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To our beloved Teaching Assistant:
Wish you and your family all the best and all is well. Pray for India.

Chenyang Zhao & Hanwen Cao
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1.

1)

$$L(1)=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}_{n\times1}=[e_1\,e_2\cdots e_n]_{n\times n}\cdot\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}_{n\times1}\quad L(x)=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}_{n\times1}=[e_1\,e_2\cdots e_n]_{n\times n}\cdot\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}_{n\times1}\quad\cdots\quad L(x^{n-1})=\begin{bmatrix}a_1^{n-1}\\a_2^{n-1}\\\vdots\\a_n^{n-1}\end{bmatrix}_{n\times1}=[e_1\,e_2\cdots e_n]_{n\times n}\cdot\begin{bmatrix}a_1^{n-1}\\a_2^{n-1}\\\vdots\\a_n^{n-1}\end{bmatrix}_{n\times1}$$

Hence $L([1\ x\ x^2\cdots x^{n-1}]_{n\times1})=[e_1\ e_2\cdots e_n]_{n\times n}\cdot\begin{bmatrix}1&a_1&a_1^2&\cdots&a_1^{n-1}\\1&a_2&a_2^2&\cdots&a_2^{n-1}\\\vdots&\vdots&\vdots&\cdots&\vdots\\1&a_n&a_n^2&\cdots&a_n^{n-1}\end{bmatrix}_{n\times n}$ It is the famous Vandermonde matrix. We name it as V.

2)

A linear map is invertible $\Leftrightarrow$ under the chosen basis for its domain and codomain, it's matrix is invertible. Hence we wanna prove V is invertible if and only if $a_i\neq a_j\ (\forall i\neq j)$

det for Vandemonde matrix $V=\begin{bmatrix}1&a_1&a_1^2&\cdots&a_1^{n-1}\\1&a_2&a_2^2&\cdots&a_2^{n-1}\\\vdots&\vdots&\vdots&\cdots&\vdots\\1&a_n&a_n^2&\cdots&a_n^{n-1}\end{bmatrix}_{n\times n}$ is $\prod\limits_{n\geq i\geq j\geq1}(a_i-a_j)$.

① L is invertible $\Rightarrow a_i\neq a_j\ (\forall i\neq j)$
L is invertible implies $\det V=\prod\limits_{n\geq i\geq j\geq1}(a_i-a_j)\neq0$. i.e. for every factor $(a_i-a_j)\ (n\geq i\geq j\geq1)$ of $\det V$. $a_i-a_j\neq0$ $\therefore a_i\neq a_j\ (\forall i\neq j)$

② $a_i\neq a_j\ (\forall i\neq j)\Rightarrow L$ is invertible
for every factor $(a_i-a_j)\ (n\geq i\geq j\geq1)$ of $\det V$. $a_i-a_j\neq0$ Hence $\det V=\prod\limits_{n\geq i\geq j\geq1}(a_i-a_j)\neq0$
Hence V is invertible $\Rightarrow L$ is invertible

3) ① $a_i\neq a_j\ (\forall i\neq j)\Rightarrow ev_{a_1}\cdots ev_{a_n}$ form a basis for V^*
We know for finite dimentional vector space V, $\dim V=\dim V^*$. Hence we need to prove $ev_{a_1}\cdots ev_{a_n}$ is linear indepedent in V^*.
Let $c_1\cdot ev_{a_1}+c_2\cdot ev_{a_2}+\cdots c_n\cdot ev_{a_n}=0$ for some $c_1,\cdots,c_n\in F$.
And $ev_{a_i}(p)=P(a_i)$ Hence:

$$(c_1\cdot ev_{a_1}+c_2\cdot ev_{a_2}+\cdots c_n\cdot ev_{a_n})(1)=c_1+\cdots+c_n=0$$
$$(c_1\cdot ev_{a_1}+c_2\cdot ev_{a_2}+\cdots c_n\cdot ev_{a_n})(x)=c_1a_1+\cdots+c_na_n=0$$
$$\vdots$$
$$(c_1\cdot ev_{a_1}+c_2\cdot ev_{a_2}+\cdots c_n\cdot ev_{a_n})(x^{n-1})=c_1a_1^{n-1}+\cdots+c_na_n^{n-1}=0$$

Hence $\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}_{n\times n} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$

from 2) we know $a_i \neq a_j \ (\forall i \neq j) \Rightarrow V$ is invertible

Hence we know L is linear bijection, $\ker L = \{0\}$ $\therefore c_1 = \cdots = c_n = 0$

Hence $ev_{a_1} \cdots ev_{a_n}$ form a basis for V^*

② $ev_{a_1} \cdots ev_{a_n}$ form a basis for $V^* \Rightarrow a_i \neq a_j \ (\forall i \neq j)$

If $\exists\, i \neq j$, s.t. $a_i = a_j$. Hence from 2) we know $\det V = \prod_{n \geq i > j \geq 1} (a_i - a_j) = 0$

Hence $\ker V \neq 0$ Hence Exist some $[c_1 \cdots c_n]^T \neq 0$

s.t. $\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}_{n\times n} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$ Hence $c_1 \cdot ev_{a_1} + c_2 \cdot ev_{a_2} + \cdots c_n \cdot ev_{a_n} = 0$

Exist a solution which not all c_i is zero

Hence $ev_{a_1} \cdots ev_{a_n}$ isn't linear indedentp in V^*. $ev_{a_1} \cdots ev_{a_n}$ cannot form a basis for V^*

So $ev_{a_1} \cdots ev_{a_n}$ form a basis for $V^* \Rightarrow a_i \neq a_j \ (\forall i \neq j)$

③ Therefore $ev_{a_1} \cdots ev_{a_n}$ form a basis for $V^* \Leftrightarrow a_i \neq a_j \ (\forall i \neq j)$

4) pick basis $\{1, x, x^2\}$ for V.

① for P_{-1}. $ev_0(P_{-1}) = P_{-1}(0) = 0$ $ev_1(P_{-1}) = P_{-1}(1) = 0$ $ev_{-1}(P_{-1}) = P_{-1}(-1) = 1$

Hence P_{-1} must be a multiple of $x(x-1)$. And we know V has degree < 3. $P_i \in V$. Hence degree of P_{-1} is less than 3. So $P_{-1} = c_1 x(x-1)$ for some $c_1 \in F$. $c_1 \neq 0$

$ev_{-1}(P_{-1}) = P_{-1}(-1) = c_1 \cdot (-1) \cdot (-2) = 1$ $\therefore c_1 = \frac{1}{2}$ $\therefore P_{-1} = \frac{1}{2} x(x-1)$

② for P_0 just like P_{-1}. $ev_0(P_0) = P_0(0) = 1$ $ev_1(P_0) = P_0(1) = 0$ $ev_{-1}(P_0) = P_0(-1) = 0$

Hence P_0 must be a multiple of $(x+1)(x-1)$ And we know V has degree < 3. $P_i \in V$. Hence degree of P_0 is less than 3. So $P_0 = c_2(x+1)(x-1)$ for some $c_2 \in F$. $c_2 \neq 0$

$ev_0(P_0) = P_0(0) = c_2(0+1)(0-1) = 1$ $\therefore c_1 = -1$ $\therefore P_{-1} = -(x+1)(x-1)$

③ for P_1 just like P_1. $ev_0(P_1) = P_1(0) = 0$ $ev_1(P_1) = P_1(1) = 1$ $ev_{-1}(P_1) = P_1(-1) = 0$

Hence P_1 must be a multiple of $x(x+1)$ And we know V has degree < 3. $P_i \in V$. Hence degree of P_1 is less than 3. So $P_1 = c_3(x+1)x$ for some $c_3 \in F$. $c_3 \neq 0$

$ev_1(P_1) = P_1(1) = c_3(1+1) 1 = 1$ $\therefore c_3 = \frac{1}{2}$ $\therefore P_1 = \frac{1}{2} x(x+1)$

5) Let $c_1ev_{-2}+c_2ev_{-1}+c_3ev_0+c_4ev_1+c_5ev_2=\vec{0}$ for some $c_1,\dots,c_5\in F$

Pick basis $\{1,x,x^2,x^3\}$ for V.

Hence

$(c_1ev_{-2}+c_2ev_{-1}+c_3ev_0+c_4ev_1+c_5ev_2)(1)=1\cdot c_1+1\cdot c_2+1\cdot c_3+1\cdot c_4+1\cdot c_5=0$

$(c_1ev_{-2}+c_2ev_{-1}+c_3ev_0+c_4ev_1+c_5ev_2)(x)=-2c_1-c_2+0\cdot c_3+1\cdot c_4+2\cdot c_5=0$

$(c_1ev_{-2}+c_2ev_{-1}+c_3ev_0+c_4ev_1+c_5ev_2)(x^2)=4c_1+1\cdot c_2+0\cdot c_3+1\cdot c_4+4\cdot c_5=0$

$(c_1ev_{-2}+c_2ev_{-1}+c_3ev_0+c_4ev_1+c_5ev_2)(x^3)=-8c_1+(-1)c_2+0\cdot c_3+1\cdot c_4+8c_5=0$

i.e. Let $A=\begin{bmatrix}1&1&1&1&1\\-2&-1&0&1&2\\4&1&0&1&4\\-8&-1&0&1&8\end{bmatrix}$, $c=\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix}$ $\quad A\cdot c=0$

Apply elementary matrix on left side of A. change A into rref(A)

$\text{rref}(A)=\begin{bmatrix}1&0&0&0&-1\\0&1&0&0&4\\0&0&1&0&-6\\0&0&0&1&4\end{bmatrix}$ $\quad A\cdot c=0\Leftrightarrow \text{rref}(A)\cdot c=0$ Hence a suitable $c=\begin{bmatrix}-1\\4\\-6\\4\\-1\end{bmatrix}$

So $-ev_{-2}+4ev_{-1}-6ev_0+4ev_1-ev_2$ sends every $v_i\in V$ to zero

And we know $ev_{-2}, ev_{-1}, ev_0, ev_1, ev_2$ is linear map. hence it's linear combination is a linear map. it is a non-trivial linear combination we wanted.

Definition 4.1.2. *Given a vector space V, its dual space V^* is the space of all linear maps from V to $\mathbb{R}$ (or to $\mathbb{C}$ if we were doing complex vector spaces).*

People call elements of V^* many things. Some popular choices are "dual vectors" and "linear functionals".

2. by definite. We need to prove α sends every elements of V to $\mathbb{R}$. and α is a linear map. Let L be the map. Yang Sir told us only need to consider $\mathbb{R}$.

1) $L(p(x))=ev_5(x+1)p(x)=6p(5)\in\mathbb{R}$. for $\forall\, p(x),q(x)\in V$ and $\forall\,\alpha,\beta\in\mathbb{R}$

$L(\alpha p(x)+\beta q(x))=ev_5[(x+1)(\alpha p(x)+\beta q(x))]=ev_5[\alpha(x+1)p(x)+\beta(x+1)q(x)]$

$=\alpha 6\cdot p(5)+\beta(6)q(5)=\alpha\cdot ev_5[(x+1)p(x)]+\beta\, ev_5[(x+1)q(x)]=\alpha L(p(x))+\beta L(q(x))$

$\therefore$ L is a linear map which send $\forall\, p(x)\in V$ to a real number

L is a dual vector

2) $L(p(x))=\lim\limits_{x\to\infty}\frac{p(x)}{x}$. let $p(x)=x^2$. $\lim\limits_{x\to\infty}\frac{p(x)}{x}=\lim\limits_{x\to\infty}x$. which doesn't exist.

it is also not a real number L is not a dual vector

3) for $\forall\, p(x)\in V$. Let $p(x)=a_2x^2+a_1x+a_0$ $\quad q(x)=b_2x^2+b_1x+b_0$

for some const $a_2,a_1,a_0,b_2,b_1,b_0\in\mathbb{R}$

$L(p(x))=\lim\limits_{x\to\infty}\frac{p(x)}{x^2}=\lim\limits_{x\to\infty}(a_2+\frac{a_1}{x}+\frac{a_0}{x^2})=a_2\in\mathbb{R}$

And for $\forall\,\alpha,\beta\in\mathbb{R}$, $L(\alpha p(x)+\beta q(x))=\lim\limits_{x\to\infty}\frac{\alpha p(x)+\beta q(x)}{x^2}=\lim\limits_{x\to\infty}\frac{(\alpha a_2+\beta b_2)x^2+(\alpha a_1+\beta b_1)x+(\alpha a_0+\beta b_0)}{x^2}$

$=\alpha a_2+\beta b_2=\alpha L(p(x))+\beta L(q(x))$

$\therefore$ L is a linear map which send $\forall\, p(x)\in V$ to a

L is a dual vector

4). $\forall P(x) \in V$. $L(p(x)) = P(3)P'(4) \in R$. Let $v=1$ $w=x$. Hence

$L(v) = v(3) \cdot v'(4) = 1 \cdot 0 = 0$ $\quad L(w) = w(3) \cdot w'(4) = 3 \cdot 1 = 3$

$L(v+w) = (v+w)(3) \cdot (v+w)'(4) = 4 \cdot 1 = 4$

Hence $L(v+w) \neq L(v) + L(w)$ so L is not a dual vector

5)

L is not linear. Let $p(x) = x^2$ $q(x) = x$ Hence $L(p(x)) = 2$ $L(q(x)) = 1$ $L(q(x)+q(x)) = \deg(x^2+x) = 2$

$L(p(x)) + L(q(x)) \neq L(p(x)+q(x))$ so L is not a dual vector

3.

$\nabla f: v \mapsto \nabla v f$. for $\forall v \in R^2$, Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $v_1, v_2 \in R$. $f = f(x,y)$. f is differetiable at P. hence

$f(p+tv) - f(p) = \left[\frac{\partial f}{\partial x}(p) \cdot v_1 + \frac{\partial f}{\partial y}(p) \cdot v_2\right] \cdot t + o(t)$ $t \to 0$ $\nabla f(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = \frac{\partial f}{\partial x}(p) v_1 + \frac{\partial f}{\partial y}(p) v_2$

$= \left[\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\therefore$ ∇f is a map sent $v \in R^2$ to R.

for $\forall$ $u, v \in R^2$. $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $u_1, u_2, v_1, v_2 \in R$. $\forall \alpha, \beta \in R$. Hence

Let $w = \alpha v + \beta u = \begin{bmatrix} \alpha v_1 + \beta u_1 \\ \alpha v_2 + \beta u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $w_1, w_2 \in R$

$\nabla f(\alpha v + \beta u) = \nabla f(w) = \left[\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left[\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p)\right] \begin{bmatrix} \alpha v_1 + \beta u_1 \\ \alpha v_2 + \beta u_2 \end{bmatrix}$

$= (\alpha v_1 + \beta u_1) \frac{\partial f}{\partial x}(p) + (\alpha v_2 + \beta u_2) \frac{\partial f}{\partial y}(p) = \alpha\left[\frac{\partial f}{\partial x}(p) v_1 + \frac{\partial f}{\partial y}(p) v_2\right] + \beta\left[\frac{\partial f}{\partial x}(p) u_1 + \frac{\partial f}{\partial y}(p) u_2\right]$

$= \alpha \nabla f(v) + \beta \nabla f(u)$

Hence ∇f is a linear map sent $\forall$ $v \in R^2$ to R. It's a dual vector space in $(R^2)^*$

Pick $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be the basis for R^2. We know the standard dual basis are $[1\ 0]$ and $[0\ 1]$ so ∇f's coordinates under the standard dual basis is $\begin{bmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial x}(p) \end{bmatrix}$

4. Reference book: Linear Algebra Done Right

1) Standing on the shoulder of Giants.

for a subspace $U \subset V$, we defined annihilator U° as: $U^\circ = \{f \in V^* : \forall x \in U, f(x) = 0\}$

first we need to prove if $U \subset V$, then $U^\circ \subset V^*$

A. $0 \in U^\circ$, which means U° includes zero vector

证明映射构成空间直接用映射性质

B. for $\forall f, \psi \in U^\circ$ and $\alpha, \beta \in F$. the $\forall u \in U$. $f(u) = \psi(u) = 0$. the $(\alpha f + \beta \psi)(u) = \alpha f(u) + \beta \psi(u) = 0$ Then $\alpha f + \beta \psi \in U^\circ$. So U° is a vector space

And by definition, $\forall v \in U^\circ$. v is also a vector of V^*. the $U^\circ \subset V^*$

The whole space for ran L is W. hence $(\text{Range } L)^\circ \subset W^*$

Then we need to prove $\ker(L^*) = (\text{Range } L)^\circ$

① $\ker(L^*) \subset (\text{Rang } L)^\circ$. $\forall f \in \ker(L^*)$. Since $L^* : W^* \to V^*$ $\therefore f \in W^*$

Definition 4.2.2. *For any linear map $L : V \to W$, we define its dual map to be the linear map $L^* : W^* \to V^*$ such that $L^*(\alpha) = \alpha \circ L$.*

Then by definition, $L^*(f) = f \circ L = 0$ So $\forall v \in V$ $L^*(f)(v) = (f \circ L)(v) = f(Lv) = 0$

For $\forall x \in \text{Range } L$, $\exists v \in V$, s.t. $Lv = x$ Hence f send $\forall x \in \text{Range } L$ to zero

$\therefore f \in (\text{Rang } L)^\circ$ $\therefore \ker(L^*) \subset (\text{Range } L)^\circ$

② $(\text{Ran } L)^\circ \subset \ker(L^*)$

$\forall f \in (\text{Ran } L)^\circ$. for $\forall v \in V$ $L(v) \in \text{Ran } L$. then $f(Lv) = 0$

$(f \circ L)(v) = 0$ $L^*(f)(v) = 0$ since v can be anything in V. and V is the domain of $L^*(f)$

So $L^*(f) = 0 \Rightarrow f \in \ker(L^*) \Rightarrow (\text{Ran } L)^\circ \subset \ker(L^*)$

③ by ① and ②. we know $(\text{ran } L)^\circ = \ker(L^*)$

2) we need to prove: for $(\ker L)$'s whole space V, $\ker L \subset V$, $(\ker L)^\circ \subset V^*$

$\text{Ran}(L^*) = (\ker L)^\circ$ We prove it in several steps

① $\dim V = \dim V^*$

Yang Sir's proof

Lemma 4.1.10. *If $\dim V = n$, then $\dim V^* = n$.*

Proof. The cheap way is to pick a basis, and pretent V is $\mathbb{R}^n$. Then it is the space of $n \times 1$ column vectors. Then the space V^* is the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}$, so it is the space of $1 \times n$ row vectors, and immediately $\dim V^* = n$. □

② if dim V is finite $U \subset V$. then $\dim U + \dim U^\circ = \dim V$

let $i : U \to V$ be the inclusion map. for $\forall u \in U$, $i(u) = u$

Then $i^* : V^* \to U^*$ $\dim \text{Ran}(i^*) + \dim \ker(i^*) = \dim(V^*)$

(Fundamental Theorem of Linear map)

$\forall \alpha \in \ker(i^*)$. $\Leftrightarrow i^*(\alpha) = 0 \Leftrightarrow \alpha \circ i = 0 \Leftrightarrow \forall u \in U$, $\alpha(iu) = 0$ but $i(u) = u$.

$\Leftrightarrow \alpha(u) = 0 \Leftrightarrow \alpha \in U^\circ$ $\therefore \ker(i^*) = U^\circ$

for any $f \in U^*$. define $\varphi \in V^*$, s.t. $\begin{cases} \varphi(x) = f(x), & x \in V \\ \varphi(x) = 0, & x \in V/U \end{cases}$

$i^*(\varphi) = \varphi \circ i : U \to R$

for $\forall u \in U$, $\varphi \circ i$ send u to $f(u)$ so $i^*(\varphi) = f$ $\therefore f \in Ran(i^*)$ $\therefore U^* \subset Ran(i^*)$

Also: $i^* : V^* \to U^*$, $Ran(i^*) \subset U^*$. Hence $Ran(i^*) = U^*$

Hence $\dim Ran(i^*) = \dim U^* = \dim U$ $\dim \ker(i^*) = \dim U^\circ$ and $\dim V = \dim V^*$

$\because \dim Ran(i^*) + \dim \ker(i^*) = \dim(V^*)$

$\therefore \dim U + \dim U^\circ = \dim V$

③ $\dim Ran(L) = \dim Ran(L^*)$

proof: $L^* : W^* \to V^*$. $\dim(W^*) = \dim Ran(L^*) + \dim \ker(L^*)$

(Fundamental Theorem of Linear map)

$\therefore \dim Ran(L^*) = \dim(W^*) - \dim \ker(L^*) = \dim W - \dim(Ran\, L)^\circ$ (from problem 1)

And $\dim Ran(L) + \dim(Ran\, L)^\circ = \dim W$ $\therefore \dim Ran(L^*) = \dim Ran(L)$

④ $\dim Ran(L^*) = \dim(\ker L)^\circ$

$\dim Ran(L^*) = \dim Ran(L)$ (from ③)

$= \dim V - \dim \ker(L)$ (Fundamental Theorem of Linear map)

$= \dim(\ker L)^\circ$ (from ②)

⑤ $Ran(L^*) \subset (\ker L)^\circ$

$\forall f \in Ran(L^*)$ $\exists \alpha \in W^*$, s.t. $L^*(\alpha) = f$. for $\forall v \in \ker(L)$. $L(v) = 0$

$f(v) = (L^*(\alpha))(v) = (\alpha \circ L)(v) = \alpha \circ (Lv) = \alpha \circ (0) = 0$

$\therefore f \in (\ker L)^\circ$ $\therefore Ran(L^*) \subset (\ker L)^\circ$

⑥ from ④. ⑤ $\dim Ran(L^*) = \dim(\ker L)^\circ$ and $Ran(L^*) \subset (\ker L)^\circ$

$\therefore Ran(L^*) = (\ker L)^\circ$

5. ① An inner product structure of a real vector space V is a map: which sent $V \times V$ to R, And it's bilinear, symmetric, and positive-definite

(1) Bilinear: for $\forall\ V, V_1, V_2, W, W_1, W_2 \in V$, $\forall \alpha, \beta \in R$

$\langle V, \alpha W_1 + \beta W_2 \rangle = V^T A (\alpha W_1 + \beta W_2) = \alpha V^T A W_1 + \beta V^T A W_2 = a \langle V, W_1 \rangle + b \langle V, W_2 \rangle$

$\langle \alpha V_1 + \beta V_2, W \rangle = (\alpha V_1 + \beta V_2)^T A W = (\alpha V_1^T + \beta V_2^T) A W = \alpha V_1^T A W + \beta V_2^T A W$

$= \alpha \langle V_1, W \rangle + \beta \langle V_2, W \rangle$

(2) Symmetric $\langle V, W \rangle = V^T A W$ Note that for $\forall\ V_1, V_2 \in R^n$, $V_1^T V_2 = V_2^T V_1$

$\langle V, W \rangle = V^T A W = (AW)^T V = W^T A^T V = W^T A V$ (Since A is symmetric) $= \langle W, V \rangle$

(3) positive-definite

Since A is linear-defined, $X^T A X = 0 \Longleftrightarrow X = 0$ $\quad X^T A X > 0 \Longleftrightarrow X \neq 0$

$\therefore \langle V, V \rangle = V^T A V \geqslant 0$. and $\langle V, V \rangle = 0$ if and only if $V = 0$

So it's a inner product

② $V^T \in (R^n)^*$. V^T is a linear map that sends $W \in R^n$ into $V^T W$.

Let the corresponding vector of V^T in R^n be X. By definition of bra map,

$V^T = \langle X |$ Then for $\forall W \in R^n$, $\langle X, W \rangle = V^T(W) = V^T W$

$\therefore X^T A W = V^T W$ for $\forall\ W \in R^n$ Let $W = e_1, e_2 \cdots e_n$. $X^T A W = 1, 2, \cdots, n$ th entry of $X^T A$

it is also the $1, 2, 3 \cdots n$ th entry of V^T

$\therefore$ i.e. every entry of $X^T A$ is equal to the same entry of V^T

$\therefore X^T A = V^T \Rightarrow A^T X = V$ $\therefore AX = V$

Because A is positive-defined, all eigenvalues of $A > 0$

$\therefore A$ is invertible Reise $(V^T) = A^{-1} V$

③ Reisz map: $\langle V | \rightarrow V$ So it's inverse: $V \rightarrow \langle V |$, the bra map

for $\forall\ W \in R^n$, $\langle V, W \rangle = V^T A W$

So the inverse of Reisz map sends V to $V^T A$

④ Reisz: $(R^n)^* \longrightarrow R^n$

Reisz*: $(R^n)^* \longrightarrow (R^n)^{**}$

We already know Reisz$(W^T) = A^{-1} W$, And Reisz$^*(V^T) \in (R^n)^{**}$, which evaluates $W^T \in (R^n)^*$

then Reisz$^*(V^T)(W^T) = V^T \cdot$ Reisz$(W^T) = V^T A^{-1} W = W^T A^{-1} V$

(A is symestric and invertible, so $(A^T)^{-1} = (A^{-1})^T = A^{-1}$)

$\therefore$ Reisz$^*(V^T) = A^{-1} V$.

Hence Reisz* = Reisz

(5) by definition and (4)

$\text{Reisz}^*: V^* \longrightarrow V$

$\text{Reisz}^* = \text{Reisz}$

$\text{Reisz}^{ad}: V \longrightarrow V^*$

$$\begin{tikzcd} V^* \arrow[r, "\text{Reisz}^*"] \arrow[d, shift left, "\text{Reisz}"] & V \arrow[d, shift left, "\text{bra}"] \\ V \arrow[r, "\text{Reisz}^{ad}"] \arrow[u, shift left, "\text{bra}"] & V^* \arrow[u, shift left, "\text{Reisz}"] \end{tikzcd}$$

Hence: for $\forall v \in V$

$$\begin{aligned} \text{Reisz}^{ad}(v) &= (\text{bra} \circ \text{Reisz}^* \circ \text{bra})(v) \\ &= (\text{bra} \circ \text{Reisz} \circ \text{bra})(v) \\ &= \text{bra}(v) = v^T A \end{aligned}$$