Solution 1

Euler Cat

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1 Solutions to exercises in chapter 1

1.1 1.2.2

Note that by definition

$$\begin{align} EX &= \int_0^\infty x * p(x)dx + \int_{-\infty}^0 x * p(x)dx \tag{1} \\ &= \int_0^\infty x * p(x)dx - \int_{-\infty}^0 (-x) * p(x)dx \tag{2} \\ & \tag{3} \end{align}$$

Note that the first part can be treated as the expectation with respect to a non-negative random variable, which is by Lemma 1.2.1 $\int_0^\infty P(X > t)dt$. And

$$\begin{align} &\int_{-\infty}^0 (-x) * p(x)dx \tag{4} \\ &= \int_0^\infty t * p(-t)dt \tag{5} \\ &= \int_0^\infty P(-X > t)dt \tag{6} \\ &= \int_0^\infty P(X < -t)dt \tag{7} \\ &= \int_{-\infty}^0 P(X < y)dy \tag{8} \end{align}$$

Where in (5) we use the replacement $t = -x$ and in (8) we use replacement again.

1.2 1.2.3

Note that

$$E|X|^p = \int_0^\infty P(|X|^p > t)dt \tag{9}$$

$$= \int_0^\infty P(|X| > t^{\frac{1}{p}})dt \tag{10}$$

$$= \int_0^\infty P(|X| > y)dy^p \tag{11}$$

$$= \int_0^\infty P(|X| > y) * p * y^{p-1}dy \tag{12}$$

1.3 1.2.6

Note that

$$P(|X-\mu| \geq t) \tag{13}$$

$$= P(|x-\mu|^2 \geq t^2) \tag{14}$$

$$\leq \frac{E|x-\mu|^2}{t^2} \tag{15}$$

$$= \frac{\sigma^2}{t^2} \tag{16}$$

1.4 1.3.3

The proof comes from link.

$$E|\frac{1}{N}\sum_{i=1}^{N}(X_i - \mu)| \tag{17}$$

$$\leq \sqrt{E|\frac{1}{N}\sum_{i=1}^{N}(X_i - \mu)|^2} \tag{18}$$

$$= \sqrt{\frac{1}{N} * Var(X_i)} \tag{19}$$

Where from (17) to (18) we use the Cauchy-Schwartz inequality.