第四周习题课　隐函数（续），空间曲线与曲面

一．隐函数的二阶（偏）导数

例 1.　设 $z=f\left(x,\varphi\left(x^2,x^2\right)\right)$，其中函数 f 于 φ 的二阶偏导数连续，求 $\dfrac{d^2z}{dx^2}$

解：$z=f(u,v)$，其中 $\begin{cases}u=x\\ v=\varphi(s,t)\end{cases}$，$\begin{cases}s=x^2\\ t=x^2\end{cases}$。

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}(u,v)\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}(u,v)\cdot\frac{dv}{dx}=\frac{\partial f}{\partial u}(u,v)\cdot 1+\frac{\partial f}{\partial v}(u,v)\cdot\frac{dv}{dx}$$

而 $\dfrac{dv}{dx}=\dfrac{\partial\varphi}{\partial s}(s,t)\cdot\dfrac{ds}{dx}+\dfrac{\partial\varphi}{\partial t}(s,t)\cdot\dfrac{dt}{dx}=\dfrac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\dfrac{\partial\varphi}{\partial t}(s,t)\cdot 2x$。所以

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}(u,v)+\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$，其中 $\begin{cases}u=x\\ v=\varphi(s,t)\end{cases}$，$\begin{cases}s=x^2\\ t=x^2\end{cases}$。

$$\frac{d^2z}{dx^2}=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)+\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)\right\}+\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}=I+II\ 。$$

$$I=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)\right\}=\frac{\partial^2 f}{\partial u^2}\cdot\frac{du}{dx}+\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{dv}{dx}$$

$$=\frac{\partial^2 f}{\partial u^2}\cdot 1+\frac{\partial^2 f}{\partial u\partial v}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]。$$

$$II=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$+\frac{\partial f}{\partial v}(u,v)\cdot\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=II_1+II_2$$

$$II_1=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=\left[\frac{\partial^2 f}{\partial v\partial u}\cdot\frac{du}{dx}+\frac{\partial^2 f}{\partial v^2}\cdot\frac{dv}{dx}\right]\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\ ;$$

$$=\left\{\frac{\partial^2 f}{\partial v\partial u}\cdot 1+\frac{\partial^2 f}{\partial v^2}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$II_2=\frac{\partial f}{\partial v}(u,v)\cdot\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x\right]+\frac{d}{dx}\left[\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2+\frac{d}{dx}\left[\frac{\partial\varphi}{\partial t}(s,t)\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\begin{array}{l}\left[\frac{\partial^2\varphi}{\partial s^2}\cdot\frac{ds}{dx}+\frac{\partial^2\varphi}{\partial t\partial s}\cdot\frac{dt}{dx}\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2\\ +\left[\frac{\partial^2\varphi}{\partial s\partial t}\cdot\frac{ds}{dx}+\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{dt}{dx}\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\end{array}\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\begin{array}{l}\left[\frac{\partial^2\varphi}{\partial s^2}\cdot 2x+\frac{\partial^2\varphi}{\partial t\partial s}\cdot 2x\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2\\ +\left[\frac{\partial^2\varphi}{\partial s\partial t}\cdot 2x+\frac{\partial^2\varphi}{\partial t^2}\cdot 2x\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\end{array}\right\}。$$

代入即可。

例 2. 设 $z=z(x,y)$ 二阶连续可微，并且满足方程

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=0$$

若令 $\begin{cases}u=x+\alpha y\\ v=x+\beta y\end{cases}$，试确定 α,β 为何值时能变原方程为 $\frac{\partial^2 z}{\partial u\partial v}=0$.

解 将 x,y 看成自变量，u,v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha^2\frac{\partial^2 z}{\partial u^2}+2\alpha\beta\frac{\partial^2 z}{\partial u\partial v}+\beta^2\frac{\partial^2 z}{\partial v^2}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha\frac{\partial^2 z}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 z}{\partial u\partial v}+\beta\frac{\partial^2 z}{\partial v^2}$$

$$=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

由此可得， $0=A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=$

$$=\left(A+2B\alpha+C\alpha^2\right)\frac{\partial^2 z}{\partial u^2}+2\left(A+B(\alpha+\beta)+C\alpha\beta\right)\frac{\partial^2 z}{\partial u\partial v}+\left(A+2B\beta+C\beta^2\right)\frac{\partial^2 z}{\partial v^2}=0$$

只要选取 α,β 使得 $\begin{cases}A+2B\alpha+C\alpha^2=0\\ A+2B\beta+C\beta^2=0\end{cases}$，可得 $\frac{\partial^2 z}{\partial u\partial v}=0$.

问题成为方程 $A+2Bt+Ct^2=0$ 有两不同实根，即要求： $B^2-AC>0$.

令 $\alpha=-B+\sqrt{B^2-AC}$，$\beta=-B-\sqrt{B^2-AC}$，即可。

此时，$\frac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\frac{\partial^2 z}{\partial u\partial v}=0\Rightarrow\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=0\Rightarrow\frac{\partial z}{\partial v}=\varphi(v)\Rightarrow z=\int\varphi(v)dv+f(u)$.

$$z=f(u)+g(v)=f(x+\alpha y)+g(x+\beta y).$$

例 3. 设$u(x,y)\in C^2$，又$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$, $u(x,2x)=x$，$u'_x(x,2x)=x^2$,求 $u''_{xx}(x,2x)$,

$u''_{xy}(x,2x)$ $u''_{yy}(x,2x)$

解:
$$\frac{\partial u}{\partial x}(x,2x)=x^2,$$

两边对x求导,

$$\frac{\partial^2 u}{\partial x^2}(x,2x)+\frac{\partial^2 u}{\partial x\partial y}(x,2x)\cdot 2=2x. \qquad (1)$$

$$u(x,2x)=x,$$

两边对x求导,

$$\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1, \qquad \frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}.$$

两再边对x求导,

$$\frac{\partial^2 u}{\partial x\partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知
$$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

(1), (2), (3) 联立可解得:

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x,\quad \frac{\partial^2 u}{\partial x\partial y}(x,2x)=\frac{5}{3}x$$

二、向量函数的微分和导数

1． 计算极坐标、柱坐标、球坐标变换的 Jacobi 矩阵和 Jacobi 行列式：

（1）平面极坐标变换 $\vec{\mathbf{f}}(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$，也即$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$；

（2）空间柱坐标变换$\vec{\mathbf{f}}(r,\theta,z)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$，也即$\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=z \end{cases}$；

（3）空间球坐标变换 $\vec{\mathbf{f}}(r,\varphi,\theta)=\begin{pmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}$，也即 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$。

解：直接计算如下

（1）$J_{\mathbf{f}}(r,\theta)=\dfrac{\partial(x,y)}{\partial(r,\theta)}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$，

$$\det J_{\mathbf{f}}(r,\theta)=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}=r\text{；}$$

（2）$J_{\mathbf{f}}(r,\theta,z)=\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$，

$$\det J_{\mathbf{f}}(r,\theta,z)=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=r\text{；}$$

（3）$J_{\mathbf{f}}(r,\varphi,\theta)=\dfrac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$，

$$\det J_{\mathbf{f}}(r,\varphi,\theta)=\begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix}=r^2\sin\varphi\text{。}$$

2. 计算向量复合函数的 Jacobi 矩阵：

（1）$\mathbf{f}(x,y)=(x,y,x^2y)$，$x=s+t$，$y=s^2-t^2$，在 $s=2,t=1$；

（2）$\mathbf{f}(x,y,z)=(x^2+y+z,2x+y+z^2,0)$，$x=uv^2w^2,y=w^2\sin v,z=u^2e^v$。

解：（1）记 $\mathbf{g}(s,t)=(x,y)$，$x=s+t$，$y=s^2-t^2$，在 $s=2,t=1$ 时 $x=y=3$，

$$J_{\mathbf{f}}(3,3)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2xy & x^2 \end{pmatrix}_{x=y=3}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 18 & 9 \end{pmatrix},$$

$$J_{\mathbf{g}}(2,1)=\begin{pmatrix} 1 & 1 \\ 2s & -2t \end{pmatrix}_{s=2,t=1}=\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix},$$

$$J_{\mathbf{f}\circ\mathbf{g}}(2,1)=J_{\mathbf{f}}(3,3)J_{\mathbf{g}}(2,1)=\begin{pmatrix}1&0\\0&1\\18&9\end{pmatrix}\begin{pmatrix}1&1\\4&-2\end{pmatrix}=\begin{pmatrix}1&1\\4&-2\\54&0\end{pmatrix}。$$

法二：由 $\mathbf{g}(s,t)=(x,y)=(s+t,s^2-t^2)$，$\mathbf{f}(x,y)=(x,y,x^2y)$ 得到

$$\mathbf{f}\circ\mathbf{g}(s,t)=(s+t,s^2-t^2,(s+t)^2(s^2-t^2))$$

$$=(s+t,s^2-t^2,s^4+2s^3t-2st^3-t^4),$$

$$J_{\mathbf{f}\circ\mathbf{g}}(s,t)=\begin{pmatrix}1&1\\2s&-2t\\4s^3+6s^2t-2t^3&2s^3-6st^2-4t^3\end{pmatrix},$$

再将 $s=2,t=1$ 带入即得……

（2）由题意 $\mathbf{g}(u,v,w)=(x,y,z)$，$x=uv^2w^2,y=w^2\sin v,z=u^2e^v$，并且

$$f_1(x,y,z)=x^2+y+z，\ f_2(x,y,z)=2x+y+z^2，\ f_3(x,y,z)=0，$$

$$J_{\mathbf{f}\circ\mathbf{g}}(u,v,w)=\begin{pmatrix}2x&1&1\\2&1&2z\\0&0&0\end{pmatrix}\begin{pmatrix}v^2w^2&2uvw^2&2uv^2w\\0&w^2\cos v&2w\sin v\\2ue^v&u^2e^v&0\end{pmatrix}$$

$$=\begin{pmatrix}2xv^2w^2+2ue^v&4xuvw^2+w^2\cos v+u^2e^v&4xuv^2w+2w\sin v\\2v^2w^2+4zue^v&4uvw^2+w^2\cos v+2zu^2e^v&4uv^2w+2w\sin v\\0&0&0\end{pmatrix}$$

$$=\begin{pmatrix}2uv^4w^4+2ue^v&4u^2v^3w^4+w^2\cos v+u^2e^v&4u^2v^4w^3+2w\sin v\\2v^2w^2+4u^3e^{2v}&4uvw^2+w^2\cos v+2u^4e^{2v}&4uv^2w+2w\sin v\\0&0&0\end{pmatrix}$$

三、切平面,切线,法平面,法线

[例 1] 求曲线 $L:\begin{cases}x^2+y^2+z^2=4\\x^2+y^2=2x\end{cases}$

在点 $M_0(1,1,\sqrt{2})$处的切线和法平面方程

解:方程两边对x求导 $\begin{cases}2x+2yy'(x)+2zz'(x)=0\\2x+2yy'(x)=2\end{cases}$,解得 $y'(1)=0, z'(1)=\dfrac{-1}{\sqrt{2}}$

故切线方程为： $\dfrac{x-1}{1}=\dfrac{y-1}{0}=\dfrac{z-\sqrt{2}}{-1/\sqrt{2}}$

法平面方程为： $(x-1)-\dfrac{1}{\sqrt{2}}(z-\sqrt{2})=0$

[例2] 设函数f可微,求证：曲面 $S: z = yf(\frac{x}{y})$ 的

所有切平面相交于一个公共点。

解：曲面 S：在点 (x, y, z) 的切平面

$Z - z = \frac{\partial z}{\partial x}(X - x) + \frac{\partial z}{\partial y}(Y - y)$，代入得

$$Z - yf(\frac{x}{y}) = f'(\frac{x}{y})(X - x) + [f(\frac{x}{y}) - \frac{x}{y}f'(\frac{x}{y})](Y - y)$$

当 $(X, Y, Z) = (0, 0, 0)$ 时,两端恒等。因此都经过原点。

[例3] 过直线 $10x + 2y - 2z = 27,\quad x + y - z = 0$作曲面
$3x^2 + y^2 - z^2 = 27$的切平面,求其方程.

解：设 $F(x, y, z) = 3x^2 + y^2 - z^2 - 27$，则 $F'_x = 6x$， $F'_y = 2y$ ，$F'_z = -2z$

过直线 $10x + 2y - 2z = 27,\quad x + y - z = 0$ 的平面束
方程为

$$10x + 2y - 2z - 27 + \lambda(x + y - z) = 0 \qquad ,$$

法向量 $\bar{n} = \{(10 + \lambda), (2 + \lambda), (-2 - \lambda)\}$ 设切点为 (x_0, y_0, z_0),则有

$$\begin{cases} 3x_0^2 + y_0^2 - z_0^2 - 27 = 0 \\ (10 + \lambda)x_0 + (2 + \lambda)y_0 - (2 + \lambda)z_0 - 27 = 0 \end{cases}$$

又因为 $\vec{n} \parallel gradF$,所以 $\frac{10 + \lambda}{6x_0} = \frac{2 + \lambda}{2y_0} = \frac{-2 - \lambda}{-2z_0}$

解得 $x_0 = -3,\quad y_0 = -17,\quad z_0 = -17,\quad \lambda = -19$

于是,所求切平面方程为 $6 \cdot 3(x - 3) + 2 \cdot 1(y - 1) + (-2) \cdot 1(z - 1) = 0$

[例4] 求证满足微分方程 $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$ 的$u(x, y)$
为$u(x, y) = f(x^2 - y^2)$，其中,f为任意一元可微
函数.

只需证明： $u=f(x^2-y^2)$ 等价于

$u=u(x,y)$满足微分方程 $y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0$ 因为$u=f(x^2-y^2)$ 等价于

在曲线 $L: x^2-y^2=C$ 上 $u(x,y)\equiv$ 常数

又等价于 $gradu(x,y)$ 与 L 切向量处处正交 .

当$\nabla F(M_0)\neq 0$时,不妨设 $F_y'\neq 0$ 确定函数：$y=f(x)$， 且$y_0=f(x_0)$

切向量为$\vec{v}=(1,\frac{dy}{dx})$，$\frac{dy}{dx}=-\frac{F_x'}{F_y'}$,代入得到

切向量$\vec{v}=(F_y',-F_x')=(-2y,-2x)//(y,x)$。