第 3 次习题课：复合函数求导、隐函数定理

1. 设 $z = f(x^2 y, \frac{y}{x})$， 其中 $f \in C^2$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial^2 z}{\partial x \partial y}$。

解：$\frac{\partial z}{\partial x} = 2xyf_1'(x^2 y, \frac{y}{x}) - \frac{y}{x^2} f_2'(x^2 y, \frac{y}{x})$， 简记为 $\frac{\partial z}{\partial x} = 2xyf_1' - \frac{y}{x^2} f_2'$；

$$\frac{\partial^2 z}{\partial x \partial y} = 2xy(f_{11}'' x^2 + \frac{1}{x} f_{12}'') + 2xf_1' - \frac{y}{x^2}(x^2 f_{21}'' + \frac{1}{x} f_{22}'') - \frac{1}{x^2} f_2'$$

$$= 2x^3 y f_{11}'' + y f_{12}'' - \frac{y}{x^3} f_{22}'' + 2xf_1' - \frac{1}{x^2} f_2'.$$

2. 可微二元函数 $f(x,y)$ 满足 $xf_x'(x,y) + yf_y'(x,y) = 0$，证明： $f(x,y)$ 恒为常数.

证法一：设 $x = r\cos\theta, y = r\sin\theta$，取 $\tau = (\cos\theta, \sin\theta)$，由 $xf_x'(x,y) + yf_y'(x,y) = 0$ 得

$$\frac{\partial f}{\partial \tau} = f_x'(r\cos\theta, r\sin\theta)\cos\theta + f_y'(r\cos\theta, r\sin\theta)\sin\theta = 0.$$

因而在任意从原点出发的射线上 $f(x,y)$ 为常数函数，故

$$f(x,y) = f(0,0), \qquad \forall (x,y) \in \mathbb{R}^2.$$

证法二：设 $u = f(x,y)$， $x = r\cos\theta, y = r\sin\theta$，由复合函数的链式法则及 $xf_x'(x,y) + yf_y'(x,y) = 0$ 得

$$u_r' = f_x'(r\cos\theta, r\sin\theta)\cos\theta + f_y'(r\cos\theta, r\sin\theta)\sin\theta = 0,$$

因此 $u = f(r\cos\theta, r\sin\theta)$ 与 r 无关。对任意 θ，令 $r \to 0^+$，由 f 的连续性得

$$f(x,y) = \lim_{r \to 0^+} f(r\cos\theta, r\sin\theta) = f(0,0).$$

证法三：令 $\varphi(t) = f(tx, ty)$，则

$$\varphi'(t) = xf_x'(tx,ty) + yf_y'(tx,ty) = \frac{1}{t}\left(txf_x'(tx,ty) + tyf_y'(tx,ty)\right) = 0, \quad \forall t > 0.$$

因而 $\varphi(t)$ 为常数， $\varphi(t) = \varphi(1), \forall t > 0$. 也即

$$f(tx, ty) = f(x,y), \quad \forall t > 0.$$

令 $t \to 0^+$，由 f 的连续性得 $f(x,y) = f(0,0)$.

3. 已知函数 $y=y(x)$ 满足方程 $ax+by=f(x^2+y^2)$，其中 a,b 是常数，求导函数 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$。

解： 方程 $ax+by=f\left(x^2+y^2\right)$ 两边对 x 求导，

$$a+b\frac{dy}{dx}=f'(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{2xf'(x^2+y^2)-a}{b-2yf'(x^2+y^2)}. \quad \square$$

4. 设函数 $x=x(z),\ y=y(z)$ 由方程组 $\begin{cases} x^2+y^2+z^2-1=0 \\ x^2+2y^2-z^2-1=0 \end{cases}$ 确定, 求 $\dfrac{dx}{dz},\ \dfrac{dy}{dz}$.

解： $\begin{cases} x^2+y^2=-z^2+1 \\ x^2+2y^2=z^2+1 \end{cases} \Rightarrow \begin{cases} 2x\dfrac{dz}{dx}+2y\dfrac{dz}{dy}=-2z \\ 2x\dfrac{dz}{dx}+4y\dfrac{dz}{dy}=2z \end{cases}$ 解方程得：

$$\begin{bmatrix} \dfrac{dx}{dz} \\ \dfrac{dy}{dz} \end{bmatrix}=-\frac{1}{4xy}\begin{bmatrix} 4y & -2y \\ -2x & 2x \end{bmatrix}\begin{bmatrix} 2z \\ -2z \end{bmatrix}=-\frac{1}{4xy}\begin{bmatrix} 12yz \\ -8xz \end{bmatrix}$$

由此得到 $\dfrac{dx}{dz}=\dfrac{3z}{x},\dfrac{dy}{dz}=-\dfrac{2z}{y}$. $\square$

5. 已知函数 $z=z(x,y)$ 由参数方程: $\begin{cases} x=u\cos v \\ y=u\sin v \\ z=uv \end{cases}$, 给定，试求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解: 这个问题涉及到复合函数微分法与隐函数微分法. x,y 是自变量, u,v 是中间变量(u,v 是 x,y 的函数)。先由 $z=uv$ 得到

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}$$

u,v 是由 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 确定的 x,y 的隐函数，在这两个等式两端分别对 x,y 求偏导数，得

$$\begin{cases} 1=\cos v\dfrac{\partial u}{\partial x}-u\sin v\dfrac{\partial v}{\partial x} \\ 0=\sin v\dfrac{\partial u}{\partial x}+u\cos v\dfrac{\partial v}{\partial x} \end{cases}, \qquad \begin{cases} 0=\cos v\dfrac{\partial u}{\partial y}-u\sin v\dfrac{\partial v}{\partial y} \\ 1=\sin v\dfrac{\partial u}{\partial y}+u\cos v\dfrac{\partial v}{\partial y} \end{cases}.$$

解得 $\dfrac{\partial u}{\partial x}=\cos v,\dfrac{\partial v}{\partial x}=\dfrac{-\sin u}{u},\dfrac{\partial u}{\partial y}=\sin v,\dfrac{\partial v}{\partial x}=\dfrac{\cos v}{u}$.

将这个结果代入前面的式子，得到

$$\frac{\partial z}{\partial x}=v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}=v\cos v-\sin v$$

$$\frac{\partial z}{\partial y}=v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}=v\sin v+\cos v. \qquad \square$$

6. 隐函数函数 $u=u(x,y)$ 由方程 $\begin{cases}u=f(x,y,z,t)\\ g(y,z,t)=0\\ h(z,t)=0\end{cases}$ 确定，求 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$

解： 函数关系分析：5（变量）– 3（方程）=2（自变量）；x,y 为自变量．后两个方程确定 2 个隐函数 $z=z(y),t=t(y)$，代入第一个方程得 $u(x,y)=f(x,y,z(y),t(y))$.

视 $g(y,z,t)=0$ 与 $h(z,t)=0$ 中 $z=z(y),t=t(y)$, 对 y 求导, 得

$$\left.\begin{aligned}\frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial g}{\partial t}\frac{\partial t}{\partial y}=0\\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial h}{\partial t}\frac{\partial t}{\partial y}=0\end{aligned}\right\}\Rightarrow\begin{pmatrix}\frac{\partial z}{\partial y}\\ \frac{\partial t}{\partial y}\end{pmatrix}=\left(\left|\frac{\partial(g,h)}{\partial(z,t)}\right|\right)^{-1}\begin{pmatrix}\frac{\partial h}{\partial t} & -\frac{\partial g}{\partial t}\\ -\frac{\partial h}{\partial z} & \frac{\partial g}{\partial z}\end{pmatrix}\begin{pmatrix}-\frac{\partial g}{\partial y}\\ 0\end{pmatrix}$$

于是

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x},$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial y}=\frac{\partial f}{\partial y}+\frac{\left(\frac{\partial f}{\partial t}\frac{\partial h}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial h}{\partial t}\right)\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}\frac{\partial h}{\partial t}-\frac{\partial g}{\partial t}\frac{\partial h}{\partial z}}. \square$$

7. $z=z(x,y)$ 由 $x^2+y^2+z^2=a^2$ 决定，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解： $2x+2z\frac{\partial z}{\partial x}=0$，$2y+2z\frac{\partial z}{\partial y}=0$

$$\frac{\partial z}{\partial x}=-\frac{x}{z},\quad \frac{\partial z}{\partial y}=-\frac{y}{z}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{y}{z^2}\cdot\frac{\partial z}{\partial x}=-\frac{xy}{z^3}. \square$$

8. $B^2-AC>0,C\neq 0,$ 设 $z=z(x,y)$ 二阶连续可微，并且满足方程

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=0.$$

若令 $\begin{cases}u=x+\alpha y\\ v=x+\beta y\end{cases}$，试确定 α,β 为何值时能变原方程为 $\frac{\partial^2 z}{\partial u\partial v}=0$.

解： 将 x,y 看成自变量，u,v 看成中间变量，利用链式法则得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha^2\frac{\partial^2 z}{\partial u^2}+2\alpha\beta\frac{\partial^2 z}{\partial u\partial v}+\beta^2\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}\right)=\alpha\frac{\partial^2 z}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 z}{\partial u\partial v}+\beta\frac{\partial^2 z}{\partial v^2}$$

由此可得， $0=A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=$

$$=\left(A+2B\alpha+C\alpha^2\right)\frac{\partial^2 z}{\partial u^2}+2\left(A+B(\alpha+\beta)+C\alpha\beta\right)\frac{\partial^2 z}{\partial u\partial v}+\left(A+2B\beta+C\beta^2\right)\frac{\partial^2 z}{\partial v^2}.$$

$B^2-AC>0$，则 $A+2Bt+Ct^2=0$ 有两不同实根。选取 α,β 为这两个实根，即

$$\alpha=-B+\sqrt{B^2-AC},\quad \beta=-B-\sqrt{B^2-AC},$$

或

$$\alpha=-B-\sqrt{B^2-AC},\quad \beta=-B+\sqrt{B^2-AC},$$

则有

$$A+2B\alpha+C\alpha^2=0,$$
$$A+2B\beta+C\beta^2=0,$$
$$A+B(\alpha+\beta)+C\alpha\beta=\frac{2(AC-B^2)}{C}\neq 0,$$

从而有 $\frac{\partial^2 z}{\partial u\partial v}=0$. □

9. 已知 $\begin{cases} w=x+y+z, \\ u=x, \\ v=x+y, \end{cases}$ $z=z(x,y)$ 二阶连续可微，化简方程

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=0,$$

以 w 为因变量，以 u,v 为自变量。

解：由已知条件可知

$$w=x+y+z(x,y)=v+z(u,v-u),$$

x,y 为中间变量，u,v 为自变量。$z=z(x,y)$ 二阶连续可微，混合偏导与求导次序无关，因而由复合函数的链式法则，有

$$\frac{\partial w}{\partial u}=\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y},\quad \frac{\partial^2 w}{\partial u^2}=\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2},$$

原方程可化简为 $\frac{\partial^2 w}{\partial u^2}+\frac{\partial w}{\partial u}=0$. □

10. 设$u(x,y)\in C^2$，又$\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}=0$, $u(x,2x)=x$，$u'_x(x,2x)=x^2$,求 $u''_{xx}(x,2x)$，

$u''_{xy}(x,2x)$，$u''_{yy}(x,2x)$.

解: $\dfrac{\partial u}{\partial x}(x,2x)=x^2$ 两边对 x 求导,得

$$\frac{\partial^2 u}{\partial x^2}(x,2x)+\frac{\partial^2 u}{\partial x\partial y}(x,2x)\cdot 2=2x. \qquad (1)$$

$u(x,2x)=x$,两边对 x 求导,得

$$\frac{\partial u}{\partial x}(x,2x)+\frac{\partial u}{\partial y}(x,2x)\cdot 2=1, \qquad \frac{\partial u}{\partial y}(x,2x)=\frac{1-x^2}{2}.$$

两再边对 x 求导,

$$\frac{\partial^2 u}{\partial x\partial y}(x,2x)+\frac{\partial^2 u}{\partial y^2}(x,2x)\cdot 2=-x. \qquad (2)$$

由已知 $$\frac{\partial^2 u}{\partial x^2}(x,2x)-\frac{\partial^2 u}{\partial y^2}(x,2x)=0, \qquad (3)$$

(1), (2), (3) 联立可解得:

$$\frac{\partial^2 u}{\partial x^2}(x,2x)=\frac{\partial^2 u}{\partial y^2}(x,2x)=-\frac{4}{3}x, \quad \frac{\partial^2 u}{\partial x\partial y}(x,2x)=\frac{5}{3}x.\square$$

11. 已知 $f(x,y)\in C^2(\mathbb{R}^2)$, $f>0$, $f''_{xy}f=f'_xf'_y$ 求证：$f(x,y)$必为分离变量型，即

$f(x,y)=u(x)v(y)$, 其中$u(\cdot),v(\cdot)$为一元函数.

证明：由已知条件可得

$$\frac{\partial}{\partial y}\left(\frac{f'_x}{f}\right)=\frac{f''_{xy}f-f'_xf'_y}{f^2}=0,$$

即$\dfrac{f'_x}{f}$与 y 无关，$\dfrac{f'_x}{f}=\varphi(x)$。而$\dfrac{\partial}{\partial x}(\ln f)=\dfrac{f'_x}{f}=\varphi(x)$，于是

$$\ln f(x,y)=\lambda(x)+\mu(y),$$

其中 $\lambda'(x)=\varphi(x)$.故 $f(x,y)=e^{\lambda(x)}e^{\mu(y)}$,为分离变量型.

12. 已知 $f(x,y)=g(x^2+y^2), g\in C^1, f(x,y)=\varphi(x)\varphi(y),\ f(0,0)=1,\ f(1,0)=e.$ 求 $f(x,y).$

解： 由已知条件得，当 $x_1^2+y_1^2=x_2^2+y_2^2$ 时，有

$$\varphi(x_1)\varphi(y_1)=f(x_1,y_1)=g(x_1^2+y_1^2)=g(x_2^2+y_2^2)=f(x_2,y_2)=\varphi(x_2)\varphi(y_2).$$

令 $r=\sqrt{x^2+y^2}$，则 $\varphi(x)\varphi(y)=\varphi^2(\frac{r}{\sqrt{2}})\geq 0, \forall(x,y)\in\mathbb{R}^2.$ 因此 $\forall x,y\in\mathbb{R},\ \varphi(x),\varphi(y)$ 不可能异号。不妨设 $\varphi\geq 0.$

由 $f(0,0)=1,\ f(1,0)=e,$ 得 $\varphi^2(0)=1,\varphi(0)\varphi(1)=e,$ 故 $\varphi(0)=1,\varphi(1)=e.$

下证 $\forall x\in\mathbb{R},\varphi(x)>0.$ 事实上, $\forall|x|\leq 1,$ 有

$$\varphi(x)\varphi(\sqrt{1-x^2})=\varphi(1)\varphi(0)=e>0,$$

因此有 $\varphi(x)>0,\forall|x|\leq 1.$ 继而 $\forall|x|\leq\sqrt{2},$ 有

$$\varphi(x)\varphi(\sqrt{2-x^2})=\varphi(1)\varphi(1)>0,$$

因此有 $\varphi(x)>0,\forall|x|\leq\sqrt{2}.$ 假设 $\varphi(x)>0,\forall|x|\leq 2^{n/2},$ 则

$$\varphi(x)\varphi(\sqrt{2^{n+1}-x^2})=\varphi(2^{n/2})\varphi(2^{n/2})>0,$$

因此有 $\varphi(x)>0,\forall|x|\leq 2^{(n+1)/2}.$ 由归纳证明法知 $\forall x\in\mathbb{R},\varphi(x)>0.$

等式 $\varphi(x)\varphi(y)=\varphi(r)\varphi(0)=\varphi(r)$ 左右两边分别对 x,y 求偏导，当 $r>0$ 时，有

$$\varphi'(x)\varphi(y)=\frac{x}{r}\varphi'(r),$$
$$\varphi(x)\varphi'(y)=\frac{y}{r}\varphi'(r).$$

若 $\exists x_0\neq 0,$ 使得 $\varphi'(x_0)=0,$ 则 $\forall y>0, r=\sqrt{x_0^2+y^2},$ 由第一式有 $\varphi'(r)=0,$ 再由第二式有 $\varphi'(y)=0,\forall y>0.$ 因此 $\varphi(y)$ 为常数，与 $\varphi(0)=1,\varphi(1)=e$ 矛盾。故 $\varphi'(x)\neq 0,\forall x\neq 0.$ 于是，当 $x\neq 0, y\neq 0$ 时，以上两式相除，得 $\dfrac{\varphi'(x)}{x\varphi(x)}=\dfrac{\varphi'(y)}{y\varphi(y)},$ 即

$$\frac{\varphi'(x)}{x\varphi(x)}\equiv c,\forall x\neq 0.$$

于是

$$\left(\ln|\varphi(x)|\right)' = cx,\ \ln|\varphi(x)| = c_1 x^2 + c_2,\ \varphi(x) = c_3 e^{c_1 x^2}.$$

再由 $\varphi(0)=1, \varphi(1)=e,$ 得

$$\varphi(x) = e^{x^2},\ f(x,y) = \varphi(x)\varphi(y) = e^{x^2+y^2}.$$