赵晨阳 软01 2020012363　　1.6 No 4,5,7,9

4. $f=(u^2-x^2,u^2-y^2,u^2-z^2)=0$ —① $u=u(x,y,z)$ 对①式左右分别对x,y,z求偏导有: $f_1'(2u\cdot u_1'-2x)+f_2'\cdot(2u\cdot u_1')+f_3'\cdot 2u\cdot u_1'=0$

$f_1'(2u\cdot u_2')+f_2'(2u u_2'-2y)+f_3' 2u u_2'=0$ $\quad f_1'(2u\cdot u_3')+f_2'(2u\cdot u_3')+f_3'(2u\cdot u_3'-2z)=0$

$u\cdot u_1'(f_1'+f_2'+f_3')-x\cdot f_1'=0$ $\quad u\cdot u_2'(f_1'+f_2'+f_3')-y\cdot f_2'=0$ $\quad u\cdot u_3'(f_1'+f_2'+f_3')-z\cdot f_3'=0$

$u_1'=x\cdot f_1'/(f_1'+f_2'+f_3')u$ $\quad u_2'=y\cdot f_2'/(f_1'+f_2'+f_3')u$ $\quad u_3'=z\cdot f_3'/(f_1'+f_2'+f_3')\cdot u$

$\therefore \frac{1}{x}u_1'+\frac{1}{y}u_2'+\frac{1}{z}u_3'=\frac{1}{u}$

5. 能确定

$z=z(u,v)$ $\quad u=u(x,y)$ $\quad v=v(x,y)$.
$$\begin{cases}u^2v^2-z=0 & —h(u,v,z)\\ u+v-x=0 & —f(u,v,x)\\ u-v-y=0 & —M(u,v,y)\end{cases}$$

注：三个未知数，三个方程

法一: $\frac{\partial(z,u,v)}{\partial(x,y)}=-\left(\frac{\partial(h,f,M)}{\partial(z,u,v)}\right)^{-1}\cdot\frac{\partial(z,u,v)}{\partial(x,y)}$

$$\frac{\partial(h,f,M)}{\partial(z,u,v)}=\begin{bmatrix}-1 & 2v^2u & 2u^2v\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

之后求逆即可

法二: $u=\frac{x+y}{2}$, $v=\frac{x-y}{2}$ $\quad z=u^2\cdot v^2=\frac{1}{16}(x^2-y^2)^2$

$\frac{\partial z}{\partial x}=\frac{1}{16}2(x^2-y^2)\cdot 2x=\frac14\cdot x\cdot(x^2-y^2)$ $\quad \frac{\partial z}{\partial y}=\frac14 y\cdot(y^2-x^2)$

7. $\begin{cases}h(x,y,z): x^2+y^2-\frac12 z^2=0\\ g(x,y,z): x+y+z-2=0\end{cases}$ $\quad \frac{\partial(x,y)}{\partial z}=-\left(\frac{\partial(h,g)}{\partial(x,y)}\right)^{-1}\cdot\frac{\partial(h,g)}{\partial z}$

$\frac{\partial(h,g)}{\partial(x,y)}=\begin{bmatrix}2x & 2y\\ 1 & 1\end{bmatrix}$ $\quad \left(\frac{\partial(h,g)}{\partial(x,y)}\right)^{-1}=\frac{1}{2x-2y}\begin{bmatrix}1 & -2y\\ -1 & 2x\end{bmatrix}$

$\frac{\partial(h,g)}{\partial z}=\begin{bmatrix}-z\\ 1\end{bmatrix}$ $\quad \therefore \frac{\partial(x,y)}{\partial z}=\begin{bmatrix}\frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial z}\end{bmatrix}=\frac{-1}{2x-2y}\begin{bmatrix}-z-2y\\ z+2x\end{bmatrix}$

$\therefore \frac{dx}{dz}=\frac{+z+2y}{2x-2y}$ $\quad \frac{dy}{dz}=\frac{z+2x}{-2x+2y}$ $\quad$ 代入点(1,-1,2)有:

$\frac{d^2x}{dz^2}=+\frac{1}{2x-2y}$ $\quad \frac{d^2y}{dz^2}=\frac{1}{-2x+2y}$

$\frac{dx}{dz}=0$ $\quad \frac{dy}{dz}=-1$ $\quad \cancel{\frac{d^2x}{dz^2}}=+\frac14$ $\quad \cancel{\frac{d^2y}{dz^2}}=-\frac14$

注意到二阶导时，x与y是z的函数，故不可粗糙处理

$$\frac{dx^2}{d^2z}=\frac{(1+2y')(2x-2y)-(2x'-2y')(z+2y)}{(2x-2y)^2}=\frac{(-1)\cdot4+2\cdot0}{(2+2)^2}=-\frac14$$

$$\frac{dy^2}{d^2z}=\frac{(1+2x')(2y-2x)+(2x'-2y')(z+2x)}{(-2x+2y)^2}=\frac{1\cdot(-4)+2\cdot4}{16}=\frac14$$

法二：$\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \Rightarrow x^2+[2-(x+z)]^2=\frac{1}{2}z^2$ 各有好坏

$\therefore 2x\cdot x'+2(2-(x+z))\cdot(-x'-1)=z \quad \therefore (2x-4+2x+2z)x'=z+4-2x-2z=\frac{4-2x-z}{4x+2z-4}$

$x'=0 \quad x''=\frac{(-2x'-1)(4x+2z-4)-(4x'+2\cdot 1)(4-2x-z)}{(4x+2z-4)^2}=\frac{-1\cdot 4}{16}=-\frac{1}{4}$

习题1.5： 4 5 7 9

4. $\frac{dz}{dx}=z_1'\cdot\frac{du}{dx}+z_2'\cdot\frac{dv}{dx}=(\ln(u-v)+\frac{u}{u-v})(-e^{-x})+u\cdot\frac{-1}{u-v}\cdot\frac{1}{x}$

$=-\ln(e^{-x}-\ln x)\cdot e^{-x}+\frac{e^{-x}}{e^{-x}-\ln x}(-e^{-x})+e^{-x}\cdot\frac{-1}{e^{-x}-\ln x}\cdot\frac{1}{x}$

$=-\left(\frac{\frac{1}{x}+e^{-x}}{e^{-x}-\ln x}+\ln(e^{-x}-\ln x)\right)e^{-x}$

5. $\frac{\partial u}{\partial r}=f_1'\cdot\frac{\partial x}{\partial r}+f_2'\cdot\frac{\partial y}{\partial r}=f_1'\cdot\cos\theta+f_2'\cdot\sin\theta \quad \frac{\partial u}{\partial\theta}=f_1'\cdot r\cdot(-\sin\theta)+f_2'\cdot r\cdot\cos\theta$

$\frac{\partial u}{\partial x}=f_1' \quad \frac{\partial u}{\partial y}=f_2' \quad \therefore$ 左式$=(f_1'\cdot\cos\theta+f_2'\cdot\sin\theta)^2+(f_2'\cdot\cos\theta-f_1'\cdot\sin\theta)^2=f_1'^2+f_2'^2$

右式$=f_1'^2+f_2'^2$. 即证毕

7. 即证 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \frac{\partial u}{\partial x}=f_1'\cdot\frac{(x^2+y^2)-(2x)\cdot x}{(x^2+y^2)^2}+f_2'\cdot\frac{-(2x)\cdot y}{(x^2+y^2)^2}=\frac{(y^2-x^2)f_1'-2xyf_2'}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x^2}$ 分子为 $[(-2x)\cdot f_1'+f_{11}''(y^2-x^2)-2y(f_2'+f_{12}''\cdot x)](x^2+y^2)^2-2(x^2+y^2)(2x)[(y^2-x^2)f_1'-2xyf_2']$

$\frac{\partial u}{\partial y}=f_1'\cdot\frac{-(2y)\cdot x}{(x^2+y^2)^2}+f_2'\cdot\frac{(x^2+y^2)-2y\cdot y}{(x^2+y^2)^2}=\frac{f_2'(x^2-y^2)-2xyf_1'}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial y^2}$ 分子为 $[(-2y)\cdot f_2'+f_{22}''(x^2-y^2)-2x(f_1'+f_{21}''\cdot y)]\cdot(x^2+y^2)^2-2(x^2+y^2)(2y)(f_2'(x^2-y^2)-2xyf_1')$

太过繁琐！无法消元. 改用换元法：不必将 $\frac{\partial s}{\partial x},\frac{\partial s}{\partial y},\frac{\partial t}{\partial x},\frac{\partial t}{\partial y}$ 全代入

令 $s=\frac{x}{x^2+y^2}$ $t=\frac{y}{x^2+y^2}$. 则 $\frac{\partial s}{\partial x}=\frac{(x^2+y^2)-2x(x)}{(x^2+y^2)^2}=-\frac{\partial t}{\partial y} \quad \frac{\partial s}{\partial y}=\frac{-2yx}{(x^2+y^2)^2}=\frac{\partial t}{\partial x}$

$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 f}{\partial s^2}\cdot\left(\frac{\partial s}{\partial x}\right)^2+\frac{\partial^2 f}{\partial t\partial s}\cdot\frac{\partial t}{\partial x}\cdot\frac{\partial s}{\partial x}+\frac{\partial^2 f}{\partial s\partial t}\cdot\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}+\frac{\partial^2 f}{\partial t^2}\cdot\left(\frac{\partial t}{\partial x}\right)^2$

同理：$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 f}{\partial t^2}\left(\frac{\partial t}{\partial y}\right)^2+\frac{\partial^2 f}{\partial s\partial t}\cdot\frac{\partial s}{\partial y}\cdot\frac{\partial t}{\partial y}+\frac{\partial^2 f}{\partial t\partial s}\cdot\frac{\partial t}{\partial y}\cdot\frac{\partial s}{\partial y}+\frac{\partial^2 f}{\partial s^2}\left(\frac{\partial s}{\partial y}\right)^2$

由 $f\in C^2(R^2)$ 有：$f_{12}''=f_{21}''$ ！

$\therefore$ 上下相加：$\left(\frac{\partial^2 f}{\partial s^2}+\frac{\partial^2 f}{\partial t^2}\right)\cdot\left[\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial t}{\partial x}\right)^2\right]+\left(\frac{\partial^2 f}{\partial t\partial s}\right)\cdot\left(\frac{\partial s}{\partial y}\right)\cdot\left(\frac{\partial t}{\partial y}+\frac{\partial s}{\partial x}\right)=0$

证毕

9. (1) $\begin{cases} y_1 - u_1 - u_2 = 0 & \text{—— } f \\ y_2 - u_1 u_2 = 0 & \text{—— } g \\ y_3 - \frac{u_2}{u_1} = 0 & \text{—— } h \end{cases}$ $\qquad \frac{\partial y}{\partial x} = -\left(\frac{\partial(f,g,h)}{\partial y}\right)^{-1}\left(\frac{\partial(f,g,h)}{\partial x}\right)$

$$\frac{\partial(f,g,h)}{\partial y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{\partial(f,g,h)}{\partial(x,y)} = \begin{pmatrix} -u_{1x}' - u_{2x}' & -u_{1y}' - u_{2y}' \\ -u_{1x}' u_2 - u_{2x}' u_1 & -u_{1y}' u_2 - u_{2y}' u_1 \\ \frac{-u_{2x}' u_1 + u_{1x}' u_2}{u_1^2} & \frac{-u_{2y}' u_1 + u_{1y}' u_2}{u_1^2} \end{pmatrix}$$

$$\therefore \begin{pmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y} \\ \frac{\partial y_3}{\partial x} & \frac{\partial y_3}{\partial y} \end{pmatrix} = \begin{pmatrix} u_{1x}' + u_{2x}' & u_{1y}' + u_{2y}' \\ u_{1x}' u_2 + u_{2x}' u_1 & u_{1y}' u_2 + u_{2y}' u_1 \\ \frac{u_{2x}' u_1 - u_{1x}' u_2}{u_1^2} & \frac{u_{2y}' u_1 - u_{1y}' u_2}{u_1^2} \end{pmatrix}$$

$$u_{1x}' = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad u_{2x}' = \frac{-2xy}{(x^2+y^2)^2} \quad u_{1y}' = \frac{-2xy}{(x^2+y^2)^2} \quad u_{2y}' = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$\therefore$ Jacobi 为:
$$\begin{bmatrix} \frac{-x^2 - 2xy + y^2}{(x^2+y^2)^2} & \frac{x^2 - 2xy - y^2}{(x^2+y^2)^2} \\ \frac{-3x^2 y + y^3}{(x^2+y^2)^3} & \frac{-3xy^2 + x^3}{(x^2+y^2)^3} \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$$

全微分:

$$dY = J(Y) \cdot dX$$

注：复合求 J(Y) 与隐函数本质一样，故殊途同归

(2) $J(Y) = J(u) \cdot J(x)$ $\qquad J(u) = \begin{pmatrix} 2u_1 & 2u_2 \\ 2u_1 & -2u_2 \end{pmatrix}$ $\qquad J(x) = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$

$$\therefore J(Y) = \begin{bmatrix} \frac{2\ln\sqrt{x^2+y^2} \cdot x - 2y \cdot \arctan\frac{y}{x}}{x^2+y^2} & \frac{2(y \cdot \ln\sqrt{x^2+y^2} + x \arctan\frac{y}{x})}{x^2+y^2} \\ \frac{2(x \ln\sqrt{x^2+y^2} + y \cdot \arctan\frac{y}{x})}{x^2+y^2} & \frac{2(y \cdot \ln\sqrt{x^2+y^2} - x \arctan\frac{y}{x})}{x^2+y^2} \end{bmatrix}$$

$$d(Y) = J(Y) \cdot dX$$

(3) $J(Y) = J(u) \cdot J(x)$ $\qquad J(u) = \begin{bmatrix} \frac{u_1}{u_1^2+u_2^2} & \frac{u_2}{u_1^2+u_2^2} \\ \frac{-u_1}{u_1^2+u_2^2} & \frac{u_2}{u_1^2+u_2^2} \end{bmatrix}$ $\qquad J(x) = \begin{bmatrix} \cos y \cdot e^x & -e^x \sin y \\ \sin y \cdot e^x & e^x \cos y \end{bmatrix}$

$$J(Y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$d(Y) = J(Y) \cdot dX$$