HW 8 Multilinear Maps

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Exercise 1.8.1

1. We first express α, β, γ in the standard dual basis. Let

$$\alpha = \alpha_1 \boldsymbol{e}_1^T + \alpha_2 \boldsymbol{e}_2^T$$

$$\beta = \beta_1 \boldsymbol{e}_1^T + \beta_2 \boldsymbol{e}_2^T$$

$$\gamma = \gamma_1 \boldsymbol{e}_1^T + \gamma_2 \boldsymbol{e}_2^T$$

Then the (i, j, k) entry of $\alpha \otimes \beta \otimes \gamma$ is

$$\alpha_1\beta_1\gamma_1 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T + \alpha_2\beta_1\gamma_1 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T + \alpha_1\beta_2\gamma_1 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T + \alpha_2\beta_2\gamma_1 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T$$
$$+\alpha_1\beta_1\gamma_2 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T + \alpha_2\beta_1\gamma_2 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T + \alpha_1\beta_2\gamma_2 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T + \alpha_2\beta_2\gamma_2 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T$$

3. For $\forall M \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^n)^*$ with rank r. Let $M' = M_E(M)$. Suppose the rank of M' is r'. From the definition of rank, we decompose $M = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i$, where $\{\alpha_i \otimes \beta_i \otimes \gamma_i\}$ are distinct rank one tensors. Then $M' = M_E(M) = \sum_{i=1}^{r} \alpha_i E \otimes \beta_i \otimes \gamma_i$. Since we have expressed M' in r rank one tensors, we know $r' \le r$.

We decompose $M' = \sum_{i=1}^{r'} \delta_i \otimes \epsilon_i \otimes \zeta_i$, where $\{\delta_i \otimes \epsilon_i \otimes \zeta_i\}$ are distinct rank one tensors. Then $M_{E^{-1}}(M') = \sum_{i=1}^{r'} \delta_i E^{-1} \otimes \epsilon_i \otimes \zeta_i$. Note that $M_{E^{-1}}(M') = M$, we have $M = \sum_{i=1}^{r'} \delta_i E^{-1} \otimes \epsilon_i \otimes \zeta_i$. Since we have expressed M in r' rank one tensors, we know $r \le r'$.

Combine the two conclusions above, we know $r = r'$, i.e. $rank(M) = rank(M_E(M))$.

4. Suppose the i-th "2D" layer of $M \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^n)^*$ has rank r. For $M(-,-,\boldsymbol{e}_i) \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^*$, we have $rank(M(-,-,\boldsymbol{e}_i)) = r$. Suppose $rank(M) = r_M$, i.e. $M = \sum_{i=1}^{r_M} \alpha_i \otimes \beta_i \otimes \gamma_i$. Then feed $\boldsymbol{e}_i$, we know $M(-,-,\boldsymbol{e}_i) = \sum_{i=1}^{r_M} \gamma_i(\boldsymbol{e}_i)\alpha_i \otimes \beta_i$. We have expressed $M(-,-,\boldsymbol{e}_i)$ in r_M rank one tensors, we get $r \le r_M$, i.e. $rank(M)$ is at least r.

5. $rank(M) = 3$.

First, add the second layer to the first layer, which is an elementry layer operation, we get two layers of $M_E(M)$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $\delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have

$$M_E(M) = \underbrace{\delta \otimes \delta \otimes \boldsymbol{e}_1^T}_{\text{the 1st layer}} + \underbrace{\boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T + \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T}_{\text{the 2nd layer}}$$

So $rank(M) = rank(M_E) \le 3$.

Suppose $rank(M) = 2$. Then let $M = \alpha_1 \otimes \beta_1 \otimes \gamma_1 + \alpha_2 \otimes \beta_2 \otimes \gamma_2$. Extract the two layers, i.e. feed $\boldsymbol{e}_1, \boldsymbol{e}_2$ respectively, we have

$$\text{1st layer} \quad \gamma_1(\boldsymbol{e}_1)\alpha_1 \otimes \beta_1 + \gamma_2(\boldsymbol{e}_1)\alpha_2 \otimes \beta_2$$

$$\text{2nd layer} \quad \gamma_1(\boldsymbol{e}_2)\alpha_1 \otimes \beta_1 + \gamma_2(\boldsymbol{e}_2)\alpha_2 \otimes \beta_2$$

Then......?

Exercise 1.8.2 We suppose $M \in (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$, then $M = \sum_{i,j,k \in \{1,2,3\}} (i+j+k) \boldsymbol{e}_i^T \boldsymbol{e}_j^T \boldsymbol{e}_k^T$.

1. By direct calculation, we have $M(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v}) = 3x^3 + 6y^3 + 9z^3 + 12x^2y + 15x^2z + 15y^2x + 21y^2z + 21z^2x + 24z^2y + 36xyz$.

2. We try to find the (i, j, k) entry of M^σ. For $\forall i, k, j \in \{1, 2, 3\}$, construct $p : \{1, 2, 3\} \to \{1, 2, 3\}$ s.t. $p(1) = i, p(2) = j, p(3) = k$. Then

$$\begin{aligned} M^\sigma(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) &= M^\sigma(\boldsymbol{e}_{p(1)}, \boldsymbol{e}_{p(2)}, \boldsymbol{e}_{p(3)}) \\ &= M(\boldsymbol{e}_{p(\sigma(1))}, \boldsymbol{e}_{p(\sigma(2))}, \boldsymbol{e}_{p(\sigma(3))}) \\ &= p \circ \sigma(1) + p \circ \sigma(2) + p \circ \sigma(3) \\ &= p(1) + p(2) + p(3) \\ &= i + j + k \end{aligned}$$

Then M and M^σ have the same entry, i.e. $M = M^\sigma$.

3. Consider $M(-, -, \boldsymbol{e}_1)$. For $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, $M(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{e}_1) = \boldsymbol{x}^T A \boldsymbol{y}$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$. Note that the row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & -1 \\ & 1 & 2 \\ & & 0 \end{bmatrix}$, so $rank(A) = 2$. From **1.8.1.4**, $rank(M) \geq 2$. Let $A = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T$, $\gamma = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \delta = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, then we know that

$$\begin{aligned} M &= (\alpha_1^T \otimes \beta_1^T + \alpha_2^T \otimes \beta_2^T) \otimes \gamma^T + \gamma^T \otimes \gamma^T \otimes \delta^T \\ &= \alpha_1^T \otimes \beta_1^T \otimes \gamma^T + \alpha_2^T \otimes \beta_2^T \otimes \gamma^T + \gamma^T \otimes \gamma^T \otimes \delta^T \end{aligned}$$

M can be expressed as the sum of three "rank-one" things, so $rank(M) \leq 3$.

$\square$

Exercise 1.8.3

1. For $\forall \lambda, \mu \in \mathbb{R}$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, we have

$$\begin{aligned} X \otimes Y(\lambda \boldsymbol{v}_1 + \mu \boldsymbol{v}_2, \boldsymbol{w}_1) &= X(\lambda \boldsymbol{v}_1 + \mu \boldsymbol{v}_2) \otimes Y(\boldsymbol{w}_1) \\ &= X(\lambda \boldsymbol{v}_1) \otimes Y(\boldsymbol{w}_1) + X(\mu \boldsymbol{v}_2) \otimes Y(\boldsymbol{w}_1) \\ &= \lambda X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_1) + \mu X \otimes Y(\boldsymbol{v}_2, \boldsymbol{w}_1) \\ X \otimes Y(\boldsymbol{v}_1, \lambda \boldsymbol{w}_1 + \mu \boldsymbol{w}_2) &= X(\boldsymbol{v}_1) \otimes Y(\lambda \boldsymbol{w}_1 + \mu \boldsymbol{w}_2) \\ &= X(\boldsymbol{v}_1) \otimes Y(\lambda \boldsymbol{w}_1) + X(\boldsymbol{v}_1) \otimes Y(\mu \boldsymbol{w}_2) \\ &= \lambda X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_1) + \mu X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_2) \end{aligned}$$

So $X \otimes Y$ is bilinear.

2. Note that trace is independent of basis, we try to solve this problem under a certain basis. Suppose $dimV = n$, $dimW = m$, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ forms a basis of V, $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m\}$ forms a basis of W. Then

$$\boldsymbol{T} : \{\underbrace{\underbrace{\boldsymbol{v}_1 \otimes \boldsymbol{w}_1, \boldsymbol{v}_1 \otimes \boldsymbol{w}_2, \cdots, \boldsymbol{v}_1 \otimes \boldsymbol{w}_n}_{n \text{ tensors as a group}}, \cdots, \underbrace{\boldsymbol{v}_m \otimes \boldsymbol{w}_1, \boldsymbol{v}_m \otimes \boldsymbol{w}_2, \cdots, \boldsymbol{v}_m \otimes \boldsymbol{w}_n}_{n \text{ tensors as a group}}}_{m \text{ groups in total}}\}$$

form a basis of $V \otimes W$, and in this problem we always arrange the basis in the same order as presented above. Suppose the matrix of X under $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is $A = [a_1, \cdots, a_n]_{n\times n}$, and the matrix of Y under $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m\}$ is $B = [b_1, \cdots, b_m]_{m\times m}$.

From the previous subproblem, we treat $X \otimes Y$ as a linear trasformation on $V \otimes W$. We know

$$X \otimes Y(\boldsymbol{v}_i \otimes \boldsymbol{w}_j) = \boldsymbol{T} a_i \otimes b_j$$

where $\boldsymbol{T} a_i \otimes b_j$ denotes using $a_i \otimes b_j$ as coefficients to form a linear compsition of basis $\boldsymbol{T}$. Thus, the matrix of $X \otimes Y$ under basis $\boldsymbol{T}$ is

$$M = [\underbrace{a_1 \otimes b_1, a_1 \otimes b_2, \cdots, a_1 \otimes b_n}_{n \text{ vectors as a group}}, \cdots, \underbrace{a_m \otimes b_1, a_m \otimes b_2, \cdots, a_m \otimes b_n}_{n \text{ vectors as a group}}]_{mn\times mn}$$

Then $trace(M) = \sum\limits_{i=1}^{m}(a_i \cdot \sum\limits_{j=1}^{n} b_j) = (\sum\limits_{i=1}^{m} a_i)(\sum\limits_{j=1}^{n} b_j) = trace(X)trace(Y)$. □

Exercise 1.8.4 Map *trace* is in the space $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

For $\forall M \in \mathbb{R}^n \otimes (\mathbb{R}^n)^*$, decompose it into the tensors of the standard basis, we have

$$M = \sum_{i,j\in\{1,\cdots,n\}} m_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j^T$$

From the definition of *trace* map, we know $trace(M) = \sum\limits_{k=1}^{n} m_{kk}$. Thus, the entries of *trace* in $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ under the standard basis is $trace = \sum\limits_{k=1}^{n} \boldsymbol{e}_k^T \otimes \boldsymbol{e}_k$. □