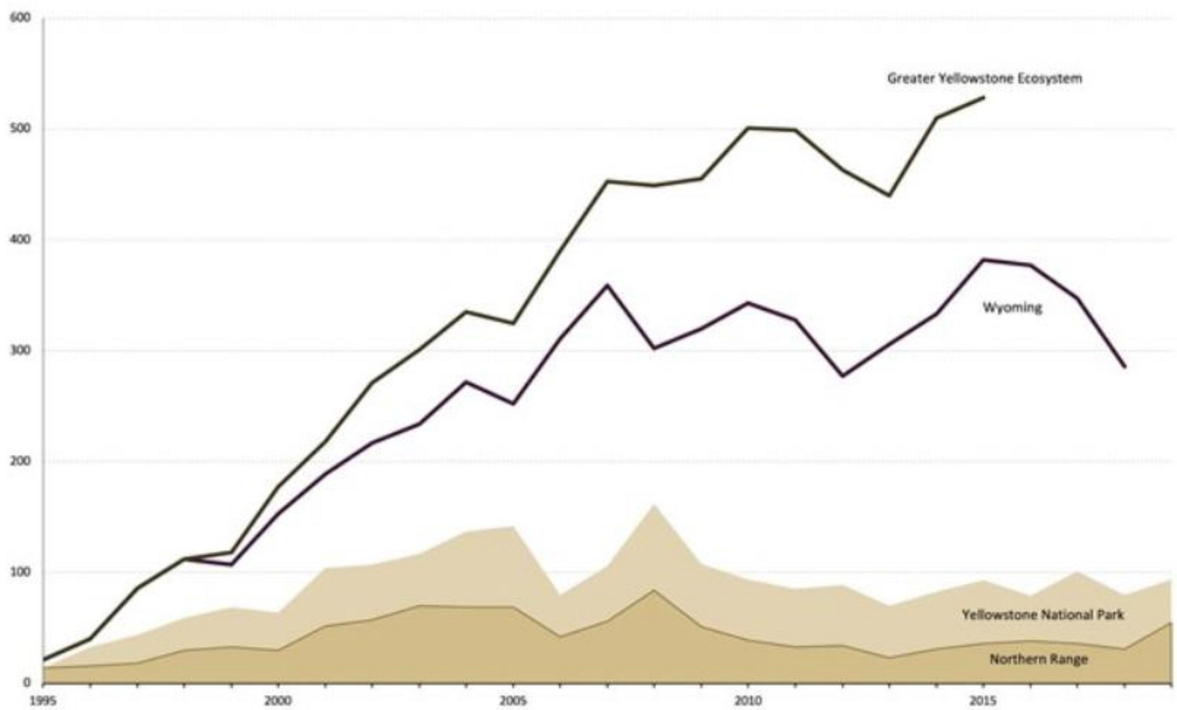
2020 Transfer Bridge Program

Math Bootcamp Project 2

# A Look at Hopf Bifurcations

In this project, you will study the Holling-Tanner predator-prey model, $N' = r_1 N\left(1 - \frac{N}{k}\right) - \frac{wN}{d+N}P$ , $P' = r_2 P\left(1 - \frac{jP}{N}\right)$. This model can exhibit equilibrium or oscillatory behavior, depending on parameter values. When a change in a parameter causes a limit cycle to form and a stable spiral equilibrium to turn into an unstable spiral, we say that a Hopf bifurcation has occurred.

In this project we will model the Yellowstone wolf population that happened between 1995-2019 as shown in the below figure. We will focus on the parameter *w*, in the Holling-Tanner model that triggered a healthy oscillatory cycle in nature. The question you want to ask is “How one could have determined the right number of wolves to release into the park to trigger the change?” We will answer this question by finding the threshold value *w* (the maximum rate at which predators can consume prey) that triggers the change.




*Greater Yellowstone wolf population, 1995–2019*

Source: https://www.nps.gov/yell/learn/nature/wolves.htm#:~:text=An%20estimated%20528%20wolves%20resided,and%20108%20wolves%20since%202009.

Watch the video:

https://www.youtube.com/watch?v=X8nyIyPZy68

## Numerical simulations

The simplest way to study the behavior of a dynamical system is just to simulate it.

**Problem 1**. (10 pts)

Make N and P symbolic variables and set the following parameters:

$r1 = 1$, $r2 = 0.1$, $k = 7$, $d = 1$, $j = 1$, $w = 0.4$. Simulate the system (show vector field) and plot a trajectory on top of it. What kind of equilibrium point is in the middle?

## Using eigenvalues to find Hopf bifurcations

Simulations are great for getting a general idea of behavior, but it's hard to use them to find the exact parameter value at which a bifurcation occurs. For this, we will rely on eigenvalues. A Hopf bifurcation occurs when the real part of a complex eigenvalue goes from negative to positive or from positive to negative. To study this, we will make heavy use of programming tools for doing symbolic calculations that would otherwise be very tedious.

First, we have to find the system's equilibrium and express them in terms of *w*. We will focus on the point at which both species have nonzero populations, since that is the one at which the bifurcation occurs.

**Problem 2.** (10 pts)

Make *w* a symbolic variable and using **solve** function, find the system's equilibria. Assign the list of equilibria to a variable and view it.

Note: You should obtain a list of several equilibria, some of which are biologically meaningful and some are not. There is only one biologically meaningful one. (Plug $w = 0.4$ in the expressions and see if you are getting the equilibrium point you see in problem 1. (Hint: Check 2$^{nd}$ element in Python, 4$^{th}$ element in Matlab))

**Problem 3.** (10 pts)

Using indexing, pull out the biologically meaningful equilibrium point and assign it to a variable. You will use it later. Make sure that the equilibrium point is a function of *w*.

We will now linearize the system around this equilibrium point and use eigenvalues to determine its stability.

**Problem 4.** (10 pts)

Using **jacobian** function and the list you defined in problem 4, find the system's Jacobian. View it and assign it to a variable.

We now need to find the eigenvalues of the Jacobian and evaluate them at the equilibrium point. Being a 2 x 2 matrix, the Jacobian has two eigenvalues but they are complex conjugates in this case (one is *a + bi*, the other is *a – bi)*. Since we are only interested in the real part, we can just focus on one of the eigenvalues.

**Problem 5.** (10 pts)

Substitute the values of the equilibrium point you found in Problem 6 into the Jacobian and assign the resulting matrix to a variable. View the resulting algebraic mess.

**Problem 6.** (10 pts)

Find the eigenvalues of the Jacobian you found in the previous exercise. Pull out one of them and assign it to a variable.

**Problem 7.** (10 pts)

Find the values of the eigenvalue at *w* = 0.4, 0.7 and 1.0.

## An algebraic shortcut

We want to examine how the real part of the eigenvalues we are studying varies with *w*. To do this, you could just modify the function you previously wrote to return just the real part of the eigenvalue. However, using such a function for multiple values of *w* turns out to be computationally intensive and therefore slow. Instead, we will use a little thought to save a lot of computation.

Recall that the eigenvalues of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are found by solving the equation $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$. Plugging this directly into the quadratic formula gives $\lambda = \frac{1}{2}\left(a + d \pm \sqrt{(-a-d)^2 - 4(ad - bc)}\right)$. Now, if $\lambda$ is complex, the imaginary part has to come from the stuff under the square root. That means that the real part is just $\frac{1}{2}(a + d)$, a considerable simplification. Even better, a+d is just the sum of the values on the main diagonal of the matrix and this quantity is common enough in math to have its own name – the *trace* of the matrix. You should be able to find a built-in function for this in the programming language you are using.

**Problem 8.** (10 pts)

Calculate 1/2 * the trace of the Jacobian and assign it to a variable. Test it on values of *w* for which you know the answer.

**Problem 9.** (10 pts)

Plot your function for values of w between 0.4 and 1.1. Be sure to label the axes.

**Problem 10.** (10 pts)

Judging by the graph, at approximately what value of *w* does the Hopf bifurcation occur? Find the exact bifurcation point by solving your equation from Problem 8. Verify your finding by showing the vector field.

**Challenge Question** (Extra Credit: 20 pts)

When the *w* value is above 8, you will see that another orbit is formed in vector space. Calculate the eigenvalue and determine the type of the equilibrium point and its location. This is called homoclinic orbit. Can you explain why this is not a healthy cycle? (Hint: What do you think you will see if you plot a time series of this orbit?)