Mathematical documentation

Version 0.1

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Chapter 1

Introduction

It has been said that the typical theorem in mathematics states that something you do not understand is equal to something else you cannot compute. There is a kernel of truth in this joke, since the rigor required when doing mathematical research has found is way into every nook and corner of every textbook and thereby to a large extend rendered them unreadable to all, except fairly expert mathematicians; and even these usually disdain from reading such texts cover to cover. The value of mathematical rigor is not in the question; but such works are condemned to remain works of reference, to be consulted, not read.

This is unfortunate since many people derive great happiness from dabbling with a little mathematics; just consider the many people who vigorously engages the daily news paper Sudoku puzzles. Such spending of time implies that many people indeed still enjoys the pleasures of the mind. However it would be foolish not to acknowledge the complexity of modern mathematical developments. After all Andrew Wiles incredible proof of Fermat's Last Theorem or Abel's remarkable proof of the in-solvability of general higher degree equations eluded many of the worlds best mathematicians for centuries. Such gems are usually only available to the person equipped with a sharp mind, a fairly developed bag of mathematical tools and a certain intellectual maturity. What I propose here is a book that will build up enough knowledge as to show the proofs of the greatest theorems in the history of mathematics; from Hippocrates Quadrature of the Lune to Lindeman's proof that Pi is transcendental. I have sought to make the text completely self contained, so the reader will be taken from simple arithmetic to measure theory and game theory.

The mathematical mature person is right to ask if such an bold attempt is

not futile and that perhaps this author is a bit naive. There goes a story that the great physicist Richard Feynman was once asked by a Caltech faculty member to explain why spin 1/2 particles obey the . He nodded and said, "I'll prepare a freshman lecture on it". But a few days later he returned and said. "You know, I couldn't do it, I couldn't reduce it to freshman level. That means we really don't understand it." Just as Feynman found in his great lectures on physics this author has also had to abandon a number of otherwise interesting subjects due to the complexity of conveying them.

The text includes a number of exercises along with answers to every one (at least briefly). I am a firm believer in learning-by-looking, also known as cheating, and thus I encourage the reader to give each problem his best shot, then check the answer. If correct then move on to a more advanced problem, if not then study the provided solution and try to solve a similar problem. If you are stuck then don't panic I have included an entire chapter devoted to the fine art of problem solving and mathematical proof.

The subject presented here is not laid out in the usual chronological manner, instead I have sought to build up mathematics as a logical progression of ever more complex ideas. Therefor instead of focusing on historical order I have strived instead to provide the clearest proofs; so for example, instead of 's original verbal proof of the divergence of the Harmonic Series I have instead included a never and simpler algebraic proof. At other times the original proof is so beautiful or provide a vital lessons in mathematical reasoning that I let it's author speak directly (such as Euclid's proof of the Pyteagorian theorem).

The knowledge and proofs contained herein is taken from a multitude of sources, many have been rewritten or reordered so to make them more accessible, but often one finds mathematics written with such breathtaking clarity as so to render any further attempt upon simplification impossible. In these cases the information is simply restated and put in context. In the end of each chapter propper credit is provided as well as suggestions on further reading for the adventures person.

Mathematics is utility and its usage is spread far and wide; from logical reason about programming languages to the physical sciences. So stories about its usages is included whenever I have found it fitting. At the same time mathematics is also history, from Archimedes war machines used against the intruding romans to everyday stories such as Newton's remark that he "do not like to be teased by foreigners about mathematical things". In the end all that I hope to achieve is to shine a little light on the fascinating field of human creativity known as mathematics, to hopefully illustrate

the meaning of the the great philosopher Spinoza's words, when he said that
"God is a mathematician".

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Lars Tackmann
Copenhagen, Denmark

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Part I

Basic Mathematics

Chapter 2

Mathematical foundations

> You just keep pushing. You just
> keep pushing. I made every
> mistake that could be made.
> But I just kept pushing.
>
> Rene Descartes

2.1 Problem solving

2.1.1 Word problems

"A bat and ball, together, cost a total of 1.10 and the bat costs 1 more
than the ball. How much is the ball?" The wrong answer is the one that
roughly one in every two people blurts out: 10 cents. The correct answer is
5 cents, since only with a bat worth 1.05 and a ball worth 5 pence are both
conditions satisfied.

2.1.2 Problem solving checklist

2.2 Proving theorems

2.2.1 Proof methods

Proof by induction

2.2.2 Proof checklist

Chapter 3

Classical geometry

Geometry arrose as practical methods in the ancient world in order to measure land, survey fields, compute the quantity of corn in containers, and to construct temples and pyramids. The theorems of Thales and Pythagoras, which are the oldest theorems of humanity and fundamental tools for geometry.

Later these methods practiced by builders and tradesmen where taken over by Euclid who replaced the geometry of nails, ropes and walls used by the temple builders with mathematical points, lines, rectangles etc., objects of pure reasoning, which require a list of definitions, axioms and postulates. This is the origin of nearly all mathematical procedures used ever since.

3.1 Classical plane geometry

The most beautiful and useful discoveries in classical geometry concerns the relations between lengths (Thales intercept theorem), angles (the central angle theorem of Euclid) areas (the Pythagorean theorem).

3.1.1 Thales theorem

Thales is the man to tell us how to measure the height of a tree without having to climb it. In figure 3.1.1 below we let side $B'C'$ be the height of the tree and AB' be its shadow. We erect a vertical stick BC in such a manner that AB is the shadow of the stick. We then measure the distance AB, say 4 metres, the distance AB', say 8 metres, and the stick BC, say 5 metres. By parallel displacements of the triangle ABC we see that, since AB measures twice AB, the height $B'C'$ will measure twice BC, hence

$B'C' = 2 \cdot 5 = 10.$
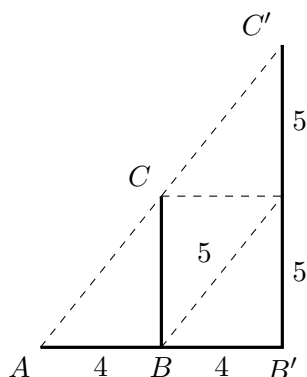
Figure 3.1: Using similar triangles to measure the height of a tree

Theorem 3.1.1. *(Thales intercept theorem). Consider an arbitrary triangle ABC (see figure 3.1.1 below) and let AC be extended to C' and AB to B' so that $B'C'$ is parallel to BC. Then the lengths of the sides satisfy the relations*

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} \quad and \; hence \quad \frac{a'}{c'} = \frac{a}{c}, \; \; \frac{c'}{b'} = \frac{c}{b}, \; \; \frac{b'}{a'} = \frac{b}{a}$$

As these proportions are also preserved when the triangle is displaced and rotated we get the following result. *If corresponding angles of two triangles are equal, then corresponding sides are proportional.* Triangles having these properties are called similar (see similarity).
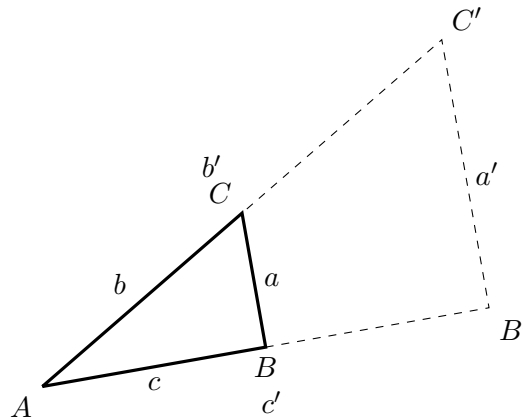
Figure 3.2: Similar triangles

3.1.2 Points, Lines and Planes

3.1.3 Angles

Radians

Degrees are often used when introducting geometry. However for more advanced work we often meassure angles in radians. Radians equals the ratio between the length of an angle arc and its radius. As the circumference of a circle equals $2\pi r$ thus an angle of $360deg$ equals $\frac{2\pi r}{r} = 2\pi$ radians. Radians allow us to use real numbers in the trigonometric functions, rather than degrees, which are an arbitrary angular measurement between $0 - 360$. The use of real numbers for meassuring angels allow is essential in more advanced mathematics, calculus for example.

3.2 Area and perimeter

The area of a parallelogram is $a \cdot h$, where h is the altitude of the parallelogram (Eucl. I.35). There are two ways to see this: (a) We cut off the triangle on the left and add it on the right to obtain a rectangle (Euclids proof, see the second figure in Fig. 1.11); (b) We cut the parallelogram parallel to AB into a large number of very slim rectangles. The area of a triangle is half the area of the parallelogram.

3.2.1 Triangles

Figure 1.2 shows a right-angled triangle (ie a triangle with one angle of 90)
with another angle denoted by the Greek letter theta . The sides of the a
right-angled triangle are called the adjacent side (next to), the opposite
side (opposite to) and the hypotenuse (opposite the right-angle).

3.2.2 Circle
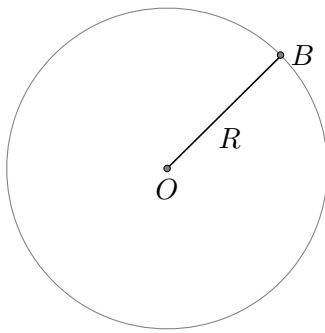
The area of a circle is

$$A = \pi \cdot r^2 \tag{3.1}$$

the perimeter (circumference) is

$$C = 2\pi \cdot r \tag{3.2}$$

3.3 Euclid Elements

He begins with some definitions of the basic concepts: point, line, circle,
triangle, quadrilateral. Euclid then states ten axioms (also called
postulates)on which all subsequent reasoning is based. We shall note these
merely to see that they do indeed describe apparently unquestionable
properties of geometric figures. The first five posgulates are:

1. Two points determine a unique straight line

2. A straight line extends indefinitely far in either direction

3. A circle may be drawn with any given center and any given radius

4. All right angles are equal

5. Given a line l (Fig. 61) and a point P not on that line, there exists in the plane of P and l and through P one and only one line m , which does not meet the given line l .

Adding four right-angled triangles with sides a and b, we arrive at and get the large square of area $(a + b)^2 = a2 + 2ab + b2$. Since the areas of the four triangles add up to $2ab$, the square with area c^2 also has area $a^2 + b^2$.

3.4 Classical solid geometry

3.4.1 Cube

3.4.2 Cone

3.4.3 Sphere

3.5 Trigonometry

3.6 Exercises

Chapter 4

Arithmetic

4.1   Numbers

Number	Name	Power	Decimal
1/10000	Ten thousandths	10^{-4}	0.0001
1/1000	Thousandths	10^{-3}	0.001
1/100	Hundredths	10^{-2}	0.01
1/10	Tenth	10^{-1}	0.1
1	Ones	10^{0}	
10	Tens	10^{1}	
100	Hundreds	10^{2}	
1000	Thousands	10^{3}	
10000	Ten thousands	10^{4}	

Using this table we can write decimal numbers as sums of integers and fractions, for example

$$5678.12 = 5 \cdot 1000 + 6 \cdot 100 + 7 \cdot 10 + 8 \cdot 1 + 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{100}$$

the class of numbers that can be written in this way is called Algebraic numbers

We also see that when a decimal is multiplied by 10, every figure moves one place to the left ($1.25 * 10 = 12.5$) and when you divide by 10, every figure moves one place to the right ($1.25 * 1/10 = 0.125$).

4.1.1   Scientific notation

Scientific notation is a way of writing numbers that are too big or too small to be conveniently written in decimal form. In scientific notation all numbers are written in the form

$$a \cdot 10^b$$

where the exponent b is chosen so its absolute value is at least one but less than ten ($1|a| < 10$). Thus 350 is written as 3.510^2. The following table are examples of scientific notation:

Decimal notation	Scientific notation
2	$2 \cdot 10^0$
300	$3 \cdot 10^2$
4,321.768	$4,321768 \cdot 10^3$
53,000	$5,3 \cdot 10^4$
6,720.000.000	$6,72 \cdot 10^9$
0.2	$2 \cdot 10^1$
0,000.000.007.51	$7,51 \cdot 10^9$

This form allows easy comparison of numbers, as the exponent b gives the number's order of magnitude. For example, the order of magnitude of 1500 is 3, since 1500 may be written as 1.510^3.

4.1.2   Rounding

Rounding a number means replacing it by another that is approximately equal but has a shorter or simpler representation. The procedure is as follows. **Pick the number to round to as the last digit to keep. Leave it the same if the next digit is less than 5 or increase it by 1 if the next digit is 5 or more. Finally turn all following digets into zeros**. The following examples demonstrate the rounding procedure:

Round to the nearest hundred 838.274 is 800 as $3 < 5$

Round to the nearest ten 838.274 is 840 as $8 > 5$

Round to the nearest one 838.274 is 838 as $2 < 5$

Round to the nearest tenth 838.274 is 838.3 as $7 > 5$

Round to the nearest hundredth 838.274 is 838.27 as $4 < 5$

Besides the standard rounding mentioned above two other methods called floor and ceiling which maps a real number to the largest previous or the smallest following integer, respectively. More precisely, $floor(x) = \lfloor x \rfloor$ is the largest integer not greater than x and $ceiling(x) = \lceil x \rceil$ is the smallest integer not less than x.

Value	Floor	Ceiling	Fractional part
2.4	2	3	0.4
2.9	2	3	0.9
-2.7	-3	-2	0.3
-2	-2	-2	0

4.1.3 Significant figures

The significant figures of a number are those digits that carry meaning contributing to its precision. The procedure for identifying significant figures is as follows

1. Identify the non-zero digits and any zeros between them. These are all significant.

2. Leading zeros are not significant.

3. If its a decimal, then trailing zeros are significant.

4.2   Addition

Addition represents the operation of adding objects to a collection. For
example, $2 + 3$ can be though of as if you have 2 apples and someone give
you 3 more, then you have 5 apples in total. Assuming b is positive then
on a number line $a + b$ represents starting on number a and moving b
places to the right. Similar if b is negative then $a + b$ represents starting
on number a and moving b places to the left. Unlike subtraction the
addition operations is both commutative (that is $a + b = b + a$) and
associativ (that is $a + (b + c) = (a + b) + c$)

Adding integers We add integers by adding the ones, tens,hundreds etc.
in each number individually, starting from the right. If the result of
addition is above 10 then we subtract 10 from the result and add one
(known as a carry) to the next group to be added. For example

$$
\begin{array}{r}
{\scriptstyle 1 \; 1} \\
1\;4\;3 \\
+\quad 8\;9 \\
\hline
2\;3\;2
\end{array}
$$

That is we add $3 + 9 = 12$ as the result is above 10 we subtract 10
from it and add one tenth to the tenths place. In the tenth place we
now add $1 + 3 + 8 = 12$ which again is above 10, so we subtract 10
from it and add 1 to the hundredts place (ten tens being one
hundred) and so finally we add $1 + 1$ in the hundreds place.

To understand why this procedure works consider that the number
$143 = 100 + 40 + 3$ and $89 = 80 + 9$ so we can do

$$
\begin{aligned}
143 + 89 &= (100 + 40 + 3) + (80 + 9) \\
&= 100 + (40 + 80) + (3 + 9) \\
&= 100 + (40 + 80) + 12 \\
&= 100 + (10 + 40 + 80) + 2 \\
&= 100 + 130 + 2 \\
&= (100 + 100) + 30 + 2 \\
&= 232
\end{aligned}
$$

Adding decimals We add decimals similar to integers by starting from
the rightmost decimal.

$$
\begin{array}{r}
{\scriptstyle 1\ 1\ 1\ 1} \\
9.0\ 8\ 7 \\
+\ \ 1\ 5.9\ 2\ 4 \\
\hline
2\ 5.0\ 1\ 1
\end{array}
$$

4.3 Subtraction

Subtraction represents the operation of removing objects from a collection. For example, $5 - 2$ can be though of as if you have 5 apples and take 2 away, then you have 3 apples left. Assuming b is positive then on a number line $a - b$ represents starting on number a and moving b places to the left. Semilar if b is negative then $a - b$ represents starting on number a and moving b places to the right.

Subtraction is not commutative, that is $a - b \neq b - a$ instead we can reverse the subtraction order by $a - b = -(b - a) = -b + a$. Further subtraction is not associative so $a - (b - c) \neq (a - b) - c$ instead it holds $a - b - c = a - (b + c)$ and $a - b - c - d = a - (b + c + d)$ and so on.

Subtracting integers We subtract numbers by subtracting each ones, tens, hundreds etc. individually, starting from the right, if the subtraction yields a result less than zero then we borrow ten from the number to the left by decreasing its value by one:

$$
\begin{array}{r}
1\ 3\ 3 \\
-\ \ \ 8\ 9 \\
\hline
4\ 4
\end{array}
$$

Subtracting decimals

4.4 Multiplication

Multiplication of two whole numbers is equivalent to the addition of one of them with itself as many times as the value of the other one; for example, $3 \cdot 4$ can be calculated by adding 4 to itself 3 times:

$$3 \cdot 4 = 4 + 4 + 4 = 12$$

it can also be calculated by adding 4 copies of 3 together:

$$3 \cdot 4 = 3 + 3 + 3 + 3 = 12$$

Formally we say that multiplication is both commutative (i.e. the order in which two numbers are multiplied does not matter $a \cdot b = b \cdot a$) and associativ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Multiplication has other important properties:

Distributive property Holds with respect to multiplication over addition. This identity is of prime importance in simplifying algebraic expressions: $x \cdot (y + z) = x \cdot y + x \cdot z$

Identity element The multiplicative identity is 1; anything multiplied by one is itself. This is known as the identity property: $x \cdot 1 = x$

Zero element Any number multiplied by zero is zero. This is known as the zero property of multiplication: $x \cdot 0 = 0$

Multiplying integers We multiply integers by multiplying the ones, tens, hundreds etc. in each number individually, starting from the right. If the result of multiplication is above 10 we subtract 10 from the result and add one (known as a carry) to the next group to be multiplied. For example

$$
\begin{array}{r}
1\ 4\ 2 \\
\times \qquad 3 \\
\hline
4\ 2\ 6
\end{array}
$$

That is we first multiply $3 \cdot 2 = 6$ in the ones place, then in the tenth place we multiply $3 \cdot 4 = 12$ which is above 10, so we subtract 10 from it and add 1 to the hundredts place and so finally we multiply $3 \cdot 1 = 3$ and add the carry of 1 to it in order to get 4 in the hundreds place.

To understand why this procedure works consider that the number $142 = 100 + 40 + 2$ so really we have

$$
\begin{aligned}
143 \cdot 3 &= (100 + 40 + 2) \cdot 3 \\
&= (100 \cdot 3) + (40 \cdot 3) + (2 \cdot 3) \\
&= (100 \cdot 3) + (10 \cdot 4 \cdot 3) + 6 \\
&= (100 \cdot 3) + (10 \cdot 12) + 6 \\
&= (100 \cdot 3) + (100 + 20) + 6 \\
&= (100 \cdot 4) + 20 + 6 \\
&= 426
\end{aligned}
$$

Multiplying decimals To multiply decimals convert them into integers, multiply them and then convert the result back to a decimal i.e.

- $8 * 0.8 = 8 * \frac{8}{10} = \frac{64}{10} = 6.4$
- $2.91 * 3.2 = \frac{291}{100} * \frac{32}{10} = \frac{291*32}{1000} = \frac{9312}{1000} = 9.312$

4.5 Fractions

In the fraction 3/4, the numerator 3 (also known as the dividend) tells us that the fraction represents 3 equal parts, and the denominator 4 (also known as the divisor) tells us that 4 parts make up a whole. The fraction 5/4 equally tells us that 4/4 makes up a whole and we have an extra 1/4 part left.

Equal fractions Fractions have the same value when they represent the same parts of a hole e.g. $\frac{a}{b} = \frac{c}{d}$ if and only if $a \cdot d = b\dot{c}$.

Ordering fractions When both denominators are positive $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. If either denominator is negative convert them into postive numbers by negating the numerator.

Additive inverse $-\left(\frac{a}{b}\right) = \frac{-a}{b} = \frac{a}{-b}$

Multiplicative inverse $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

Simplifying Fractions (TODO examples 78/52

Adding fractions $\frac{a}{b} + \frac{c}{d} = \frac{a*d+b*c}{b*d}$

Subtracting fractions $\frac{a}{b} - \frac{c}{d} = \frac{a*d-c*c}{b*d}$

Multiplying fractions $\frac{a}{b} * \frac{c}{d} = \frac{a*b}{c*d}$

Dividing fractions Division is equivalent to multiplying by the reciprocal of the divisor fraction: $\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

Exponentiation to integer power If n is a non-negative integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and if $a \neq 0$ then $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n}$.

Converting decimals to fractions Place the decimal in the numerator and one in the denominator and multiply both with multiples of ten until the decimal point disapears e.g.
$0.024 = 0.024/1 = 0.24/10 = 2.4/100 = 24/1000$

4.5.1   Mixed numbers

Impropper fractions such as 5/4, where the numerator is larger than the denominator, are commonly represented as a combination of a whole number and a proper fraction called a mixed number e.g. $5/4 = 1\frac{1}{4}$.

Converting fractions to mixed numbers Convert the numerator to a sum where the first part can be divided by the denominator and the last part has a lower value than the denominator e.g.
$\frac{64}{5} = \frac{60+4}{5} = \frac{60}{5} + \frac{4}{5} = 12\frac{4}{5}$

Converting mixed numbers to fractions Multiply the leading integer with the denominator and add it to the numerator e.g.
$12\frac{2}{3} = \frac{3*12+2}{3} = \frac{38}{3}$

Adding and subtracting mixed numbers A mixed number $a\frac{b}{c}$ is a short hand for $a + \frac{b}{c}$ so to add or subtract mixed numbers perform the operation step wise on the integer and fractional part of the numbers e.g. $6\frac{6}{12} - 3\frac{3}{5} = 6 + \frac{6}{12} - 3 - \frac{3}{5} = 2\frac{9}{10}$.

Multiplying mixed numbers Convert the mixed number to a fraction and multiply these e.g. $1\frac{1}{4} * 4 = \frac{20}{4} = 5$

4.6   Division

Dividing improper fractions Division is really counting the number of repeated subtractions of the divisor into the dividend e.g. $3024/42 = 72$ as shown below

			0	0	7	2
4	2)		3	0	2	4
			2	9	4	
					8	4
					8	4

Dividing propper fractions As propper fractions always represent values parts less than 1 whole we know the result must be a decimal

begning with 0.

```
                  0,    7    0    3
      2    7)   1    9,   0    0    0
                1    8    9
                          1    0
                          0
                          1    0    0
                               8    1
                               1    9
```

Dividing decimals To divide decimals first convert the denominator into a integer then divide the resulting fraction as shown above e.g. $3.3534/0.81 = 335.34/81$

```
                  0    0    4,   1    4
      8    1)   3    3    5,   3    4
                3    2    4
                          1    1    3
                          8    1
                          3    2    4
                          3    2    4
```

4.6.1 Divisibility tests

The following rules can test numbers for divisibility

Divisible by 2 if the last didget is divisible by 2.

Divisible by 3 if the sum of the digits is divisible by 3.

Divisible by 4 if the number formed by the last two digits is divisible by 4.

Divisible by 5 if the last digit is either 0 or 5.

Divisible by 6 if divisible by 2 and 3.

Divisible by 9 if the sum of the digits is divisible by 9.

Divisible by 10 if the last digit is 0.

4.6.2   Remainder

The remainder from the division a/b is represented mathematically as $a \bmod b$, for example $9/2 = 4$ and $9 \bmod 2 = 1$. In general to the find the result of $a \bmod b$ we follow these steps:

1. Construct a clock with size b

2. Start at 0 and move around the clock a steps (If the number is positive we step clockwise, if it's negative we step counter-clockwise).

3. Wherever we land is our solution.

For example $7 \bmod 2 = 1$ since we can make a clock with numbers $0, 1$ then start at 0 and go through 7 numbers in a clockwise sequence $1, 0, 1, 0, 1, 0, 1$. Similar $-5 \bmod 3 = 1$ since we make a clock with numbers $0, 1, 2$ then start at 0 and go through 5 numbers in counter-clockwise sequence $2, 1, 0, 2, 1$

4.7   Percents

4.8   Exponents and roots

If we multiply 3 by itself 4 times we get

$$3 \cdot 3 \cdot 3 \cdot 3 = 81$$

A more concise way of writing this is to say

$$3^4 = 81$$

where 3 is the base and the superscript 4 is called the power or exponent.

Squares b^2 means $b \cdot b$ and is read <u>b squared</u> because b^2 is the area of a sqaure whoose side has lenght b.

Cubes b^3 means $b \cdot b \cdot b$ and is read <u>b cube</u> because b^3 is the volume of a cube whoose side has lenght b.

Powes of 10 If the power is positive the result isone followed by as manyzeros as the number in the exponent $10^7 = 10000000$. If the exponent is negative we have as many zeroes as the exponent followed by one with the first zero being before the comma the rest after $10^{+}7 = 0.0000001$

Negative powers $a^{-n} = \frac{1}{a^n}$

Products of powers $a^n \cdot a^m = a^{n+m}$ this can be seen as
$5^3 \cdot 5^2 = (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5) = 5^5$

Products of unlike powers $a^n \cdot b^n = (a \cdot b)^n$ this can be seen as
$(2 \cdot 2 \cdot 2) \cdot (3 \cdot 3 \cdot 3) = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) = (2 \cdot 3)^3$

Fractions of powers $\frac{a^n}{a^m} = a^{n-m}$ this can be seen as $\frac{5^3}{5^2} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5} = 5^1$

Powers of fractions $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ this can be seen as $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2^3}{3^3}$

Powers of powers $(a^m)^n = a^{(m \cdot n)}$ this can be seen as
$(5^2)^3 = 5^2 \cdot 5^2 \cdot 5^2 = 5^6$

Fractional powers $a^{\frac{x}{y}} = (a^{\frac{1}{y}})^x$. This can be related to roots since
$\sqrt{a} \cdot \sqrt{a} = a$ can be satiesfied as $a^x \cdot a^x = a^{x+x} = a^1$ only when
$x = 1/2$. That is $\sqrt{a} = a^{1/2}$. Generally we have $\sqrt[n]{a} = a^{1/n}$.

Simplifying radicals Radicals such as $\sqrt{32}$ are simplified by rewriting
them to lower terms using the above rules e.g.
$3\sqrt{8} - 6\sqrt{32} = 3\sqrt{4}\sqrt{2} - 6\sqrt{16}\sqrt{2} = 6\sqrt{2} - 24\sqrt{2} = -18\sqrt{2}$. Often it
can be constructive to rewrite radicals using their prime factorization
e.g. $\sqrt[3]{81} = \sqrt[3]{9 \cdot 9} = \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 3} = 3\sqrt[3]{3}$

4.8.1   Estimating roots

Square roots

Cube roots find the prime factors and see if any come three times. For
example $\sqrt[3]{3430}$ is divisble by 10 and there for have 5 and 2 as prime
facors, $343 = 7^3$ so we have $\sqrt[3]{3430} = \sqrt[3]{2 \cdot 5 \cdot 7 \cdot 7 \cdot 7} = 7 \cdot \sqrt[3]{10}$

4.9   Order of operators

The order of arithmetic operators is as follows

1. Parentheses and exponents

2. Multiplication and division

3. Addition and subtraction When multiple operations exists on the
   same level you do the one that it closest on the left, for example

   $2 - 10 + 8 - (-2) + (-10) = 2 - 10 + 8 + 2 - 10 = (2 - 10 + 8) + 2 - 10 = (-8 + 8) + 2 - 10 = 0 + 2 - 10 = -8$

4.10   Exercises

Ex. 1 —   Reduce the following fractions

1. $88/72$
2. $\frac{2}{\frac{4}{5}}$
3. $\frac{\frac{-7}{2}}{\frac{4}{9}}$

Ex. 2 —   If it takes 36 minutes for 5 people to paint 9 walls. How many minutes does it take 10 people to paint 7 walls?

Ex. 3 —   Philip is going on a 4000-kilometer road trip with three friends. The car consumes 6 liters of gas per 100 kilometers, and gas costs 1.50 per liter. If Philip and his friends want to split the cost of gas evenly, how much should they each pay?.

Ex. 4 —   What is the value of

1. $7 + 7/7 + 7 \cdot 7 - 7$
2. $6 - 1 \cdot 0 + 2/2$
3. $-2 + (-3) + 4 - (-3) - 5$
4. $3 + (-4) - 8 - (-1) - 1$

Ex. 5 —   Reduce the following

1. Rewrite $\frac{4^6}{4^{-4}}$ in the form 4^n
2. Rewrite $(5^{-12})^{-9}$ in the form 5^n
3. Rewrite $((5^{-10})(9^9))^7$ in the form $5^n \cdot 9^m$
4. Rewrite $\sqrt{11} + \sqrt{44} + \sqrt{99}$ in the form $n \cdot \sqrt{m}$

Chapter 5

Basic algebra

Algebraic expressions are expressions that involves unknown variables, for example

$$8x - 6$$
$$7x^2 + 4x - 10$$
$$\frac{x - 1}{x^2 + 12}$$

are all expressions that contain the unknown variable "x". If we substitute x with a value we can simplify it further, for example if we replace x with 2 in each of the above expressions we get

$$8 \cdot 2 - 6 = 10$$
$$7 \cdot 2^2 + 4 \cdot 2 - 10 = 26$$
$$\frac{2 - 1}{2^2 + 12} = \frac{1}{16}$$

5.1   Linear equations

Algebraic expressions may look abstract but can in fact be used to represent everyday phenomenon. For example consider the statement the "price of apples went up by 0.75 per pund so now 3 punds of apples cost 5.88". If we let p denote the apple price prior to the increase then we can write the expression

$$5.88 = 3(p + 0.75) \tag{5.1}$$

which represents that the price of one pund of apples p went up by 0.75 $(p + 0.75)$ and that 3 punds of apples at the new price $(3(p + 0.75))$ now

costs 5.88. Equation 5.1 is called a linear equation or first-degree equation because the unknown p appears in the first degree only ($p^1 = p$).

Algebraic expressions can look different but actually be equivalent. If you plug the same value into equivalent expressions they will each give you the same value. For example these expressions

$$2(4x + 2y)$$
$$4(2x + y)$$
$$8x + 4y$$

yields the same value when you plug in 2 for the unknown variable x and 3 for y.

$$2(4 \cdot 2 + 2 \cdot 3) = 28$$
$$4(2 \cdot 2 + 3) = 28$$
$$8 \cdot 2 + 4 \cdot 3 = 28$$

However replacing unknown variables with actual variable values is tedious and only serves to show that two algebraic expressions will yield the same value for a specific variable value. It does not prove that the expressions will always yield equivalent results for all possible values. To prove that two algebraic expressions are equivalent we must first learn how to manipulate algebraic expressions into a form that will allow us to assertion that the expressions are in fact identical.

5.1.1   Manipulating lienar equations

Algebraic manipulation concerns the procedures we are allowed to perform on algebraic expressions. Generally we are allowed to perform all the arithmetical operations we learned in chapter 4 as long as we perform them on both sides on the equal sign in the expression. Formally the following algebraic manipulation procedures are allowed:

Addition Adding on both sides is usually done to remove negative terms, for example if we add 1 on both sides of $x - 1 = 0$ we get $x = 1$.

Subtraction Subtracting on both sides is usually done to remove positive terms, for example if we subtract b on both sides of $x + b = b + 2$ we get $x = 2$.

Multiplication Multiplying on both sides is usually done to remove fractional terms, for example if we multiply with 4 on both sides of $\frac{y}{4} = \frac{3}{4}$ we get $y = 3$.

Division Dividing on both sides is usually done to remove multiplication terms, for example if we divide with 5 on both sides of $5z = 20$ we get $z = 4$.

Replace with equals Replacing with equal expressions is usually done to rewrite part of the expressions into a form that is easier to manipulate. For example in $x - \frac{x}{2} = 1$ we wish to put x and $\frac{x}{2}$ on a common denominator. As $\frac{2x}{2} = x$ we can replace the first x with this

$$
\begin{aligned}
x - \frac{x}{2} &= 1 \\
\frac{2x}{2} - \frac{x}{2} &= 1 && \text{repalce } x = \frac{2x}{2} \\
\frac{2x - x}{2} &= 1 && \text{use common denominator} \\
\frac{x}{2} &= 1 && \text{simplify numerator} \\
x &= 2 && \text{multiply with 2}
\end{aligned}
$$

Distributive rule $m(a + b) = ma + mb$ is called the distributive rule since we "distributed" m to a and b.

When performing algebraic manipulation we are of cause allowed to make use of all the above procedures in order to simplify a expression. For example consider expression 5.1 above that we used to represent the price of apples after a price hike:

$$
\begin{aligned}
3(p + 0.75) &= 5.88 \\
\frac{3(p + 0.75)}{3} &= \frac{5.88}{3} && \text{divide with 3} \\
p + 0.75 &= \frac{5.88}{3} && \text{cancel 3 out} \\
p &= \frac{5.88}{3} - 0.75 && \text{subtract 0.75} \\
p &= 1.21 && \text{calculate value}
\end{aligned}
$$

So we now know that the price of one pund of apples prior to the price increase is 1.21. Mastering algebraic manipulation is the foundation of all modern mathematics so we will show another example to get a feel for it.

Consider the expression $2(x+1) = \frac{1}{5}x + 3$

$$
\begin{aligned}
2(x+1) &= \frac{1}{5}x + 3 & \\
2x + 2 &= \frac{1}{5}x + 3 & \text{use distributive rule} \\
2x &= \frac{1}{5}x + 1 & \text{subtract } 2 \\
2x - \frac{1}{5}x &= 1 & \text{subtract } \frac{1}{5}x \\
\frac{10x}{5} - \frac{1}{5}x &= 1 & \text{replace } 2x = \frac{10x}{5} \\
\frac{10x - x}{5} &= 1 & \text{use common denominator} \\
\frac{9x}{5} &= 1 & \text{simplify numerator} \\
9x &= 5 & \text{multiply with } 5 \\
x &= \frac{5}{9} & \text{divide with } 9
\end{aligned}
$$

before continuing with other chapters in this book be sure to familiarize yourself with algebraic manipulation by going through the exercises in 5.6.

5.2   Linear inequalities

Just as we are allowed to have have expressions with unknown variables we can have unknown variables in inequalities. For example the inequality $x < 5$ states that x is a value below 5. Inequalities may be compound such as $1 \le y \le 4$ which states that y is any value between (and including) 1 and 4. However not all equalities are as easy to analyse as the ones above. Consider $-0.5z \le 7.5$ here we may be tempted to divide both sides with -0.5 in order to isolate z, however it turns out that special rules governs algebraic manipulation of inequalities. To see why this is true observe what happens when we multiply a inequality with a negative number

$$
\begin{aligned}
1 &< 2 & \\
-1 &< -2 & \text{multiply with } -1
\end{aligned}
$$

clearly the last expressions is false as $-1 \not< -2$. In fact it turns out that when multiplying or dividing inequalities with negative numbers we must

flip the inequality $(-1 > -2)$. Observing this rule we can now manipulate $-0.5z \leq 7.5$ into a more manageable form

$$-2 \cdot -0.5z \geq -2 \cdot 7.5 \qquad \text{multiply with } -2 \text{ and flip sign}$$
$$z \geq -15 \qquad\qquad\qquad \text{simplify}$$

5.2.1 Manipulating linear inequalities

As shown above the rules for manipulating linear inequalities are sometimes different than when manipulating linear equations. However it turns out that some operations are actually safe to perform. To see this consider that when we add 1 to both sides of $1 < 2$ we get $2 < 3$ which is clearly still true. Formally the rules governing inequalities can be split between those that does not require changing the inequality sign and those who do.

Operations that preserves the inequality sign :

Addition Adding on both sides of a inequality is usually done to remove negative terms, for example if we add 1 on both sides of $x - 1 < 0$ we get $x < 1$.

Subtraction Subtracting on both sides of a inequality is usually done to remove positive terms, for example if we subtract b on both sides of $x + b > b + 2$ we get $x > 2$.

Multiplication with a positive number Multiplying a inequality with a positive number is usually done to remove fractional terms, for example if we multiply with 4 on both sides of $\frac{y}{4} \leq \frac{3}{4}$ we get $y \leq 3$.

Division with a positive number Dividing a inequality with a positive number is usually done to remove multiplication terms, for example if we divide with 5 on both sides of $5z \geq 20$ we get $z \geq 4$.

Operations that requires flipping the inequality sign :

Multiplication with a negative number Multiplying a inequality with a negative number requires flipping the inequality for example if we multiply with -4 on both sides of $-\frac{y}{4} \leq \frac{3}{4}$ we get $y \geq -3$.

Division with a negative number Dividing a inequality with a
negative number requires flipping the inequality for example if
we divide with -5 on both sides of $-5z \geq 20$ we get $z \leq -4$.

Swapping left and right hand sides When we swap the left and
right hand sides, we must also change the direction of the
inequality, for example $2y + 7 < 12$ becomes $12 > 2y + 7$.

as with linear equations we may combine the above rules to solve
complicated inequalities like $3x - 3 > 6x + 3$

$$
\begin{aligned}
3x - 3 &> 6x + 3 \\
3x &> 6x + 6 & \text{add } 3 \\
-3x &> 6 & \text{subtract } 6x \\
x &< -\frac{6}{3} & \text{divide with } -3 \text{ and flip inequality} \\
x &< -2 & \text{simplify fraction}
\end{aligned}
$$

So we now kno

5.3   Functions

As menitoned earlier when we replace a variable x in expressions such as
$x + 2$ we get a value, for example using $x = 2$ gives us $2 + 2 = 4$ and using
$x = 3$ results in $3 + 2 = 5$. To formalize the concept of replacing a variable
x in a expression and computing a value we introduce the concept of a
function. In the above example we may write

$$f(x) = x + 2 \tag{5.2}$$

That is $f(x)$ (read "f of x") is a function of one variable x when we provide
a actual value for x such as $f(1)$we replace all occurences of x in $x + 2$ with
the provided value 1 and computes the functions value. A couple of
examples serves to clarify this concept

$$
\begin{aligned}
f(1) &= 1 + 3 = 4 \\
f(2) &= 2 + 3 = 5 \\
f(3) &= 3 + 3 = 6
\end{aligned}
$$

the actual name of the function f does not matter, we only use it to refer
to the function body $x + 2$ without having to write $x + 2$ all the time. We
might as well call it $g(x) = x + 2$ in which case we just write $g(2) = 5$.

In 5.2 above the value of f only depends on x, we say f is a function of one variable. Alternatively, if f is a function of several variables, we can write $f(a, b)$ meaning f is a function of the two variables a, b. For now however we shall restrict ourselfs to functions of one variable.

It's common to write $y = f(x)$ that is to say the function f returns a value y when we plug a value into x. Assigning $y = f(x)$ gives a relation between y and x and y is then known as a dependent variable and x is an independent variable. Generally the "dependent variable" represents the output of a function. The "independent variables" represent the inputs to the function.
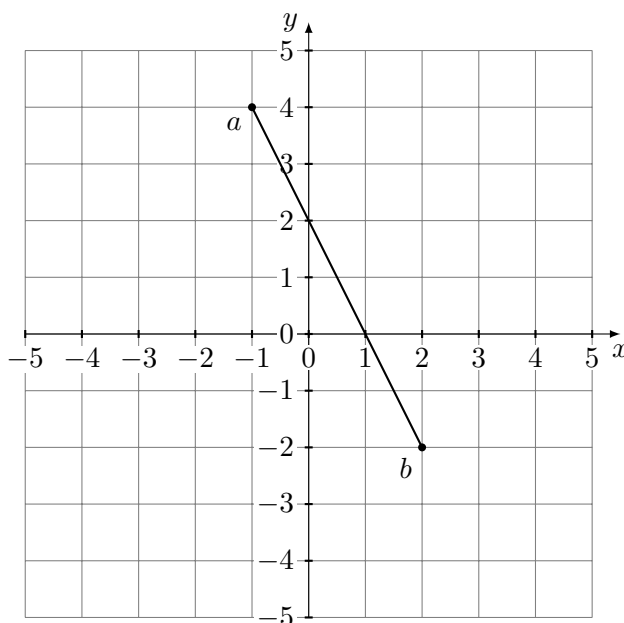
5.3.1 Function graphs

If we enumerate the x and y values of 5.2 into ordered pairs of (x, y) above we get

x	y = f(x)	(x,y)
-2	$-2 + 2 = 0$	$(-2, 0)$
-1	$-1 + 2 = 1$	$(-1, 1)$
0	$0 + 2 = 2$	$(0, 2)$
1	$1 + 2 = 3$	$(1, 3)$
2	$2 + 2 = 4$	$(2, 4)$

Carteseian coodinates (sample Xes and plot out a line)

5.4   Monomials and binominals

A , also called , is a product of powers of variables with nonnegative
integer exponents. Examples include

$$-5, 3, 21x, -2x^2, 3a^2b^3$$

here -5 and 3 are monomial of x^0 for any variable x

5.4.1   Factoring and multiplying binominals

When we multiply two binomials such as $(x + 4)$ and $(x + 2)$
With factoring one usually starts with an expression such as $x^2 + 6x + 8$
and seeks to transform it into a product of two binominals of the form
$(x + a)(x + b)$. The original expression is said to be of second degree
because it contains x^2 but no higher power of x. The factors are
first-degree expressions because each contains x but no higher power of x.
The problem is to find the correct values of a and b so that the product
$(x + a)(x + b)$ will equal the original expression. We know that

$$x^2 + (a + b)x + ab = (x + a)(x + b) \qquad (5.3)$$

Hence to factor the second-degree expression, we should look for two numbers a and b whose sum is the coefficient of x and whose product is the constant. Thus to factor $x^2 + 6x + 8$, we look for two numbers whose sum is 6 and whose product is 8. By mere trial of the possible factors of 8 we see that $a = 4$ and $b = 2$ will meet the requirement; that is:
$x^2 + 6x + 8 = (x + 4)(x + 2)$.

5.4.2 Quadratic equations

5.5 Polynomials

5.5.1 Polynominal multiplication

5.5.2 Polynominal division

5.6 Exercises

Ex. 6 — Use the distributive property to expand these expressions

1. $-(6 - \frac{z}{4})$
2. $-(\frac{1}{2}r + 4)$
3. $-5(3n + \frac{1}{2})$

Ex. 7 — Use the distributive property to collect these expressions so no fractions are within the parentheses

1. $\frac{3}{5}z - \frac{6}{5}$
2. $\frac{3}{2} + \frac{7}{8}c$

Ex. 8 — Determine if the following expressions are equivalent

1. $9x + 6 = 3x + 2$
2. $\frac{5}{2}x - 3 = x + 5$
3. $\frac{\frac{2}{b} + \frac{2}{a}}{\frac{2}{ab}} = a + b$
4. $\frac{\frac{a}{b} + 1}{\frac{b}{a} - 1} = \frac{a(a+b)}{b(b-a)}$

Ex. 9 — Expand the expression and combine the like terms:

1. $6 + 5(7n + 2)$
2. $(15 + 2a) + 4(8a6)$
3. $6(2 + 10k) + 6(5k3)$

    4. $1/5 - 2z + z + 2/3$

    5. $7n - (4n - 3)$

Ex. 10 —— Simplify the following rational expressions by removing their fractions

    1. $\frac{35n^3}{10n^4}$

    2. $\frac{44a^3}{55a^3}$

    3. $\frac{28y^5}{7y^3}$

Ex. 11 —— Write and solve equations to represent the following statements

1. One half of the quantity represented by 6 less than a number n is equal to 17. Write an equation to represent this statement and find the value of n.

2. The sum of 3 consecutive odd numbers is 69. What is the first number in this sequence?

3. 130% of the sum of 7 and a number n is equal to 91. Write an equation to represent this statement and find the value of the number.

4. The radiator of a car contains 10 gallons of liquid 20 percent of which is alcohol. We want to draw off liquid and replace it with alcohol so the resulting mixture contains 50 percent alcohol. How many gallons of liquid should he draw off?

Ex. 12 —— Solve the following linear equations

    1. $\frac{12}{x-12} = \frac{6}{5}$

    2. $\frac{3}{5} = \frac{18}{a+11}$

    3. $\frac{5}{6} = \frac{k-7}{18}$

    4. $\frac{3}{8} = \frac{15}{t-1}$

    5. $\frac{y-8}{25} = \frac{7}{5}$

Ex. 13 —— Solve these linear inequalities for x:

    1. $-12x < 1$

    2. $-18x > -10$

    3. $16x \geq 13$

   4. $12b - 15 > 21$

   5. $14 - 3x < -1$

Ex. 14 — Simplify and solve these linear inequalities for x:

   1. $9 - 4d \geq -3$

   2. $9x - 3 < 7x + 4$

   3. $2x + 8 \geq 4x + 5$

   4. $9x + 1 \geq 4x + 2$

   5. $3x + 9 \geq 9x + 10$

Ex. 15 — Name the quadrants of these coordinates

   1. $(1, 1)$

   2. $(-1, 1)$

   3. $(-1, -1)$

   4. $(1, -1)$

   5. $(0, 0)$

Ex. 16 — Multiply these binomials

   1. $(x - 3)^2$

   2. $(x - 3)(x + 10)$

   3. $(x + 8)^2$

   4. $(x + 1)(x - 4)$

   5. $(x + 9)^2$

Ex. 17 — Factor the following expressions.

   1. $x^2 + 9x + 20$

   2. $x^2 + 5x + 6$

   3. $x^2 - 5x + 6$

   4. $x^2 - 9$

   5. $x^2 + 7x - 18$

Chapter 6

Elementary number theory

In number theory we deal with the properties of various classes of numbers such as the integers, which is any number that can be written without a fractional or decimal component and is denoted with the the the symbol $\mathbb{Z}$. Thus, -7, 0, and 2 are integers; 1.7 and $\frac{1}{7}$ are not. Incidentally the Latin word "integer" literally means "untouched", hence "whole". Of particular interests is the set of all positive integers, also known as the natural numbers and symbolised by $\mathbb{N}$.

6.1 Numeral systems

Generally nu
The base (or radix) of a number system is the number of different symbols available to represent any digit within that system.
For example, the decimal system (Base 10) has a radix of 10, that is any diget in a number can have values 0-9. In contrary Computers work using electronic circuits which can only be switched to be on or off and thus digets can only be represented with radix 2 with the values 0 and 1.
Other commonly used number systems are hexadecimal wich uses radix 16 with digets represented with the numbers 0-9 and letters A-F.

6.1.1 Decimal representation

6.1.2 Conversion between systems

6.2 Basic properties of Integers

Let us first consider that integers are either *even* or *odd*, formally we write:

Definition 6.2.1. The *even* numbers are the numbers written on the form $2k$ and the *odd* numbers are the numbers written on the form $2k + 1$ where k is any integer.

some arithmetic can convince us that this indeed yields the desired numbers:

$$
\begin{aligned}
2 \cdot 0 &= 0 \\
2 \cdot 0 + 1 &= 1 \\
2 \cdot 1 &= 2 \\
2 \cdot 1 + 1 &= 3 \\
&\vdots
\end{aligned}
$$

The *odd* numbers may also be expressed as being either one more than a multiple of 4 or three more than a multiple of 4. Symbolically, we can say that they are either of the form $4k + 1$ or of the form $4k + 3$. Again some examples clarifies the smoke

$$
\begin{aligned}
4 \cdot 0 + 1 &= 1 \\
4 \cdot 0 + 3 &= 3 \\
4 \cdot 1 + 1 &= 5 \\
4 \cdot 1 + 3 &= 7 \\
&\vdots
\end{aligned}
$$

incidentally this shows that when we divide a *odd* number by 4, we must get a remainder of either 1 or 3 . Armed with this knowledge we can shine some light on the properties of *odd* numbers:

Corollary 6.2.1. *The square of an odd number is odd*

Proof. Consider, by 6.2.1, the algebraic form of the square of an odd number

$$
\begin{aligned}
(2k + 1)^2 &= (2k)^2 + 2(2k) + 1 \\
&= 4(k^2 + k) + 1
\end{aligned}
$$

The last expression being on the form $4k + 1$ yields the desired result. $\square$

Integers can be used to construct another class of numbers, known as the rational numbers. Here we say a 'rational number' is a fraction $\frac{a}{b}$, where a and b are integers; we may suppose that a and b have no common factor (since if the had we could remove it) and that b does not equal zero. We denote these numbers by $\mathbb{Q}$ for quotient. With only these properties at hand we can move on to prove one of mathematics first great results; that $\sqrt{2}$ is irrational.

Proposition 6.2.2. $\sqrt{2}$ *is irrational.*

Proof. To say that $\sqrt{2}$ is irrational is the same as saying 2 cannot be expressed in the form $\left(\frac{a}{b}\right)^2$ or that the equation

$$a^2 = 2b^2 \tag{6.1}$$

cannot be satisfied by integral values of a and b without any common factor. Let us for sake of eventual contradiction suppose that 6.1 is true for some a and b, then it follows that a^2 is even (since $2b^2$ is divisible by 2) and thus that a is even (since the square of an even number is even). If a is even then

$$a = 2c$$

for some integral value of c; and therefore

$$a^2 = 2b^2 = (2c)^2 = 4c^2$$

or

$$b^2 = 2c^2$$

Hence b^2 is also even and therefor b is even. Now since both a and b are even they must have the common factor 2. This contradicts our hypothesis, and therefore the hypothesis is false $\qquad \square$

Notice how this proof is by <u>reductio ad absurdum</u>, a proof style much beloved by the greeks and one of mathematics finest weapons. In these types of proofs we attack our theorem by taking an assumption and then show that it leads to a contradiction and hence is false.

Now since we have proved that $\sqrt{2}$ cannot be written as a fraction can we then find another way to express it ?. By definition, a number is *algebraic* (denoted by $\mathbb{A}$) if it is the solution to some polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

where all the coefficients $a_n, a_{n-1}, \cdots, a_2, a_1$ and a_0 are integers. Now as $\sqrt{2}$ is the solution to the polynomial $x^2 - 2 = 0$ it follows that it is a irrational algebraic number. Also as any integer k is a solution to $x - k = 0$ we see that all integers are algebraic and hence $\mathbb{Z} \in \mathbb{A}$. Later we shall meet another class of numbers known as transcendental numbers $\mathbb{T}$ which is simply the set of all non-algebraic numbers. As it turns out almost all irrational numbers are transcendental and all transcendental numbers are irrational. The box below should hopefully clarify the

Real Numbers ($\mathbb{R}$)

Algebraic Numbers ($\mathbb{A}$) $\sqrt{2}$	Transcendental Numbers ($\mathbb{T}$) π
Integers ($\mathbb{Z}$)	
Rational Numbers ($\mathbb{Q}$)	

relationships between these various sets of numbers Note how all of these numbers are contained in the set of real numbers $\mathbb{R}$. The real numbers thus includes both integers and rational numbers, such as 42 and $\frac{1}{3}$, and transcendental numbers, such as π and irrational algebraic numbers such as $\sqrt{2}$. Informally we may think of real numbers as any number with or without a decimal point that may contain an infinite decimal tail that continue in some way.

6.3   Divisibility

The integer n is divisible by m if the ratio n/m is an integer. Hence we write $m|n$ when m divides n evenly and define

$$m|n \iff m > 0 \text{ and } n = mk \text{ for some integer } k \qquad (6.2)$$

How do we know if a number is divisible by some other number ?. Here we are out of luck, in general we can only answer this by doing full division and checking if the remainder is zero. Mathematics contains many functions that are easy to do (such as multiplication and differentiation) but hard to undo (such as division and integration). Later on we will see that exactly this proporty of division can be used to craft a simple scheme for secure communication. For now let us extract some useful properties from the above definition of divisibility.

Proposition 6.3.1. *If $a > b$ and n divides a and b evenly then it also divide their difference.*

Proof. By 6.2 we have $a = nx$ and $b = ny$ for some integral value of x and y. Thus $a - b = nx - ny = n(x - y)$ as desired. □ □

As remarked on page 42 not all numbers divides evenly:

Proposition 6.3.2. *If a and b are integers with $b \neq 0$, then there is a unique pair of integers q and r such that*

$$a = qb + r \text{ and } 0 \leq r \leq |b|$$

Note that

$$\frac{a}{b} = q + \frac{r}{b}$$

where for $b > 0$ we have $0 \leq \frac{r}{b} < 1$ and for $b < 0$ we have $0 \geq \frac{r}{b} > -1$. As example the division $9/4$ with $a = 9$ and $b = 4$ becomes $9 = 2 \cdot 4 + 1$ giving us the quotient $q = 2$ and remainder $r = 1$

Proof. Our job is to prove that the numbers q and r exists, known as existence and that they are unique, also known as uniqueness. First we prove existence. Let

$$S = \{a - nb | n \in \mathbb{Z}\} = a, a \pm b, a \pm 2b, ...$$

□ □

6.3.1 Euclid's Algorithm

Euclid devised a clever algorithm for locating the greatest common divisor.

6.4 Primes

The *prime numbers* or *primes* are the numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37...$$

which cannot be resolved into smaller factors[1]. The primes are the material out of which all numbers are build up by multiplication

$$666 = 2 \cdot 3 \cdot 3 \cdot 37$$

indeed the following holds true:

Proposition 6.4.1. *Every integer n > 1 is divisible by at least one prime*

Proof. Let A be a composite number (i.e. non prime). By definition there must be some smaller number B dividing evenly into A, where $1 < B < A$. Now either B is prime or it is not. If B is prime, then the original number A indeed has a prime divisor, as claimed. Otherwise, B is not prime and so has a divisor, say C, with $1 < C < B < A$. If C is prime, we are done, for C divides evenly into B, and B divides evenly into A. If C is composite then it must have a proper divisor, D, and thus we continuously arrive at:

$$A > B > C > D > ... > 1$$

But all of these are positive integers, so we must reach a point in which we find a prime. Therefor any number, which is not prime itself, is divisible by at least one prime (usually, of cause, by several).   □

If one surveys the list of the first 50 primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83,
89, 97, 101, 103, 107, 109, 113, 127, 131,
137, 139, 149, 151, 157, 163, 167, 173, 179,
181, 191, 193, 197, 199, 211, 223, 227, 229

then apparently the primes seams to be getting scarcer as the numbers go larger. Indeed between the numbers 1 and 1000 there are 168 primes, whereas between 9000 and 10.000 there is but 112. At first this seams logical enough since small numbers only have few possible factors and thus higher likelihood of being primes, yet one cannot help ask if we will ever reach such large numbers that the primes may eventually run out completely, rendering all subsequent numbers composite. Luckily this question had already been answered by the great greek mathematician Euclid who 300 B.C. in his IX book as proposition 20 indeed claimed:

Proposition 6.4.2. *Prime numbers are more than any assigned multitude of prime numbers.*

Euclid's terminology sounds strange, but what he is really proposing is that the sequence of primes does not end.

Proof. Let us for sake of eventual contradiction suppose that there is only a finite number of primes and that

$$2, 3, 5, ..., P$$

is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula

$$Q = (2 \cdot 3 \cdot 5 \cdots P) + 1$$

It is plain that Q is not divisible by any of $2, 3, 5, ..., P$; for it leaves the remainder 1 when divided by any one of these numbers. But, if Q is not prime, then by 6.4.1 it is divisible by some prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P; and therefore this hypothesis is false. $\square$

Finding and counting primes is a long standing hobby for mathematicians, and one might ask if it can be automated. One such method was discovered by the greek mathematician Eratosthenes (ca. 284-192 B.C.) who as the chief librarian at the great Library at Alexandria spend much of his time studying mathematics, astronomy and philosophy. His method (known as Eratosthenes sieve) is as follows; write a list of all the consecutive integers, starting with 2, among which you want to find the primes. As 2 is prime, cross all subsequent multiples off. The next integer that has not yet been crossed of (3) must then also be prime and so we can eliminate all of its multiples. As we continue in this manner we will clearly sieve out all composite numbers and be left with the primes.

> Sift the Twos and sift the Threes,
> The Sieve of Eratosthenes.
> When the multiples sublime,
> The numbers that remain are Prime.

although this method will indeed generate the primes, it is not as efficient as one might have liked (although optimised and more complex methods exists). Later we shall encounter the prime counting function $\pi(n)$, which outputs the number of primes less than or equal to a given number n (the use of π in this function is an unfortunate standard as it has nothing to do with the well known geometric constant 3.1415...).

6.5 Modulo arithmetic

As mentioned before integers can be broken up into the classes of even $..6, 4, 2, 0, 2, 4, 6..$ and odd numbers $..5, 3, 1, 1, 3, 5..$ There are certain generalisations we can make about the arithmetic of numbers based on

which of these two classes they come from. For example, we know that the sum of two even numbers is even. The sum of an even number and an odd number is odd. The sum of two odd numbers is even. The product of two even numbers is even, etc.

Now we represent each of our two classes by a single symbol. We let the symbol "0" mean "the class of all even numbers" and the symbol "1" mean "the class of all odd numbers". The statement "the sum of two even numbers is even" can then be expressed by the following:

$$0 + 0 \equiv 0 \bmod 2.$$

Here, the "$\equiv$" symbol is not equality but congruence, and the "mod2" just signifies that our modulus is 2. The above statement is read "Zero plus zero is congruent to zero, modulo two." The statement "the sum of an even number and an odd number is odd" is represented by

$$0 + 1 \equiv 1 \bmod 2.$$

Those examples are natural enough. But when we try to express the statement "the sum of two odd numbers is even"? we get this strange looking expression

$$1 + 1 \equiv 0 \bmod 2.$$

We have analogous statements for multiplication:

$$0 \cdot 0 \equiv 0 \bmod 2$$
$$0 \cdot 1 \equiv 0 \bmod 2$$
$$1 \cdot 1 \equiv 1 \bmod 2$$

Basically we now have a number system with addition and multiplication but in which the only numbers that exist are 0 and 1. You may ask what use this has. Well, our number system is the system of integers modulo 2, and because of the previous six properties, any arithmetic done in the integers translates to arithmetic done in the integers modulo 2.

Since any integer solution of an equation reduces to a solution modulo 2, it follows that if there is no solution modulo 2, then there is no solution in integers. For example, assume that a is an integer solution to

$$2a3 = 12,$$

which reduces to

$$0 \cdot a + 1 \equiv 0 \bmod 2,$$

or $1 \equiv 0 \bmod 2$. This is a contradiction since no even number is an odd number. Therefore the above congruence has no solution, so a couldn't have been an integer. This proves that the equation $2a3 = 12$ has no integer solution.

6.6 Exercises

Ex. 18 — What is the lowest number divisible by both 12 and 20

Ex. 19 — In 2000 the Jordanian mathematician, Murad A. AlDamen devised the following method for testing divisibility [2]

1. Use Murad's method to determine if 61 divides 3598207
2. Prove Murad's method.

Notes

[1]There are technical reasons for not counting 1 as a prime

[2]Found on the prime puzzlers project, at: `http://www.primepuzzles.net/puzzles/puzz_101.htm`

Chapter 7

Combinatorics

Combinatorics is one of the most fascinating and frustrating branches of mathematics. Many excellent mathematics students find it impossibly difficult while some find it as easy as breathing. Classically, combinatorics deals with finite sets of objects and the various ways their subsets and their elements can be counted and ordered.

One of the first formal attempts of combinatorial thinking seems to have come around 960, when correctly enumerated the 56 outcomes that can arise when three dice are thrown simultaneously: $\{1, 1, 1; 1, 1, 2; 1, 1, 3, ...\}$ and so on. A thirteenth-century Latin poem, , listed the $216(= 6 \cdot 6 \cdot 6)$ outcomes that may result when three dice are thrown in succession.

If the results 56 and 216 seams somewhat unclear, don't dispair this chapter will teach you how to arrive at these results safely on your own.

So, in Mathematics we use more precise language:

If the order doesn't matter, it is a Combination. If the order does matter it is a Permutation.

A Permutation is an ordered Combination.

7.1   Permutations

In other words:

You've decided to take 3 steps and randomly choose left or right as the direction each time. How many different outcomes exists?. The mathematical branch of combinatorics helps us answers questions such as these and its mastery is essential to study statistics. Returning to our example we notice that you have two possibilities for the first step, two possibilities for the second and again two possibilities for the third yielding

$2 \cdot 2 \cdot 2 = 8$. The figure below shows the situation:
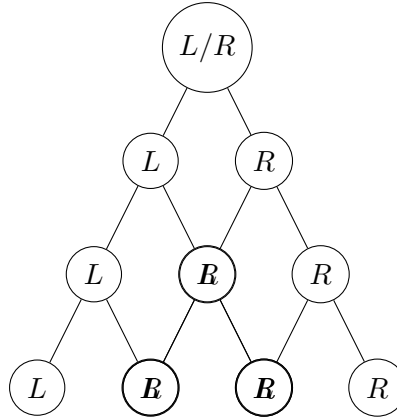
Suppose we have before us 2 red aces and 2 red kings from the usual deck of 52 cards. How many different pairs consisting of one ace and one king can be put together? Since each ace can be paired with either of 2 kings, there are 2 different pairs for any one ace. Since we have 2 aces, there are $2 \cdot 2$ or 4 different pairs.

7.2 Counting selections

Consider in how many ways can we arrange the set $\{1, 2, 3\}$:

$$\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}$$

that is we have 3 ways of choosing the first element, and $3 - 1$ options for the second (as we cannot repeat the first choise) and $3 - 2$ for the last (as non of the former two choices may be used), therefor we can arrange them in $3 \cdot 2 \cdot 1 = 6$ ways. In general we define the number of permutations of a set with n elements as the faculty function $n!$

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots (n - (n - 1)) = 1 \cdot 2 \cdots n = \prod_{k=1}^{n} k,$$

the last notation simply means that we multiply all the values of k from 1 to n. A recursive version (a self-calling function) can be constructed by noting that $n! = n(n - 1)!$.

Now consider a slightly more complex question: in how many ways can we pick two elements from the set $\{1, 2, 3\}$, the answer turns out to be

three ways:

$$\{1,2\}, \{1,3\}, \{2,3\}$$

To generalize this we can ask in how many ways can we pick a unordered subset with k elements from a set with n elements ?. To answer this, observe there are n choices for the first element and for each of these there are $n-1$ choices for the second; and so on until there are $n-k+1$ for the k'th element. That is we have $n \cdot (n-1) \cdots (n-k+1)$ ways for picking a k-element subset from n. As we are picking unordered sets and since each k-element subset has exactly $k!$ different orderings, we get our answer by dividing with $k!$:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \qquad \text{integer } n \geq k \geq 0 \qquad (7.1)$$

and if we multiply the numerator and denominator of 7.1 by $(n-k)!$ we get:

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!}$$

The symbol $\binom{n}{k}$ is known as a binomial coefficient and we read it "n choose k" (the number of ways to select k elements from n element set). Using this result we can directly answer the above question of how many ways to choose two elements from a three element set:

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3$$

Interestingly when determining the number of ways to pick k elements we have in effect also specified the $n-k$ unchosen things, therefore

$$\binom{n}{k} = \binom{n}{n-k} \qquad (7.2)$$

this symmetry can be observed in the above example where we have 3 ways of picking 2 elements from a 3-element set but then also 3 ways of picking the remaining element. It also follows directly from the definition of the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$$

Another rather remarkable result, known as Pascal's rule, can be derived by marking a particular object ϵ in an n-element set. Now if we

pick k objects from n, then either ϵ is picked or it is not. If ϵ is in the subset then there is $k-1$ objects left to choose among the $n-1$ elements, or more formally $\binom{n-1}{k-1}$. If ϵ is not in the subset, we need to choose all the k elements in the subset from the $n-1$ objects that are not ϵ; formally that is $\binom{n-1}{k}$. Thus there are

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

ways to choose k elements from n depending on whether ϵ is included in each selection or not. But this number must be the same as the binomial coefficient and thus we get:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{7.3}$$

this result can also be arrived at by algebraic manipulation

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!((n-1)-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-(k+1))!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}
$$

From Pascal's rule we can construct his famous triangle. Observe that according to this rule (see 7.3) you can calculate $\binom{5}{3}$ as:

$$
\begin{aligned}
\binom{5}{3} &= \binom{4}{3} + \binom{4}{2} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{3}{1} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}
\end{aligned}
$$

as $\binom{2}{0} = 1$, we can stop the expansion. In general as $\binom{n}{n}$ and $\binom{n}{0}$ are always 1 (when $n \geq 0$) we are guarantied that the expansion always terminates.

Thus we have found a convenient way of expressing higher binomial coefficients as sums of consecutive smaller ones:

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	2	1	0	0	0
3	1	3	3	1	0	0
4	1	4	6	4	1	0
5	1	5	10	10	5	1

discarding the zero's we can arrange these numbers into a triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\end{array}
$$

and so on

where the next row must contain:

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

but we already know that expressions like $\binom{6}{1}$ can be rewritten to $\binom{5}{1} + \binom{5}{0} = 5 + 1$. So we see that each entry in the new row is obtained simply by adding the numbers in the row above to the left and right.

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

To generate the triangle, you start with a 1, and then immediately below it, you put two 1s, one to either side. Then for each successive row, you put a new 1 at either end and complete the row between them by adding together each adjacent pair of entries in the row above and putting their sum halfway between them.

7.3   Newton's Binomial Theorem

In mathematics a binomial is a polynomial with two terms, i.e such as these

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
\vdots & \qquad \vdots
\end{aligned}
$$

or in general

$$
(a+b)^n = \overbrace{(a+b)(a+b)\cdots(a+b)}^{n \text{ factors}}
$$

the relationship between such polynomials and combinatorics stems from the so called binomial theorem.

1. A girl has 3 hats, 2 dresses, and 2 pairs of shoes. How many different costumes does she have? (TODO describe why $3 \cdot 2 \cdot 2 = 12$)

2.A deck of 52 cards contains 4 different aces and 4 different kings. How many different pairs of cards, each pair consisting of one ace and one king, can be formed from the aces and kings? (TODO $4 \cdot 4$)

3. in how many different ways can 7 people sit in 3 chairs ? (7 different ways to pick the first chair, leaving 6 different people to pick the second chair, leaving 5 different people to pick the fifth chair i.e. $7 * 6 * 5 = 210$).

Chapter 8

Trigonometry

the last great creation of the Greek period, plane and spherical trigonometry by Hipparchus and Ptolemy and their application to one of the dreams of mankind, understanding the movements of the heavenl bodies. This gave rise to modern astronomy and the physical sciences. Trigonometric functions are functions of angles. Figure 1.2 shows a right-angled triangle (ie a triangle with one angle of 90) with another angle denoted by the Greek letter theta . The sides of the a right-angled triangle are called the adjacent side (next to), the opposite side (opposite to) and the hypotenuse (opposite the right-angle). For any right-angled triangle, the ratios of the various sides are constant for any particular value of . The basic trigonometric functions are:

8.0.1 Trigonometric relations

$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \tag{8.1}$$

The reciprocals of the *cosine*, *sine*, and *tangent* are known as *secant* (*sec*), *cosecant* (*csc*), and *cotangent* (*cot*):

$$sec(\theta) = \frac{1}{sin(\theta)} \tag{8.2}$$

$$csc(\theta) = \frac{1}{cos(\theta)} \tag{8.3}$$

$$cot(\theta) = \frac{1}{tan(\theta)} \tag{8.4}$$

8.0.2   Inverse trigonometric functions

Chapter 9

Analytical geometry

Analytic geometry, also known as coordinate geometry, or Cartesian geometry, is the study of geometry using a coordinate system.

Chapter 10

Calculus

10.1 Series

10.2 Logarithm

A logarithm (log or logs for short) goes in the opposite direction to a power by asking the question: what power produced this number? So, if 2x = 32, we are asking, what is the logarithm of 32 to base 2? We know the answer: it's 5, because 25 = 32, so we say the logarithm of 32 to base 2 is 5. In general terms, if , then we say the logarithm of a to base x equals p, or For example, 104 = 10,000, so we say the logarithm of 10,000 to base 10 equals 4, or We can take logarithms of any positive number, not just whole ones. So, as 103.4321 = 2704.581 we say the logarithm of 2704.581 to base 10 equals 3.4321, or Logarithms to base 10 are called common logarithms. Older readers may remember, many years ago - after the dinosaurs, but before calculators and computers were widely available - doing numerical calculations laboriously by hand using tables of common logarithms and anti-logarithms. The properties of logarithms are based on the aforementioned rules for working with powers. Assuming that a ¿ 0 and b ¿ 0 we can say: logx(ab) = logxa + logxb, eg log10(1000 100) = log10(100,000) = 5 = log10(1000) + log10(100) = 3 + 2. logx(1/a) = -logxa, eg log3(1/27) = -log327 = -3. logx(a/b) = logxa - logxb, eg log2(128/8) = log2(16) = 4 = log2(128) - log2(8) = 7 - 3. logx(ay) = ylogxa, eg log5(253) = log515,625 = 6 = 3 log525 = 3 2. 1.8.3

10.2.1   Natural logarithm

Say we invest 1.00 in an exceedingly generous bank that pays 100%
interest per annum. If the bank calculated and credited the interest at the
end of one year, our investment would then be worth $1 + 1 = 2.00$. But
what if the bank credits the interest more frequently than once a year? If
interest is calculated and added every six months, at the end of that period
the balance would equal and at the end of one year the total amount would
be . Calculated three times a year, the final balance would be . In general,
if interest is calculated n times a year, the balance x after one year is

$$f(k) = \left(1 + \frac{1}{k}\right)^k$$

if we plug in some values we observe

$$f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25$$

$$f(5) = \left(1 + \frac{1}{5}\right)^5 = 2.248832$$

$$f(10) = \left(1 + \frac{1}{10}\right)^{10} = 2.59374...$$

n 1 2 2 2.25 3 2.37037 4 2.44141 5 2.48832 10 2.59374 100 2.70481 1000
2.71692 100,000 2.71827 1,000,000 2.71828 10,000,000 2.71828
We can see that as n increases, the value of the function appears to settle
down to a number approximately equal to 2.71828. It can be shown that
as n becomes infinitely large, it does indeed equal the constant e. The
mathematically succinct way of saying this introduces the important idea
of a limit and we say where means the limit of what follows (ie) as n
approaches infinity (symbol). In other words, e approaches the value of as
n approaches infinity.

$$e = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k \tag{10.1}$$

10.3   The exponential function

The exponential function f(x) = ex, often written as exp x (see Figure 1.7)
arises whenever a quantity grows or decays at a rate proportional to its
size: radioactive decay, population growth and continuous interest, for

example. The exponential function is defined, using the concept of a limit, as

10.3.1 Taylor series

$$cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \theta^{2n}}{2n!} \qquad (10.2)$$

10.3.2 Geometric Series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\frac{1 - r^n}{1 - r} \qquad (10.3)$$

10.4 Integration

10.4.1 Laplace Transform

10.5 Numerical integration

10.5.1 Trapez

Chapter 11

Linear algebra

11.1   Vectors

Addition

Inner product (dot product)

Pointwise vector division

Pointwise vector multiplication

Saxpy

Scalar vector multiplication

11.2   Matrices

Addition

Pointwise matrix division (denominator matrix must have non-zero entries)

Pointwise matrix multiplication

Rank The rank of a matrix is the number of linearly independent columns (which is equal to the number of linear independent rows)

Scalar matrix multiplication

Chapter 12

Set theory

12.1 Sets

A indexed set is a collection of values associated with indices. For example

- An ordered pair is a family indexed by the two element set $2 = \{1, 2\}$.

- An n-tuple is a family indexed by n.

the set that whose members label (or index) members of a family is called an index set.

Chapter 13

Probability and statistics

Probability and statistics are among the most commonly used branches of mathematics. Politicians try to predict the voters opinion of important matters by polling samples of the total population, medical researchers try to infer what percentage of a users of a medicine will be impacted by side effects, economists try to gauge the effectiveness of fiscal policy studying sample data, physics studies dynamic systems, in which the behavior of its part cannot be known in detail but only assigned a probabilities.

The ability to calculate probabilities transformed the practice of statistics, changing it from the mere collection and tabulation of data to the use of data to draw inferences and make informed decisions.

The pundits and pollsters who today tell us who is likely to win the next election make direct use of mathematical techniques developed by Pascal and Fermat. In modern medicine, future-predictive statistical methods are used all the time to compare the benefits of various drugs and treatments with their risks.

In short probability is an enormously useful subject. In spite of its usefulness the formal study of probabilities is rather new branch of matematics. The earlist attempts of describing probability using formal mathematics comes from gambling. The sixteenth century polymath Gerolamo Cardano demonstrated the efficacy of defining odds on games as the ratio of favorable to unfavorable outcome. Later these ideas was picked up and expanded upon by Pierre de Fermat and Blaise Pascal (1654). In recent years the study of probability and statistics has gained pace as computers allows us to perform studies on and make inference from large samples.

In 1494, in his book Summa de arithmetica, geometrica, proportioni et

proportionalita (Everything About Arithmetic, Geometry, and Proportions), Luca Pacioli first put into print the problem that Pascal and Fermat would solve two centuries later. Known as the problem of the unfinished game, it asks how the stakes should be divided when a game of several rounds must be abandoned unfinished. The puzzle is also known as the problem of the points, since rather than counting rounds, we can assign each player points for winning each round.

Suppose two playerscall them Blaise and Pierreplace equal bets on who will win the best of five tosses of a fair coin. They start the game, but then have to stop before either player has won. How do they divide the pot? If each has won one toss when the game is abandoned after two throws, then clearly, they split the pot evenly, and if they abandon the game after four tosses when each has won twice, they do likewise. But what if they stop after three tosses, with one player ahead 2 to 1?.

Pacioli, the man who first wrote about the problem, considered a version in which the game is played until one player has won six rounds, but play is abandoned when the score is 5 to 2. He suggested that the solution is to divide the pot according to the current state of play, namely, 5 to 2, but this reasoning is incorrect as was demonsgrated in 1539 by the itallian polymath Gerolamo Cardano. Cardano noted, correctly, that the apportionment of the pot depended not on how many rounds each player had already won (as Pacioli thought) but on how many each player must still win in order to win the contest.

In this version, the players, Blaise and Pierre, place equal bets on who will win the best of five tosses of a fair coin. Well suppose that on each round, Blaise chooses heads, Pierre tails. Now suppose they have to abandon the game after three tosses, with Blaise ahead 2 to 1. How do they divide the pot? The idea is to look at all possible ways the game might have turned out had they played all five rounds. Since Blaise is ahead 2 to 1 after round three, the first three rounds must have yielded two heads and one tail. The remaining two throws can yield H H H T T H T T Each of these four is equally likely. In the first (H H), the final outcome is four heads and one tail, so Blaise wins; in the second and the third (H T and T H), the final outcome is three heads and two tails, so again Blaise wins; in the fourth (T T), the final outcome is two heads and three tails, so Pierre wins. This means that in three of the four possible ways the game could have ended, Blaise wins, and in only one possible play does Pierre win. Blaise has a 3-to-1 advantage over Pierre when they abandon the game; therefore, the pot should be divided 3/4 for Blaise and 1/4 for Pierre. Today, we are so used to the idea that the future comes with

probabilitiesthe chance that it will rain tomorrow, that house prices will rise over the coming months, that an individual will die within the next five years. people no longer saw the future as completely unpredictable and beyond their control. They could compute the likelihoods of various things happening and plan their activitiesand their livesaccordingly. In short, we can manage risk.

Cardanos Liber de ludo aleae (Book of Games of Chance) was published in 1663, In this book he noted

"For example, I can as easily throw one, three or five as two, four or six. The wagers there are laid in accordance with this equality if the die is honest."

thus for the first time formulating what we now call the probability of an event as a fraction: the number of ways the event can occur divided by the total number of possible outcomes. More penetrating, he observed that to obtain the probability of getting a certain outcome on two successive throws, you square the probability of getting it on a single throw. For example, the probability of getting a 6 twice in succession is $1/6 \ 1/6 = 1/36$. Similarly, the probability of getting an even number twice in succession is $1/2 \ 1/2 = 1/4$. Extending the reasoning, the probability of getting a 6 three times in a row will be $1/6 \ 1/6 \ 1/6 = 1/216$, and the probability of rolling three even numbers is $1/2 \ 1/2 \ 1/2 = 1/8$. This assumes that the first action does not influence the nextin modern parlance, the two occurrences of the action have to be independent.

For example, the probability of rolling a 6 on the first throw and any even number on the second is $1/6 \ 1/2 = 1/12$. (If you remove the restriction on the order in which the 6 and an even number occur, there are more possibilities and, accordingly, the probability is bigger. The easiest way to calculate the probability in this case is to do what Fermat did in considering the problem of the points, and enumerate all the favorable outcomes. They are 2-6, 6-2, 4-6, 6-4, 6-6. Since there are 36 possible outcomes altogether, this gives the answer 5/36.)

More tricky is Cardanos next result: he calculated the probability of throwing a 1 or a 2 with not one die but a pair of dice. The probability of throwing a 1 or a 2 with a single die is 1/3, so the naive intuition is that with two dice, you double your chancesthat is, to a probability of 2/3. But as Cardano observed, this is incorrect. The problem is that a 1 or a 2 could come up on both throws, and by adding the two individual probabilities, you would be counting this possibility twice. The probability of not rolling a 1 or a 2 with a single die is 4/6, or 2/3, so the probability of not rolling a 1 or a 2 with two dice is $2/3 \ 2/3 = 4/9$. Hence, since the

probability of rolling something is 1, the probability of rolling a 1 or a 2 must be 1 - 4/9 = 5/9.)

13.1   Probability

Probability is a way of attaching a number that describes the possibility of a certain outcome to events that are fundamentally random. For example a coin toss with a fair coin (a coin that has equal chance of landing on each side) would have 50% chance of heads and 50% chance of tails.
The probability of a event occurring is calculated in different ways based on the nature of the event at hand, in general we speak of events being simple, compound or complementary
(The first rule tells you that if you throw a die, the odds of getting a 1 or an even number are 1/6 + 1/2 = 2/3; the second rule tells you that when you throw a die twice, the odds of getting a 1 followed by an even number are 1/6 1/2 = 1/12.)

13.1.1   Simple events

When finding the probability of an event occurring, we use the formula:

$$\frac{\text{Number of events that meet constraint}}{\text{Number of equally likely events}}$$

The following examples illustrate how to calculate simple probability using the above formula

- A toss of a fair coin have two possible outcomes $\{H, T\}$ so the probability of heads is $P(H) = \frac{1}{2}$.

- A toss of a fair die has 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$ so the probability of the die showing six is $P(6) = \frac{1}{6}$, the probability of getting one or six is $P(1|6) = \frac{2}{6}$ and probability of even number as $P(even) = \frac{3}{6}$

13.1.2   Compound events

The way Compound events works is that the likelihood of a series of independent events occurring is the multiplication of the likelihood of each individual event. So, if the probability of you finishing this chapter is 9 out of 10 (9/10), and the probability of you finishing the next one is 9/10, the total probability of you finishing both chapters isn't 9/10: it's
$9/10 * 9/10 = 81/100$.

13.1.3 Complementary events

Complementary events are events that add together to equal a whole. For example, if the probability of it raining today were 2/5, the probability of it not raining would be $1 - 2/5 = 3/5$.

13.2 Data visualization

When studying data in order to make better decisions we often start by looking for patterns of behavior. The simplest form of pattern is the number of times a specific data value occurs also known as its frequency. For example when schools create learning plans for classes they analyze frequency of the class grades. Armed with this knowledge special measures can be taken, such as splitting the class into multiple groups to be thought separately.

For example, if four students receives an F in mathematics, then the grade F is said to have a frequency of 4. The frequency of a data is often represented in a table by arranging data values in ascending order of magnitude with their corresponding frequencies. For example on a A, B, C, D, F grading scale the grades $A, A, B, B, B, C, C, D, F$ produces the following frequency table

Grade	Frequency
A	2
B	3
C	2
D	1
F	1

the table above can be easily visualized in a frequency plot, where each dot represents a occurrence of the value below
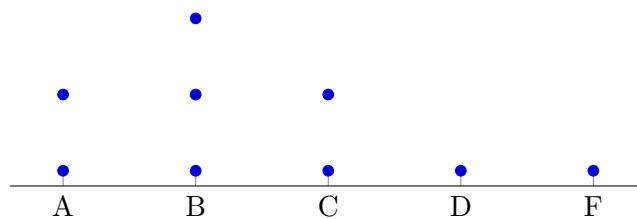
from this frequency plot we can easily deduce that the most common grade in the exam was B.

While frequency tables are convenient they often become hard to read if the data set contains more than a handful of observations. For these case we often resort to making a histogram or stem-and-leaf plot for the values.

13.2.1   Histograms

Often data is not repetitive enough for the same value to be repeated enough times to draw conclusions from a frequency table. In these cases we instead split the data into groups and count the frequencies of values within each group. For example if test scores are reassured between $0 - 100$ we may split the scores $15, 72, 55, 29, 67, 89, 33, 65, 78, 90$ in five groups.
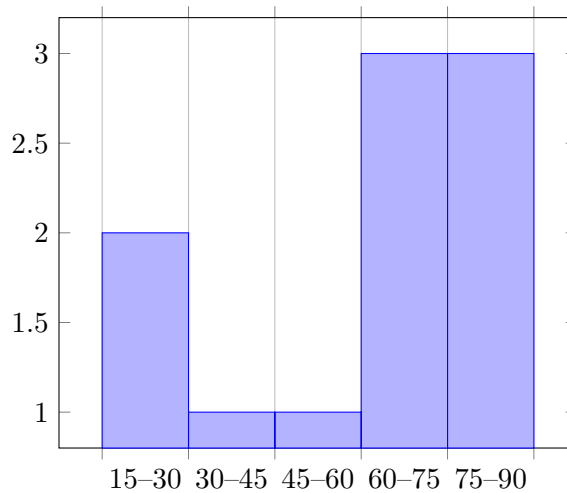
13.2.2   Steam and leaf plots

Steam and leaf plots (The left column of the stem and leaf plot represents the tens place; each number on the right side represents the ones place for the number of pairs of jeans at a department store.)

Next, it must be determined what the stems will represent and what the leaves will represent. Typically, the leaf contains the last digit of the number and the stem contains all of the other digits. In the case of very large numbers, the data values may be rounded to a particular place value (such as the hundreds place) that will be used for the leaves. The remaining digits to the left of the rounded place value are used as the stem.

In this example, the leaf represents the ones place and the stem will represent the rest of the number (tens place and higher).

13.2.3   Quartiles

When using histograms for data analysis we may split the data into any number of groups we deem necessary. This arbitrary split can lead to results that are hard to compare as different researchers may group their data differently. Thus to compare data its useful to group data using a stansafized method, one such method is that of quartiles. With quartiles the data is split into four groups, known as a quartile. Each quartile contains 25% of the total observations. Generally, the data is ordered from smallest to largest with those observations falling below 25% of all the data analysed allocated within the $1st$ quartile, observations falling between 25.1% and 50% and allocated in the $2nd$ quartile, then the observations falling between 51% and 75% allocated in the $3rd$ quartile, and finally the remaining observations allocated in the $4th$ quartile. When splitting data sets into their quartiles we do so by calculating the boundary values of each quartile known as Q_1, Q_2 and Q_3. Such that $1st$ quartile $v \in S where v < Q_1$ $2nd$ quartile $v \in S where v > Q_1 and v < Q_2$ $3rd$ quartile $v \in S where v > Q_2 and v < Q_3$ $4th$ quartile $v \in S where v > Q_3$ to calculate these cuts we use the following method

1. Sort data values and find its median (main median)

2. Q_1 is the median of the data left of the main median

3. Q_2 is the main median

4. Q_3 is the median of the data right of the main median

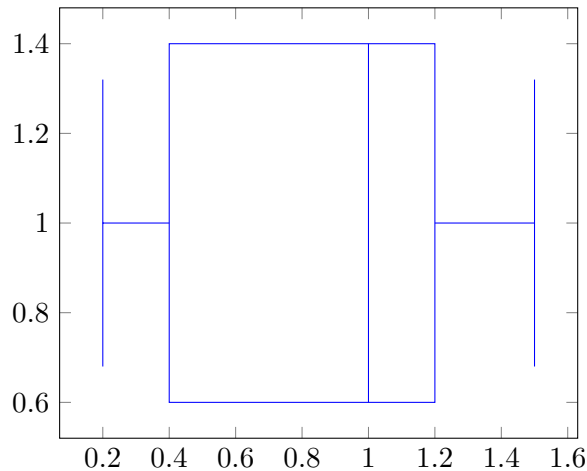For example the data set $\{0, 1.5, 2.5, 3, 4, 4, 4, 7, 7.5\}$ has the median $4 = Q_2$. The data to the left of the median is $\{0, 1.5, 2.5, 3\}$ which has $2 = Q_1$ as its median. The data to the right is $\{4, 4, 7, 7.5\}$ which has $5.5 = Q_3$ as its median.

Five number summary

Inter quartile range

The inter quartile range is the range between Q_3 ans Q_1 or formally $Q_3 - Q_1$

Box plots

13.2.4   Distributions

13.3   Mean, Median and Mode

Mean, median, and mode are three kinds of "averages".

Mean (arithmetic mean) The sum of numbers divided by the number
of numbers e.g. the mean of $\{3, 3, 5, 9, 11\}$ is
$(3 + 3 + 5 + 9 + 11)/5 = 6.2$.

Median The "middle" value of the numbers. To find the median
arranging all the observations from lowest to highest and pick the
middle one (e.g., the median of $\{3, 3, 5, 9, 11\}$ is 5). If there is an even
number of observations then the median is the mean of the two
middle values (the median of $\{3, 5, 7, 9\}$ is $(5 + 7)/2 = 6$).

Mode The "mode" is the value that occurs most often. If no number is
repeated, then there is no mode. e.g. the mode of $\{3, 3, 5, 9, 11\}$ is 3.
In the data set: $\{1, 1, 1, 2, 2, 2, 3\}$ we see that 1 and 2 both have the
most occurrences, so they are both modes.

Range The "range" is the difference between the largest and smallest
values. e.g. the range of $\{3, 3, 5, 9, 11\}$ is $11 - 3 = 8$.

Mid-range the mid-range is the arithmetic mean of the maximum and

minimum values in a data set

$$\text{MidRange} = \frac{min(x) + max(x)}{2}$$

13.3.1 Deviation

Mean absolute deviation The mean of the distances of each value from their mean.

1. Find the mean of all values
2. Find the distance of each value from that mean (subtract the mean from each value, ignore minus signs
3. Then find the mean of those distances

For example the Mean Deviation of $3, 6, 6, 7, 8, 11, 15, 16$ is 3.75 as the mean is 9 and the distances from this are $6, 3, 3, 2, 1, 2, 6, 7$

13.4 Regression

Regression is the main statistical technique used to quantify the relationship between two or more variables. A regression analysis would show a positive relationship between height and weight, for example. The measure of the accuracy of a regression is called R-squared. A perfect relationship, with no error, would have an R-squared of 1.00 or 100. Strong relationships, like height and weight, would have an R-squared of around 70 percent. A meaningless relationship, like hair color and weight, would have an R-squared of zero.

13.5 Random samples

A random sample of 25% of a schools pupules shoved that 16 pupiles had red hair. Based on the data, what is the most reasonable estimate of the total number of pupiles with red hair on the school?. 25% is the same as 1/4 of the pupiles at the school so the most resonable estimate is 4 times the number from the sample $4 \cdot 16 = 62$

13.6 Conditional probabilities

The most famous confusing problem of conditional probabilities is probably the Monty Hall Problem. At the last round of a game show,

youre faced with three curtains. Behind one of the curtains, there is a car, behind the two others there is a goat. Youre asked by the presenter to make a first choice. He then reveals one of the curtains you havent chosen which contains a goat. The presenter then gives a chance to change your mind and switch curtain.

door 1	door 2	door 3
stay	switch	switch
switch	stay	switch
switch	switch	stay

that is if its behind dore 1 and you chose door 1 you should stay if you chose door 2 or 3 you shoukd awitch, equally for door two and three. that is you should switch 6 out of 9 times.
if you take a card from a deck it has 1/52 chance of bejng the ace of spades. If you flip 50 of the of the remaining 51 cards in the deck and none of them turns out to be the ace of spades then the remaining card now has 51/52 chances to be the ace of spades, that is significanly more likely than our initial draw.

13.7   Probability space

A probability spaces is a way to models processes consisting of states that occur randomly. For example in a deck of 52 cards the sample space is a 52-element set, as each card is a possible outcome. Since there can be many outcomes (even infinitely many), outcomes elements are grouped into sets which are called "events. For our deck of cards, possible events may include

- **The 5 of Hearts** (1 element),

- **A King** (4 elements),

- **A Face card** (12 elements),

- **A card** (52 elements).

More formally an event, is any subset of the sample space we like to consider in our model, including the empty set (an impossible event, with probability zero) and the sample space itself (a certain event, with probability one).
To sum up a probability space consists of three parts

- The sample space Ω which is a set of all possible outcomes.

- The σ-algebra $\mathcal{F}$ which is a collection of the events we would like to consider.

- The probability measure P which is a function returning an event's probability $P : F \to [0, 1]$.

For example if the experiment consists of just one flip of a perfect coin, then the outcomes are either heads or tails: $\Omega = H, T$. The σ-algebra $\mathcal{F} = 2^\Omega$ contains $2^2 = 4$ events, namely:

- $\{H\}$ "heads"

- $\{T\}$ "tails",

- $\{\}$ "neither heads nor tails"

- $\{H, T\}$ "either heads or tails"

So, $\mathcal{F} = \{\{\}, \{H\}, \{T\}, \{H, T\}\}$. Since there is a fifty percent chance of tossing heads, and fifty percent for tails the probability measure in this example is $P(\{\}) = 0, P(\{H\}) = 0.5, P(\{T\}) = 0.5, P(\{H, T\}) = 1$.

13.7.1 Random variable

A random variable (or stochastic variable) is a variable whose value is subject to variations due to chance. A random variable's possible values might represent the possible outcomes of a yet-to-be-performed experiment The mathematical function describing the possible values of a random variable and their associated probabilities is known as a probability distribution. Random variables can be discrete, that is, taking any of a specified finite list of values; or continuous, taking any numerical value in an interval.

13.8 Exercises

Ex. 20 — A gardening store offers 4 different flowers $\{f_1, f_2, f_3, f_4\}$ tand 4 different types of pots $\{p_1, p_2, p_3, p_4\}$, how many different arrangement of pot and plant can you make

Ex. 21 — If you role a fair six sided die and a fair four sided die what is the probabilityb that neither will shown one

Ex. 22 —   You are given two dice to roll. One is black with six sides; the other is white with four sides. For a given roll, what is the probability the black die is even and the white die is 2

Ex. 23 —   You are going to randomly select a marble from a bag of marbles that contains 3 blue marbles, 4 green marbles, and 5 red marbles. What is P(blue marble)

Ex. 24 —   A store offers sells four types of cloth: shirts, pants, socks and hats and offers each in three colors orange", purple and blue. If you randomly pick the piece of clothing and the color, what is the probability that you'll end up with an orange hat

Ex. 25 —   If you flip a fair coin 1200 times. What is the best prediction for the number of times that the coin will land heads up?

Ex. 26 —   If you toss a fair die 180 times what is the best prediction of the number of times you will get more than 4

Ex. 27 —   You've decided to flip 3 coins, how many possible outcomes are there.

Ex. 28 —   If a pirate and a navy boat fires at each other simultanously and the navy boat has 3/5 probability of a hit and the pirate 1/3. What is then the probability that the navy hits and the pirate misses ?

Ex. 29 —   In a class of 7, there are 3 students who have red hair. If the teacher randomly chooses 2 students, what is the probability that neither of them have red hair.

Ex. 30 —   Three dice are thrown simultaneously. Find the probability that:

1. All show distinct faces
2. Two of them show the same face

Appendices

Appendix A

Solutions to the exercises

Each exercise is numbered in accordance with the chapter where it is to be found, thus exercise two in chapter one is referred to as 1.2.

Answer (Ex. 1) — Canceling out common terms we get

1. $88/72 = (11 * 8)/(9 * 8) = 11/9$.

2. $\frac{2}{\frac{4}{5}} = 2 \cdot \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$

3. $\frac{\frac{-7}{2}}{\frac{4}{9}} = \frac{-7}{-2} \cdot \frac{9}{4} = \frac{63}{8}$

Answer (Ex. 2) — $36/9$ is the time it takes for 5 people to paint one wall, therefor $5 * 36/9$ is the time it takes for one person to paint one wall. Thus it will take $7 * ((5 * 36)/9) = 140$ for one person to paint 7 walls and therefor $140/10 = 14$ minutes for 10 people

Answer (Ex. 3) — The car will spend $4000/100 * 6 = 240$ liters of gas at a price of $240 * 1.5 = 360$, as there are four people in the car the cost per person becomes $360/4 = 90$.

Answer (Ex. 4) — Using the PEMDAS meme (parentheses, exponents, multiplication, division, addition, subtraction) listed in 4.9 we get:

1. $7 + 7/7 + 7cdot7 - 7 = 7 + (7/7) + (7 \cdot 7) - 7 = 50$.

2. $6 - 1 \cdot 0 + 2/2 = 6 + -1 \cdot 0 + 2/2 = 6 + 0 + 2/2 = 7$

3. $-2 + (-3) + 4 - (-3) - 5 = -2 - 3 + 4 + 3 - 5 = -5 + 7 - 5 = -3$

4. $3 + (-4) - 8 - (-1) - 1 = 3 - 4 - 8 + 1 - 1 = -1 - 8$

Answer (Ex. 5) — Using the exponent rules from 4.8 we get

1. $\frac{4^6}{4^{-4}} = 4^{6--4} = 4^{10}$

2. $(5^{-12})^{-9} = 5^{108}$

3. $((5^{-10})(9^9))^7 = 5^{-70} \cdot 9^{63}$

4. $\sqrt{11} + \sqrt{44} + \sqrt{99} = \sqrt{11} + \sqrt{4 \cdot 11} + \sqrt{9 \cdot 11} = 6\sqrt{11}$

Answer (Ex. 6) — Noting that $-(x + y) = -1 \cdot (x + y) = -x - y$ we see

1. $-(6 - \frac{z}{4}) = -6 + \frac{z}{4}$

2. $-(\frac{1}{2}r + 4) = -\frac{1}{2}r - 4$

3. $-5(3n + \frac{1}{2}) = -15n - \frac{5}{2}$

Answer (Ex. 7) — We get

1. $\frac{3}{5}z - \frac{6}{5} = \frac{3}{5}(z + 2)$

2. $\frac{3}{2} + \frac{7}{8}c = \frac{1}{8}(12 + 7c)$

Answer (Ex. 8) — We determine equivalent expressions by subtracting them from each other in order to see if the result becomes zero

1.   Not equal: $9x + 6 - (3x + 2) = 6x - 4$

2.   Not equal: $\frac{5}{2}x - 3 - (x + 5) = \frac{3/2}{x} - 7$

3.   Equal:
$$\frac{\frac{2}{b} + \frac{2}{a}}{\frac{2}{ab}} - (a + b) = \left(\frac{2a + 2b}{ab}\right)\left(\frac{ab}{2}\right) - (a + b) = 0$$

4.   Equal:
$$\frac{\frac{a}{b} + 1}{\frac{b}{a} - 1} - \frac{a(a + b)}{b(b - a)} = \frac{\frac{a}{b} + \frac{b}{b}}{\frac{b}{a} - \frac{a}{a}} - \frac{a(a + b)}{b(b - a)} =$$
$$\left(\frac{a + b}{b}\right)\left(\frac{a}{b - a}\right) - \frac{a(a + b)}{b(b - a)} = 0$$

Answer (Ex. 9) — The expressions simplifies to

1. $6 + 5(-7n + 2) = -35n + 16$
2. $-(-15 + 2a) + 4(8a - 6) = 30a - 9$
3. $6(-2 + 10k) + 6(5k - 3) = 90k - 30$
4. $1/5 - 2z + z + 2/3 = 13/15 - z$
5. $7n - (4n - 3) = 3n + 3$

Answer (Ex. 10) — Using the exponent rules from 4.8 we get

1. $\frac{35n^3}{10n^4} = 3.5n^{-1}$
2. $\frac{44a^3}{55a^3} = \frac{4}{5}$
3. $\frac{28y^5}{7y^3} = 4y^3$

Answer (Ex. 11) — We formulate the expressions in terms of their unknown and solve for it.

1. The equation is $17 = \frac{n-6}{2}$ and $n = 2 \cdot 17 + 6 = 40$.
2. We have $n + (n + 2) + (n + 4) = 69$ and $n = \frac{63}{3} = 21$.
3. The equation is $\frac{13}{10}(7 + n) = 91$ and $n = 63$
4. Let x be the amount to be drawn off. Then $10 - x$ is the gallons remaining of which $\frac{1}{5}$ is alcohol that is $1/5(10 - x)$ is alcohol. After the x gallons are replaced with alcohol, the amount of alcohol in the tank will be $\frac{1}{5}(10 - x) + x$. As we want this to be 5 gallons (50 percent alcohol of 10 gallons) we end up with

$$\frac{1}{5}(10 - x) + x = 5$$

Which reduces to

$$3 \cdot \frac{5}{4} = \frac{15}{4} = \frac{12}{4} + \frac{3}{4} = 3\frac{3}{4}$$

Answer (Ex. 12) — The solutioms are

1. $x = \frac{12 \cdot 5}{6} + 12 = 22$.

2. $a = \frac{18 \cdot 5}{3} - 11 = 19$.

3. $k = \frac{5 \cdot 18}{6} + 7 = 22$.

4. $t = \frac{15 \cdot 8}{3} + 1 = 41$.

5. $y = \frac{7 \cdot 25}{5} + 8 = 43$.

Answer (Ex. 13) — x is

1. $x > -\frac{1}{12}$.

2. $x < \frac{5}{9}$.

3. $x \leq -\frac{13}{16}$.

4. $b > 3$.

5. $x > 5$.

Answer (Ex. 14) — x is

1. $d \leq 3$.

2. $x < \frac{7}{2}$.

3. $x \leq \frac{3}{2}$.

4. $x \geq \frac{1}{5}$

5. $x \leq -\frac{1}{6}$

Answer (Ex. 15) — The quadrants are

1.   First quadrant

2.   Second quadrant

3.   Third quadrant

4.   Fourth quadrant

5.   The origin has no quadrant.

Answer (Ex. 16) — Using 5.3 we get the following quadratics

1. $(x - 3)^2 = x^2 + (-3 + -3)x + (-3 \cdot -3) = x^2 - 6x + 9$

2. $(x - 3)(x + 10) = x^2 + (-3 + 10)x + (-3 \cdot 10) = x^2 + 7x - 30$

3. $(x + 8)^2 = x^2 + (8 + 8)x + (8 \cdot 8) = x^2 + 16x + 64$

4. $(x + 1)(x - 4) = x^2 + (1 + -4)x + (1 \cdot -4) = x^2 - 3x - 4$

5. $(x + 9)^2 = x^2 + (9 + 9)x + (9 \cdot 9) = x^2 + 18x + 81$

Answer (Ex. 17) — Using 5.3 we get the following factors

1. $x^2 + 9x + 20 = x^2 + (4 + 5)x + (4 \cdot 5) = (x + 4)(x + 5)$.

2. $x^2 + 5x + 6 = x^2 + (3 + 2)x + (2 \cdot 3) = (x + 2)(x + 3)$.

3. $x^2 - 5x + 6 = x^2 + (-3 + -2)x + (-2 \cdot -3) = (x - 2)(x - 3)$.

4. $x^2 - 9 = x^2 + (-3 + 3)x + (-3 \cdot 3) = (x - 3)(x + 3)$.

5. $x^2 + 7x - 18 = x^2 + (-2 + 9)x + (-2 \cdot 9) = (x + 9)(x - 2)$.

Answer (Ex. 18) — Numbers divisible by 12 and 20 also have to be divisible by each of their prime factors $12 = 2 \cdot 2 \cdot 3$ and $20 = 2 \cdot 2 \cdot 5$ thus $2 \cdot 2 \cdot 3 \cdot 5 = 60$.

Answer (Ex. 19) — TODO

Answer (Ex. 20) — Each of the four plants can be placed in four different pots yielding $4 \cdot 4 = 16$ possible arrangements

Answer (Ex. 21) —
$P(\text{six sided not not one}) \cdot P(\text{four sided not not one}) = 5/6 * 3/4 = 5/8$

Answer (Ex. 22) —
$P(\text{even black die}) \cdot P(\text{white die is 2}) = 3/6 * 1/4 = 3/24$

Answer (Ex. 23) — $P(bluemarble) = \frac{\text{number of blue marbles}}{\text{total number of marbles}} = 3/12 = 0.25$

Answer (Ex. 24) — $1/3 \cdot 1/4 = 1/12$

Answer (Ex. 25) — $1/2 \cdot 1200 = 600$

Answer (Ex. 26) — $2/6 \cdot 180 = 60$

Answer (Ex. 27) — There are two possible outcomes for the first flip, two for the second and two for the thrid, thus $2 \cdot 2 \cdot 2 = 8$ possible outomes.

Answer (Ex. 28) — Our scenario has probability
$P(\text{navy hits}) * P(\text{pirate misses})$ which is
$3/5 \cdot (1 - 1/3) = 3/5 \cdot 2/3 = 6/15 = 2/5$.

Answer (Ex. 29) — there is a $4/7$ chance that the first student is not a red hair and $3/6$ chance that the second student is not so
$4/7 * 3/6 = 12/42 = (2 * 2 * 3)/(2 * 3 * 7) = 2/7$.

Answer (Ex. 30) — TODO

Appendix B

Glossary

B.1 Glossary

associativity operation is associativ if within an expression containing two or more occurrences in a row of the same associative operator, the order in which the operations are performed does not matter as long as the sequence of the operands is not changed. ii

axiom ii, 11

cohort A group of subjects who have shared a particular event together during a particular time span (e.g., people born in Europe between 1918 and 1939; survivors of an aircrash; truck drivers who smoked between age 30 and 40) ii

commutativity Commutativity operation is commutative if changing the order of the operands does not change the result. ii

congruent Two figures are congruent if they have the same shape and size, but are in different positions ii

constant a number, term or expression that doesn't change. If $y = x + 7$ then 7 is a constant. The area of a circle equals $2\pi r$ where r is the radius (a variable), and π is a constant. ii

cube the cube of a number is the number multiplied by itself twice, eg 3 cubed $= 3^3 = 333 = 27$. ii

denominator ii

factor a number or expression that divides exactly into another number
   or expression. 1,2 and 5 are factors of 10; x and $(2x + 3)$ are factors
   of 2x2 + 3x because x multiplied by $(2x + 3) = 2x2 + 3x$. ii

fraction a number such as 1/2, 2/3, 17/2. Can also be written with a
   horizontal line, eg . The number above the line (called the
   numerator) is divided by the number below the line (called the
   denominator). ii

frustrum A solid, usually a pyramid or a cone, with its top cut of by a
   plane, parallel to its base. ii

integer Positive and negative whole numbers such as -34, -5, 0, 1, 17,
   1021 ii

irrational number A number that is not rational, eg π, Euler's number
   e. ii

isosceles triangle Isosceles triangle is a triangle that has two sides of
   equal length. ii

linear equation A algebraic expression, usually of the form

$$ax = b$$

   in which each term is either a constant or the product of a constant
   and (the first power of) a single variable (that is no variables in the
   expression contains exponents, like the 2 in x^2, square roots, cube
   roots, etc.) ii, 30

measure A measure on a set is a systematic way to assign a number to
   each suitable subset of that set, intuitively interpreted as its size. In
   this sense, a measure is a generalisation of the concepts of length,
   area, and volume ii

numerator   ii

postulate . ii, 11, see axiom

product The result of multiplying together two or more numbers or
   things, eg the product of 4 and 5 is 20. ii

QED <u>Quod erat demonstrandum</u>, Latin for <u>which was to be proved</u> ii

quadratic polynomial A polynomial with degree 2 such as
$7x^2 + 4x - 10$. ii

quotient The result of dividing one number by another. The quotient of
14 and 7 is 2 as $14/7 = 2$. ii

rational expressions A expression involving fractions of polynomials
such as $\frac{x-1}{x^2+12}$. ii

rational number A number that can be made by dividing one integer by
another, eg 1/2, 5/7, 12/108, 12/1. ii

real number the normal numbers we use, including decimals, fractions,
integers, negative numbers, positive numbers, irrational numbers,
etc. They are called real because they aren't complex numbers
(numbers that involve the square root of -1) ii

similarity Two geometrical objects are called similar if they both have
the same shape, that is one can be obtained from the other by
uniformly scaling (enlarging or shrinking) ii, 12

square the square of a number is the number multiplied by itself, eg 5
squared $= 52 = 25$ ii

square root the square root of a number is a number that when
multiplied by itself gives the original number. The square root of 36
is 6. The square root symbol is , so . ii

statement <u>If P, then Q</u>, where its **converse** has the following form: <u>if Q,</u>
<u>then P</u> ii

term ii

variable a quantity that may change, eg the circumference of a circle
equals 2r where (pi) is a constant and r, the radius, is a variable. ii