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↳ My girlfriend ^^!

Make it clear and specific

1.3.1

(1) $V_1=[1,0]^T$ $V_2=[i,0]^T$ $V_3=[0,1]^T$ $V_4=[0,i]^T$ $T=[V_1\ V_2\ V_3\ V_4]$ (And T is basis for V)

$\therefore$ Hence $A(V_1)=\begin{bmatrix}1\\0\end{bmatrix}=T\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $A(V_2)=\begin{bmatrix}-i\\1+i\end{bmatrix}=T\cdot\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$ $A(V_3)=\begin{bmatrix}-1\\-1\end{bmatrix}=T\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}$

$A(V_4)=\begin{bmatrix}0\\-i\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$ $\Longrightarrow A(T)=[AV_1\ AV_2\ AV_3\ AV_4]=T\cdot B$

And: $B=\begin{bmatrix}1&0&-1&0\\0&-1&0&0\\0&1&-1&0\\0&1&0&-1\end{bmatrix}$ $|B-\lambda I|=\begin{vmatrix}1-\lambda&0&-1&0\\0&-1-\lambda&0&0\\0&1&-1-\lambda&0\\0&1&0&-1-\lambda\end{vmatrix}=(-1-\lambda)\cdot\begin{vmatrix}1-\lambda&0&-1\\0&-1-\lambda&0\\0&1&-1-\lambda\end{vmatrix}$

$=(-1-\lambda)(1-\lambda)\cdot\begin{vmatrix}1-\lambda&0\\1&1-\lambda\end{vmatrix}=(\lambda-1)(\lambda+1)^3$ $\therefore \lambda_1=1$ (Algebraic multiplicity is 1) $\lambda_2=-1$ (Algebraic multiplicity is 3)

$(B-\lambda_1 I)X=0\Rightarrow\begin{bmatrix}0&0&-1&0\\0&-2&0&0\\0&1&-2&0\\0&1&0&-2\end{bmatrix}X=0\Rightarrow X=k_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ let $X_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$(B-\lambda_2 I)X=0\Rightarrow\begin{bmatrix}2&0&-1&0\\0&0&0&0\\0&1&0&0\\0&1&0&0\end{bmatrix}X=0\Rightarrow X=k_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+k_2\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$

Hence we choose $X_1=\begin{bmatrix}1\\0\\2\\2\end{bmatrix}$ $X_3=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

λ_2 has algebraic multiplicity 3, Hence $\dim N(B-\lambda_1 I)=3$
$k_1X_1+k_2X_3$ ($k_1, k_2\in R$) represents all real vectors in $N(B-\lambda_1 I)$

Then we must have a vector X_2 which suits:
$(B-\lambda_1 I)X_2=k_1X_1+k_2X_3$ (k_1 and k_3 are not all 0)
Let $k_1=1$. And we have

$(B-\lambda_2 I)X_2 = X_1 \Longrightarrow \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} \Rightarrow$ A suitable $X_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$V_1 = \text{span}(X_1, X_2)$ $V_2 = \text{span}(X_3)$ $V_3 = \text{span}(X_4)$

Thus: $R^4 = V_1 \oplus V_2 \oplus V_3$

$(B-\lambda_1 I) X_3 = 0$ $(B-\lambda_1 I) X_2 = X_1$ $(B-\lambda_1 I) X_1 = 0$

So under the basis $\{X_1, X_2, X_3\}$ $B-\lambda_1 I$ has

Jordan form: $\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \Rightarrow B$ has $\begin{bmatrix} -1 & 1 & \\ & -1 & \\ & & -1 \end{bmatrix}$

As the same, under basis $\{X_4\}$, $(B-\lambda_2 I) X_4 = 0$

$B-\lambda_2 I$ has Jordan block $[0] \Rightarrow B$ has $[1]$

So we have $P = [X_1 X_2 X_3 X_4] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ $J = \begin{bmatrix} -1 & 1 & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$

$P^{-1}AP = J$ basis is $\{TX_1, TX_2, TX_3, TX_4\}$

$TX_1 = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix}$ $TX_2 = \begin{bmatrix} 2 \\ -i \end{bmatrix}$ $TX_3 = \begin{bmatrix} 0 \\ i \end{bmatrix}$ $TX_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

1.3.1.2: basis $T=\{X^4, X^3, X^2, X, 1\}$ $A(X^4)=4X^3=T\cdot[0\,4\,0\,0\,0]^T$

$A(X^3)=3X^2=T\cdot[0\,0\,3\,0\,0]^T$ $A(X^2)=2X=T[0\,0\,0\,2\,0]^T$

$A(X)=1+X^2=T[0\,0\,1\,0\,1]^T$ $A(1)=1=T[0\,0\,0\,0\,1]^T$

So $A(T)$

$=[A(X^4), A(X^3), A(X^2), A(X)\ A(1)]$

$=T\cdot B$

$$B=\begin{bmatrix}0&0&0&0&0\\4&0&0&0&0\\0&3&0&1&0\\0&0&2&0&0\\0&0&0&1&1\end{bmatrix}\quad |B-\lambda I|=\begin{vmatrix}-\lambda&0&0&0&0\\4&-\lambda&0&0&0\\0&3&-\lambda&1&0\\0&0&2&-\lambda&0\\0&0&0&1&1-\lambda\end{vmatrix}=(1-\lambda)(\lambda^2-2)\lambda^2$$

$\lambda_1=1$ (Algebraic multiplicity is 1)

$\lambda_2=-\sqrt{2}$ (Algebraic multiplicity is 1)

$\lambda_3=\sqrt{2}$ (Algebraic multiplicity is 1)

$\lambda_4=0$ (Algebraic multiplicity is 2)

for λ_4: $(B-\lambda_4 I)X=0 \Rightarrow \ker(B)=\text{span}([0\,1\,0\,-3\,3]^T)$ Let $X_1=\begin{bmatrix}0\\4\\0\\-12\\12\end{bmatrix}$

The Algebraic multiplicity is 2, so $N_\infty(B)=2$

There must be a vector X_2, which suits:

$BX_2=X_1 \Rightarrow$ a suitable $X_2=[1\,0\,-6\,0\,12]^T$

for λ_3: $(B-\lambda_3 I)X=0 \Rightarrow$ a suitable $X_3=[0\,0\,1\,\sqrt{2}\ \ 2+\sqrt{2}]^T$

for λ_2: $(B-\lambda_4 I)X=0 \Rightarrow$ a suitable $X_4=[0\,0\,-\sqrt{2}-1\ \ 2+\sqrt{2}\ \ -\sqrt{2}]^T$

for λ_1: $(B-\lambda_1 I)X=0 \Rightarrow$ a suitable $X_5=[0\,0\,0\,0\,1]^T$

So Let $\{TX_1, TX_2, TX_3, TX_4, TX_5\}$ as a basis

And $P=[X_1\ X_2\ X_3\ X_4\ X_5]$

$$P^{-1}BP=\begin{bmatrix}0&1&&&\\&0&&&\\&&\sqrt{2}&&\\&&&-\sqrt{2}&\\&&&&1\end{bmatrix}$$

basis is $TX_1=4X^3-12X+12$ $TX_4=(-\sqrt{2}-1)X^2+(2+\sqrt{2})X-\sqrt{2}$

$TX_2=X^4-6X^2+12$ $TX_5=1$

$TX_3=X^2+\sqrt{2}X+2+\sqrt{2}$

1.3.1.3 We discuss these situation

① $a_1a_2a_3a_4\neq 0$

② $a_1a_4=0$ $a_2a_3\neq 0$

$a_1=0$, $a_2a_3a_4\neq 0$

$a_4=0$, $a_1a_2a_3\neq 0$

$a_1=a_4=0$, $a_2a_3\neq 0$

③ $a_2a_3=0$, $a_1a_4\neq 0$

$a_2=0$, $a_1a_3a_4\neq 0$

$a_3=0$, $a_1a_2a_4\neq 0$

$a_2=a_3=0$, $a_1a_4\neq 0$

④ $a_1a_4=a_2a_3=0$

$a_1=a_2=0$ $a_3a_4\neq 0$

$a_1=a_3=0$ $a_2a_4\neq 0$

$a_4=a_2=0$ $a_1a_3\neq 0$

$a_4=a_3=0$ $a_1a_2\neq 0$

⑤ $a_1=a_2=a_4=0$ $a_3\neq 0$

$a_1=a_3=a_4=0$ $a_2\neq 0$

$a_2=a_3=a_4=0$ $a_1\neq 0$

$a_1=a_2=a_3=0$ $a_4\neq 0$

⑥ $a_1=a_2=a_3=a_4=0$

1.3.1.3: $|A-\lambda I| = (\lambda^2 - a_1a_4)(\lambda^2 - a_2a_3)$ $\quad A = \begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$

① If $a_1a_4 \neq 0$ and $a_2a_3 \neq 0$. Then $\lambda_{1,2} = \pm\sqrt{a_1a_4}$ $\lambda_{3,4} = \pm\sqrt{a_2a_3}$
note that a_1a_4 may equal to a_2a_3, but that doesn't matter

eigenvector for λ_1, i.e.: $\begin{bmatrix} -\sqrt{a_1a_4} & & & a_1 \\ & -\sqrt{a_1a_4} & a_2 & \\ & a_3 & -\sqrt{a_1a_4} & \\ a_4 & & & -\sqrt{a_1a_4} \end{bmatrix} X = 0 \Rightarrow$ a suitable $X_1 = \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ \sqrt{a_4} \end{bmatrix}$

Just like λ_1, eigenvector for λ_2: $X_2 = \begin{bmatrix} \sqrt{a_1} \\ 0 \\ 0 \\ -\sqrt{a_4} \end{bmatrix}$

eigenvector for λ_3: $X_3 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ +\sqrt{a_3} \\ 0 \end{bmatrix}$ eigenvector for λ_4; $X_4 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ -\sqrt{a_3} \\ 0 \end{bmatrix}$

under basis: $[X_1\ X_2\ X_3\ X_4]$ Jordan form is $\begin{bmatrix} \sqrt{a_1a_4} & & & \\ & -\sqrt{a_1a_4} & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}$

② If $a_1a_4 = 0$ but $a_2a_3 \neq 0$.

(2.1). $a_1 = 0$, $a_2a_3a_4 \neq 0$, then $A = \begin{bmatrix} & & & 0 \\ & & a_2 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$

eigenvector for $\lambda_1 = 0$: a suitable $X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix}$

$AX_2 = X_1 \Rightarrow$ a suitable $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

eigenvector for $\lambda_{3,4} = \pm\sqrt{a_2a_3}$ is still the same. $X_3 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ \sqrt{a_3} \\ 0 \end{bmatrix}$ $X_4 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ -\sqrt{a_3} \\ 0 \end{bmatrix}$

∴ under basis $\{X_1\ X_2\ X_3\ X_4\}$ Jordan Form $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}$

(2.2) $a_4 = 0$ and $a_1a_2a_3 \neq 0$ $A = \begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & a_3 & & \\ 0 & & & \end{bmatrix}$

eigenvector for $\lambda_{1,2} = 0$ is: a suitable $X_1 = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$AX_2 = X_1$: a suitable $X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

eigenvector for $\lambda_{3,4} = \pm\sqrt{a_2a_3}$ is still the same. $X_3 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ \sqrt{a_3} \\ 0 \end{bmatrix}$ $X_4 = \begin{bmatrix} 0 \\ \sqrt{a_2} \\ -\sqrt{a_3} \\ 0 \end{bmatrix}$

∴ under basis $\{X_1\ X_2\ X_3\ X_4\}$ Jordan Form $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3} \end{bmatrix}$

(2.3) $a_1=a_4=0$. $a_2a_3\neq 0$ $\quad A=\begin{bmatrix} & & & 0\\ & & a_2 & \\ & a_3 & & \\ 0 & & & \end{bmatrix}$

two suitable eigenvector for $\lambda_{1,2}=0$ are $X_{1,2}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ $\quad X_{3,4}$ are still $X_3=\begin{bmatrix}0\\\sqrt{a_2}\\\sqrt{a_3}\\0\end{bmatrix}$ $X_4=\begin{bmatrix}0\\\sqrt{a_2}\\-\sqrt{a_3}\\0\end{bmatrix}$

Jordan block is $\begin{bmatrix}0 & & & \\ & 0 & & \\ & & \sqrt{a_2a_3} & \\ & & & -\sqrt{a_2a_3}\end{bmatrix}$ basis is $\{X_1\ X_2\ X_3\ X_4\}$

(3) $a_2a_3=0$ and $a_1a_4\neq 0$

(3.1) if $a_2=0$ and $a_3a_1a_4\neq 0$ $\quad A=\begin{bmatrix} & & & a_1\\ & & 0 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix}$

a suitable eigenvector for $\lambda_{1,2}=0$ is: $X_1=\begin{bmatrix}0\\0\\a_3\\0\end{bmatrix}$

$AX_2=X_1$ a suitable $X_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

two suitable eigenvectors for $\lambda_{3,4}$ are $X_3=\begin{bmatrix}\sqrt{a_1}\\0\\0\\\sqrt{a_4}\end{bmatrix}$ $X_4=\begin{bmatrix}\sqrt{a_1}\\0\\0\\-\sqrt{a_4}\end{bmatrix}$

basis is $\{X_1 X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix}0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4}\end{bmatrix}$

(3.2) if $a_3=0$ $a_1a_2a_4\neq 0$ $\quad A=\begin{bmatrix} & & & a_1\\ & & a_2 & \\ & 0 & & \\ a_4 & & & \end{bmatrix}$

a suitable eigenvector for $\lambda_{1,2}=0$ is: $X_1=\begin{bmatrix}0\\a_2\\0\\0\end{bmatrix}$

$AX_2=X_1$ a suitable $X_2=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

two suitable eigenvectors for $\lambda_{3,4}$ are $X_3=\begin{bmatrix}\sqrt{a_1}\\0\\0\\\sqrt{a_4}\end{bmatrix}$ $X_4=\begin{bmatrix}\sqrt{a_1}\\0\\0\\-\sqrt{a_4}\end{bmatrix}$

basis is $\{X_1 X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix}0 & 1 & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4}\end{bmatrix}$

(3.3) $a_2=a_3=0$ $\quad a_1a_4\neq 0$

two suitable eigenvector for $\lambda_{1,2}=0$ are $X_{1,2}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $\quad X_{3,4}$ are still

$X_3=\begin{bmatrix}\sqrt{a_1}\\0\\0\\\sqrt{a_4}\end{bmatrix}$ $X_4=\begin{bmatrix}\sqrt{a_1}\\0\\0\\-\sqrt{a_4}\end{bmatrix}$

Jordan block is $\begin{bmatrix}0 & & & \\ & 0 & & \\ & & \sqrt{a_1a_4} & \\ & & & -\sqrt{a_1a_4}\end{bmatrix}$ basis is $\{X_1\ X_2\ X_3\ X_4\}$

④ $a_1 a_4 = a_2 a_3 = 0$

(4.1) $a_1 = a_2 = 0 \quad a_4 a_3 \neq 0 \qquad A = \begin{bmatrix} & & & 0 \\ & & 0 & \\ & a_3 & & \\ a_4 & & & \end{bmatrix} \quad \lambda_{1,2,3,4} = 0$

$AX = 0$: suitable $X_1 = \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix}$

$AX_2 = X_1$: suitable $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad AX_4 = X_3$: suitable $X_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

basis is $\{X_1, X_2, X_3, X_4\}$ Jordan block is: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

(4.2) $a_1 = a_3 = 0 \quad a_2 \cdot a_4 \neq 0 \qquad A = \begin{bmatrix} & & & 0 \\ & & a_2 & \\ & 0 & & \\ a_4 & & & \end{bmatrix} \quad \lambda_{1,2,3,4} = 0$

$AX = 0$: suitable $X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix}$

$AX_2 = X_1$: suitable $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad AX_4 = X_3$: suitable $X_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

basis is $\{X_1, X_2, X_3, X_4\}$ Jordan block is: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

(4.2) $a_4 = a_2 = 0 \quad a_1 \cdot a_3 \neq 0 \qquad A = \begin{bmatrix} & & & a_1 \\ & & 0 & \\ & a_3 & & \\ 0 & & & \end{bmatrix} \quad \lambda_{1,2,3,4} = 0$

$AX = 0$: suitable $X_1 = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix}$

$AX_2 = X_1$: suitable $X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad AX_4 = X_3$: suitable $X_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

basis is $\{X_1, X_2, X_3, X_4\}$ Jordan block is: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

(4.2) $a_4 = a_3 = 0 \quad a_1 a_2 \neq 0 \qquad A = \begin{bmatrix} & & & a_1 \\ & & a_2 & \\ & 0 & & \\ 0 & & & \end{bmatrix} \quad \lambda_{1,2,3,4} = 0$

$AX = 0$: suitable $X_1 = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix}$

$AX_2 = X_1$: suitable $X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad AX_4 = X_3$: suitable $X_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

basis is $\{X_1, X_2, X_3, X_4\}$ Jordan block is: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

(5.1) $a_1 \neq 0 \quad a_2=a_3=a_4=0 \quad \lambda_{1,2,3,4}=0 \qquad A=\begin{bmatrix} & & & a_1\\ & & 0 & \\ & 0 & & \\ 0 & & & \end{bmatrix}$

$AX=0$ three suitable eigenvectors

are $x_1=\begin{bmatrix} a_1\\0\\0\\0\end{bmatrix}$ $x_3=\begin{bmatrix} 0\\1\\0\\0\end{bmatrix}$ $x_4=\begin{bmatrix} 0\\0\\1\\0\end{bmatrix}$

$AX_2=x_1 \therefore$ a suitable $X_2=\begin{bmatrix} 0\\0\\0\\1\end{bmatrix}$

basis: $\{X_1\ X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{bmatrix}$

(5.2) $a_2 \neq 0 \quad a_1=a_3=a_4=0 \quad \lambda_{1,2,3,4}=0 \qquad A=\begin{bmatrix} & & & 0\\ & & a_2 & \\ & 0 & & \\ 0 & & & \end{bmatrix}$

$AX=0$ three suitable eigenvectors

are $x_1=\begin{bmatrix} 0\\a_2\\0\\0\end{bmatrix}$ $x_3=\begin{bmatrix} 1\\0\\0\\0\end{bmatrix}$ $x_4=\begin{bmatrix} 0\\0\\0\\1\end{bmatrix}$

$AX_2=x_1 \therefore$ a suitable $X_2=\begin{bmatrix} 0\\0\\1\\0\end{bmatrix}$

basis: $\{X_1\ X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{bmatrix}$

(5.3) $a_3 \neq 0 \quad a_1=a_2=a_4=0 \quad \lambda_{1,2,3,4}=0 \qquad A=\begin{bmatrix} & & & 0\\ & & 0 & \\ & a_3 & & \\ 0 & & & \end{bmatrix}$

$AX=0$ three suitable eigenvectors

are $x_1=\begin{bmatrix} 0\\0\\a_3\\0\end{bmatrix}$ $x_3=\begin{bmatrix} 0\\0\\0\\1\end{bmatrix}$ $x_4=\begin{bmatrix} 1\\0\\0\\0\end{bmatrix}$

$AX_2=x_1 \therefore$ a suitable $X_2=\begin{bmatrix} 0\\1\\0\\0\end{bmatrix}$

basis: $\{X_1\ X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{bmatrix}$

(5.4) $a_4 \neq 0 \quad a_1=a_2=a_3=0 \quad \lambda_{1,2,3,4}=0 \qquad A=\begin{bmatrix} & & & 0\\ & & 0 & \\ & 0 & & \\ a_4 & & & \end{bmatrix}$

$AX=0$ three suitable eigenvectors

are $x_1=\begin{bmatrix} 0\\0\\0\\a_4\end{bmatrix}$ $x_3=\begin{bmatrix} 0\\1\\0\\0\end{bmatrix}$ $x_4=\begin{bmatrix} 0\\0\\1\\0\end{bmatrix}$

$AX_2=x_1 \therefore$ a suitable $X_2=\begin{bmatrix} 1\\0\\0\\0\end{bmatrix}$

basis: $\{X_1\ X_2\ X_3\ X_4\}$ Jordan form: $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{bmatrix}$

(6) $a_1=a_2=a_3=a_4=0$

eigenvectors are: $\begin{array}{cccc} X_1 & X_2 & X_3 & X_4 \\ \begin{bmatrix} 0\\0\\0\\1\end{bmatrix} & \begin{bmatrix} 0\\0\\1\\0\end{bmatrix} & \begin{bmatrix} 0\\1\\0\\0\end{bmatrix} & \begin{bmatrix} 1\\0\\0\\0\end{bmatrix}\end{array}$

basis: $\{X_1, X_2, X_3, X_4\}$ Jordan form $\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0\end{bmatrix}$

1.3.2.1

They are transposition to each other.

proof is in the next page.

$$
\begin{array}{lllllllllll}
\ker(\sigma)-\ker(\sigma^0) & x_1^{(1)} & \cdots & x_1^{(d_p)} & x_1^{(d_p+1)} & \cdots & x_1^{(d_p+d_{p-1})} & \cdots & x_1^{(t_i-d_1-d_2+1)} \cdots x_1^{(t_i-d_1)} & x_1^{(t_i-d_1+1)} \cdots x_1^{(t_i)} \\
\ker(\sigma^2)-\ker(\sigma) & x_2^{(1)} & \cdots & x_2^{(d_p)} & x_2^{(d_p+1)} & \cdots & x_2^{(d_p+d_{p-1})} & \cdots & x_2^{(t_i-d_1-d_2+1)} \cdots x_2^{(t_i-d_1)} & \\
 & \vdots & & \vdots & \vdots & & \vdots & & & \\
 & x_{p-1}^{(1)} & \cdots & x_{p-1}^{(d_p)} & x_{p-1}^{(d_p+1)} & \cdots & x_{p-1}^{(d_p+d_{p-1})} & & & \\
 & x_p^{(1)} & \cdots & x_p^{(d_p)} & & & & & & \\
 & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{d_p\text{个}p\text{维循环子空间}}} & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{d_{p-1}\text{个}p-1\text{维循环子空间}}} & \cdots & \underbrace{\qquad\qquad}_{d_2\text{个二维循环子空间}} & \underbrace{\qquad\qquad}_{d_1\text{个一维循环子空间}}
\end{array}
\qquad (9.
$$

循环子空间的维数
= Nj的大小

于是

$$t_i = \dim\ker(\sigma-\lambda_i\varepsilon), \qquad (9.$$

且

$$\dim\ker(\sigma-\lambda_i\varepsilon)^2-\dim\ker(\sigma-\lambda_i\varepsilon)=t_i-d_1, \qquad (9.6$$

$$\dim\ker(\sigma-\lambda_i\varepsilon)^3-\dim\ker(\sigma-\lambda_i\varepsilon)^2=t_i-d_1-d_2. \qquad (9.7$$

(9.6)式减去(9.7)式，有

$$d_2=2\dim\ker(\sigma-\lambda_i\varepsilon)^2-\dim\ker(\sigma-\lambda_i\varepsilon)-\dim\ker(\sigma-\lambda_i\varepsilon)^3.$$

推广到一般情形，得到求 j 维循环子空间的数目 d_j 的公式：

$$d_j=2\dim\ker(\sigma-\lambda_i\varepsilon)^j-\dim\ker(\sigma-\lambda_i\varepsilon)^{j-1}-\dim\ker(\sigma-\lambda_i\varepsilon)^{j+1}. \qquad (9.8$$

以上的关系对于矩阵来说，相应地有

$$(\boldsymbol{A}-\lambda_i\boldsymbol{I})\boldsymbol{x}_1^{(j)}=\boldsymbol{0},$$

$$(\boldsymbol{A}-\lambda_i\boldsymbol{I})\boldsymbol{x}_2^{(j)}=\boldsymbol{x}_1^{(j)},$$

$$(\boldsymbol{A}-\lambda_i\boldsymbol{I})^2\boldsymbol{x}_2^{(j)}=\boldsymbol{0},$$

$$\vdots$$

再利用核与值域的维数关系（见第 6 章），以及值域的维数就是矩阵的秩（见第 6 章），就得到

$$d_j=\text{秩}(\boldsymbol{A}-\lambda_i\boldsymbol{I})^{j-1}+\text{秩}(\boldsymbol{A}-\lambda_i\boldsymbol{I})^{j+1}-2\,\text{秩}(\boldsymbol{A}-\lambda_i\boldsymbol{I})^j. \qquad (9.9)$$

一般来说，利用(9.9)式求 d_j 比用(9.8)式要方便些.

综合以上的分析，对于给定的 n 阶方阵 $\boldsymbol{A}$，主对角元为 λ_i 的若尔当块的块数 t_i 就是 $\sigma-\lambda_i\varepsilon$ 的零度((9.5)式)，写成矩阵形式即

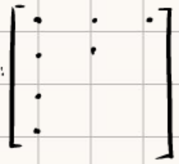
$$t_i=n-\text{秩}(\boldsymbol{A}-\lambda_i\boldsymbol{I}). \qquad (9.10)$$

再利用(9.9)式依次求出各阶若尔当块的块数，(d_j 为主对角元是 λ_i 的 j 阶若尔当块的个数)，就能得到 $\boldsymbol{A}$ 的若尔当标准形.

二阶若尔当方阵

eg: the example 2.2.10 in our class:

$$\left(\begin{array}{l|lll} \ker(A)-\ker(A^0) & A^3V_1 & AV_2 & V_3 \\ \ker(A^2)-\ker(A) & A^2V_1 & V_2 & \\ \ker(A^3)-\ker(A^2) & AV_1 & & \\ \ker(A^4)-\ker(A^3) & V_1 & & \\ & a_1 & a_2 & a_3 \end{array}\right) \Rightarrow \text{for } n=a_1+a_2+a_3 \quad 7=4+2+1$$

ie: 

However, for $n=(\ker(A)-\ker(A^0))+(\ker(A^2)-\ker(A))+(\ker(A^3)-\ker(A^2))+(\ker(A^4)-\ker(A^3))$

we have $7=3+2+1+1$. ie: 

They are clearly transposition to each other.

proof: For a nilpotent matrix: A. $A^m=0$ and $A^{m-1}\neq 0$. (m is an integer)

P_i is the dimention of the i-th column.

dp_i is the number of P_i-dimentional cyclic subspaces. And $P_1=P_2=\cdots=P_{dp_1-1}$

K_i is the dimention of $\ker(A^i)$

Hence we come to a map:

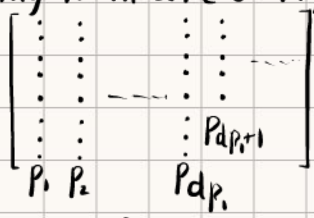
$\ker(A)-\ker(A^0)$	$A^{m-1}X_1 \cdots A^{m-1}X_{dp}$	$A^{m-2}X_{dp+1} \cdots A^{m-2}X_{dp+dp-1}$	----
$\ker(A^2)-\ker(A)$	$A^{m-2}X_1 \cdots A^{m-2}X_{dp}$	$A^{m-3}X_{dp+1} \cdots A^{m-3}X_{dp+dp-1}$	----
$\vdots$	$\cdots$	$\cdots$	----
$\ker(A^{m-1})-\ker(A^{m-2})$	$AX_1\ AX_2 \cdots AX_{dp}$	$X_{dp+1}\ X_{dp+2} \cdots X_{dp+dp-1}$	----
$\ker(A^m)-\ker(A^{m-1})$	$X_1\ X_2 \cdots X_{dp_1}$		

① the i-th row has vector $\{V_1, V_2, \cdots, V_{K_i-K_{i-1}}\}$ as basis for $\ker(A^i)-\ker(A^{i-1})$

② each column is a basis for an p dimensional cyclic subspace

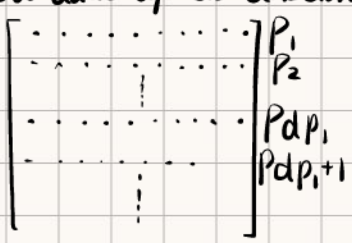
Our jordan block: $n=a_1+a_2+a_3+\cdots+a_K$ is the dimention of these subspaces

ie: $n=P_1+P_2+\cdots+P_K$. (And P_i is decreasing from left to right)

replace every vector with a dot: Hence: 

And note that the i-th line of (*) are a basis of $(\ker(A^i)-\ker(A^{i-1}))$

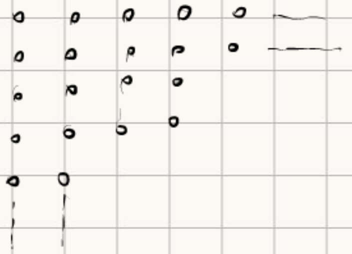
Vectors in the first i row adds up as a basis for $\ker(A^i)$

It gives us a graph: 

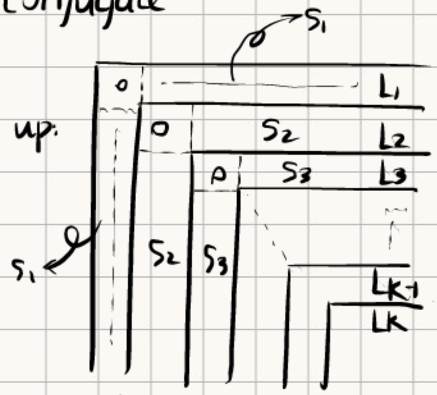
They are clearly transposition.

1.3.2: for every fix int n: a self-conjugate

Then n can be write as:



Then we block it up:



for every V_i ($i=1,2,\cdots,k$): ① the number of each L_i is odd (ie: $2\cdot S_i+1$ is odd)

② if $i>j$, the number of V_i is always less than V_j. (because S_i is decreasing)

③ If two self-conjugate partitions are different, then their distinct odd positive number partition are different

So we know dot number for $V_1 \cdots V_k$ are all distinct odd number.

and every self-conjugate partition has a coordinate distinct odd positive number partition

And. Suppose n has a odd positive number partition

$\partial_1+\partial_2+\cdots+\partial_k$ ($\partial_k < \partial_{k-1} \cdots < \partial_1$)

we can create $L_1, L_2, \cdots, L_k$ with $s_1=\frac{\partial_1-1}{2}$ $\cdots\cdots$ $S_k=\frac{\partial_k-1}{2}$

Every odd positive number partition implies a self-conjugate partition

Then we get a bijection.

So the number of odd positive number partition is equal to the number of self-conjugate partition

1.3.3. A is likely to be $\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{bmatrix}$

① For $4=3+1$. It implies. $\ker(A)=2$. $\ker(A^2)=3$ $\ker(A^3)=4$

$\operatorname{rank}(A)=2$ $\operatorname{rank}(A^2)=1$ $\operatorname{rank}(A^3)=0$

② For $4=2+1+1$. $\ker(A)=3$ $\ker(A^2)=4$

Hence $\operatorname{rank}(A)=1$ $\operatorname{rank}(A^2)=0$

③ For $4=1+1+1+1$. $\ker(A)=4$.

Hence $\operatorname{rank}(A)=0$

For ②. ③. ④. There must be at least one a_{ij} equal to 0.

But $P(a_{ij}=0)=0$ (The posibility of choosing a specific number from a interval is 0, though it sometimes occurs) So $P(②)=P(③)=P(①)=0$

For $4=4$. It implies that $\ker(A)=1$. $\ker(A^2)=2$ $\ker(A^3)=3$ $\ker(A^4)=4$

We know that $1=P(①)+P(②)+P(③)+P(④)$ (We only have these possibilities) So $P(④)=1-P(①)-P(②)-P(③)=1$