第六章、数值积分和数值微分、

(1) $f(x)$ 在 a 点 Taylor 展开得 $f(x)=f(a)+\frac{(x-a)f'(\xi)}{1!}$ $\xi\in(a,b)$

则余项 $R_0=\int_a^b f(x)dx-f(a)(b-a)=\int_a^b f(x)-f(a)dx=\int_a^b (x-a)f'(\xi_x)dx=f'(\eta)\frac{(b-a)^2}{2}$

(2) $f(x)$ 在点 $\frac{a+b}{2}$ Taylor 展开得 $f(x)=f(\frac{a+b}{2})+(x-\frac{a+b}{2})f'(\xi)$ $\xi\in(a,b)$

则余项 $R=\int_a^b f(x)dx-f(\frac{a+b}{2})(b-a)=\int_a^b f(x)-f(\frac{a+b}{2})dx=\int_a^b(x-\frac{a+b}{2})f'(\xi_x)dx=$

2. $\cancel{I(1)-I_n(1)=(b-a)f(1)-3hf(1)=0}$

$\cancel{I(x)-I_n(x)=\int_a^b xdx-[\frac{9}{4}h(a+h)+\frac{3}{4}hb]=\frac{(b-a)^2}{2}-\frac{1}{2}(b-a)^2=0}$

$\cancel{I(x^2)-I_n(x^2)=\int_a^b x^2dx-[\frac{9}{4}h(a+h)^2+\frac{3}{4}hb^2]=}$

$I(1)-I_n(1)=(b-a)f(1)-3hf(1)=0$

$I(x)-I_n(x)=\int_a^b xdx-[\frac{9}{4}h(a+h)+\frac{3}{4}hb]=\frac{b^2-a^2}{2}-\frac{1}{2}(b-a)(a+b)=0$

$I(x^2)-I_n(x^2)=\int_a^b x^2dx-[\frac{9}{4}h(a+h)^2+\frac{3}{4}hb^2]=\frac{b^3-a^3}{3}-[(b-a)(b^2+ab+a^2)]=0$

$I(x^3)-I_n(x^3)=\int_a^b x^3dx-[\frac{9}{4}h(a+h)^3+\frac{3}{4}hb^3]\neq 0$

故求积公式的代数精度为2.

3、以 $x_0=-h$ $x_1=0$ $x_2=2h$ 为节点构造二次插值多项式 $L_2(x)=\sum_{i=0}^{2}a_if(x_i)$.

则 $I\approx I=\int_a^b[l_0(x)f(-h)+l_1(x)f(0)+l_2(x)f(2h)]dx$.

则 $\alpha_{-1}=\int_a^b l_0(x)dx=\int_a^b\frac{(x-0)(x-2h)}{-h\cdot(-h-2h)}dx=0$

$\alpha_0=\int_{-h}^{2h}l_1(x)dx=\int_{-h}^{2h}\frac{(x+h)(x-2h)}{h\cdot-2h}=\frac{9}{4}h$

$\alpha_1=\int_{-h}^{2h}l_2(x)dx=\int_{-h}^{2h}\frac{(x+h)x}{(2h+h)2h}=\frac{3}{4}h$

则数值积分公式为 $I(f)=\frac{9}{4}hf(0)+\frac{3}{4}hf(2h)$.

4. (1) $\int_a^b f(x)dx=\int_0^1\sqrt{x^4+1}\,dx\approx(1-0)\cdot\frac{f(0)+f(1)}{2}=\frac{1+\sqrt{2}}{2}$

(2) $\int_1^2(x^2-x)dx\approx I_1[f(x)]=(2-1)\frac{f(1)+f(2)}{2}=\frac{3}{2}$

5 (1) $\int_0^{\frac{\pi}{6}}\sqrt{2-\sin^2x}\,dx=I_2[f(x)]=\frac{\frac{\pi}{6}-0}{6}[f(0)+4f(\frac{\pi}{12})+f(\frac{\pi}{6})]=\frac{\pi}{36}(\sqrt{2}+4\times1.3903+1.3229)$

$=0.7242$

(2) $\int_1^2(x^2-x)dx=I_2[f(x)]=\frac{2-1}{6}[f(1)+4f(\frac{3}{2})+f(2)]=\frac{1}{6}(0+4\times\frac{3}{4}+3)=1$

6.(1). 3阶代数精度的Gauss-Legendre 积分公式共需两个积分节点..

令 $L_2(x) = \frac{1}{2^2 2!}\frac{d^2(x^2-1)^2}{dx^2} = 0$ 得 $x_0 = -\frac{\sqrt{3}}{3}$ $x_1 = \frac{\sqrt{3}}{3}$

积分系数: $\alpha_0 = \int_{-1}^{1} \frac{x-x_1}{x_0-x_1}dx = \int_{-1}^{1} \frac{x-\frac{\sqrt{3}}{3}}{-\frac{2\sqrt{3}}{3}}dx = 1$

$\alpha_1 = \int_{-1}^{1} \frac{x-x_0}{x_1-x_0}dx = \int_{-1}^{1} \frac{x+\frac{\sqrt{3}}{3}}{\frac{2\sqrt{3}}{3}}dx = 1$

故在[-1,1]上的Gauss积分公式为 $G_2(f) = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) = 2$

?(2) $\alpha_0 = \int_{-3}^{1} \frac{x-x_1}{x_0-x_1}dx = \int_{-3}^{1} \frac{x-\frac{\sqrt{3}}{3}}{-\frac{2\sqrt{3}}{3}}dx = 2+2\sqrt{3}$

$\alpha_1 = \int_{-3}^{1} \frac{x-x_0}{x_1-x_0}dx = \int_{-3}^{1} \frac{x+\frac{\sqrt{3}}{3}}{\frac{2\sqrt{3}}{3}}dx = 2-2\sqrt{3}$

则在[-3,1]上的Gauss积分公式为 $G_2(f) = 2[(2+2\sqrt{3})f(-1+2\frac{-\sqrt{3}}{3}) + (2-2\sqrt{3})f(-1+\frac{2\sqrt{3}}{3})]$

$= -531.556$

11. 向前差商: $f'(0.02) = \frac{f(0.04)-f(0.02)}{0.02} = \frac{7-9}{0.02} = 100$.

向后差商: $f'(0.06) = \frac{f(0.06)-f(0.04)}{0.02} = \frac{10-7}{0.02} = 150$

12. $f'(0.10) = \frac{f(0.10)-f(0.00)}{0.10} = \frac{1.50-1.70}{0.10} = -2$.

$f'(0.20) = \frac{f(0.20)-f(0.10)}{0.10} = \frac{1.60-1.50}{0.1} = 1$

$f''(0.20) = \frac{f'(0.20)-f'(0.10)}{0.1} = 30$

$f'(0.30) = \frac{f(0.30)-f(0.20)}{0.10} = \frac{2.00-1.60}{0.1} = 4$

$f'(0.40) = \frac{f(0.40)-f(0.30)}{0.10} = \frac{1.90-2.00}{0.1} = -1$

$f''(0.40) = \frac{f'(0.40)-f'(0.30)}{0.1} = -30$

14. 三点公式: 取 $x_0=0.51$ $x_1=0.53$ $x_2=0.55$.

则 $f'(x) = L_2'(x) = \frac{f(x_0)}{2h^2}(2x-x_1-x_2) - \frac{f(x_1)}{2h^2}(2x-x_0-x_2) + \frac{f(x_2)}{2h^2}(2x-x_0-x_1)$.

则 f