Lars Tackmann

Concepts of Mathematics

Volume 1

Fundemental Mathematics

First Edition

Contents

Contents 2

1 Introduction 3

2 Mathematical foundations 1
2.1 Problem solving 1
2.2 Proving theorems 1

3 Classical geometry 3
3.1 Classical plane geometry 3
3.2 Area and perimeter 5
3.3 Euclid Elements 6
3.4 Classical solid geometry 7
3.5 Trigonometry 7
3.6 Exercises 7

4 Arithmetic 9
4.1 Hindu numeral system 10
4.2 Addition 15
4.3 Subtraction 16
4.4 Multiplication 16
4.5 Fractions 18
4.6 Division 19
4.7 Percents 21
4.8 Exponents and roots 21
4.9 Order of operators 22
4.10 Exercises 23

5 Basic algebra 25
5.1 Linear equations 25

5.2 Linear inequalities . . . 28
5.3 Functions . . . 30
5.4 Monomials and binominals . . . 32
5.5 Polynomials . . . 32
5.6 Exercises . . . 32

6 Elementary number theory **37**
6.1 Basic properties of Integers . . . 37
6.2 Divisibility . . . 39
6.3 Primes . . . 41
6.4 Modulo arithmetic . . . 43
6.5 Exercises . . . 44
6.6 Notes . . . 45

7 Combinatorics **47**
7.1 Permutations . . . 47
7.2 Counting selections . . . 48
7.3 Newton's Binomial Theorem . . . 52
7.4 Exercises . . . 52

8 Trigonometry **53**

9 Analytical geometry **55**

10 Calculus **57**
10.1 Series . . . 57
10.2 Logarithm . . . 57
10.3 The exponential function . . . 58
10.4 Integration . . . 59
10.5 Numerical integration . . . 59

11 Linear algebra **61**
11.1 Vectors . . . 61
11.2 Matrices . . . 61

12 Set theory **63**
12.1 Sets . . . 63

13 Basic probability **65**
13.1 Events . . . 66
13.2 Conditional probabilities . . . 68

13.3 Probability space 69
13.4 Propability distributions 70
13.5 Exercises 70

14 Introduction to statistics 73
14.1 Data visualization 73
14.2 Standard Data points 75
14.3 Regression 78
14.4 Random samples 78
14.5 Exercises 79

A Solutions to the exercises 81

Index 89

Chapter 1

Introduction

It has been said that the typical theorem in mathematics states that something you do not understand is equal to something else you cannot compute. There is a kernel of truth in this joke, since the rigor required when doing mathematical research has found is way into every nook and corner of every textbook and thereby to a large extend rendered them unreadable to all, except fairly expert mathematicians; and even these usually disdain from reading such texts cover to cover. The value of mathematical rigor is not in the question; but such works are condemned to remain works of reference, to be consulted, not read.

This is unfortunate since many people derive great happiness from dabbling with a little mathematics; just consider the many people who daily tries to solve news paper Sudoku puzzles. Such activities implies that many people enjoys the pleasures of the mind and might even find a little mathematical education fulfilling. However it would be foolish not to acknowledge the complexity of modern mathematics. After all Andrew Wiles incredible proof of Fermat's Last Theorem or Abel's remarkable proof of the in-solvability of general higher degree equations eluded many of the worlds best mathematicians for centuries. Such gems are usually only available to the person equipped with a sharp mind, a fairly developed bag of mathematical tools and a certain intellectual maturity. What I propose here is a book that will build up enough knowledge to show the proofs of some of the greatest theorems in the history of mathematics; from Hippocrates Quadrature of the Lune to Lindeman's proof that π is transcendental. I have sought to make the text completely self contained, so the reader will be taken from simple arithmetic to measure theory and game theory.

The mathematical mature person is right to ask if such an bold attempt is not futile and perhaps its author is a bit naive. There goes a story that the great physicist Richard Feynman was once asked by a Caltech faculty member to explain why spin 1/2 particles obey the . He nodded and said, "I'll prepare a freshman lecture on it". But a few days later he returned and said. "You know, I couldn't do it, I couldn't reduce it to freshman level. That means we really don't understand it." Just as Feynman found out this author has also had to abandon a number of otherwise interesting subjects due to the complexity of conveying them within a reasonable span of pages.

The text includes a number of exercises along with answers to every one (at least briefly). I am a firm believer in learning-by-looking, also known as cheating, and thus I encourage the reader to give each problem his best shot, then check the answer. If correct then move on to a more advanced problem, if not then study the provided solution and try to solve a similar problem. If you are stuck then don't panic I have included an entire chapter devoted to the fine art of problem solving and mathematical proof.

The subject presented here is not laid out in the usual chronological manner, instead I have sought to build up mathematics as a logical progression of ever more complex ideas. Therefor instead of focusing on historical order I have strived instead to provide the clearest proofs; so for example, instead of 's original verbal proof of the divergence of I have instead included a new simple algebraic proof. At other times the original proof is so beautiful or provide a vital lessons in mathematical reasoning that I let it's author speak directly (such as Euclid's proof of the Pyteagorian theorem).

The knowledge and proofs contained herein is taken from a multitude of sources, many have been rewritten or reordered so to make them more accessible, but often one finds mathematics written with such breathtaking clarity as so to render any further attempt upon simplification impossible. In these cases the information is simply restated and put in context.

Mathematics is utility and its usage is spread far and wide; from logical reason about programming languages to the physical sciences. So stories about its usages is included whenever I have found it fitting. At the same time mathematics is also history, from Archimedes war machines used against the intruding romans to everyday stories such as Newton's remark that he "do not like to be teased by foreigners about mathematical things". In the end all that I hope to achieve is to shine a little light on the fascinating field of human creativity known as mathematics, to hopefully illustrate the meaning of the the great philosopher Spinoza's words, when he said that "God is a mathematician".

Lars Tackmann

Copenhagen, Denmark

Chapter 2

Mathematical foundations

> You just keep pushing. You just keep pushing. I made every mistake that could be made. But I just kept pushing.
>
> Rene Descartes

2.1 Problem solving

Word problems

"A bat and ball, together, cost a total of 1.10 and the bat costs 1 more than the ball. How much is the ball?" The wrong answer is the one that roughly one in every two people blurts out: 10 cents. The correct answer is 5 cents, since only with a bat worth 1.05 and a ball worth 5 pence are both conditions satisfied.

Problem solving checklist

2.2 Proving theorems

Proof methods

Proof by induction

Proof checklist

Chapter 3

Classical geometry

Geometry arrose as practical methods in the ancient world in order to measure land, survey fields, compute the quantity of corn in containers, and to construct temples and pyramids. The theorems of Thales and Pythagoras, which are the oldest theorems of humanity and fundamental tools for geometry.
Later these methods practiced by builders and tradesmen where taken over by Euclid who replaced the geometry of nails, ropes and walls used by the temple builders with mathematical points, lines, rectangles etc., objects of pure reasoning, which require a list of definitions, s and . This is the origin of nearly all mathematical procedures used ever since.

3.1 Classical plane geometry

The most beautiful and useful discoveries in classical geometry concerns the relations between lengths (Thales intercept theorem), angles (the central angle theorem of Euclid) areas (the Pythagorean theorem).

Thales theorem

Thales is the man to tell us how to measure the height of a tree without having to climb it. In figure 3.1 below we let side $B'C'$ be the height of the tree and AB' be its shadow. We erect a vertical stick BC in such a manner that AB is the shadow of the stick. We then measure the distance AB, say 4 metres, the distance AB', say 8 metres, and the stick BC, say 5 metres. By parallel displacements of the triangle ABC we see that, since AB measures twice AB, the height $B'C'$ will measure twice BC, hence $B'C' = 2 \cdot 5 = 10$.

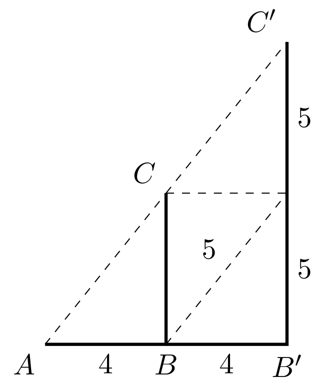


Figure 3.1: Using similar triangles to measure the height of a tree

Theorem 3.1.1. *(Thales intercept theorem). Consider an arbitrary triangle ABC (see figure 3.1 below) and let AC be extended to C' and AB to B' so that $B'C'$ is parallel to BC. Then the lengths of the sides satisfy the relations*

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} \quad \textit{and hence} \quad \frac{a'}{c'} = \frac{a}{c}, \quad \frac{c'}{b'} = \frac{c}{b}, \quad \frac{b'}{a'} = \frac{b}{a}$$

As these proportions are also preserved when the triangle is displaced and rotated we get the following result. *If corresponding angles of two triangles are equal, then corresponding sides are proportional.* Triangles having these properties are called similar (see).

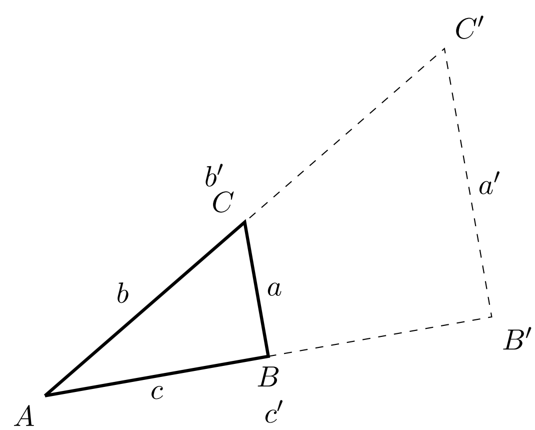


Figure 3.2: Similar triangles

Points, Lines and Planes

Angles

Radians

Degrees are often used when introducting geometry. However for more advanced work we often meassure angles in radians. Radians equals the ratio between the length of an angle arc and its radius. As the circumference of a circle equals $2\pi r$ thus an angle of $360deg$ equals $\frac{2\pi r}{r} = 2\pi$ radians. Radians allow us to use real numbers in the trigonometric functions, rather than degrees, which are an arbitrary angular measurement between $0 - 360$. The use of real numbers for meassuring angels allow is essential in more advanced mathematics, calculus for example.

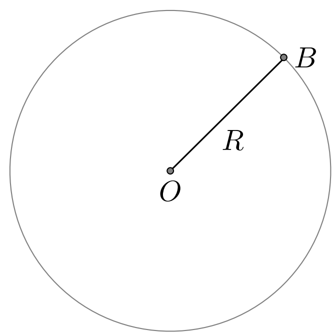
3.2 Area and perimeter

The area of a parallelogram is $a \cdot h$, where h is the altitude of the parallelogram (Eucl. I.35). There are two ways to see this: (a) We cut off the triangle on the left and add it on the right to obtain a rectangle (Euclids proof, see the second figure in Fig. 1.11); (b) We cut the parallelogram parallel to AB into a large number of very slim rectangles. The area of a triangle is half the area of the parallelogram.

Triangles

Figure 1.2 shows a right-angled triangle (ie a triangle with one angle of 90) with another angle denoted by the Greek letter theta . The sides of the a right-angled triangle are called the adjacent side (next to), the opposite side (opposite to) and the hypotenuse (opposite the right-angle).

Circle



The area of a circle is

$$A = \pi \cdot r^2 \tag{3.1}$$

the perimeter (circumference) is

$$C = 2\pi \cdot r \tag{3.2}$$

3.3 Euclid Elements

He begins with some definitions of the basic concepts: point, line, circle, triangle, quadrilateral. Euclid then states ten axioms (also called postulates)on which all subsequent reasoning is based. We shall note these merely to see that they do indeed describe apparently unquestionable properties of geometric figures. The first five posgulates are:

1. Two points determine a unique straight line
2. A straight line extends indefinitely far in either direction
3. A circle may be drawn with any given center and any given radius
4. All right angles are equal
5. Given a line l (Fig. 61) and a point P not on that line, there exists in the plane of P and l and through P one and only one line m , which does not meet the given line l .

Adding four right-angled triangles with sides a and b, we arrive at and get the large square of area $(a + b)^2 = a2 + 2ab + b2$. Since the areas of the four triangles add up to $2ab$, the square with area c^2 also has area $a^2 + b^2$.

3.4 Classical solid geometry

Cube

Cone

Sphere

3.5 Trigonometry

3.6 Exercises

Chapter 4

Arithmetic

Numbers are inherited in nature, a farmer may own six animals, a day can be split into 24 equal parts, each of our hands has five fingers and so on. Thus even without any formal knowledge of arithmetic, the oldest and most elementary branch of mathematics, humans has an inherited understanding of numbers. Arithmetic consist of the study of numbers, especially the properties of the operations between them addition, subtraction, multiplication and division.

The earliest formal concepts of numbers dates back to the Babylonians, Egyptians and later Greek civilizations. Here tax collectors used tablets to record tax payments of citizens. When counting the tax payments its easy to imagine that various ways of shorthand representation was invented. If a citizen payed five animals a tax collector may have written IIIII. Each I representing one animal. If later additional five animals where collected he may have written X instead of IIIIIIIIII as its easer to read and deal with. From these early tabulation shorthands rose the first numerical systems where it was agreed upon to let specific symbols represent a quantity. By the far the most successful of these was the, still familiar, roman symbols I, II, III, IV, V, VI, VII, VIII, IX and X. Once the numbers was established various properties of them was researched, for example the greeks discovered the primes, which bool VII of Euclid's elements deals extensively with. The concept of rational (fractional) numbers was also quickly arrived at as a useful device for division, such as dividing the land belonging to a village evenly among its inhabitants for harvesting. Egyptians used the word "nfr" to denote zero balance in accounting and later during the $7th$ century AD, negative numbers were used in India to represent debts.

The use of numbers in trading and science quickly outgrew the capabilities of the early numerical systems. The romans for example had no concept of number position, that is the use of the same symbol for the different orders of magnitude (for example, the "ones place", "tens place", "hundreds place"). This lack of position required them to either come up with new symbols for ever greater numbers (for example C for 100 and M for 1000) thus greatly increasing the complexity of the system. Or to simply keep count using the symbols they already had (for example writing 11111 as MMMMMMMMMMMCXI). These shortcomings eventually led to efforts of creating a new better numeral system and around $400AD$ a new system began surfacing in India. This new system used the position of ten numbers $0-9$ to denote any possible integral value.

4.1 Hindu numeral system

The Hindu numeral system was invented between the $1st$ and $4th$ centuries by Hindu mathematicians. The system was adopted, by Persian and Arab mathematicians in the 9th century and from the $14th$ century on, Roman numerals began to be replaced in most contexts by it; however, this process was gradual, and the use of Roman numerals in some minor applications continues to this day.
The main advantage of the Hindu numeral system is it use of position to denote order of magnitude thus requiring far fewer different symbols to represents numbers as well as greatly simplifying operations such as adding and subtracting those numbers. Thus the number 2506 is represented as

$$2506 = 2 \times 1000 + 5 \times 100 + 0 \times 10 + 6 \times 1.$$

Using the exponentiation notation

$$x^n = \underbrace{x \times \cdots \times x}_{n}$$

and noting that $10^0 = 1$ (see section 4.8 for formal proof of this) we can rewrite the above as

$$2506 = 2 \times 10^3 + 5 \times 10^2 + 0 \times 10^1 + 6 \times 10^0.$$

further taking advantage that $x^{-n} = \frac{1}{x^n}$ we see that $0.1 = \frac{1}{10} = 1 \times 10^{-1}$ and $0.02 = \frac{2}{100} = 2 \times 10^{-2}$ using this knowledge we
numbers as sums of integers and fractions, for example

$$5678.12 = 5 \times 10^3 + 6 \times 10^2 + 7 \times 10^1 + 8 \times 10^0 + 1 \times 10^{-1} + 2 \times 10^{-2}$$

Generally we represent a number with n integer digits $a_0 \cdots a_n$ and m decimal digits $b_0 \cdots b_m$ as

$$a_0 \cdots a_n.b_0 \cdots b_m = a0 = a^0 \times 10^n + \cdots + a^n \times 10^0.b^0 \times 10^{-1} + \cdots + b^m \times 10^{-m}$$

the class of numbers that can be written in this way is called Algebraic numbers. From the above formula we also see the well known fact that when a decimal number is multiplied by 10, every figure moves one place to the left

$$\begin{aligned} 1.25 \times 10 =& (1 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2}) \times 10 \\ =& 1 \times 10^1 + 2 \times 10^0 + 5 \times 10^{-1} \\ =& 12.5 \end{aligned}$$

and when you divide by 10, every figure moves one place to the right i.e

$$\begin{aligned} \frac{1.25}{10} =& 1.25 \times 10^{-1} \\ =& (1 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2}) \times 10^{-1} \\ =& 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3} \\ =& 0.125 \end{aligned}$$

General Numeral systems

As time passed positional numbers systems was generalized such that the digits ($0 - 9$ in the decimal system) could be changed to accomedate different purposes. For example electric current can be in only two states on (power is flowing) or off (no power). Thus in cumputers its convinient to represent numbers using only two digits 0 and 1. To represent higher numbers multiple wires in either on or off states can be used for example if we have 4 wires and all are on we can represent

$$1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 = 1 + 2 + 4 + 8 = 15$$

If all wires are off we have

$$0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 = 0 + 0 + 0 + 0 = 0$$

Thus with a four digit number in a binary number system we can represent the integers $0 - 15$ or 2^4 different values.

In the generalized positional number system we use the (or) to represent the different symbols available to each digit within that system. For example, the decimal system has a radix of 10, that is any diget in a number can have values $0 - 9$. In contrary computers work using electronic circuits and thus digets are represented with radix 2. Other commonly used number systems are hexadecimal wich uses radix 16 with digets represented with the numbers $0 - 9$ and letters $A - F$. Generally we have

Decimal representation

$$(a_n a_{n-1} \cdots a_1 a_0 . c_1 c_2 \cdots c_{m-1} c_m) = \sum_{k=0}^{n} a_k b^k + \sum_{k=1}^{m} c_k b^{-k}$$

The numbers b^k and b_k are the weights of the corresponding digits. The position k is the logarithm of the corresponding weight w, that is
$k = \log_b w = \log_b b^k$
For example to write 1/3 in base 16 we get

Conversion between systems

Scientific notation

Scientific notation is a way of writing numbers that are too big or too small to be conveniently written in decimal form. In scientific notation all numbers are written in the form

$$a \times 10^b$$

where the exponent b is chosen so its absolute value is at least one but less than ten ($1 \leq |a| < 10$). Thus 350 is written as 3.510^2. The following table are examples of scientific notation:

Decimal notation	**Scientific notation**
2	2×10^0
300	3×10^2
4,321.768	$4,321768 \times 10^3$
53,000	$5,3 \times 10^4$
6,720.000.000	$6,72 \times 10^9$
0.2	2×10^1
0,000.000.007.51	$7,51 \times 10^9$

This form allows easy comparison of numbers, as the exponent b gives the number's order of magnitude. For example, the order of magnitude of 1500 is 3, since 1500 may be written as 1.510^3.

Rounding

Rounding a number means replacing it by another that is approximately equal but has a shorter or simpler representation. The procedure is as follows. **Pick the number to round to as the last digit to keep. Leave it the same if the next digit is less than 5 or increase it by 1 if the next digit is 5 or more. Finally turn all following digets into zeros**. The following examples demonstrate the rounding procedure:

Round to the nearest hundred 838.274 is 800 as $3 < 5$

Round to the nearest ten 838.274 is 840 as $8 > 5$

Round to the nearest one 838.274 is 838 as $2 < 5$

Round to the nearest tenth 838.274 is 838.3 as $7 > 5$

Round to the nearest hundredth 838.274 is 838.27 as $4 < 5$

Besides the standard rounding mentioned above two other methods called floor and ceiling which maps a real number to the largest previous or the smallest following integer, respectively. More precisely, $floor(x) = \lfloor x \rfloor$ is the largest integer not greater than x and $ceiling(x) = \lceil x \rceil$ is the smallest integer not less than x.

Value	**Floor**	**Ceiling**	**Fractional part**
2.4	2	3	0.4
2.9	2	3	0.9
-2.7	-3	-2	0.3
-2	-2	-2	0

Significant figures

The significant figures of a number are those digits that carry meaning contributing to its precision. The procedure for identifying significant figures is as follows

1. Identify the non-zero digits and any zeros between them. These are all significant.

2. Leading zeros are not significant.
3. If its a decimal, then trailing zeros are significant.

4.2 Addition

Addition represents the operation of adding objects to a collection. For example, $2+3$ can be though of as if you have 2 apples and someone give you 3 more, then you have 5 apples in total. Assuming b is positive then on a number line $a+b$ represents starting on number a and moving b places to the right. Similar if b is negative then $a+b$ represents starting on number a and moving b places to the left. Unlike subtraction the addition operations is both commutative (that is $a+b=b+a$) and associative (that is $a+(b+c)=(a+b)+c$)

Adding integers We add integers by adding the ones, tens,hundreds etc. in each number individually, starting from the right. If the result of addition is above 10 then we subtract 10 from the result and add one (known as a carry) to the next group to be added. For example

$$\begin{array}{r} {\scriptstyle 1\ 1\ \ } \\ + \begin{array}{r} 1\ 4\ 3 \\ 8\ 9 \end{array} \\ \hline 2\ 3\ 2 \end{array}$$

That is we add $3+9=12$ as the result is above 10 we subtract 10 from it and add one tenth to the tenths place. In the tenth place we now add $1+3+8=12$ which again is above 10, so we subtract 10 from it and add 1 to the hundreds place (ten tens being one hundred) and so finally we add $1+1$ in the hundreds place.

To understand why this procedure works consider that the number $143 = 100+40+3$ and $89 = 80+9$ so we can do

$$\begin{aligned} 143+89 &= (100+40+3)+(80+9) \\ &= 100+(40+80)+(3+9) \\ &= 100+(40+80)+12 \\ &= 100+(10+40+80)+2 \\ &= 100+130+2 \\ &= (100+100)+30+2 \\ &= 232 \end{aligned}$$

Adding decimals We add decimals semilar to integers by starting from the rightmost decimal.

$$\begin{array}{r} {\scriptstyle 1\,1\,1\,1} \\ 9.0\,8\,7 \\ +\;1\,5.9\,2\,4 \\ \hline 2\,5.0\,1\,1 \end{array}$$

4.3 Subtraction

Subtraction represents the operation of removing objects from a collection. For example, $5 - 2$ can be though of as if you have 5 apples and take 2 away, then you have 3 apples left. Assuming b is positive then on a number line $a - b$ represents starting on number a and moving b places to the left. Similar if b is negative then $a - b$ represents starting on number a and moving b places to the right.
Subtraction is not commutative, that is $a - b \neq b - a$ instead we can reverse the subtraction order by $a - b = -(b - a) = -b + a$. Further subtraction is not associative so $a - (b - c) \neq (a - b) - c$ instead it holds $a - b - c = a - (b + c)$ and $a - b - c - d = a - (b + c + d)$ and so on.

Subtracting integers We subtract numbers by subtracting each ones, tens, hundreds etc. individually, starting from the right, if the subtraction yields a result less than zero then we borrow ten from the number to the left by decreasing its value by one:

$$\begin{array}{r} 1\,3\,3 \\ -\quad 8\,9 \\ \hline 4\,4 \end{array}$$

Subtracting decimals

4.4 Multiplication

Multiplication of two whole numbers is equivalent to the addition of one of them with itself as many times as the value of the other one; for example, $3 \cdot 4$ can be calculated by adding 4 to itself 3 times:

$$3 \cdot 4 = 4 + 4 + 4 = 12$$

it can also be calculated by adding 4 copies of 3 together:

$$3 \cdot 4 = 3 + 3 + 3 + 3 = 12$$

Formally we say that multiplication is both commutative (i.e. the order in which two numbers are multiplied does not matter $a \cdot b = b \cdot a$) and associative $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Multiplication has other important properties:

Distributive property Holds with respect to multiplication over addition. This identity is of prime importance in simplifying algebraic expressions: $x \cdot (y + z) = x \cdot y + x \cdot z$

Identity element The multiplicative identity is 1; anything multiplied by one is itself. This is known as the identity property: $x \cdot 1 = x$

Zero element Any number multiplied by zero is zero. This is known as the zero property of multiplication: $x \cdot 0 = 0$

Multiplying integers We multiply integers by multiplying the ones, tens, hundreds etc. in each number individually, starting from the right. If the result of multiplication is above 10 we subtract 10 from the result and add one (known as a carry) to the next group to be multiplied. For example

$$\begin{array}{r} 1\,4\,2 \\ \times \quad\;\; 3 \\ \hline 4\,2\,6 \end{array}$$

That is we first multiply $3 \cdot 2 = 6$ in the ones place, then in the tenth place we multiply $3 \cdot 4 = 12$ which is above 10, so we subtract 10 from it and add 1 to the hundreds place and so finally we multiply $3 \cdot 1 = 3$ and add the carry of 1 to it in order to get 4 in the hundreds place.

To understand why this procedure works consider that the number $142 = 100 + 40 + 2$ so really we have

$$\begin{aligned} 143 \cdot 3 &= (100 + 40 + 2) \cdot 3 \\ &= (100 \cdot 3) + (40 \cdot 3) + (2 \cdot 3) \\ &= (100 \cdot 3) + (10 \cdot 4 \cdot 3) + 6 \\ &= (100 \cdot 3) + (10 \cdot 12) + 6 \\ &= (100 \cdot 3) + (100 + 20) + 6 \\ &= (100 \cdot 4) + 20 + 6 \\ &= 426 \end{aligned}$$

Multiplying decimals To multiply decimals convert them into integers, multiply them and then convert the result back to a decimal i.e.

- $8 * 0.8 = 8 * \frac{8}{10} = \frac{64}{10} = 6.4$
- $2.91 * 3.2 = \frac{291}{100} * \frac{32}{10} = \frac{291*32}{1000} = \frac{9312}{1000} = 9.312$

4.5 Fractions

In the fraction 3/4, the numerator 3 (also known as the dividend) tells us that the fraction represents 3 equal parts, and the denominator 4 (also known as the divisor) tells us that 4 parts make up a whole. The fraction 5/4 equally tells us that 4/4 makes up a whole and we have an extra 1/4 part left.

Equal fractions Fractions have the same value when they represent the same parts of a hole e.g. $\frac{a}{b} = \frac{c}{d}$ if and only if $a \cdot d = b\dot{c}$.

Ordering fractions When both denominators are positive $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. If either denominator is negative convert them into postive numbers by negating the numerator.

Additive inverse $-\left(\frac{a}{b}\right) = \frac{-a}{b} = \frac{a}{-b}$

Multiplicative inverse $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

Simplifying Fractions (TODO examples 78/52

Adding fractions $\frac{a}{b} + \frac{c}{d} = \frac{a*d+b*c}{b*d}$

Subtracting fractions $\frac{a}{b} - \frac{c}{d} = \frac{a*d-c*c}{b*d}$

Multiplying fractions $\frac{a}{b} * \frac{c}{d} = \frac{a*b}{c*d}$

Dividing fractions Division is equivalent to multiplying by the reciprocal of the divisor fraction: $\frac{a}{b}/\frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

Exponentiation to integer power If n is a non-negative integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and if $a \neq 0$ then $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \frac{1}{\frac{a^n}{b^n}} = \frac{b^n}{a^n}$.

Converting decimals to fractions Place the decimal in the numerator and one in the denominator and multiply both with multiples of ten until the decimal point disapears e.g.
$0.024 = 0.024/1 = 0.24/10 = 2.4/100 = 24/1000$

Mixed numbers

Improper fractions such as 5/4, where the numerator is larger than the denominator, are commonly represented as a combination of a whole number and a proper fraction called a mixed number e.g. $5/4 = 1\frac{1}{4}$.

Converting fractions to mixed numbers Convert the numerator to a sum where the first part can be divided by the denominator and the last part has a lower value than the denominator e.g.
$\frac{64}{5} = \frac{60+4}{5} = \frac{60}{5} + \frac{4}{5} = 12\frac{4}{5}$

Converting mixed numbers to fractions Multiply the leading integer with the denominator and add it to the numerator e.g.
$12\frac{2}{3} = \frac{3*12+2}{3} = \frac{38}{3}$

Adding and subtracting mixed numbers A mixed number $a\frac{b}{c}$ is a short hand for $a + \frac{b}{c}$ so to add or subtract mixed numbers perform the operation step wise on the integer and fractional part of the numbers e.g. $6\frac{6}{12} - 3\frac{3}{5} = 6 + \frac{6}{12} - 3 - \frac{3}{5} = 2\frac{9}{10}$.

Multiplying mixed numbers Convert the mixed number to a fraction and multiply these e.g. $1\frac{1}{4} * 4 = \frac{20}{4} = 5$

4.6 Division

Dividing improper fractions Division is really counting the number of repeated subtractions of the divisor into the dividend e.g.
$3024/42 = 72$ as shown below

$$\begin{array}{rr|rrrr}
 & & 0 & 0 & 7 & 2 \\ \cline{3-6}
4 & 2) & 3 & 0 & 2 & 4 \\ \cline{1-2}
 & & 2 & 9 & 4 & \\ \cline{3-5}
 & & & & 8 & 4 \\
 & & & & 8 & 4 \\ \cline{5-6}
\end{array}$$

Dividing proper fractions As proper fractions always represent values parts less than 1 whole we know the result must be a decimal beginning with 0.

			0,	7	0	3
2	7)	1	9,	0	0	0
		1	8	9		
				1	0	
					0	
				1	0	0
					8	1
					1	9

Dividing decimals To divide decimals first convert the denominator into a integer then divide the resulting fraction as shown above e.g. $3.3534/0.81 = 335.34/81$

		0	0	4,	1	4
8	1)	3	3	5,	3	4
		3	2	4		
			1	1	3	
				8	1	
				3	2	4
				3	2	4

Divisibility tests

The following rules can test numbers for divisibility

Divisible by 2 if the last didget is divisible by 2.

Divisible by 3 if the sum of the digits is divisible by 3.

Divisible by 4 if the number formed by the last two digits is divisible by 4.

Divisible by 5 if the last digit is either 0 or 5.

Divisible by 6 if divisible by 2 and 3.

Divisible by 9 if the sum of the digits is divisible by 9.

Divisible by 10 if the last digit is 0.

Remainder

The remainder from the division a/b is represented mathematically as $a\mathrm{mod}b$, for example $9/2 = 4$ and $9mod2 = 1$. In general to the find the result of $a\mathrm{mod}b$ we follow these steps:

1. Construct a clock with size b
2. Start at 0 and move around the clock a steps (If the number is positive we step clockwise, if it's negative we step counter-clockwise).
3. Wherever we land is our solution.

For example $7mod2 = 1$ since we can make a clock with numbers $0, 1$ then start at 0 and go through 7 numbers in a clockwise sequence $1, 0, 1, 0, 1, 0, 1$. Similar $-5mod3 = 1$ since we make a clock with numbers $0, 1, 2$ then start at 0 and go through 5 numbers in counter-clockwise sequence $2, 1, 0, 2, 1$

4.7 Percents

4.8 Exponents and roots

If we multiply 3 by itself 4 times we get

$$3 \cdot 3 \cdot 3 \cdot 3 = 81$$

A more concise way of writing this is to say

$$3^4 = 81$$

where 3 is the base and the superscript 4 is called the power or exponent.

Squares b^2 means $b \cdot b$ and is read b squared because b^2 is the area of a square whose side has length b.

Cubes b^3 means $b \cdot b \cdot b$ and is read b cube because b^3 is the volume of a cube whose side has length b.

Powers of 10 If the power is positive the result is one followed by as many zeros as the number in the exponent $10^7 = 10000000$. If the exponent is negative we have as many zeroes as the exponent followed by one with the first zero being before the comma the rest after $10^+7 = 0.0000001$

Negative powers $a^{-n} = \frac{1}{a^n}$

Products of powers $a^n \cdot a^m = a^{n+m}$ this can be seen as $5^3 \cdot 5^2 = (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5) = 5^5$

Products of unlike powers $a^n \cdot b^n = (a \cdot b)^n$ this can be seen as $(2 \cdot 2 \cdot 2) \cdot (3 \cdot 3 \cdot 3) = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) = (2 \cdot 3)^3$

Fractions of powers $\frac{a^n}{a^m} = a^{n-m}$ this can be seen as $\frac{5^3}{5^2} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5} = 5^1$

Powers of fractions $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ this can be seen as $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2^3}{3^3}$

Powers of powers $(a^m)^n = a^{(m \cdot n)}$ this can be seen as $(5^2)^3 = 5^2 \cdot 5^2 \cdot 5^2 = 5^6$

Fractional powers $a^{\frac{x}{y}} = (a^{\frac{1}{y}})^x$. This can be related to roots since $\sqrt{a} \cdot \sqrt{a} = a$ can be satisfied as $a^x \cdot a^x = a^{x+x} = a^1$ only when $x = 1/2$. That is $\sqrt{a} = a^{1/2}$. Generally we have $\sqrt[n]{a} = a^{1/n}$.

Simplifying radicals Radicals such as $\sqrt{32}$ are simplified by rewriting them to lower terms using the above rules e.g. $3\sqrt{8} - 6\sqrt{32} = 3\sqrt{4}\sqrt{2} - 6\sqrt{16}\sqrt{2} = 6\sqrt{2} - 24\sqrt{2} = -18\sqrt{2}$. Often it can be constructive to rewrite radicals using their prime factorization e.g. $\sqrt[3]{81} = \sqrt[3]{9 \cdot 9} = \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 3} = 3\sqrt[3]{3}$

Estimating roots

Square roots

Cube roots find the prime factors and see if any come three times. For example $\sqrt[3]{3430}$ is divisible by 10 and there for have 5 and 2 as prime factors, $343 = 7^3$ so we have $\sqrt[3]{3430} = \sqrt[3]{2 \cdot 5 \cdot 7 \cdot 7 \cdot 7} = 7 \cdot \sqrt[3]{10}$

4.9 Order of operators

The order of arithmetic operators is as follows

1. Parentheses and exponents
2. Multiplication and division
3. Addition and subtraction When multiple operations exists on the same level you do the one that it closest on the left, for example

   $$2-10+8-(-2)+(-10) = 2-10+8+2-10 = (2-10+8)+2-10 = (-8+8)+2-10 = 0+2-10 =$$

4.10 Exercises

Ex. 1 — Reduce the following fractions

1. 88/72
2. $\frac{2}{\frac{4}{5}}$
3. $\frac{\frac{-7}{2}}{\frac{4}{9}}$

Ex. 2 — If it takes 36 minutes for 5 people to paint 9 walls. How many minutes does it take 10 people to paint 7 walls?

Ex. 3 — Philip is going on a 4000-kilometer road trip with three friends. The car consumes 6 liters of gas per 100 kilometers, and gas costs 1.50 per liter. If Philip and his friends want to split the cost of gas evenly, how much should they each pay?.

Ex. 4 — What is the value of

1. $7 + 7/7 + 7 \cdot 7 - 7$
2. $6 - 1 \cdot 0 + 2/2$
3. $-2 + (-3) + 4 - (-3) - 5$
4. $3 + (-4) - 8 - (-1) - 1$

Ex. 5 — Reduce the following

1. Rewrite $\frac{4^6}{4^{-4}}$ in the form 4^n
2. Rewrite $(5^{-12})^{-9}$ in the form 5^n
3. Rewrite $((5^{-10})(9^9))^7$ in the form $5^n \cdot 9^m$
4. Rewrite $\sqrt{11} + \sqrt{44} + \sqrt{99}$ in the form $n \cdot \sqrt{m}$

Chapter 5

Basic algebra

Algebraic expressions are expressions that involves unknown variables, for example

$$8x - 6$$
$$7x^2 + 4x - 10$$
$$\frac{x-1}{x^2+12}$$

are all expressions that contain the unknown variable "x". If we substitute x with a value we can simplify it further, for example if we replace x with 2 in each of the above expressions we get

$$8 \cdot 2 - 6 = 10$$
$$7 \cdot 2^2 + 4 \cdot 2 - 10 = 26$$
$$\frac{2-1}{2^2+12} = \frac{1}{16}$$

5.1 Linear equations

Algebraic expressions may look abstract but can in fact be used to represent everyday phenomenon. For example consider the statement the "price of apples went up by 0.75 per pund so now 3 punds of apples cost 5.88". If we let p denote the apple price prior to the increase then we can write the expression

$$5.88 = 3(p + 0.75) \tag{5.1}$$

which represents that the price of one pund of apples p went up by 0.75 ($p + 0.75$) and that 3 punds of apples at the new price ($3(p + 0.75)$) now

costs 5.88. Equation 5.1 is called a or first-degree equation because the unknown p appears in the first degree only ($p^1 = p$).
Algebraic expressions can look different but actually be equivalent. If you plug the same value into equivalent expressions they will each give you the same value. For example these expressions

$$\begin{aligned} 2(4x + 2y) \\ 4(2x + y) \\ 8x + 4y \end{aligned}$$

yields the same value when you plug in 2 for the unknown variable x and 3 for y.

$$\begin{aligned} 2(4 \cdot 2 + 2 \cdot 3) = 28 \\ 4(2 \cdot 2 + 3) = 28 \\ 8 \cdot 2 + 4 \cdot 3 = 28 \end{aligned}$$

However replacing unknown variables with actual variable values is tedious and only serves to show that two algebraic expressions will yield the same value for a specific variable value. It does not prove that the expressions will always yield equivalent results for all possible values. To prove that two algebraic expressions are equivalent we must first learn how to manipulate algebraic expressions into a form that will allow us to assertion that the expressions are in fact identical.

Manipulating lienar equations

Algebraic manipulation concerns the procedures we are allowed to perform on algebraic expressions. Generally we are allowed to perform all the arithmetical operations we learned in chapter 4 as long as we perform them on both sides on the equal sign in the expression. Formally the following algebraic manipulation procedures are allowed:

Addition Adding on both sides is usually done to remove negative terms, for example if we add 1 on both sides of $x - 1 = 0$ we get $x = 1$.

Subtraction Subtracting on both sides is usually done to remove positive terms, for example if we subtract b on both sides of $x + b = b + 2$ we get $x = 2$.

Multiplication Multiplying on both sides is usually done to remove fractional terms, for example if we multiply with 4 on both sides of $\frac{y}{4} = \frac{3}{4}$ we get $y = 3$.

Division Dividing on both sides is usually done to remove multiplication terms, for example if we divide with 5 on both sides of $5z = 20$ we get $z = 4$.

Replace with equals Replacing with equal expressions is usually done to rewrite part of the expressions into a form that is easier to manipulate. For example in $x - \frac{x}{2} = 1$ we wish to put x and $\frac{x}{2}$ on a common denominator. As $\frac{2x}{2} = x$ we can replace the first x with this

$$\begin{aligned}
x - \frac{x}{2} &= 1 \\
\frac{2x}{2} - \frac{x}{2} &= 1 && \text{repalce } x = \frac{2x}{2} \\
\frac{2x - x}{2} &= 1 && \text{use common denominator} \\
\frac{x}{2} &= 1 && \text{simplify numerator} \\
x &= 2 && \text{multiply with 2}
\end{aligned}$$

Distributive rule $m(a + b) = ma + mb$ is called the distributive rule since we "distributed" m to a and b.

When performing algebraic manipulation we are of cause allowed to make use of all the above procedures in order to simplify a expression. For example consider expression 5.1 above that we used to represent the price of apples after a price hike:

$$\begin{aligned}
3(p + 0.75) &= 5.88 \\
\frac{3(p + 0.75)}{3} &= \frac{5.88}{3} && \text{divide with 3} \\
p + 0.75 &= \frac{5.88}{3} && \text{cancel 3 out} \\
p &= \frac{5.88}{3} - 0.75 && \text{subtract 0.75} \\
p &= 1.21 && \text{calculate value}
\end{aligned}$$

So we now know that the price of one pund of apples prior to the price increase is 1.21. Mastering algebraic manipulation is the foundation of all modern mathematics so we will show another example to get a feel for it.

Consider the expression $2(x+1) = \frac{1}{5}x + 3$

$$
\begin{aligned}
2(x+1) =& \frac{1}{5}x + 3 \\
2x + 2 =& \frac{1}{5}x + 3 && \text{use distributive rule} \\
2x =& \frac{1}{5}x + 1 && \text{subtract 2} \\
2x - \frac{1}{5}x =& 1 && \text{subtract } \frac{1}{5}x \\
\frac{10x}{5} - \frac{1}{5}x =& 1 && \text{replace } 2x = \frac{10x}{5} \\
\frac{10x - x}{5} =& 1 && \text{use common denominator} \\
\frac{9x}{5} =& 1 && \text{simplify numerator} \\
9x =& 5 && \text{multiply with 5} \\
x =& \frac{5}{9} && \text{divide with 9}
\end{aligned}
$$

before continuing with other chapters in this book be sure to familiarize yourself with algebraic manipulation by going through the exercises in 5.6.

5.2 Linear inequalities

Just as we are allowed to have have expressions with unknown variables we can have unknown variables in inequalities. For example the inequality $x < 5$ states that x is a value below 5. Inequalities may be compound such as $1 \leq y \leq 4$ which states that y is any value between (and including) 1 and 4. However not all equalities are as easy to analyse as the ones above. Consider $-0.5z \leq 7.5$ here we may be tempted to divide both sides with -0.5 in order to isolate z, however it turns out that special rules governs algebraic manipulation of inequalities. To see why this is true observe what happens when we multiply a inequality with a negative number

$$
\begin{aligned}
1 <& 2 \\
-1 <& -2 && \text{multiply with } -1
\end{aligned}
$$

clearly the last expressions is false as $-1 \not< -2$. In fact it turns out that when multiplying or dividing inequalities with negative numbers we must flip the inequality $(-1 > -2)$. Observing this rule we can now manipulate $-0.5z \leq 7.5$ into a more manageable form

$$\begin{aligned} -2 \cdot -0.5z \geq& -2 \cdot 7.5 && \text{multiply with } -2 \text{ and flip sign} \\ z \geq& -15 && \text{simplify} \end{aligned}$$

Manipulating linear inequalities

As shown above the rules for manipulating linear inequalities are sometimes different than when manipulating linear equations. However it turns out that some operations are actually safe to perform. To see this consider that when we add 1 to both sides of $1 < 2$ we get $2 < 3$ which is clearly still true. Formally the rules governing inequalities can be split between those that does not require changing the inequality sign and those who do.

Operations that preserves the inequality sign :

Addition Adding on both sides of a inequality is usually done to remove negative terms, for example if we add 1 on both sides of $x - 1 < 0$ we get $x < 1$.

Subtraction Subtracting on both sides of a inequality is usually done to remove positive terms, for example if we subtract b on both sides of $x + b > b + 2$ we get $x > 2$.

Multiplication with a positive number Multiplying a inequality with a positive number is usually done to remove fractional terms, for example if we multiply with 4 on both sides of $\frac{y}{4} \leq \frac{3}{4}$ we get $y \leq 3$.

Division with a positive number Dividing a inequality with a positive number is usually done to remove multiplication terms, for example if we divide with 5 on both sides of $5z \geq 20$ we get $z \geq 4$.

Operations that requires flipping the inequality sign :

Multiplication with a negative number Multiplying a inequality with a negative number requires flipping the inequality for example if we multiply with -4 on both sides of $-\frac{y}{4} \leq \frac{3}{4}$ we get $y \geq -3$.

Division with a negative number Dividing a inequality with a negative number requires flipping the inequality for example if we divide with -5 on both sides of $-5z \geq 20$ we get $z \leq -4$.

Swapping left and right hand sides When we swap the left and right hand sides, we must also change the direction of the inequality, for example $2y + 7 < 12$ becomes $12 > 2y + 7$.

as with linear equations we may combine the above rules to solve complicated inequalities like $3x - 3 > 6x + 3$

$$\begin{aligned}
3x - 3 >& 6x + 3 && \\
3x >& 6x + 6 && \text{add 3} \\
-3x >& 6 && \text{subtract } 6x \\
x <& -\frac{6}{3} && \text{divide with } -3 \text{ and flip inequality} \\
x <& -2 && \text{simplify fraction}
\end{aligned}$$

So we now kno

5.3 Functions

As menitoned earlier when we replace a variable x in expressions such as $x + 2$ we get a value, for example using $x = 2$ gives us $2 + 2 = 4$ and using $x = 3$ results in $3 + 2 = 5$. To formalize the concept of replacing a variable x in a expression and computing a value we introduce the concept of a function. In the above example we may write

$$f(x) = x + 2 \tag{5.2}$$

That is $f(x)$ (read "f of x") is a function of one variable x when we provide a actual value for x such as $f(1)$we replace all occurences of x in $x + 2$ with the provided value 1 and computes the functions value. A couple of examples serves to clarify this concept

$$\begin{aligned}
f(1) =& 1 + 3 = 4 \\
f(2) =& 2 + 3 = 5 \\
f(3) =& 3 + 3 = 6
\end{aligned}$$

the actual name of the function f does not matter, we only use it to refer to the function body $x + 2$ without having to write $x + 2$ all the time. We might as well call it $g(x) = x + 2$ in which case we just write $g(2) = 5$.
In 5.2 above the value of f only depends on x, we say f is a function of one variable. Alternatively, if f is a function of several variables, we can write

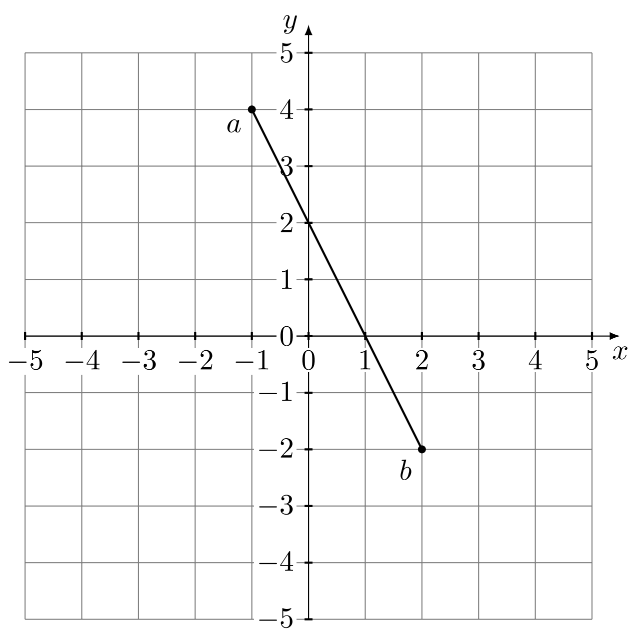
$f(a, b)$ meaning f is a function of the two variables a, b. For now however we shall restrict ourselfs to functions of one variable.
It's common to write $y = f(x)$ that is to say the function f returns a value y when we plug a value into x. Assigning $y = f(x)$ gives a relation between y and x and y is then known as a dependent variable and x is an independent variable. Generally the "dependent variable" represents the output of a function. The "independent variables" represent the inputs to the function.

Function graphs

If we enumerate the x and y values of 5.2 into ordered pairs of (x, y) above we get

x	**y = f(x)**	**(x,y)**
-2	$-2+2=0$	$(-2,0)$
-1	$-1+2=1$	$(-1,1)$
0	$0+2=2$	$(0,2)$
1	$1+2=3$	$(1,3)$
2	$2+2=4$	$(2,4)$

Carteseian coodinates (sample Xes and plot out a line)



5.4 Monomials and binominals

A , also called , is a product of powers of variables with nonnegative integer exponents. Examples include

$$-5, 3, 21x, -2x^2, 3a^2b^3$$

here -5 and 3 are monomial of x^0 for any variable x

Factoring and multiplying binominals

When we multiply two binomials such as $(x+4)$ and $(x+2)$
With factoring one usually starts with an expression such as x^2+6x+8 and seeks to transform it into a product of two binominals of the form $(x+a)(x+b)$. The original expression is said to be of second degree because it contains x^2 but no higher power of x. The factors are first-degree expressions because each contains x but no higher power of x. The problem is to find the correct values of a and b so that the product $(x+a)(x+b)$ will equal the original expression. We know that

$$x^2 + (a+b)x + ab = (x+a)(x+b) \tag{5.3}$$

Hence to factor the second-degree expression, we should look for two numbers a and b whose sum is the coefficient of x and whose product is the constant. Thus to factor x^2+6x+8, we look for two numbers whose sum is 6 and whose product is 8. By mere trial of the possible factors of 8 we see that $a = 4$ and $b = 2$ will meet the requirement; that is:
$x^2 + 6x + 8 = (x+4)(x+2)$.

Quadratic equations

5.5 Polynomials

Polynominal multiplication

Polynominal division

5.6 Exercises

Ex. 6 — Evaluate the expression values of

1. $-1-(-z)-5-(-3)$ where $z=-2$.
2. $x-(-y)$ where $x=-2$ and $y=5$.

3. $3 - (-6) + (-h) + (-4)$ where $h = -7$

Ex. 7 — Use the distributive property to expand these expressions

1. $-(6 - \frac{z}{4})$
2. $-(\frac{1}{2}r + 4)$
3. $-5(3n + \frac{1}{2})$

Ex. 8 — Use the distributive property to collect these expressions so no fractions are within the parentheses

1. $\frac{3}{5}z - \frac{6}{5}$
2. $\frac{3}{2} + \frac{7}{8}c$

Ex. 9 — Determine if the following expressions are equivalent

1. $9x + 6 = 3x + 2$
2. $\frac{5}{2}x - 3 = x + 5$
3. $\frac{\frac{2}{b}+\frac{2}{a}}{\frac{2}{ab}} = a + b$
4. $\frac{\frac{a}{b}+1}{\frac{b}{a}-1} = \frac{a(a+b)}{b(b-a)}$

Ex. 10 — Expand the expression and combine the like terms:

1. $6 + 5(7n + 2)$
2. $(15 + 2a) + 4(8a6)$
3. $6(2 + 10k) + 6(5k3)$
4. $1/5 - 2z + z + 2/3$
5. $7n - (4n - 3)$

Ex. 11 — Simplify the following rational expressions by removing their fractions

1. $\frac{35n^3}{10n^4}$
2. $\frac{44a^3}{55a^3}$
3. $\frac{28y^5}{7y^3}$

Ex. 12 — Write and solve equations to represent the following statements

1. One half of the quantity represented by 6 less than a number n is equal to 17. Write an equation to represent this statement and find the value of n.
2. The sum of 3 consecutive odd numbers is 69. What is the first number in this sequence?
3. 130% of the sum of 7 and a number n is equal to 91. Write an equation to represent this statement and find the value of the number.
4. The radiator of a car contains 10 gallons of liquid 20 percent of which is alcohol. We want to draw off liquid and replace it with alcohol so the resulting mixture contains 50 percent alcohol. How many gallons of liquid should he draw off?

Ex. 13 — Solve the following linear equations

1. $\frac{12}{x-12} = \frac{6}{5}$
2. $\frac{3}{5} = \frac{18}{a+11}$
3. $\frac{5}{6} = \frac{k-7}{18}$
4. $\frac{3}{8} = \frac{15}{t-1}$
5. $\frac{y-8}{25} = \frac{7}{5}$

Ex. 14 — Solve these linear inequalities for x:

1. $-12x < 1$
2. $-18x > -10$
3. $16x \geq 13$
4. $12b - 15 > 21$
5. $14 - 3x < -1$

Ex. 15 — Simplify and solve these linear inequalities for x:

1. $9 - 4d \geq -3$
2. $9x - 3 < 7x + 4$
3. $2x + 8 \geq 4x + 5$
4. $9x + 1 \geq 4x + 2$
5. $3x + 9 \geq 9x + 10$

Ex. 16 — Name the quadrants of these coordinates

1. $(1, 1)$

2. $(-1, 1)$
3. $(-1, -1)$
4. $(1, -1)$
5. $(0, 0)$

Ex. 17 — Multiply these binomials

1. $(x-3)^2$
2. $(x-3)(x+10)$
3. $(x+8)^2$
4. $(x+1)(x-4)$
5. $(x+9)^2$

Ex. 18 — Factor the following expressions.

1. $x^2+9x+20$
2. x^2+5x+6
3. x^2-5x+6
4. x^2-9
5. $x^2+7x-18$

Chapter 6

Elementary number theory

In number theory we deal with the properties of various classes of numbers such as the integers, which is any number that can be written without a fractional or decimal component and is denoted with the the symbol $\mathbb{Z}$. Thus, -7, 0, and 2 are integers; 1.7 and $\frac{1}{7}$ are not. Incidentally the Latin word "integer" literally means "untouched", hence "whole". Of particular interests is the set of all positive integers, also known as the natural numbers and symbolised by $\mathbb{N}$.

6.1 Basic properties of Integers

Let us first consider that integers are either *even* or *odd*, formally we write:

Definition 6.1.1. The *even* numbers are the numbers written on the form $2k$ and the *odd* numbers are the numbers written on the form $2k+1$ where k is any integer.

some arithmetic can convince us that this indeed yields the desired numbers:

$$\begin{array}{lcl} 2\cdot 0 & = & 0 \\ 2\cdot 0+1 & = & 1 \\ 2\cdot 1 & = & 2 \\ 2\cdot 1+1 & = & 3 \\ \vdots & & \vdots \end{array}$$

The *odd* numbers may also be expressed as being either one more than a multiple of 4 or three more than a multiple of 4. Symbolically, we can say that they are either of the form $4k+1$ or of the form $4k+3$. Again some examples clarifies the smoke

$$\begin{array}{rcl} 4 \cdot 0 + 1 & = & 1 \\ 4 \cdot 0 + 3 & = & 3 \\ 4 \cdot 1 + 1 & = & 5 \\ 4 \cdot 1 + 3 & = & 7 \\ \vdots & & \vdots \end{array}$$

incidentally this shows that when we divide a *odd* number by 4, we must get a remainder of either 1 or 3 . Armed with this knowledge we can shine some light on the properties of *odd* numbers:

Corollary 6.1.1. *The square of an odd number is odd*

Proof. Consider, by 6.1.1, the algebraic form of the square of an odd number

$$\begin{aligned} (2k+1)^2 &= (2k)^2 + 2(2k) + 1 \\ &= 4(k^2 + k) + 1 \end{aligned}$$

The last expression being on the form $4k + 1$ yields the desired result. □

Integers can be used to construct another class of numbers, known as the rational numbers. Here we say a 'rational number' is a fraction $\frac{a}{b}$, where a and b are integers; we may suppose that a and b have no common factor (since if the had we could remove it) and that b does not equal zero. We denote these numbers by $\mathbb{Q}$ for quotient. With only these properties at hand we can move on to prove one of mathematics first great results; that $\sqrt{2}$ is irrational.

Proposition 6.1.2. $\sqrt{2}$ *is irrational.*

Proof. To say that $\sqrt{2}$ is irrational is the same as saying 2 cannot be expressed in the form $\left(\frac{a}{b}\right)^2$ or that the equation

$$a^2 = 2b^2 \tag{6.1}$$

cannot be satisfied by integral values of a and b without any common factor. Let us for sake of eventual contradiction suppose that 6.1 is true for some a and b, then it follows that a^2 is even (since $2b^2$ is divisible by 2) and thus that a is even (since the square of an even number is even). If a is even then

$$a = 2c$$

for some integral value of c; and therefore

$$a^2 = 2b^2 = (2c)^2 = 4c^2$$

or

$$b^2 = 2c^2$$

Hence b^2 is also even and therefor b is even. Now since both a and b are even they must have the common factor 2. This contradicts our hypothesis, and therefore the hypothesis is false □

Notice how this proof is by reductio ad absurdum, a proof style much beloved by the greeks and one of mathematics finest weapons. In these types of proofs we attack our theorem by taking an assumption and then show that it leads to a contradiction and hence is false.

Now since we have proved that $\sqrt{2}$ cannot be written as a fraction can we then find another way to express it ?. By definition, a number is *algebraic* (denoted by $\mathbb{A}$) if it is the solution to some polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0$$

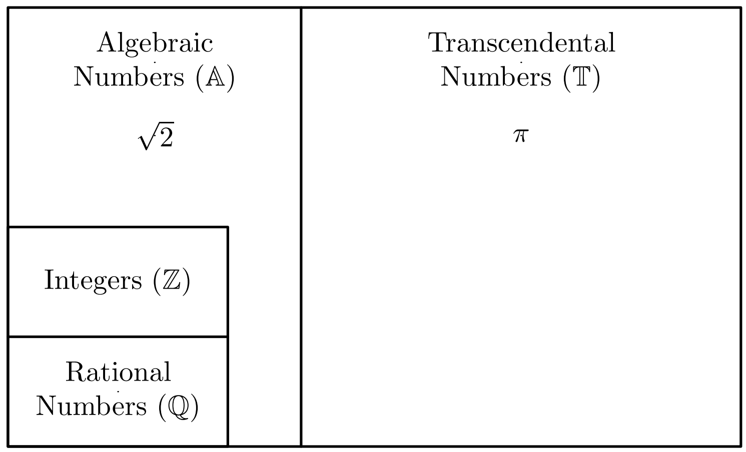
where all the coefficients $a_n, a_{n-1}, \cdots, a_2, a_1$ and a_0 are integers. Now as $\sqrt{2}$ is the solution to the polynomial $x^2 - 2 = 0$ it follows that it is a irrational algebraic number. Also as any integer k is a solution to $x - k = 0$ we see that all integers are algebraic and hence $\mathbb{Z} \in \mathbb{A}$. Later we shall meet another class of numbers known as transcendental numbers $\mathbb{T}$ which is simply the set of all non-algebraic numbers. As it turns out almost all irrational numbers are transcendental and all transcendental numbers are irrational. The box below should hopefully clarify the relationships between these various sets of numbers Note how all of these numbers are contained in the set of real numbers $\mathbb{R}$. The real numbers thus includes both integers and rational numbers, such as 42 and $\frac{1}{3}$, and transcendental numbers, such as π and irrational algebraic numbers such as $\sqrt{2}$. Informally we may think of real numbers as any number with or without a decimal point that may contain an infinite decimal tail that continue in some way.

6.2 Divisibility

The integer n is divisible by m if the ratio n/m is an integer. Hence we write $m|n$ when m divides n evenly and define

$$m|n \iff m > 0 \text{ and } n = mk \text{ for some integer } k \tag{6.2}$$

Real Numbers ($\mathbb{R}$)



How do we know if a number is divisible by some other number ?. Here we are out of luck, in general we can only answer this by doing full division and checking if the remainder is zero. Mathematics contains many functions that are easy to do (such as multiplication and differentiation) but hard to undo (such as division and integration). Later on we will see that exactly this proporty of division can be used to craft a simple scheme for secure communication. For now let us extract some useful properties from the above definition of divisibility.

Proposition 6.2.1. *If $a > b$ and n divides a and b evenly then it also divide their difference.*

Proof. By 6.2 we have $a = nx$ and $b = ny$ for some integral value of x and y. Thus $a - b = nx - ny = n(x - y)$ as desired. $\square$

As remarked on page 38 not all numbers divides evenly:

Proposition 6.2.2. *If a and b are integers with $b \neq 0$, then there is a unique pair of integers q and r such that*

$$a = qb + r \text{ and } 0 \leq r \leq |b|$$

Note that

$$\frac{a}{b} = q + \frac{r}{b}$$

where for $b > 0$ we have $0 \leq \frac{r}{b} < 1$ and for $b < 0$ we have $0 \geq \frac{r}{b} > -1$. As example the division $9/4$ with $a = 9$ and $b = 4$ becomes $9 = 2 \cdot 4 + 1$ giving us the quotient $q = 2$ and remainder $r = 1$

Proof. Our job is to prove that the numbers q and r exists, known as existence and that they are unique, also known as uniqueness. First we prove existence. Let

$$S = \{a - nb | n \in \mathbb{Z}\} = a, a \pm b, a \pm 2b, ...$$

□

Euclid's Algorithm

Euclid devised a clever algorithm for locating the greatest common divisor.

6.3 Primes

The *prime numbers* or *primes* are the numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37...$$

which cannot be resolved into smaller factors[1]. The primes are the material out of which all numbers are build up by multiplication

$$666 = 2 \cdot 3 \cdot 3 \cdot 37$$

indeed the following holds true:

Proposition 6.3.1. *Every integer $n > 1$ is divisible by at least one prime*

Proof. Let A be a composite number (i.e. non prime). By definition there must be some smaller number B dividing evenly into A, where $1 < B < A$. Now either B is prime or it is not. If B is prime, then the original number A indeed has a prime divisor, as claimed. Otherwise, B is not prime and so has a divisor, say C, with $1 < C < B <$ A. If C is prime, we are done, for C divides evenly into B, and B divides evenly into A. If C is composite then it must have a proper divisor, D, and thus we continuously arrive at:

$$A > B > C > D > ... > 1$$

But all of these are positive integers, so we must reach a point in which we find a prime. Therefor any number, which is not prime itself, is divisible by at least one prime (usually, of cause, by several). □

If one surveys the list of the first 50 primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83,
89, 97, 101, 103, 107, 109, 113, 127, 131,
137, 139, 149, 151, 157, 163, 167, 173, 179,
181, 191, 193, 197, 199, 211, 223, 227, 229

then apparently the primes seams to be getting scarcer as the numbers go larger. Indeed between the numbers 1 and 1000 there are 168 primes, whereas between 9000 and 10.000 there is but 112. At first this seams logical enough since small numbers only have few possible factors and thus higher likelihood of being primes, yet one cannot help ask if we will ever reach such large numbers that the primes may eventually run out completely, rendering all subsequent numbers composite. Luckily this question had already been answered by the great greek mathematician Euclid who 300 B.C. in his IX book as proposition 20 indeed claimed:

Proposition 6.3.2. *Prime numbers are more than any assigned multitude of prime numbers.*

Euclid's terminology sounds strange, but what he is really proposing is that the sequence of primes does not end.

Proof. Let us for sake of eventual contradiction suppose that there is only a finite number of primes and that

$$2, 3, 5, ..., P$$

is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula

$$Q = (2 \cdot 3 \cdot 5 \cdots P) + 1$$

It is plain that Q is not divisible by any of $2, 3, 5, ..., P$; for it leaves the remainder 1 when divided by any one of these numbers. But, if Q is not prime, then by 6.3.1 it is divisible by some prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P; and therefore this hypothesis is false. □

Finding and counting primes is a long standing hobby for mathematicians, and one might ask if it can be automated. One such method was discovered by the greek mathematician Eratosthenes (ca. 284-192 B.C.) who as the chief librarian at the great Library at Alexandria spend much of his time studying mathematics, astronomy and philosophy. His method (known as Eratosthenes sieve) is as follows; write a list of all the consecutive integers, starting with 2, among which you want to find the primes. As 2 is prime, cross all subsequent multiples off. The next integer that has not yet been crossed of (3) must then also be prime and so we can eliminate all of its multiples. As we continue in this manner we will clearly sieve out all composite numbers and be left with the primes.

Sift the Twos and sift the Threes,
The Sieve of Eratosthenes.
When the multiples sublime,
The numbers that remain are Prime.

although this method will indeed generate the primes, it is not as efficient as one might have liked (although optimised and more complex methods exists). Later we shall encounter the prime counting function $\pi(n)$, which outputs the number of primes less than or equal to a given number n (the use of π in this function is an unfortunate standard as it has nothing to do with the well known geometric constant 3.1415...).

6.4 Modulo arithmetic

As mentioned before integers can be broken up into the classes of even $..6, 4, 2, 0, 2, 4, 6..$ and odd numbers $..5, 3, 1, 1, 3, 5..$ There are certain generalisations we can make about the arithmetic of numbers based on which of these two classes they come from. For example, we know that the sum of two even numbers is even. The sum of an even number and an odd number is odd. The sum of two odd numbers is even. The product of two even numbers is even, etc.
Now we represent each of our two classes by a single symbol. We let the symbol "0" mean "the class of all even numbers" and the symbol "1" mean "the class of all odd numbers". The statement "the sum of two even numbers is even" can then be expressed by the following:

$$0 + 0 \equiv 0 \text{ mod } 2.$$

Here, the "$\equiv$" symbol is not equality but congruence, and the "mod2" just signifies that our modulus is 2. The above statement is read "Zero plus zero is congruent to zero, modulo two." The statement "the sum of an even number and an odd number is odd" is represented by

$$0 + 1 \equiv 1 \bmod 2.$$

Those examples are natural enough. But when we try to express the statement "the sum of two odd numbers is even"? we get this strange looking expression

$$1 + 1 \equiv 0 \bmod 2.$$

We have analogous statements for multiplication:

$$0 \cdot 0 \equiv 0 \bmod 2$$
$$0 \cdot 1 \equiv 0 \bmod 2$$
$$1 \cdot 1 \equiv 1 \bmod 2$$

Basically we now have a number system with addition and multiplication but in which the only numbers that exist are 0 and 1. You may ask what use this has. Well, our number system is the system of integers modulo 2, and because of the previous six properties, any arithmetic done in the integers translates to arithmetic done in the integers modulo 2.
Since any integer solution of an equation reduces to a solution modulo 2, it follows that if there is no solution modulo 2, then there is no solution in integers. For example, assume that a is an integer solution to

$$2a3 = 12,$$

which reduces to

$$0 \cdot a + 1 \equiv 0 \bmod 2,$$

or $1 \equiv 0 \bmod 2$. This is a contradiction since no even number is an odd number. Therefore the above congruence has no solution, so a couldn't have been an integer. This proves that the equation $2a3 = 12$ has no integer solution.

6.5 Exercises

Ex. 19 — What is the lowest number divisible by both 12 and 20

Ex. 20 — In 2000 the Jordanian mathematician, Murad A. AlDamen devised the following method for testing divisibility [2]

1. Use Murad's method to determine if 61 divides 3598207
2. Prove Murad's method.

6.6 Notes

1. There are technical reasons for not counting 1 as a prime
2. Found on the prime puzzlers project, at:
`http://www.primepuzzles.net/puzzles/puzz_101.htm`

Chapter 7

Combinatorics

Combinatorics is one of the most fascinating and frustrating branches of mathematics. Many excellent mathematics students find it impossibly difficult while some find it as easy as breathing. Classically, combinatorics deals with finite sets of objects and the various ways their subsets and their elements can be counted and ordered.

One of the first formal attempts of combinatorial thinking seems to have come around 960, when correctly enumerated the 56 outcomes that can arise when three dice are thrown simultaneously: $\{1, 1, 1; 1, 1, 2; 1, 1, 3, ...\}$ and so on. A thirteenth-century Latin poem, , listed the $216 (= 6 \cdot 6 \cdot 6)$ outcomes that may result when three dice are thrown in succession.

If the results 56 and 216 seams somewhat unclear, don't dispair this chapter will teach you how to arrive at these results safely on your own.

So, in Mathematics we use more precise language:

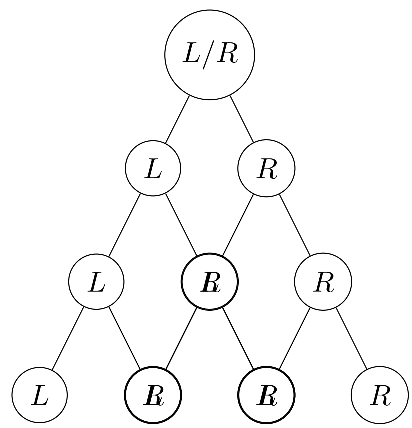
If the order doesn't matter, it is a Combination. If the order does matter it is a Permutation.

A Permutation is an ordered Combination.

7.1 Permutations

In other words:

You've decided to take 3 steps and randomly choose left or right as the direction each time. How many different outcomes exists?. The mathematical branch of combinatorics helps us answers questions such as these and its mastery is essential to study statistics. Returning to our example we notice that you have two possibilities for the first step, two possibilities for the second and again two possibilities for the third yielding $2 \cdot 2 \cdot 2 = 8$. The figure below shows the situation:



Suppose we have before us 2 red aces and 2 red kings from the usual deck of 52 cards. How many different pairs consisting of one ace and one king can be put together? Since each ace can be paired with either of 2 kings, there are 2 different pairs for any one ace. Since we have 2 aces, there are $2 \cdot 2$ or 4 different pairs.

7.2 Counting selections

Consider in how many ways can we arrange the set $\{1, 2, 3\}$:

$$\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}$$

that is we have 3 ways of choosing the first element, and $3 - 1$ options for the second (as we cannot repeat the first choise) and $3 - 2$ for the last (as non of the former two choices may be used), therefor we can arrange them in $3 \cdot 2 \cdot 1 = 6$ ways. In general we define the number of permutations of a set with n elements as the faculty function $n!$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (n-(n-1)) = 1 \cdot 2 \cdots n = \prod_{k=1}^{n} k,$$

the last notation simply means that we multiply all the values of k from 1 to n. A recursive version (a self-calling function) can be constructed by noting that $n! = n(n-1)!$.

Now consider a slightly more complex question: in how many ways can we pick two elements from the set $\{1, 2, 3\}$, the answer turns out to be three ways:

$$\{1, 2\}, \{1, 3\}, \{2, 3\}$$

To generalize this we can ask in how many ways can we pick a unordered subset with k elements from a set with n elements ?. To answer this, observe there are n choices for the first element and for each of these there are $n - 1$ choices for the second; and so on until there are $n - k + 1$ for the k'th element. That is we have $n \cdot (n - 1) \cdots (n - k + 1)$ ways for picking a k-element subset from n. As we are picking unordered sets and since each k-element subset has exactly $k!$ different orderings, we get our answer by dividing with $k!$:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \qquad \text{integer } n \geq k \geq 0 \tag{7.1}$$

and if we multiply the numerator and denominator of 7.1 by $(n - k)!$ we get:

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!}$$

The symbol $\binom{n}{k}$ is known as a binomial coefficient and we read it "n choose k" (the number of ways to select k elements from n element set). Using this result we can directly answer the above question of how many ways to choose two elements from a three element set:

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1} = 3$$

Interestingly when determining the number of ways to pick k elements we have in effect also specified the $n - k$ unchosen things, therefore

$$\binom{n}{k} = \binom{n}{n-k} \tag{7.2}$$

this symmetry can be observed in the above example where we have 3 ways of picking 2 elements from a 3-element set but then also 3 ways of picking the remaining element. It also follows directly from the definition of the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$$

Another rather remarkable result, known as Pascal's rule, can be derived by marking a particular object ϵ in an n-element set. Now if we pick k objects from n, then either ϵ is picked or it is not. If ϵ is in the subset then there is $k-1$ objects left to choose among the $n-1$ elements, or more formally $\binom{n-1}{k-1}$. If ϵ is not in the subset, we need to choose all the k elements in the subset from the $n-1$ objects that are not ϵ; formally that is $\binom{n-1}{k}$. Thus there are

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

ways to choose k elements from n depending on whether ϵ is included in each selection or not. But this number must be the same as the binomial coefficient and thus we get:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{7.3}$$

this result can also be arrived at by algebraic manipulation

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!((n-1)-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-(k+1))!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

From Pascal's rule we can construct his famous triangle. Observe that according to this rule (see 7.3) you can calculate $\binom{5}{3}$ as:

$$\begin{aligned}
\binom{5}{3} &= \binom{4}{3} + \binom{4}{2} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{3}{1} \\
&= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}
\end{aligned}$$

as $\binom{2}{0} = 1$, we can stop the expansion. In general as $\binom{n}{n}$ and $\binom{n}{0}$ are always 1 (when $n \geq 0$) we are guarantied that the expansion always terminates. Thus we have found a convenient way of expressing higher binomial coefficients as sums of consecutive smaller ones:

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	2	1	0	0	0
3	1	3	3	1	0	0
4	1	4	6	4	1	0
5	1	5	10	10	5	1

discarding the zero's we can arrange these numbers into a triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\end{array}$$

and so on

where the next row must contain:

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

but we already know that expressions like $\binom{6}{1}$ can be rewritten to $\binom{5}{1} + \binom{5}{0} = 5 + 1$. So we see that each entry in the new row is obtained simply by adding the numbers in the row above to the left and right.

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

To generate the triangle, you start with a 1, and then immediately below it, you put two 1s, one to either side. Then for each successive row, you put a new 1 at either end and complete the row between them by adding together each adjacent pair of entries in the row above and putting their sum halfway between them.

7.3 Newton's Binomial Theorem

In mathematics a binomial is a polynomial with two terms, i.e such as these

$$\begin{array}{rcl}
(a+b)^0 & = & 1 \\
(a+b)^1 & = & a+b \\
(a+b)^2 & = & a^2+2ab+b^2 \\
(a+b)^3 & = & a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 & = & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
\vdots & & \vdots
\end{array}$$

or in general

$$(a+b)^n = \overbrace{(a+b)(a+b)\cdots(a+b)}^{n \text{ factors}}$$

the relationship between such polynomials and combinatorics stems from the so called binomial theorem.

7.4 Exercises

Ex. 21 — Count the following

1. A gardening store offers 4 different flowers and 4 different types of pots how many different arrangement of pot and plant can you make
2. A girl has 3 hats, 2 dresses, and 2 pairs of shoes. How many different costumes does she have?
3. A deck of 52 cards contains 4 different aces and 4 different kings. How many different pairs of cards, each pair consisting of one ace and one king, can be formed from the aces and kings?
4. In how many different ways can 7 people sit in 3 chairs ?
5. There are 7 people in a room. If everyone shakes everyone else's hand exactly once, how many handshakes occur?

Chapter 8

Trigonometry

the last great creation of the Greek period, plane and spherical trigonometry by Hipparchus and Ptolemy and their application to one of the dreams of mankind, understanding the movements of the heavenl bodies. This gave rise to modern astronomy and the physical sciences.

Trigonometric functions are functions of angles. Figure 1.2 shows a right-angled triangle (ie a triangle with one angle of 90) with another angle denoted by the Greek letter theta . The sides of the a right-angled triangle are called the adjacent side (next to), the opposite side (opposite to) and the hypotenuse (opposite the right-angle). For any right-angled triangle, the ratios of the various sides are constant for any particular value of . The basic trigonometric functions are:

Trigonometric relations

$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \tag{8.1}$$

The reciprocals of the *cosine*, *sine*, and *tangent* are known as *secant* (*sec*), *cosecant* (*csc*), and *cotangent* (*cot*):

$$sec(\theta) = \frac{1}{sin(\theta)} \tag{8.2}$$

$$csc(\theta) = \frac{1}{cos(\theta)} \tag{8.3}$$

$$cot(\theta) = \frac{1}{tan(\theta)} \tag{8.4}$$

Inverse trigonometric functions

Chapter 9

Analytical geometry

Analytic geometry, also known as coordinate geometry, or Cartesian geometry, is the study of geometry using a coordinate system.

Chapter 10

Calculus

10.1 Series

10.2 Logarithm

A logarithm (log or logs for short) goes in the opposite direction to a power by asking the question: what power produced this number? So, if 2x = 32, we are asking, what is the logarithm of 32 to base 2? We know the answer: it's 5, because 25 = 32, so we say the logarithm of 32 to base 2 is 5. In general terms, if , then we say the logarithm of a to base x equals p, or For example, 104 = 10,000, so we say the logarithm of 10,000 to base 10 equals 4, or We can take logarithms of any positive number, not just whole ones. So, as 103.4321 = 2704.581 we say the logarithm of 2704.581 to base 10 equals 3.4321, or Logarithms to base 10 are called common logarithms. Older readers may remember, many years ago - after the dinosaurs, but before calculators and computers were widely available - doing numerical calculations laboriously by hand using tables of common logarithms and anti-logarithms. The properties of logarithms are based on the aforementioned rules for working with powers. Assuming that a ¿ 0 and b ¿ 0 we can say: logx(ab) = logxa + logxb, eg log10(1000 100) = log10(100,000) = 5 = log10(1000) + log10(100) = 3 + 2. logx(1/a) = -logxa, eg log3(1/27) = -log327 = -3. logx(a/b) = logxa - logxb, eg log2(128/8) = log2(16) = 4 = log2(128) - log2(8) = 7 - 3. logx(ay) = ylogxa, eg log5(253) = log515,625 = 6 = 3 log525 = 3 2. 1.8.3

Natural logarithm

Say we invest 1.00 in an exceedingly generous bank that pays 100% interest per annum. If the bank calculated and credited the interest at the end of one year, our investment would then be worth 1 + 1 = 2.00. But what if the bank credits the interest more frequently than once a year? If interest is calculated and added every six months, at the end of that period the balance would equal and at the end of one year the total amount would be . Calculated three times a year, the final balance would be . In general, if interest is calculated n times a year, the balance x after one year is

$$f(k) = \left(1 + \frac{1}{k}\right)^k$$

if we plug in some values we observe

$$f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25$$
$$f(5) = \left(1 + \frac{1}{5}\right)^5 = 2.248832$$
$$f(10) = \left(1 + \frac{1}{10}\right)^{10} = 2.59374...$$

n 1 2 2 2.25 3 2.37037 4 2.44141 5 2.48832 10 2.59374 100 2.70481 1000 2.71692 100,000 2.71827 1,000,000 2.71828 10,000,000 2.71828
We can see that as n increases, the value of the function appears to settle down to a number approximately equal to 2.71828. It can be shown that as n becomes infinitely large, it does indeed equal the constant e. The mathematically succinct way of saying this introduces the important idea of a limit and we say where means the limit of what follows (ie) as n approaches infinity (symbol). In other words, e approaches the value of as n approaches infinity.

$$e = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k \tag{10.1}$$

10.3 The exponential function

The exponential function f(x) = ex, often written as exp x (see Figure 1.7) arises whenever a quantity grows or decays at a rate proportional to its size: radioactive decay, population growth and continuous interest, for example. The exponential function is defined, using the concept of a limit, as

Taylor series

$$cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \theta^{2n}}{2n!} \tag{10.2}$$

Geometric Series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r} \tag{10.3}$$

10.4 Integration

Laplace Transform

10.5 Numerical integration

Trapez

Chapter 11

Linear algebra

11.1 Vectors

Addition

Inner product (dot product)

Pointwise vector division

Pointwise vector multiplication

Saxpy

Scalar vector multiplication

11.2 Matrices

Addition

Pointwise matrix division (denominator matrix must have non-zero entries)

Pointwise matrix multiplication

Rank The rank of a matrix is the number of linearly independent columns (which is equal to the number of linear independent rows)

Scalar matrix multiplication

Chapter 12

Set theory

12.1 Sets

A indexed set is a collection of values associated with indices. For example

- An ordered pair is a family indexed by the two element set $2 = \{1, 2\}$.
- An n-tuple is a family indexed by n.

the set that whose members label (or index) members of a family is called an index set.

Chapter 13

Basic probability

Probability is the foundation for statistics one of the most useful branches of mathematics. The ability to calculate probabilities transformed statistics from the mere collection data to the use of data to draw inferences and make informed decisions. Today politicians try to predict the voters opinion through opionen polling. Marketing departments in big corporations study their prospective customers to figure out how best to target new products and sales campaings to each segment and in modern medicine, statistical methods are used to compare the benefits of various drugs and treatments with their risks.

In spite of the obvious usefulness probability and statistics are rather new branches of matematics. Some of the earlist studies of probability comes from gambling. The sixteenth century polymath demonstrated the efficacy of defining odds on games as the ratio of favorable to unfavorable outcome. Later these ideas was picked up and expanded upon by and . In recent years the study of probability and statistics has gained pace as computers allows us to perform studies on and make inference from large samples.

In 1494 first put into print the challange that Pascal and Fermat would solve two centuries later and with it usher in the area of probability. The challange is known as the problem of the unfinished game. Suppose two players $\{A, B\}$ bets on who will win the best of five tosses of a fair coin, but then have to stop before either player has won. How do they divide the pot?. If each has won the same number of throws then clearly the pot is split evenly. But what if they stop after three tosses, with player A ahead 2 to 1?. Pacioli, the man who first wrote about the problem, suggested that the solution is to divide the pot according to the current state of play, namely, 2 to 1. But this reasoning is incorrect as was demonsgrated in 1539 by

Gerolamo Cardano who noted that splitting the pot depended not on how many rounds each player had already won (as Pacioli thought) but on how many each player must still win in order to win the contest. To see this consider that since A is ahead 2-to-1, the first three rounds must have yielded two heads and one tail. The remaining two throws can yield $\{H, H\}$, $\{H, T\}$, $\{T, H\}$, $\{T, T\}$. In the first scenario $\{H, H\}$, the final score is four heads and one tail, so player A wins; in the second and the third ($\{H, T\}$ and $\{T, H\}$), the final outcome is three heads and two tails, so again player A wins. Only in the fourth scenario with $\{T, T\}$ is the final outcome two heads and three tails, so player B wins. This means that player A wins in three of the four possible ways the game could have ended and thus the pot should be divided 3/4 for A and 1/4 for for B.

13.1 Events

> ”I can as easily throw one, three or five as two, four or six. The wagers there are laid in accordance with this equality if the die is honest.”
>
> Gerolamo Cardano, Liber de ludo aleae (Book of Games of Chance)

In the quote above Gerolamo Cardano states that with a fair die the probability of getting $\{1, 3, 5\}$ is the same as getting $\{2, 4, 6\}$. He thus formulated one of the earlist known examples of what we now call the probability of an event as a fraction: the number of events that meets a constraint (such as a die being even or odd) divided by the total number of possible outcomes (such as the six different faces of a die).

$$P(event) = \frac{\text{Number of events that meet constraint}}{\text{Number of equally likely events}}$$

For example a toss of a fair coin have two possible outcomes $\{H, T\}$ so the probability of heads is $P(H) = \frac{1}{2}$. Semilarly a toss of a fair die has 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$ so the probability of the die showing six is $P(6) = \frac{1}{6}$, the probability of getting one or six is $P(1 \text{ or } 6) = \frac{2}{6}$ and probability of even number as $P(even) = \frac{3}{6}$

Cardano also observed that the probability of getting a certain outcome on two successive throws is the square of the probability of getting it on a single throw. For example, the probability of getting a 6 twice is $1/6 \times 1/6 = 1/36$. Similarly, the probability of getting three even numbers is $1/2 \times 1/2 \times 1/2 = 1/8$ (this assumes that is the events are independent events, i.e. that the first throw does not influence the second). That is the probability of two independendt events A and B occuring is

$$P(A \text{ and } B) = P(A) \times P(B)$$

Such events are known as a compound events. With compound events the likelihood of a series of independent events occurring is the multiplication of the likelihood of each individual event. So, if the probability of you finishing this chapter is 9 out of 10 (9/10), and the probability of you finishing the next one is 9/10, the total probability of you finishing both chapters isn't 9/10: it's $9/10 \times 9/10 = 81/100$. Note if you remove the restriction on the order in which the events occures then there are more possibilities. For example, the probability of rolling a 6 followed by an even number is $1/6 \times 1/2 = 1/12$ but without the restriction of order it becomes 5/36. The easist way to see this is to count all the possible outomes of throwing two dies $6 \times 6 = 36$ and then count the favorable outcomes (the die is even or six) $\{2, 6\}$, $\{6, 2\}$, $\{4, 6\}$, $\{6, 4\}$, and $\{6, 6\}$. As there are five such outcomes so we arrive at 5/36. Cardano also considered examples where we are interested in any of two possible events, such as the odds of getting a 1 or an even number are $1/6 + 1/2 = 2/3$. That is the probability of either of twi independendt events A and B occuring is

$$P(A \text{ or } B) = P(A) + P(B)$$

Cardanos also calculated the probability of throwing a 1 or a 2 with a pair of dice. The probability of throwing a 1 or a 2 with a single die is 1/3, so the naive answer would be that with two dice the probability is 2/3. Cardano notes this was incorrect as the probability of not rolling 1 or 2 with a single die is $4/6 = 2/3$, so the probability of not rolling it with two dice is $2/3 \times 2/3 = 4/9$. Hence the probability of rolling a 1 or a 2 must be $1 - 4/9 = 5/9$. This last scenario is a example of . Complementary events are events that when add together to equal a whole. For example, if the probability of it raining today were 2/5, the probability of it not raining would be $1 - 2/5 = 3/5$. From Cardano's early results it was made clear that probabilities functioned very differently if events where indepdendnet or depdendet

Independent events are events not affected by any other events. For example a coin toss is a independt event with $\{H, T\}$ each having 50% chance.

Dependent events are events that are affected by previous events. For example if we have 2 blue and 3 red marbles in a bag, then the chance of getting a blue marble is $2/5$. But after taking one marble out the chances change. If we got a red marble then chance of picking a blue is $2/4$, but if we got a blue then the chance of another is $1/4$. Thhus the probability of drawing two blue marbles is $2/5 \times 1/4 = 2/20 = 1/10$.

Understanding these distictions lets you use the correct values for calculating probability. For example if you take a card from a deck it has $1/52$ chance of bejng the ace of spades. If you flip 50 of the of the remaining 51 cards and none are the ace of spades then the remaining card now has $51/52$ chances to be the ace of spades, that is significanly more likely than our initial draw.

13.2 Conditional probabilities

A conditional probability is the probability of an event given that another event has occurred. For example, the probability that any given person has a cough on any given day may be only 5%. But if we assume the person has a cold, then they are much more likely to be coughing. The most famous problem of conditional probabilities is likely the Monty Hall problem. At the last round of a game show, youre faced with three curtains. Behind one there is a car but behind the two others there is a goat. Youre asked by the presenter to make a first choice. He then reveals one of the curtains you havent chosen which contains a goat. The presenter then offers you a chance to change your mind and switch curtain. Should you switch your choice or stick to the origiginal one. To most peoples surprise the correct answer is that you should switch your choice after being given this additional information. To see this consider the following table of posible actions

door 1	**door 2**	**door 3**
stay	switch	switch
switch	stay	switch
switch	switch	stay

that is if the car is behind dore one and you chose door one you should stay, if you chose door two or three you shoukd switch, equally for door two and three. From this table its easy to count that you should swich 6 out of 9 times which is therefor the best strategy.

13.3 Probability space

A probability spaces is a way to models processes consisting of states that occur randomly. For example in a deck of 52 cards the sample space is a 52-element set, as each card is a possible outcome. Since there can be many outcomes (even infinitely many), outcomes elements are grouped into sets which are called "events. For our deck of cards, possible events may include

- The 5 of Hearts (1 element),

- A King (4 elements),

- A Face card (12 elements),

- A card (52 elements).

More formally an event, is any subset of the sample space we like to consider in our model, including the empty set (an impossible event, with probability zero) and the sample space itself (a certain event, with probability one).
To sum up a probability space consists of three parts

- The sample space Ω which is a set of all possible outcomes.
- The σ-algebra $\mathcal{F}$ which is a collection of the events we would like to consider.
- The probability measure P which is a function returning an event's probability $P : F \to [0, 1]$.

For example if the experiment consists of just one flip of a perfect coin, then the outcomes are either heads or tails: $\Omega = H, T$. The σ-algebra $\mathcal{F} = 2^{\Omega}$ contains $2^2 = 4$ events, namely:

- $\{H\}$ "heads"
- $\{T\}$ "tails",
- $\{\}$ "neither heads nor tails"
- $\{H, T\}$ "either heads or tails"

So, $\mathcal{F} = \{\{\}, \{H\}, \{T\}, \{H, T\}\}$. Since there is a fifty percent chance of tossing heads, and fifty percent for tails the probability measure in this example is $P(\{\}) = 0, P(\{H\}) = 0.5, P(\{T\}) = 0.5, P(\{H, T\}) = 1$.

Random variable

A random variable (or stochastic variable) is a variable whose value is subject to variations due to chance. A random variable's possible values might represent the possible outcomes of a yet-to-be-performed experiment The mathematical function describing the possible values of a random variable and their associated probabilities is known as a probability distribution. Random variables can be discrete, that is, taking any of a specified finite list of values; or continuous, taking any numerical value in an interval.

13.4 Propability distributions

Binomial distribution

Normal distribution

13.5 Exercises

Ex. 22 — If you role a fair six sided die and a fair four sided die what is the probability that neither will shown one

Ex. 23 — You are given two dice to roll. One is black with six sides; the other is white with four sides. For a given roll, what is the probability the black die is even and the white die is 2

Ex. 24 — You are going to randomly select a marble from a bag of marbles that contains 3 blue marbles, 4 green marbles, and 5 red marbles. What is $P(\text{blue marble})$

Ex. 25 — A store offers sells four types of cloth: shirts, pants, socks and hats and offers each in three colors orange", purple and blue. If you randomly pick the piece of clothing and the color, what is the probability that you'll end up with an orange hat

Ex. 26 — If you flip a fair coin 1200 times. What is the best prediction for the number of times that the coin will land heads up?

Ex. 27 — If you toss a fair die 180 times what is the best prediction of the number of times you will get more than 4

Ex. 28 — You've decided to flip 3 coins, how many possible outcomes are there.

Ex. 29 — If a pirate and a navy boat fires at each other simultanously and the navy boat has 3/5 probability of a hit and the pirate 1/3. What is then the probability that the navy hits and the pirate misses ?

Ex. 30 — In a class of 7, there are 3 students who have red hair. If the teacher randomly chooses 2 students, what is the probability that neither of them have red hair.

Ex. 31 — Three dice are thrown simultaneously. Find the probability that:

1. All show distinct faces
2. Two of them show the same face

Chapter 14

Introduction to statistics

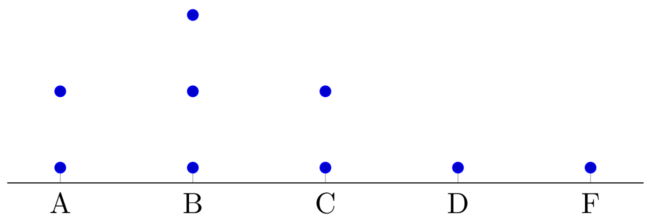
14.1 Data visualization

When studying data in order to make better decisions we often start by looking for patterns of behavior. The simplest form of pattern is the number of times a specific data value occurs also known as its frequency. For example when schools create learning plans for classes they analyze frequency of the class grades. Armed with this knowledge special measures can be taken, such as splitting the class into multiple groups to be thought separately.

For example, if four students receives an F in mathematics, then the grade F is said to have a frequency of 4. The frequency of a data is often represented in a table by arranging data values in ascending order of magnitude with their corresponding frequencies. For example on a A, B, C, D, F grading scale the grades $A, A, B, B, B, C, C, D, F$ produces the following frequency table

Grade	**Frequency**
A	2
B	3
C	2
D	1
F	1

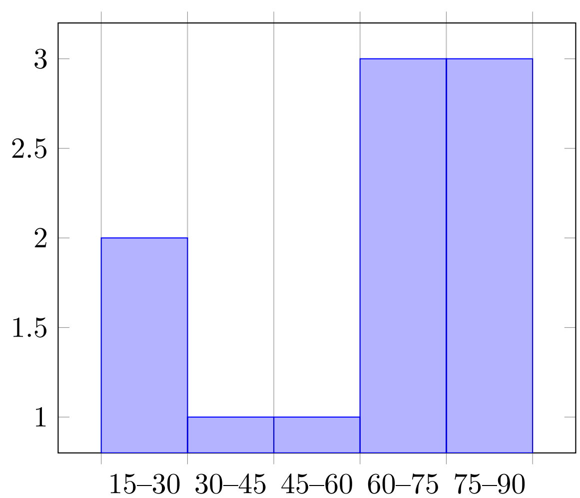
the table above can be easily visualized in a frequency plot, where each dot represents a occurrence of the value below



from this frequency plot we can easily deduce that the most common grade in the exam was B.
While frequency tables are convenient they often become hard to read if the data set contains more than a handful of observations. For these case we often resort to making a histogram or stem-and-leaf plot for the values.

Histograms

Often data is not repetitive enough for the same value to be repeated enough times to draw conclusions from a frequency table. In these cases we instead split the data into groups and count the frequencies of values within each group. For example if test scores are meassured in values between $0 - 100$ we may split the scores $15, 72, 55, 29, 67, 89, 33, 65, 78, 90$ in five groups.



Steam and leaf plots

Steam and leaf plots (The left column of the stem and leaf plot represents the tens place; each number on the right side represents the ones place for the number of pairs of jeans at a department store.)

Next, it must be determined what the stems will represent and what the leaves will represent. Typically, the leaf contains the last digit of the number and the stem contains all of the other digits. In the case of very large numbers, the data values may be rounded to a particular place value (such as the hundreds place) that will be used for the leaves. The remaining digits to the left of the rounded place value are used as the stem.
In this example, the leaf represents the ones place and the stem will represent the rest of the number (tens place and higher).

14.2 Standard Data points

When performing statistical analysis its often helpful to calculate a number of standard data points that helps uncover the nature of the data. Of these the mean, median and mode are all estimates of where the "middle" or "average" of the data is, once these values have been established we can take any observation and see how it relates to the average, for example if we have established that the everage grade is B then a student getting C is below average and a A student is above.

Mean Is the sum of the observations divided by the number of observations e.g. the mean of $\{3, 3, 5, 9, 11\}$ is $(3 + 3 + 5 + 9 + 11)/5 = 6.2$. more formally suppose we have a data set containing the values $a_1, \ldots, a_n$, then the arithmetic mean A is defined as

$$A = \frac{1}{n}\sum_{i=1}^{n} a_i \tag{14.1}$$

Median The "middle" value of the numbers. To find the median arranging all the observations from lowest to highest and pick the middle one (e.g., the median of $\{3, 3, 5, 9, 11\}$ is 5). If there is an even number of observations then the median is the mean of the two middle values (the median of $\{3, 5, 7, 9\}$ is $(5 + 7)/2 = 6$).

Mode The "mode" is the value that occurs most often. If no number is repeated, then there is no mode. e.g. the mode of $\{3, 3, 5, 9, 11\}$ is 3. In the data set: $\{1, 1, 1, 2, 2, 2, 3\}$ we see that 1 and 2 both have the most occurrences, so they are both modes.

Two less frequently used values are the range and mid-range

Range The "range" is the difference between the largest and smallest values. e.g. the range of $\{3, 3, 5, 9, 11\}$ is $11 - 3 = 8$.

Mid-range the mid-range is the arithmetic mean of the maximum and minimum values in a data set

$$\text{MidRange} = \frac{min(x) + max(x)}{2}$$

The standard deviation is the average distance between the actual data and the mean.

Quartiles

When using histograms for data analysis we may split the data into any number of groups we deem necessary. This arbitrary split can lead to results that are hard to compare as different researchers may group their data differently. Thus to compare data its useful to group data using a stansafized method, one such method is that of quartiles. With quartiles the data is split into four groups, known as a quartile. Each quartile contains 25% of the total observations. Generally, the data is ordered from smallest to largest with those observations falling below 25% of all the data analysed allocated within the $1st$ quartile, observations falling between 25.1% and 50% and allocated in the $2nd$ quartile, then the observations falling between 51% and 75% allocated in the $3rd$ quartile, and finally the remaining observations allocated in the $4th$ quartile.
When splitting data sets into their quartiles we do so by calculating the boundary values of each quartile known as Q_1, Q_2 and Q_3. Such that

- $1st$ quartile $v \in S$ where $v < Q_1$
- $2nd$ quartile $v \in S$ where $v > Q_1$ and $v < Q_2$
- $3rd$ quartile $v \in S$ where $v > Q_2$ and $v < Q_3$
- $4th$ quartile $v \in S$ where $v > Q_3$

to calculate these we use the following method

1. Sort data values and find its median, This is Q_1.
2. Divide the data set into two groups: a low group and a high group. When n is odd, the median should be placed in both groups.

3. Find the median of the low group. This is Q_1.

4. Find the median)of the high group. This is Q_3.

Once the quartiles have been determined, its common to calculate the inter-quartile range (IQR)

$$IQR = Q_3 - Q_1 \tag{14.2}$$

A good summary of locations in the distribution is provided by the points that divide the data it into four equally-sized groups. This is the 5-point summary is made of:

1. Quartile 0 the minimum

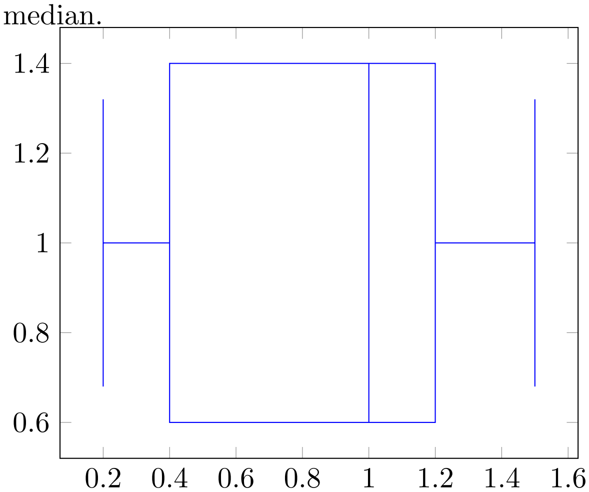
2. Quartile 1 (bigger than 25% of the data points)

3. Quartile 2 (the median)

4. Quartile 3 (bigger than 75% of the data points)

5. Quartile 4 (the maximum)

The five-number summary gives information about the location (from the median), spread (from the quartiles) and range (from the sample minimum and maximum) of the observations
For example the data set $\{0, 1.5, 2.5, 3, 4, 4, 4, 7, 7.5\}$ has the median $4 = Q_2$. The data to the left of the median is $\{0, 1.5, 2.5, 3\}$ which has $2 = Q_1$ as its median. The data to the right is $\{4, 4, 7, 7.5\}$ which has $5.5 = Q_3$ as its median.



Distributions

Deviation and variance

deviation is a measure of difference between the observed value of a variable and some other value, often that variable's mean

Mean absolute deviation The mean of the distances of each value from their mean.

1. Find the mean of all values
2. Find the distance of each value from that mean (subtract the mean from each value, ignore minus signs
3. Then find the mean of those distances

For example the Mean Deviation of $3, 6, 6, 7, 8, 11, 15, 16$ is 3.75 as the mean is 9 and the distances from this are $6, 3, 3, 2, 1, 2, 6, 7$

The Variance is defined as:
The average of the squared differences from the Mean.
To calculate the variance follow these steps:
Work out the Mean (the simple average of the numbers) Then for each number: subtract the Mean and square the result (the squared difference). Then work out the average of those squared differences.

14.3 Regression

Regression is the main statistical technique used to quantify the relationship between two or more variables. A regression analysis would show a positive relationship between height and weight, for example. The measure of the accuracy of a regression is called R-squared. A perfect relationship, with no error, would have an R-squared of 1.00 or 100. Strong relationships, like height and weight, would have an R-squared of around 70 percent. A meaningless relationship, like hair color and weight, would have an R-squared of zero.

14.4 Random samples

A random sample of 25% of a schools pupules shoved that 16 pupiles had red hair. Based on the data, what is the most reasonable estimate of the total number of pupiles with red hair on the school?. 25% is the same as 1/4 of the pupiles at the school so the most resonable estimate is 4 times the number from the sample $4 \cdot 16 = 62$

14.5 Exercises

Ex. 32 — Calculate the quartiles of the following data sets

1. $\{0, 0, 1, 2, 2, 3, 3, 4\}$
2. $\{3, 4, 4, 5, 5, 5, 6, 7\}$
3. $\{7, 9, 9, 10, 10, 10, 11, 12, 12, 14\}$
4. $\{0, 1, 1, 3, 3, 3, 4, 5, 7\}$

Ex. 33 — Use the normal distrubhtion to answer

1. normally distributed random variable X has a mean of 20 and a standard deviation of 4. Determine the probability that a randomly selected x-value is between 15 and 22.
2. The final exam scores in a statistics class were normally distributed with a mean of 58 and a standard deviation of 4. Find the probability that a randomly selected student scored more than 62 on the exam

Appendix A

Solutions to the exercises

Each exercise is numbered in accordance with the chapter where it is to be found, thus exercise two in chapter one is referred to as 1.2.

Answer (Ex. 1) — Canceling out common terms we get

1. $88/72 = (11 * 8)/(9 * 8) = 11/9$.
2. $\frac{2}{\frac{4}{5}} = 2 \cdot \frac{5}{4} = \frac{10}{4} = \frac{5}{2}$
3. $\frac{\frac{-7}{2}}{\frac{4}{9}} = \frac{-7}{-2} \cdot \frac{9}{4} = \frac{63}{8}$

Answer (Ex. 2) — $36/9$ is the time it takes for 5 people to paint one wall, therefor $5 * 36/9$ is the time it takes for one person to paint one wall. Thus it will take $7 * ((5 * 36)/9) = 140$ for one person to paint 7 walls and therefor $140/10 = 14$ minutes for 10 people

Answer (Ex. 3) — The car will spend $4000/100 * 6 = 240$ liters of gas at a price of $240 * 1.5 = 360$, as there are four people in the car the cost per person becomes $360/4 = 90$.

Answer (Ex. 4) — Using the PEMDAS meme (parentheses, exponents, multiplication, division, addition, subtraction) listed in 4.9 we get:

1. $7 + 7/7 + 7cdot7 - 7 = 7 + (7/7) + (7 \cdot 7) - 7 = 50$.
2. $6 - 1 \cdot 0 + 2/2 = 6 + -1 \cdot 0 + 2/2 = 6 + 0 + 2/2 = 7$

3. $-2 + (-3) + 4 - (-3) - 5 = -2 - 3 + 4 + 3 - 5 = -5 + 7 - 5 = -3$
4. $3 + (-4) - 8 - (-1) - 1 = 3 - 4 - 8 + 1 - 1 = -1 - 8$

Answer (Ex. 5) — Using the exponent rules from 4.8 we get

1. $\frac{4^6}{4^{-4}} = 4^{6--4} = 4^{10}$
2. $(5^{-12})^{-9} = 5^{108}$
3. $((5^{-10})(9^9))^7 = 5^{-70} \cdot 9^{63}$
4. $\sqrt{11} + \sqrt{44} + \sqrt{99} = \sqrt{11} + \sqrt{4 \cdot 11} + \sqrt{9 \cdot 11} = 6\sqrt{11}$

Answer (Ex. 6) — First we simply each expression and then replace the variables with its value in order to calculate the expression value

1. $-1 - (-z) - 5 - (-3) = -1 + z - 5 + 3 = -1 - 2 - 5 + 3 = -5$
2. $x - (-y) = x + y = -2 + 5 = 3$
3. $3 - (-6) + (-h) + (-4) = 3 + 6 - h - 4 = 5 - h = 5 - (-7) = 12$

Answer (Ex. 7) — Noting that $-(x + y) = -1 \cdot (x + y) = -x - y$ we see

1. $-(6 - \frac{z}{4}) = -6 + \frac{z}{4}$
2. $-(\frac{1}{2}r + 4) = -\frac{1}{2}r - 4$
3. $-5(3n + \frac{1}{2}) = -15n - \frac{5}{2}$

Answer (Ex. 8) — We get

1. $\frac{3}{5}z - \frac{6}{5} = \frac{3}{5}(z + 2)$
2. $\frac{3}{2} + \frac{7}{8}c = \frac{1}{8}(12 + 7c)$

Answer (Ex. 9) — We determine equivalent expressions by subtracting them from each other in order to see if the result becomes zero

1. Not equal: $9x + 6 - (3x + 2) = 6x - 4$
2. Not equal: $\frac{5}{2}x - 3 - (x + 5) = \frac{3/2}{x} - 7$

3. Equal:

$$\frac{\frac{2}{b}+\frac{2}{a}}{\frac{2}{ab}}-(a+b)=\left(\frac{2a+2b}{ab}\right)\left(\frac{ab}{2}\right)-(a+b)=0$$

4. Equal:

$$\frac{\frac{a}{b}+1}{\frac{b}{a}-1}-\frac{a(a+b)}{b(b-a)}=\frac{\frac{a}{b}+\frac{b}{b}}{\frac{b}{a}-\frac{a}{a}}-\frac{a(a+b)}{b(b-a)}=$$
$$\left(\frac{a+b}{b}\right)\left(\frac{a}{b-a}\right)-\frac{a(a+b)}{b(b-a)}=0$$

Answer (Ex. 10) — The expressions simplifies to

1. $6+5(-7n+2)=-35n+16$
2. $-(-15+2a)+4(8a-6)=30a-9$
3. $6(-2+10k)+6(5k-3)=90k-30$
4. $1/5-2z+z+2/3=13/15-z$
5. $7n-(4n-3)=3n+3$

Answer (Ex. 11) — Using the exponent rules from 4.8 we get

1. $\frac{35n^3}{10n^4}=3.5n^{-1}$
2. $\frac{44a^3}{55a^3}=\frac{4}{5}$
3. $\frac{28y^5}{7y^3}=4y^3$

Answer (Ex. 12) — We formulate the expressions in terms of their unknown and solve for it.

1. The equation is $17=\frac{n-6}{2}$ and $n=2\cdot 17+6=40$.
2. We have $n+(n+2)+(n+4)=69$ and $n=\frac{63}{3}=21$.
3. The equation is $\frac{13}{10}(7+n)=91$ and $n=63$

4. Let x be the amount to be drawn off. Then $10 - x$ is the gallons remaining of which $\frac{1}{5}$ is alcohol that is $1/5(10 - x)$ is alcohol. After the x gallons are replaced with alcohol, the amount of alcohol in the tank will be $\frac{1}{5}(10 - x) + x$. As we want this to be 5 gallons (50 percent alcohol of 10 gallons) we end up with

$$\frac{1}{5}(10 - x) + x = 5$$

Which reduces to

$$3 \cdot \frac{5}{4} = \frac{15}{4} = \frac{12}{4} + \frac{3}{4} = 3\frac{3}{4}$$

Answer (Ex. 13) — The solutioms are

1. $x = \frac{12 \cdot 5}{6} + 12 = 22.$
2. $a = \frac{18 \cdot 5}{3} - 11 = 19.$
3. $k = \frac{5 \cdot 18}{6} + 7 = 22.$
4. $t = \frac{15 \cdot 8}{3} + 1 = 41.$
5. $y = \frac{7 \cdot 25}{5} + 8 = 43.$

Answer (Ex. 14) — x is

1. $x > -\frac{1}{12}.$
2. $x < \frac{5}{9}.$
3. $x \leq -\frac{13}{16}.$
4. $b > 3.$
5. $x > 5.$

Answer (Ex. 15) — x is

1. $d \leq 3.$
2. $x < \frac{7}{2}.$
3. $x \leq \frac{3}{2}.$
4. $x \geq \frac{1}{5}$
5. $x \leq -\frac{1}{6}$

Answer (Ex. 16) — The quadrants are

1. First quadrant
2. Second quadrant
3. Third quadrant
4. Fourth quadrant
5. The origin has no quadrant.

Answer (Ex. 17) — Using 5.3 we get the following quadratics

1. $(x-3)^2 = x^2 + (-3+-3)x + (-3\cdot -3) = x^2 - 6x + 9$
2. $(x-3)(x+10) = x^2 + (-3+10)x + (-3\cdot 10) = x^2 + 7x - 30$
3. $(x+8)^2 = x^2 + (8+8)x + (8\cdot 8) = x^2 + 16x + 64$
4. $(x+1)(x-4) = x^2 + (1+-4)x + (1\cdot -4) = x^2 - 3x - 4$
5. $(x+9)^2 = x^2 + (9+9)x + (9\cdot 9) = x^2 + 18x + 81$

Answer (Ex. 18) — Using 5.3 we get the following factors

1. $x^2 + 9x + 20 = x^2 + (4+5)x + (4\cdot 5) = (x+4)(x+5)$.
2. $x^2 + 5x + 6 = x^2 + (3+2)x + (2\cdot 3) = (x+2)(x+3)$.
3. $x^2 - 5x + 6 = x^2 + (-3+-2)x + (-2\cdot -3) = (x-2)(x-3)$.
4. $x^2 - 9 = x^2 + (-3+3)x + (-3\cdot 3) = (x-3)(x+3)$.
5. $x^2 + 7x - 18 = x^2 + (-2+9)x + (-2\cdot 9) = (x+9)(x-2)$.

Answer (Ex. 19) — Numbers divisible by 12 and 20 also have to be divisible by each of their prime factors $12 = 2\cdot 2\cdot 3$ and $20 = 2\cdot 2\cdot 5$ thus $2\cdot 2\cdot 3\cdot 5 = 60$.

Answer (Ex. 20) — TODO

Answer (Ex. 21) — Using the rules of counting we find

1. Each of the four plants can be placed in four different pots yielding $4\cdot 4 = 16$ possible arrangements

2. TODO describe why $3 \cdot 2 \cdot 2 = 12$
3. TODO describe why $4 \cdot 4$
4. 7 different ways to pick the first chair, leaving 6 different people to pick the second chair, leaving 5 different people to pick the fifth chair i.e. $7 * 6 * 5 = 210$
5. $\frac{7*6}{2}$ handshakes will occur as each of the seven people will shake hands with six other people but since one person shaking with another is counted twice we divide with two.

Answer (Ex. 22) —
$P(\text{six sided not not one}) \cdot P(\text{four sided not not one}) = 5/6 * 3/4 = 5/8$

Answer (Ex. 23) —
$P(\text{even black die}) \cdot P(\text{white die is 2}) = 3/6 * 1/4 = 3/24$

Answer (Ex. 24) — $P(bluemarble) = \frac{\text{number of blue marbles}}{\text{total number of marbles}} = 3/12 = 0.25$

Answer (Ex. 25) — $1/3 \cdot 1/4 = 1/12$

Answer (Ex. 26) — $1/2 \cdot 1200 = 600$

Answer (Ex. 27) — $2/6 \cdot 180 = 60$

Answer (Ex. 28) — There are two possible outcomes for the first flip, two for the second and two for the thrid, thus $2 \cdot 2 \cdot 2 = 8$ possible outomes.

Answer (Ex. 29) — Our scenario has probability
$P(\text{navy hits}) * P(\text{pirate misses})$ which is
$3/5 \cdot (1 - 1/3) = 3/5 \cdot 2/3 = 6/15 = 2/5$.

Answer (Ex. 30) — there is a 4/7 chance that the first student is not a red hair and 3/6 chance that the second student is not so
$4/7 * 3/6 = 12/42 = (2 * 2 * 3)/(2 * 3 * 7) = 2/7$.

Answer (Ex. 31) — TODO

Answer (Ex. 32) — As described in 14.2 we first calculate Q_2 and use its value to find Q_1 and Q_3

1. $Q_2 = 2, Q_1 = 1/2, Q_3 = 3$
2. $Q_2 = 5, Q_1 = 4, Q_3 = 5.5$
3. $Q_2 = 10, Q_1 = 9, Q_3 = 12$
4. $Q_2 = 3, Q_1 = 1, Q_3 = 4.5$

Answer (Ex. 33) — TODO

Index

Algebraic numbers, 11
axiom, 3

base, 12
Bishop Wibold of Cambrai, 47
Blaise Pascal, 65

complementary events, 67
compound event, 67

DeVetula, 47

Fermi-Dirac statistics, 4

Gerolamo Cardano, 65

harmonic series, 4

independent events, 67

linear equation, 26
Luca Pacioli, 65

monomial, 32

Oresme, 4

Pierre de Fermat, 65
postulate, 3
power product, 32

radix, 12

similarity, 4