第四周习题课　　隐函数（续），空间曲线与曲面

一．隐函数的二阶（偏）导数

例 1.　　设 $z=f\left(x,\varphi\left(x^2,x^2\right)\right)$，其中函数 f 于 φ 的二阶偏导数连续，求 $\dfrac{d^2z}{dx^2}$

解：$z=f(u,v)$，其中 $\begin{cases}u=x\\ v=\varphi(s,t)\end{cases}$，$\begin{cases}s=x^2\\ t=x^2\end{cases}$。

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}(u,v)\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}(u,v)\cdot\frac{dv}{dx}=\frac{\partial f}{\partial u}(u,v)\cdot 1+\frac{\partial f}{\partial v}(u,v)\cdot\frac{dv}{dx}$$

而 $\dfrac{dv}{dx}=\dfrac{\partial\varphi}{\partial s}(s,t)\cdot\dfrac{ds}{dx}+\dfrac{\partial\varphi}{\partial t}(s,t)\cdot\dfrac{dt}{dx}=\dfrac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\dfrac{\partial\varphi}{\partial t}(s,t)\cdot 2x$。所以

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}(u,v)+\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$，其中 $\begin{cases}u=x\\ v=\varphi(s,t)\end{cases}$，$\begin{cases}s=x^2\\ t=x^2\end{cases}$。

$$\frac{d^2z}{dx^2}=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)+\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)\right\}+\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}=I+II$$ 。

$$I=\frac{d}{dx}\left\{\frac{\partial f}{\partial u}(u,v)\right\}=\frac{\partial^2 f}{\partial u^2}\cdot\frac{du}{dx}+\frac{\partial^2 f}{\partial u\partial v}\cdot\frac{dv}{dx}$$

$$=\frac{\partial^2 f}{\partial u^2}\cdot 1+\frac{\partial^2 f}{\partial u\partial v}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$。

$$II=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$+\frac{\partial f}{\partial v}(u,v)\cdot\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=II_1+II_2$$

$$II_1=\frac{d}{dx}\left\{\frac{\partial f}{\partial v}(u,v)\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=\left[\frac{\partial^2 f}{\partial v\partial u}\cdot\frac{du}{dx}+\frac{\partial^2 f}{\partial v^2}\cdot\frac{dv}{dx}\right]\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$ ；

$$=\left\{\frac{\partial^2 f}{\partial v\partial u}\cdot 1+\frac{\partial^2 f}{\partial v^2}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}\cdot\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$II_2=\frac{\partial f}{\partial v}(u,v)\cdot\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\cdot 2x\right]+\frac{d}{dx}\left[\frac{\partial\varphi}{\partial t}(s,t)\cdot 2x\right]\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\frac{d}{dx}\left[\frac{\partial\varphi}{\partial s}(s,t)\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2+\frac{d}{dx}\left[\frac{\partial\varphi}{\partial t}(s,t)\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\begin{aligned}&\left[\frac{\partial^2\varphi}{\partial s^2}\cdot\frac{ds}{dx}+\frac{\partial^2\varphi}{\partial t\partial s}\cdot\frac{dt}{dx}\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2\\&+\left[\frac{\partial^2\varphi}{\partial s\partial t}\cdot\frac{ds}{dx}+\frac{\partial^2\varphi}{\partial t^2}\cdot\frac{dt}{dx}\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\end{aligned}\right\}$$

$$=\frac{\partial f}{\partial v}(u,v)\cdot\left\{\begin{aligned}&\left[\frac{\partial^2\varphi}{\partial s^2}\cdot 2x+\frac{\partial^2\varphi}{\partial t\partial s}\cdot 2x\right]\cdot 2x+\frac{\partial\varphi}{\partial s}(s,t)\cdot 2\\&+\left[\frac{\partial^2\varphi}{\partial s\partial t}\cdot 2x+\frac{\partial^2\varphi}{\partial t^2}\cdot 2x\right]\cdot 2x+\frac{\partial\varphi}{\partial t}(s,t)\cdot 2\end{aligned}\right\}。$$

代入即可。

例 2. 设 $z=z(x,y)$ 二阶连续可微，并且满足方程

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=0$$

若令 $\begin{cases}u=x+\alpha y\\ v=x+\beta y\end{cases}$，试确定 α,β 为何值时能变原方程为 $\dfrac{\partial^2 z}{\partial u\partial v}=0$.

解　将 x,y 看成自变量，u,v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha^2\frac{\partial^2 z}{\partial u^2}+2\alpha\beta\frac{\partial^2 z}{\partial u\partial v}+\beta^2\frac{\partial^2 z}{\partial v^2}=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)^2 z$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha\frac{\partial^2 z}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 z}{\partial u\partial v}+\beta\frac{\partial^2 z}{\partial v^2}$$

$$=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)z$$

由此可得，$0=A\dfrac{\partial^2 z}{\partial x^2}+2B\dfrac{\partial^2 z}{\partial x\partial y}+C\dfrac{\partial^2 z}{\partial y^2}=$

$$=\left(A+2B\alpha+C\alpha^2\right)\frac{\partial^2 z}{\partial u^2}+2\left(A+B(\alpha+\beta)+C\alpha\beta\right)\frac{\partial^2 z}{\partial u\partial v}+\left(A+2B\beta+C\beta^2\right)\frac{\partial^2 z}{\partial v^2}=0$$

只要选取 α,β 使得 $\begin{cases}A+2B\alpha+C\alpha^2=0\\ A+2B\beta+C\beta^2=0\end{cases}$，可得 $\dfrac{\partial^2 z}{\partial u\partial v}=0$.

问题成为方程 $A+2B\,t+C\,t^2=0$ 有两不同实根，即要求：$B^2-AC>0$.

令 $\alpha=-B+\sqrt{B^2-AC}$，$\beta=-B-\sqrt{B^2-AC}$，即可。

此时，$\dfrac{\partial^2 z}{\partial u \partial v}=0 \Rightarrow \dfrac{\partial^2 z}{\partial u \partial v}=0 \Rightarrow \dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right)=0 \Rightarrow \dfrac{\partial z}{\partial v}=\varphi(v) \Rightarrow z=\int \varphi(v) dv+f(u)$.

$$z=f(u)+g(v)=f(x+\alpha y)+g(x+\beta y).$$

例 3. 设$u(x,y)\in C^2$，又$\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}=0$，$u(x,2x)=x$，$u'_x(x,2x)=x^2$，求 $u''_{xx}(x,2x)$，

$u''_{xy}(x,2x)$ $u''_{yy}(x,2x)$

解:
$$\frac{\partial u}{\partial x}(x,2x)=x^2,$$

两边对 x 求导,

$$\frac{\partial^2 u}{\partial x^2}(x,2x)+\frac{\partial^2 u}{\partial x \partial y}(x,2x)\cdot 2=2x. \qquad (1)$$

$$u(x,2x)=x,$$

两边对 x 求导,

$$\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1, \qquad \frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}.$$

两再边对 x 求导,

$$\frac{\partial^2 u}{\partial x \partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知
$$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

(1), (2), (3) 联立可解得:

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x, \quad \frac{\partial^2 u}{\partial x \partial y}(x,2x)=\frac{5}{3}x$$

二、向量函数的微分和导数

1． 计算极坐标、柱坐标、球坐标变换的 Jacobi 矩阵和 Jacobi 行列式：

（1）平面极坐标变换 $\vec{\mathbf{f}}(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$，也即$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$；

（2）空间柱坐标变换$\vec{\mathbf{f}}(r,\theta,z)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$，也即$\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=z \end{cases}$；

（3）空间球坐标变换$\vec{\mathbf{f}}(r,\varphi,\theta)=\begin{pmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}$，也即$\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$。

2. 计算向量复合函数的 Jacobi 矩阵：

（1）$\mathbf{f}(x,y)=(x,y,x^2y)$， $x=s+t$， $y=s^2-t^2$，在$s=2,t=1$；

（2）$\mathbf{f}(x,y,z)=(x^2+y+z,2x+y+z^2,0)$， $x=uv^2w^2, y=w^2\sin v, z=u^2e^v$。

三、切平面,切线,法平面,法线

[例 1] 求曲线 L : $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=2x \end{cases}$

在点 $M_0(1,1,\sqrt{2})$处的切线和法平面方程

[例2] 设函数f可微, 求证：曲面 $S: z=yf(\dfrac{x}{y})$ 的

所有切平面相交于一个公共点。

[例3] 过直线 $10x+2y-2z=27,\ \ x+y-z=0$作曲面 $3x^2+y^2-z^2=27$的切平面, 求其方程.

[例4] 求证满足微分方程 $y\dfrac{\partial u}{\partial x}+x\dfrac{\partial u}{\partial y}=0$ 的$u(x,y)$ 为$u(x,y)=f(x^2-y^2)$, 其中, f为任意一元可微函数.