frac{\arctan bx - \arctan ax}{x} dx \quad (b > a > 0).$

解:  $I = \int_0^{+\infty} \frac{\arctan bx - \arctan ax}{x} dx.$

$$\because \int_a^b \frac{1}{1+t^2x^2} dt = \frac{\arctan bx - \arctan ax}{x}$$

$$\therefore I = \int_0^{+\infty} \frac{\arctan bx - \arctan ax}{x} dx = \int_0^{+\infty} \int_a^b \frac{1}{1+t^2x^2} dt dx$$

$\int_0^{+\infty} \frac{1}{1+t^2x^2} dx$  对  $t \in [a, b]$  一致收敛 (Weierstrass)

$$\therefore I = \int_0^{+\infty} \int_a^b \frac{1}{1+t^2x^2} dt dx = \int_a^b dt \int_0^{+\infty} \frac{1}{1+t^2x^2} dx = \int_a^b \frac{1}{t} \frac{\pi}{2} dt = \frac{\pi}{2} \ln(b/a)$$



课后作业  $I = \int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} \cos cxdx \quad (b > a > 0).$

答案:  $I = \frac{1}{2} \ln \frac{b^2 + c^2}{a^2 + c^2}$

# 4 / 含参数积分

- 含参数定积分:  $\int_{\alpha}^{\beta} g(t, x) dx$
- 含参数广义积分:  $\int_{\alpha}^{+\infty} g(t, x) dx$
- 无论是含参数定积分还是含参数的广义积分, 本质上都是关于参数  $t$  的函数
- 两大工具:
  - 积分号下求导: 导数好积分;
  - 导数和原函数的关系: 计算定积分用含参积分
  - 交换积分次序: 换完之后好积分;
  - 给你一个定积分, 要知道把它转化为含参积分
- 被积函数的连续性和可导性+一致收敛性是这两个工具能够使用的条件

例.  $\alpha, \beta > 0$ , 计算Laplace积分

$$I(\beta) = \int_0^{+\infty} \frac{\cos \beta x}{\alpha^2 + x^2} dx, J(\beta) = \int_0^{+\infty} \frac{x \sin \beta x}{\alpha^2 + x^2} dx.$$

解:  $\int_0^{+\infty} \frac{x \sin \beta x}{\alpha^2 + x^2} dx$  关于  $\beta \geq b > 0$  一致收敛 (Dirichlet).

故  $I'(\beta) = -J(\beta)$ . (再在积分下求导是不允许的?)

$$\text{已知 } \beta > 0 \text{ 时, } \int_0^{+\infty} \frac{\sin \beta x}{x} dx = \int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

$$\text{两式相加, 得 } I'(\beta) + \frac{\pi}{2} = \alpha^2 \int_0^{+\infty} \frac{\sin \beta x}{x(\alpha^2 + x^2)} dx.$$

求导得  $I''(\beta) = \alpha^2 I(\beta).$

此微分方程通解为  $I = c_1 e^{-\alpha\beta} + c_2 e^{\alpha\beta}.$

因为  $|I| \leq \int_0^{+\infty} \frac{dx}{\alpha^2 + x^2} = \frac{\pi}{2\alpha}, \lim_{\alpha \rightarrow +\infty} I = 0,$

所以  $c_2 = 0, I = c_1 e^{-\alpha\beta}.$

又  $\lim_{\beta \rightarrow 0+} I = \lim_{\beta \rightarrow 0+} \int_0^{+\infty} \frac{\cos \beta x}{\alpha^2 + x^2} dx = \int_0^{+\infty} \frac{1}{\alpha^2 + x^2} dx = \frac{\pi}{2\alpha}.$

所以  $c_1 = \frac{\pi}{2\alpha}, I(\beta) = \frac{\pi}{2\alpha} e^{-\alpha\beta},$

$J(\beta) = -I'(\beta) = -\frac{\pi}{2} e^{-\alpha\beta}. \square$