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5.1

$-(A-I)(A^T-I) = -(AA^T - AI - IA^T + I^2)$ Since A is orthogonal, So $AA^T = A^TA = I$

Hence $-(A-I)(A^T-I) = -(I - A - A^T + I) = A + A^T - 2I$

5.2

Since $f(t)$ is an orthognal matrix, So from 5.1. $-(f(t)-I)(f(t)^T-I) = f(t) + f(t)^T - 2I$.

$$\lim_{t\to 0} \frac{-(f(t)-I)(f(t)^T-I)}{t} = \lim_{t\to 0} \frac{(f(t)+f(t)^T-2I)}{t} = \lim_{t\to 0} \frac{(f(t)-I)+(f(t)^T-I)}{t}$$

And $f(0) = f(0)^T = I$ Then it equals $= \lim_{t\to 0} \frac{(f(t)-f(0)) + (f(t)^T - f(0))}{t}$

we know that if $\lim\limits_{t\to 0} A_t = M$ and $\lim\limits_{t\to 0} B_t = N$. ($M$ and N exist)

then $\lim\limits_{t\to 0}(A_t + B_t) = \lim\limits_{t\to 0} A_t + \lim\limits_{t\to 0} B_t = M + N$

And we know $\lim\limits_{dt\to 0} \frac{f(t+dt)-f(t)}{dt}$ exists at all t. Hence $\lim\limits_{t\to 0} \frac{(f(t)-f(0))}{t}$ exists and equal to $f(0)$

and $\lim\limits_{t\to 0} \frac{(f(t)^T - f(0))}{t}$ exists and equal to $f(0)^T$ Then $\lim\limits_{t\to 0} \frac{(f(t)-f(0)) + (f(t)^T - f(0))}{t} = f(0) + f(0)^T$

5.3

$$\lim_{t\to 0} \frac{-(f(t)-I)(f(t)^T-I)}{t} = \lim_{t\to 0} \frac{-(f(t)-I)(f(t)^T-I)\cdot t}{t\cdot t} = \lim_{t\to 0}\left(\frac{-(f(t)-I)}{t} \cdot \frac{(f(t)^T-I)}{t} \cdot t\right)$$

And we know if $\lim\limits_{t\to 0} A_t = M$ and $\lim\limits_{t\to 0} B_t = N$. ($M$ and N exist)

then $\lim\limits_{t\to 0}(A_t \cdot B_t) = \lim\limits_{t\to 0} A_t \cdot \lim\limits_{t\to 0} B_t = M \cdot N$ And $\lim\limits_{t\to 0} \frac{f(t)^T - I}{t} = \lim\limits_{t\to 0} \frac{f(t)^T - f(0)}{t} = f(0)^T$

$\lim\limits_{t\to 0} t = 0$ $\lim\limits_{t\to 0} \frac{f(t) - I}{t} = \lim\limits_{t\to 0} \frac{f(t) - f(0)}{t} = f(0)$ Then $\lim\limits_{t\to 0}\left(\frac{f(t)^T - I}{t} \cdot t\right) = f(0)^T \cdot 0 = 0$

Then $\lim\limits_{t\to 0}\left(\frac{-(f(t)-I)}{t} \cdot \frac{(f(t)^T-I)}{t} \cdot t\right) = \lim\limits_{t\to 0} \frac{-(f(t)-I)}{t} \cdot \lim\limits_{t\to 0}\left(\frac{f(t)^T-I}{t} \cdot t\right) = -f(0)\cdot 0 = 0$

And from 5.2. $\lim\limits_{t\to 0} \frac{-(f(t)-I)(f(t)^T-I)}{t} = f(0) + f(0)^T$ So $f(0) + f(0)^T = 0$

Hence $f(0)$ must be skew symmetric

5.4: We know that $f(t)$ is orthogonal Then $f(t)\cdot f(t)^T = I$

Suppose $F(t) = f(t)\cdot f(t)^T$ $G(t) = I$. We know $\lim_{dt\to 0}\frac{f(t+dt)-f(t)}{dt}$ exists at all t

then $F'(t) = f'(t)\cdot f(t)^T + f(t)\cdot f'(t)^T$ $G'(t) = 0$

Hence $f'(t) f(t)^T + f(t) f'(t)^T = 0$ $(f'(t)\cdot f(t)^T) + (f'(t)\cdot f(t)^T)^T = 0$ Hence $f'(t)\cdot f(t)^T$ is skew symmetric

Hence $f(t)^{-1} = f(t)^T$ Then $f'(t)\cdot f(t)^{-1} = f'(t)\cdot f(t)^T$, which is skew symmetric

5.5

$f'(t)\cdot f(t)^{-1} = A$ Then $f'(t) = A\cdot f(t)$. Note that this is a first-order Homogenous differential equation(一阶齐次微分方程)

Corollary 3.3.4. *Let $\mathbf{v}(t)$ be a vector of functions, i.e., each coordinate may change as t change. Suppose it satisfy the differential equation $\mathbf{v}'(t) = A\mathbf{v}(t)$ for some linear transformation A. Then $e^{At}\mathbf{c}$ is a solution for any constant vector $\mathbf{c}$. (In fact $\mathbf{v}(0) = \mathbf{c}$, so it is the initial condition.)*

linear combination of all columns of e^{At}.

I claim that this is in fact the only solution.

Proposition 3.3.5. *$\boxed{\text{The solution space to } \mathbf{v}'(t) = A\mathbf{v}(t) \text{ is } n \text{ dimensional}}$ where n is the dimension of the domain. (So columns of e^{At} form a basis.)*

$\frac{df(A)}{dt} = A\cdot f(t)$ And e^{At} is a soultion to it. So then general solution to $f(t)$ is $e^{At}\cdot B$

where B is a constant matrix depending on other initial conditions

$f(0) = e^{A0}\cdot B = B$ And $f: R \to SO_n$ Then $B = f(0) \in SO_n$

1.1 Proof:

$A^{k+1}v - A^k v = A^k v - A^{k-1} v$ for $\forall k \geq 1$. Hence $A^2 v - Av = Av - v$ $\quad A^2 v - 2Av + v = 0$

$(A^2 - 2A + I)v = 0 \Rightarrow (A-I)^2 v = 0$ Hence v is a generlized eigenvector for egenvalue 1.

1.2 False.

If $A^2 v = Av + v$. Then $(A^2 - A - I)v = 0 \Leftrightarrow (A - \frac{1+\sqrt{5}}{2} I)(A - \frac{1-\sqrt{5}}{2} I)v = 0$

Let $A = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2} \end{bmatrix}$ take $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Let $B = A - \frac{1+\sqrt{5}}{2} I = \begin{bmatrix} 0 & \\ & -\sqrt{5} \end{bmatrix}$ Then $B^k = \begin{bmatrix} 0 & \\ & (-\sqrt{5})^k \end{bmatrix}$ $k \in \mathbb{Z}^+$

Then $B^k \cdot v = \begin{bmatrix} 0 \\ (-\sqrt{5})^k \end{bmatrix} \neq 0$ let $C = A - \frac{1-\sqrt{5}}{2} I = \begin{bmatrix} \sqrt{5} & \\ & 0 \end{bmatrix}$ $\quad C^k = \begin{bmatrix} (\sqrt{5})^k & \\ & 0 \end{bmatrix}$ $k \in \mathbb{Z}^+$. $C^k v = \begin{bmatrix} (\sqrt{5})^k \\ 0 \end{bmatrix} \neq 0$

A only get eigenvalue $\frac{1 \pm \sqrt{5}}{2}$ so v is not generlised eigenvector for A.

But $(A - \frac{1-\sqrt{5}}{2} I)(A - \frac{1+\sqrt{5}}{2} I)v = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence $A^2 v - Av - v = 0$ Apply A^k to it. $A^{k+2} v - A^{k+1} v - A^k v = 0$ $(k \geq 0)$

$A^{k+2} v = A^{k+1} v + A^k v$. So this A and v is a counter example

1.3 True

Suppose $P(x)$ is an eigenvector for M. $P(x) \neq 0$; $\quad$ Then let I be the Idential map

So $(M - \lambda I)^k P(x) = 0$ which means $(x - \lambda)^k P(x) = 0$ for certain λ and some k

That that only happens if and only if $x - \lambda = 0$ or $P(x) = 0$ But $x - \lambda$ and $P(x)$ are not zero vector in V. So we get a controdiction.

1.4 True

$A^5 = 0$ A is nilpotent. Then Suppose A's Jordan block $N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix}_{n \times n}$. N_i is all nilpotent $A = PNP^{-1}$

$A^5 = P \begin{bmatrix} N_1^5 & & & \\ & N_2^5 & & \\ & & \ddots & \\ & & & N_k^5 \end{bmatrix}_{n \times n} P^{-1}$ Then $\forall N_i$. $N_i^5 = 0$

But N_i is nilpotent jordan block. for a $k \times k$ nilpotent jordan block. The minimal killing polynomial is x^k. Hence Size of $N_i \leq 5$.

So the number of N_i (i.e. k) $\geq \frac{n}{5}$. k is also the geometric multiplicity of eigenvalue 0, which is equal to $\dim \ker(A)$. Hence $\dim \ker(A) \geq \frac{n}{5}$

1.5 True

① Lemma 1 : for $\forall$ $m\times n$ matrix A. $A: Ran(A^T) \rightarrow Ran(A)$ is bijection.

$A_{m\times n} X_n = Y_m$

$$\begin{tikzcd}
V_r \quad Ran(A^T) \arrow[r, "T_1"] \arrow[dr, "T_2"'] & Ran(A) \quad W_r \\
V_{n-r} \quad Ker(A) \arrow[r, "T_4"'] \arrow[ur, "T_3"] & Ker(A^T) \quad W_{m-r}
\end{tikzcd}$$

We block $A: R^n \rightarrow R^m$ into four smaller linear transformation T_1, T_2, T_3, T_4

$T_1: R(A^T) \rightarrow R(A)$ Choose the bases of $R(A^T)$. $V_1, V_2 \cdots V_r$ bases of $R(A)$: $W_1 \cdots W_r$

$T_1([V_1\ V_2 \cdots V_r]) = [W_1\ W_2 \cdots W_r]\ A_{11\ r\times r}$ Chose Any vector $X \in R^r$. Hence we have

$T_1([V_1 \cdots V_n] \cdot X) = [W_1 W_2 \cdots W_r] \cdot A_{11} \cdot X$, $\Rightarrow T_1(V_r \cdot X) = W_r \cdot A_{11} X$

And $T_2(V_r X) = W_{m-r} \cdot A_{21} \cdot X$ $\quad T_3(V_{n-r} \cdot X) = W_r \cdot A_{12} \cdot X$ $\quad T_4(V_{n-r} \cdot X) = W_{m-r} \cdot A_{22} X$

target ①: T_1 is bijection.

1) T_1 is injection. That is to prove if $T_1(A^T X) = 0 \Rightarrow A^T X = 0$

$A \cdot A^T X = T_1(A^T X) + T_2(A^T X)$, if $T_1(A^T X) = 0$

$\Rightarrow A \cdot A^T X = T_2(A^T X) \in N(A^T)$. But $A \cdot A^T X \in Ran(A)$

$Ran(A) \cap N(A^T) = 0$ $\therefore A \cdot A^T X = 0$

$\therefore A^T X \in N(A)$ and $A^T X \in Ran(A^T)$ $N(A) \cap Ran(A^T) = 0$

$\therefore A^T X = 0$

So T_1 is injection

2) T_1 is surjection.

$A \cdot A^T X \in Ran(A)$ $\therefore T_2(A^T X) \in N(A^T)$ $\quad T_1(A^T X) \in R(A)$ $\quad$ proof is attached

Hence $T_2(A^T X)$ must be 0

For any $V \in Ran(A)$ $V = A \cdot X$. $Ran(A) = Ran(A \cdot A^T)$ $\therefore V = AX = A \cdot A^T y$

every V from $Ran(A)$ we have a vector $A^T y \in Ran(A^T)$ as its preimage

So T_1 is bijection. So lemma 1 is proved

proof: $ran(A) = ran(A \cdot A^T)$

which is to say: $N(A^T) = N(A \cdot A^T)$

if $A^T X = 0$, it is obvious $A \cdot A^T X = 0$

if $A \cdot A^T X = 0 \Rightarrow X^T A \cdot A^T X = 0 \Rightarrow (A^T X)^T \cdot AX = 0 \Rightarrow AX = 0$

So $ran(A) = ran(A \cdot A^T)$

② Corallery: for $\forall$ $n\times n$ matrix X. $X: Ran(X^T) \rightarrow Ran(X)$ is bijection.

③ Corallery: for $\forall$ $n\times n$ matrix X. $X^T: Ran(X) \rightarrow Ran(X^T)$ is bijection.

④ Corallery: for $\forall$ $n\times n$ matrix X. $X \circ X^T: Ran(X) \to Ran(X^T) \to Ran(X)$ is bijection.

So $A \cdot A^T: Ran(A) \to Ran(A)$ in this problem is a bijection. Let $f: Ran(A) \to Ran(A)$ be the inverse of $A \cdot A^T$ and let it coresponding matrix be B. $\forall V \in Ran(A)$

$BAA^TV = AA^TBV = V$ Then $\forall$ $n\times 1$ vector V, $AV \in Ran(A)$ $\therefore BAA^TAV = AA^TBAV = AV$

Apply A on the left of either side of the equation. $AAA^TBAV = A^2V$

Apply BAA^T on the left of either side of the equation. $BAA^TAAA^TBAV = BAA^TA^2V$

$A^2V \in Ran(A)$ So $BAA^TA^2V = A^2V$. And $AA^TAAA^T = 0$ $\therefore A^2V = BAA^TAAA^TBAV = 0$

for $\forall$ $n\times 1$ vector V. $\Rightarrow A^2 = 0$

The same as 1.4, for $n\times n$ matrix A.

$A^2=0$ A is nilpotent. Then Suppose A's Jordan block $N = \begin{bmatrix} N_1 & & \\ & N_2 & \\ & & \ddots \\ & & & N_k \end{bmatrix}_{n\times n}$. N_i is all nilpotent $A = PNP^{-1}$

$A^2 = P\begin{bmatrix} N_1^2 & & \\ & N_2^2 & \\ & & \ddots \\ & & & N_k^2 \end{bmatrix}_{n\times n} P^{-1}$ Then $\forall N_i$. $N_i^2 = 0$

But N_i is nilpotent jordan block. for a $k\times k$ nilpotent jordan block. The minimal killing polynomial is X^k. Hence Size of $N_i \le 2$.

So the number of N_i (i.e. k) $\geq \frac{n}{2}$. k is also the geometric multiplicity of eigenvalue 0, which is equal to $\dim \ker(A)$. Hence $\dim\ker(A) \geq \frac{n}{2}$

3.1. assume the matrix that we are searching for has eigenvalue $a+bi$ ($a, b \in \mathbb{R}, b \neq 0$)
Since it has no 1×1 jordan block, so it must have algebraic multiplicity greater than 1.
So as $a-bi$. Since complex eigenvalues come in pairs.
So the matrix has even size number, it is at leat 4×4.

Let $A=\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ $\det(A-\lambda I)=\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ -1 & -\lambda & 0 & 1 \\ & & -\lambda & 1 \\ & & -1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 0 & 1 \\ & -\lambda & 1 \\ & -1 & -\lambda \end{vmatrix} + (-1)\cdot(-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & 0 \\ & -\lambda & 1 \\ & -1 & -\lambda \end{vmatrix}$

$= -\lambda^2 \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} + \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = (\lambda^2+1)^2$

Hence $\lambda_1 = i$ $\lambda_2 = +i$ $\lambda_3 = -i$ $\lambda_4 = -i$

$A-iI=\begin{bmatrix} -i & 1 & 1 & 0 \\ -1 & -i & 0 & 1 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix}$ $AX_1=0 \Rightarrow X_1=\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$ $AX_2 = X_1 \Rightarrow X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$

$A+iI=\begin{bmatrix} i & 1 & 1 & 0 \\ -1 & i & 0 & 1 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & i \end{bmatrix}$ $AX_3=0 \Rightarrow X_3=\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$ $AX_4 = X_3 \Rightarrow X_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$

basis $P=\begin{bmatrix} 1 & 0 & 1 & 0 \\ i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & i \end{bmatrix}$ $A = P^{-1}\begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{bmatrix} P$ A satisfies these requirements

3.2

$A=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $AB=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $BA=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ AB and BA are both already in Jordan form. $AB \neq BA$
Hence AB are not similar.

3.3

$A=\begin{bmatrix} & i \\ -i & \end{bmatrix}$ $|A-\lambda I| = \begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = (\lambda^2 + i^2) = \lambda^2 - 1$
$\lambda_A = \pm 1$. It's all real.

3.4 ⊫ there may be some problems. see page 8 for more
Suppose A is already in Jordan-normal form. $e^A - A - I = 0$ then $e^\lambda - \lambda - 1 = 0$
Let $f(x) = e^x - x - 1$ $f(x)$ is derivable. $f'(x) = e^x - 1$. Hence $f(x)$ decrease when $x \in (-\infty, 0]$
$f(x)$ increases when $x \in [0, +\infty)$ $f(x)_{min} = f(0) = 0$
Hence $e^\lambda - \lambda - 1 = 0 \Rightarrow \lambda = 0$ Hence A is nilpotent and A is 4×4
Then $A^4 = 0$ $e^A = I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n$. And $A^n = 0$ when $n \geq 4$ and $N \in \mathbb{Z}^+$.
Hence $e^A - A - I = 0 \Rightarrow \frac{1}{2}A^2 + \frac{1}{6}A^3 = 0$ i.e. $3A^2 + A^3 = 0$
if is a 4×4 Jordan block i.e. $A=\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$ But $3A^2 + A^3 = \begin{bmatrix} 0 & 0 & 3 & 1 \\ & 0 & 0 & 3 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \neq 0$

Then if A has 3×3 Jordan block. i.e. $A=\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$ A_1 is $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$

$$3A^2+A^3=\begin{bmatrix} 3A_1^2 & \\ & 3A_2^2 \end{bmatrix}+\begin{bmatrix} A_1^3 & \\ & A_2^3 \end{bmatrix}=\begin{bmatrix} 0&0&3&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\neq 0.$$

$n\times n$ Nilportent jordan N satisfies $N^n=0$.

And A has no 4×4 and 3×3 Jordan block Hence A has jordan block not Larger than 2×2. Then $A^2=0$ which already implies $3A^2+A^3=0$

Hence

$$A_1=\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\quad A_2=\begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\quad A_3=\begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}\quad A_4=\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}\quad A_5=\begin{bmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$

There maybe some problem in 3.5.

Because when I thought about $e^x - x - 1 = 0$. emm.....

Will there be solutions like $a+bi$ ($a,b \in \mathbb{R}$ and $b \neq 0$)

$\lambda = a+bi$.

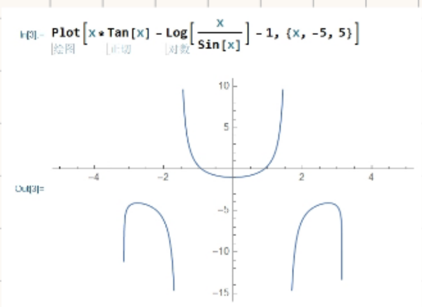
$e^\lambda - \lambda - 1 = 0$

$e^a e^{bi} - a - bi - 1 = 0$

$e^a(\cos b + i\sin b) - a - bi - 1 = 0$

$$\begin{cases} e^a\cos b - a - 1 = 0 \\ e^a \sin b - b = 0. \end{cases}$$

What! Then I use mathmatica



Okay, there are solution more that $x=0$

Oh no. I gonna do something more. Then I discuss with lmd. Here it comes.

Let $A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$ Then $e^A = \begin{bmatrix} e^0 & \frac{de^x}{dx}\big|_{x=0} & & \\ & e^0 & & \\ & & e^0 & \frac{de^x}{dx}\big|_{x=0} \\ & & & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} = A + I$

Let $A = XJX^{-1}$. J is A's Jordan form. $e^A = A + I \iff e^J = J + I$.

When J is 1×1, then we have $e^\lambda = \lambda + 1$. J is 2×2, we have $\begin{bmatrix} e^\lambda & e^\lambda \\ & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda+1 & 1 \\ & \lambda+1 \end{bmatrix}$

$\Rightarrow e^\lambda = 1$ $e^\lambda = \lambda + 1$. $\lambda = 0$. And J is nilpotent 2×2 matrix.

When J is 3×3. $\begin{bmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2} \\ & e^\lambda & e^\lambda \\ & & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda+1 & 1 & \\ & \lambda+1 & 1 \\ & & \lambda+1 \end{bmatrix} \Rightarrow \begin{cases} e^\lambda = 1 \\ e^\lambda = \lambda+1 \\ \frac{e^\lambda}{2} = 0 \end{cases} \Rightarrow$ No solution even in $\mathbb{C}$.

J has no 3×3 single Jordan block. Similarly there is no 4×4 single jordan block

Take μ_i: $e^\mu - \mu - 1 = 0$ and $\mu_i \in \mathbb{C}$. μ_i can be the same or different

A's jordan are

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} \mu_1 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \mu_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \mu_1 & \\ & & & \mu_2 \end{bmatrix} \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \mu_3 & \\ & & & \mu_4 \end{bmatrix}$$

2.1 let $A=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}$ $B=\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}$

$$A\otimes B=\begin{bmatrix}a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2\\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4\\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2\\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4\end{bmatrix} \qquad B\otimes A=\begin{bmatrix}a_1b_1 & a_2b_1 & a_1b_2 & a_2b_2\\ a_3b_1 & a_4b_1 & a_3b_2 & a_4b_2\\ a_1b_3 & a_2b_3 & a_1b_4 & a_2b_4\\ a_3b_3 & a_4b_3 & a_3b_4 & a_4b_4\end{bmatrix}$$

Let $P=\begin{bmatrix}1 & & & \\ 0 & 0 & 1 & \\ & 1 & 0 & \\ & & & 1\end{bmatrix}$ Hence $P(A\otimes B)=\begin{bmatrix}a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2\\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2\\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4\\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4\end{bmatrix}$

$$(B\otimes A)P=\begin{bmatrix}a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2\\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2\\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4\\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4\end{bmatrix}=P(A\otimes B)$$

Then $P(A\otimes B)P^{-1}=B\otimes A$. $P=\begin{bmatrix}1 & & & \\ 0 & 0 & 1 & \\ & 1 & 0 & \\ & & & 1\end{bmatrix}$

2.2

$I\otimes e^B=\begin{bmatrix}e^B & \\ & e^B\end{bmatrix}$. By definition $f\left(\begin{bmatrix}A_1 & & \\ & \ddots & \\ & & A_k\end{bmatrix}\right)=\begin{bmatrix}f(A_1) & & \\ & \ddots & \\ & & f(A_k)\end{bmatrix}$

Hence $e^{I\otimes B}=e^{\begin{bmatrix}B & \\ & B\end{bmatrix}}=\begin{bmatrix}e^B & \\ & e^B\end{bmatrix}$ Hence $I\otimes e^B=e^{I\otimes B}$

2.2

from 2.1 We have a matrix P satisfies $P(I\otimes A)P^{-1}=A\otimes I$

Hence $e^{A\otimes I}=e^{P(I\otimes A)P^{-1}}=P\cdot e^{I\otimes A}\cdot P^{-1}=P(I\otimes e^A)P^{-1}=e^A\otimes I$

2.3

$e^{A\oplus B}=e^{A\otimes I+I\otimes B}$. $(A\otimes I)(I\otimes B)=(AI)\otimes(IB)=(IA)\otimes(BI)=(I\otimes B)(A\otimes I)$

Since $A\otimes I$ commute with $I\otimes B$

Hence $e^{A\oplus B}=e^{A\otimes I+I\otimes B}=e^{A\otimes I}\cdot e^{I\otimes B}=(e^A\otimes I)(I\otimes e^B)$

$=(e^AI)\otimes(Ie^B)=e^A\otimes e^B$

2.4:

let $A=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}$ $A\otimes I=\begin{bmatrix}a_1 & & a_2 & \\ & a_1 & & a_2\\ a_3 & & a_4 & \\ & a_3 & & a_4\end{bmatrix}$ $I\otimes B=\begin{bmatrix}B & \\ & B\end{bmatrix}$

$\mathrm{trace}(A\oplus B)=\mathrm{trace}(A\otimes I+I\otimes B)=\mathrm{trace}(A\otimes I)+\mathrm{trace}(I\otimes B)=2(a_1+a_2)+2\,\mathrm{trace}(B)$

$=2\,\mathrm{trace}(A)+2\,\mathrm{trace}(B)$

2.5:

① If A is diagonalizable with eigenvalue λ_1, λ_2. A is invertible, so $\lambda_1\lambda_2 \neq 0$

Hence we get an invertible matrix P. s.t. $A = P\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}P^{-1}$

Any non-zero complex number z is equal to e^{μ} for a certain μ

Then suppose $e^{\mu_1} = \lambda_1$. $e^{\mu_2} = \lambda_2$. $\mu_1, \mu_2 \in \mathbb{C}$. Then $A = P\begin{bmatrix}e^{\mu_1} & \\ & e^{\mu_2}\end{bmatrix}P^{-1}$

let $X = P\begin{bmatrix}\mu_1 & \\ & \mu_2\end{bmatrix}P^{-1}$ $e^X = P\begin{bmatrix}e^{\mu_1} & \\ & e^{\mu_2}\end{bmatrix}P^{-1} = A$.

② If A is not diagnizable then A has an eigenvalue λ_3 with algebraic multiplicity 2. A is invertible, hence $\lambda_3 \neq 0$.

$\exists P$. s.t. $A = P\begin{bmatrix}1 & \lambda_3 \\ & 1\end{bmatrix}P^{-1}$. suppose $\lambda_3 = e^{\mu}$. $(\mu_3 \in \mathbb{C})$

Then let $X = P\begin{bmatrix}\frac{1}{\lambda_3} & \\ & 1\end{bmatrix}\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}\begin{bmatrix}\lambda_3 & \\ & 1\end{bmatrix}P^{-1}$ Then $e^X = e^{P\begin{bmatrix}\frac{1}{\lambda_3} & \\ & 1\end{bmatrix}\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}\begin{bmatrix}\lambda_3 & \\ & 1\end{bmatrix}P^{-1}}$ And $P\begin{bmatrix}\frac{1}{\lambda_3} & \\ & 1\end{bmatrix}\cdot\begin{bmatrix}\lambda_3 & \\ & 1\end{bmatrix}P^{-1} = I$

So let $P\begin{bmatrix}\frac{1}{\lambda_3} & \\ & 1\end{bmatrix} = M$

Then $e^X = e^{M\cdot\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}M^{-1}} = M\cdot e^{\begin{bmatrix}\mu & 1 \\ & \mu\end{bmatrix}} M^{-1} = M\begin{bmatrix}e^{\mu} & \frac{e^{\mu}}{1!} \\ & e^{\mu}\end{bmatrix}M^{-1}$

$= P\begin{bmatrix}\frac{1}{\lambda_3} & \\ & 1\end{bmatrix}\begin{bmatrix}\lambda_3 & \lambda_3 \\ & \lambda_3\end{bmatrix}\begin{bmatrix}\lambda_3 & \\ & 1\end{bmatrix}P^{-1} = P\begin{bmatrix}1 & 1 \\ & \lambda_3\end{bmatrix}\begin{bmatrix}\lambda_3 & \\ & 1\end{bmatrix}P^{-1} = P\begin{bmatrix}\lambda_3 & 1 \\ & \lambda_3\end{bmatrix}P^{-1} = A$

2.6.

① If A and B are both invertible then $\exists$ matrix X, Y. $e^X = A$ $e^Y = B$.

from 2.3. $A \otimes B = e^X \otimes e^Y = e^{X \otimes Y}$

$\det(A \otimes B) = \det(e^{X \otimes Y})$. for Any matrix A with eigenvalue $\lambda_1, \cdots \lambda_k$ (counting algebraic multiplicity. e^A has eigenvalue $e^{\lambda_1} \cdots e^{\lambda_k}$

$\det(e^A) = e^{\lambda_1} \cdot e^{\lambda_2} \cdots \lambda_k = e^{\sum_{i=1}^{k}\lambda_i} = e^{\operatorname{trace}(A)}$

$\det(A \otimes B) = \det(e^{X \otimes Y}) = e^{\operatorname{trace}(X \otimes Y)} = e^{2\operatorname{trace}(X) + 2\operatorname{trace}(Y)} = (e^{\operatorname{trace}(X)})^2 \cdot (e^{\operatorname{trace}(Y)})^2$

$= \det(e^X)^2 \cdot \det(e^Y)^2 = \det(A)^2 \cdot \det(B)^2$

4.1 $M(1)=x$ $M(x)=x^2$ $M(x^2)=x^3$ $M(x^3)=x^4=(x^4+ax^3+bx^2+cx+d)-ax^3-bx^2-cx-d$

under basis $[1\ x\ x^2\ x^3]$

A is $\begin{bmatrix} 0 & 0 & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}$ i.e. $M([1\ x\ x^2\ x^3])=[1\ x\ x^2\ x^3]\begin{bmatrix} 0 & 0 & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}$

4.2 For $\forall$ polynomial $r(x), \eta(x)$ $\exists m(x), n(x)$ s.t.

$r(x)=m(x)P(x)+r(x) \bmod P(x)$ $\eta(x)=n(x)\cdot P(x)+\eta(x) \bmod P(x)$

$[r(x)+\eta(x)]=[m(x)+n(x)]P(x)+r(x) \bmod P(x)+\eta(x) \bmod P(x)$

from the requirements. $\deg[r(x) \bmod P(x)+\eta(x) \bmod P(x)]$ is strictly less than 4.

Hence $[r(x)+\eta(x)] \bmod P(x)=r(x) \bmod P(x)+\eta(x) \bmod P(x)$

$r(x)\cdot\eta(x)=[m(x)P(x)+r(x) \bmod P(x)]\cdot[n(x)\cdot P(x)+\eta(x) \bmod P(x)]$

$=[m(x)n(x)+(r(x) \bmod P(x))\cdot n(x)+[\eta(x) \bmod P(x)]m(x)]P(x)+(\eta(x) \bmod P(x))(r(x) \bmod P(x))$

$r(x) \bmod P(x)$ and $\eta(x) \bmod P(x)$ don't contain any factor that can devide $P(x)$

$\therefore [r(x)\cdot\eta(x)] \bmod P(x)=[r(x) \bmod P(x)][\eta(x) \bmod P(x)]$

for $\forall P$. $Mp=xP(x) \bmod q(x)$ $M^kP=M(M^{k-1}P)=x\cdot M^{k-1}P(x) \bmod q(x)$

$=(x \bmod P(x))(M^{k-1}P(x) \bmod q(x))=x\cdot[(x\cdot M^{k-2}P(x) \bmod q(x)) \bmod q(x)]$

$=x\cdot[x\cdot M^{k-2}P(x) \bmod q(x)] \bmod q(x)$ Because $[P(x) \bmod q(x)] \bmod q(x)=P(x) \bmod q(x)$

So $M^kP=x^2\cdot[M^{k-2}P(x) \bmod q(x)]$. By induction, $M^kP(x)=x^k[P(x) \bmod q(x)]$

Suppose M as a operator that sends $P(x)$ to $xP(x)$ mode $q(x)$

Then for any polynomial of M, it's linear combination of some powers of M.

$f(x)=a_0+a_1x+\cdots+a_kx^k$. Then $f(M)P(x)=a_0P(x) \bmod q(x)+a_1xP(x) \bmod q(x)+\cdots+a_kx^kP(x) \bmod q(x)$

$=f(x)P(x) \bmod q(x)$

$\therefore q(M)P(x)=q(x)P(x) \bmod q(x)=0$ for All polynomial of $P(x)$ Thus $q(M)$ sends any polynomial in V to 0. $\Rightarrow q(M)$ is zero map. $q(A)=0$

Also for $\forall$ polynomial $Q(x)$ s.t. $Q(M)=0$, we have $Q(x)P(x) \bmod q(x)=0$ for any $P(x)$ $\therefore Q(x)$ is a multiple of $q(x)$. Then $q(x)$ is minimal polynomial

On the other side, the charateristic polynomial is a kill polynomial, i.e. $P(x)$ is it factor. Note that $q(x)$ has degree 4. and charateristic polynomial has degree 4.

Thus the charateristic polynomial contains no factor more than $q(x)$

And their coeffients of x^4 are both 1.

So they are the same

We wanna prove every eigenvalue of A has at most one Jordan block.
If not so, the suppose eigenvalue λ has more than one Jordan block
The the size of biggest λ-block is strictly smaller than algebraic multipulicity of λ
Then the degree of factor $(x-\lambda)$ in minimal polynomial is strictly less than that of characteristic polynomial, this is contradict with 4.2

Reffering to our lecture notes, so we are done.

Proposition 3.5.4. *Suppose A has a single Jordan block for each eigenvalue. (I.e., all geometric multiplicity are one.) The $AB = BA$ implies that $B = p(A)$ for some polynomial p.*

Proof. Suppose A is a single nilpotent Jordan block. Then $AB = BA$ means entries of B shifted up and entries of B shifted right shall have the same results. Use this and you can show that we must have

$$B = \begin{bmatrix} a_0 & a_1 \dots & a_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_1 \\ & & & a_0 \end{bmatrix} = a_0 I + a_1 A + \cdots + a_{n-1}A^{n-1}.$$ We are good.

iBij

Now suppose A is a single λ Jordan block. Then $A = \lambda I + N$ for a nilpotent Jordan block N. So $AB = BA$ implies that $(\lambda I + N)B = B(\lambda I + N)$, and simplification gives $NB = BN$. So $B = p(N)$ for some polynomial p. Let $q(x) = p(x - \lambda)$, then $B = p(A - \lambda I) = q(A)$.

Now consider the generic case. By changing basis, I assume that A is in Jordan canonical form, say $A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix}$. Now we write B into a block matrix in the same manner, and let the (i,j)-block be B_{ij}. Then $AB = BA$ implies that $A_i B_{ij} = B_{ij} A_j$. But by our assumption, A_i, A_j has no common eigenvalues! Hence the only solution to the Sylvester's equation $A_i X - X A_j = 0$ is zero. So B is block diagonal as well.

A_i and B_i commute $\Rightarrow$ B_i is a polynomial of A_i

So $B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{bmatrix}$, and we have $A_i B_i = B_i A_i$. Since each A_i is a single Jordan block, we see that $B_i = p_i(A_i)$ for some polynomial p_i. So our goal is now the following: we want to find a polynomial $p(x)$ such that $p(A_i) = p_i(A_i)$ for all i.

So we want to find $p(x)$ such that $\boxed{p \equiv p_i \text{ modulus the killing polynomial of } A_i}$. We are done by the lemma below. ☐

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Lemma 3.5.5. *Given polynomials $q_1, \dots, q_k$ and $p_1, \dots, p_k$, there is a polynomial p such that $p \equiv p_i \pmod{q_i}$ for all i. (We can in fact require this polynomial to have degree less than $\sum \deg(p_i)$, and in this case such p is unique.)*

p整除 q_i 之后余 p_i.

4.4

from 4.2 we get that $x(x-1)(x-2)(x-3)$ is A's characteristic polynomial.
So the Jordan-block is $J(A) = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$ If we choose other basis.
it will switch the place of four Jordan form.

4.5 $x^2(x-1)(x-2)$ is minimal polynomial, the size of 0-Jordan block is 2.

$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$ If we choose other basis.
it will switch the place of three Jordan form.