Review

- 向量值函数在一点可微及微分的定义

$$f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,\ \lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0$$

- $f=\left(f_1,f_2,\cdots,f_m\right)^{\mathrm{T}}:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$在$x_0$可微

$\Leftrightarrow n$元函数$f_i:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}$在$x_0$可微,$i=1,2,\cdots,m.$

•Chain Rule

$u = g(x): \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, y = f(u): g(\Omega) \subset \mathbb{R}^m \to \mathbb{R}^k,$

$g(x)$在$x_0 \in \Omega$可微, $f(u)$在$u_0 = g(x_0)$可微,则

$$J(f \circ g)\big|_{x_0} = J(f)\big|_{u_0} \cdot J(g)\big|_{x_0},$$

即$$\left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0} = \left.\frac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{u_0} \cdot \left.\frac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0},$$

简记为$$\left.\frac{\partial y}{\partial x}\right|_{x_0} = \left.\frac{\partial y}{\partial u}\right|_{u_0} \cdot \left.\frac{\partial u}{\partial x}\right|_{x_0}.$$

$k = 1$时, $$\frac{\partial y}{\partial x_i} = \frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i}, i = 1, 2, \cdots, n.$$

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§6. 隐函数定理与反函数定理

曲线$x^2+y^2=1$在$(0,1)$的某个邻域中可表示为

$y=\sqrt{1-x^2}$, 且$y'(x)=\dfrac{-x}{\sqrt{1-x^2}}$; 在$(1,0)$的某个邻域

中可表示为$x=\sqrt{1-y^2}$, 且$x'(y)=\dfrac{-y}{\sqrt{1-y^2}}$.

Question: (1) $f(x,y)=0$何时确定隐函数$y=y(x)$?

(2)如何通过$f(x,y)$的性质研究隐函数$y=y(x)$的性质, 如连续性, 可微性?

(3)如何计算隐函数的(偏)导数和(全)微分?

1. 一个方程确定的隐函数

设$f(x,y)=0, f(x_0,y_0)=0$.若存在连续可微的隐函数$y=y(x)$, $y(x_0)=y_0$,满足$f(x,y(x))\equiv 0$, 两边对x求导,有

$$f_1'(x,y(x))+f_2'(x,y(x))\cdot y'(x)=0.$$

若$f_2'(x_0,y_0)\neq 0$,则在x_0的某个邻域中,

$$y'(x)=-\frac{f_1'(x,y(x))}{f_2'(x,y(x))}=-\frac{\partial f(x,y(x))}{\partial x}\Big/\frac{\partial f(x,y(x))}{\partial y}.$$

(这里求偏导函数时x,y相互独立!)

Thm. 设F在$(x_0, y_0)\in\mathbb{R}^2$的某个邻域W中有定义,且

(1)$F(x_0, y_0)=0$,

(2) $F(x,y)\in C^1(\mathrm{W})$,即$F_x', F_y'$在W中连续,

(3)$F_y{}'(x_0, y_0)\neq 0$.

则存在$\delta>0$以及$I=(x_0-\delta, x_0+\delta)$上定义的函数

$y=y(x)$,满足

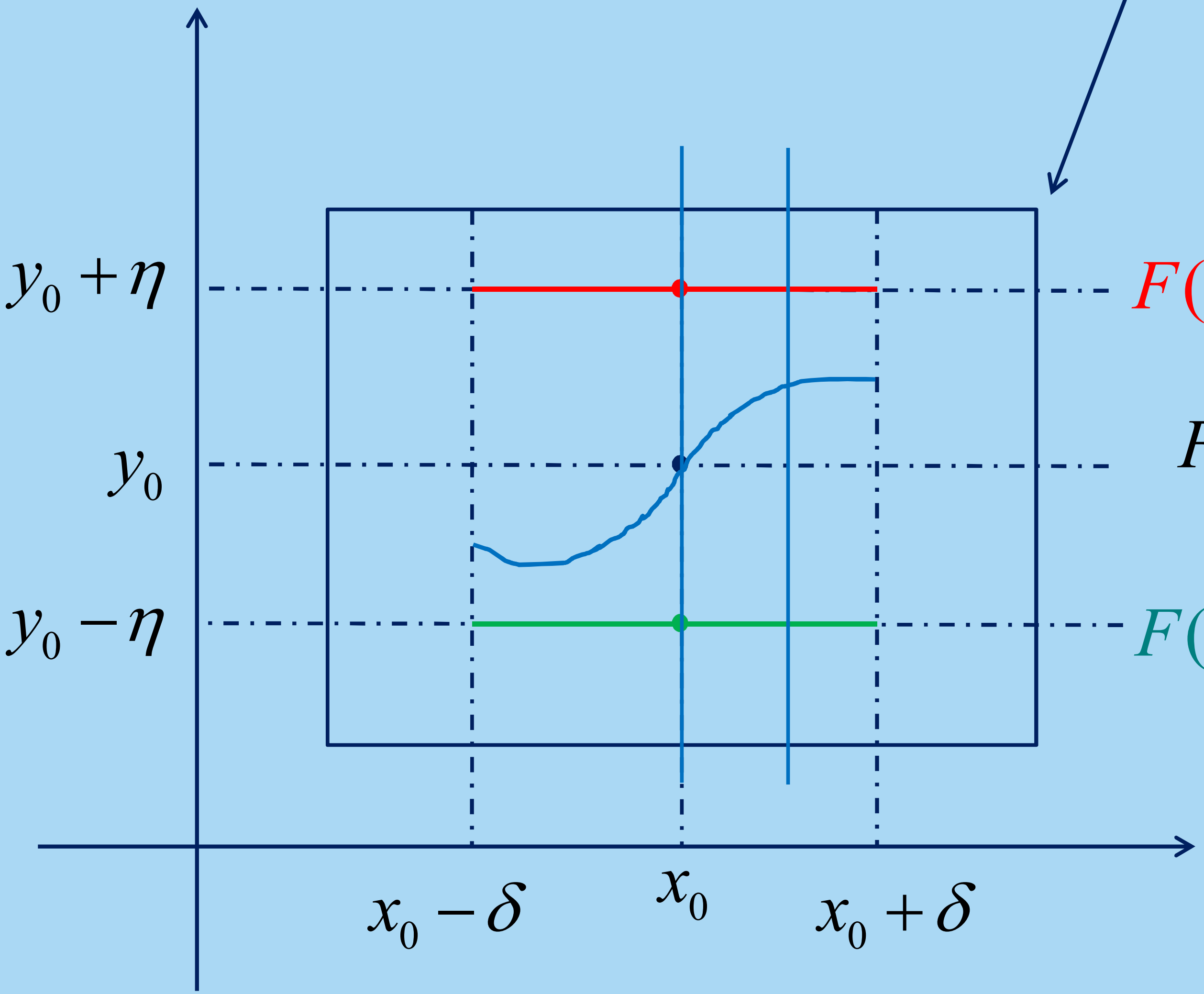
(1)$y(x_0)=y_0$,且$F(x,y(x))\equiv 0, \forall x\in I$,

(2)$y=y(x)\in C^1(I)$,即$y'(x)$在I上连续,

(3)$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\partial F(x,y(x))}{\partial x}\Big/\frac{\partial F(x,y(x))}{\partial y}, \forall x\in I.$$

(这里求偏导函数时x, y相互独立!)

$$F'_y(x_0, y_0) > 0 \Rightarrow F'_y(x, y) > 0, \forall (x, y) \in \mathrm{W}_1$$



$$F(x, y_0+\eta) > 0,$$
$$\forall |x-x_0| < \delta$$
$$\Uparrow$$
$$F(x_0, y_0+\eta) > 0$$

$$F(x_0, y_0) = 0$$

$$F(x_0, y_0-\eta) < 0$$
$$\Downarrow$$
$$F(x, y_0-\eta) < 0,$$
$$\forall |x-x_0| < \delta$$

Proof. (1)先证隐函数的存在性.

因$F_y'(x_0, y_0) \neq 0$,不妨设$F_y'(x_0, y_0) > 0$.$F \in C^1(W)$,则

$\exists a, b > 0, s.t.$ $F_y'(x, y) > 0, \forall |x - x_0| < a, |y - y_0| < b.$ (*)

$F(x_0, y)$对y连续,由(*)及$F(x_0, y_0) = 0$,给定$\eta \in (0, b)$,有

$$F(x_0, y_0 - \eta) < 0 < F(x_0, y_0 + \eta).$$

由F的连续性,$\exists \delta \in (0, a), s.t.$

$$F(x, y_0 - \eta) < 0 < F(x, y_0 + \eta),\ \forall |x - x_0| < \delta.$$

由(*)知,任意给定$|x - x_0| < \delta$,$F(x, y)$是y的增函数.结合连续函数的介值定理, $\forall |x - x_0| < \delta, \exists! y = y(x) \in (y_0 - \eta, y_0 + \eta), s.t.\ F(x, y) = 0.$

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(2)记(1)中构造的隐函数为$y=f(x)$,下证其连续性.

由(1)中证明知,不论$\eta>0$取多小,都$\exists\delta>0$,当$|x-x_0|<\delta$时,必有$|y-y_0|<\eta$. 因此$y=f(x)$在x_0连续.

任给$x_1\in(x_0-\delta,x_0+\delta)$,记$y_1=f(x_1)$,则$|y_1-y_0|<\eta$, $F(x_1,y_1)=0, F'_y(x_1,y_1)>0$.即$F$在$(x_1,y_1)$与$(x_0,y_0)$满足相同的条件.由前面的证明, F在(x_1,y_1)的充分小邻域中确定了同一个隐函数$y=f(x)$,且f在x_1连续.

(3)最后证隐函数$y=y(x)$的可导公式及连续可微性.

任意给定$x\in(x_0-\delta,x_0+\delta)$,由隐函数的连续性, 当$\Delta x\to 0$时,$\Delta y=y(x+\Delta x)-y(x)\to 0$.由隐函数的定义及$F$的连续可微性知,

$$0=F(x+\Delta x,y(x)+\Delta y)-F(x,y(x))$$

$$=F'_x(x,y(x))\Delta x+F'_y(x,y(x))\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y,$$

其中, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\varepsilon_i=0, i=1,2.$

而$F'_y(x,y(x))>0$,于是有

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\lim_{\Delta x\to 0}\frac{F_x'(x,y(x))+\varepsilon_1}{F_y'(x,y(x))+\varepsilon_2}$$

$$=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall |x-x_0|<\delta.$$

即 $$y'(x)=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))},\qquad \forall |x-x_0|<\delta.$$

由F的连续可微性知, $y'(x)$在$(x_0-\delta,x_0+\delta)$上连续.□

Remark:$F_y'(x_0,y_0)\neq 0$不是隐函数存在的必要条件.

设$f(x_1,\cdots,x_n,y)=0, f(x_1^0,\cdots,x_n^0,y_0)=0$确定了连续可微的隐函数 $y=y(x_1,\cdots,x_n),\ y_0=y(x_1^0,\cdots,x_n^0)$,满足

$$f(x_1,\cdots,x_n,y(x_1,\cdots,x_n))\equiv 0,$$

两边对x_i求偏导,有

$$f_i'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))\cdot 1+f_{n+1}'\cdot y_{x_i}'=0.$$

若$f_{n+1}'(x_1^0,\cdots,x_n^0,y_0)\neq 0$,则在$(x_1^0,\cdots,x_n^0)$的某邻域中,

$$y_{x_i}'(x_1,x_2,\cdots,x_n)=-\frac{f_i'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}{f_{n+1}'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}$$

$$=-\frac{f_{x_i}'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}{f_y'(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}.$$

右端求偏导函数时$x_1,\cdots,x_n,y$相互独立

Thm. 设函数$F(x_1,x_2,\cdots,x_n,y)$在点$(x_1^0,x_2^0,\cdots,x_n^0,y_0)$ $\in\mathbb{R}^{n+1}$的某个邻域W中有定义, 且

$$(1)F(x_1^0,x_2^0,\cdots,x_n^0,y_0)=0,$$

$$(2)F(x_1,x_2,\cdots,x_n,y)\in C^1(\mathrm{W}),$$

$$(3)\left.\frac{\partial F}{\partial y}\right|_{(x_1^0,x_2^0,\cdots,x_n^0,y_0)}\neq 0.$$

则存在点$(x_1^0,x_2^0,\cdots,x_n^0)\in\mathbb{R}^n$的一个邻域U,以及定义在U上的$n$元函数$y=y(x_1,x_2,\cdots,x_n)$,满足

(1) $y_0 = y(x_1^0, x_2^0, \cdots, x_n^0)$, 且当 $(x_1, x_2, \cdots, x_n) \in U$ 时,

$$F\left(x_1, x_2, \cdots, x_n, y(x_1, x_2, \cdots, x_n)\right) \equiv 0;$$

(2) $y = y(x_1, x_2, \cdots, x_n) \in C^1(\mathrm{U})$, 即 y'_{x_i} 在U中连续,

$i = 1, 2, \ldots \cdots, n$;

(3) $$y'_{x_i}(x_1, \cdots, x_n) = -\frac{F'_{x_i}(x_1, \cdots, x_n, y(x_1, \cdots, x_n))}{F'_y(x_1, \cdots, x_n, y(x_1, \cdots, x_n))}.$$

右端求偏导函数时 $x_1, \cdots, x_n, y$ 相互独立!

Remark: $F'_y(x_1^0, x_2^0, \cdots, x_n^0, y_0) \neq 0$ 不是隐函数存在的必要条件.

2. 方程组确定的隐函数

设 $F_i(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m)=0, i=1,2,\cdots,m.$

$$F_i(x_1^0,x_2^0,\cdots,x_n^0,y_1^0,y_2^0,\cdots,y_m^0)=0, i=1,2,\cdots,m.$$

若存在连续可微的隐函数

$$y_i=y_i(x_1,x_2,\cdots,x_n), i=1,2,\cdots,m.$$

满足

$$y_i(x_1^0,x_2^0,\cdots,x_n^0)=y_i^0, i=1,2,\cdots,m.$$

$$F_i(x_1,x_2,\cdots,x_n,y_1(x_1,x_2,\cdots,x_n),y_2(x_1,x_2,\cdots,x_n),$$

$$\cdots,y_m(x_1,x_2,\cdots,x_n))=0,\qquad i=1,2,\cdots,m$$

简记为 $F(x,y)=0, F(x_0,y_0)=0,$ $\left(\begin{array}{l} x\in\mathbb{R}^n, y\in\mathbb{R}^m, \\ F:\mathbb{R}^{n+m}\to\mathbb{R}^m. \end{array}\right)$

$F(x,y(x))=0, y(x_0)=y_0.$

由复合映射的链式法则, 有 $\dfrac{\partial F}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial x}=0,$

(求 $\dfrac{\partial F}{\partial(x,y)}$ 时 x,y 相互独立!)

即 $\begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{pmatrix}\begin{pmatrix} \dfrac{\partial x}{\partial x} \\ \dfrac{\partial y}{\partial x} \end{pmatrix}=0,\ \dfrac{\partial F}{\partial x}I_n+\dfrac{\partial F}{\partial y}\dfrac{\partial y}{\partial x}=0,$

若 $\dfrac{\partial F}{\partial y}$ 可逆,则 $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}.$

Thm. $F(x,y):\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$在$(x_0,y_0)$的邻域W中有定义,且满足 (1)$F(x_0,y_0)=0$, (2)$F\in C^q(\mathrm{W})$,即$F$的各分量函数在W中$q$阶连续可微, (3) $\dfrac{\partial F}{\partial y}(x_0,y_0)$可逆,则存在$x_0$的某个邻域$\mathrm{U}\in\mathbb{R}^n$,以及定义在U上的向量值函数$y=y(x)$,满足

(1) $y(x_0)=y_0, F(x,y(x))=0, \forall x\in\mathrm{U}$;

(2) $y(x)$在U上q阶连续可微;

(3) $\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$. 求$\dfrac{\partial F}{\partial x},\dfrac{\partial F}{\partial y}$时 x,y 相互独立!

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Remark: $\dfrac{\partial F}{\partial y}(x_0, y_0)$可逆不是隐函数存在的必要条件.

Remark: $F:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m,(x,y)\mapsto F(x,y)$,若$\dfrac{\partial F}{\partial y}$可逆,

则$F(x,y)=0$确定隐"函数"$y=y(x)$,求$\dfrac{\partial y}{\partial x}$有两种方法:

- 套用定理:$\dfrac{\partial y}{\partial x}=-\left(\dfrac{\partial F}{\partial y}\right)^{-1}\dfrac{\partial F}{\partial x}$.

这里求Jaccobi矩阵时 x,y 相互独立!

- 将$F(x,y)=0$中y视为$y=y(x)$,利用复合映射的链式法则,方程组 $F(x,y(x))=0$两边对x求Jaccobi矩阵.

Remark: 对具体的例子, 不必死记硬背隐函数定理中的公式, 只要将某些变量视为其它变量的隐函数, 再利用复合函数的求导法则即可.

Remark: m个方程确定m个隐函数, 将某m个变量看成函数, 其它变量相互独立.

例. φ可微, $x^2+z^2=y\varphi\left(\frac{z}{y}\right)$确定隐函数$z=z(x,y)$.求$z'_x, z'_y$.

解: 视$x^2+z^2=y\varphi(z/y)$中$z=z(x,y)$为隐函数. 两边分别对x,y求偏导, 有

$$2x+2zz'_x=y\varphi'(z/y)\cdot\frac{1}{y}z'_x,$$

$$2zz'_y=\varphi(z/y)+y\varphi'(z/y)\cdot\frac{1}{y^2}(yz'_y-z).$$

求解得

$$z'_x=\frac{2x}{\varphi'(z/y)-2z},\ z'_y=\frac{y\varphi(z/y)-\varphi'(z/y)}{2yz-y\varphi'(z/y)}.\square$$

例. $u=f(x,y,z)$有连续偏导数,且$z=z(x,y)$由方程$xe^x-ye^y=ze^z$所确定,求du.

解:方程$xe^x-ye^y=ze^z$两边分别对x,y求偏导,有

$$\left.\begin{aligned} e^x+xe^x&=z'_x e^z+zz'_x e^z \\ -e^y-ye^y&=z'_y e^z+zz'_y e^z \end{aligned}\right\}\Rightarrow\begin{cases} z'_x=\dfrac{1+x}{1+z}e^{x-z}, \\ z'_y=\dfrac{-(1+y)}{1+z}e^{y-z}. \end{cases}$$

于是,

$$du=u'_x dx+u'_y dy=\left(f'_x+f'_z z'_x\right)dx+\left(f'_y+f'_z z'_y\right)dy$$

$$=\left(f'_x+\frac{1+x}{1+z}e^{x-z}f'_z\right)dx+\left(f'_y-\frac{1+y}{1+z}e^{y-z}f'_z\right)dy.\square$$

Remark: $du = f'_x dx + f'_y dy + f'_z dz$

$$= f'_x dx + f'_y dy + f'_z (z'_x dx + z'_y dy)$$

$$= \left(f'_x + f'_z z'_x \right) dx + \left(f'_y + f'_z z'_y \right) dy.$$

一阶微分的形式不变性

例. $u=f\left(x-ut,y-ut,z-ut\right),g(x,y,z)=0$,求$u'_x,u'_y$.

分析: 五个变量x,y,z,t,u,两个方程, 确定两个隐函数 $z=z(x,y,t)=z(x,y),u=u(x,y,t)$.

解法一: 视$u=f\left(x-ut,y-ut,z-ut\right)$中$z=z(x,y)$为隐函数, 两边分别对$x,y$求偏导, 有

$$u'_x=(1-tu'_x)f_1'+(-tu'_x)f_2'+(z'_x-tu'_x)f_3',$$
$$u'_y=(-tu'_y)f_1'+(1-tu'_y)f_2'+(z'_y-tu'_y)f_3'.$$

其中 f_1',f_2',f_3'在$(x-ut,y-ut,z-ut)$处取值.

视$g(x,y,z)=0$中$z=z(x,y)$, 两边对x,y求偏导,有

$$\begin{cases} g_x' + g_z' z_x' = 0, \\ g_y' + g_z' z_y' = 0, \end{cases} \Rightarrow \begin{cases} z_x' = -g_x' / g_z', \\ z_y' = -g_y' / g_z'. \end{cases}$$

代入前两式,求解得

$$u_x' = \frac{f_1' + f_3' z_x'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_1' g_z' - f_3' g_x'}{[1 + t(f_1' + f_2' + f_3')] g_z'}$$

$$u_y' = \frac{f_2' + f_3' z_y'}{1 + t(f_1' + f_2' + f_3')} = \frac{f_2' g_z' - f_3' g_y'}{[1 + t(f_1' + f_2' + f_3')] g_z'}.$$

解法二：套用隐函数定理.

$$h(x,y,z,u,t) \triangleq f(x-ut, y-ut, z-ut) - u = 0,$$

$$g(x,y,z) = 0.$$

$$\frac{\partial(u,z)}{\partial(x,y,t)} = -\left(\frac{\partial(h,g)}{\partial(u,z)}\right)^{-1} \frac{\partial(h,g)}{\partial(x,y,t)}$$

$$= \begin{pmatrix} 1+t(f_1'+f_2'+f_3') & -f_3' \\ 0 & -g_z' \end{pmatrix}^{-1} \frac{\partial(h,g)}{\partial(x,y,t)}$$

$$= \frac{-1}{[1+t(f_1'+f_2'+f_3')]g_z'} \begin{pmatrix} -g_z' & f_3' \\ 0 & 1+t(f_1'+f_2'+f_3') \end{pmatrix} \frac{\partial(h,g)}{\partial(x,y,t)}$$

于是 $(u_x', u_y') = \dfrac{(g_z' \quad -f_3')}{[1+t(f_1'+f_2'+f_3')]g_z'} \begin{pmatrix} f_1' & f_2' \\ g_x' & g_y' \end{pmatrix}$.□

3. 逆映射定理

Thm. (逆映射的微分) $f:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^n$连续可微, $x_0\in\Omega$.若 $J(f)\big|_{x_0}$ 可逆,则存在$y_0=f(x_0)$的某个邻域U,使得U上定义了映射$y=f(x)$的逆映射$x=f^{-1}(y), x_0=f^{-1}(y_0)$,且$x=f^{-1}(y)$在$y_0$可微,

$$\frac{\partial\left(x_1,x_2,\cdots,x_n\right)}{\partial\left(y_1,y_2,\cdots,y_n\right)}=\left(\frac{\partial\left(y_1,y_2,\cdots,y_n\right)}{\partial\left(x_1,x_2,\cdots,x_n\right)}\right)^{-1},$$

即$J(f^{-1})=\left(J(f)\right)^{-1}$.

Proof: 考虑方程组$F(x,y)\triangleq f(x)-y=0$,有

$$F(x_0,y_0)=0,\text{且}\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{\partial f}{\partial x}\right|_{x_0}\text{可逆.}$$

由隐函数定理,存在$y_0=f(x_0)$的邻域U及U上定义的函数$x=x(y)\triangleq f^{-1}(y)$,满足

$$f(x(y))-y\equiv 0, x(y_0)=x_0,$$

由复合映射的链式法则,有

$$\frac{\partial f}{\partial x}(x(y))\cdot\frac{\partial x}{\partial y}(y)-I=0,\quad \forall y\in U.$$

即$J(f)\cdot J(f^{-1})=\mathrm{I}$, $J(f^{-1})=\left(J(f)\right)^{-1}$. □

作业：习题**1.6 No. 4,5,7,9.**