1. n个节点的树有n-1条边

总度数为2(n-1)

$\therefore \sum_{i=1} n_i \cdot i = 2(n-1)$ $\quad\therefore n_1 = 2(n-1) - 2n_2 - \cdots - kn_k$

$= 2(n_1+n_2+\cdots+n_k-1) - 2n_2 \cdots - kn_k = 2n_1 - 2 + \sum_{i=2}^{k}(2-i)n_i$

$\therefore n_1 = \sum_{i=2}^{k}(i-2)n_i + 2$

8. 完全二分图：$V=V_1\cup V_2$，$V_1\cap V_2=\phi$ V₁中任一点与V₂中所有点相连

不妨令所有边由V_1指向V_2. $|V_1|=m$；$|V_2|=n$.

$$B=\begin{vmatrix} 1\ 1 \cdots 1 & & & \\ & 1\ 1\cdots 1 & & \\ & & \cdots & \\ & & & 1\ 1\cdots 1 \\ \hline -I_n & -I_n & -I_n & -I_n \end{vmatrix}_{(m+n)\times mn}$$

删去第一行：$B_k=$

$$\begin{array}{c} m-1\{ \\ n\{ \end{array}\begin{vmatrix} & 1\ 1\cdots 1 & & \\ & & \cdots & \\ & & & 1\ 1\cdots 1 \\ \hline -I_n & -I_n & -I_n & -I_n \end{vmatrix}_{(m+n-1)\times mn}$$

$$\det(B_k\cdot B_k^T)=\begin{vmatrix} n & & & 1 & \cdots & 1 \\ & \ddots & & \vdots & & \vdots \\ & & n & 1 & \cdots & 1 \\ 1 & \cdots & 1 & m & & \\ \vdots & & \vdots & & \ddots & \\ 1 & \cdots & 1 & & & m \end{vmatrix}_{(m+n-1)\times(m+n-1)} \begin{array}{l} \}m-1 \\ \}n \end{array} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

A可逆 故

$\det(B_k\cdot B_k^T)=|A|\cdot|D-CA^{-1}B|$

$|A|=n^{m-1}$

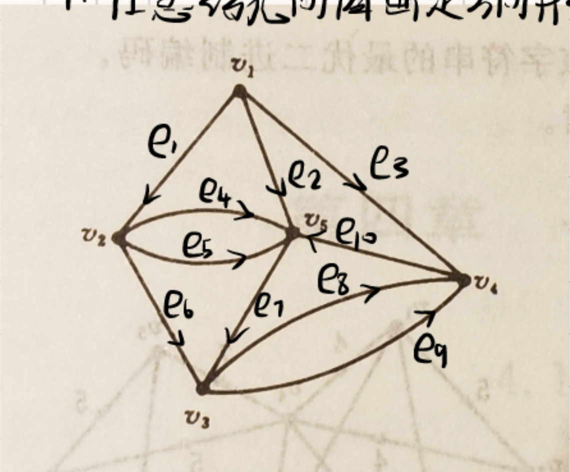
$$|D-CA^{-1}B| = |mI - C_{n\times(m-1)}\tfrac{1}{n}I\cdot B_{(m-1)\times n}| = \begin{vmatrix} \frac{mn-m+1}{n} & \frac{1-m}{n} & \cdots & \frac{1-m}{n} \\ \frac{1-m}{n} & \ddots & & \vdots \\ \vdots & & & \frac{1-m}{n} \\ \frac{1-m}{n} & \cdots & \frac{1-m}{n} & \frac{mn-m+1}{n} \end{vmatrix}$$

$$= \begin{vmatrix} m & 0 & 0 & \cdots & -m \\ 0 & m & 0 & \cdots & -m \\ 0 & 0 & m & & \vdots \\ 0 & 0 & & \ddots & \\ \frac{1-m}{n} & \frac{1-m}{n} & \cdots & \frac{1-m}{n} & \frac{mn-m+1}{n} \end{vmatrix} = \begin{vmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & -m \\ \vdots & & \ddots & -m \\ \frac{1-m}{n} & \frac{1-m}{n} & \cdots & \frac{mn-2m+2}{n} \end{vmatrix} = m\cdot\begin{vmatrix} m & 0 & -m \\ & \ddots & -m \\ \frac{1-m}{n} & \cdots & \frac{mn-2m+2}{n} \end{vmatrix} = m^2\cdot\begin{vmatrix} m & 0 & -m \\ & \ddots & -m \\ \frac{1-m}{n} & \cdots & \frac{mn-3m+3}{n} \end{vmatrix}$$

$$= m^3\cdot\begin{vmatrix} m & 0 & -m \\ & \ddots & -m \\ \frac{1-m}{n} & \cdots & \frac{mn-4m+4}{n} \end{vmatrix} = \cdots = m^{n-2}\begin{vmatrix} m & -m \\ \frac{1-m}{n} & \frac{mn-(n-1)m+(n-1)}{n} \end{vmatrix} = m^{n-2}\cdot\begin{vmatrix} m & 0 \\ \frac{1-m}{n} & \frac{n}{n} \end{vmatrix} = m^{n-1}$$

$$\therefore \det(B_k\cdot B_k^T)=\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)=|A|\cdot|D-CA^{-1}B| = n^{m-1}\cdot m^{n-1}$$

4. 任意给无向图画定方向并编号，关联矩阵为:



题图 3.4

$$
\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}
\end{array}
$$

删去 V_5: $B_K=$

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix}
$$

树的数目 $=\det(B_K B_K^T)$

$$
=\begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{vmatrix}
=\begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & -4 & -1 & 4 \end{vmatrix}
$$

$$
=\begin{vmatrix} 0 & 11 & -3 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ 0 & -4 & -1 & 4 \end{vmatrix}
=(-1)\cdot(-1)^{2+1}\cdot\begin{vmatrix} 11 & -3 & -1 \\ -1 & 4 & -2 \\ -4 & -1 & 4 \end{vmatrix}
=11\cdot4\cdot4+(-3)\cdot(-2)\cdot(-4)+(-1)(-1)(-1)-4^2-22-12
$$

$$
=176-24-1-16-34=101
$$

删去 e_2: (V_1,V_5):

$$
\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1
\end{bmatrix}
\end{array}
$$

$B_K=$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix}
$$

不含 e_2 的树: $\det(B_K\cdot B_K^T)=\begin{vmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{vmatrix}
=\begin{vmatrix} 2 & -1 & 0 & -1 \\ 0 & 4 & 1 & -4 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{vmatrix}
=\begin{vmatrix} 0 & -1 & -4 & 7 \\ 0 & 4 & 1 & -4 \\ 0 & -1 & 4 & -2 \\ -1 & 0 & -2 & 4 \end{vmatrix}
=(-1)^{4+1}\cdot(-1)\cdot\begin{vmatrix} -1 & -4 & 7 \\ 4 & 1 & -4 \\ -1 & 4 & -2 \end{vmatrix}$

$$
\begin{vmatrix} -1 & -4 & 7 \\ 4 & 1 & -4 \\ -1 & 4 & -2 \end{vmatrix} = 2+(-16)+7\times16+7-16-32=57
$$

$\therefore$ 必含 e_2 的树 $=101-57=44$

删去 e_{10}: (V_4,V_5) 之后.

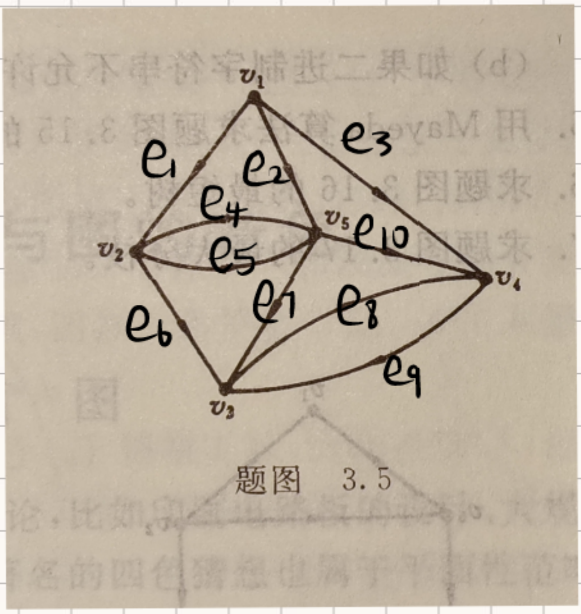
$$
\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
B_K=\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1
\end{bmatrix}
$$

不含e_{10}的树 $=\det|B_k\cdot B_k^T| = 60$

$$\begin{vmatrix}3&-1&0&-1\\-1&4&-1&0\\0&-1&4&-2\\-1&0&-2&3\end{vmatrix}=\begin{vmatrix}3&-1&0&-1\\-1&4&-1&0\\0&-1&4&-2\\0&-4&-1&3\end{vmatrix}=\begin{vmatrix}0&11&-3&-1\\-1&4&-1&0\\0&-1&4&-2\\0&-4&-1&3\end{vmatrix}=(-1)^{2+1}\cdot(-1)\cdot\begin{vmatrix}11&-3&-1\\-1&4&-2\\-4&-1&3\end{vmatrix}=132-24-1-16-22-9=60$$

5.



题图 3.5

关联矩阵如下:

$$\begin{array}{c}\\ V_1\\V_2\\V_3\\V_4\\V_5\end{array}\begin{array}{c}\begin{array}{cccccccccc}e_1&e_2&e_3&e_4&e_5&e_6&e_7&e_8&e_9&e_{10}\end{array}\\ \begin{bmatrix}1&1&1&0&0&0&0&0&0&0\\-1&0&0&-1&1&1&0&0&0&0\\0&0&0&0&0&-1&-1&1&-1&0\\0&0&-1&0&0&0&0&-1&1&-1\\0&-1&0&1&-1&0&1&0&0&1\end{bmatrix}\end{array}$$

删去V_1后:

$$\begin{bmatrix}-1&0&0&-1&1&1&0&0&0&0\\0&0&0&0&0&-1&-1&1&-1&0\\0&0&-1&0&0&0&0&-1&1&-1\\0&-1&0&1&-1&0&1&0&0&1\end{bmatrix}=B_k$$

$$\overrightarrow{B_k}=\begin{bmatrix}-1&0&0&-1&0&0&0&0&0&0\\0&0&0&0&0&-1&-1&0&-1&0\\0&0&-1&0&0&0&0&-1&0&-1\\0&-1&0&0&-1&0&0&0&0&0\end{bmatrix}$$

以V_1为根的树为: $\det(\overrightarrow{B_k}\cdot B_k^T)=\begin{vmatrix}2&0&0&-1\\-1&3&-1&-1\\0&-1&3&-1\\-1&0&0&2\end{vmatrix}=\begin{vmatrix}0&0&0&-1\\-3&3&-1&-1\\-2&-1&3&-1\\3&0&0&2\end{vmatrix}$

$$=(-1)\cdot(-1)^{1+4}\cdot\begin{vmatrix}-3&3&-1\\-2&-1&3\\3&0&0\end{vmatrix}=3\times(-1)^{3+1}\cdot\begin{vmatrix}3&-1\\-1&3\end{vmatrix}=24$$

(2)

删去e_2后 $B_k=$

$$\begin{bmatrix}-1&0&-1&1&1&0&0&0&0\\0&0&0&0&-1&-1&1&-1&0\\0&-1&0&0&0&0&-1&1&-1\\0&0&1&-1&0&1&0&0&1\end{bmatrix}$$

$\overrightarrow{B_k}=$

$$\begin{bmatrix}-1&0&-1&0&0&0&0&0&0\\0&0&0&0&-1&-1&0&-1&0\\0&-1&0&0&0&0&-1&0&-1\\0&0&0&-1&0&0&0&0&0\end{bmatrix}$$

以V_1为根且不含e_2:

$$\det(\overrightarrow{B_k}\cdot B_k^T)=\begin{vmatrix}2&0&0&-1\\-1&3&-1&-1\\0&-1&3&-1\\-1&0&0&1\end{vmatrix}=\begin{vmatrix}0&0&0&-1\\-3&3&-1&-1\\-2&-1&3&-1\\1&0&0&+1\end{vmatrix}=(-1)\cdot(-1)^{1+4}\begin{vmatrix}-3&3&-1\\-2&-1&3\\1&0&0\end{vmatrix}=1\cdot(-1)^{3+1}\cdot 8=8$$

以V_1为根且不含e_2 = 8棵

(3) 删去v_1与e_6后:

$$B_K=\begin{bmatrix} -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \vec{B_K}=\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

以v_1为根且不含e_6的树:

$$\det(\vec{B_K}\cdot B_K^T)=\begin{vmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 2 \end{vmatrix}=\begin{vmatrix} 0 & 0 & 0 & -1 \\ -2 & 2 & -1 & -1 \\ -2 & -1 & 3 & -1 \\ 3 & 0 & 0 & 2 \end{vmatrix}=(-1)\cdot(-1)^{1+4}\begin{vmatrix} -2 & 2 & -1 \\ -2 & -1 & 3 \\ 3 & 0 & 0 \end{vmatrix}=3\times 5=15$$

$\therefore$ 以v_1为根且必含e_6的根树有$24-15=9$个.