第三次习题课　空间曲线与曲面

一、向量函数的微分和导数

1．计算极坐标、柱坐标、球坐标变换的 Jacobi 矩阵和 Jacobi 行列式：

（1）平面极坐标变换 $\vec{\mathbf{f}}(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$，也即 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$；

$dxdy = |J_f(r,\theta)|\,dr\,d\theta$

（2）空间柱坐标变换 $\vec{\mathbf{f}}(r,\theta,z)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$，也即 $\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=z \end{cases}$；

④ $\int_0^R \int_0^{\sqrt{R^2-x^2}} dxdy$

（3）空间球坐标变换 $\vec{\mathbf{f}}(r,\varphi,\theta)=\begin{pmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}$，也即 $\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$。

解：直接计算如下

（1）$J_{\mathbf{f}}(r,\theta)=\dfrac{\partial(x,y)}{\partial(r,\theta)}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$，

$\int_0^R \int_0^{2\pi} r\,dr\,d\theta$

$= \frac{1}{2}R^2\cdot 2\pi = \pi R^2$

$$\det J_{\mathbf{f}}(r,\theta)=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}=r；$$

（2）$J_{\mathbf{f}}(r,\theta,z)=\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$，

$\int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\varphi\, dr\, d\theta\, d\varphi$

$= \frac{1}{3}R^3\cdot 2\pi\cdot 2 = \frac{4}{3}\pi R^3$

$$\det J_{\mathbf{f}}(r,\theta,z)=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=r；$$

（3）$J_{\mathbf{f}}(r,\varphi,\theta)=\dfrac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$，

$$\det J_{\mathbf{f}}(r,\varphi,\theta)=\begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix}=r^2\sin\varphi。$$

2．计算向量复合函数的 Jacobi 矩阵：

（1）$\mathbf{f}(x,y)=(x,y,x^2y)$，$x=s+t$，$y=s^2-t^2$，在 $s=2,t=1$；

（2）$\mathbf{f}(x,y,z)=(x^2+y+z,2x+y+z^2,0)$， $x=uv^2w^2, y=w^2\sin v, z=u^2e^v$。

解：（1）记$\mathbf{g}(s,t)=(x,y)$， $x=s+t$， $y=s^2-t^2$，在$s=2,t=1$时$x=y=3$，

$$J_{\mathbf{f}}(3,3)=\begin{pmatrix}1&0\\0&1\\2xy&x^2\end{pmatrix}_{x=y=3}=\begin{pmatrix}1&0\\0&1\\18&9\end{pmatrix},$$

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$J_{\mathbf{g}}(2,1)=\begin{pmatrix}1&1\\2s&-2t\end{pmatrix}_{s=2,t=1}=\begin{pmatrix}1&1\\4&-2\end{pmatrix},$$

$$J_{\mathbf{f}\circ\mathbf{g}}(2,1)=J_{\mathbf{f}}(3,3)J_{\mathbf{g}}(2,1)=\begin{pmatrix}1&0\\0&1\\18&9\end{pmatrix}\begin{pmatrix}1&1\\4&-2\end{pmatrix}=\begin{pmatrix}1&1\\4&-2\\54&0\end{pmatrix}。$$

法二：由 $\mathbf{g}(s,t)=(x,y)=(s+t,s^2-t^2)$， $\mathbf{f}(x,y)=(x,y,x^2y)$ 得到

$$\mathbf{f}\circ\mathbf{g}(s,t)=(s+t,s^2-t^2,(s+t)^2(s^2-t^2))$$
$$=(s+t,s^2-t^2,s^4+2s^3t-2st^3-t^4),$$

$$J_{\mathbf{f}\circ\mathbf{g}}(s,t)=\begin{pmatrix}1&1\\2s&-2t\\4s^3+6s^2t-2t^3&2s^3-6st^2-4t^3\end{pmatrix},$$

再将$s=2,t=1$带入即得……

（2）由题意 $\mathbf{g}(u,v,w)=(x,y,z)$， $x=uv^2w^2, y=w^2\sin v, z=u^2e^v$，并且

$f_1(x,y,z)=x^2+y+z$， $f_2(x,y,z)=2x+y+z^2$， $f_3(x,y,z)=0$，

$$J_{\mathbf{f}\circ\mathbf{g}}(u,v,w)=\begin{pmatrix}2x&1&1\\2&1&2z\\0&0&0\end{pmatrix}\begin{pmatrix}v^2w^2&2uvw^2&2uv^2w\\0&w^2\cos v&2w\sin v\\2ue^v&u^2e^v&0\end{pmatrix}$$

$$=\begin{pmatrix}2xv^2w^2+2ue^v&4xuvw^2+w^2\cos v+u^2e^v&4xuv^2w+2w\sin v\\2v^2w^2+4zue^v&4uvw^2+w^2\cos v+2zu^2e^v&4uv^2w+2w\sin v\\0&0&0\end{pmatrix}$$

$$=\begin{pmatrix}2uv^4w^4+2ue^v&4u^2v^3w^4+w^2\cos v+u^2e^v&4u^2v^4w^3+2w\sin v\\2v^2w^2+4u^3e^{2v}&4uvw^2+w^2\cos v+2u^4e^{2v}&4uv^2w+2w\sin v\\0&0&0\end{pmatrix}$$

二、切平面，切线，法平面，法线

[例1] 求曲线$L:\begin{cases}x^2+y^2+z^2=4\\x^2+y^2=2x\end{cases}$

在点$M_0(1,1,\sqrt{2})$处的切线和法平面方程

$$\boxed{\frac{x-x_0}{1}=\frac{y-y_0}{y'(x_0)}}=\frac{z-z_0}{z'(x_0)}$$

$x - x_0 + y'(x_0)(y-y_0) + z'(x_0)(z-z_0) = 0$

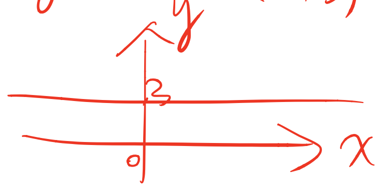
$y'(x_0)(x-x_0) = y - y_0$　　$z'(x_1)$

0

解：方程两边对x求导 $\begin{cases} 2x + 2yy'(x) + 2zz'(x) = 0 \\ 2x + 2yy'(x) = 2 \end{cases}$，解得 $y'(1) = 0, z'(1) = \dfrac{-1}{\sqrt{2}}$

故切线方程为：$\dfrac{x-1}{1} = \dfrac{y-1}{0} = \dfrac{z-\sqrt{2}}{-\frac{1}{\sqrt{2}}}$

$y=3$　　$(0,3)$



法平面方程为：$(x-1) - \dfrac{1}{\sqrt{2}}(z-\sqrt{2}) = 0$

$\dfrac{x-x_0}{1} = \dfrac{y-y_0}{y'(x_0)}$

[例2] 设函数f可微，求证：曲面 $S: z = yf(\dfrac{x}{y})$ 的所有切平面相交于一个公共点。

$\dfrac{x-0}{1} = \dfrac{y-3}{0}$

解：曲面 S：在点 (x, y, z) 的切平面

$Z - z = \dfrac{\partial z}{\partial x}(X - x) + \dfrac{\partial z}{\partial y}(Y - y)$，代入得

$Z - yf(\dfrac{x}{y}) = f'(\dfrac{x}{y})(X - x) + [f(\dfrac{x}{y}) - \dfrac{x}{y}f'(\dfrac{x}{y})](Y - y)$

当 $(X, Y, Z) = (0, 0, 0)$ 时，两端恒等。因此都经过原点。

[例3] 过直线 $10x + 2y - 2z = 27,\ \ x + y - z = 0$ 作曲面 $3x^2 + y^2 - z^2 = 27$ 的切平面，求其方程.

曲面法向量：$grad(F) = (F_x', F_y', F_z')^T$

解：设 $F(x, y, z) = 3x^2 + y^2 - z^2 - 27$，则 $F_x' = 6x$，$F_y' = 2y$，$F_z' = -2z$

过直线 $10x + 2y - 2z = 27,\ \ x + y - z = 0$ 的平面束方程为

$$\boxed{10x + 2y - 2z - 27 + \lambda(x + y - z) = 0}$$

$\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

法向量 $\vec{n} = \{(10+\lambda), (2+\lambda), (-2-\lambda)\}$ 设切点为 (x_0, y_0, z_0)，则有

$\begin{cases} 3x_0^2 + y_0^2 - z_0^2 - 27 = 0 \\ (10+\lambda)x_0 + (2+\lambda)y_0 - (2+\lambda)z_0 - 27 = 0 \end{cases}$

①：切点在曲面上.
②：切点在平面上

又因为 $\vec{n} \parallel gradF$, 所以 $\frac{10+\lambda}{6x_0} = \frac{2+\lambda}{2y_0} = \frac{-2-\lambda}{-2z_0}$ ②：法向量平行

解得 $x_0 = -3,\quad y_0 = -17,\quad z_0 = -17,\quad \lambda = -19$

于是,所求切平面方程为 $6\cdot 3(x-3) + 2\cdot 1(y-1) + (-2)\cdot 1(z-1) = 0$

$x^2-y^2=C$ 时, $u(x,y)=g(C)$

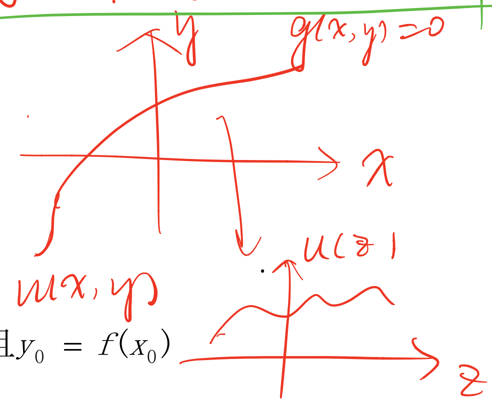
①

[例4] 求证满足微分方程 $\boxed{y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0}$ 的 $u(x,y)$ 为 $u(x,y) = f(x^2-y^2)$, 其中, f为任意一元可微函数.

$u(x,y)$ 在 $x^2-y^2=C$ 的曲线上的方向导数恒为0.

只需证明： $u = f(x^2-y^2)$ 等价于

$u = u(x,y)$满足微分方程 $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$



因为$u = f(x^2-y^2)$ 等价于

在曲线 $L: x^2-y^2 = C$ 上 $u(x,y) \equiv$ 常数

又等价于 $gradu(x,y)$ 与 L 切向量处处正交

当$\nabla F(M_0) \neq 0$时, 不妨设 $F'_y \neq 0$ 确定函数: $y = f(x)$, 且$y_0 = f(x_0)$

切向量为$\vec{v} = (1, \frac{dy}{dx})$, $\frac{dy}{dx} = -\frac{F'_x}{F'_y}$,代入得到

$(1, \frac{dy}{dx}) \Rightarrow (1, \frac{x}{y})$

切向量 $\vec{v} = (F'_y, -F'_x) = (-2y, -2x) // (y, x)$。

$\left(\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y}\right)\begin{pmatrix}1 \\ \frac{dy}{dx}\end{pmatrix} = 0$

$x^2-y^2=C$

$2x-2y\cdot y'_x=0$

$y'_x=\frac{x}{y}$

$\left(\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y}\right)\begin{pmatrix}1 \\ \frac{x}{y}\end{pmatrix} = \frac{\partial u}{\partial x} + \frac{x}{y}\frac{\partial u}{\partial y} = 0$

若存在 $k>0$, s.t.

$\lim\limits_{x\to x_0} \frac{f(x)}{\rho^k} = C(\neq 0)$成立,

那么 $f(x)$ 的阶为 k.

$\rho=|x-x_0|$

当$\rho\to 0$时

1. 当$(x,y)\to(0,0)$时，讨论 $(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}$ 的阶.

对所有的 $k>0$ $\lim\limits_{x\to x_0}\frac{f(x)}{\rho^k}\neq C$ ($C\neq 0$)

$\exists k$ s.t. $\lim\limits_{\rho\to 0}\frac{\rho^2\sin\frac{1}{\rho}}{\rho^k}=C\neq 0$? ①

$\rho\to 0$时，$\rho^k\to 0$ ($k>0$)

$\rho^{2-k}\sin\frac{1}{\rho}$

① $0<k<2$时. ① $=0$

② $k=2$时 ① 极限不存在.

③ $k>2$时. ① 极限不存在 ($\to\infty$). $\frac{\sin\frac{1}{\rho}}{\rho^{k-2}}\to\infty$

2. 设 $\frac{\partial f}{\partial x}(x_0,y_0)$ 存在，$\frac{\partial f}{\partial y}(x,y)$ 连续，证明$f(x,y)$在$f(x_0,y_0)$处可微.

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\Delta f-\frac{\partial f}{\partial x}(x_0,y_0)\Delta x-\frac{\partial f}{\partial y}(x_0,y_0)\Delta y}{\rho}$$

$$=\lim\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)+f(x_0+\Delta x,y_0)-f(x_0,y_0)-\frac{\partial f}{\partial x}(\ )\Delta x-\frac{\partial f}{\partial y}(\ )\Delta y}{\rho}$$

$$=\lim\frac{\boxed{\frac{\partial f}{\partial y}(x_0+\Delta x,y_0)}\Delta y+o(\Delta y)+\boxed{\frac{\partial f}{\partial x}(x_0,y_0)\Delta x}+o(\Delta x)-\boxed{\frac{\partial f}{\partial x}(\ )\Delta x}-\frac{\partial f}{\partial y}(\ )\Delta y}{\rho}$$

$$=\lim\frac{\frac{\partial f}{\partial y}(x_0,y_0)+\overset{\to 0}{a(\Delta x)}\Delta y+o(\Delta y)+o(\Delta x)-\frac{\partial f}{\partial y}(\ )\Delta y}{\rho}$$

$$=\lim_{\rho\to 0}\frac{a(\Delta x)\Delta y+o(\Delta y)+o(\Delta x)}{\rho}=0$$

$\rho=\sqrt{\Delta x^2+\Delta y^2}$ $\left|\frac{\Delta y}{\rho}\right|\leq 1$

$0\leftarrow\frac{o(\Delta y)}{\Delta y}\geqslant\frac{o(\Delta y)}{\rho}$

$$\lim_{\rho \to 0} \frac{\partial f}{\partial y}(x_0+\Delta x, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

$$\lim_{a \to 0} a = 0$$

$$\lim_{a \to 0} \frac{a}{a} = \frac{0}{a} = 0 \quad \times$$