第四章. 求解线性方程组的直接法

1. (1) Gauss消元法.

$$\begin{pmatrix} 0.002 & 87.13 & 87.15 \\ 4.453 & -7.26 & 37.27 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.002 & 87.13 & 87.15 \\ 0 & -1.940\times10^5 & -1.940\times10^5 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 10.00 \\ x_2 = 1.000 \end{cases}$$

列主元法

$$\begin{pmatrix} 0.002 & 87.13 & 87.15 \\ 4.453 & -7.26 & 37.27 \end{pmatrix} \Rightarrow \begin{pmatrix} 4.453 & -7.26 & 37.27 \\ 0.002 & 87.13 & 87.15 \end{pmatrix} \Rightarrow \begin{pmatrix} 4.453 & -7.26 & 37.27 \\ 0 & 87.13 & 87.1 \end{pmatrix}$$

得 $\begin{cases} x_1 = 10.00 \\ x_2 = 1.000 \end{cases}$

(2) Gauss消元法.

$$\begin{pmatrix} 0.01 & -69.47 & -137.93 \\ 2.01 & 3.51 & 15.01 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.01 & -69.47 & -138.93 \\ 0 & -1.396\times10^4 & -2.791\times10^4 \end{pmatrix} \Rightarrow \begin{cases} x_1 = -5.947 \\ x_2 = 1.999. \end{cases}$$

列主元法

$$\begin{pmatrix} 0.01 & -69.47 & -137.93 \\ 2.01 & 8.51 & 15.01 \end{pmatrix} = \begin{pmatrix} 2.01 & 8.51 & 15.01 \\ 0.01 & -69.47 & -138.93 \end{pmatrix} = \begin{pmatrix} 2.01 & 8.51 & 15.01 \\ 0 & -69.43 & -138.8 \end{pmatrix}$$

得 $\begin{cases} x_1 = -0.9958 \\ x_2 = 1.999 \end{cases}$

2. 略.

3. (1)
$$\begin{vmatrix} -1 & 3 & 2 \\ 2 & 1 & -2 \\ 3 & 6 & 2 \end{vmatrix} \begin{matrix} \\ \times 2 \\ \times 3 \end{matrix} \Rightarrow \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 15 & 8 \end{vmatrix} \times\left(-\frac{15}{7}\right) \Rightarrow \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 0 & \frac{26}{7} \end{vmatrix} = -1\times7\times\frac{26}{7} = -26$$

(2)
$$\begin{vmatrix} 10 & -2 & -1 \\ -2 & 10 & -1 \\ -1 & -2 & 5 \end{vmatrix} \begin{matrix} \\ \times\frac{1}{5} \\ \times\frac{1}{10} \end{matrix} \Rightarrow \begin{vmatrix} 10 & -2 & -1 \\ 0 & \frac{48}{5} & -1 \\ 0 & -\frac{11}{5} & 5 \end{vmatrix} \times\frac{11}{48} \Rightarrow \begin{vmatrix} 10 & -2 & -1 \\ 0 & \frac{48}{5} & -1 \\ 0 & 0 & \end{vmatrix}$$

3 (1)
$$\begin{vmatrix} -1 & 3 & 2 \\ 2 & 1 & -2 \\ 3 & 6 & 2 \end{vmatrix} \xrightarrow{\times 2,\ \times 3} = \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 15 & 8 \end{vmatrix} \xrightarrow{\times(-\frac{15}{7})} \Rightarrow \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 0 & \frac{26}{7} \end{vmatrix} = -1\times 7\times \frac{26}{7} = -26$$

(2)
$$\begin{vmatrix} 10 & -2 & -1 \\ -2 & 10 & -1 \\ -1 & -2 & 5 \end{vmatrix} \xrightarrow{\times \frac{1}{5},\ \times\frac{1}{10}} = \begin{vmatrix} 10 & -2 & -1 \\ 0 & \frac{48}{5} & -\frac{6}{5} \\ 0 & -\frac{11}{5} & \frac{49}{10} \end{vmatrix} \xrightarrow{\times \frac{11}{48}} = \begin{vmatrix} 10 & -2 & -1 \\ 0 & \frac{48}{5} & -\frac{6}{5} \\ 0 & 0 & \frac{37}{8} \end{vmatrix} = 10\times\frac{48}{5}\times\frac{37}{8} = 444$$

4. (1)
$$\begin{pmatrix} 5 & 2 & 2 & 1 \\ -1 & 3 & 0 & 7 \\ 1 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{\times\frac{1}{5},\ \times(-\frac{1}{5})} \Rightarrow \begin{pmatrix} 5 & 2 & 2 & 1 \\ 0 & \frac{17}{5} & \frac{2}{5} & \frac{36}{5} \\ 0 & \frac{3}{5} & \frac{8}{5} & \frac{14}{5} \end{pmatrix} \xrightarrow{\times(-\frac{3}{17})} \Rightarrow \begin{pmatrix} 5 & 2 & 2 & 1 \\ 0 & \frac{17}{5} & \frac{2}{5} & \frac{36}{5} \\ 0 & 0 & \frac{26}{17} & \frac{26}{17} \end{pmatrix} \xrightarrow{-\frac{10}{17}} \Rightarrow \begin{pmatrix} 5 & 0 & \frac{30}{17} & -\frac{55}{17} \\ 0 & \frac{17}{5} & \frac{2}{5} & \frac{36}{5} \\ 0 & 0 & \frac{26}{17} & \frac{26}{17} \end{pmatrix} \xrightarrow{-\frac{15}{13},\ \times(-\frac{17}{65})}$$

$$\Rightarrow \begin{pmatrix} 5 & & & -5 \\ & \frac{17}{5} & & \frac{34}{5} \\ & & \frac{26}{17} & \frac{26}{17} \end{pmatrix} \quad 得 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 5 & 2 & 2 & -4 \\ 0 & 2 & 1 & 5 \\ 1 & -1 & 3 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 2 & 1 & 5 \\ 5 & 2 & 2 & -4 \end{pmatrix} \xrightarrow{\times(-5)} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 2 & 1 & 5 \\ 0 & 7 & -13 & 1 \end{pmatrix} \xrightarrow{\times(-\frac{7}{2})} \Rightarrow \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -\frac{33}{2} & -\frac{33}{2} \end{pmatrix} \xrightarrow{\times\frac{1}{2}}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{7}{2} & \frac{3}{2} \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -\frac{33}{2} & -\frac{33}{2} \end{pmatrix} \xrightarrow{\times\frac{7}{33},\ \times\frac{2}{33}} \Rightarrow \begin{pmatrix} 1 & & & -2 \\ & 2 & & 4 \\ & & -\frac{33}{2} & -\frac{33}{2} \end{pmatrix} \quad 得 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

5 (1) 系数矩阵的Doolittle分解为

$$\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & -1 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ & 3 & 1 \\ & & 3 \end{pmatrix} = LU$$

解 $Ly=b$ 即 $\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 18 \\ -39 \\ 24 \end{pmatrix}$ 得 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 18 \\ -21 \\ 27 \end{pmatrix}$

解 $Ux=y$ 即 $\begin{pmatrix} 2 & 1 & 2 \\ & 3 & 1 \\ & & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 18 \\ -21 \\ 27 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cancel{18}\ 5 \\ -10 \\ 9 \end{pmatrix}$

(2) 系数矩阵的Doolittle分解为

$$\begin{pmatrix} 3 & 1 & 2 \\ -3 & 1 & -1 \\ 6 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ & 2 & 1 \\ & & 1 \end{pmatrix} = LU$$

解 $Ly=b$ 即 $\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 23 \\ -10 \\ 12 \end{pmatrix}$ 得 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 23 \\ 13 \\ 5 \end{pmatrix}$

解 $Ux=y$ 即 $\begin{pmatrix} 3 & 1 & 2 \\ & 2 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

6 (1) 系数矩阵的Crout分解为

$$\begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 5 & & \\ 1 & \frac{14}{5} & \\ 2 & \frac{13}{5} & \frac{11}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{5} & \frac{2}{5} \\ & 1 & -\frac{1}{2} \\ & & 1 \end{pmatrix} = LU$$

解 $Ly=b$ 即 $\begin{pmatrix} 5 & & \\ 1 & \frac{14}{5} & \\ 2 & \frac{13}{5} & \frac{11}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ 15 \end{pmatrix}$ 得 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

解 $Ux=y$ 即 $\begin{pmatrix} 1 & \frac{1}{5} & \frac{2}{5} \\ & 1 & -\frac{1}{2} \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(2) 系数矩阵的Crout分解为

$$\begin{pmatrix} 2 & 4 & 6 \\ 1 & 4 & 7 \\ 3 & 8 & 12 \end{pmatrix} = \begin{pmatrix} 2 & & \\ 1 & 2 & \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{pmatrix} = LU$$

解 LY=B 即 $\begin{pmatrix} 2 & & \\ 1 & 2 & \\ 3 & 2 & -1 \end{pmatrix} Y = \begin{pmatrix} 26 & 40 \\ 25 & 34 \\ 46 & 71 \end{pmatrix}$ 得 $Y = \begin{pmatrix} 13 & 20 \\ 6 & 7 \\ 5 & 3 \end{pmatrix}$

解 UX=Y 即 $\begin{pmatrix} 1 & 2 & 3 \\ & 1 & 2 \\ & & 1 \end{pmatrix} X = \begin{pmatrix} 13 & 20 \\ 6 & 7 \\ 5 & 3 \end{pmatrix}$ 得 $X = \begin{pmatrix} 6 & 9 \\ -4 & 1 \\ 5 & 3 \end{pmatrix}$

7. 系数矩阵的Doolittle分解为

$$A = \begin{pmatrix} -6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{pmatrix} = LU = \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -\frac{1}{3} & \frac{4}{13} & 1 \end{pmatrix}\begin{pmatrix} -6 & 3 & 2 \\ & \frac{13}{2} & 2 \\ & & \frac{236}{39} \end{pmatrix}$$

求解 Lz=b 即 $\begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -\frac{1}{3} & \frac{4}{13} & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 11 \\ -8 \end{pmatrix} \Rightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ -\frac{472}{39} \end{pmatrix}$

求解 DY=Z 即 $\begin{pmatrix} -6 & & \\ & \frac{13}{2} & \\ & & \frac{236}{39} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ -\frac{472}{39} \end{pmatrix}$ 得 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{18}{13} \\ -2 \end{pmatrix}$.

求解 $L^TX=Y$ 即 $\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ & 1 & \frac{4}{13} \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{18}{13} \\ -2 \end{pmatrix}$ 得 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

(2) 系数矩阵的Doolittle分解为:

$$\begin{pmatrix}1&2&3\\2&1&-2\\3&-2&1\end{pmatrix}=\begin{pmatrix}1&&\\2&1&\\3&\frac{8}{3}&1\end{pmatrix}\begin{pmatrix}1&2&3\\&-3&-8\\&&\frac{40}{3}\end{pmatrix}=LU$$

解 $LZ=b$ 即 $\begin{pmatrix}1&&\\2&1&\\3&\frac{8}{3}&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}-3\\10\\7\end{pmatrix}$ 得 $\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}-3\\16\\-\frac{20}{3}\end{pmatrix}$

解 $DY=Z$ 即 $\begin{pmatrix}1&&\\&-3&\\&&\frac{40}{3}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}-3\\16\\-\frac{20}{3}\end{pmatrix}$ 得 $\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}-3\\-\frac{16}{3}\\-2\end{pmatrix}$

解 $L^TX=Y$ 即 $\begin{pmatrix}1&2&3\\&1&\frac{8}{3}\\&&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-3\\-\frac{16}{3}\\-2\end{pmatrix}$ 得 $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}$

8 (1) $\alpha_1=a_1=1$　　$\alpha_2=a_2-c_2\beta_1=7-2\times3=1$　　$\alpha_3=a_3-c_3\beta_2=7-2\times3=1$

$\beta_1=b_1/\alpha_1=3/1=3$　　$\beta_2=b_2/\alpha_2=3$　　$\beta_3=b_3/\alpha_3=3$

$y_1=(f_1-c_1y_0)/\alpha_1=-2$　　$y_2=(f_2-c_2y_1)/\alpha_2=-4$　　$y_3=(f_3-c_3y_2)/\alpha_3=2$

$\alpha_4=a_4-c_4\beta_3=7-2\times3=1$　　$x_4=y_4=1$　　$x_3=y_3-\beta_3x_4=2-3\times1=-1$

$\beta_4=b_4/\alpha_4=0$　　$x_2=y_2-\beta_2x_3=-4-(3\times-1)=-1$

$y_4=(f_4-c_4y_3)/\alpha_4=1$　　$x_1=y_1-\beta_1x_2=-2-3\times(-1)=1$

(2) $\alpha_1=a_1=10$　　$\alpha_2=a_2-c_2\beta_1=2-2\times\frac{1}{2}=1$　　$\alpha_3=a_3-c_3\beta_2=10-1\times1=9$

$\beta_1=b_1/\alpha_1=5/10=\frac{1}{2}$　　$\beta_2=b_2/\alpha_2=1$　　$\beta_3=b_3/\alpha_3=5/9$

$y_1=(f_1-c_1y_0)/\alpha_1=\frac{1}{2}$　　$y_2=(f_2-c_2y_1)/\alpha_2=2$　　$y_3=(f_3-c_3y_2)/\alpha_3=\frac{25}{9}$

$\alpha_4=a_4-c_3\beta_3=1-2\times\frac{5}{9}=-\frac{1}{9}$　　$x_4=y_4=-4$　　$x_3=y_3-\beta_3x_4=\frac{25}{9}-\frac{5}{9}\times-4$
$=5$

$\beta_4=0$

$y_4=(f_4-c_4y_3)/\alpha_4=-4$　　$x_2=y_2-\beta_2x_3=2-1\times5=-3$

$x_1=y_1-\beta_1x_2=\frac{1}{2}-\frac{1}{2}\times-3=2$