第五章 求解线性方程组的迭代方法.

1.(1) $\|A\|_1=2$　$\|A\|_\infty=2$.

$$A^TA=\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$$ 令 $|A^TA-\lambda I|=0$ 得.

A^TA的特征值为 $\lambda_1=\frac{3-\sqrt5}{2}$　$\lambda_2=\frac{3+\sqrt5}{2}$. 则 $\|A\|_2=\sqrt{\frac{3+\sqrt5}{2}}=\frac{\sqrt5+1}{2}$

(2) $\|A\|_1=7$　$\|A\|_\infty=8$

$$A^TA=\begin{pmatrix}5&0&-1\\1&3&1\\1&0&6\end{pmatrix}\begin{pmatrix}5&1&1\\0&3&0\\-1&1&6\end{pmatrix}=\begin{pmatrix}26&4&-1\\4&11&7\\-1&7&37\end{pmatrix}$$ 令 $|A^TA-\lambda I|=0$ 得.

A^TA的特征值为 $\lambda_1=26.959$　$\lambda_2=8.276$　$\lambda_3=38.765$　则得 $\|A\|=\sqrt{38.765}=6.226$.

(3) $\|A\|_1=2$　$\|A\|_\infty=2$

$$A^TA=\cancel{\begin{pmatrix}0&0&1\\0&1&1\\1&0&0\end{pmatrix}}\begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&1\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&1\\0&1&2\end{pmatrix}$$ 令 $|A^TA-\lambda I|=0$ 得

A^TA的特征值为 $\lambda_1=1$　$\lambda_2=\frac{3-\sqrt5}{2}$　$\lambda_3=\frac{3+\sqrt5}{2}$　则 $\|A\|_2=\sqrt{\frac{3+\sqrt5}{2}}=\frac{\sqrt5+1}{2}$.

2.(1) $|B-\lambda I|=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}=(3-\lambda)^2-1=0$ 得 $\lambda_1=2$ $\lambda_2=4$

则 $\rho(B)=4$

(2) $|B-\lambda I|=\begin{vmatrix}5-\lambda&2&2\\2&6-\lambda&0\\2&0&4-\lambda\end{vmatrix}=2\times2(\lambda-6)+(4-\lambda)[(5-\lambda)(6-\lambda)-4]=0$.

得 $\lambda_1=2$　$\lambda_2=8$　$\lambda_3=5$　则 $\rho(B)=8$

3.(1) $\begin{cases} x_1^{(k+1)} = -\frac{1}{10}(-x_2^{(k)} - 1) \\ x_2^{(k+1)} = -\frac{1}{10}(-x_1^{(k)} - x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{10}(-x_2^{(k)} - x_4^{(k)} - 1) \\ x_4^{(k+1)} = -\frac{1}{10}(-x_3^{(k)} - 2) \end{cases}$

$x^{(1)} = (\frac{1}{10}, 0, \frac{1}{10}, \frac{1}{5})^T$

$x^{(2)} = (\frac{1}{10}, \frac{1}{50}, \frac{3}{25}, \frac{21}{100})^T$

$x^{(3)} = (\frac{51}{500}, \frac{11}{500}, \frac{123}{1000}, \frac{53}{250})^T$.

(2). $\begin{cases} x_1^{(k+1)} = -\frac{1}{10}(-x_2^{(k)} - 1) \\ x_2^{(k+1)} = -\frac{1}{10}(-x_1^{(k+1)} - x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{10}(-x_2^{(k+1)} - x_4^{(k)} - 1) \\ x_4^{(k+1)} = -\frac{1}{10}(-x_3^{(k+1)} - 2) \end{cases}$

$x^{(1)} = (\frac{1}{10}, \frac{1}{100}, \frac{11}{100}, \frac{211}{1000})^T$

$x^{(2)} = (\frac{101}{1000}, \frac{221}{10^4}, 0.12321, 0.212321)^T$

$x^{(3)} = (0.10211, 0.022532, 0.1234835, 0.21234853)$

(3) Jacobi 迭代矩阵为:

$$G = I - D^{-1}A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} & & & \\ & \frac{1}{10} & & \\ & & \frac{1}{10} & \\ & & & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 10 & -1 & & \\ -1 & 10 & -1 & \\ & -1 & 10 & -1 \\ & & -1 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$$

解 $|G - \lambda I| = \begin{vmatrix} -\lambda & 0.1 & 0 & 0 \\ 0.1 & -\lambda & 0.1 & 0 \\ 0 & 0.1 & -\lambda & 0.1 \\ 0 & 0 & 0.1 & -\lambda \end{vmatrix} = 0$ 得 $\lambda_1 = 0.162$ $\lambda_2 = 0.0618$ $\lambda_3 = -0.0618$ $\lambda_4 = -0.162$

$\rho(G) = 0.162 < 1$ 故其 Jacobi 迭代收敛

Gauss-Seidel 迭代矩阵为

$$G = I - Q^{-1}A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \left[\begin{pmatrix} 10 & & & \\ -1 & 10 & & \\ & -1 & 10 & \\ & & -1 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 10 & -1 & & \\ -1 & 10 & -1 & \\ & -1 & 10 & -1 \\ & & -1 & 10 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 0.1 & 0 & 0 \\ 0 & 0.01 & 0.1 & 0 \\ 0 & 1\times10^{-3} & 0.01 & 0.1 \\ 0 & 1\times10^{-4} & 1\times10^{-3} & 0.01 \end{pmatrix}$$

解 $|G - \lambda I| = 0$ 得.

$\lambda_1 = 0.004$ $\lambda_2 = 0.026$ $\lambda_3 = 1.7\times10^{-18}$

$\rho(G) = 0.026 < 1$ 故其 Gauss-Seidel 迭代收敛

4.(1) 其Jacobi的迭代格式为

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{2}(-x_2^{(k)}+x_3^{(k)}+1) \\ x_2^{(k+1)} = -\frac{1}{3}(3x_1^{(k)}+9x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{5}(3x_1^{(k)}+3x_2^{(k)}-4) \end{cases}$$

则 $x^{(1)}=(-\frac{1}{2}, 0, \frac{4}{5})$

$x^{(2)}=(-\frac{9}{10}, -\frac{19}{10}, \frac{11}{10})$

(2) 其Jacobi迭代格式为

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{5}(-x_2^{(k)}-x_3^{(k)}+1) \\ x_2^{(k+1)} = -\frac{1}{6}(3x_1^{(k)}+2x_3^{(k)}) \\ x_3^{(k+1)} = -\frac{1}{2}(x_1^{(k)}-x_2^{(k)}-4) \end{cases}$$

$x^{(1)}=(-\frac{1}{5}, 0, 2)$

$x^{(2)}=(\frac{1}{5}, -\frac{17}{30}, \frac{21}{10})$

(1) 5.

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{10}(-2x_2^{(k)}-x_3^{(k)}) \\ x_2^{(k+1)} = -\frac{1}{10}(-2x_1^{(k+1)}-x_3^{(k)}+21) \\ x_3^{(k+1)} = -\frac{1}{5}(-x_1^{(k+1)}-2x_2^{(k+1)}+20) \end{cases}$$

取 $x^{(0)}=(0,0,0)^T$

则 $x^{(1)}=(-\frac{1}{10}, -\frac{106}{50}, \frac{-1217}{250})^T=(-0.1, -2.12, -4.868)^T$

$x^{(2)}=(-0.9108, -2.76896, -5.28974)^T$

$x^{(4)}=(-1.10458, -2.35639, -5.36347)^T$ $x^{(3)}=(-1.08277, -2.34553, -5.35477)^T$

$x^{(5)}=(-1.10762, -2.35787, -5.36467)^T$ $x^{(6)}=(-1.10804, -2.35808, -5.36484)^T$

$x^{(7)}=(-1.10804, -2.35809, -5.36484)^T$ 则 $\|x^{(7)}-x^{(6)}\|_\infty=10^{-5}<10^{-4}$.

(2)

$$\begin{cases} x_1^{(k+1)} = -\frac{1}{5}(-x_2^{(k)}-x_3^{(k)}-16) \\ x_2^{(k+1)} = -\frac{1}{6}(3x_1^{(k+1)}+2x_3^{(k)}-11) \\ x_3^{(k+1)} = -\frac{1}{2}(x_1^{(k+1)}-x_2^{(k+1)}+2) \end{cases}$$

取 $x^{(0)}=(1,1,1)^T$

$x^{(1)}=(3.2, 0.5667, -2.3167)^T$ $x^{(2)}=(2.85, 1.1806, -1.8347)$

$x^{(3)}=(3.0692, 0.9103, -2.0795)$ $x^{(4)}=(2.9662, 1.0434, -1.9614)$

$\cancel{x^{(4)}=(3.4010,}$ $x^{(5)}=(3.0164, 0.9784, -2.0188)$ $x^{(6)}=(2.9920, 1.0102, -1.9909)$

$x^{(7)}=(3.0039, 0.9950, -2.0045)$ $x^{(8)}=(2.9981, 1.0025, -1.9978)$

$x^{(9)}=(3.0009, 0.9988, -2.00101)$ $x^{(10)}=(2.9995, 1.0006, -1.9995)$

$x^{(11)}=(3.0002, 0.9997, -2.0003)$ $x^{(12)}=(3.0000, 1.0001, 2.0000)$

则 $\|x^{(12)}-x^{(11)}\|_\infty=4\times10^{-4}<10^{-3}$

6(1) Jacobi 迭代的迭代矩阵为：

$$G = I - D^{-1}A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} 1 & \\ & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & t \\ t & 2 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ -\frac{t}{2} & 0 \end{pmatrix}$$ 令 $|G-\lambda I|=0$ 得

$\lambda^2 - \frac{t^2}{2} = 0$ 　$\lambda_1 = \frac{\sqrt{2}}{2}t$ 　$\lambda_2 = -\frac{\sqrt{2}}{2}t$

若迭代收敛，则 $\rho(G) = \frac{\sqrt{2}}{2}|t| < 1$ 即 $-\sqrt{2} < t < \sqrt{2}$

(2) Gauss-Seidel 迭代的迭代矩阵为

$$G = I - Q^{-1}A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} 1 & \\ t & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & t \\ t & 2 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ 0 & \frac{t^2}{2} \end{pmatrix}$$ 令 $|G-\lambda I|=0$ 得.

$-\lambda(\frac{t^2}{2} - \lambda) = 0$ 得 $\lambda_1 = 0$ 　$\lambda_2 = \frac{t^2}{2}$ 若迭代收敛则 $\rho(G) = \frac{t^2}{2} < 1$ 则 $-\sqrt{2} < t < \sqrt{2}$

7(1) $$G = I - D^{-1}A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

令 $|G-\lambda I|=0$ 得 $\lambda_1 = \lambda_2 = \lambda_3 = 0$ 　$\rho(G) = 0 < 1$ Jacobi 迭代收敛

其 Gauss-Seidel 迭代矩阵为：

$$G = I - Q^{-1}A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 2 & 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

令 $|G-\lambda I|=0$ 得 $\lambda(2-\lambda)^2 = 0$ 得 $\lambda_1 = 0$ 　$\lambda_2 = \lambda_3 = 2$ 　$\rho(G) = 2 > 1$ 故 Gauss-Seidel 迭代不收敛

(2) 其 Jacobi 迭代矩阵为：

$$G = I - D^{-1}A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & & \\ & 1 & \\ & & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & -0.5 \\ -1 & 0 & -1 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

令 $|G-\lambda I|=0$ 得 $\lambda_1 = 0$ 　$\lambda_2 = \frac{\sqrt{5}}{2}i$ 　$\lambda_3 = -\frac{\sqrt{5}}{2}i$ 　$\rho(G) = \frac{\sqrt{5}}{2} > 1$ 故 Jacobi 迭代不收敛

其 Gauss-Seidel 矩阵为：

$$G = I - Q^{-1}A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 2 & & \\ 1 & 1 & \\ 1 & 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & -0.5 \end{pmatrix}$$

令 $|G-\lambda I|=0$ 得 $-\lambda(0.5+\lambda)^2 = 0$ 　$\lambda_1 = 0$ 　$\lambda_2 = \lambda_3 = -0.5$ 　$\rho(G) = 0.5 < 1$ 故 Gauss-Seidel 迭代收敛

8. 略.