赵晨阳 软01/计06 2020012363

习题4.7 3,5~7

3.(1) $\vec{v}=(x^3,y^3,z^3)$ 则 $\vec{v}$ 在S内部Ω内连续可微，在Ω与S^+上连续

$$\oiint_{S^+}\vec{v}\,d\vec{S}=\iiint_{\Omega}(3x^2+3y^2+3z^2)\,dxdydz=9\iiint_{x^2+y^2+z^2\le a^2}x^2\,dxdydz$$

换用球坐标 $\begin{cases}x=r\sin\varphi\cos\theta & r\in[0,a]\quad \varphi\in[0,\pi]\quad \theta\in[0,2\pi]\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}$

原式 $=9\int_0^a dr\int_0^{\pi}d\varphi\int_0^{2\pi}r^2\sin^2\varphi\cos^2\theta\cdot r^2\sin\varphi\,d\theta$

$=9\pi\int_0^a dr\int_0^{\pi}r^4\sin^3\varphi\,d\varphi=9\pi\int_0^a r^4dr\int_0^{\pi}(\cos^2\varphi-1)\,d\cos\varphi$

$=9\pi\left(\frac{1}{3}\cos^3\varphi-\cos\varphi\right)\Big|_0^{\pi}\cdot\frac{1}{5}r^5\Big|_0^a=9\pi\cdot\frac{4}{3}\cdot\frac{1}{5}a^5=\frac{12}{5}\pi a^5$

3.(2) $\vec{v}=(xy-xz,0,x-y)$ 则 $\vec{v}$ 在S内部Ω内连续可微，在Ω与S^+上连续

$\oiint_{S^+}\vec{v}\,d\vec{S}=\iiint_{\Omega}(y-z)\,dxdydz$ 换用柱坐标 $\begin{cases}x=r\cos\theta & r\in[0,1]\\ y=r\sin\theta & \theta\in[0,2\pi]\\ z=t & t\in[0,1]\end{cases}$

$=\int_0^1dt\int_0^{2\pi}d\theta\int_0^1(r\sin\theta-t)r\,dr$

$=\int_0^1dt\int_0^{2\pi}\left(\frac{1}{3}\sin\theta-\frac{1}{2}t\right)d\theta=\int_0^1-\pi t\,dt=-\frac{\pi}{2}$

3.(3) $\vec{v}=(y+2z,z^2-1,x+2y+3z)$ 则 $\vec{v}$ 在S内部Ω内连续可微，在Ω与S^+上连续

$\oiint_{S^+}\vec{v}\,d\vec{S}=\iiint_{\Omega}3\,dxdydz=3\times1\times1\times\frac{1}{2}\times1\times\frac{1}{3}=\frac{1}{2}$

3.(4) 令S_1^+为 $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1$ xoy平面朝下为正. 则 $\vec{v}=(a^2b^2z^2x,\ b^2c^2x^2y,\ c^2a^2y^2z)$
则 $\vec{v}$ 在$S^+\cup S_1^+$内部Ω内连续可微，在Ω与$S^+\cup S_1^+$上连续

$$\oiint_{S^+\cup S_1^+}\vec{v}\,d\vec{S}=\iiint_{\Omega}(a^2b^2z^2+b^2c^2x^2+c^2a^2y^2)\,dxdydz$$

柱坐标不好算

~~令$x=ar\cos\theta$ $y=br\sin\theta$ $z=t$ $r\in[0,1]$ $\theta\in[0,2\pi]$ $t\in[0,c^2-\frac{c^2}{a^2}x^2-\frac{c^2}{b^2}y^2]$ 即 $t\in[0,c^2(1-r^2)]$~~

$$\cancel{\frac{\partial(x,y,z)}{\partial(r,\theta,t)}=\begin{vmatrix}a\cos\theta & -ar\sin\theta & 0\\ b\sin\theta & br\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=abr}$$

$$\cancel{\therefore\iiint_{\Omega}(a^2b^2z^2+b^2c^2x^2+c^2a^2y^2)\,dxdydz=\iiint_V(a^2b^2t^2+a^2b^2c^2r^2)\cdot abr\,drd\theta dt}$$

$$\cancel{=\iint_{D_{\theta r}}d\theta dr\int_0^{c^2(1-r^2)}(a^3b^3t^2r+a^3b^3c^2r^3)\,dt}$$

令$x=ar\sin\varphi\cos\theta$ $y=br\sin\varphi\sin\theta$ $z=cr\cos\varphi$ $r\in[0,1]$ $\theta\in[0,2\pi]$ $\varphi\in[0,\frac{\pi}{2}]$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\theta & 0 & -cr\sin\theta\end{vmatrix}=abcr^2\sin\varphi$$

$\theta\in[0,2\pi]$ $\varphi\in[0,\frac{\pi}{2}]$ $r\in[0,1]$

原式 $=\iiint(a^2b^2c^2r^2\cos^2\varphi+a^2b^2c^2r^2\sin^2\varphi\cos^2\theta+a^2b^2c^2r^2\sin^2\varphi\sin^2\theta)abc\,r^2\sin\varphi\,d\theta d\varphi dr$

$=\iiint a^3b^3c^3r^4\sin\varphi\,d\theta dr d\varphi=2\pi\int_0^1 dr\int_0^{\frac{\pi}{2}}a^3b^3c^3r^4\sin\varphi d\varphi=\frac{2}{5}\pi a^3b^3c^3$

而在 S_1^+ 面上 $\iint_{S_1^+}\vec{v}\,d\vec{s}=-\iint_{S_1}c^2a^2y^2z\,dxdy=0$ $\therefore\iint_{S_-^+}\vec{v}d\vec{s}=\frac{2}{5}\pi a^3b^3c^3$

故 $\iint_{S^+}\vec{v}\,d\vec{s}=-\iint_{S_-^+}\vec{v}\,d\vec{s}=-\frac{2}{5}\pi a^3b^3c^3$

(5) 取S内部一小圆面 S_1^+: $x^2+y^2+z^2=d^2$. $d>0$ 且足够小. 取外侧为正

则 $\vec{v}=\frac{\vec{r}}{\|\vec{r}\|^3}$ 在 $S^+\cup S_1^-$ 内部 Ω 内连续可微, 在 Ω 与 $S^+\cup S_1^-$ 上连续

由高斯公式有: $\oiint_{S_1^-\cup S^+}A\,d\vec{s}=\iiint_\Omega\nabla\cdot(\frac{\vec{r}}{r^3})dxdydz$ $\vec{r}=(x,y,z)$ $|r|=(x^2+y^2+z^2)^{\frac{3}{2}}$

$\nabla\cdot\frac{\vec{r}}{r^3}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ 而 $\vec{A}=\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\frac{\partial A_x}{\partial x}=\frac{(x^2+y^2+z^2)^{\frac{3}{2}}-\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2m^2}{(x^2+y^2+z^2)^3}$

$\therefore\nabla\cdot\frac{\vec{r}}{r^3}=\frac{3(x^2+y^2+z^2)^{\frac{3}{2}}-3(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}=0$ $\therefore\iiint_\Omega\nabla\cdot(\frac{\vec{r}}{r^3})dxdydz=0$

$\therefore\iint_{S^+}A\,d\vec{s}=\iint_{S_1}A\,d\vec{s}=\iint_{S_1}\frac{\vec{r}}{r^3}\cdot\vec{n}\,ds=\iint_{S_1}\frac{1}{r^2}ds=4\pi r^2\cdot\frac{1}{r^2}=4\pi$

5. 记 L^+ 在 $x+y+z=0$ 上围成曲面为 S^+. 以上侧为正. 则单位正法向量为 $\vec{n}=\frac{(1,1,1)}{\sqrt{3}}$

$\nabla\times\vec{v}=\det\begin{vmatrix}i&j&k\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y&z&x\end{vmatrix}=(-1,-1,-1)$ $\therefore\int_{L^+}\vec{v}\,d\vec{r}=\iint_{S^+}\nabla\times\vec{v}\,d\vec{s}=\iint_{D_{xy}}(-1\cdot\frac{\sqrt{3}}{3}-1\cdot\frac{\sqrt{3}}{3}-1\cdot\frac{\sqrt{3}}{3})dxdy$

$=-\sqrt{3}\sigma(D_{xy})$

$=-\sqrt{3}\pi R^2$

(2) 记L在 $x+z=1$ 面上围成的曲面为 S^+, 以上侧为正, 则

单位正法向量为 $\vec{n}=\frac{(1,0,1)}{\sqrt{2}}$

$\nabla\times\vec{v}=\det\begin{vmatrix}i&j&k\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y-x&z-y&x-z\end{vmatrix}=(-1,-1,-1)$ $\therefore\int_{L^+}\vec{v}\,d\vec{r}=\iint_{S^+}\nabla\times\vec{v}\cdot\vec{n}\,ds=\iint_{D_{xy}}((-1)\times1+(-1)\times1)\,dxdy$

$=-2\sigma(D_{xy})=-2\pi$

(3) L围成面 S^+: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ 以上侧为正. 则单位法向量为 $\vec{n}=\frac{(\frac{1}{a},\frac{1}{b},\frac{1}{c})}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

$\nabla\times\vec{v}=\det\begin{vmatrix}i&j&k\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y^2&z^2&x^2\end{vmatrix}=(-2z,-2x,-2y)$ $I=\int_{L^+}\vec{v}\,d\vec{r}=\iint_{S^+}\nabla\times\vec{v}\cdot\vec{n}\,ds$

$=-\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\iint_S(\frac{z}{a}+\frac{x}{b}+\frac{y}{c})ds=-\frac{2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\iint_S(\frac{c}{a}-\frac{cx}{a^2}-\frac{cy}{ab}+\frac{x}{b}+\frac{y}{c})\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}dxdy$

$=-2c\int_0^a dx\int_0^{b-\frac{b}{a}x}\frac{c}{a}+(\frac{1}{b}-\frac{c}{a^2})x+(\frac{1}{c}-\frac{c}{ab})y\,dy$

$=-2c\int_0^a\frac{bc}{a}-\frac{bc}{a^2}x+x-\frac{1}{a}x^2-\frac{bc}{a^2}x+\frac{bc}{a^3}x^2+(\frac{1}{c}-\frac{c}{ab})\cdot\frac{1}{2}(b-\frac{b}{a}x)^2dx$

$=-2c(bc-\frac{bc}{a^2}\cdot\frac{1}{2}a^2+\frac{1}{2}a^2-\frac{1}{a}\cdot\frac{a^3}{3}-\frac{bc}{a^2}\cdot\frac{a^2}{2}+\frac{bc}{a^3}\cdot\frac{a^3}{3}+(\frac{1}{c}-\frac{c}{ab})\cdot\frac{1}{6}\cdot b^3\cdot\frac{a}{b})$

$=-2c(\frac{1}{6}a^2+\frac{bc}{3}+\frac{1}{6}\cdot\frac{1}{c}\cdot ab^2-\frac{1}{6}bc)=\frac{-(a^2c+b^2a+c^2b)}{3}$

6. $\nabla\times\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{x^2} & -\frac{z}{x} & -\frac{y}{x}\end{vmatrix}=(0,0,0)$ 故 $\mathrm{rot}\,V=\vec{0}$，故存在 $u(x,y,z)$ 的全微分为该式. $du=\frac{yz}{x^2}dx-\frac{z}{x}dy-\frac{y}{x}dz$ $\quad \frac{\partial u}{\partial x}=\frac{yz}{x^2}$

$\therefore u=-\frac{yz}{x}+f(y,z)$ $\quad \frac{\partial u}{\partial y}=-\frac{z}{x}+f'_y=\frac{z}{x}$ $\quad \therefore f'_y=0$

$\therefore f=f(z)$ $\quad \frac{\partial u}{\partial z}=-\frac{y}{x}+f'_z=-\frac{y}{x}$ $\quad \therefore f'_z=0$ $\quad \therefore f=C$ (C为一常数)

$\therefore u=-\frac{yz}{x}+C$ (C为一常数)

(2)

$\nabla\times\vec{V}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x+z)^2+y^2} & \frac{-z-x}{(x+z)^2+y^2} & \frac{y}{(x+z)^2+y^2}\end{vmatrix}=(0,0,0)$ 故 $\mathrm{rot}\,V=\vec{0}$，故存在 $u(x,y,z)$ 的全微分为该式.

$du=\frac{y}{(x+z)^2+y^2}dx+\frac{-z-x}{(x+z)^2+y^2}dy+\frac{y}{(x+z)^2+y^2}dz$ $\quad \frac{\partial u}{\partial x}=\frac{1}{(x+z)^2+y^2}$

$\therefore u=\arctan\frac{x+z}{y}+f(y,z)$ $\quad \frac{\partial u}{\partial y}=\frac{-z-x}{(x+z)^2+y^2}+f'_y=\frac{-z-x}{(x+z)^2+y^2}$ $\quad \therefore f'_y=0$

$\frac{\partial u}{\partial z}=\frac{y}{(x+z)^2+y^2}+f'_z=\frac{y}{(x+z)^2+y^2}$ $\quad \therefore f'_z=0$ $\quad \therefore f=C$ (C为一常数)

$\therefore u=\arctan\frac{x+z}{y}+C$ (C为一常数)

7.

$\nabla\times\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y\end{vmatrix}=(0,0,0)$ 故积分与路径无关

原式 $=\int_{(0,0,0)}^{(1,0,0)}(y+z)dx+\int_{(1,0,0)}^{(1,2,0)}(z+x)dy+\int_{(1,2,0)}^{(1,2,1)}(x+y)dz$

$=0+\int_0^2 dy+\int_0^1 3dz=5$

(2)

$\nabla\times\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & 2xyz & xy^2\end{vmatrix}=(0,0,0)$ 故积分与路径无关

原式 $=\int_{(1,-1,1)}^{(1,1,1)}2xyz\,dy+\int_{(1,1,1)}^{(1,1,-1)}xy^2dz=2\int_{-1}^{1}y\,dy+\int_1^{-1}dz=-2$

习题5.1

2. 设 $\lim\limits_{n\to+\infty}S_{2n+1}=A$，则 $\forall\varepsilon>0,\exists N_1\in N$ s.t. $\forall n>N_1$, $|S_{2n+1}-A|\le\frac{\varepsilon}{2}$ 又 $\lim\limits_{n\to+\infty}u_n=0$
则 $\exists N_2\in N$ s.t. $\forall n>N_2$, $|u_{2n+2}|\le\frac{\varepsilon}{2}$. 令 $N_0=N_1+N_2$ 则 $\forall\varepsilon>0$, $\exists N_0\in N$
$|S_{2n+2}-A|=|S_{2n+1}+u_{2n+2}-A|\le|S_{2n+1}-A|+|u_{2n+2}|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le\varepsilon$
又 $\because n>N_0>N_1$，故 $|S_{2n+1}-A|\le\frac{\varepsilon}{2}<\varepsilon$ 故 $\lim\limits_{n\to+\infty}S_{2n+1}=\lim\limits_{n\to+\infty}S_{2n+2}=A$.
$\therefore \lim\limits_{n\to+\infty}S_n=A$. 即 $\sum\limits_{n=1}^{+\infty}u_n$ 收敛

5.

① $\sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)$ 收敛 $\Rightarrow \sum_{n=1}^{+\infty}u_n$ 收敛.

设 $\sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)=A$. 则 $A=\lim\limits_{n\to+\infty}2(u_2-u_1)+3(u_3-u_2)+\cdots+(n+1)(u_{n+1}-u_n)$

$=\lim\limits_{n\to+\infty}(n+1)u_{n+1}-(u_n+u_{n-1}+u_{n-2}+\cdots+u_1)-u_1$ 而 $\lim\limits_{n\to+\infty}n\cdot u_n=0$. 故:

$A=-\lim\limits_{n\to+\infty}(u_1+\cdots+u_n)-u_1$ ∴ $\lim\limits_{n\to+\infty}(u_1+u_2+\cdots+u_n)$ 收敛 即 $\sum_{n=1}^{+\infty}u_n$ 收敛.

② $\sum_{n=1}^{+\infty}u_n$ 收敛 $\Rightarrow \sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)$ 收敛

设 $\sum_{n=1}^{+\infty}u_n=A$ 则 $\sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)=\lim\limits_{n\to+\infty}2(u_2-u_1)+\cdots+(n+1)(u_{n+1}-u_n)$

$=\lim\limits_{n\to+\infty}(n+1)u_{n+1}-(u_n+u_{n-1}+u_{n-2}+\cdots+u_1)-u_1$ 而 $\lim\limits_{n\to+\infty}n\cdot u_n=0$. 故:

$\sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)=-\lim\limits_{n\to+\infty}(u_1+\cdots+u_n)-u_1=-A-u_1\in R$.

故 $\sum_{n=1}^{+\infty}(n+1)(u_{n+1}-u_n)$ 收敛

6. (1) 设通项为 a_k. 部分和为 $S_n=\sum_{k=1}^{n}a_k$ $\lim\limits_{n\to+\infty}S_n=\lim\limits_{n\to\infty}100[(\frac{1}{4})^0+(\frac{1}{4})^1+\cdots+(\frac{1}{4})^{n-1}]$

$=\lim\limits_{n\to+\infty}100\cdot(\frac{1}{4})^0\cdot\frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}=\frac{400}{3}$ ∴ $\sum_{n=1}^{+\infty}a_n$ 存在, 且 $\sum_{n=1}^{+\infty}a_n=\frac{400}{3}$

(3) 设通项为 a_k. 部分和为 $S_n=\sum_{k=1}^{n}a_k$ $\lim\limits_{n\to+\infty}S_n=\lim\limits_{n\to+\infty}(\frac{1}{1\times5}+\frac{1}{3\times7}+\cdots+\frac{1}{(2n-1)(2n+3)})$

$=\frac{1}{4}\lim\limits_{n\to+\infty}\frac{1}{1}-\frac{1}{5}+\frac{1}{3}-\frac{1}{7}+\cdots+\frac{1}{2n-1}-\frac{1}{2n+3}=\frac{1}{4}\lim\limits_{n\to+\infty}1+\frac{1}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}=\frac{1}{4}\cdot\frac{4}{3}=\frac{1}{3}$

∴ $\sum_{n=1}^{+\infty}a_n$ 存在, 且 $\sum_{n=1}^{+\infty}a_n=\frac{1}{3}$

(5) 设通项为 a_k. 部分和为 $S_n=\sum_{k=1}^{n}a_k$ 若 $\lim\limits_{n\to+\infty}S_n$ 存在. 则 $\lim\limits_{n\to+\infty}S_{2n+2}-S_{2n+1}=0$.

也即 $\lim\limits_{n\to+\infty}a_{2n+2}=0$. 而 $\lim\limits_{n\to+\infty}a_{2n+2}=\lim\limits_{n\to+\infty}\frac{(-1)^{2n+2}\cdot(2n+2)^3}{2\cdot(2n+2)^3+(2n+2)}=\frac{1}{2}\neq0$

故矛盾. ∴ $S_n=\sum_{k=1}^{n}a_k$ 不收敛

(7) 设通项为 a_k. 部分和为 $S_n=\sum_{k=1}^{n}a_k$ $\lim\limits_{n\to+\infty}a_n=\lim\limits_{n\to+\infty}\arctan\frac{1}{2n^2}=\frac{1}{2n^2}+O(\frac{1}{n^2})$ $n\to+\infty$

∴ $\exists N_0\in N$, s.t. $\forall n>N_0$, $\arctan\frac{1}{2n^2}<\frac{1}{n^2}$

$\lim\limits_{n\to+\infty}S_n=\sum_{n=1}^{N_0}a_n+\sum_{n=N_0+1}^{+\infty}a_n<\sum_{n=1}^{N_0}a_n+\sum_{n=N_0+1}^{+\infty}\frac{1}{n^2}<\sum_{n=1}^{N_0}a_n+\sum_{n=N_0+1}^{+\infty}\frac{1}{(n-1)n}=\sum_{n=1}^{N_0}a_n+\lim\limits_{n\to+\infty}\frac{1}{N_0}-\frac{1}{n}$

$=\sum_{n=1}^{N_0}a_n+\frac{1}{N_0}$ 故 $\lim\limits_{n\to+\infty}S_n$ 存在. 即 $\sum_{n=1}^{+\infty}a_n$ 收敛

又 $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$ 令 $x=\arctan\alpha$. $y=\arctan\beta$. 则:

$\tan(\arctan\alpha-\arctan\beta)=\frac{\tan(\arctan\alpha)-\tan(\arctan\beta)}{1+\tan(\arctan\alpha)\cdot\tan(\arctan\beta)}=\frac{\alpha-\beta}{1+\alpha\beta}$

∴ $\arctan\alpha-\arctan\beta=\arctan\frac{\alpha-\beta}{1+\alpha\beta}$ 故令 $\alpha=2n+1$, $\beta=2n-1$ 则

$\arctan(2n+1)-\arctan(2n-1)=\arctan\frac{1}{2n^2}$

$\therefore \lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to+\infty} \arctan3 - \arctan1 + \cdots + \arctan(2n+1) - \arctan(2n-1)$

$= \lim\limits_{n\to+\infty} \arctan(2n+1) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ $\therefore \sum\limits_{n=1}^{+\infty} a_n$存在，且$\sum\limits_{n=1}^{+\infty} a_n = \frac{\pi}{4}$

(9) 设通项为a_k，部分和为$S_n = \sum\limits_{k=1}^{n} a_k$ $\because \lim\limits_{n\to+\infty} \sqrt[n]{n} = 1$ $\therefore \exists N_0 \in N, s.t. \forall n > N_0, \sqrt[n]{n} > \frac{1}{2}$

$\therefore \lim\limits_{n\to+\infty} S_n = \lim\limits_{n\to+\infty} (\sum\limits_{k=1}^{N_0} a_k + \sum\limits_{k=N_0+1}^{n} a_k) > \sum\limits_{k=1}^{N_0} a_k + \lim\limits_{n\to+\infty} \frac{1}{2}(n-N_0)$ 发散

$\therefore S_n = \sum\limits_{k=1}^{n} a_k$ 不收敛

7. $m>0$且$m\in Z$ 故$\forall n\in Z^+$, $\frac{1}{n(n+m)} < \frac{1}{n^2}$ 又$\because \sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$收敛 由比较判别法有

$\sum\limits_{n=1}^{+\infty} \frac{1}{n(n+m)}$ 收敛 $\therefore \sum\limits_{n=1}^{+\infty} \frac{1}{n(n+m)} = \lim\limits_{n\to+\infty} \frac{1}{1\cdot(1+m)} + \frac{1}{2\cdot(2+m)} + \cdots + \frac{1}{n(n+m)}$

$= \frac{1}{m} \lim\limits_{n\to+\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \frac{1}{1+m} - \frac{1}{2+m} - \cdots - \frac{1}{n+m})$ 又$\because n\to+\infty$，故$n > m+1$

$\therefore$ 原式 $= \frac{1}{m} \lim\limits_{n\to+\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \frac{1}{n+m} - \frac{1}{n+m-1} - \cdots - \frac{1}{n+1})$

$= \frac{1}{m} \lim\limits_{n\to+\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - 0) = \frac{1}{m} \sum\limits_{n=1}^{m} \frac{1}{n}$

习题5.2

1.(1) 设通项为a_k，部分和为$S_n = \sum\limits_{k=1}^{n} a_k$ 则$\lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{2n-1}{2n+1} \cdot \frac{1}{2} = \frac{1}{2} < 1$

由D'Alembert判别法有：$\sum\limits_{k=1}^{\infty} a_k$ 收敛

(3) $\lim\limits_{n\to+\infty} \frac{\frac{1}{\ln^n}}{\frac{1}{n}} = \lim\limits_{n\to+\infty} \frac{n}{\ln^n} = +\infty$ 且$\sum\limits_{n=1}^{\infty} \frac{1}{n}$发散，故由比较判别法 $\sum\limits_{n=1}^{\infty} \frac{1}{\ln^n}$ 发散

(5) $\lim\limits_{n\to+\infty} \frac{\left(\frac{1+n^2}{1+n^3}\right)^2}{\frac{1}{n^2}} = \lim\limits_{n\to+\infty} \frac{n^2(n^4+2n^2+1)}{n^6+2n^3+1} = 1$ 且$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$收敛

故由比较判别法 $\sum\limits_{n=1}^{\infty} \left(\frac{1+n^2}{1+n^3}\right)^2$ 收敛

(7) $\lim\limits_{n\to+\infty} \frac{\frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)}{n^{-\frac{3}{2}}} = \lim\limits_{n\to+\infty} \ln(1+\frac{2}{n-1}) \cdot n = \lim\limits_{n\to+\infty} \frac{2n}{n-1} = 2$ 且$\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ 收敛

由比较判别法 $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right)$ 收敛

2. 设通项为a_k，部分和为$S_n = \sum\limits_{k=1}^{n} a_k$

(1) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{2^{n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{2^n} = \lim\limits_{n\to+\infty} \frac{1}{n(2n+1)} = 0$

$(1+\frac{1}{t})^t$

$\therefore$ 由D'Alembert判别法有：$\sum\limits_{k=1}^{\infty} a_k$ 收敛

(3) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to+\infty} \frac{3^{n+1}(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n \cdot n} = 3 \lim\limits_{n\to+\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n} = 3 \lim\limits_{n\to+\infty} \left(1 - \frac{1}{n+1}\right)^{-(n+1)\cdot\frac{n}{-(n+1)}} \cdot \frac{1}{n}$

$= 3 \lim\limits_{n\to\infty} \frac{1}{e} \cdot \frac{1}{n} = 0 < 1$ $\therefore$ 由D'Alembert判别法有：$\sum\limits_{k=1}^{\infty} a_k$ 收敛

(5) $\because \lim\limits_{n\to+\infty} \dfrac{n^3 \cdot \sin\frac{\pi}{3^n}}{\frac{1}{n^2}} = \lim\limits_{n\to+\infty} n^5 \cdot \frac{\pi}{3^n} = \pi \lim\limits_{n\to+\infty} \frac{n^5}{3^n} = 0$ 又 $\because \sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$ 收敛

$\therefore$ 由 D'Alembert 判别法有 $\sum\limits_{k=1}^{\infty} a_k$ 收敛

王振波老师指出此类题仅考虑参数 p, q, r 为正

3. 设通项为 a_k. 部分和为 $S_n = \sum\limits_{k=1}^{n} a_k$

(1) $\because \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = \lim\limits_{n\to+\infty} \frac{2}{\sqrt{n}} = 0 < 1$ $\therefore$ 由 Cauchy 根式判别法有 $\sum\limits_{k=1}^{\infty} a_k$ 收敛

(3) ① $p > 1$ 时, $0 < a_n < \frac{1}{n^p}$ 且 $\sum\limits_{n=3}^{\infty} \frac{1}{n^p}$ 收敛 故由比较判别法 $\sum\limits_{n=3}^{\infty} a_n$ 收敛

② $p < 1$ 时, $\lim\limits_{n\to+\infty} \dfrac{a_n}{\frac{1}{n}} = \lim\limits_{n\to+\infty} \dfrac{n^{1-p}}{(\ln n)^q (\ln\ln n)^r} = +\infty$ 且 $\sum\limits_{n=3}^{\infty} \frac{1}{n}$ 发散

故由比较判别法 $\sum\limits_{n=3}^{\infty} a_n$ 发散

③ $p=1$ 时, 先讨论 $q=1$. 即考虑 $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \cdot \ln n \cdot (\ln\ln n)^r}$

(3.1) $r=1$ 时, 由积分判别法 $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \cdot \ln n \cdot \ln(\ln n)}$ 与 $\int_3^{+\infty} \dfrac{dn}{n \cdot \ln n \cdot \ln(\ln n)}$ 同敛散

且 $\int_3^{+\infty} \dfrac{dn}{n \cdot \ln n \cdot \ln(\ln n)} = \int_3^{+\infty} \dfrac{d\ln n}{\ln n (\ln \ln n)} = \int_{\ln 3}^{+\infty} \dfrac{dt}{t \cdot \ln t} = \int_{\ln\ln 3}^{+\infty} \dfrac{dy}{y} = \ln y \Big|_{\ln\ln 3}^{+\infty}$

故 $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \cdot \ln n \cdot \ln(\ln n)}$ 发散, 即 $r=1$ 时, $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \cdot \ln n \cdot (\ln\ln n)^r}$ 发散

(3.2) $r<1$ 时, $\lim\limits_{n\to+\infty} \dfrac{\frac{1}{n \cdot \ln n \cdot (\ln\ln n)^r}}{\frac{1}{n \cdot \ln n \cdot \ln(\ln n)}} = \lim\limits_{n\to+\infty} (\ln\ln n)^{1-r} = +\infty$ 故而

由比较判别法, $r<1$ 时, $\sum\limits_{n=3}^{\infty} a_n$ 发散

(3.3) $r>1$ 时, $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \cdot \ln n \cdot (\ln\ln n)^r}$ 与 $\int_3^{+\infty} \dfrac{dn}{n \cdot \ln n \cdot (\ln\ln n)^r}$

而 $\int_3^{+\infty} \dfrac{dn}{n \cdot \ln n \cdot (\ln\ln n)^r} = \int_3^{+\infty} \dfrac{d\ln n}{\ln n (\ln\ln n)^r} = \int_{\ln 3}^{+\infty} \dfrac{dt}{t (\ln t)^r} = \int_{\ln\ln 3}^{+\infty} \dfrac{dy}{y^r}$ 收敛

故 $r>1$ 时, $\sum\limits_{n=3}^{\infty} a_n$ 收敛

④ $p=1, q>1$ 时, $\because r>0$, 故当 $n > e^e$ 后, $(\ln\ln n)^r > 1$ 故取 $N_0 = [e^e]+1$, 则

$\sum\limits_{n=3}^{\infty} a_n$ 与 $\sum\limits_{n=N_0}^{+\infty} a_n$ 同敛散. 而对后者, $n \geq N_0$ 后, $0 \leq a_n \leq \dfrac{1}{n (\ln n)^q}$.

而 $\sum\limits_{n=N_0}^{+\infty} \dfrac{1}{n(\ln n)^q}$ 与 $\int_{N_0}^{+\infty} \dfrac{dn}{n(\ln n)^q}$ 同敛散 $\int_{N_0}^{+\infty} \dfrac{dn}{n(\ln n)^q} = \int_{N_0}^{+\infty} \dfrac{d\ln n}{(\ln n)^q} = \int_{\ln N_0}^{+\infty} \dfrac{dt}{t^q}$, 收敛

故 $p=1, q>1$ 时, $\sum\limits_{n=3}^{\infty} a_n$ 收敛

⑤ $p=1, q<1$ 时 $\lim\limits_{n\to+\infty}\dfrac{\frac{1}{n(\ln n)^q(\ln\ln n)^r}}{\frac{1}{n\ln n}}=\lim\limits_{n\to+\infty}\dfrac{(\ln n)^{1-q}}{(\ln\ln n)^r}=+\infty \quad \forall r>0$

而 $\sum\limits_{n=3}^{+\infty}\dfrac{1}{n\ln n}$ 与 $\int_3^{+\infty}\dfrac{dx}{x\ln x}$ 同敛散 $\int_3^{+\infty}\dfrac{dx}{x\ln x}=\int_3^{+\infty}\dfrac{d\ln x}{\ln x}=\ln\ln x\Big|_3^{+\infty}=+\infty$

$\therefore p=1, q<1$ 时，$\sum\limits_{n=3}^{\infty}a_n$ 发散

$\therefore$ 综上所述 $\sum\limits_{n=3}^{\infty}a_n\begin{cases}\text{收敛} & p>1\\ \text{收敛} & p=1,\ q>1\\ \text{收敛} & p=1,\ q=1,\ r>1\\ \text{发散} & \text{其他}\end{cases}$

(5) $\lim\limits_{n\to+\infty}\sqrt[n]{a_n}=\lim\limits_{n\to+\infty}\sin(\frac{\pi}{4}+\frac{1}{n})=\frac{\sqrt{2}}{2}<1$ 由Cauthy根式判别法有：$\sum\limits_{k=1}^{\infty}a_k$ 收敛

(7) $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to+\infty}e^{-\frac{n^2+2n+2}{n+2}+\frac{n^2+1}{n+1}}=\lim\limits_{n\to+\infty}e^{\frac{-n^2-3n}{(n+1)(n+2)}}=\frac{1}{e}<1$

由 D'Alembert 判别法有：$\sum\limits_{k=1}^{\infty}a_k$ 收敛

(9) $\because 0\le\dfrac{3n-1}{2^n+2^{-n}}\le\dfrac{4n}{2^n}$ 而 $\sum\limits_{k=1}^{\infty}\dfrac{4k}{2^k}$ 与 $\int_1^{+\infty}\dfrac{4x}{2^x}dx$ 同敛散，且 $\int_1^{+\infty}\dfrac{4x}{2^x}dx$ 收敛

故由比较判敛法知：$\sum\limits_{k=1}^{\infty}a_k$ 收敛

5. $\because\{nu_n\}$ 有界，设 $\forall n\in N^*, |n\cdot u_n|\le M$，而 $u_n>0$，故 $n\cdot u_n\le M$.

$\therefore\forall n\in N^*$，$\dfrac{u_n}{n}\le\dfrac{M}{n^2}$. 又因为 $\sum\limits_{n=1}^{\infty}\dfrac{M}{n^2}$ 收敛，故由比较判敛法知：$\sum\limits_{n=1}^{\infty}\dfrac{u_n}{n}$ 收敛

8.(1) $n(\dfrac{a_n}{a_{n+1}}-1)=n\left(\dfrac{\sqrt{n!}}{(1+\sqrt{1})\cdots(1+\sqrt{n})}\cdot\dfrac{(1+\sqrt{1})(1+\sqrt{2})\cdots(1+\sqrt{n+1})}{\sqrt{(n+1)!}}-1\right)=n\cdot\left(\dfrac{1+\sqrt{n+1}}{\sqrt{n+1}}-1\right)=\dfrac{n}{\sqrt{n+1}}$

$\therefore\lim\limits_{n\to+\infty}n(\dfrac{a_n}{a_{n+1}}-1)=\lim\limits_{n\to+\infty}\dfrac{n}{\sqrt{n+1}}=+\infty>1$ 故：$\sum\limits_{k=1}^{\infty}a_k$ 收敛

8.(2) $n(\dfrac{a_n}{a_{n+1}}-1)=n\left(\dfrac{n!\cdot n^{-p}}{q(q+1)\cdots(q+n)}\cdot\dfrac{q(q+1)\cdots(q+n+1)}{(n+1)!\cdot(n+1)^{-p}}-1\right)=n\cdot\left((\frac{n}{n+1})^{-p}\cdot\dfrac{q+n+1}{n+1}-1\right)$

$\therefore\lim\limits_{n\to+\infty}n(\dfrac{a_n}{a_{n+1}}-1)=\lim\limits_{n\to+\infty}n\cdot\left((\frac{n}{n+1})^{-p}\cdot\dfrac{q+n+1}{n+1}-1\right)=\lim\limits_{n\to+\infty}(\frac{n}{n+1})^{1-p}\cdot(q+n+1)-n=q+1>1$

故：$\sum\limits_{k=1}^{\infty}a_k$ 收敛

9.

(1) $\lim\limits_{n\to+\infty}\dfrac{\frac{1}{\sqrt{n}}-\sqrt{\ln\frac{n+1}{n}}}{n^{-\frac{3}{2}}}=\lim\limits_{n\to+\infty}\dfrac{(\frac{1}{n})^{\frac{1}{2}}-(\ln(1+\frac{1}{n}))^{\frac{1}{2}}}{n^{-\frac{3}{2}}}=\lim\limits_{n\to+\infty}\dfrac{(\frac{1}{n})^{\frac{1}{2}}-[\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2})]^{\frac{1}{2}}}{n^{-\frac{3}{2}}}$

$=\lim\limits_{n\to+\infty}\dfrac{\frac{1}{2n^2}+o(\frac{1}{n^2})}{\sqrt{\frac{1}{n}}+\sqrt{\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2})}}\cdot n^{\frac{3}{2}}=\lim\limits_{n\to+\infty}\frac{1}{4}(n^{-\frac{3}{2}})\cdot n^{\frac{3}{2}}=\frac{1}{4}$

由比较判敛法知：$\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\frac{3}{2}}}$ 收敛 $\Rightarrow\sum\limits_{k=1}^{\infty}a_k$ 收敛

10. 设 $a_n=\frac{u_n}{u_n+1}$，一方面 $a_n=\frac{u_n}{u_n+1}\le\frac{u_n+u_n^2}{u_n+1}=u_n$

故由比较判别法 $\begin{cases}\sum_{n=1}^{+\infty}u_n\text{收敛}\Rightarrow\sum_{n=1}^{+\infty}a_n\text{收敛}\\ \sum_{n=1}^{+\infty}a_n\text{发散}\Rightarrow\sum_{n=1}^{+\infty}u_n\text{发散}\end{cases}$

另若 $\sum_{n=1}^{\infty}a_n$ 收敛，则 $\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}\frac{u_n}{u_n+1}=0$ 且 $u_n>0$ 故 $\lim_{n\to+\infty}u_n=0$ (1)

即 $\lim_{n\to+\infty}u_n=0$ $\therefore\exists N_0\in N$, s.t. $\forall n>N_0$, $u_n<1$ $\therefore\forall n>N_0$, $\frac{u_n}{2}=\frac{\frac{1}{2}u_n^2+\frac{1}{2}u_n}{u_n+1}<\frac{u_n}{u_n+1}=a_n$

故由比较判别法有 $\sum_{n=1}^{\infty}u_n$ 收敛，而由逆否命题有：若 $\sum_{n=1}^{\infty}u_n$ 发散，则 $\sum_{n=1}^{\infty}a_n$ 发散

综上所述：$\sum_{n=1}^{\infty}u_n$ 与 $\sum_{n=1}^{\infty}a_n$ 同敛散（敛散性相同）

11. 设 $a_n=\frac{n!}{n^n}$ 则 $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to+\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to+\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to+\infty}\left(1-\frac{1}{n+1}\right)^{-(n+1)\cdot\frac{-n}{n+1}}$

$=e^{-1}<1$. 故由比值判断法有 $\sum_{n=1}^{\infty}a_n$ 收敛. 而有级数收敛必要条件：$\lim_{n\to+\infty}\frac{n!}{n^n}=0$

(2) 设 $b_n=\frac{n^4}{a^n}$ 则 $\lim_{n\to+\infty}\sqrt[n]{b_n}=\lim_{n\to+\infty}\frac{\sqrt[n]{n^4}}{a}=\frac{1}{a}<1$，故由 Cauchy 根式判定法有 $\sum_{n=1}^{\infty}b_n$ 收敛

而有级数收敛必要条件：$\lim_{n\to+\infty}\frac{n^4}{a^n}=0$