HW for 3.25

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Ex. 1.3.1

(1) $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\i\end{pmatrix} \right\}^T$ = basis T

$\forall \vec{v} \in \mathbb{C}^2$, $\vec{v} = \vec{e}^T \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$A\vec{v} = A\vec{e}^T \begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \vec{e}^T \begin{pmatrix}1&0&-1&0\\0&-1&0&0\\0&-1&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$|B-\lambda I| = \begin{vmatrix} 1-\lambda & & -1 & \\ & -1-\lambda & 0 & \\ & -1 & -1-\lambda & \\ & 1 & & -1-\lambda \end{vmatrix}$

$= (1-\lambda)\cdot(-\lambda-1)^3 = (1+\lambda)^3(\lambda-1)$

$\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\lambda_4 = 1$

$B+I = \begin{pmatrix}2&0&-1&0\\0&0&0&0\\0&-1&0&0\\0&1&0&0\end{pmatrix}$ $(B+I)^2 = \begin{pmatrix}4&1&-2&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \Rightarrow$ Ker: $\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \\ \begin{pmatrix}-\frac{1}{4}\\1\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2}\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\end{array}$

$(B+I)(v_1 v_2 v_3) = \begin{pmatrix}-\frac{1}{2}&0&0\\0&0&0\\-1&0&0\\1&0&0\end{pmatrix}$

A has block form $\begin{pmatrix}-1&1&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}$

2. Let basis $T = \{x^4, x^3, x^2, x, 1\}$

$A(x^4) = 4x^3 = T\begin{pmatrix}0\\4\\0\\0\\0\end{pmatrix}$

$A(x^3) = 3x^2 = T\begin{pmatrix}0\\0\\3\\0\\0\end{pmatrix}$

$A(x^2) = 2x = T\begin{pmatrix}0\\0\\0\\2\\0\end{pmatrix}$

$A(x) = 1 = T\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$

$A(1) = \vec{0}$

So $AT = TB$

$= T\begin{pmatrix}0&&&&\\4&0&&&\\&3&0&&\\&&2&0&\\&&&1&0\end{pmatrix}$

$|B-\lambda I| = \begin{vmatrix}-\lambda&&&&\\4&-\lambda&&&\\&3&-\lambda&&\\&&2&-\lambda&\\&&&1&-\lambda\end{vmatrix}$

$= \lambda^2(1-\lambda)(\lambda-\sqrt{2})(\lambda+\sqrt{2})$

for $\lambda=0$. $B\vec{e_5}=0$. $B^2\vec{e_4} = B\vec{e_5} = 0$

B's block form: $\begin{pmatrix}1&&&&\\&\sqrt{2}&&&\\&&-\sqrt{2}&&\\&&&0&1\\&&&&0\end{pmatrix}$

3. $|A-\lambda I| = \begin{vmatrix}-\lambda&&&a_1\\&-\lambda&a_2&\\&a_3&-\lambda&\\a_4&&&-\lambda\end{vmatrix}$

$= (\lambda^2 - a_1a_4)(\lambda^2 - a_2a_3) = 0$

$\pm\sqrt{a_1a_2}$ $\pm\sqrt{a_3a_4}$ are eigenvalues of A

no zero

① if $a_1 \dots a_4 \neq 0$ $A \to \begin{pmatrix}\sqrt{a_1a_4}&&&\\&\sqrt{a_1a_4}&&\\&&\sqrt{a_2a_3}&\\&&&-\sqrt{a_2a_3}\end{pmatrix}$

one zero

② if $a_2=0$. $a_3a_1a_4 \neq 0$ then A is like $\begin{pmatrix}&&&a_1\\&&0&\\&a_3&&\\a_4&&&\end{pmatrix}$

eigenvalues: $\pm\sqrt{a_1a_4}$, 0

A send $\vec{e_3}$ to $\vec{0}$

send $\vec{e_2}$ to $\vec{e_3}$

So A has block form: $\begin{pmatrix}\sqrt{a_1a_4}&&&\\&-\sqrt{a_1a_4}&&\\&&0&1\\&&&0\end{pmatrix}$

* $a_3=0$, $a_2a_1a_4 \neq 0$ has similar situation.

* $a_1=0$ $a_2a_3a_4 \neq 0$, $a_4=0$ $a_1a_2a_3 \neq 0$ has block form $\begin{pmatrix}\sqrt{a_2a_3}&&&\\&-\sqrt{a_2a_3}&&\\&&0&1\\&&&0\end{pmatrix}$

two zero

③ if $a_2=a_3=0$, $a_1a_4 \neq 0$

A is like $\begin{pmatrix}&&&a_4\\&0&&\\&&0&\\a_1&&&\end{pmatrix}$ $\lambda = \pm\sqrt{a_1a_4}$,

$\lambda_3 = \lambda_4 = 0$. have eigenvectors $\vec{e_2}, \vec{e_3}$

A has block form $\begin{pmatrix}\sqrt{a_1a_4}&&&\\&-\sqrt{a_1a_4}&&\\&&0&\\&&&0\end{pmatrix}$

* similarly, $a_1=a_4=0$. $a_2a_3 \neq 0$ has $\begin{pmatrix}\sqrt{a_2a_3}&&&\\&-\sqrt{a_2a_3}&&\\&&0&\\&&&0\end{pmatrix}$

⑥ if $a_1=a_2=0$. $a_3a_4 \neq 0$

A $\begin{pmatrix}&&&0\\&&0&\\&a_3&&\\a_4&&&\end{pmatrix}$ $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$

eigenvectors: $\vec{e_3}, \vec{e_4}$

$A\vec{e_1} = e_4$, $A\vec{e_2} = \vec{e_3}$

A has block form: $\begin{pmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{pmatrix}$

* similarly. $a_1=a_3=0$, $a_2a_4 \neq 0$

$a_2=a_4=0$ $a_1a_3 \neq 0$ have block form $\begin{pmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{pmatrix}$

$a_3=a_4=0$ $a_1a_2 \neq 0$

three zero

④ $a_1=a_2=a_3=0$, $a_4 \neq 0$

A $\begin{pmatrix}&&&0\\&&0&\\&0&&\\a_4&&&\end{pmatrix}$ eigenvalues are all 0

eigenvectors: $\vec{e_2}, \vec{e_3}, \vec{e_4}$

$A\vec{e_1} = \vec{e_4}$

A has block form: $\begin{pmatrix}0&&&\\&0&&\\&&0&1\\&&&0\end{pmatrix}$

* $a_2 \neq 0$, $a_3 \neq 0$, $a_1 \neq 0$ have same block form.

⑤ all zero A $\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}$ eigenvalues are all 0

eigenvectors: $\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}$

A has block form: $\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}$

Ex. 1.3.2

1. the 2 dot diagram is identical.

$A = \begin{bmatrix} A^{k}v_k & \cdots & Av_k & v_k \end{bmatrix}$ $\begin{bmatrix} J_{a_1} & \\ & J_{a_2} \end{bmatrix}$

$A^k v_k = 0$, $\dim = k$

$\dim Ker(A) = K$. $\dim Ker(A^2)$ = the amount of blocks $j \times j$ where $j \geq 2$

suppose the diagram of dig (, , ,) is

← these vectors can be killed by 1 operation → dim Ker(A)

← these vectors can be killed by at least 2 operations → dim Ker(A^2)

$k = 6$ $n = 6+4+3+3+2+1 = 19$

then the diagram of $\sum$ dim =

the second diagram is the transpose of the first one.

2. the diagram is like (symmetric)

considering the dots either on the 1st column / row (or both)

suppose the ith row/column has a_i dots.

then the sum of dots will be $2a_i - 1$, which is an odd number.

notice that when the sum $2a_1-1, \dots, 2a_k-1$ are determined, the diagram is determined. the reverse statement is also correct.

$\sum_{i=1}^{k} (2a_i - 1) = n$. so one partition of n into distinct odd number corresponds to one self-conjugate partition.

3. A is upper triangular so the eigenvalues are on its diagonal.

A is nilpotent so all $\lambda = 0$.

A is like $\begin{bmatrix}0&a_{12}&a_{13}&a_{14}\\&0&a_{23}&a_{24}\\&&0&a_{34}\\&&&0\end{bmatrix}$

$A\vec{e_1} = 0$, $A^2\vec{e_2} = A(a_{12}\vec{e_1}) = \vec{0}$

$A^3\vec{e_3} = A^2(a_{13}\vec{e_1} + a_{23}\vec{e_2}) = 0 + \vec{0} = 0$

$A^4\vec{e_4} = A^3(a_{14}\vec{e_1} + a_{24}\vec{e_2} + a_{34}\vec{e_3}) = 0$.

so the block form of A must be $\begin{bmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{bmatrix}$