Parallel Algorithms & the PRAM Model

Advanced Topics Spring 2009

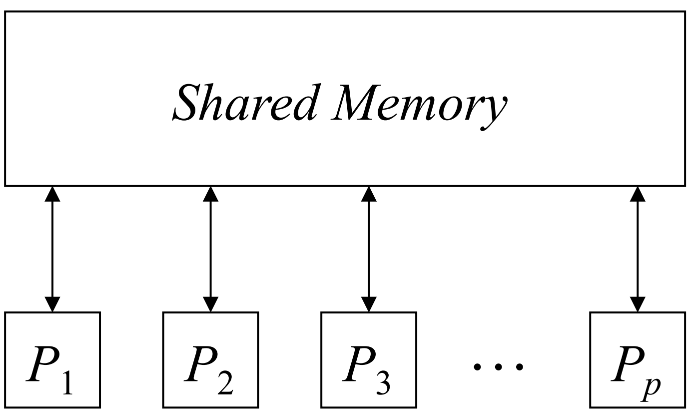
Prof. Robert van Engelen

Overview

- The PRAM model of parallel computation
- Simulations between PRAM models
- Work-time presentation framework of parallel algorithms
- Design and analysis of parallel algorithms

The PRAM Model of Parallel Computation

- Parallel Random Access Machine (PRAM)
- Natural extension of RAM: each processor is a RAM
- Processors operate synchronously
- Earliest and best-known model of parallel computation



Shared memory with m locations

p processors, each with private memory

All processors operate synchronously, by executing load, store, and operations on data

Synchronous PRAM versus Asynchronous PRAM

- The **synchronous PRAM** model has a similarity with data-parallel execution on a SIMD machine
  - All processors execute the same program
  - All processors execute the same PRAM step instruction stream in "lock-step"
  - Effect of operation depends on local data
  - Instructions can be selectively disabled (for if-then-else flow)

- The **asynchronous PRAM** model
  - Several competing models
  - No lock-step

Classification of PRAM Model

- A PRAM step ("clock cycle") consists of three phases
  1. *Read*: each processor may read a value from shared memory
  2. *Compute*: each processor may perform operations on local data
  3. *Write*: each processor may write a value to shared memory
- Model is refined for concurrent read/write capability
  - Exclusive Read Exclusive Write (EREW)
  - Concurrent Read Exclusive Write (CREW)
  - Concurrent Read Concurrent Write (CRCW)
- CRCW PRAM: what to do with concurrent writes?
  - Common CRCW: all processors must write the same value
  - Arbitrary CRCW: one of the processors succeeds in writing
  - Priority CRCW: processor with highest priority succeeds in writing

Comparison of PRAM Models

- A model *A* is less powerful compared to model *B* if either
  - The time complexity is asymptotically less in model *B* for solving a problem compared to *A*
  - Or the time complexity is the same and the work complexity is asymptotically less in model *B* compared to *A*
- From weakest to strongest:
  - EREW
  - CREW
  - Common CRCW
  - Arbitrary CRCW
  - Priority CRCW

Simulations Between PRAM Models

- An algorithm designed for a weaker model can be executed within the same time complexity and work complexity on a stronger model

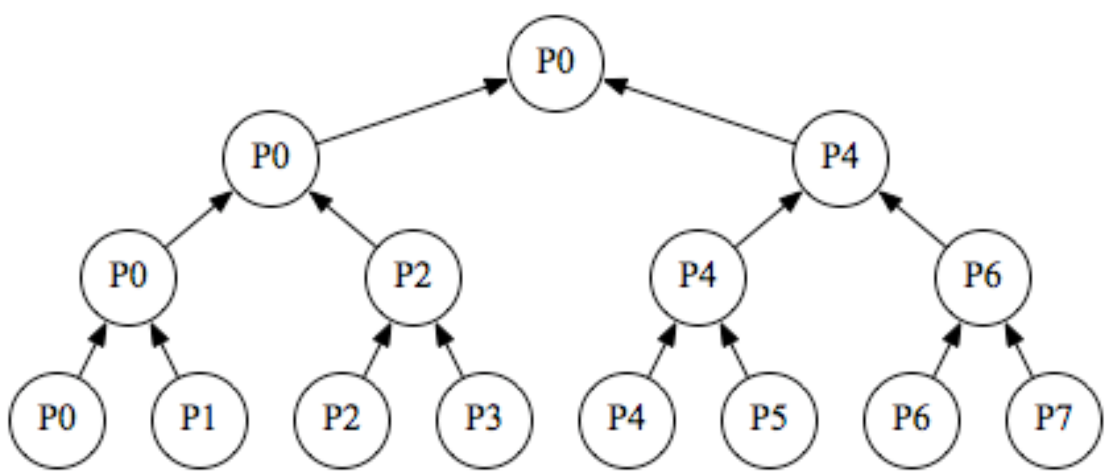
- An algorithm designed for a stronger model can be *simulated* on a weaker model, either with
  - Asymptotically more processors (or more work by the same number of processors)
  - Or asymptotically more time

Simulating a Priority CRCW on an EREW PRAM

- Theorem: An algorithm that runs in T time on the p-processor priority CRCW PRAM can be simulated by EREW PRAM to run in $O(T \log p)$ time
  - A concurrent read or write of an p-processor CRCW PRAM can be implemented on a p-processor EREW PRAM to execute in $O(\log p)$ time
  - $Q_1,\ldots,Q_p$ CRCW processors, such that Q_i has to read (write) $M[j_i]$
  - $P_1,\ldots,P_p$ EREW processors
  - $M_1,\ldots,M_p$ denote shared memory locations for special use
  - P_i stores $<j_i,i>$ in M_i
  - Sort pairs in lexicographically non-decreasing order in $O(\log p)$ time using EREW merge sort algorithm
  - Pick representative from each block of pairs that have same first component in $O(1)$ time
  - Representative P_i reads (writes) from $M[k]$ with $<k,\_>$ in M_i and copies data to each M in the block in O(log p) time using EREW segmented parallel prefix algorithm
  - P_i reads data from M_i

Example 1:
Reduction on the EREW PRAM

- Reduce (sum) p values on the p-processor EREW PRAM in $O(\log p)$ time
- Reduction algorithm uses exclusive reads and writes
- Algorithm is the basis of other EREW algorithms



Example 1

Sum of n values using n processors (i)
Each processor i, $1 \leq i \leq n$, executes:
Input: $A[1,\ldots,n]$, $n = 2^k$
Output: sum $S = \sum_{j=1..n} A[j]$
begin
  $B[i] := A[i]$
  for $h = 1$ **to** $\log n$ **do**
    if $i \leq n/2^h$ **then**
      $B[i] := B[2i-1] + B[2i]$
  if $i = 1$ **then**
    $S := B[i]$
end

How much time?

How many operations?

Example 2: Matrix Multiply on the CREW PRAM

- Consider $n \times n$ matrix multiplication $C = A\, B$ using n^3 processors

- Each element of C

  $$c_{ij} = \sum_{k=1..n} a_{ik}\, b_{kj}$$

  can be computed on the CREW PRAM in parallel using n processors in $\mathrm{O}(\log n)$ time

- All c_{ij} can be computed using n^3 processors in $\mathrm{O}(\log n)$ time

Example 2

Matrix multiply with n^3 processors (i,j,l)

Each processor (i,j,l) executes:

Input: $n \times n$ matrices A and B, $n = 2^k$

Output: $C = A\ B$

begin

  $C'[i,j,l] := A[i,l]B[l,j]$

  for $h = 1$ **to** $\log n$ **do**

    if $i \leq n/2^h$ **then**

      $C'[i,j,l] := C'[i,j,2l-1] + C'[i,j,2l]$

  if $l = 1$ **then**

    $C[i,j] := C'[i,j,1]$

end

O(log n) *time*

How many operations?

Example 2:
CREW versus EREW PRAM

- Algorithm on the CREW PRAM requires $O(\log n)$ time and $O(n^3)$ operations (n^2 processors perform $O(n)$ ops)

- On the EREW PRAM, the exclusive reads of a_{ij} and b_{ij} values can be satisfied by making n copies of a and b, which takes $O(\log n)$ time with n processors (broadcast tree)
- Total time is still $O(\log n)$
- But requires more work and total memory requirement is huge!

The WT Scheduling Principle

- The **work-time (WT) scheduling principle** schedules p processors to execute an algorithm
  - Algorithm has $T(n)$ time steps and $W(n)$ total operations
  - A time step can be parallel, i.e. **pardo**
- We can adapt the algorithm to run on the p-processor PRAM in $\leq \lfloor W(n)/p \rfloor + T(n)$ steps
- Proof
  - Let $W_i(n)$ be the number of operations (work) performed in time unit i, $1 \leq i \leq T(n)$
  - Simulate each set of $W_i(n)$ operations in $\lceil W_i(n)/p \rceil$ parallel steps, for each $1 \leq i \leq T(n)$
  - The number of steps on the p-processor PRAM takes $\sum_i \lceil W_i(n)/p \rceil \leq \sum_i (\lfloor W_i(n)/p \rfloor + 1) \leq \lfloor W(n)/p \rfloor + T(n)$

Work-Time Presentation

- The WT presentation can be used to determine the time and operation requirements of an algorithm
- The upper-level WT presentation framework describes the algorithm in terms of a sequence of time units
  - From which we can determine $T(n)$ and $W(n)$
- The lower-level follows the WT scheduling principle
  - p-processor PRAM requires $\leq \lfloor W(n)/p \rfloor + T(n)$ steps

Example 1 Revisited: WT Presentation

Input: $A[1,\ldots,n]$, $n = 2^k$
Output: sum $S = \sum_{j=1..n} A[j]$
begin
  for $1 \leq i \leq n$ **pardo**
    $B[i] := A[i]$
  for $h = 1$ **to** $\log n$ **do**
    for $1 \leq i \leq n/2^h$ **pardo**
      $B[i] := B[2i-1] + B[2i]$
  if $i = 1$ **then**
    $S := B[1]$
end

Do you spot any
concurrent reads?
concurrent writes?

$T(n) = O(\log n)$
$W(n) = O(n)$

WT scheduling principle:
total time $\leq O(n/p + \log n)$

Example 2 Revisited: WT-Presentation

Input: $n \times n$ matrices A and B, $n = 2^k$
Output: $C = A\ B$
begin
  for $1 \leq i, j, l \leq n$ **pardo**
    $C'[i,j,l] := A[i,l]B[l,j]$
  for $h = 1$ **to** $\log n$ **do**
    for $1 \leq i, j \leq n,\ 1 \leq l \leq n/2^h$ **pardo**
      $C'[i,j,l] := C'[i,j,2l-1] + C'[i,j,2l]$
  for $1 \leq i, j \leq n$ **pardo**
    $C[i,j] := C'[i,j,1]$
end

$T(n) = \mathrm{O}(\log n)$
$W(n) = n^3$

WT scheduling principle:
total time $\leq \mathrm{O}(n^3/p + \log n)$

Example 3: PRAM Recursive Prefix Sum Algorithm

Input: Array of $(x_1, x_2, \ldots, x_n)$ elements, $n = 2^k$
Output: Prefix sums s_i, $1 \leq i \leq n$
begin
  if n = 1 **then** $s_1 = x_1$; **exit**
  for $1 \leq i \leq n/2$ **pardo**
    $y_i := x_{2i-1} + x_{2i}$
  Recursively compute prefix sums of y and store in z
  for $1 \leq i \leq n$ **pardo**
   if i is even **then** $s_i := z_{i/2}$
   if $i > 1$ is odd **then** $s_i := z_{(i-1)/2} + x_i$
   if $i = 1$ **then** $s_1 := x_1$
end

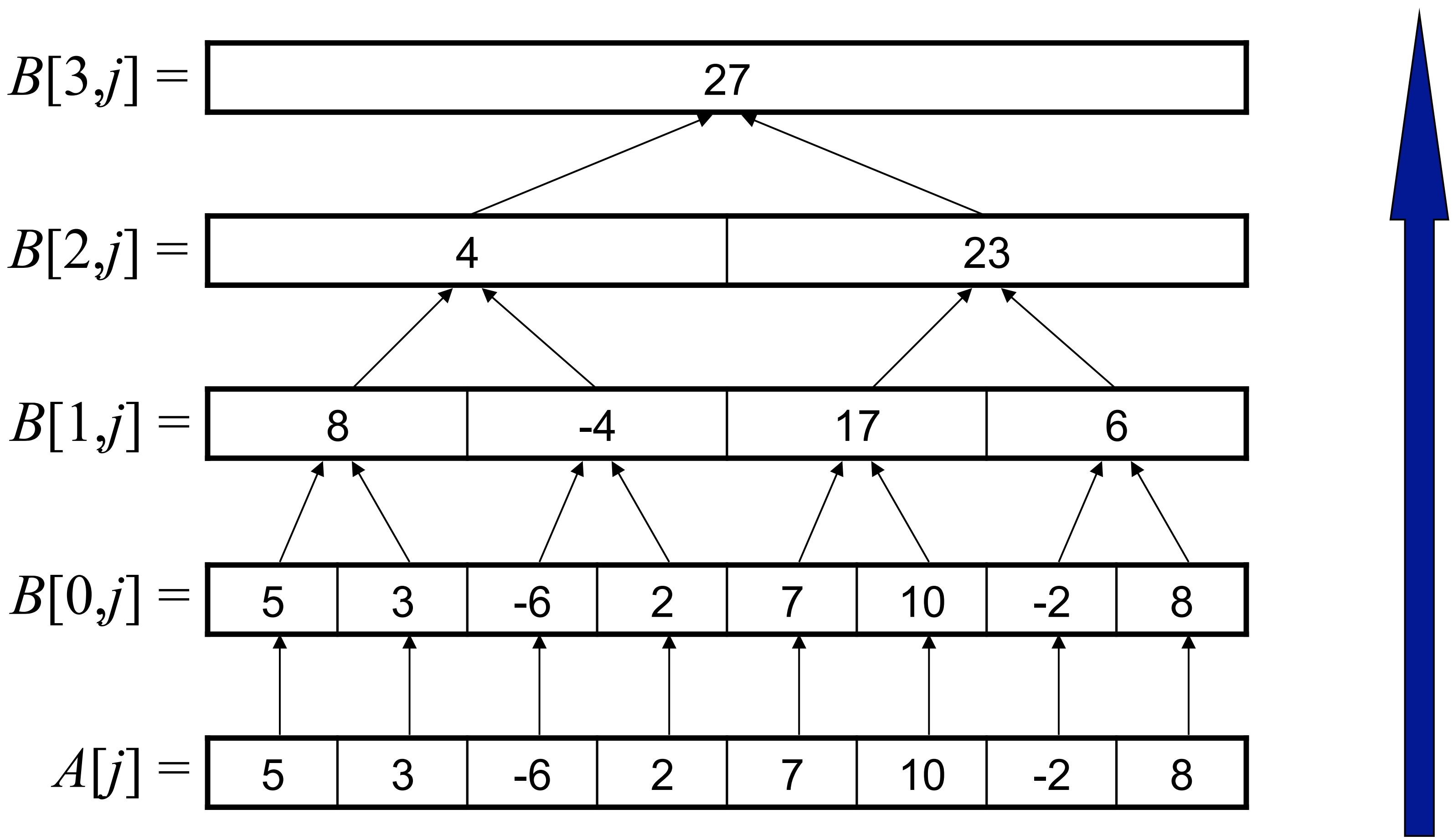
Proof of Work Optimality

- **Theorem**: The PRAM prefix sum algorithm correctly computes the prefix sum and takes $T(n) = \mathrm{O}(\log n)$ time using a total of $W(n) = \mathrm{O}(n)$ operations
- **Proof** by induction on k, where input size $n = 2^k$
  - Base case $k = 0$: $s_1 = x_1$
  - Assume correct for $n = 2^k$
  - For $n = 2^{k+1}$
    - For all $1 \leq j \leq n/2$ we have $z_j = y_1 + y_2 + \ldots + y_j = (x_1 + x_2) + (x_3 + x_4) + \ldots + (x_{2j\text{-}1} + x_{2j})$
    - Hence, for $i = 2j \leq n$ we have $s_i = s_{2j} = z_j = z_{i/2}$
    - And $i = 2j+1 \leq n$ we have $s_i = s_{2j+1} = s_{2j} + x_{2j+1} = z_j + x_{2j+1} = z_{(i-1)/2} + x_i$
- $T(n) = T(n/2) + a \quad \Rightarrow T(n) = \mathrm{O}(\log n)$
- $W(n) = W(n/2) + bn \quad \Rightarrow W(n) = \mathrm{O}(n)$

PRAM Nonrecursive Prefix Sum

Input: Array A of size $n = 2^k$
Output: Prefix sums in $C[0,j]$, $1 \leq j \leq n$
begin
  for $1 \leq j \leq n$ **pardo**
    $B[0,j] := A[j]$
  for $h = 1$ **to** $\log n$ **do**
    for $1 \leq j \leq n/2^h$ **pardo**
      $B[h,j] := B[h-1,2j-1] + B[h-1,2j]$
  for $h = \log n$ **to** 0 **do**
    for $1 \leq j \leq n/2^h$ **pardo**
      if j is even **then** $C[h,j] := C[h+1,j/2]$
      else if $i = 1$ **then** $C[h,1] := B[h,1]$
      else $C[h,j] := C[h+1,(j-1)/2] + B[h,j]$
end

First Pass: Bottom-Up

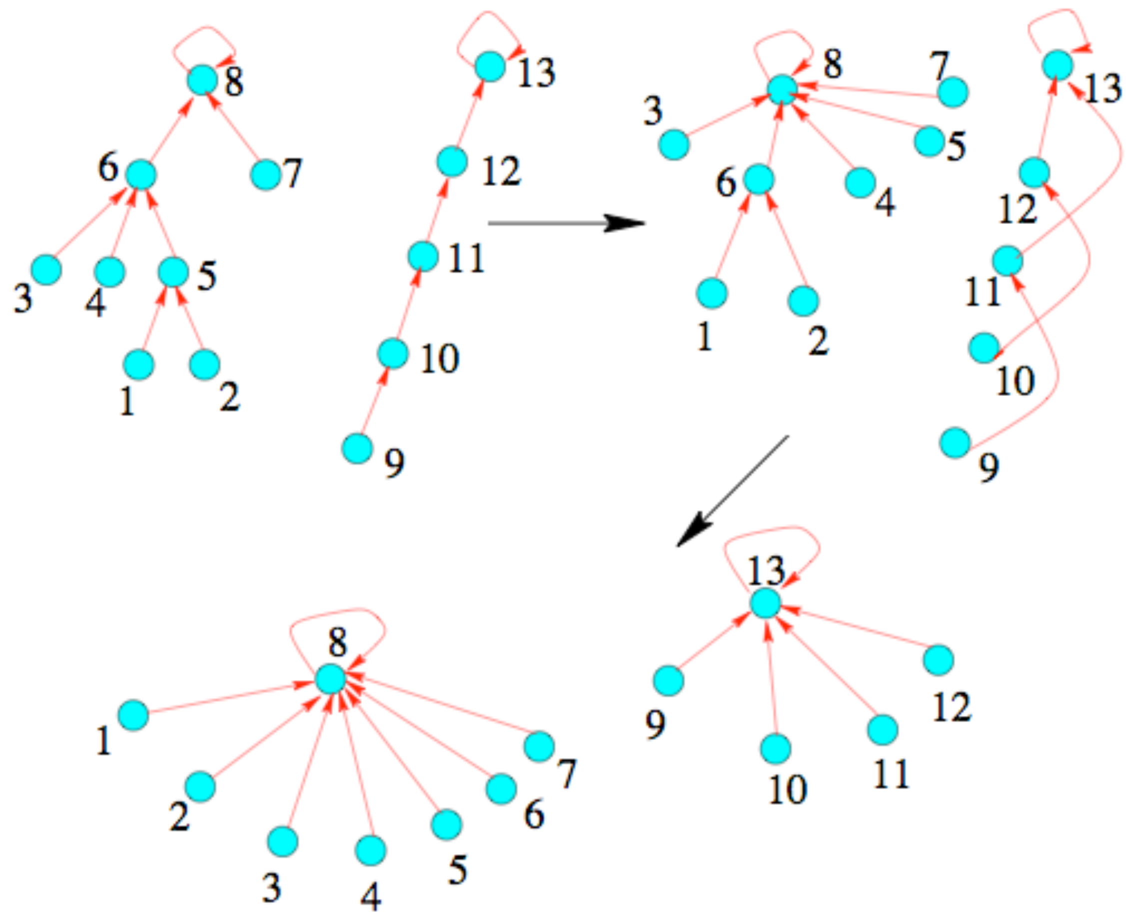


Second Pass: Top-Down

<table>
<tr><td>$B[3,j] =$</td><td colspan="8">27</td></tr>
<tr><td>$C[3,j] =$</td><td colspan="8">27</td></tr>
<tr><td>$B[2,j] =$</td><td colspan="4">4</td><td colspan="4">23</td></tr>
<tr><td>$C[2,j] =$</td><td colspan="4">4</td><td colspan="4">27</td></tr>
<tr><td>$B[1,j] =$</td><td colspan="2">8</td><td colspan="2">-4</td><td colspan="2">17</td><td colspan="2">6</td></tr>
<tr><td>$C[1,j] =$</td><td colspan="2">8</td><td colspan="2">4</td><td colspan="2">21</td><td colspan="2">27</td></tr>
<tr><td>$B[0,j] =$</td><td>5</td><td>3</td><td>-6</td><td>2</td><td>7</td><td>10</td><td>-2</td><td>8</td></tr>
<tr><td>$C[0,j] =$</td><td>5</td><td>8</td><td>2</td><td>4</td><td>11</td><td>21</td><td>19</td><td>27</td></tr>
<tr><td>$A[j] =$</td><td>5</td><td>3</td><td>-6</td><td>2</td><td>7</td><td>10</td><td>-2</td><td>8</td></tr>
</table>

Example 4: Pointer Jumping

- Finding the roots of a forest using pointer-jumping



Pointer Jumping on the CREW PRAM

Input: A forest of trees, each with a self-loop at its root, consisting of arcs $(i,P(i))$ and nodes i, where $1 \leq i \leq n$
Output: For each node i, the root $S[i]$
begin
  for $1 \leq i \leq n$ **pardo**
    $S[i] := P[i]$
    while $S[i] \neq S[S[i]]$ **do**
      $S[i] := S[S[i]]$
end

$T(n) = O(\log h)$ with h the maximum height of trees
$W(n) = O(n \log h)$

PRAM Model Summary

- PRAM removes algorithmic details concerning synchronization and communication, allowing the algorithm designer to focus on problem properties
- A PRAM algorithm includes an explicit understanding of the operations performed at each time unit and an explicit allocation of processors to jobs at each time unit
- PRAM design paradigms have turned out to be robust and have been mapped efficiently onto many other parallel models and even network models
  - A SIMD network model considers *communication diameter*, *bisection width*, and *scalability* properties of the network topology of a parallel machine such as a mesh or hypercube

Design and Analysis of Parallel Algorithms

- Arithmetic problems:
  - Polynomial evaluation: first-order linear recurrence
  - Polynomial multiplication: FFT
  - Lagrange interpolation
- Planar geometry:
  - The convex hull problem revisited: constant-time computation of the upper common tangent

First-Order Linear Recurrences

- Consider the first-order linear recurrence:

  $$y_1 = b_1$$
  $$y_i = a_i\, y_{i-1} + b_i \qquad \text{for } 2 \leq i \leq n$$

- At first sight this seems impossible to parallelize, at least in its current form
- However, note that the prefix sum

  $$y_i = \textstyle\sum_{j=1..i} b_j$$

  is a special case of a first-order linear recurrence where $a_i = \mathbf{1}$ (the multiplicative unit element)
- We know how to parallelize the prefix sum

Divide and Conquer Parallelization

- Rewrite $y_i = a_i\, y_{i-1} + b_i$ into $y_i = a_i\,(a_{i-1}\, y_{i-2} + b_{i-1}) + b_i$
- This equation defines a linear recurrence of size $n/2$ for even index i

$$z_1 = b_1'$$
$$z_i = a_i'\, z_{i-1} + b_i' \qquad 2 \le i \le n/2$$

1. Let

$$a_i' = a_{2i}\, a_{2i-1}$$
$$b_i' = a_{2i}\, b_{2i-1} + b_{2i}$$

2. Solve z_i recursively
3. For $1 \le i \le n$ set

$$\begin{array}{ll} y_i = z_{i/2} & \text{if } i \text{ is even} \\ y_i = a_i\, z_{(i-1)/2} + b_i & \text{if } i \text{ is odd} > 1 \\ y_i = b_1 & \text{if } i = 1 \end{array}$$

First-Order Linear Recurrence

Input: Arrays $B = (b_1, b_2, \dots, b_n)$ and $A = (a_1 = 0, a_2, \dots, a_n)$, $n = 2^k$
Output: The y_i values such that $y_i = a_i y_{i-1} + b_i$
begin
  if $n = 1$ **then** $y_1 := b_1$; **exit**
  for $1 \le i \le n/2$ **pardo**
    $a_i' := a_{2i} a_{2i-1}$
    $b_i' := a_{2i} b_{2i-1} + b_{2i}$
  Recursively solve the recurrence z_i defined by
      $z_1 = b_1'$ and $z_i = a_i' z_{i-1} + b_i'$ for $2 \le i \le n/2$
  for $1 \le i \le n$ **pardo**
    if i is even **then** $y_i := z_{i/2}$
    if $i > 1$ is odd **then** $y_i := a_i z_{(i-1)/2} + b_i$
    if $i = 1$ **then** $y_1 := b_1$
end

Parallel Time and Work

- From the algorithm we observe
  - $T(n) = T(n/2) + O(1)$ therefore total parallel time $T(n) = O(\log n)$
  - $W(n) = W(n/2) + O(n)$ therefore total operations $W(n) = O(n)$

Polynomial Evaluation

- We wish to evaluate the polynomial

$$p(x) = b_1 x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \ldots + b_n$$

- Two steps:

1. Use prefix sum
   - Compute the $x^{n-i} = [1, x, x^2, x^3, \ldots, x^{n-1}]$ concurrently for all i, which takes $\mathrm{O}(\log n)$ time and $\mathrm{O}(n)$ work
2. Use a tree reduction to compute the sum
   - Parallel sum $b_i x^{n-i}$ takes $\mathrm{O}(\log n)$ parallel time and $\mathrm{O}(n)$ work

Polynomial Evaluation (cont'd)

- We wish to evaluate the polynomial

$$p(x) = b_1 x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \ldots + b_n$$

- Horner's rule

$$p(x) = (((b_1\, x + b_2)\, x + b_3)\, x + \ldots + b_{n-1})\, x + b_n$$

gives a first-order linear recurrence with $a_i = x$

  - Takes $O(\log n)$ total parallel time with $O(n)$ total operations

Polynomial Multiplication

- Consider the polynomials

$$p(x) = a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + \ldots + a_{n-1}$$
$$q(x) = b_0 x^{m-1} + b_1 x^{m-2} + b_2 x^{m-3} + \ldots + b_{m-1}$$

- We wish to compute the product

$$r(x) = p(x)q(x) = c_0 x^{n+m-2} + c_1 x^{n+m-3} + c_2 x^{n+m-4} + \ldots + c_{n+m-2}$$

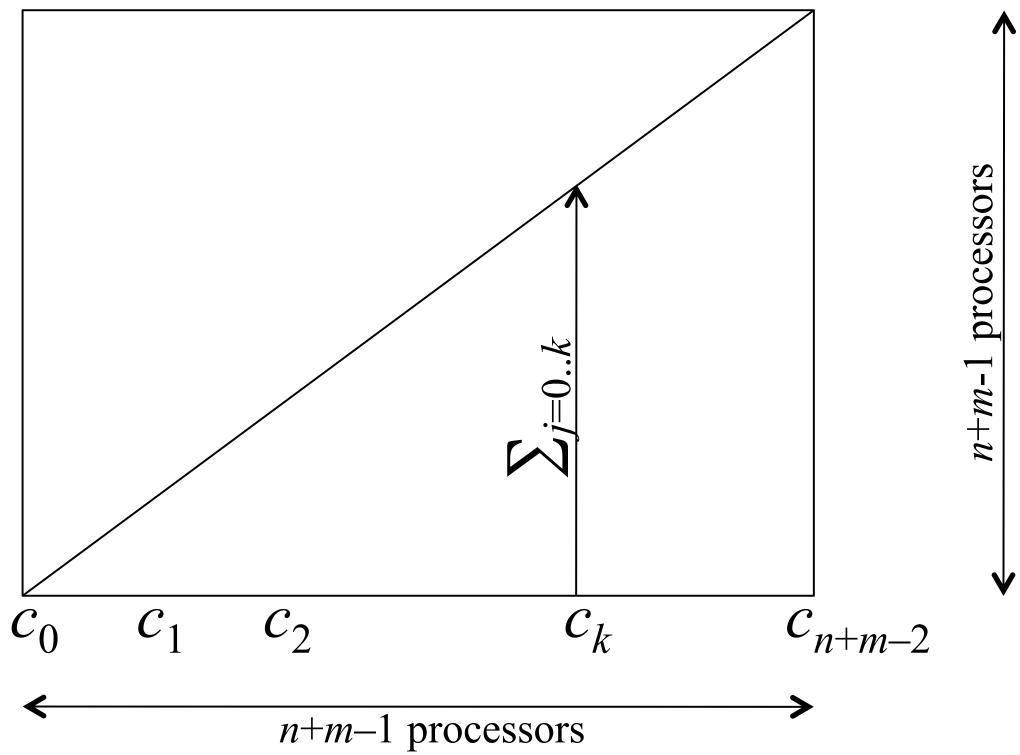
where

$$c_k = \textstyle\sum_{\mathrm{j}=0..k} a_j \, b_{k-j}$$

(we take $a_j = 0$ for $j \geq n$ and $b_{k-j} = 0$ for $k-j \geq m$)

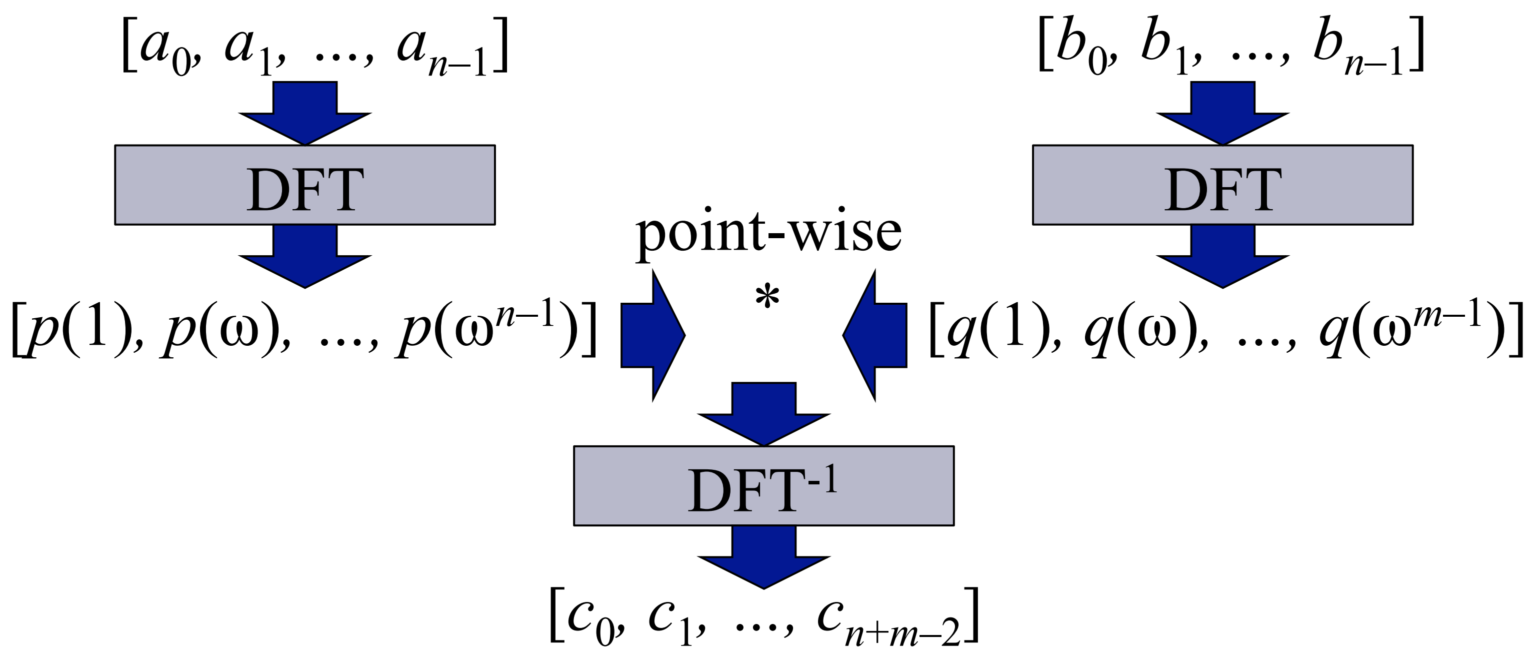
Polynomial Multiplication (cont'd)

- We can compute all $c_k = \sum_{j=0..k} a_j\, b_{k-j}$ in $O(\log(n+m))$ parallel time
  - Takes $O((n+m-1)^2/2) = O(nm)$ operations



Polynomial Multiplication & FFT

- Convolution theorem: polynomial multiplication with FFT
  - $O(\log(n+m))$ parallel time, with a simple use of the FFT algorithm reduces the total number of operations to $O((n+m)\log(n+m))$
  - The FFT of the coefficients a_i of p and a_j of q gives the values of the product polynomial $r(\omega^j) = p(\omega^j)q(\omega^j)$ at the distinct roots of unity ω^j



Polynomial Multiplication & FFT

Input: Polynomial coeff. $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$ and $\boldsymbol{b} = (b_0, b_1, \ldots, b_{m-1})$
Output: $\boldsymbol{c} = (c_0, c_1, \ldots, c_{n+m-2})$ such that $c_k = \sum_{j=0..k} a_j\, b_{k-j}$
begin
1. Find integer $l = 2^s$ such that $n + m - 2 < l \leq 2(n + m - 2)$
2. Use FFT to compute $\boldsymbol{y} = \mathrm{DFT}_l(\boldsymbol{a})$ and $\boldsymbol{z} = \mathrm{DFT}_l(\boldsymbol{b})$
3. Compute $u_j = y_j\, z_j$ for all $j = 0, \ldots, l$-1
4. Use FFT^{-1} to compute $\boldsymbol{c} = DFT_l^{-1}(\boldsymbol{u})$ giving $\boldsymbol{c} = (c_0, c_1, \ldots, c_{l-1})$

end

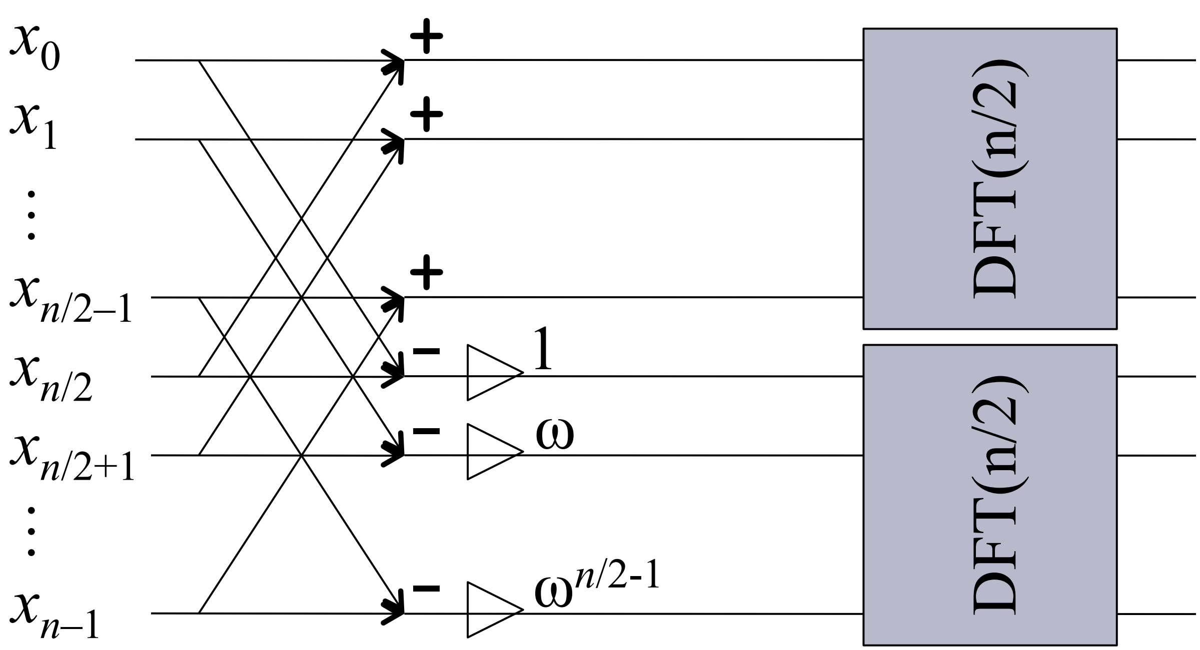
Steps 2, 4 take O(log $(n + m)$) parallel time and O($(n + m)$ log $(n + m)$) operations

Step 3 takes O(1) parallel time and O($n + m$) total operations

Note: $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ vectors are implicitly padded with 0s, e.g. $a_i = 0$ for all $i \geq n$

Parallel FFT

- The FFT is easily parallelizable, since the fast sequential algorithm reduces the $O(n^2)$ problem into a $O(n \log n)$ problem using a divide-and-conquer strategy



Parallel FFT

Input: $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$, $n = 2^k$, $\omega = e^{i2\pi/n}$, where $i = \sqrt{-1}$
Output: $\boldsymbol{y} = \mathrm{DFT}_n(\boldsymbol{x})$
begin
1. **if** $n = 2$ **then**
   $y_1 := x_1 + x_2$; $y_2 := x_1 - x_2$; **exit**
2. **for** $0 \leq j \leq n/2 - 1$ **pardo**
   $u_j := x_j + x_{n/2+j}$
   $v_j := \omega^j (x_j - x_{n/2+j})$
3. Recursively compute $\boldsymbol{z} := \mathrm{DFT}_{n/2}(\boldsymbol{u})$ and $\boldsymbol{z}' := \mathrm{DFT}_{n/2}(\boldsymbol{v})$
4. **for** $0 \leq j \leq n - 1$ **pardo**
   if j is even **then** $y_j := z_{j/2}$
   if j is odd **then** $y_j := z'_{(j-1)/2}$

end

Lagrange Interpolation

- Given a set of n points $\{(\alpha_j, \beta_j)\}_{j=0..n-1}$ determine the polynomial p of degree $n - 1$ such that for all $j = 0, \ldots, n-1$

$$p(\alpha_j) = \beta_j$$

- Lagrange interpolation specifies p as follows

$$p(x) = \sum_{j=0}^{n-1} \beta_j \frac{\prod_{l=0,l\neq j}^{n-1} (x - \alpha_j)}{\prod_{l=0,l\neq j}^{n-1} (\alpha_j - \alpha_l)}$$

Lagrange Interpolation (cont'd)

- Divide-and-conquer strategy: rearrange terms
- Define

  $q_l = x - \alpha_l$

  and

  $$Q(x) = \prod_{l=0}^{n-1} q_l = \prod_{l=0}^{n-1} (x - \alpha_j)$$

  then the derivative of Q at point α_j is

  $$Q'(\alpha_j) = \prod_{l=0,l\neq j}^{n-1} (\alpha_j - \alpha_l)$$

  which can be evaluated $\gamma_j = Q'(\alpha_j)$, $c_j = \beta_j/\gamma_j$ giving

  $$p(x) = \sum_{j=0}^{n-1} \beta_j \frac{Q(x)/(x - \alpha_j)}{Q'(\alpha_j)} = Q(x) \sum_{j=0}^{n-1} \frac{c_j}{x - \alpha_j}$$

Lagrange Interpolation (cont'd)

- A balanced tree can be used to compute the sum in

$$p(x) = Q(x)\sum_{j=0}^{n-1}\frac{c_j}{x-\alpha_j}$$

  and use FFT-based polynomial multiplication

- There is another way: note that

$$\frac{p(x)}{Q(x)} = \frac{p_{k-1,0}(x)}{Q_{k-1,0}(x)} + \frac{p_{k-1,1}(x)}{Q_{k-1,1}(x)} = \frac{p_{k-1,0}(x)Q_{k-1,1}(x) + p_{k-1,1}(x)Q_{k-1,0}(x)}{Q(x)}$$

  where

$$p_{k-1,0}(x) = Q_{k-1,0}\sum_{j=0}^{n/2-1}\frac{c_j}{x-\alpha_j} \qquad Q_{k-1,0}(x) = \prod_{j=0}^{n/2-1} q_l(x)$$

$$p_{k-1,1}(x) = Q_{k-1,1}\sum_{j=n/2}^{n-1}\frac{c_j}{x-\alpha_j} \qquad Q_{k-1,1}(x) = \prod_{j=n/2}^{n-1} q_l(x)$$

Lagrange Interpolation (cont'd)

Input: Set of pairs (α_j, β_j) for $j = 0, \ldots, n-1$, $n = 2^k$
Output: The n coefficients of $p(x) = p_{k,0}(x)$ such that $p(\alpha_j) = \beta_j$
begin

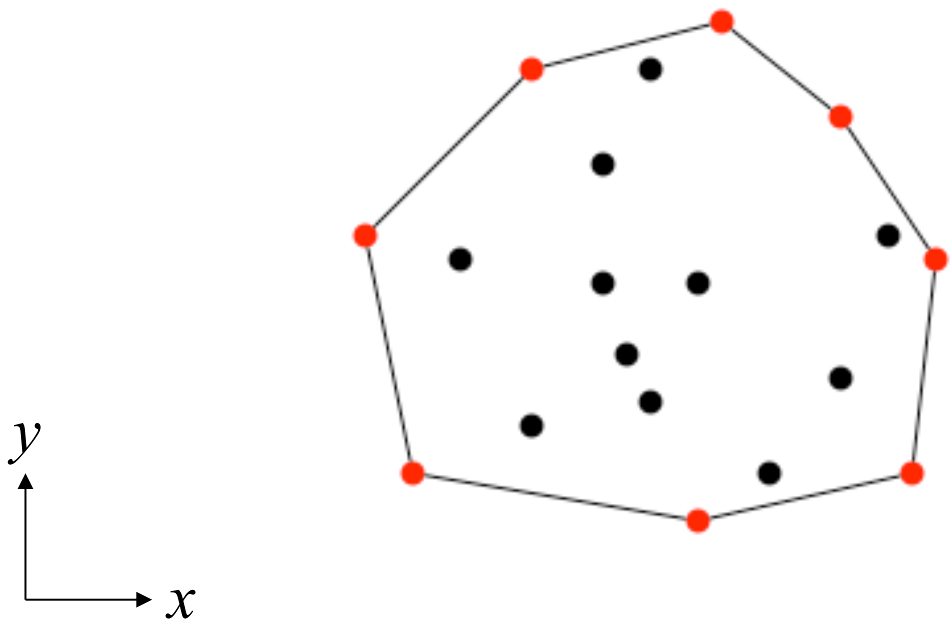
1. **for** $0 \le j \le n-1$ **pardo**
   $Q_{0,j}(x) := x - \alpha_j$
2. **for** $h = 1$ **to** $\log n$ **do**
   for $0 \le j \le n/2^h - 1$ **pardo**
   $Q_{h,j}(x) := Q_{h-1,2j}(x) \times Q_{h-1,2j+1}(x)$
3. Compute $Q'_{0,j}(x)$ and $\gamma_j := Q'_{0,j}(\alpha_j)$ for all $j = 0, \ldots, n-1$
4. **for** $0 \le j \le n-1$ **pardo**
   $p_{0,j}(x) := \beta_j/\gamma_j$
5. **for** $h = 1$ **to** $\log n$ **do**
   for $0 \le j \le n/2^h - 1$ **pardo**
   $p_{h,j}(x) := p_{h-1,2j}(x) \times Q_{h-1,2j+1}(x) + p_{h-1,2j+1}(x) \times Q_{h-1,2j}(x)$

end

Step	$T(n)$	$W(n)$
1	$O(1)$	$O(n)$
2	$O(\log^2 n)$	$O(n \log^2 n)$
3	$O(\log n)$	$O(n^2)$
4	$O(1)$	$O(n)$
5	$O(\log^2 n)$	$O(n \log^2 n)$

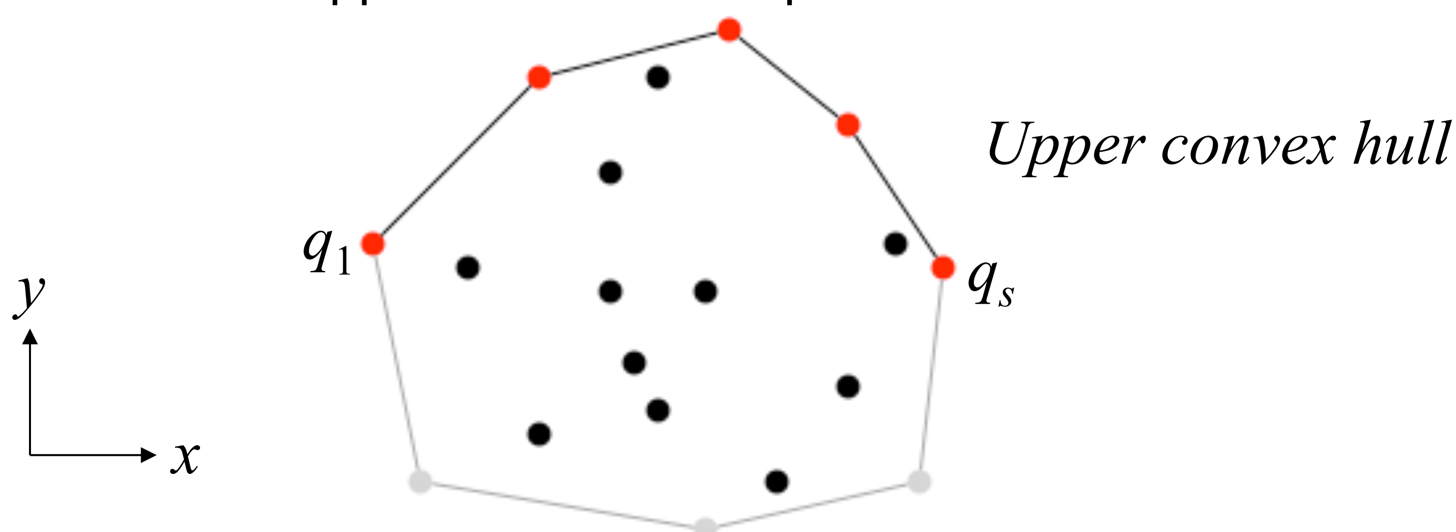
Convex Hull Problem Revisited

- The *planar convex hull* of a set of points $S = \{p_1, p_2, \ldots, p_n\}$ of $p_i = (x,y)$ coordinates is the smallest convex polygon that encompasses all points S on the x-y plane



Convex Hull Problem Revisited

- The *upper convex hull* spans points $\{q_1, \ldots, q_s\} \subseteq S$ from point q_1 with minimum x to q_s with maximum x
- The *convex hull* = *upper convex hull* + *lower convex hull*
- Problem:
  - Given points $S = \{p_1, \ldots, p_n\}$ such that $x(p_1) < x(p_2) < \ldots < x(p_n)$, construct the upper convex hull in parallel

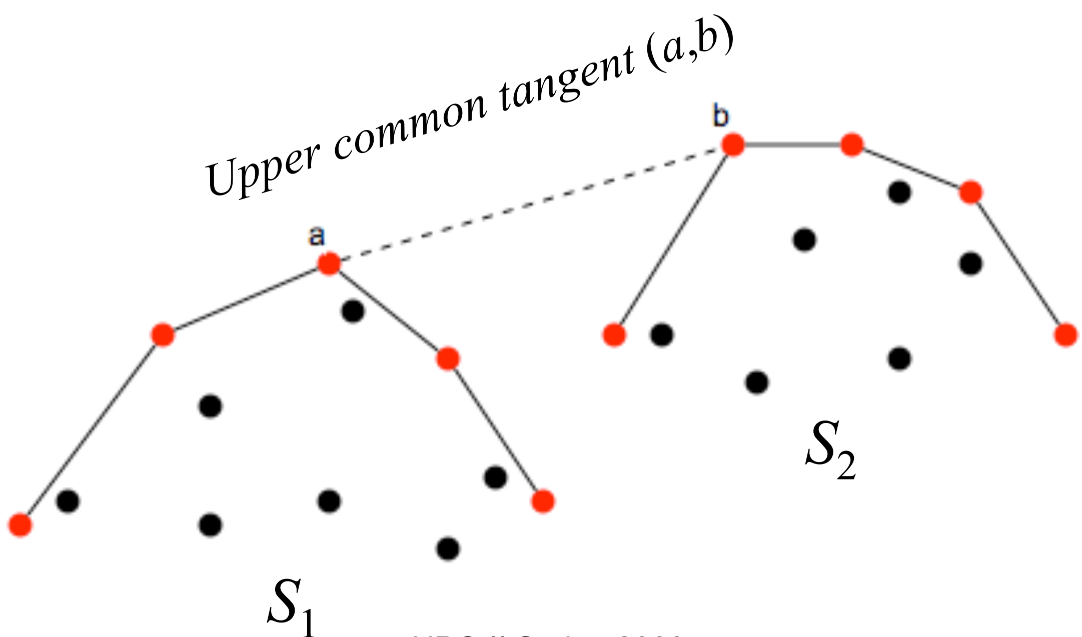


Convex Hull Problem Revisited

- Points $S = \{p_1, \ldots, p_n\}$ may have duplicate x-coordinate values
- Sort the points $x(p_1) \leq x(p_2) \leq \ldots \leq x(p_n)$ in $O(\log n)$ parallel time and $O(n \log n)$ operations (pipelined merge sort)
- Then, if two or more points have the same x coordinate:
  - Keep the point with the largest y coordinate for the UCH
  - Keep the point with the smallest y coordinate for the LCH
- We can now assume that $x(p_1) < x(p_2) < \ldots < x(p_n)$ to compute the UHS (and similarly the LHS)

Convex Hull Problem Revisited

- Parallel convex hull:
  1. Divide the x-sorted points S into sets S_1 and S_2 of equal size
  2. Compute upper convex hull recursively on S_1 and S_2
  3. Combine $\mathrm{UCH}(S_1)$ and $\mathrm{UCH}(S_2)$ by computing the upper common tangent to form $\mathrm{UCH}(S)$



Convex Hull Problem Revisited

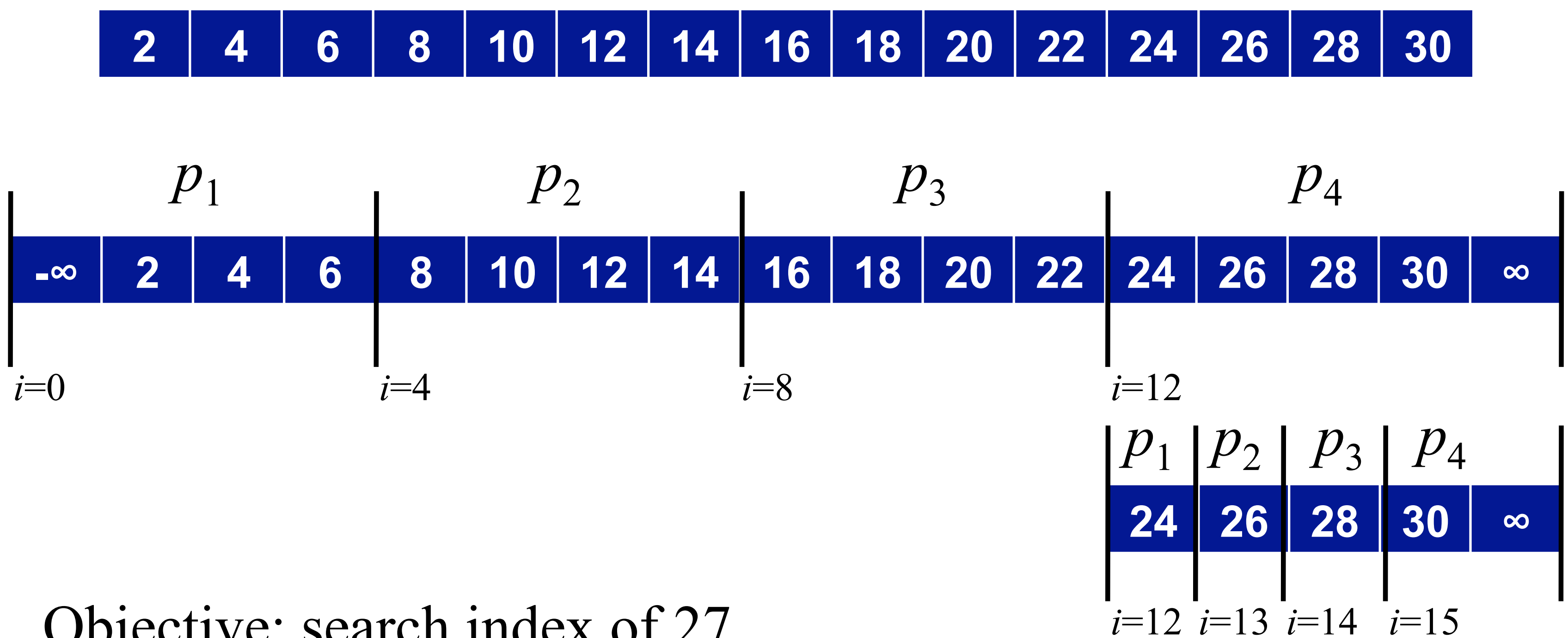
- Base case of recursion: two points, which are returned as $\mathrm{UCH}(S)$
- *Revisit the common tangent computation*:
  - The line segment (a,b) can be computed sequentially in $O(\log n)$ time with $n = |\mathrm{UCH}(S_1) + \mathrm{UCH}(S_2)|$ using a binary search method
- *And replace with parallel computation*:
  - The line segment (a,b) can be computed in $O(1)$ parallel time
- Line segments can be implemented as linked list of points, thus $\mathrm{UCH}(S_1)$ and $\mathrm{UCH}(S_2)$ can be connected using one pointer change of a to point to b in $O(1)$ time

Intermezzo: Parallel Search

- Let $X = (x_1, x_2, \ldots, x_n)$ be n distinct elements from a set S such that $x_1 < x_2 < \ldots < x_n$
- Given $y \in S$, find the index i for which $x_i \leq y < x_{i+1}$ where we added $x_0 = -\infty$ and $x_{n+1} = +\infty$

- Parallel search with p processors:
  - Split X in p segments of (almost) equal length
  - Each processor verifies if y is in its segment
  - If so, restrict search to the segment containing y and repeat

Intermezzo: Parallel Search



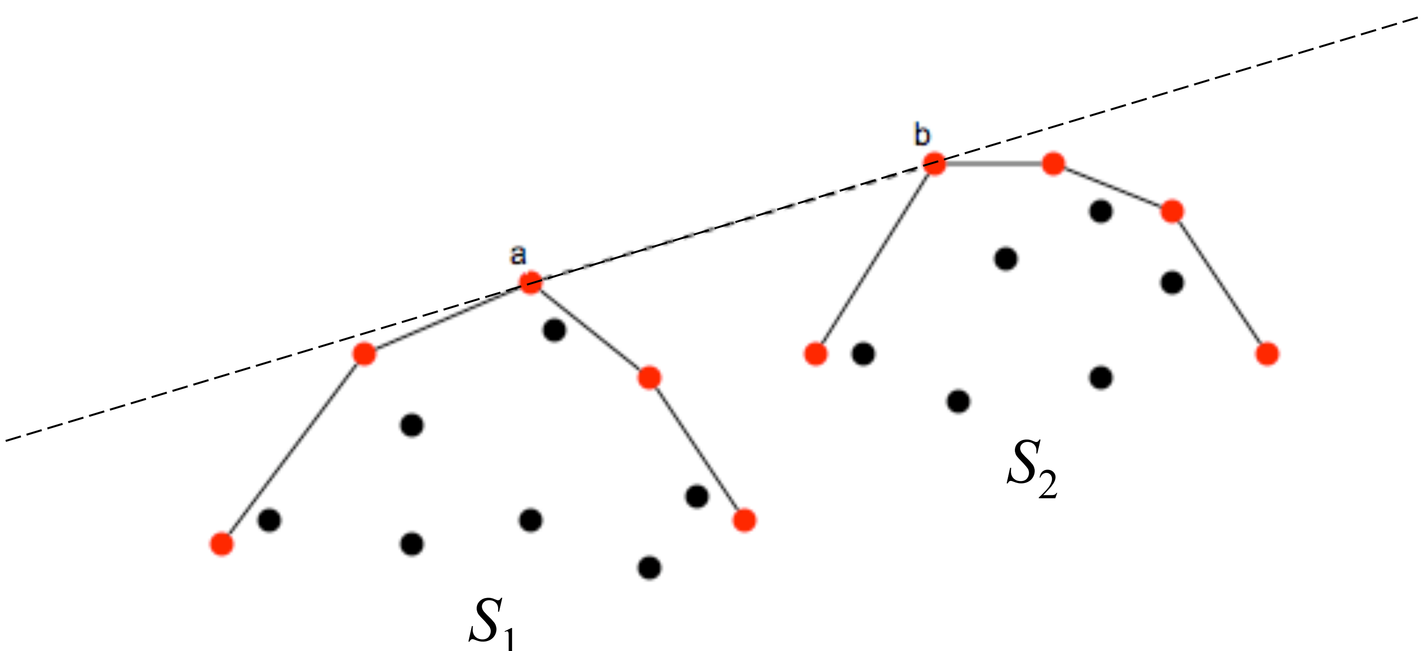
Objective: search index of 27

Parallel time = $O\left(\frac{\log(n+1)}{\log(p+1)}\right)$

p_2 found i=13

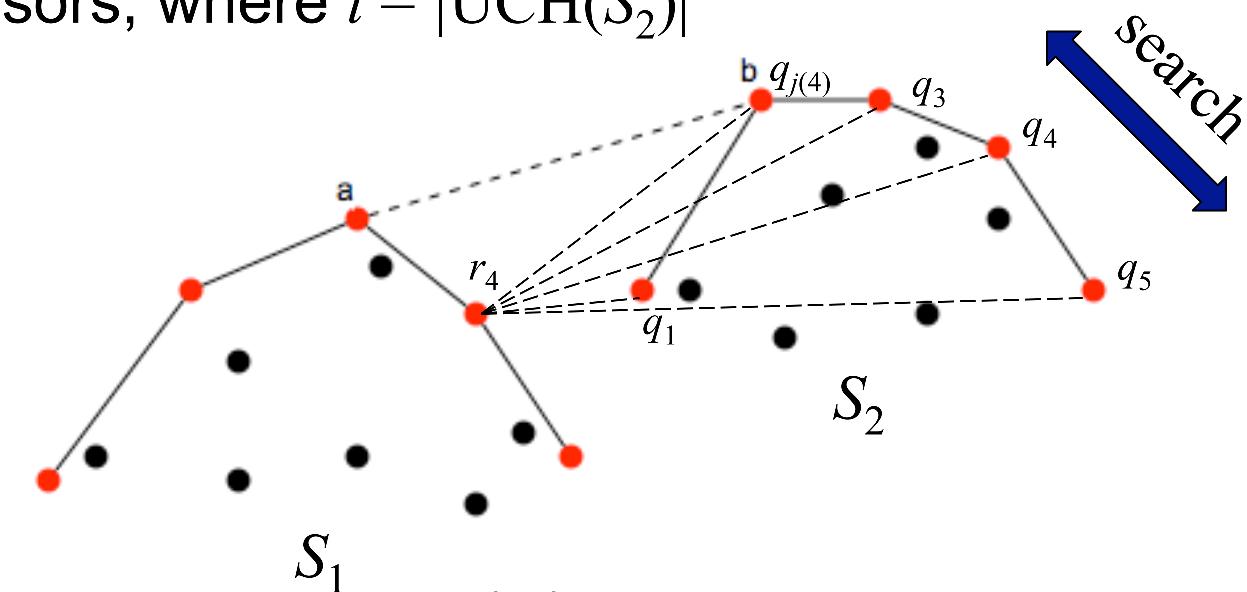
Convex Hull Problem Revisited: Using Parallel Search

- Let $\mathrm{UCH}(S_1) = (r_1, \ldots, r_s)$ and $\mathrm{UCH}(S_2) = (q_1, \ldots, q_t)$
- We need to determine points $a = r_i$ and $b = q_{j(i)}$ such that all points in S are below the line through points a and b



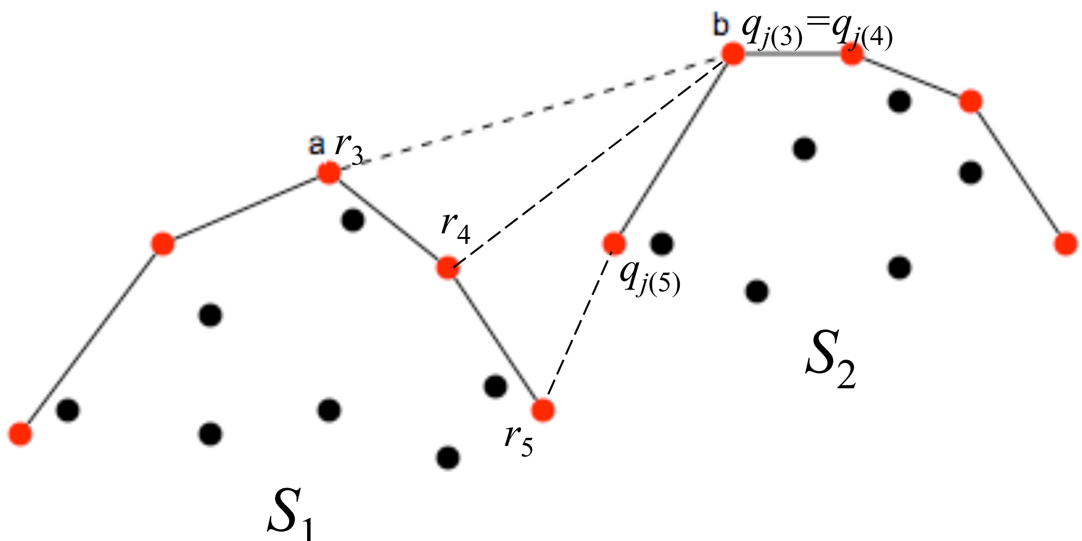
Convex Hull Problem Revisited: Using Parallel Search

- Given a point $r_i \in \mathrm{UCH}(S_1)$ then for any $q_k \in \mathrm{UCH}(S_2)$ we can determine in $\mathrm{O}(1)$ sequential time if $q_k = q_{j(i)}$, or $q_{j(i)}$ is to the left of q_k or $q_{j(i)}$ is to the right of q_k
- Thus, using parallel search, we can determine for point r_i the tangent $(r_i, q_{j(i)})$ in $\mathrm{O}(\log t / \log p)$ parallel time using p processors, where $t = |\mathrm{UCH}(S_2)|$



Convex Hull Problem Revisited: Using Parallel Search

- Given a point $r_i \in \mathrm{UCH}(S_1)$ and $q_{j(i)} \in \mathrm{UCH}(S_2)$, then we can determine in $\mathrm{O}(1)$ sequential time if $r_i = a$, or a is to the left of r_i or a is to the right of r_i
- Thus, using parallel search, we can determine the tangent (a, b) in $\mathrm{O}(\log(st) / \log p)$ parallel time using p processors, where $s = |\mathrm{UCH}(S_1)|$ and $t = |\mathrm{UCH}(S_2)|$



Convex Hull Problem Revisited: Using Parallel Search

- Take $p = \sqrt{s}\sqrt{t}$ then

  $$O(\log (st) / \log (\sqrt{s}\sqrt{t})) = O(\log (st) / \tfrac{1}{2}\log (st)) = O(1)$$

  parallel time and $O(\sqrt{s}\sqrt{t}) = O(n)$ operations

1. Choose $\sqrt{s}$ points from $\mathrm{UCH}(S_1)$ thereby dividing the set $\mathrm{UCH}(S_1)$ into (almost) equal blocks of size $\sqrt{s}$ each
2. Find the $q_{j(k\sqrt{s})}$ for each $r_{k\sqrt{s}}$, $k = 1, \ldots, \sqrt{s}$, using $p = \sqrt{s}\sqrt{t}$ processors in $O(1)$ parallel time
3. Deduce the block $B_k = (r_{k\sqrt{s}+1}, \ldots, r_{(k+1)\sqrt{s}-1})$ that contains a
4. For each r_i in block B_k, determine $q_{j(i)}$ and search $a = r_i$ using $p = \sqrt{s}\sqrt{t}$ processors in $O(1)$ parallel time
5. Set $b = q_{j(i)}$

Convex Hull Problem Revisited: Putting it Together

- Preprocess the points by sorting in $O(\log n)$ parallel time (pipelined merge sort), such that $x(p_1) \leq x(p_2) \leq \ldots \leq x(p_n)$
- Remove duplicates $x(p_i) = x(p_j)$ (for UCH and LCH)
- Divide-and-conquer:
  1. Split S into S_1 S_2 and recursively compute the UCH of S_1 and S_2
  2. Combine $\mathrm{UCH}(S_1)$ and $\mathrm{UCH}(S_2)$ by computing the upper common tangent in $O(1)$ time to form $\mathrm{UCH}(S)$
- Repeat to compute the LCH
- Parallel time (assuming $p = O(n)$ processors)

  $$T(n) = T(n/2) + O(1)$$

  gives

  $$T(n) = O(\log n)$$

Further Reading

- An Introduction to Parallel Algorithms, by J. JaJa, 1992