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1.1 Let e_1^T and e_2^T be the basis of $(\mathbb{R}^2)^*$

$\alpha = a_1 e_1^T + a_2 e_2^T \quad \beta = b_1 e_1^T + b_2 e_2^T \quad r = c_1 e_1^T + c_2 e_2^T$

Since $L(\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$ is a multilnear space

Hence $\alpha \otimes \beta \otimes r = a_1 b_1 c_1 e_1^T \otimes e_1^T \otimes e_1^T + a_1 b_1 c_2 e_1^T \otimes e_1^T \otimes e_2^T$
$+ a_1 b_2 c_1 e_1^T \otimes e_2^T \otimes e_1^T + a_1 b_2 c_2 e_1^T \otimes e_2^T \otimes e_2^T + a_2 b_1 c_1 e_2^T \otimes e_1^T \otimes e_1^T$
$+ a_2 b_1 c_2 e_2^T \otimes e_1^T \otimes e_2^T + a_2 b_2 c_1 e_2^T \otimes e_2^T \otimes e_1^T + a_2 b_2 c_2 e_2^T \otimes e_2^T \otimes e_2^T$

1.3 Let $M = \sum_{i=1}^{r} m_i (\alpha_i^T \otimes \beta_i^T \otimes r_i^T)$, where $\alpha_i^T \otimes \beta_i^T \otimes r_i^T$ are different rank 1 tensor and $m_i \in \mathbb{R}$ and $m_i \neq 0$

from 1.2, we know if $\alpha \otimes \beta \otimes r$ is rank 1, then $M_E(\alpha \otimes \beta \otimes c)$ is also rank 1

$M_E(M) = \sum_{i=1}^{r} m_i ((\alpha_i^T E) \otimes \beta_i^T \otimes r_i^T)$ Then $M_E(M)$ is decomposed into r rank 1 tensors. But these tensors may not be completely distinct, hence

$\mathrm{Rank}(M_E(M)) \le \mathrm{Rank}(M)$

So let $M_E(M) = N = \sum_{i=1}^{r} n_i a_i^T \otimes b_i^T \otimes c_i^T$, where $\alpha_i^T \otimes \beta_i^T \otimes r_i^T$ are different rank 1 tensor, and $n_i \in \mathbb{R}$ and $n_i \neq 0$

But we know M_E is bijection, hence M_E is invertible. So let $(M_E)^{-1} = M_{E^{-1}}$. Hence we know if $\alpha \otimes \beta \otimes r$ is rank 1, then $M_{E^{-1}}(\alpha \otimes \beta \otimes c)$ is also rank 1

So $M = M_{E^{-1}}(N) = \sum_{i=1}^{r} n_i (a_i^T E^{-1}) \otimes b_i^T \otimes c_i^T$ So M is decomposed into r rank 1 tensors. But these tensors may not be completely distinct, hence

$\mathrm{Rank}(M) \le \mathrm{Rank}(N) = \mathrm{Rank}(M_E(M))$

Then from all above. $\mathrm{Rank}(M) = \mathrm{Rank}(M_E(M))$

i.e. the elementary layer operation peseve rank.

1.4. without lose of genrality, let's suppose the i-th layer of $M \in ((\mathbb{R}^2)^*)^{\otimes 3}$ has rank r

i.e. $M(-,-,e_i) \in (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)^*$ have rank r. let $\mathrm{Rank}(M) = R$.

Then $M = \sum_{i=1}^{R} \alpha^T \otimes \beta^T \otimes r^T$. feed M the e_i vector.

$M(e_i) = \sum_{i=1}^{R} r^T(e_i)\, \alpha^T \otimes \beta^T$ $M(-,-,e_i)$ is rank r. r is the smallest possible integer such that $M(e_i)$ can be written as the linear combination of r rank one tensor.

And $M(e_i)$ can also be written as the linear combination of R rank one tensor.

$\therefore r \le R$

1.2. We define $M_E: ((\mathbb{R}^2)^*)^{\otimes 3} \longrightarrow ((\mathbb{R}^2)^*)^{\otimes 3}$

$\forall\ \alpha\otimes\beta\otimes\gamma \in ((\mathbb{R}^2)^*)^{\otimes 3}$, $M_E(\alpha\otimes\beta\otimes\gamma) = (\alpha E)\otimes\beta\otimes\gamma$

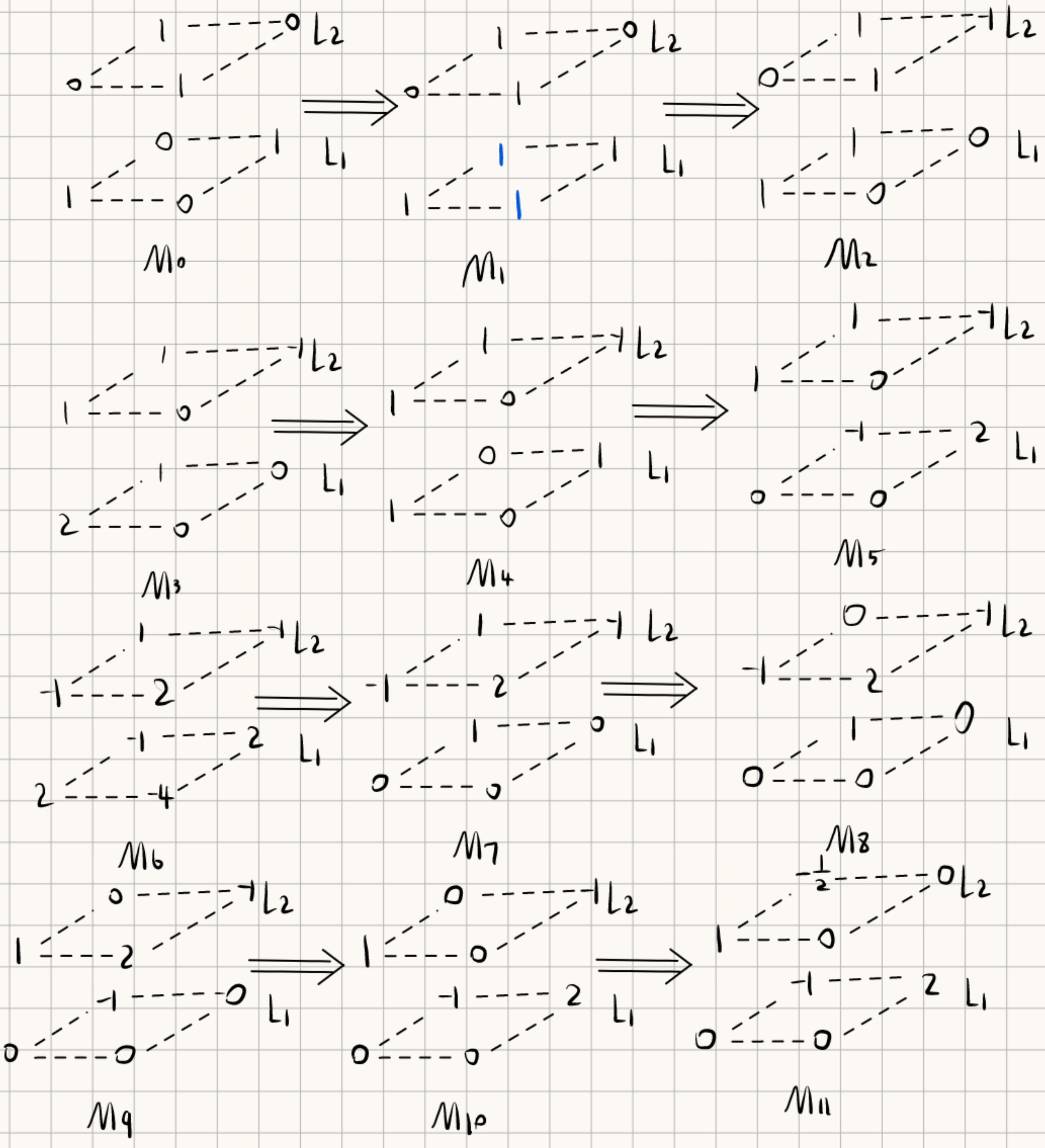
Since All elements of $((\mathbb{R}^2)^*)^{\otimes 3}$ are a linear combination of rank 1 tensors $\alpha\otimes\beta\otimes\gamma$ for some $\alpha, \beta, \gamma \in (\mathbb{R}^2)^*$

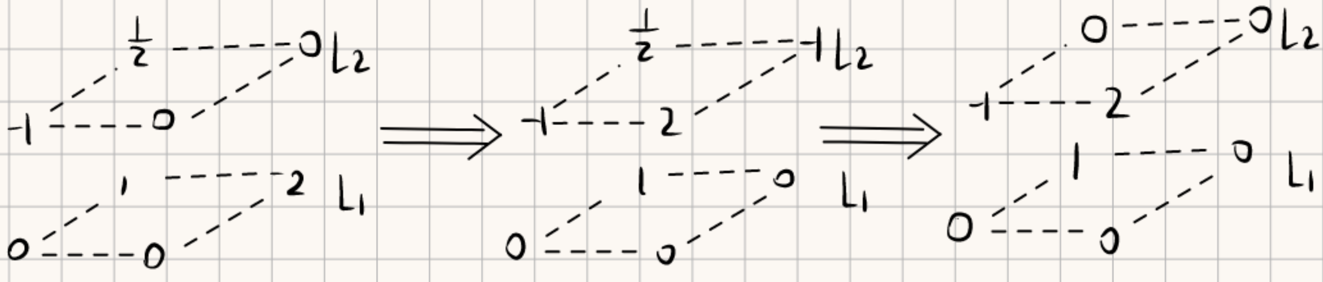
Hence $M \in ((\mathbb{R}^2)^*)^{\otimes 3}$, $M = \sum_{i=1}^{k} a_i\, \alpha_i\otimes\beta_i\otimes\gamma_i$ for some $a_1 \cdots a_k \in \mathbb{R}$.

let $M_E(M) = \sum_{i=1}^{k} a_i\, (\alpha_i E)\otimes\beta_i\otimes\gamma_i$

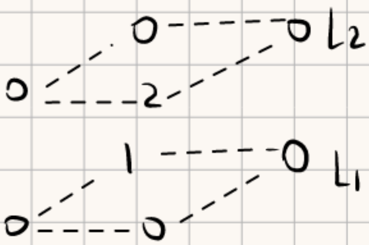
M_E is a linear map from $((\mathbb{R}^2)^*)^{\otimes 3}$ to $((\mathbb{R}^2)^*)^{\otimes 3}$

1.5. Eelementary layer operation perserves rank. Hence:





M_{12} M_{13} M_{14}



M_{15}

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have rank 2. hence from 1.4, we know that our tensor has rank ≥ 2.

$$\text{Rank}(M_{15}) = \text{Rank}(M) \geq 2$$

$$M_{15} = 1\, e_1^T \otimes e_1^T \otimes e_1^T + 2\, e_2^T \otimes e_2^T \otimes e_2^T$$

$$\therefore \text{Rank}(M) = \text{Rank}(M_{15}) = 2$$

2.1 $M=\sum_{i,j,k=1}^{i,j,k=3}(i+j+k)\,e_i^T\otimes e_j^T\otimes e_k^T$

$$\begin{array}{ccc} \text{layer1} & \text{layer2} & \text{layer3} \\ \begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix} & \begin{bmatrix}4&5&6\\5&6&7\\6&7&8\end{bmatrix} & \begin{bmatrix}5&6&7\\6&7&8\\7&8&9\end{bmatrix} \\ \parallel & \parallel & \parallel \\ A_1 & A_2 & A_3 \end{array}$$

$$V^TA_1V=[x\ y\ z]\begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=[3x+4y+5z\quad 4x+5y+6z\quad 5x+6y+7z]\begin{bmatrix}x\\y\\z\end{bmatrix}=3x^2+8xy+10xz+5y^2+7z^2+12yz$$

$$V^TA_2V=[x\ y\ z]\begin{bmatrix}4&5&6\\5&6&7\\6&7&8\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=[4x+5y+6z\quad 5x+6y+7z\quad 6x+7y+8z]\begin{bmatrix}x\\y\\z\end{bmatrix}=4x^2+10xy+12xz+6y^2+8z^2+14yz$$

$$V^TA_2V=[x\ y\ z]\begin{bmatrix}5&6&7\\6&7&8\\7&8&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=[5x+6y+7z\quad 6x+7y+8z\quad 7x+8y+9z]\begin{bmatrix}x\\y\\z\end{bmatrix}=5x^2+12xy+14xz+7y^2+9z^2+16yz$$

$$M(V,V,V)=[V^TA_1V\quad V^TA_2V\quad V^TA_3V]\cdot V$$

$$=[3x^2+8xy+10xz+5y^2+7z^2+12yz\quad 4x^2+10xy+12xz+6y^2+8z^2+14yz\quad 5x^2+12xy+14xz+7y^2+9z^2+16yz]\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$=3x^3+8x^2y+10x^2z+5xy^2+7xz^2+12xyz+4x^2y+10xy^2+12xyz+6y^3+8yz^2+14y^2z+5x^2z$$
$$+14xz^2+12xyz+16yz^2+7y^2z+9z^3$$
$$=3x^3+12x^2y+15x^2z+15xy^2+21xz^2+36xyz+24yz^2+21y^2z+21y^2z+9z^3+6y^2$$
$$=3x^2+6y^2+9z^2+12x^2y+21y^2z+21z^2x+15xy^2+24yz^2+15zx^2+36xyz$$

2.2

Let $P:\{1,2,3\}\to\{1,2,3\}$ $P(1)=i,\ P(2)=j,\ P(3)=k.$ $i\neq j\neq k.$ $i,j,k\in\{1,2,3\}$
$\quad\ \sigma:\{1,2,3\}\to\{1,2,3\}$ $\sigma(1)=m,\ \sigma(2)=n,\ \sigma(3)=q.$ $m\neq n\neq q.$ $m,n,q\in\{1,2,3\}$

Hence $M^\sigma(e_i,e_j,e_k)=M^\sigma(e_{p(1)},e_{p(2)},e_{p(3)})=M(e_{p(\sigma(1))}\,e_{p(\sigma(2))}\,e_{p(\sigma(3))})$
$=p(\sigma(1))+p(\sigma(2))+p(\sigma(3))=P(m)+P(n)+P(q).=P(1)+P(2)+P(3)$ (in certain order)
$=i+j+k.$ Hence M and M^σ have the same entry. i.e. $M=M^\sigma$

2.3

$$\begin{array}{c}\text{layer1}\\ \begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}\\ \parallel \\ A_1\end{array}\longrightarrow\begin{bmatrix}3&4&5\\4&5&6\\8&10&12\end{bmatrix}\longrightarrow\begin{bmatrix}3&4&5\\4&5&6\\0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}3&4&5\\1&1&1\\0&0&0\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}0&1&2\\1&1&1\\&&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&2\\0&1&2\\0&0&0\end{bmatrix}$$ Hence $\mathrm{Rank}(A_1)=2.$

from 1.4. $\mathrm{Rank}(M)\geqslant\mathrm{Rank}(A_2)=2$ And let $A=\alpha_1\beta_1^T+\alpha_2\beta_2^T$

$r=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\delta=\begin{bmatrix}0\\1\\2\end{bmatrix}$ Hence we know $M=(\alpha_1^T\otimes\beta_1^T+\alpha_2^T\otimes\beta_2^T)\otimes r^T+r^T\otimes r^T\otimes\delta^T$
$=\alpha_1^T\otimes\beta_1^T\otimes r^T+\alpha_2^T\otimes\beta_2^T\otimes r^T+r^T\otimes r^T\otimes\delta^T$

Hence Rank $M\leq 3$

4. Trace map $\mathbb{R}^n \otimes (\mathbb{R}^n)^* \to \mathbb{R}$. Trace is an element of $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$

i.e feed it a $n\times n$ matrix it will give you a number

Basis for $\mathbb{R}^n \otimes (\mathbb{R}^n)^*$ is all $e_i \otimes e_j^T$. entries in the matrix is simply the cordinates under the standard basis.

for example. $a_{ij}\ e_i \otimes e_j^T$ means the coefficient in the entry in the (i,j) Location of Matrix M

Any $n\times n$ matrix M in the space of $\mathbb{R}^n \otimes (\mathbb{R}^n)^*$, it is a linear combination of the tensor basis, i.e $e_i \otimes e_j^T$. $(i,j=1,\cdots,n)$

let $M = \sum_{i,j=1}^{n} a_{ij} e_i \otimes e_j^T$, under the stander basis, M is: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$

Similarly, $e_i^T \otimes e_j$ forms the basis of $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. $e_i^T \otimes e_j$ means taking the entry in the (i,j) location of input matrix M.

By definition of trace map, trace $M = \sum_{i=1}^{n} a_{ii}$. for All $M_{n\times n}$.

i.e trace map takes the entry on the diagonal of matrix M.

3.1: For $\forall v_1, v_2 \in$, $w_1, w_2 \in W$, $\mu, \eta \in \mathbb{R}$

Then $(X\otimes Y)(\mu v_1 + \eta v_2, w_1) = X(\mu v_1 + \eta v_2) \otimes Y w_1 = (\mu X v_1 + \eta X v_2) \otimes Y w_1$

Kronecker product is linear. this equals to $\mu X v_1 \otimes Y w_1 + \eta X v_2 \otimes Y w_1$

$= \mu (X\otimes Y)(v_1, w_1) + \eta (X \otimes Y)(v_2, w_1)$

Similarly

$(X\otimes Y)(v_1, \mu w_1 + \eta w_2) = X v_1 \otimes Y(\mu w_1 + \eta w_2) = X v_1 \otimes (\mu Y w_1 + \eta Y w_2)$

$= \mu X v_1 \otimes Y w_1 + \eta X v_1 \otimes Y w_2 = \mu (X\otimes Y)(v_1, w_1) + \eta (X\otimes Y)(v_1, w_2)$

Hence $X \otimes Y$ is a bilinear

3.2

Since the trace of a linear map is independent of basis. We only need to verify these under certain basis. Suppose that V is n-dimensional and has basis $\{v_1, v_2, \cdots, v_n\}$

W is m-dimensional and has basis $\{w_1, w_2, \cdots, w_m\}$

under the basis $\{v_1, v_2, \cdots, v_n\}$, X has matrix $A = [a_1\ a_2 \cdots a_n]$ a_i is a $n\times 1$ vector

under the basis $\{w_1, w_2, \cdots, w_m\}$, Y has matrix $B = [b_1\ b_2 \cdots b_m]$ b_i is a $m\times 1$ vector

Then $\forall v_i, w_j$, $X(v_i) = [v_1 \cdots v_n]_{n\times n}\, a_i$, $Y(w_j) = [w_1 \cdots w_m]_{m\times m}\, b_j$

from 3.1, we know that we can think $X\otimes Y$ as a linear map that sends $v\otimes w$ in $V\otimes W$ to $Xv \otimes Yw$ in $V\otimes W$

Let $T = [v_1\otimes w_1, v_1 \otimes w_2 \cdots v_1 \otimes w_m, v_2 \otimes w_1, v_2 \otimes w_2, \cdots v_2 \otimes w_m \cdots v_n \otimes w_1, v_n \otimes w_2, \cdots v_n \otimes w_m]_{mn\times mn}$

be the basis of the space $V\otimes W$

$[v_1 \cdots v_n] \otimes [w_1 \cdots w_n] = [v_1 \otimes w_1, \cdots v_1 \otimes w_m, \cdots v_n \otimes w_1, \cdots v_n \otimes w_m]_{mn \times mn}$

Then $(X \otimes Y)(v_i \otimes w_j) = Xv_i \otimes Yw_j = [v_1 \cdots v_n] a_i \otimes [w_1 \cdots w_m] b_j =$

$([v_1 \cdots v_n] \otimes [w_1 \cdots w_m])(a_i \otimes b_j) = T(a_i \otimes b_j)$, the reason is following

Corollary 3.5.20. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Proof. For any basis vector $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, then $((AC) \otimes (BD))(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = (AC\boldsymbol{e}_i) \otimes (BD\boldsymbol{e}_j)$, while $(A \otimes B)(C \otimes D)(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = (A \otimes B)((C\boldsymbol{e}_i) \otimes (D \otimes \boldsymbol{e}_j)) = (AC\boldsymbol{e}_i) \otimes (BD\boldsymbol{e}_j)$. So the two agree on a basis. They must be the same map.

Alternatively, we can also prove this using the matrix interpretation of the input $\sum x_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Let this corresponds to the matrix X. Then $(A \otimes B)(C \otimes D)$ sends this to the matrix $A(CXD^{\mathrm{T}})B^{\mathrm{T}}$, while $(AC) \otimes (BD)$ sends this to the matrix $(AC)X(BD)^{\mathrm{T}}$. You can see that the two resulting image are the same. $\square$

Then by listing all $v_i \otimes w_j$ in the same squential manner of T, we know:
$(X \otimes Y)(T) = T \cdot (A \otimes B)_{mn \times mn}$. The matrix for $X \otimes Y$ under T basis is $A \otimes B$

let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{n1} & \cdots & & b_{nn} \end{bmatrix}$

Thus $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}B & \cdots & & a_{nn}B \end{bmatrix}$

$\mathrm{trace}(A \otimes B)$
$= a_{11}\mathrm{trace}(B) + \cdots + a_{nn}\mathrm{trace}(B)$
$= \sum_{i=1}^{n} a_{ii} \cdot \mathrm{trace}(B) = \mathrm{trace}(A) \, \mathrm{trace}(B)$

Hence $\mathrm{trace}(X \otimes Y) = \mathrm{trace}(X) \otimes \mathrm{trace}(Y)$